



UNIVERSITY OF NAIROBI

**RELATIVISTIC DYNAMICS IN A MATTER-DOMINATED
FRIEDMANN UNIVERSE**

BY

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for Award of the Degree of Master of Science in Physics of the University of
Nairobi**

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Declaration

I declare that this project is my original work and has not been submitted elsewhere for examination, award of a degree or publication. Where other people's work or my own has been used, this has properly been acknowledged and referenced in accordance with the University of Nairobi's requirements.

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Dedication

I dedicate this research work to a friend to my heart, my dear loving sister Rhoda Riziki Langa (Nyojwang'o) and my unfailing friend John Odindo Anyango.

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God is ever great because He is the author of knowledge and wisdom.

Abstract

Relativistic dynamics based on the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric is considered for a matter-dominated Friedmannian universe driven by dark energy. Relevant Einstein field equations for open, closed and flat universe are derived and used to establish the relationship between various astronomical quantities, that is, light intensity-redshift and number density-redshift relations. The analytical results are prepared in a suitable form for comparison with future experimental results by making matlab computational plots for both matter-dominated Friedmann universe driven with and without dark energy. It is possible to judge from the results whether the universe is open, closed or flat and whether it is friedmann or not on large scales.

Key words: *Dark energy, Redshift, Number density, Light intensity, Friedmann, Fractal*

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List of Abbreviations/Acronyms and Symbols

ALMA	The Atacama Large Millimeter/submillimeter Array
ESA	European Space Agency
FLRW	Friedmann-Lemaitre-Robertson-Walker
IAU	International Astronomical Union
SDSS	Sloan Digital Sky Survey
SKA	Square Kilometer Array
Spitzer	Space (Infrared) Telescope
matlab	Matrix laboratory
κ	Space curvature
z	Redshift
R	Ricci scalar
λ	Cosmological constant
$R^{\mu\nu}$	Ricci curvature tensor
$T^{\mu\nu}$	Stress energy tensor
β	Beta
G	Newton's Gravitational constant
c	Speed of light
n	Number density of galaxies
I	Light intensity
$G^{\mu\nu}$	Einstein tensor
$g^{\mu\nu}$	Metric tensor
g	Friedmann metric
$\rho(t)$	Matter density of the universe
$P(t)$	Pressure of the universe
Mpc	Mega persecs

Chapter One: Introduction

In this chapter, we give an overview of the problem posed by tridimensional maps of the universe with regard to homogeneity and isotropy of the universe. We give a brief background to this problem and justify the importance of our study. We also outline clearly our overall goal and specific objectives.

1.1 Introduction

The goal of modern cosmology is to find the scale matter distribution and spacetime structure of the universe from astronomical observations. Several theories and models that explain matter distribution in the universe have been postulated. However, none of these has given an agreeable solution on the matter distribution in the universe (Marcelo & Alexandre, 1998).

Looking back to 1915, Einstein theory of gravity (Einstein, 1915) was introduced, making most of us to believe that it marked the beginning of modern cosmology. Currently, we are consumed with the authenticity of the so called Cosmological Principle which explains the aspect of the universe being the same in every point of which, mathematically, it means that the universe is homogeneous and isotropic.

We can not fail to applaud Albert Einstein with his theory of General Relativity (Einstein, 1915). In the formulation of the same theory, it marked the eye opener for the cosmologists to be able to advance their approach on comological relativistic dynamics. He predicted a general expansion of the universe from the same Theory of General Relativity (Gomez, 2011). Mathematically, the employment of Einstein Equivalence Principle could allow us to construct the metric and the equation of motion by transforming from freely-falling to an accelerating frame. It can be mathematically expressed by the assumption that all matter fields are minimally coupled to a single metric tensor, $g_{\mu\nu}$ (Marcelo, 1998). To explain the correspondence of the energy and the curvature of spacetime of an object, he used his original field equations widely known as Einstein field equations (Einstein, 1915). This original work was faced with some shortcoming of expansion or deflation which showed an almost impossible result to avoid. To our interest of the day, the work progressed by introducing the cosmological constant (Ofer Lahav, 2017) as a result of dark matter and dark energy to form part of the Einstein field equations: $R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \lambda g^{\mu\nu} = \beta T^{\mu\nu}$, where λ is the cosmological term or constant, $R^{\mu\nu}$ is the Ricci curvature tensor, R is the Ricci scalar, $T^{\mu\nu}$ is the stress energy tensor, $\beta = \frac{8\pi G}{c^4}$; G is

the gravitational constant and c is the speed of light (Einstein, 1917).

The contribution of Alexander Friedmann, de Sitter, Lemaitre and others (Lemaitre, 1925, Gomez, 2011, Friedmann, 1922) by solving Einstein field equations in a cosmological setting confirmed an expanding universe. Together with the distance-redshift relation (Hubble law), cosmological principle led us to an expansion scale which is homogeneous and isotropic. As a result of the evolved scale with time, there has been always strong arguments of whether the universe is of the homogeneity or fractality nature.

One of the oldest and most widely accepted models used for the description of matter in the universe is the Friedmann model. The model describes a dynamic and isotropic universe. In this universe the cosmological principle is implemented as follows: there is a preferred time coordinate t such that the t is equal to constant slices which are homogenous and isotropic spaces. There are basically three simply connected isotropic and homogenous spaces: the sphere (constant positive curvature), flat space (no curvature), and hyperbolic space (constant negative curvature). By means of stereographic projection, these cases can be handled in a uniform manner (Wamalwa, 2016).

Nevertheless, the whole matter-dominated Friedmannian universe rely on the particular model of the formation of structure. Such that, whenever the current surveys happen to be below the scale of homogeneity, then the distribution of the galaxies can be estimated by a fractal, just as the main redshift survey indicates (Amendola Luca, 1998). Therefore, intensive research on statistics of fractals can reveal to us much understanding of the large scale structure of the universe. We understand that SDSS is about $200Mpc$. We hope that it will go up hence better data is expected of over $200Mpc$ in future.

At around 1908, Carl Charlier (Carl Charlier, 1908) proposed to abandon the uniformity idea and settled on a hierarchical model with the stars arranged in such a way that their density seen from any star decreases with increase in distance since in his model the density decreased radially from every star. This defines and records the first fractal model of the universe.

Going with the recent catalogues already compiled which list astronomical objects in the sky, showing their distances from us, and direction from which their light reaches us, we get a

3-dimensional view of the luminous matter in the universe. Here, the distribution seems to be inhomogeneous in most of the scales (Wamalwa, 2016). This would require us to consider cosmic distance measurements. Knowing the distance of an astrophysical object from us would confirm to us the homogeneity of the universe. However, the measurements are also challenged by lots of uncertainties. Using distance ladder of measurements, it gives the small distances to larger distances and within the range we assume that the universe is Friedmannian, which is still questionable (Wamalwa, 2016).

In this project, we adopt the view that the universe is homogeneous (Friedmann) in line with the widely accepted picture of the standard model that embodies the cosmological principle. However, we shall prepare the result in a suitable form for comparison with experimental data from more accurate and reliable future galaxy surveys.

We shall describe the matter distribution in the universe based on the Friedmann-Lemaitre-Robertson-Walker (FLRW) model for describing dynamics and evolution of the universe by applying analytical methods to solve the Friedmann's equations in the presence of dark energy effects. This shall involve measurable astronomical quantities such as the redshift(z), light intensity(I) and number density(n) of galaxies.

1.2 Statement of the problem

The standard model is the simplest and widely accepted model that has successfully described structure formation in the universe in line with observations. This model pictures the universe as isotropic and homogeneous on large scales. However, as galaxy redshift surveys probe deeper into the universe, they uncover more and more inhomogeneous structures with no clear tendency to homogeneity. Such inhomogeneous structures point to a fractal universe. As a result, homogeneity and fractality of the matter-dominated universe is a major debate in cosmology today. A lot of controversies keep on streaming from various researchers on the question as to whether or not the universe is homogeneous on large scales. Nevertheless, these galaxy surveys provide only limited statistical data and are also depended on our ability to accurately measure distance. The uncertainties associated with cosmic distance measurements are huge and unresolved to date while the availability of huge observational data will wait for the next generation

advanced techniques. It is therefore the goal of this project to investigate as to whether or not our universe is homogeneous on large scale and, consequently, provide a road map for the fractal debate.

1.3 Justification

The Friedmann model embodies the cosmological principle that is widely accepted by physicists so that the idea of abandoning it would seem revolutionary to us besides creating the need to search for new models of dynamics and evolution of the universe.

The solution to our task gives us an insight into the large scale distribution of galaxies in the universe. It will also put us in a position to test as to whether or not the galaxies are really distributed homogeneously in the universe.

1.4 Overall Objective

To describe dynamics and evolution of the matter-dominated Friedmann universe by solving Einstein field equations based on FLRW.

1.4.1 Specific Objectives

1. Derive the Einstein field equations describing dynamics and evolution of the matter-dominated universe based on the Friedmann metric.
2. Explain the role of the cosmological constant in accelerated expansion of the universe.
3. Derive the relationship between light intensity and number density of galaxies with red-shift using Friedmann metric and,
4. Relate it to the type of the universe.

Chapter Two: Literature Review

In this chapter, we give a historical background to the problem as to whether or not our universe is Friedmann on large scales. We shall consider relevant Literature to our problem so as to put it into perspective. In particular, literature on fractal and homogeneous universe is discussed in detail.

2.1 Homogeneity and Fractality

The history of cosmology as a science can be compared to the construction of a pyramid that you do not know how high it will go, and whose base has to be so robust as to support it even if it is increased. Thus, gradually, cosmology is being built, where new problems arise on each move, so that the theory has to be amplified with new elements in order to achieve the explanations of the problems arisen, what usually drive us to new physical aspects. Some of the difficulties faced with the theoretical physics nowadays has something to do with cosmology, whose unresolved problems have motivated a lot of research works (Gomez, 2011).

Still up to date, there is a serious disagreement or rather controversy on the homogeneous and the fractal view of the world even though their history appear to begin almost at the same time. The first light was cast by Newton in 1692 while trying to respond to the cosmological properties of the law of gravitation (Gomez, 2011). He proposed that the system of stars resists the gravitational collapse by virtue of specially arranged some initial conditions, for instance, if all stars are equally spaced, by God's power and intervention, then each star feels the same attraction in all direction, and therefore remain stable. Later on, after further research, the model was made accurate. He proposed that all stars have equal intrinsic brightness, and that those of first magnitude lie equally spaced on a shell at a unit distance from us, those of second magnitude lie similarly on a shell at double distance, and so on through the faintest stars (Amendola Luca, 1998). This model is clearly homogeneous averaging over a few unit distances, and also more or less accounts for the relative number of stars at various magnitudes.

The first inhomogeneous model was proposed by J.H Lambert (Lambert, 1750) at around 1750. This explained about a ring of stars rotating around a central obscure body, perhaps part of a system of rings rotating around another body, and so on. Almost at the same time,

another inhomogeneous model came on board being suggested by Immanuel Kant giving details on a distribution radially decreasing of worlds, to be identified with galaxies. In other words, evolution starts from the center of galaxies to the outer parts (Amendola Luca, 1998).

In 1820, Heinrich Olbers presented the famous Olbers' paradox, already formulated by Halley in 1721. Any homogeneous, static and infinite model should be in equilibrium everywhere at the temperature of the star's surfaces, though it faces a challenge of the dark night sky (Amendola Luca, 1998).

Carl Charlier, at around 1908, proposed to abandon the homogeneity idea and settled on a hierarchical model with the stars arranged in such a way that their density seen from any star decreases with distance. Since his model the density decreased radially from every star. This recorded the first fractal model of the Universe (Carl Charlier, 1908).

The powerful action of Einstein's equations, solved in a cosmological setting by Einstein himself and by Friedmann (Friedmann, 1922), de Sitter and Lemaitre (Lemaitre, 1925) along with the discoveries of Edwin Hubble in the twenties, gave to the Cosmological Principle the status of almost a dogma among cosmologists. According to this principle, the universe is homogeneous and isotropic when averaged over some scale. This scale of homogeneity has evolved in time, from a few megaparsecs initially to a few tens of megaparsecs. The cosmological models based on the Cosmological Principle, that is, based on Friedmann equations, had tremendous success in explaining the observed facts and in anticipating new discoveries (Lemaitre, 1925, Peebles, 1989, Marcelo, 2008).

In 1980's the redshift campaigns were completed (Perseus-Pisces, CfA1), which clearly exposed the disuniformity of our Universe. This could require another scale up to about thirty megaparsecs or more from which the Friedmann's equations are supposed to hold (Marcelo, 2008).

Thereafter at around 1983, Benoit Mandelbrot brought back fractals. He introduced fractals as a mathematical tool to investigate the properties of a vast class of phenomena. In 70's, he proposed that even the distribution of galaxies may follow a fractal law, that is density scaling with distance. Then, the power law decrease of the correlation function with distance was proposed in particular by Peebles (Peebles, 1989). The reasoning was drawn from the 2-point

angular correlation function (Marcelo, 2008). This could be rejected later when it was realized that the small amplitude of the angular correlation function proved large scale homogeneity.

Later Pietronero and coworkers (Pietronero, 1987) addressed the criticisms based on the angular correlation function whereby they revived the fractal hypothesis after shifting the ideas to redshift surveys. It showed strong inhomogeneities and density scaling. However, still fractality scaled up also up to tens of megaparsecs.

Nevertheless, a major concern still piles up. There is a claim that the large scale distribution of matter or galaxies in the Universe is inhomogeneous, from the smallest to the largest observed scales and perhaps indefinitely (Marcelo, 1998).

However, the principal endeavor to build a reasonable cosmological model in the frame of General Relativity depended on an altogether different thought from what we think now, it was accepted on a static universe with no start and end. With a specific end goal to accomplish such sort of question, Einstein presented, surprisingly, the cosmological constant in his field equation, which yields, under specific conditions, a temperamental static universe. In the meantime, Willem de Sitter realized an expanding cosmology with the presence of a cosmological constant, however, this sort of cosmology was disregarded along the years. Later in 1922, Alexander Friedmann found an answer of the Einstein's field equation that proposes an expanding universe. Simultaneously, Georges Lemaitre proposed, surprisingly, a creation occasion as the start of the universe extension, being the primary model that later would be known as the Big Bang model, and proposed the distance-redshift relation that would clarify the development and expansion of the space (Einstein, 1915, Lemaitre, 1998).

These proposition, together with the metric given by Howard Percy Robertson and Arthur Geoffrey Walker (Gomez, 2011), give the name of what we know these days as the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric or cosmology. We trust it is the best fit to depict our universe advancement. They freely demonstrated a universe with the premise that the matter in it would be observed to be uniform from anywhere in it.

2.2 Review on Cosmic Distance Measurements

In 1929, Edwin Hubble proposed the relation between the distance and redshift by utilizing the observational data accounted by Vesto Slipher on cosmic systems spectra a few years prior, which he demonstrated the reality of the expansion of the universe.

Astronomers tirelessly research to know the distances for the farthest possible galaxies from us and their distribution in the universe. The European Space Agency (ESA) in 1989 were able to bring on board Hipparcos satellite (Heck & Caputo, 1999). This enabled the first star chart to be formed allowing the astronomers to determine stellar distances out of several hundred parsecs.

IAU symposium 289 addressed the physics underlying methods of distance determination across the universe, exploring the various approaches involved and acknowledging that the controversy has, thus, not been solved, and all methods applied to date are affected by their own unique sets of uncertainties (Richard, 2012).

Sloan Digital Sky Survey(SDSS) which constructs the largest map of the universe only maps about a quarter of the sky in 3-dimension (Kazuhiro,2003). However, this volume-limited latest SDSS data reveal clustering of galaxies that point towards inhomogeneities even to the largest scales (Joyce et al; 2005).

The assumption is that such a challenge can only be tackled with huge and direct observational data that does not involve distance measurements. With that, we may be in a position to tell the nature of the universe, that is, if the distribution of galaxies lies with the Friedmann universe (Wamalwa, 2016).

These models of accelerated cosmological expansion always raise a variety of interesting mathematical questions. Therefore, we lay the basis for responding to this challenge by developing a functional relationships between light intensity(I) and redshift(z) as well as between number density(n) and redshift(z) in the presence of dark matter and dark energy, that is, including cosmological constant, in our mathematical equations. The analytical theoretical results are prepared in a suitable form for comparison with experimental results by making matlab computational plots.

Chapter Three: Methodology

In this chapter, we clearly outline the method that we shall use to solve our problem. In particular, we assume that the Friedmann model is correct and therefore proceed to describe dynamics and evolution of the universe basen on FLRW metric. To do this, we derive and solve Einstein's field equations based on the Friedmann metric. This is rendered possible by developing a functional relationship between light intensity (I) and redshift (z) in the presence of dark energy. We further derive number density-redshift relation on the platform of this metric.

3.1 Einstein Field Equations based on Friedmann Metric

Einstein's field equations based on Friedmann metric are considered in this section.

Consider the second rank covariant metric in matrix form:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

The terms in the main diagonal are nonvanishing, otherwise the rest vanish, that is,

$$g_{\mu\nu} = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & \frac{-R(t)^2}{(1+\kappa r^2)^2} & 0 & 0 \\ 0 & 0 & \frac{-R(t)^2}{(1+\kappa r^2)^2} & 0 \\ 0 & 0 & 0 & \frac{-R(t)^2}{(1+\kappa r^2)^2} \end{pmatrix}$$

so that

$$g = c^2 dt^2 - \frac{R(t)^2}{(1 + \kappa r^2)^2} (dx^2 + dy^2 + dz^2) \quad (3.1)$$

Equation (3.1) is the Friedmann-Lemaitre-Robertson-Walker metric that we shall use in this work with

$$g_{00} = c^2 \quad (3.2)$$

and

$$g_{11} = -\frac{R(t)^2}{(1 + \kappa r^2)^2} = g_{22} = g_{33} \quad (3.3)$$

Raising of the indices in equations (3.2) and (3.3) above gives

$$g^{00} = \frac{1}{c^2} \quad (3.4)$$

and

$$g^{11} = -\frac{(1 + \kappa r^2)^2}{R(t)^2} = g^{22} = g^{33} \quad (3.5)$$

respectively.

We now need the curvature scalar and components of the Ricci tensor.

Let us, therefore, consider non-vanishing values of the Ricci tensor, that is, R_{11} , R_{22} , R_{33} and R_{00} with their respective raised indices as already derived (Wamalwa, 2016) as

$$R_{00} = -\frac{3R''(t)}{R(t)}$$

$$R_0^0 = -\frac{3R''(t)}{c^2 R(t)} \quad (3.6)$$

$$R^{00} = -\frac{3R''(t)}{c^4 R(t)} \quad (3.7)$$

and also

$$R_{11} = R_{22} = R_{33} = \frac{R(t)R''(t) + 2R'(t)^2 + 8\kappa c^2}{c^2(1 + \kappa r^2)^2} \quad (3.8)$$

$$R_1^1 = \frac{R(t)R''(t) + 2R'(t)^2 + 8\kappa c^2}{c^2 R(t)^2} = R_2^2 = R_3^3 \quad (3.9)$$

$$R^{11} = \frac{8\kappa c^2 + R(t)R''(t) + 2R'(t)^2}{c^2 R(t)^4} (1 + \kappa r^2)^2 = R^{22} = R^{33} \quad (3.10)$$

Therefore, let us now express the curvature scalar as

$$R = R_u^u = (R_0^0, R_1^1, R_2^2, R_3^3) \quad (3.11)$$

so that on using equations (3.6) and (3.9), we obtain

$$R = R_u^u = -\frac{3(8\kappa c^2 + 2R(t)R''(t) + 2R'(t)^2)}{c^2 R(t)^2} \quad (3.12)$$

We are interested in describing the dynamics and evolution of the universe considering the effects of dark energy hence, we consider the Einstein field equations with cosmological constant (Einstein, 1917) as

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \lambda g^{\mu\nu} = \beta T^{\mu\nu} \quad (3.13)$$

where $\beta = \frac{8\pi G}{c^4}$, G is the gravitational constant, λ is the cosmological constant, $g^{\mu\nu}$ is the metric tensor, $R^{\mu\nu}$ is the Ricci tensor, R is Ricci scalar, $T^{\mu\nu}$ is the stress-energy tensor of the matter content in the universe and c is the speed of light. This matter content in the universe must be uniformly distributed if the universe is homogeneous. Furthermore, the matter content must be at rest with respect to the coordinates otherwise direction of velocity would break isotropy of the universe.

Just like Ricci tensor components above, the stress-energy tensor also has the components with raised indices (Wamalwa, 2016) in the form:

$$T^{00} = \rho(t) \quad (3.14)$$

and

$$T^{11} = T^{22} = T^{33} = \frac{(1 + \kappa r^2)^2}{R(t)^2} P(t) \quad (3.15)$$

where $\rho(t)$ and $P(t)$ are mass density and pressure of the universe respectively.

Equation (3.13) can be rewritten as

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = \beta T^{\mu\nu} - \lambda g^{\mu\nu} \quad (3.16)$$

The stress energy tensor is related to the Einstein tensor by the Einstein equation:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = \beta T^{\mu\nu} - \lambda g^{\mu\nu} \quad (3.17)$$

In the next chapter, this equation will yield our specific Einstein field equations for describing dynamics and evolution in a matter-dominated universe.

Chapter Four: Results and Discussion

In the last chapter, we introduced our method adopted for use in this work and wrote down general Einstein field equations. We now proceed to derive Einstein field equations of dynamics and evolution of the matter-dominated universe.

For $\mu = \nu = 0$, equation (3.17) becomes

$$G^{00} = R^{00} - \frac{1}{2}Rg^{00} = \beta T^{00} - \lambda g^{00} \quad (4.1)$$

Substituting equations (3.4), (3.7), (3.12) and (3.14) into equation (4.1), it yields

$$G^{00} = \frac{-3R''(t)R(t)}{c^4 R(t)^2} + \frac{12\kappa c^2 + 3R'(t)^2}{c^4 R(t)^2} + \frac{3R''(t)R(t)}{c^4 R(t)^2} = \beta \rho(t) - \frac{\lambda}{c^2}$$

or

$$12\kappa c^2 + 3R'(t)^2 = \beta c^4 R(t)^2 \rho(t) - \lambda c^2 R(t)^2 \quad (4.2)$$

Similarly, for $\mu = \nu = 1, 2, 3$, we have

$$G^{11} = R^{11} - \frac{1}{2}Rg^{11} = \beta T^{11} - \lambda g^{11} = G^{22} = G^{33} \quad (4.3)$$

We then apply equations (3.5), (3.10), (3.12) and (3.15) in equation (4.3) to get

$$G^{11} = \frac{(8\kappa c^2 + R(t)R''(t) + 2R'(t)^2)}{c^2 R(t)^4} - \frac{(12\kappa c^2 + 3R(t)R''(t) + 3R'(t)^2)}{c^2 R(t)^4} = \frac{\beta P(t)}{R(t)^2} + \frac{\lambda}{R(t)^2}$$

or

$$4\kappa c^2 + 2R(t)R''(t) + R'(t)^2 = -\beta c^2 R(t)^2 P(t) - \lambda c^2 R(t)^2 \quad (4.4)$$

Equations (4.2) and (4.4) form our main equations for describing the dynamics and evolution of the universe. They are more general than field equations obtained earlier (Wamalwa, 2016) while ignoring the effects of dark energy. Clearly, these equations reduce to familiar equations without dark energy effects when we let $\lambda = 0$. We now proceed to obtain a conservation law based on these equations.

4.1 Coservation law

We begin by differentiating equation (4.2) with respect to t which gives

$$6R'(t)R''(t) = 2\beta c^4 R(t)R'(t)\rho(t) + \beta c^4 R(t)^2 \rho'(t) - 2\lambda c^2 R(t)R'(t)$$

where $R''(t) = \frac{dR'(t)}{dt}$ and $R'(t) = \frac{dR(t)}{dt}$

Multiply through by $R(t)$ to get

$$6R(t)R'(t)R''(t) = 2\beta c^4 R(t)^2 R'(t)\rho(t) + \beta c^4 R(t)^3 \rho'(t) - 2\lambda c^2 R(t)^2 R'(t) \quad (4.5)$$

Then let us multiply equation (4.4) by 3 to obtain

$$12\kappa c^2 + 6R(t)R''(t) + 3R'(t)^2 = -3\beta c^2 R(t)^2 P(t) - 3\lambda c^2 R(t)^2$$

Rearranging, it yields

$$6R(t)R''(t) = -(12\kappa c^2 + 3R'(t)^2) - 3\beta c^2 R(t)^2 P(t) - 3\lambda c^2 R(t)^2 \quad (4.6)$$

Applying equation (4.2) in equation (4.6), we get

$$6R(t)R''(t) = -\beta c^4 R(t)^2 \rho(t) + \lambda c^2 R(t)^2 - 3\beta c^2 R(t)^2 P(t) - 3\lambda c^2 R(t)^2$$

Multiplying this equation by $R'(t)$ gives

$$6R(t)R'(t)R''(t) = -\beta c^4 R'(t)R(t)^2 \rho(t) - 3\beta c^2 R'(t)R(t)^2 P(t) - 2\lambda c^2 R'(t)R(t)^2 \quad (4.7)$$

Subtracting equation (4.7) from equation (4.5), yields

$$3c^2 R'(t)R(t)^2 \rho(t) + c^2 R(t)^3 \rho'(t) = -3R'(t)R(t)^2 P(t)$$

or

$$\frac{d}{dt}(c^2 \rho(t) R(t)^3) = -P(t) \frac{d}{dt} R(t)^3 \quad (4.8)$$

Let us consider a pressureless matter-dominated universe so that $P(t) = 0$. Equation (4.2) becomes

$$\frac{d}{dt}(c^2 \rho(t) R(t)^3) = 0$$

or

$$\rho(t) R(t)^3 = \alpha \tag{4.9}$$

where α is a constant.

Equation (4.2) can be rewritten as

$$12\kappa c^2 + 3R'(t)^2 = \frac{\beta c^4 R(t)^3 \rho(t)}{R(t)} - \frac{\lambda c^2 R(t)^3 \rho(t)}{R(t) \rho(t)}$$

so that upon using equation (4.9), we obtain

$$12\kappa c^2 + 3R'(t)^2 = \frac{\beta\alpha c^4}{R(t)} - \frac{\lambda\alpha c^2}{R(t)\rho(t)}$$

or

$$4\kappa c^2 + R'(t)^2 = \frac{\beta c^4 \alpha}{3R(t)} - \frac{\lambda \alpha c^2}{3R(t)\rho(t)}$$

or

$$R'(t)^2 = \frac{\beta c^4 \alpha}{3R(t)} - \frac{\lambda \alpha c^2}{3R(t)\rho(t)} - 4\kappa c^2$$

which also can be expressed as

$$\frac{dR}{dt} = \sqrt{\frac{\beta c^4 \alpha}{3R(t)} - \frac{\lambda \alpha c^2}{3R(t)\rho(t)} - 4\kappa c^2}$$

so that

$$dt = \frac{dR}{\sqrt{\frac{\beta c^4 \alpha}{3R(t)} - \frac{\lambda \alpha c^2}{3R(t)\rho(t)} - 4\kappa c^2}} \quad (4.10)$$

Equation (4.10) has an additional term $(\lambda\alpha c^2/3R(t)\rho(t))$ carrying the effect of dark energy which is usually absent in Einstein Field equations without the cosmological constant.

4.2 Light intensity-redshift relation

Suppose light from an astronomical object starts at $r(t_e)$ and travels towards the origin such that at time $t = t_o$, it reaches the origin ($r(t_o) = 0$). The Friedmann metric (see equation (3.1)), can be rewritten for null geodesics as

$$0 = c^2\dot{t}^2 - \frac{R(t)^2}{(1 + \kappa r^2)^2}\dot{r}^2$$

where $dt^2 = \dot{t}^2$ and $dr^2 = \dot{r}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$

$$c^2\dot{t}^2 = \frac{R(t)^2\dot{r}^2}{(1 + \kappa r^2)^2}$$

The square root of the above equation gives

$$c\dot{t} = \pm \frac{R(t)\dot{r}}{1 + \kappa r^2} \quad (4.11)$$

By assumption, \dot{t} is positive while \dot{r} is negative. Therefore,

$$cdt = -\frac{R(t)}{1 + \kappa r^2} dr$$

It also can be expressed as

$$\frac{c}{R(t)} dt = -\frac{1}{1 + \kappa r^2} dr \quad (4.12)$$

Performing integration of equation (4.12) over (t_e, t_o) and $(r(t_e), r(t_o))$, we get

$$\int_{t_e}^{t_o} \frac{c}{R(t)} dt = -\int_{r(t_e)}^{r(t_o)} \frac{1}{1 + \kappa r^2} dr \quad (4.13)$$

Substituting equation (4.10) into equation (4.13) yields

$$\int_{R(t_e)}^{R(t_o)} \frac{c\sqrt{R}dR}{R(t)\sqrt{\frac{\beta c^4 \alpha}{3} - \frac{\lambda \alpha c^2}{3\rho(t)} - 4\kappa c^2 R}} = -\int_{r(t_e)}^{r(t_o)} \frac{1}{1 + \kappa r^2} dr$$

or

$$\int_{R(t_e)}^{R(t_o)} \frac{dR}{\sqrt{R} \sqrt{\frac{\beta c^2 \alpha \rho(t) - \lambda \alpha}{3\rho(t)} - 4\kappa R(t)}} = - \int_{r(t_e)}^{r(t_o)} \frac{1}{1 + \kappa r^2} dr \quad (4.14)$$

This equation is more general and suitable for describing dynamics and evolution of the universe as it contains an extra cosmological term on its L.H.S earlier present in equation (4.10). We can solve equation (4.14) for three different cases of κ , that is, $\kappa = -1$ (open universe), $\kappa = 0$ (flat universe) and $\kappa = 1$ (closed universe).

4.2.1 Flat Universe ($\kappa = 0$)

Equation (4.14) becomes

$$- \int_{r(t_e)}^{r(t_o)} dr = \int_{R(t_e)}^{R(t_o)} \frac{dR}{\sqrt{R} \sqrt{\frac{\beta c^2 \alpha \rho - \lambda \alpha}{3\rho(t)}}}$$

This equation can be integrated as follows:

$$r(t_e) - r(t_o) = \int_{R(t_e)}^{R(t_o)} \frac{R^{-\frac{1}{2}}}{\sqrt{\frac{\beta c^2 \alpha \rho - \lambda \alpha}{3\rho(t)}}} dR$$

$$= \frac{1}{\sqrt{\frac{\beta c^2 \alpha \rho - \lambda \alpha}{3\rho(t)}}} \int_{R(t_e)}^{R(t_o)} R^{-\frac{1}{2}} dR$$

$$= \frac{1}{\sqrt{\frac{\beta c^2 \alpha \rho - \lambda \alpha}{3\rho(t)}}} \frac{R^{\frac{1}{2}}}{\frac{1}{2}} \Big|_{R(t_e)}^{R(t_o)}$$

$$r(t_e) - r(t_o) = \sqrt{R} \sqrt{\frac{12\rho(t)}{\beta c^2 \alpha \rho - \lambda \alpha}} \Big|_{R(t_e)}^{R(t_o)}$$

$$= \frac{\sqrt{12\rho(t)R}}{\sqrt{\beta c^2 \alpha \rho(t) - \lambda \alpha}} \Big|_{R(t_e)}^{R(t_o)}$$

$$r(t_e) - r(t_o) = \frac{\sqrt{12\rho(t)R(t_o)}}{\sqrt{\beta c^2 \alpha \rho(t) - \lambda \alpha}} - \frac{\sqrt{12\rho(t)R(t_e)}}{\sqrt{\beta c^2 \alpha \rho(t) - \lambda \alpha}}$$

Now using the relation

$$R(t_e) = \frac{R(t_o)}{1+z} \quad (4.15)$$

and setting $r(t_o) = 0$, we obtain

$$r(t_e) = \sqrt{\frac{12\rho(t)R(t_o)}{\beta c^2 \alpha \rho(t) - \lambda \alpha}} - \sqrt{\frac{12\rho(t)R(t_o)}{(\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z)}} \quad (4.16)$$

4.2.2 Closed Universe ($\kappa=1$)

Equation (4.14) now becomes

$$-\int_{r(t_e)}^{r(t_o)} \frac{1}{1+r^2} dr = \int_{R(t_e)}^{R(t_o)} \frac{dR}{\sqrt{R} \sqrt{\frac{\beta c^2 \alpha \rho(t) - \lambda \alpha}{3\rho(t)} - 4R}} \quad (4.17)$$

Equation (4.17) can also be integrated as follows:

$$\tan^{-1} r \Big|_{r(t_e)}^{r(t_o)} = \int_{R(t_e)}^{R(t_o)} \frac{dR}{\sqrt{R} \sqrt{\frac{\beta c^2 \alpha \rho(t) - \lambda \alpha}{3\rho(t)} \left(1 - \frac{12\rho(t)R}{\beta c^2 \alpha \rho(t) - \lambda \alpha}\right)}}$$

$$\tan^{-1} r(t_e) = \int_{R(t_e)}^{R(t_o)} \frac{dR}{\sqrt{R} \sqrt{\frac{\beta c^2 \alpha \rho(t) - \lambda \alpha}{3\rho(t)} \left(1 - \frac{12\rho(t)R}{\beta c^2 \alpha \rho(t) - \lambda \alpha}\right)}}$$

$$\tan^{-1} r(t_e) = \int_{R(t_e)}^{R(t_o)} \frac{dR}{\sqrt{R} \sqrt{\frac{\beta c^2 \alpha \rho(t) - \lambda \alpha}{3\rho(t)}} \sqrt{1 - \frac{12\rho(t)R}{\beta c^2 \alpha \rho(t) - \lambda \alpha}}} \quad (4.18)$$

Now let $\frac{12\rho(t)R}{\beta c^2 \alpha \rho(t) - \lambda \alpha} = \sin^2 \theta$

Such that $R = \left(\frac{\beta c^2 \rho(t) \alpha - \lambda \alpha}{12\rho(t)} \right) \sin^2 \theta$

Thus, $\sqrt{R} = \sqrt{\frac{\beta c^2 \rho(t) \alpha - \lambda \alpha}{12\rho(t)}} \sin \theta$

and $dR = \frac{\beta c^2 \rho(t) \alpha - \lambda \alpha}{12\rho(t)} 2 \sin \theta \cos \theta d\theta$

Therefore,

$$\sqrt{1 - \frac{12\rho(t)R}{\beta c^2 \alpha \rho(t) - \lambda \alpha}} = \sqrt{1 - \sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta \quad (4.19)$$

Performing substitution by using above equations, equation (4.18) becomes

$$\begin{aligned}
\tan^{-1} r(t_e) &= \int_{\theta(t_e)}^{\theta(t_o)} \frac{2 \sin \theta \cos \theta \left(\frac{\beta c^2 \rho(t) \alpha - \lambda \alpha}{12 \rho(t)} \right)}{\sqrt{\frac{\beta c^2 \rho(t) \alpha - \lambda \alpha}{12 \rho(t)}} \sin \theta \sqrt{\frac{\beta c^2 \rho(t) \alpha - \lambda \alpha}{3 \rho(t)}} \cos \theta} d\theta \\
&= \int_{\theta(t_e)}^{\theta(t_o)} \frac{2(\beta c^2 \rho(t) \alpha - \lambda \alpha) 6 \rho(t)}{12 \rho(t) (\beta c^2 \rho(t) \alpha - \lambda \alpha)} d\theta \\
&= \theta \Big|_{\theta(t_e)}^{\theta(t_o)}
\end{aligned} \tag{4.20}$$

or

$$\theta = \sin^{-1} \sqrt{\frac{12 \rho(t) R}{\beta c^2 \rho(t) \alpha - \lambda \alpha}} \tag{4.21}$$

Therefore,

$$\tan^{-1} r(t_e) = \sin^{-1} \sqrt{\frac{12\rho(t)R}{\beta c^2 \alpha \rho(t) - \lambda \alpha}} \Big|_{R(t_e)}^{R(t_o)} = \sin^{-1} \sqrt{\frac{12\rho(t)R(t_o)}{\beta c^2 \alpha \rho(t) - \lambda \alpha}} - \sin^{-1} \sqrt{\frac{12c^2 \rho(t)R(t_e)}{\beta c^2 \alpha \rho(t) - \lambda \alpha}} \quad (4.22)$$

Applying equation (4.15), we obtain

$$\tan^{-1} r(t_e) = \sin^{-1} \sqrt{\frac{12\rho(t)R(t_o)}{\beta c^2 \alpha \rho(t) - \lambda \alpha}} - \sin^{-1} \sqrt{\frac{12\rho(t)R(t_o)}{(\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z)}} \quad (4.23)$$

Taking the tangent of both sides of equation (4.23), we use the identity

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \pm \tan A \tan B}$$

to get

$$r(t_e) = \frac{\tan \sin^{-1} \sqrt{\frac{12\rho(t)R(t_o)}{\beta c^2 \alpha \rho(t) - \lambda \alpha}} - \tan \sin^{-1} \sqrt{\frac{12\rho(t)R(t_o)}{(\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z)}}}{1 + \tan \sin^{-1} \sqrt{\frac{12\rho(t)R(t_o)}{\beta c^2 \alpha \rho(t) - \lambda \alpha}} \tan \sin^{-1} \sqrt{\frac{12\rho(t)R(t_o)}{(\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z)}}} \quad (4.24)$$

Further use of the identities;

$$\tan = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1 - \sin^2\theta}},$$

Equation (4.24) can, therefore, be rewritten as

$$r(t_e) = \frac{\sqrt{\frac{12\rho(t)R(t_o)}{\beta c^2 \rho(t)\alpha - \lambda\alpha}} \frac{1}{\sqrt{1 - \frac{12\rho(t)R(t_o)}{\beta c^2 \rho(t)\alpha - \lambda\alpha}}} - \sqrt{\frac{12\rho(t)R(t_o)}{(\beta c^2 \rho(t)\alpha - \lambda\alpha)(1+z)}} \frac{1}{\sqrt{1 - \frac{12\rho(t)R(t_o)}{(\beta c^2 \rho(t)\alpha - \lambda\alpha)(1+z)}}}{1 + \sqrt{\frac{12\rho(t)R(t_o)}{\beta c^2 \rho(t)\alpha - \lambda\alpha}} \frac{1}{\sqrt{1 - \frac{12\rho(t)R(t_o)}{\beta c^2 \rho(t)\alpha - \lambda\alpha}}} \sqrt{\frac{12\rho(t)R(t_o)}{(\beta c^2 \rho(t)\alpha - \lambda\alpha)(1+z)}} \frac{1}{\sqrt{1 - \frac{12\rho(t)R(t_o)}{(\beta c^2 \rho(t)\alpha - \lambda\alpha)(1+z)}}}$$

which can be simplified to

$$r(t_e) = \frac{\sqrt{\frac{12\rho(t)R(t_o)}{\beta c^2 \rho(t)\alpha - \lambda\alpha - 12\rho(t)R(t_o)}} - \sqrt{\frac{12\rho(t)R(t_o)}{(\beta c^2 \rho(t)\alpha - \lambda\alpha)(1+z) - 12\rho(t)R(t_o)}}}{1 + 12\rho(t)R(t_o) \frac{1}{\sqrt{\beta c^2 \rho(t)\alpha - \lambda\alpha - 12\rho(t)R(t_o)}} \frac{1}{\sqrt{(\beta c^2 \rho(t)\alpha - \lambda\alpha)(1+z) - 12\rho(t)R(t_o)}}}$$

This equation can also be written as

$$r(t_e) = \frac{\sqrt{12\rho(t)R(t_o)} \left[\sqrt{(\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z) - 12\rho(t)R(t_o)} - \sqrt{\beta c^2 \alpha \rho(t) - \lambda \alpha - 12\rho(t)R(t_o)} \right]}{\sqrt{\beta c^2 \alpha \rho(t) - \lambda \alpha - 12\rho(t)R(t_o)} \sqrt{(\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z) - 12\rho(t)R(t_o)} + 12\rho(t)R(t_o)} \quad (4.25)$$

4.2.3 Open Universe ($\kappa = -1$)

In this case, equation (4.14) now becomes

$$-\int_{r(t_e)}^{r(t_o)} \frac{1}{1-r^2} dr = \int_{R(t_e)}^{R(t_o)} \frac{dR}{\sqrt{R} \sqrt{\frac{\beta c^2 \alpha \rho(t) - \lambda \alpha}{3\rho(t)} + 4R}} \quad (4.26)$$

Let us perform integration of equation (4.26) as follows:

$$\tanh^{-1} r \Big|_{r(t_e)}^{r(t_o)} = \int_{R(t_e)}^{R(t_o)} \frac{dR}{\sqrt{R} \sqrt{\frac{\beta c^2 \alpha \rho(t) - \lambda \alpha}{3\rho(t)} \left(1 + \frac{12\rho(t)R}{\beta c^2 \alpha \rho(t) - \lambda \alpha} \right)}}$$

$$\tanh^{-1} r(t_e) = \int_{R(t_e)}^{R(t_o)} \frac{dR}{\sqrt{R} \sqrt{\frac{\beta c^2 \alpha \rho(t) - \lambda \alpha}{3\rho(t)} \left(1 + \frac{12\rho(t)R}{\beta c^2 \alpha \rho(t) - \lambda \alpha}\right)}}$$

$$\tanh^{-1} r(t_e) = \int_{R(t_e)}^{R(t_o)} \frac{dR}{\sqrt{R} \sqrt{\frac{\beta c^2 \alpha \rho(t) - \lambda \alpha}{3\rho(t)} \sqrt{1 + \frac{12\rho(t)R}{\beta c^2 \alpha \rho(t) - \lambda \alpha}}}} \quad (4.27)$$

Let $\frac{12\rho(t)R}{\beta c^2 \rho(t)\alpha - \lambda \alpha} = \sinh^2 \theta$

Such that $R = \left(\frac{\beta c^2 \rho(t)\alpha - \lambda \alpha}{12\rho(t)} \right) \sinh^2 \theta$

Thus, $\sqrt{R} = \sqrt{\frac{\beta c^2 \rho(t)\alpha - \lambda \alpha}{12\rho(t)}} \sinh \theta$

and $dR = \frac{\beta c^2 \rho(t)\alpha - \lambda \alpha}{12\rho(t)} 2\sinh\theta \cosh\theta d\theta$

Therefore,

$$\sqrt{1 + \frac{12\rho(t)R}{\beta c^2 \alpha \rho(t) - \lambda \alpha}} = \sqrt{1 + \sinh^2 \theta} = \sqrt{\cosh^2 \theta} = \cosh \theta \quad (4.28)$$

Performing substitutions, equation (4.27) becomes

$$\begin{aligned}
\tanh^{-1} r(t_e) &= \int_{\theta(t_e)}^{\theta(t_o)} \frac{2 \sinh \theta \cosh \theta \left(\frac{\beta c^2 \rho(t) \alpha - \lambda \alpha}{12 \rho(t)} \right)}{\sqrt{\frac{\beta c^2 \rho(t) \alpha - \lambda \alpha}{12 \rho(t)}} \sinh \theta \sqrt{\frac{\beta c^2 \rho(t) \alpha - \lambda \alpha}{3 \rho(t)}} \cosh \theta} d\theta \\
&= \int_{\theta(t_e)}^{\theta(t_o)} \frac{2(\beta c^2 \rho(t) \alpha - \lambda \alpha) 6 \rho(t)}{12 \rho(t) (\beta c^2 \rho(t) \alpha - \lambda \alpha)} d\theta \\
&= \theta \Big|_{\theta(t_e)}^{\theta(t_o)}
\end{aligned} \tag{4.29}$$

However, we know that

$$\theta = \sinh^{-1} \sqrt{\frac{12 \rho(t) R}{\beta c^2 \rho(t) \alpha - \lambda \alpha}} \tag{4.30}$$

Therefore,

$$\tanh^{-1} r(t_e) = \sinh^{-1} \sqrt{\frac{12\rho(t)R}{\beta c^2 \alpha \rho(t) - \lambda \alpha}} \Big|_{R(t_e)}^{R(t_o)} = \sinh^{-1} \sqrt{\frac{12\rho(t)R(t_o)}{\beta c^2 \alpha \rho(t) - \lambda \alpha}} - \sinh^{-1} \sqrt{\frac{12\rho(t)R(t_e)}{\beta c^2 \alpha \rho(t) - \lambda \alpha}} \quad (4.31)$$

Applying equation (4.15) in equation (4.31), we achieve

$$\tanh^{-1} r(t_e) = \sinh^{-1} \sqrt{\frac{12\rho(t)R(t_o)}{\beta c^2 \alpha \rho(t) - \lambda \alpha}} - \sinh^{-1} \sqrt{\frac{12\rho(t)R(t_o)}{(\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z)}} \quad (4.32)$$

We can now consider the following identity to be applied in equation (4.32)

$$\tanh(A \pm B) = \frac{\tanh A \pm \tanh B}{1 \pm \tanh A \tanh B}$$

to obtain

$$r(t_e) = \frac{\tanh \sinh^{-1} \sqrt{\frac{12\rho(t)R(t_o)}{\beta c^2 \alpha \rho(t) - \lambda \alpha}} - \tanh \sinh^{-1} \sqrt{\frac{12\rho(t)R(t_o)}{(\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z)}}}{1 + \tanh \sinh^{-1} \sqrt{\frac{12\rho(t)R(t_o)}{\beta c^2 \alpha \rho(t) - \lambda \alpha}} \tanh \sinh^{-1} \sqrt{\frac{12\rho(t)R(t_o)}{(\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z)}}} \quad (4.33)$$

Using the given trigonometric identities, $\tanh = \frac{\sinh \theta}{\cosh \theta} = \frac{\sinh \theta}{\sqrt{1 + \sinh^2 \theta}}$, we rewrite and simplify equation (4.33) as

$$r(t_e) = \frac{\sqrt{\frac{12\rho(t)R(t_o)}{\beta c^2 \rho(t)\alpha - \lambda \alpha}} \frac{1}{\sqrt{1 + \frac{12\rho(t)R(t_o)}{\beta c^2 \rho(t)\alpha - \lambda \alpha}}} - \sqrt{\frac{12\rho(t)R(t_o)}{(\beta c^2 \rho(t)\alpha - \lambda \alpha)(1+z)}} \frac{1}{\sqrt{1 + \frac{12\rho(t)R(t_o)}{(\beta c^2 \rho(t)\alpha - \lambda \alpha)(1+z)}}}}{1 + \sqrt{\frac{12\rho(t)R(t_o)}{\beta c^2 \rho(t)\alpha - \lambda \alpha}} \frac{1}{\sqrt{1 + \frac{12\rho(t)R(t_o)}{\beta c^2 \rho(t)\alpha - \lambda \alpha}}} \sqrt{\frac{12\rho(t)R(t_o)}{(\beta c^2 \rho(t)\alpha - \lambda \alpha)(1+z)}} \frac{1}{\sqrt{1 + \frac{12\rho(t)R(t_o)}{(\beta c^2 \rho(t)\alpha - \lambda \alpha)(1+z)}}}}$$

or

$$r(t_e) = \frac{\sqrt{\frac{12\rho(t)R(t_o)}{\beta c^2 \rho(t)\alpha - \lambda \alpha + 12\rho(t)R(t_o)}} - \sqrt{\frac{12\rho(t)R(t_o)}{(\beta c^2 \rho(t)\alpha - \lambda \alpha)(1+z) + 12\rho(t)R(t_o)}}}{1 + 12\rho(t)R(t_o) \frac{1}{\sqrt{\beta c^2 \rho(t)\alpha - \lambda \alpha + 12\rho(t)R(t_o)}} \frac{1}{\sqrt{(\beta c^2 \rho(t)\alpha - \lambda \alpha)(1+z) + 12\rho(t)R(t_o)}}}}$$

which can also be written, after being reduced as in the closed universe ($\kappa = 1$) case above, as

$$r(t_e) = \frac{\sqrt{12\rho(t)R(t_o)} \left[\sqrt{(\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z) + 12\rho(t)R(t_o)} - \sqrt{\beta c^2 \alpha \rho(t) - \lambda \alpha + 12\rho(t)R(t_o)} \right]}{\sqrt{\beta c^2 \alpha \rho(t) - \lambda \alpha + 12\rho(t)R(t_o)} \sqrt{(\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z) + 12\rho(t)R(t_o)} + 12\rho(t)R(t_o)} \quad (4.34)$$

All the three cases, that is, equations (4.16), (4.25) and (4.34) can be written in form of one equation as

$$r(t_e) = \frac{\sqrt{12\rho(t)R(t_o)} \left[\sqrt{(\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z) - 12\kappa\rho(t)R(t_o)} - \sqrt{\beta c^2 \alpha \rho(t) - \lambda \alpha - 12\kappa\rho(t)R(t_o)} \right]}{\sqrt{\beta c^2 \alpha \rho(t) - \lambda \alpha - 12\kappa\rho(t)R(t_o)} \sqrt{(\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z) - 12\kappa\rho(t)R(t_o)} + 12\kappa\rho(t)R(t_o)} \quad (4.35)$$

where $\kappa = -1, 0, 1$.

Let us now define

$$a = \beta c^2 \alpha \rho(t) - \lambda \alpha - 12\kappa\rho(t)R(t_o) \quad (4.36)$$

$$b = (\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z) - 12\kappa\rho(t)R(t_o) \quad (4.37)$$

We consider $r(t_e)$ as a function of $r(z)$ since t_e depends on z , hence equation (4.35) becomes

$$r(z) = \frac{\sqrt{12\rho(t)R(t_o)}\sqrt{b} - \sqrt{a}}{\sqrt{ab} + 12\kappa\rho(t)R(t_o)} \quad (4.38)$$

Suppose our astronomical object, for instance, star or galaxy located at $r = 0$ is emitting light at an absolute power L . In the interval of time of emission t_e , we consider the light emitted dt_e . At an observation time t_o , an observer measures the brightness of I of that light which he receives at a redshift z . The observer's position of reception of this light is given by equation (4.35). As photons pass through space, they get redshifted. This means that the energy that passes through the sphere of radius $r = r(z)$ during some time interval is the same as $1/(1+z)$. With this, we can express light intensity (I) dependent on luminosity (L) of luminous matter in the universe as

$$I = \frac{Ldt_e}{(1+z)S_r(z)} \quad (4.39)$$

where $S_r(z)$ denotes the surface area of the sphere of radius, $r = r(z)$ at time $t = t_o$.

From equation (4.12) we can perform integration from t_e to t_o and from the coordinates radius $r = 0$ to $r = r(z)$ to give us

$$\int_{t_e}^{t_o} \frac{c}{R(t)} dt = \int_0^{r(z)} \frac{1}{1 + \kappa r^2} dr \quad (4.40)$$

In terms of time intervals, we can write this equation as

$$\int_{t_e+dt_e}^{t_o+dt_o} \frac{c}{R(t)} dt = \int_0^{r(z)} \frac{1}{1 + \kappa r^2} dr \quad (4.41)$$

or

$$\frac{cdt_o}{R(t_o)} - \frac{cdt_e}{R(t_e)} + \int_{t_e}^{t_o} \frac{c}{R(t)} dt = \int_0^{r(z)} \frac{1}{1 + \kappa r^2} dr \quad (4.42)$$

Substituting equation (4.40) into (4.42), we obtain

$$\frac{dt_o}{R(t_o)} = \frac{dt_e}{R(t_e)}$$

or

$$\frac{dt_e}{dt_o} = \frac{R(t_e)}{R(t_o)} = \frac{1}{1 + z} \quad (4.43)$$

where we have applied equation (4.15).

Taking

$$S_r(z) = \frac{4\pi r(z)^2 R(t_o)^2}{(1 + \kappa r(z)^2)^2} \quad (4.44)$$

and substituting equation (4.43) into equation (4.39) and using equation (4.44), we obtain

$$I = \frac{L}{(1+z) \frac{(1+z)(4\pi r(z)^2 R(t_o)^2)}{(1+\kappa r(z)^2)^2}}$$

hence,

$$I = \frac{L(1 + \kappa r(z)^2)^2}{(1+z)(1+z)(4\pi r(z)^2 R(t_o)^2)} \quad (4.45)$$

Application of equation (4.38) in equation (4.45), we finally achieve

$$I = \frac{L \left[1 + \kappa \left(\frac{\sqrt{12\rho(t)R(t_o)}\sqrt{b-a}}{\sqrt{ab+12\kappa\rho(t)R(t_o)}} \right)^2 \right]^2}{(1+z)^2 4\pi \left(\frac{\sqrt{12\rho(t)R(t_o)}\sqrt{b-a}}{\sqrt{ab+12\kappa\rho(t)R(t_o)}} \right)^2 R(t_o)^2} \quad (4.46)$$

4.3 Number density-redshift relation

Suppose that our astronomical objects (for instance, supernova or galaxies) under consideration are uniformly distributed in the universe so that we count how many galaxies we see in a given redshift interval. If N is the number of galaxies per unit volume of the space with metric $(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)/(1 + \kappa r^2)^2$ and $r^2 \sin \theta d\theta d\phi dr/(1 + \kappa r^2)^3$ is the volume element, then the number of galaxies between r and dr is $4\pi r^2 dr/(1 + \kappa r^2)^3 N$ (Wamalwa, 2016).

Let us consider equation (4.35) as a function of z ,

$$r(z) = \frac{\sqrt{12\rho(t)R(t_o)} \left[\sqrt{(\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z) - 12\kappa\rho(t)R(t_o)} - \sqrt{\beta c^2 \alpha \rho(t) - \lambda \alpha - 12\kappa\rho(t)R(t_o)} \right]}{\sqrt{\beta c^2 \alpha \rho(t) - \lambda \alpha - 12\kappa\rho(t)R(t_o)} \sqrt{(\beta c^2 \alpha \rho(t) - \lambda \alpha)(1+z) - 12\kappa\rho(t)R(t_o)} + 12\kappa\rho(t)R(t_o)} \quad (4.47)$$

We find that $r(z)$ is proportional to z , that is, after application of expansion in powers of z , it reduces to

$$r(z) = \sqrt{\frac{3R(t_o)}{\beta c^2 \alpha \rho(t) - 12\kappa \rho(t) R(t_o)}}(z) \quad (4.48)$$

We can now differentiate equation (4.47) with respect to z

$$\frac{dr}{dz} = \frac{(\beta c^2 \alpha \rho(t))^2 \sqrt{3R(t_o)}}{\sqrt{a} \left(\sqrt{ab} + 12\kappa \rho(t) R(t_o) \right)^2} \quad (4.49)$$

where $a = \beta c^2 \alpha \rho(t) - \lambda \alpha - 12\kappa \rho(t) R(t_0)$ and $b = (\beta c^2 \alpha \rho(t) - \lambda \alpha)(1 + z)) - 12\kappa \rho(t) R(t_0)$

Further, we can assume the number of galaxies to be enclosed within the coordinate hyperspheres $r(z)$ and $r(z + dz)$ as

$$n(z)dz = \frac{4\pi r(z)^2 N r'(z) dz}{(1 + \kappa(r)^2)^3} \quad (4.50)$$

where $r'(z) = \frac{dr}{dz}$

Therefore, substituting equations (4.47) and (4.49) into equation (4.50), we get

$$n(z) = \frac{48\pi NR(t_o)(\beta c^2 \alpha \rho(t))^2 \sqrt{3R(t_o)} (\sqrt{b} - \sqrt{a})^2}{\left[1 + \kappa \left(\frac{\sqrt{b} - \sqrt{a}}{\sqrt{ab} + 12\kappa \rho(t) R(t_o)}\right)^2\right]^3 \left[\sqrt{ab} + 12\kappa \rho(t) R(t_o)\right]^4} \quad (4.51)$$

This equation relates how the number density of galaxies evolves with redshift. Together with equation (4.46), they constitute our important result in this research work. These results are more general than the results obtained using Einsteins field equations of General Relativity without the effects of dark energy or without the cosmological constant.

4.4 Graphical Results

Generally, we are now in a position to apply our results to enable us determine or approximate the number of galaxies in the observable universe, to which extend we can estimate their scales of distribution, and also appropriate methods by which galaxies are evolving in number density. The main outcome from this work is that we now have theoretical or mathematical results of the dynamics and evolution the number density of galaxies up to redshift $z = 5$.

In the last section, we established analytically the relationship between number density of galaxies, redshift and light intensity. Let us now consider the graphical evaluation of our results by writing and running simple computer matlab programs and plotting a few results for light intensity-redshift and number density-redshift relations based on equations (4.46) and (4.51) respectively for our analysis. In our matlab program, we have used the values of redshift from $z = 0$ to $z = 5$. These redshift values are in line with available statistics. Density of

the universe used varies from $\rho(t) = 3e^{-25}kgm^{-3}$ to $\rho(t) = 5e^{-27}kgm^{-3}$ while the speed of light used in the program is $c = 3 \times 10^8 m/s$. The cosmic scale factor used is $R(t_o) = 9e^{25}m$ and the gravitational constant, $G = 6.67 \times 10^{-11}m^3kg^{-1}s^{-2}$. The curvature of the universe, $\kappa = -1, 0, 1$ and the cosmological constant, $\lambda = 1.19e^{-52}m^{-2}$. Furthermore, for better results, we have taken logarithm of values of light intensity and number density. Since we are interested in the choice of parameters $\rho(t)$ and $R(t_o)$ that would give us the shape of the curve that would fit the experimental result (using unlimited observational data that does not have assumptions about the background geometry), we can assign the number N of galaxies per unit volume of our metric and the constant absolute power L of a galaxy or star, the arbitrary value 1. If we run our program based on equation (4.46) for various values of $\rho(t)(m)$ and $R(t_o)(kg/m^3)$, we obtain the following results (figure 4.1 to figure 4.6) for light intensity:

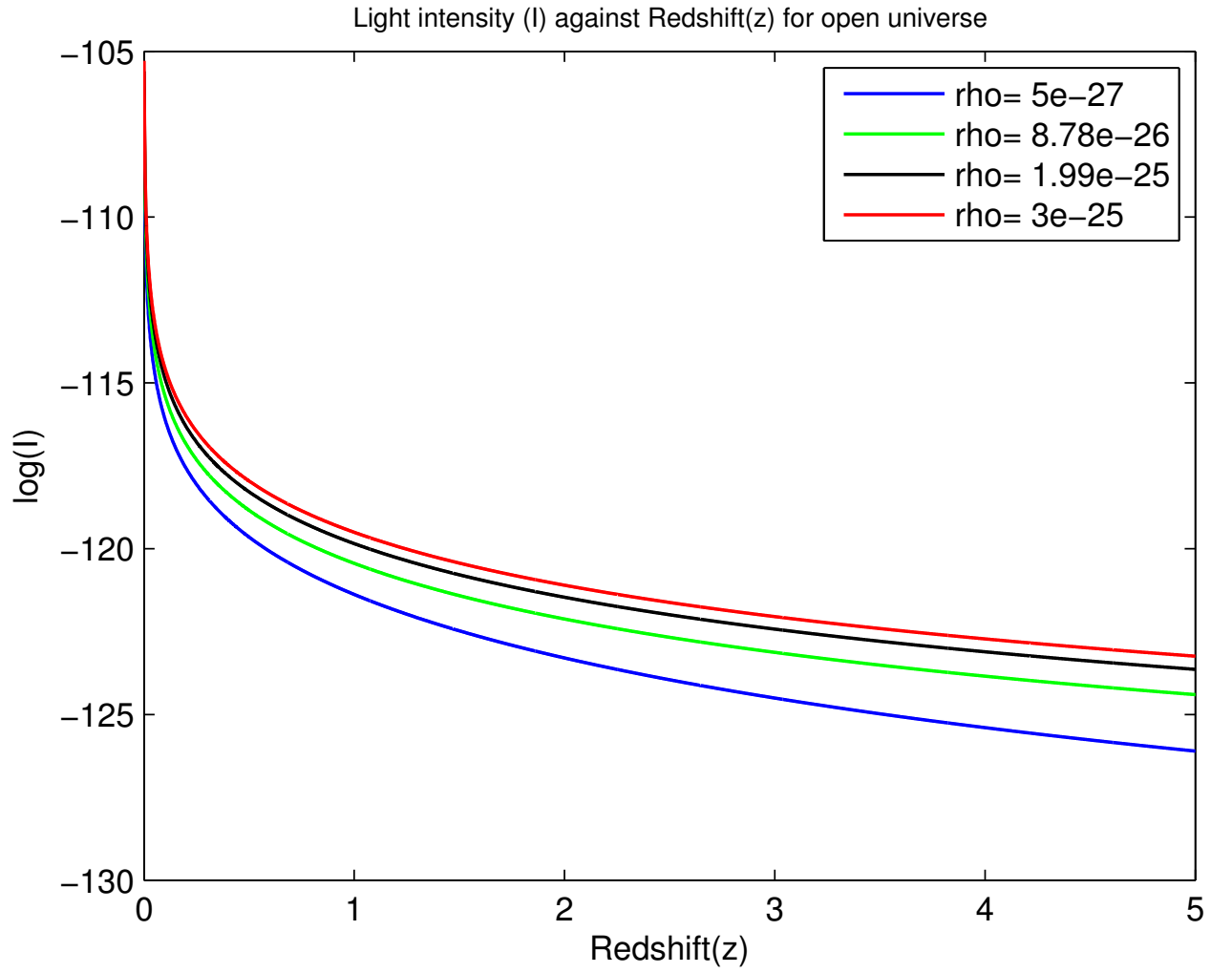


Figure 4.1: Plot of $\log(I)$ against z for $0 \leq z \leq 5$ and $\kappa = -1$ without cosmological constant(λ)

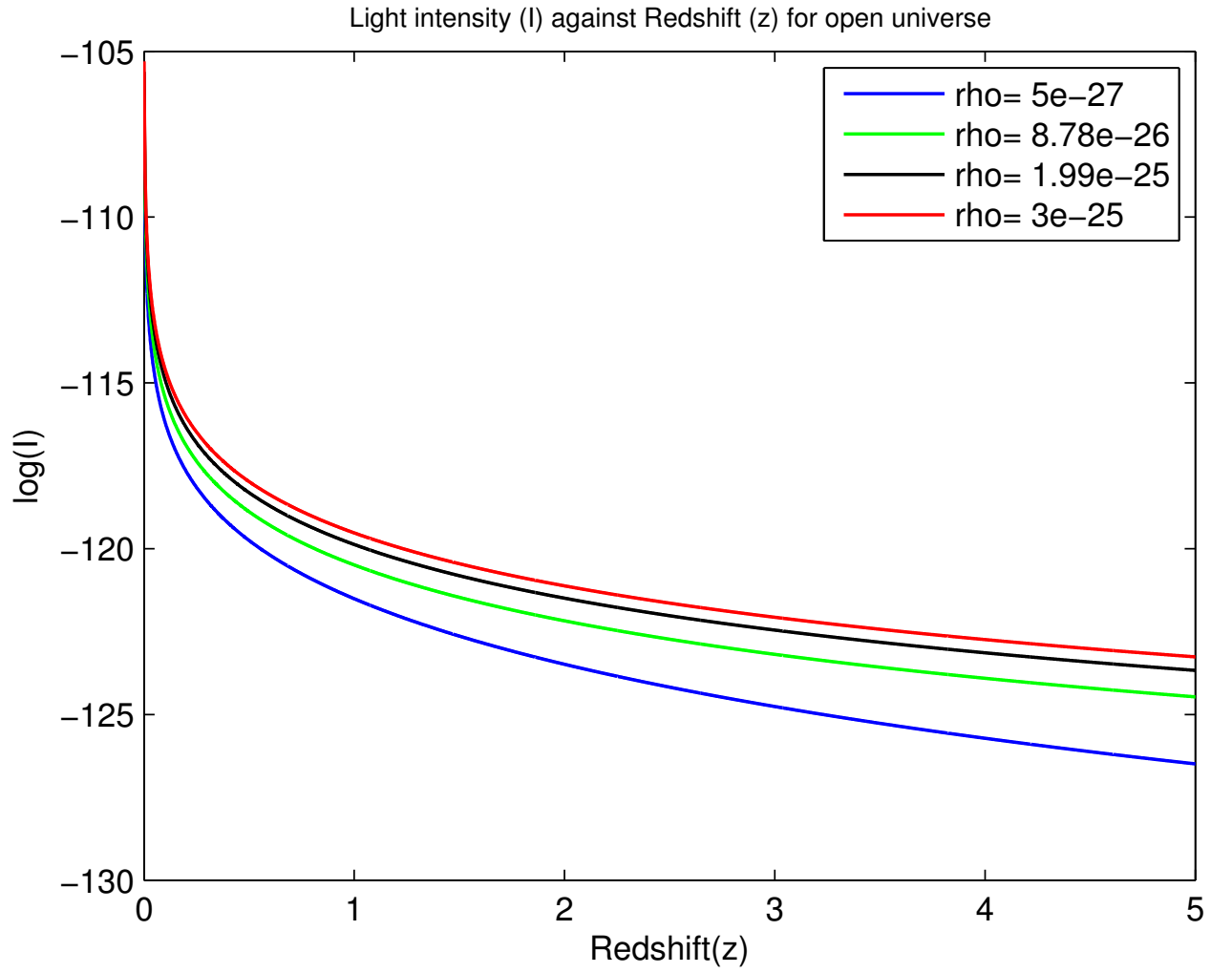


Figure 4.2: Plot of $\log(I)$ against z for $0 \leq z \leq 5$ and $\kappa = -1$ with cosmological constant(λ)

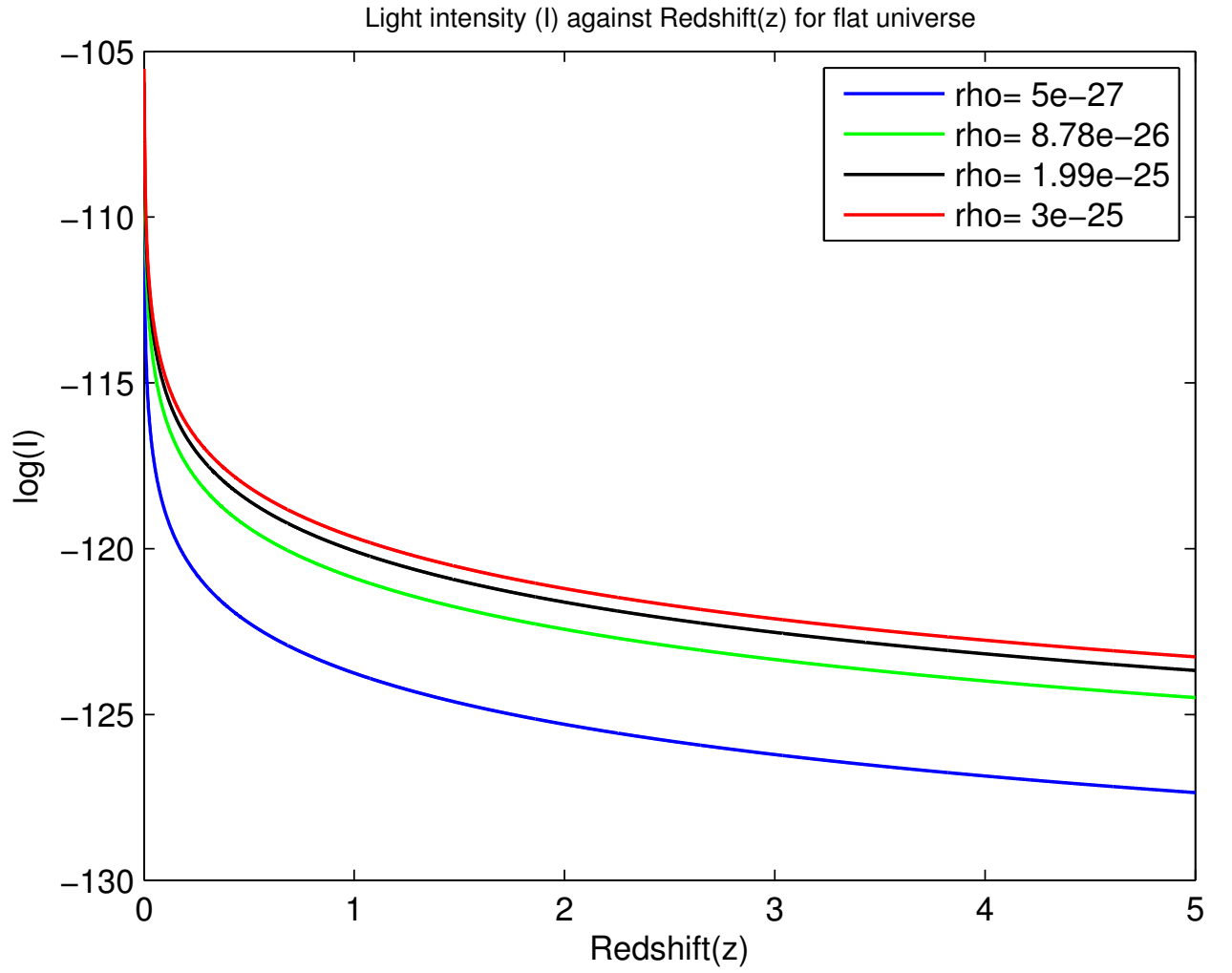


Figure 4.3: Plot of $\log(I)$ against z for $0 \leq z \leq 5$ and $\kappa = 0$ without cosmological constant(λ)

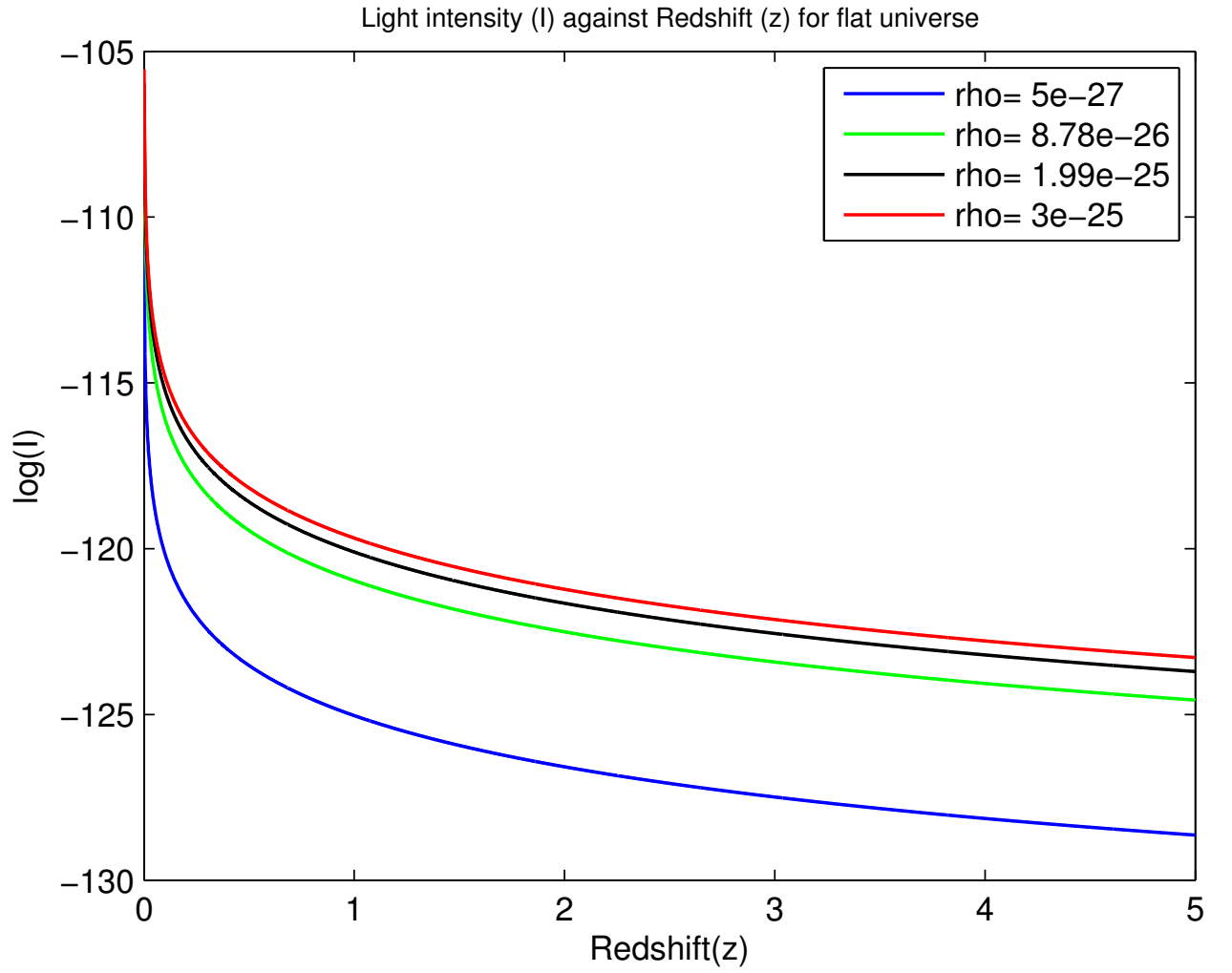


Figure 4.4: *Plot of $\log(I)$ against z for $0 \leq z \leq 5$ and $\kappa = 0$ with cosmological constant(λ)*

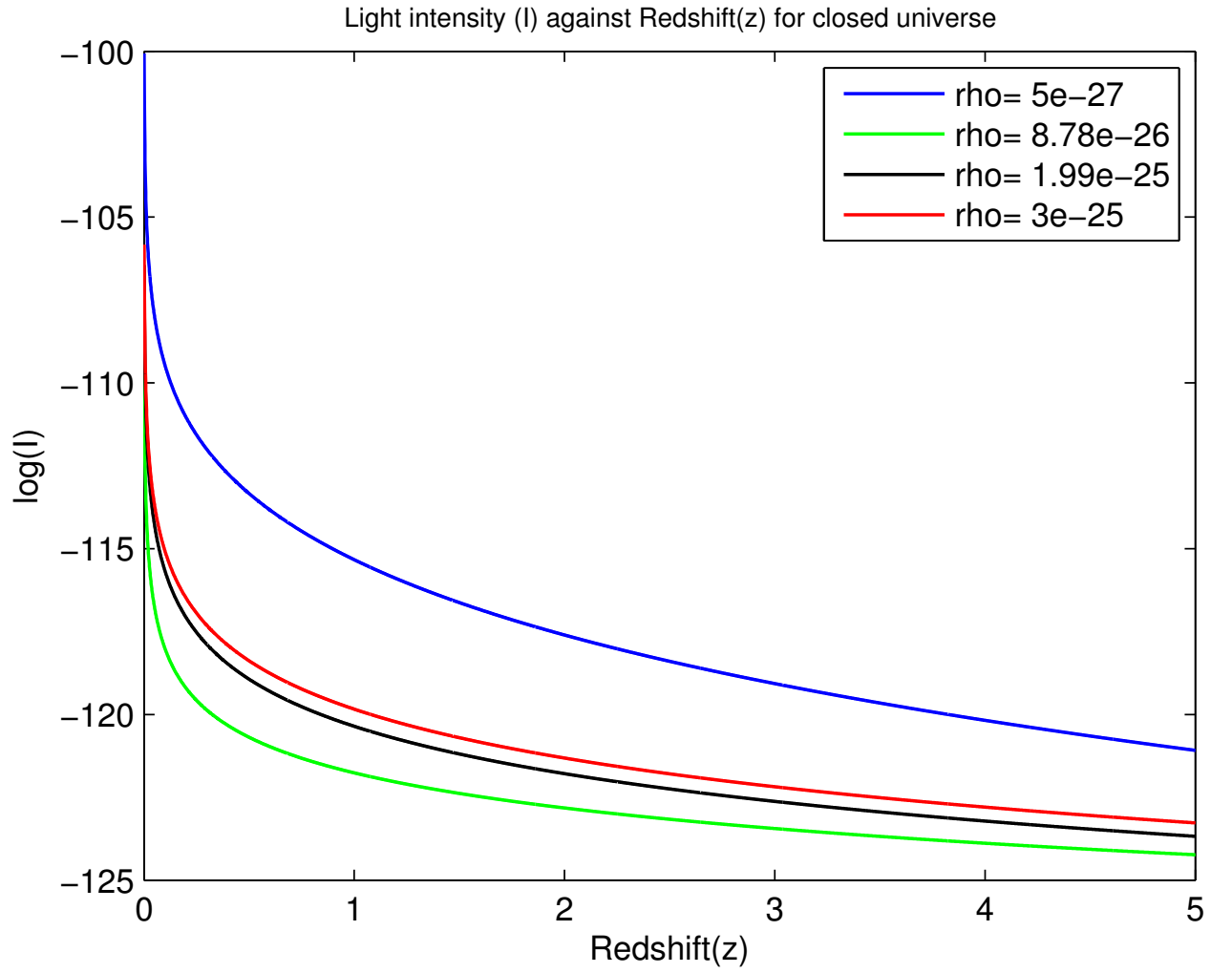


Figure 4.5: Plot of $\log(I)$ against z for $0 \leq z \leq 5$ and $\kappa = 1$ without cosmological constant(λ)

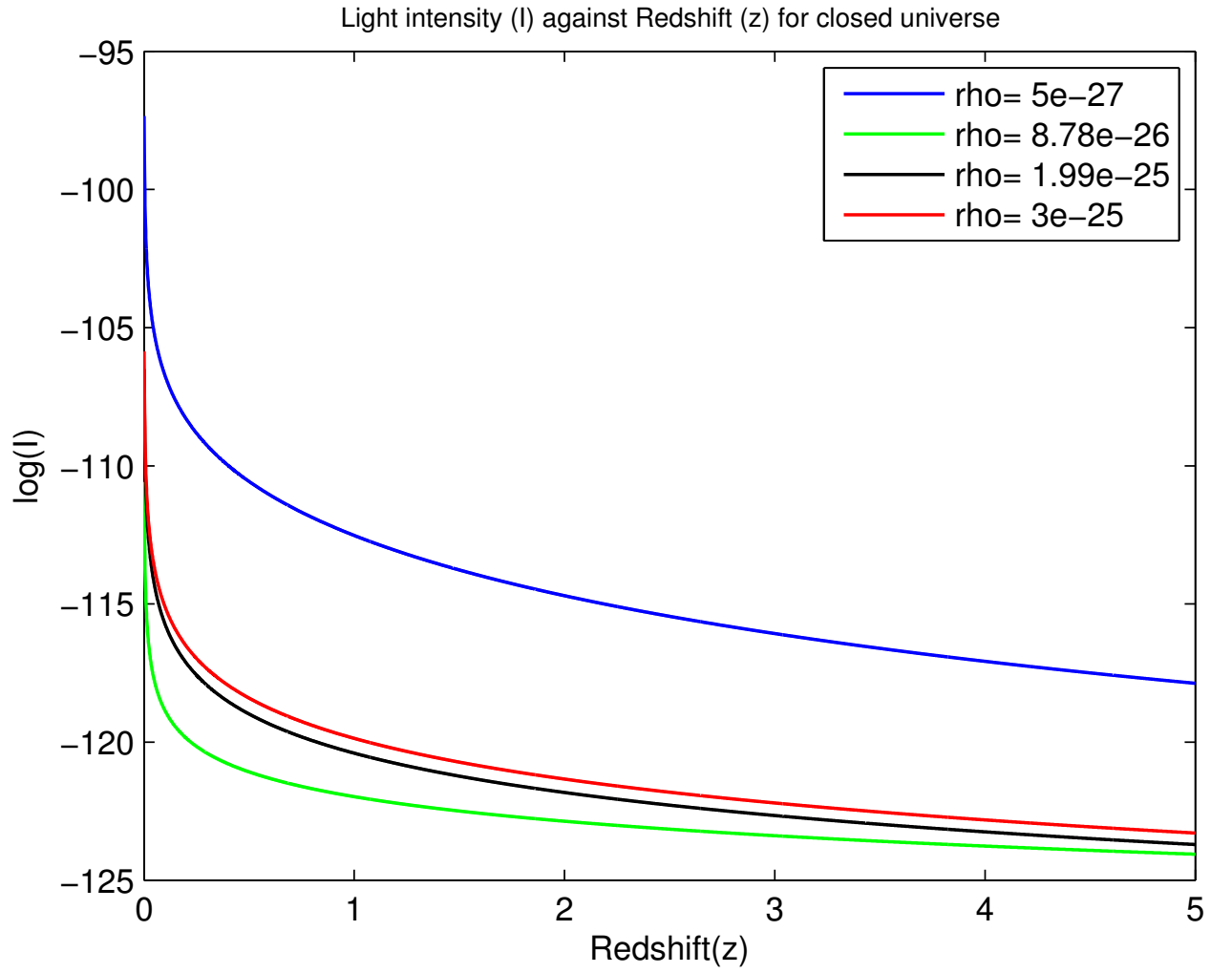


Figure 4.6: Plot of $\log(I)$ against z for $0 \leq z \leq 5$ and $\kappa = 1$ with cosmological constant(λ)

From figure 4.1 to figure 4.6, we observe that light intensity generally decrease with redshift in accordance with our classical expectation. As redshift increases, the ionizing sources decrease because structure formation becomes less advanced. The case of positive curvature describes the closed universe, whose three dimensional space is analogous to the surface of a sphere. As the coordinate of a sphere ranges from zero to one, the r-sphere sweeps out the entire universe leaving it unbounded. We also see the light curves tend to ultimately converge for this case. The converse is visible for open universe where the light curves tend to ultimately diverge. The light curves for flat universe neither converge nor diverge. These effects are clear when the value of (z) is increased in the program. From our results, it is also clear that light curves are affected by dark energy, density and curvature of the universe e.g., light curves corresponding to density of $5e^{-27}kg/m^3$ decrease faster in a flat and open universe than in a closed universe. This effect is more pronounced in a flat and open universe driven by dark energy effects, as seen from figure 4.1 to figure 4.4, than one without dark energy.

Let us now look at the graphical results of the evolution of the number density of galaxies, n with redshift, z as given by equation (4.51).

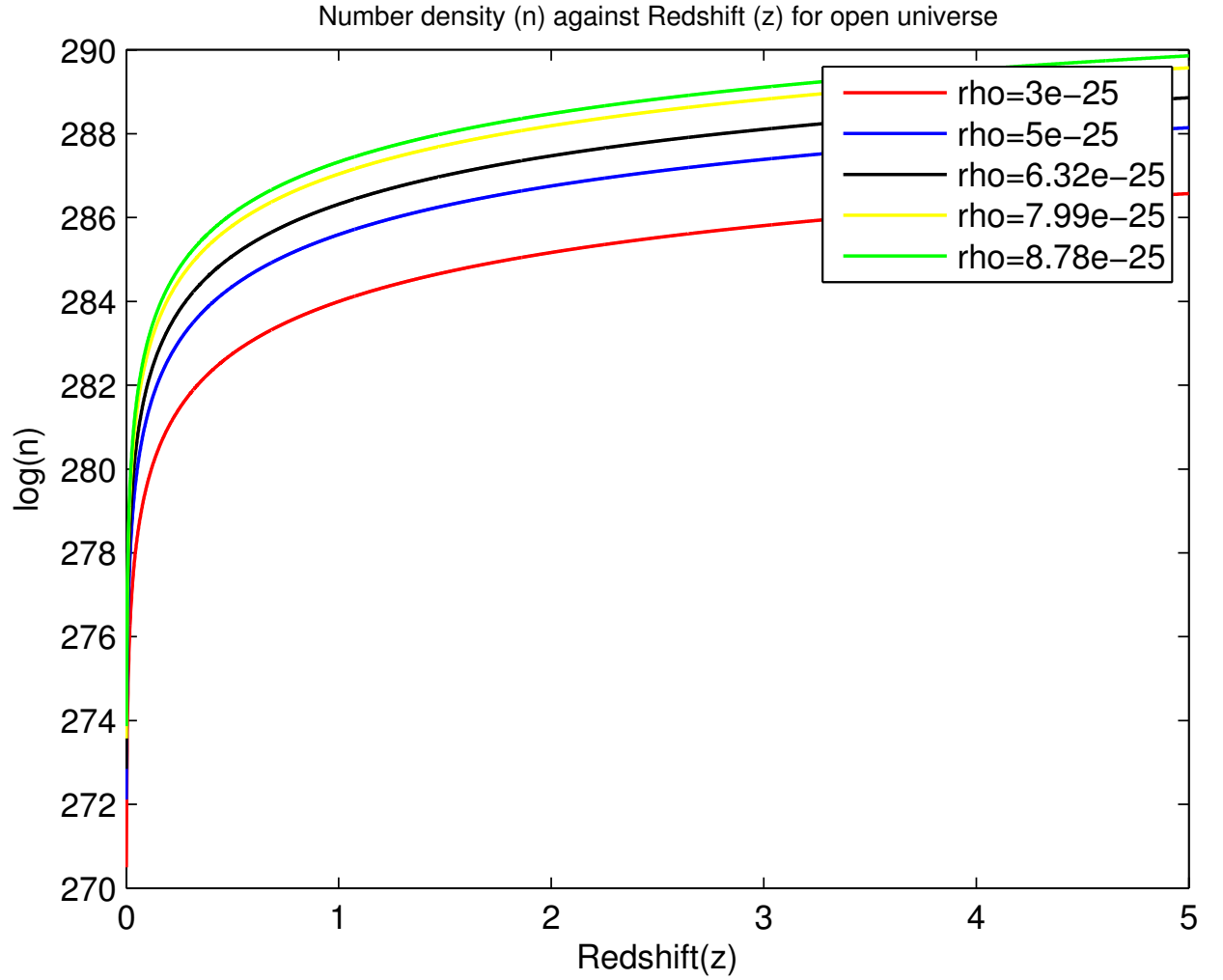


Figure 4.7: Plot of $\log(n)$ against z for $0 \leq z \leq 5$ and $\kappa = -1$ without cosmological constant(λ)

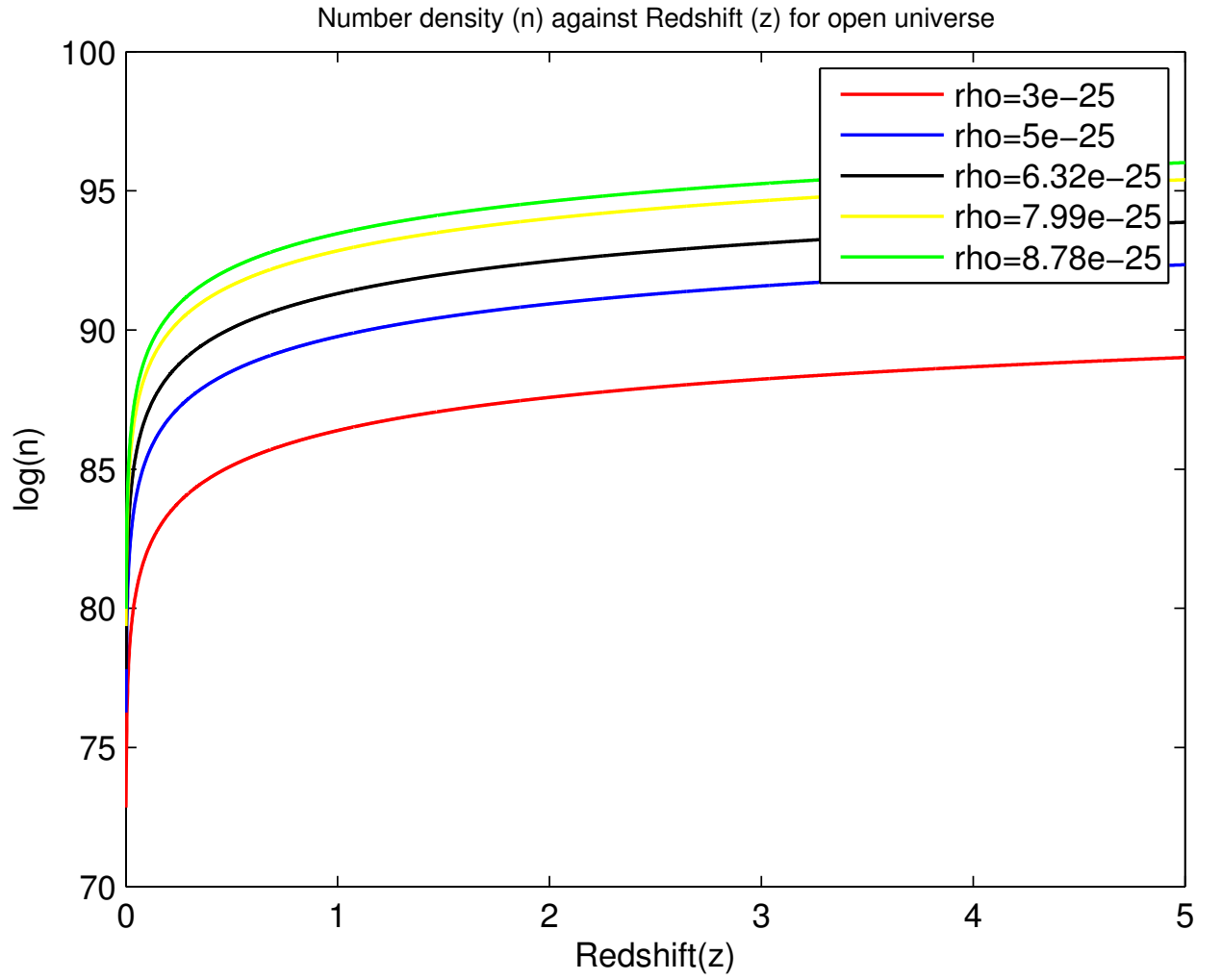


Figure 4.8: Plot of $\log(n)$ against z for $0 \leq z \leq 5$ and $\kappa = -1$ with cosmological constant(λ)

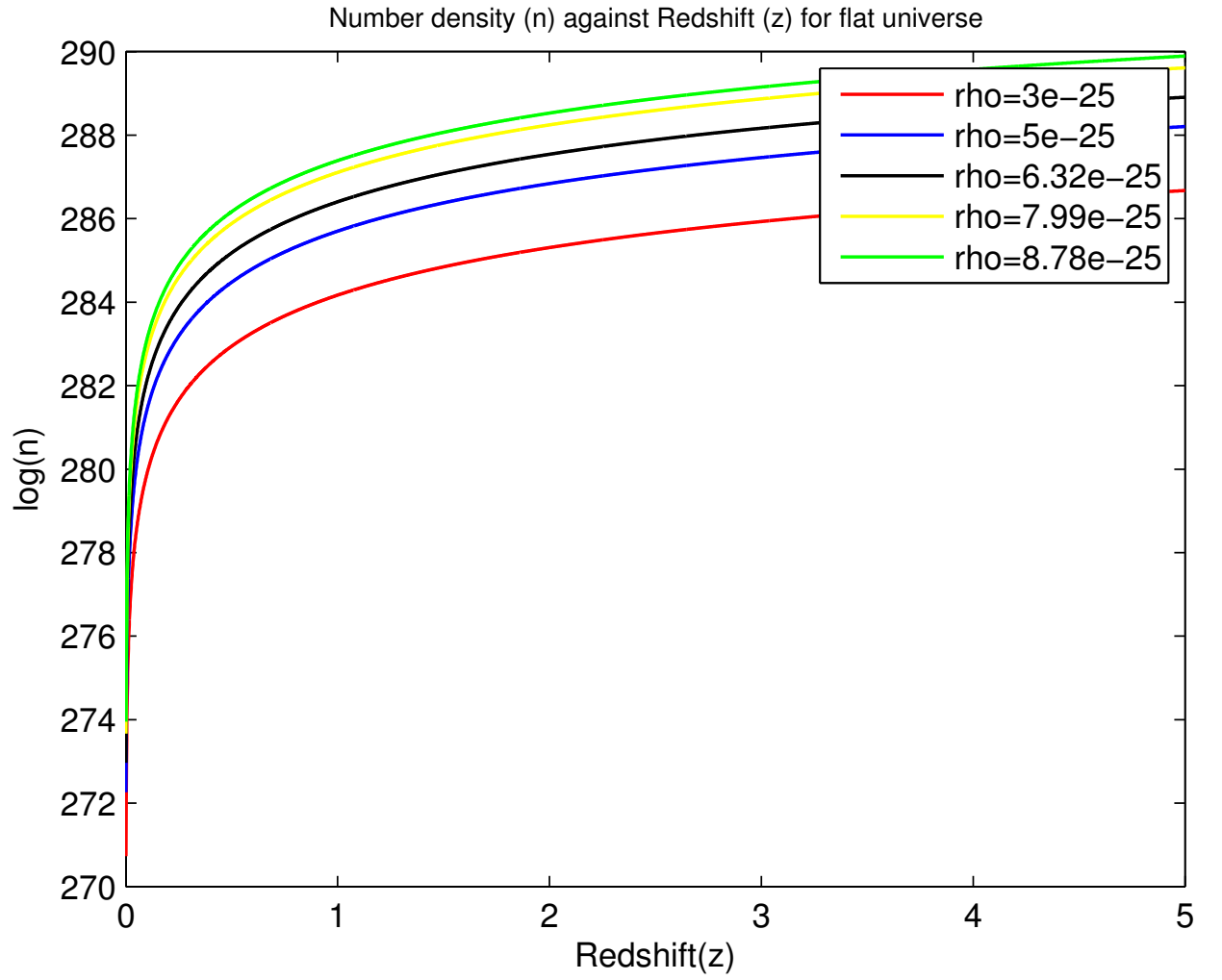


Figure 4.9: Plot of $\log(n)$ against z for $0 \leq z \leq 5$ and $\kappa = 0$ without cosmological constant(λ)

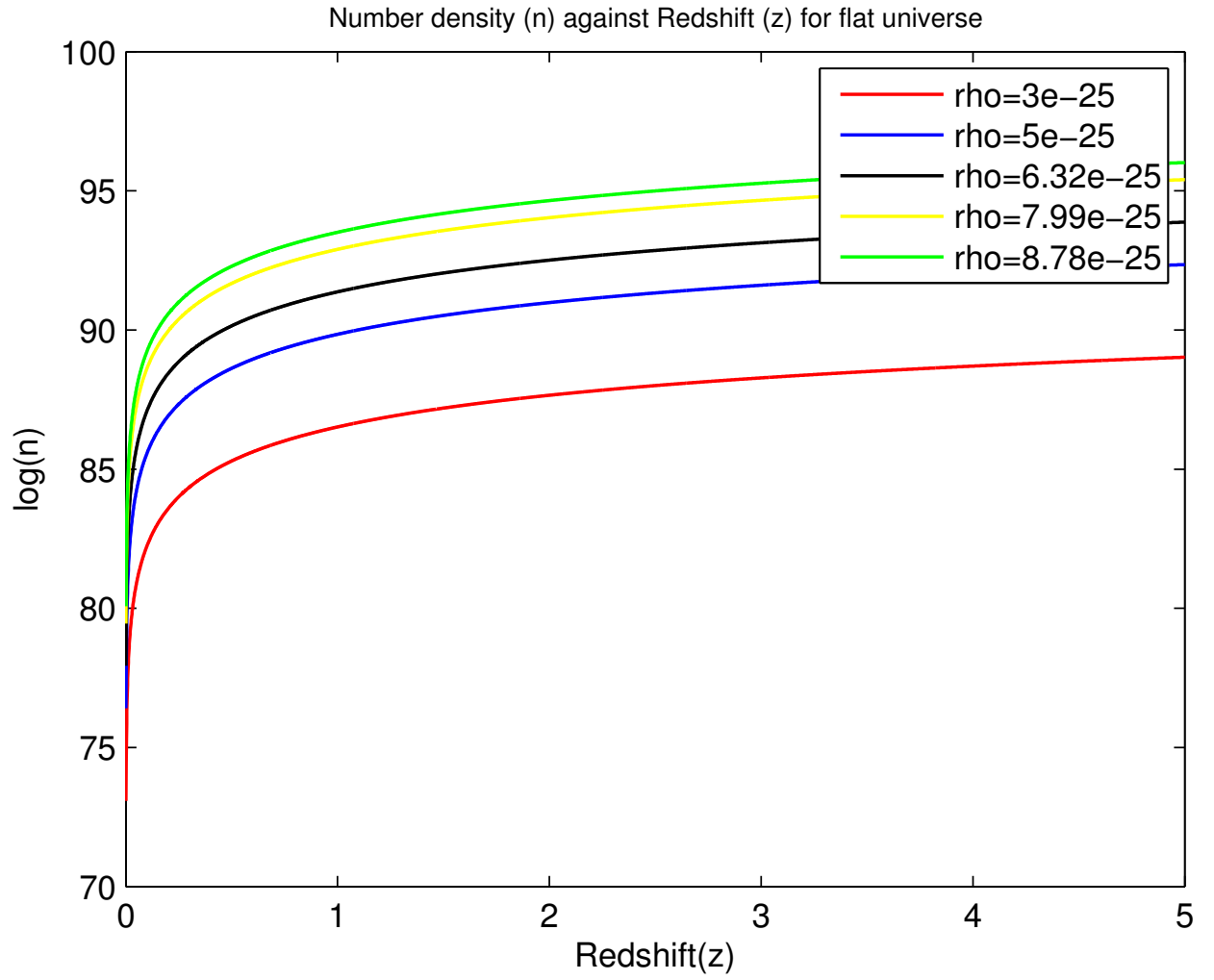


Figure 4.10: Plot of $\log(n)$ against z for $0 \leq z \leq 5$ and $\kappa = 0$ with cosmological constant(λ)

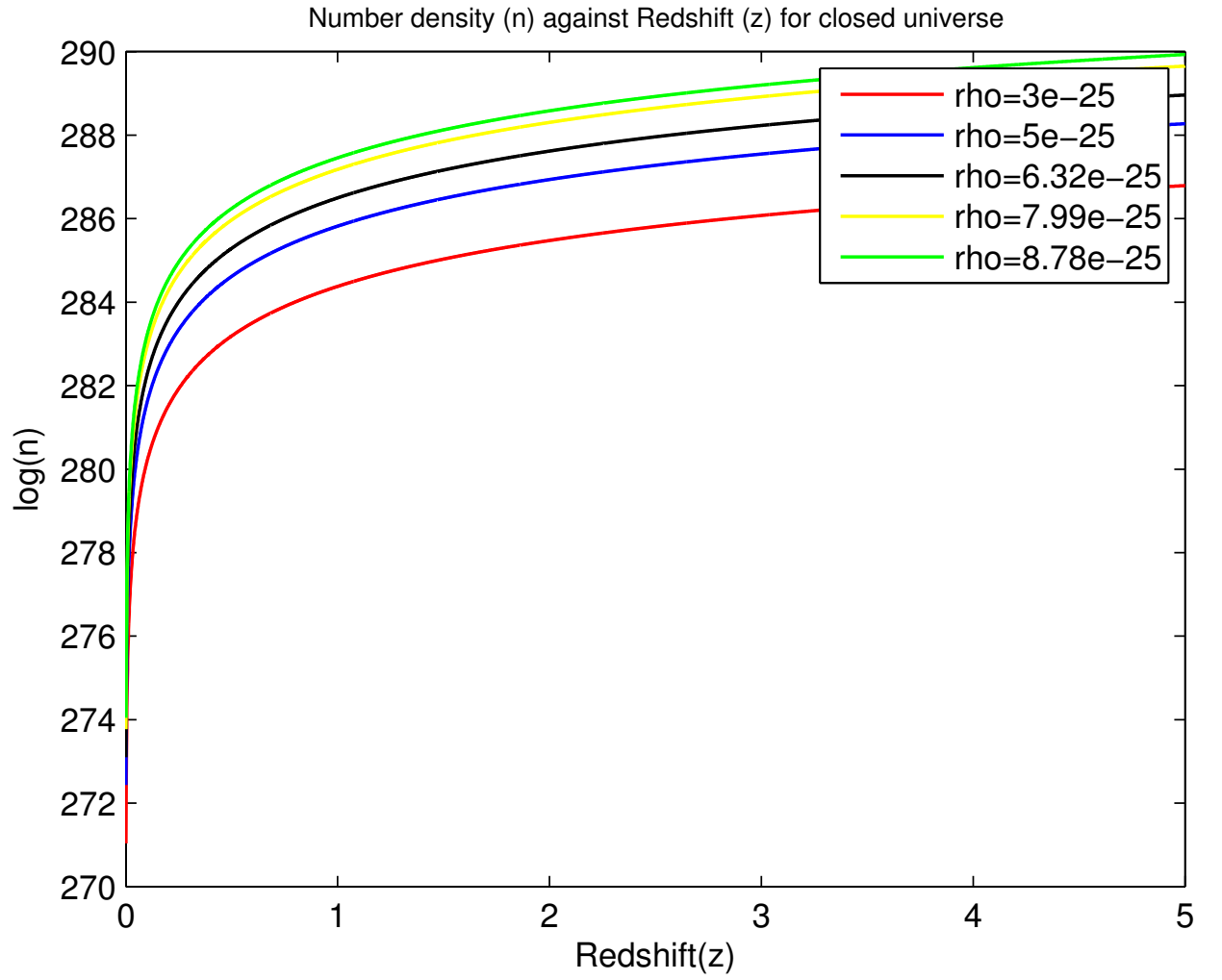


Figure 4.11: Plot of $\log(n)$ against z for $0 \leq z \leq 5$ and $\kappa = 1$ without cosmological constant(λ)

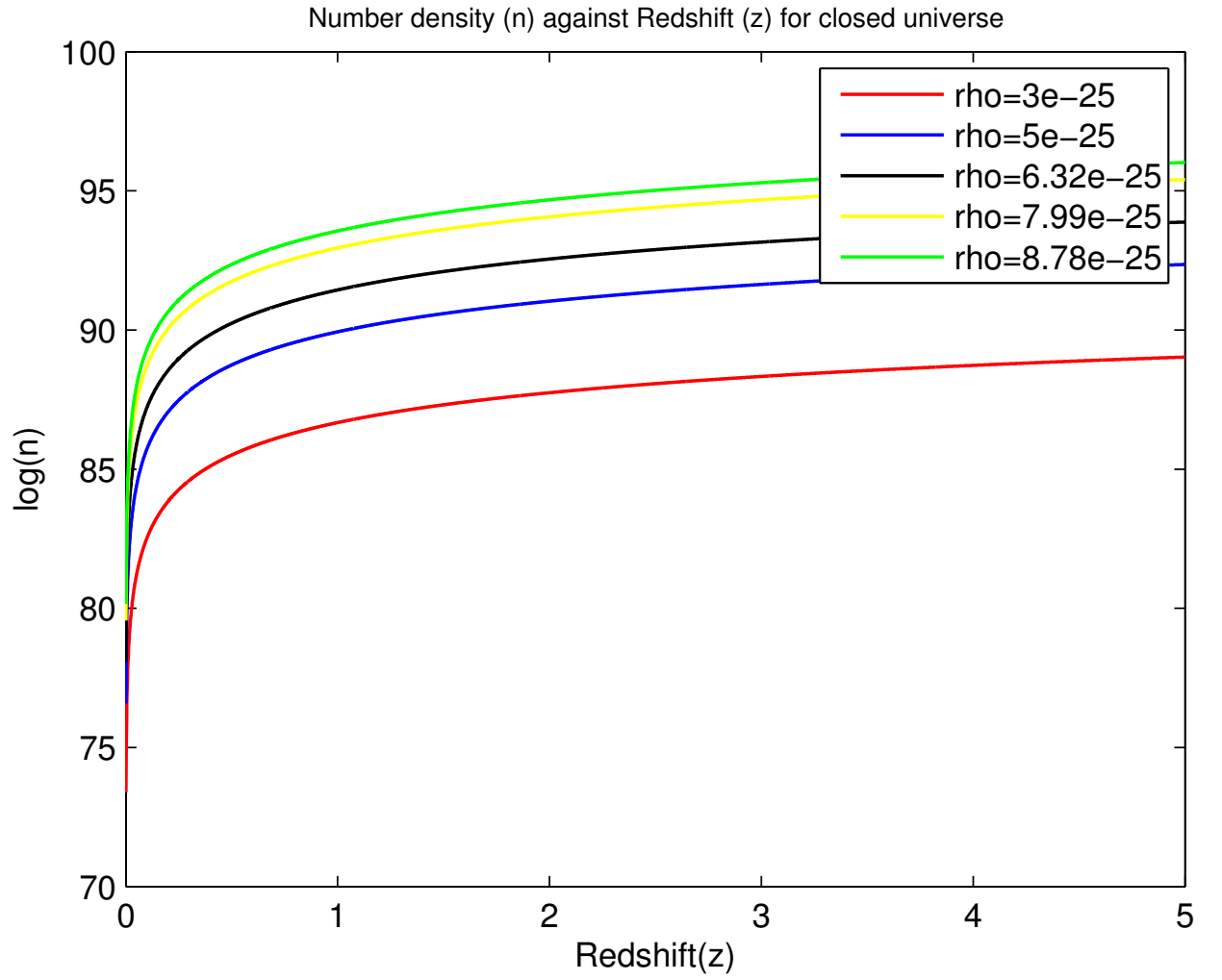


Figure 4.12: Plot of $\log(n)$ against z for $0 \leq z \leq 5$ and $\kappa = 1$ with cosmological constant(λ)

Plots for the number density of galaxies as shown from figure 4.7 to figure 4.12 reveal that galaxies seem to have formed at a faster rate at the beginning of the universe than at much later time. Structure formation seem to have grown at a fast rate from $z = 0$ until $z \approx 1$ when it started to slow down. However, the rate of structure or galaxy formation seems to be more for a universe without dark energy as compared to one with dark energy. This may be due to continued accelerated expansion of the universe caused by dark energy. The overall expansion of the universe reduces structure formation exponential law and hence inhibits structure formation. For plots of κ that give almost similar curves, the universe may probably be unstable for those values of densities and cosmic scale factor although appropriate scaling can help in differentiating them.

4.5 How to compare our theoretical and experimental results

We have already obtained our graphical results for both light intensity and number density as shown above. Suppose we have accurate observational data (huge enough data and free of errors associated with cosmic distance measurements) measured for redshift, number density and light intensity, then we can compare the experimental curve with our theoretical curve by plotting them on the same scale. In this way, we can judge which experimental curve does match our theoretical curve for both light intensity and number density. It is possible that the experimental curve may not fit any of our theoretical curves but lie in between two theoretical curves. If this is the case, then one can choose a new value, say $R(t_o)$, that lie in between the two values of the theoretical curves sandwiching the experimental curve and use it in the corresponding program of light intensity or number density. The process can be repeated until we get a perfect theoretical curve that fits the experimental curve. At this point, we can regard parameter values $R(t_o)$, $\rho(t)$ and κ as the radius of the universe, density of the universe and type of universe, that is, flat, open or closed respectively. However, if we cannot get any experimental curve that fits our theoretical curve no matter the choice of parameters, then we can disregard the Friedmann model in favour of other models like the fractal model.

Chapter Five: Conclusions and Recommendations

For a long time, scientists have successfully described structure formation in the universe based on the standard model. However, current tridimensional maps of the universe cast doubt on the validity of this model, in favour of the fractal model, as galaxy redshift surveys probe deeper into the universe uncovering more inhomogeneous structures on large scales. These maps depend on our ability to measure cosmic distance which is associated with a lot of errors. Furthermore, the limited available observational data may not be relied upon for accurate results.

5.1 Conclusions

In this work, since the standard model is based on the Friedmann universe which embodies the cosmological principle, we have derived Einstein field equations for a matter-dominated Friedmann universe while considering the effects of dark energy. Analytical and Computational results for the evolution of light intensity and number density of galaxies with redshift have been obtained in a suitable form for comparison with future observed dependencies that are not associated with uncertainties. From our results, light intensity from astronomical objects falls off with redshift and hence, by Hubble law, distance. This is in agreement with the classical results. Thus, the laws of classical physics holds true for open, closed and flat universe. From our number density-redshift relation, there was increased activity of galaxy or structure formation at the beginning of the universe than at later times for open, closed and flat universe. This activity seems to have slowed down at around $z \approx 1$ but is more pronounced in structures for a universe without dark energy than one without. This may be due to the continued accelerated expansion of the universe caused by dark energy that inhibits structure growth. This structure formation rate is in accordance with astrophysical observations and structure formation theory of the standard model. We, therefore, hold a view that the Friedmann model adopted in this work is correct. Nevertheless, we have prepared our results for experimental test based on future accurate observational data. Subject to availability of this data, we hope, this rests the fractal debate on whether or not the matter-dominated Friedmann universe is homogeneous on large scales.

5.2 Recommendations

This project reveals the need for accurate and intensive astrophysical data analysis of the observable matter-dominated universe. Certainly, multi-band and multi-facilities like ALMA, Spitzer, SKA and others, should be incorporated in order to probe the universe to higher redshifts so as to give us deeper insight about structure formation.

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