ON SOME ASPECTS OF SPECTRAL THEORY OF OPERARORS IN HILBERT SPACES

By

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DECLARATION

This thesis is my original work and has not been presented for a degree award in any other University.

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ABSTRACT

The property of quasi similarity of operators in Hilbert spaces has been studied by anumber of outhors; However the consideration of equality of spectrum and essential spectrum for such operators is yet to be fully exploited.Moreover the equality of spectrum and essential spectrum is scarcely mentioned in the extant literature of the subject.

The study is devoted to filling up this knowledge gap more specifically we consider classes of operators together with conditions under which two quasisimilar operator have either equal spectra or equal essential spectra. This study will help in investigating how an operator behaves in different aspects of spectral theory more specifically on quasisimilarity and spectrum of the opetator. This knowledge will be further extended to the study of quasisimilarity,essential spectrum and CI operators. Finally it will be agreat achievement to show the equality of spectrum and essential spectra of an operator and under which conditions they can be equal i.e $\sigma_e(A) = \sigma(A)$.

This thesis is organized as follows:

Chapter 1 is on introduction,Literature review, notations, terminologies, inclusions of classes of operators and decomposition of spectrum of an operator.

Chapter 2 is on quasisimilarity and spectrum of the operator.In this chapter we study under condition invertibility relates with condition of quasisimilarity of operators.The study will be extended to the product of the operators and to the classes of operators eg hyponomal, p-hyponomal, dominant operator etc. We also study quasisimilarity and CI operators and give condition of how the spectrum will behave if an operator is M-hyponomal,quasiinvertible and its dominant.

Chapter 3 deals with the quasisimilarity and essential spectrum.Here we pay attention to *Putnam-Fuglede* property to see how the spectrum of operator behaves together with its adjoint.This will be extended the study of essential spectrum of apositive operator.

Chapter 4 is on equality of spectra and essential spectra.Here we study the spectrum under the condition that the operator is Compact and we also set case of pure dominant operators and study undr which conditions is the spectra is equal to essential spectra.

Chapter 5 is on the summary of chapters and generalize the results of the study. Finally we briefly present problems of interest for possible future research.

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DEDICATION

This work is dedicated to my lovely wife Alice Githara, my children(Joram,Joyce and Monicah), and to the memory of my late parents namely Joram Rugiri Gakabu and Joyce Wanja Rugiri.

List of abbreviations

H: infinite dimensional separable Hilbert space.

***B*(*H*):**algebra of bounded linear operators on H.

SVEP: Single Valued Extension Property.

Q.E.D.: Quite Easily Done.

CI: Consistent in invertibility.

K:Compact Operator.

r(A): The spectral radius of A.

CHAPTER ONE
PRELIMINARIES

1.0 Introduction

From a simple study of a general spectrum, we decompose the spectrum and see the composition of a spectrum trying to answer the question whether the spectrum is empty or not ,from study of the general spectrum.

Since much has been done on this classical spectrum i.e. point spectrum, approximate point spectrum, residue ,compression spectrum and general study of a resolvent set we need to make link with the new concepts under . The aspects we focus on are similarity and quasisimilarity of operators in Hilbert spaces. In this thesis, we intend to do more studies in Hilbert spaces (spectral quassiffinity, isometry and quasisimilarity of operators. It will be very important to make study on other spectrums (which are not classical) where little research has been done specifically on the Essential spectrum and study how it relates with various aspects of the spectrum of operator. Quasisimilarity was introduced by Nagy and Foias [20] in their theory on infinite-dimensional analogue of the Jordan form for certain classes of contractions as a means of studying their invariant subspace structures. It replaces the familiar notion of similarity which is the appropriate equivalence relation to use with finite dimensional Hilbert spaces. In finite dimensional spaces quasisimilarity is the same thing as similarity, but in infinite dimensional spaces, it is a much weaker relation.

Two operators A and B are said to be *almost-similar* (denoted $(A \approx B)$ if there exists an invertible operators N such that the following two conditions are satisfied.

$$A^*A = N^{-1}(B^*B)N$$

$$A^*+A=N^{-1}(B^*+B)N$$

An attempt will be made to study the relationship between the Essential spectrum and general spectrum. This helps to see what we can borrow in solving any mathematical Problem, especially in the study of operators. Spectral analysis is a very powerful tool in functional analysis and we need to do more research in this area.

1.1 Literature review

The spectral theorem for compact Hermitian operators was essentially proved by the Hilbert in a paper published in 1906, though his view point and terminology were different . One could say that this was the point at which functional analysis crystallized, out of the confidence of geometry topology and analysis is in the words of J.Dieudonne [4] .The work of many mathematicians had led to development of the study of operators and it was until (1973). Williams[28], later introduced essential spectrum as an image of A in Calkin algebra and considered approximate equivalence: $A \sim B \Rightarrow \sigma_e(A)=\sigma_e(B)$ and hence strengthening the results of Duggal [7]. The spectral theorem of compact Hermitian operators was the first substantial result in the branch of functional analysis known as operator theory. It was soon followed by a satisfactory spectral theorem for Hermitian operators which are bounded but not compact. Such operators need not have eigenvalue and so one has to use a more subtle notion involving measure theory This line of investigation was completed by M.HStone and Von Neumann in the late 1920` s with the theory of compact non-Hermitian operators.Many operators are neither compact nor Hermitian, so operator theories have had plenty to occupy them over the past eight decades.This is an enormous, rich and expensive theory covering many classes in operators.Practically every problem in analysis can be reformulated in operator theoretical terms, so it will surely never come about that there is no more to say about operators. Galido (1991) managed compose operators in essential of partial isometries.The new definition of essential is well illustrated in wikipedia, 2005.The new spectrum is studied separately in isolation of the classical spectra hence study needs to give a further study of the spectrum and hence beef up more materials on the spectral theory .

Further the concept of quasisimilarity particularly with respect to equality of spectra has been studied by a number of authors among them W.C Clary [2]; who showed that quasisimilar hyponormal operators have equal spectra. J.M Khalagai and B. Nyamai [19] showed that if A and B are quasisimilar operators with A dominant and B* is M-hyponormal then A and B have same spectra. J.P. William [27] showed that there are several cases which imply that A and B have equal essential spectra. For example if A and B are both hyponormal operators or are both partial isometries or quasinormal operators etc. B.P. Duggal[7] proved that if A_i i=1,2 are quasisimilar p-hyponormal operators such that U_i is unitary in the polar decomposition $A_i=U_i|A_i|$, then A_1 and A_2 have same spectra and also same essential spectra. Luketero *etal[18]*, have most recently worked on the concept of Quasisimilarity in relation to essential spectrum.

1.2 Notation and Terminology

Throughout this study let H denote an infinite dimensional separable Hilbert space and $\boldsymbol{B(H)}$**:**algebra of bounded linear operators on H. If $T \in B(H)$ write $N(T)$ and $R(T)$ for the null space and range of T: $\rho\,(T)$ for the resolvent set of T; $\sigma(T)$ for the spectrum of $T; \pi_0(T)$ for the eigenvalues of $T; \pi_{0f}(T)$ for the eigenvalues of finite multiplicity; $\pi_{0i}(T)$ for the eigenvaluesof infinite multiplicity. $V_{\mathcal{H}}$ will denote unilateral shift on $\mathcal{H}$. It is familiar that if $T \in B(H)$ then T is regular if and only if T has closed range. An operator $T \in B(H)$ is called *upper semi-Fredholm* if it has closed range with finite-dimensional null space and *lower semi-Fredholm* if it has closed range with its range of finite co-dimension. If T is either upper or lower upper or lower Semi-Fredholm, we call it Semi-Fredholm and if T is both upper and *lower semi-Fredholm*, we call it Fredholm. An operator $T \in B(H)$ is called *Weyl* if it is Fredholm of index zero. An operator $T \in B(H)$is called Browder if it is Fredholm "of finiteascent and descent": The essential spectrum $\sigma_e(T)$, the Weyl spectrum σ_w and the Browder spectrum $\sigma_b(T)$ of $T \in B(H)$ are defined by ;

$$\sigma_e(T) = \{\lambda \in \mathbb{C}: T - \lambda I \text{ is not Fredholm}\};$$

$$\sigma_w = \{\lambda \in \mathbb{C}: T - \lambda I \text{ is not Weyl}\};$$

$$\sigma_b(T) = \{\lambda \in \mathbb{C}: T - \lambda I \text{ is not Browder}\}:$$

then

$$\sigma_e(T) \subseteq \omega(T) \subseteq \sigma_b(T) \cup \text{acc}\,\sigma(T) \subseteq \eta\,\sigma_e(T),$$

Where we write acc$K$ and $Conv(\,K)$ for the *accumulation points* and the *polynomially-convex hull* of K respectively, of $K \subseteq \mathbb{C}$. If we write iso $K = K \setminus acc\ K$, and ∂K for the topological boundary of K, and

$\pi_{00}(T) := \{\lambda \in \ iso\ \sigma(T): 0 < \dim(T - \lambda I)^{-1}(0) < \infty\}$

For the isolated eigenvalues of finite multiplicity, and $p_{00}(T) := \sigma(T) \setminus \sigma_b(T)$

for the Riesz points of $\sigma(T)$, then by the punctured neighborhood theorem, i.e., iso $\sigma(T) \setminus \sigma_e(T) =$ iso $\sigma(T) \setminus \omega(T) = p_{00}(T) \subseteq \pi_{00}(T)$.

If $T \in B(H)$, write $W(T)$ for the numerical range of T. It is also familiar that $W(T)$ is convex and conv $\sigma(T) \subseteq W(T)$The essential spectrum of T, denoted by $\sigma_e\,(T)$, is defined by $\sigma_e(T) = \{\lambda \in C: T - \lambda I$ is not fredholm$\}$.

A Fredholm operator of index Zero is called a weyl operator.Ran (T), $Ker(T)$, range and kernel of T, respectively. We reserve the symbols $\mathbb{Z}$,$\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{D}$ for the sets of integers, positive integers, real numbers, complex numbers, open unit disc in $\mathbb{C}$ respectively. $\sigma(T)$, $W(T)$, $\omega(T)$, $r(T)$ denote the spectrum, numerical range, numerical radius and spectral radius of T, respectively where $\sigma(T) = \{\lambda \in \mathbb{C}: \lambda I - T$ is not invertible$\}$.

We denote by $\left(\sigma_p(T)\right) = \{\lambda \in \mathbb{C}: \mathrm{Ker}(\lambda \mathrm{I} - \mathrm{T}) \neq 0\}$, which is the set of all eigenvalues of T and is called the point spectrum of T. If $(\lambda I - T)$ has an inverse that is not densely defined, then λ belongs to the residual spectrum: $\sigma_R(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\}\}$ *and* $\overline{Ran\ (\lambda I - T)} \neq$ H. The parts $\sigma_p(T), \sigma_C(T)$ *and* $\sigma_R(T)$ are pair wise disjoint and $\sigma(T) = \sigma_p(T) \cup \sigma_C(T) \cup \sigma_R(T)$. We also define the approximate point spectrum of T: $\sigma_{ap}(T) = \{\lambda \in \mathbb{C}: (\lambda I - T)$ is not bounded below$\}$. Any other notation used is in the list of the abbreviations and also not limited to the definations that follow.

CLASSES OF OPERATORS

An operator T is said to be:

- **unitary** if $T^*T = TT^* = I$,
- **normal** if $T^*T = TT^*$,
- **2-normal** if $T^*T^2 = T^2T^*$.
- **Hyponormal** if $T^*T \geq TT^*$,
- **self-adjoint** if $T = T^*$,
- **a projection** if $T^2 = T \; and \; T^* = T$,
- **an involution** if $T^2 = I$,
- **an isometry** if $T^*T = I$,
- **co-isometry** if $TT^* = I$,
- **a partial isometry** if $T = TT^*T$,
- **Compact** if for each bounded sequence $\{x_n\}$ in the domain $\mathcal{H}$, the sequence Tx_n contains a subsequence converging to some limit in the range,
- **Seminormal** if either T or T^* is hyponormal,
- **p-hyponormal** if $(T^*T)^p \geq (T^*T)^p$, where $0 < p \leq 1$,
- **semi-hyponormal** if $(T^*T)^{\frac{1}{2}} \geq (TT^*)^{\frac{1}{2}}$,
- **quasihyponormal** if $T^{*2}T^2 - (T^*T)^2 \geq 0$,
- M**hyponormal** if $\|(zI - T)^*x\| \leq M\|(zI - T)x\|$, for all complex numbers z and for all $x \in \mathcal{M}(\mathcal{H})$ and M some positive number (i.e. $M > 0$),
- **paranormal** if $\|Tx\|^2 \leq \|T^2x\|$, for all unit vectors $x \in \mathcal{H}$, or $\|Tx\|^2 \leq \|T^2x\|\|x\|$
- **dominant** if for each $\lambda \in \mathbb{C}$ there corresponds a number $M_\lambda \geq 1$ such that

  $\|(T - \lambda I)^*x\| \leq M_\lambda\|(T - \lambda I)x\|$, for all $x \in \mathcal{H}$,

1.3 Inclusions of classes of operators

It is well known that the following inclusions hold and are proper:

Unitary ⊊ Normal⊊ Quasinormal ⊊ Binormal

Projection ⊊ Self - adjoint ⊊ Normal ⊊ Hyponormal

Normal ⊊ Quasinormal ⊊ Subnormal ⊊ hyponormal ⊊ M – hyponormal

Hyponormal ⊊ p-hyponormal $(½ < p < 1)$ ⊆ semi-hyponormal ⊆

p-hyponormal$(0 < p < ½)$

Normal ⊆ M - hyponormal ⊊ Dominant

Unitary ⊊ Isometry ⊊ Partial Isometry ⊊ Contraction

$$\text{Hyponormal} \subseteq \begin{cases} k - quasihyponormal \\ k - paranormal \end{cases}$$

p-quasihyponormal ⊆ $(p, k) -$ quasihyponormal

Normal ⊊ Quasinormal ⊊ Subnormal ⊊ Hyponormal ⊊ Paranormal ⊆ Hilbert – Schmidt ⊊ Compact.

1.4 Decomposition and properties of spectrum

It is a well known fact in operator Theory that if A and B are operators with at least one of them invertible then AB and BA are similar operators by Khalagai[19]. We introduce scalars $\lambda \in \mathbb{C}$, by stating the following definition. $\sigma(A) = \{ \lambda \in \mathbb{C}, (A - \lambda I)$ is not invertible in $B(H) \}$

An element of $\sigma(A)$ is called a spectral value of A and hence finding the spectrum of a bounded operator involves invertibility of certain elements of B(H) (where B(H) is a set of bounded operators in a complex Hilbert space

Remark 1.4.1

Since an element of $\sigma(A)$ is a spectral value of A, we need to study more about these elements of A .

Definition 1.4.2

Let A be a transformation from v to v where v is a finite – dimensional vector space, let $v \in V$ such that $Av = \lambda v$ where λ is a scalar for all $v \in v$, λ is called an eignenvalue corresponding to the linear transformation A. If $v \in v$ such that $v \neq O$ and $Av = \lambda v$ is referred to as eigenvector corresponding to *eigenvalue* .

Definition 1.4.3

The set of all eigenvalues of $A \in B(H)$ is called the eigenspectrum.

Theorem 1.4.4[25]

Every non-zero spectrum value of A is an eigenvalue of A where $A \in B(H)$ finite dimensional.

Proof

Let $0 \neq \lambda \in \mathbb{C}$ and $\lambda \in \sigma(A)$

Since $A - \lambda I$ is not invertible, it is either not one to one or onto. Then $A - \lambda I$ is not one to one hence $\lambda \in$ eigenspectrum.

The name eigenspectrum arises from the following physical considerations. If a physical quality (like position, momentum or energy) represented by an operator A is measured in an experiment, then the result of the measurement is one of the eigenvalues of A. In an atomic quantum mechanical system if A is the energy operator of an atom, then the differences of the various eigenvalues of A are the amounts of energy emitted by the atom as it undergoes a transition. The amounts are seen in form of electromagnetic waves, which constitute the optical spectrum of that atom.

Theorem 1.4.5[25]

Let *H* be an n-dimensional Hilbert space. Then the spectrum of every operator on H consists of n eigenvalues. If H is a Hilbert space over R, then the spectrum of every self-adjoint operator on H consists of n real eigenvalues.

Definition 1.4.6

Let us consider an equation of the form f(x) = 0 where $f(x) = a_o x^n + a_1 x^{n-1} +\text{----} + a_n$, $a_o \neq 0$. In some algebraic extension field K of k f(x) can be factored.

$$f(x) = a_o(x-\lambda_1)(x-\lambda_2)\text{---------}(x-\lambda_n)$$

λ_1, -------, λn are called the roots of equation of degree n has exactly n roots. If λ appears ρ times in λ_1, ----------λn, we say that λ is a p- tuple root, and ρ is called the multiplicity of the root λ. When $\rho=1$, λ is called a simple root when $\rho \geq 2$, λ is called a multiple root.

Definition 1.4.7

The dimension of the eigenspace corresponding to the eigenvalue λ_0 is called the multiplicity of the eigenvlaue λ_0. Eigenvalues which occurs exactly n – times is said to be of multiplicity n.

Example 1.4.8

If the characteristic equation of a given 6x6 matrix is $(3-\lambda)^2 (1+\lambda)^3 (2-\lambda) = 0$

Then 3 is an eigenvalue with multiplicity 2, and –1 is an eigenvalue with multiplicity 3(and 2 is a ‘normal’ eigenvlaue).

Theorem 1.4.9[1]

If the operator A has distinct eigenvalues then there exists a complete set of linearly independent eigenvectors.

Remark 1.4.10

We may not be particularly interested in the value of eigenvalues but rather in its relative position in the spectrum. We now run down some results on the properties of the spectrum of the operators.

Definitions 1.4.11

We say that an operator A is bounded below on H if there is a constant $\propto>0$ such that

$\propto\|x\| \leq \|A(x)\|$ for all $x\in H$

Lemma 1.4.12 [1]

$A\in B(H)$ is invertible in B(H) if A is bounded below and the range of A is dense in H.

Proof

Let A be invertible for $xo\in H$, if $A(x)=y$,

Then $\|x\| = \|A^{-1}(y)\| \leq \|A^{-1}\| \|y\| = \|A^{-1}\| \|A(x)\|$.

Hence A is bounded below. Also, the range of A is dense in H, since it is, infact, equal to H.

Conversely, assume that;

$\propto\|x\| \leq \|A(x)\|$ for all $x\in H$ and some constant ;

$\propto > 0$, and that the range A is dense in H. To show that A is onto, it is enough to prove that the range of A is closed in H.

Let $A(x_n)\rightarrow y$ in H

Then with $y_n = A(x_n)$ we have for all n, m,

$\|x_n - x_m\| \leq (1/\propto|)\| A(x_n - x_m)\| = 1/\propto \|y_n - y_m\|$

Hence (x_n) is a cauchy sequence in H, since H is complete, let $x_n\rightarrow x\in H$. Then, by continuity of A, $A(x_n) \rightarrow A(x)$ so that $y = A(x)$ is in the range of A. Thus, A is onto. Since A is bounded below, it is clearly one to one. Let A^{-1} be the set theoretic inverse of A on H. Then A^{-1} is seen to be automatically linear. Also, for any $y \in H$,

If y = A(x) then;

$\|A^{-1}(y)\| = \|x\| \leq {}^{1}/_{\propto}) \|A(x)\| = ({}^{1}/_{\propto})\|y\|$

Thus, A^{-1} is a bounded operator on H; ie, A is invertible in B(H).

Remark 1.4.13

It is clear that for $A \in B(H)$ which is bounded below and has a dense range is invertible;Lemma 1.4.12 and so we further decompose the spectrum of the operator.

Definition 1.4.14

If λ_0 is such that the range R $\sigma(A)$ is dense and λI-A has a continuous inverse,

we say that λ_0 is in the resolvent set denoted $\rho(A)$ of A, The complex number λ is called a regular point of the operator A if the operator $(A-\lambda I)$ has an inverse $(A-\lambda I)^{-1}$. In the opposite case, λ is called a point of the spectrum of the operator A.

The regular points form an open set in the complex plane; the spectrum is closed this is by Halmos[14].

All points lying outside the circle of radius $\|A\|$ with center at the origin are regular. All points of the spectrum are in the disk;

$|\lambda| \leq \|A\|$.

The radius of the smallest disk with center at the origin containing the spectrum of the operator A is called spectral radius of the operator A.

The series for the resolvent will converge if $r_A < |\lambda|$ and diverge if $r_A > |\lambda|$ in particular, the series.

$$(I-A)^{-1} = -R_1 = I + A + A^2 + \text{----} + A^n + \text{-----}$$

Converges if $r_A<1$ and diverges if $r_A>1$

If the spectrum of an arbitrary bounded operator is anon-empty set then

$r_A = \lim_{n\to\infty} \sqrt{\|A^n\|} = 0$; then

the spectrum consists of one point, $\lambda = 0$

Definition 1.4.15

An operator-valued function of λ is called analytic at λ_0 if it can be expanded in a neighbourhood of λ_0 in a series of positive integral powers of $(\lambda-\lambda_0)$ which converges with respect to the operator norm.

The resolvent is an operator-valued function of λ which is analytic in the region consisting of regular points of the operator A.

Example1.4.16[16]

For $x \in A$, the function $\rho_\lambda(x)$, whose domain consists of regular points of A, is analytic function with values in A.

Let λ_0 be a pole of analytic function ρ_λ. Then any element ρ_λ (x) has an expansion

$= \quad f_0 + f_1 (\lambda-\lambda_0) +\text{------}+ f_n(\lambda-\lambda_0)^n +\text{----------}$

The element $\rho_o = \lambda\rho_o(x)$ satisfies the equation $A\rho_o = \lambda_o\rho_o$

And is called an eigenvector of the operator A corresponding to the eigenvalue λ_o

Theorem 1.4 .17 [5]

A complex number $\lambda \in \sigma_p$ (where $\sigma_p(A)$ is point spectrum equivalent to eigenspectrum) iff the equation $Ax = \lambda x$ ___________ (a) has anon-zero solution of x.

Proof

Let $\lambda \in \sigma_P(A)$ then its very clear

That $(\lambda I - A)^{-1}$ does not exist and

So $(\lambda I-A)$ is not one to one so

$(\lambda I-A)x = 0$ for some $x \neq 0$

Therefore $Ax=\lambda x$ has a non-zero solution in x

Conversely, if $Ax = \lambda x$ for some $x \neq 0$

But $(\lambda I-A)x = 0$

Therefore $\lambda I-A$ is not one to one ie $(\lambda I-A)^{-1}$ does not exist

ie $\lambda \in \sigma_P(A)$

Remark 1.4.18

The above theorem exhibits a very important property of the spectrum, ie should have anon zero solution ,theorem 1.4.17.

Theorem 1.4.22[5]

The spectrum of an operator $A \in B(H)$ is not empty.

Proof

Suppose that $\sigma(A) = \phi$, let $R(\lambda)$ denote the element of resolvent at λ.

Since $R(\lambda)$ is an analytic function on the resolvent which now the entire plane,

$R(\lambda)$ is an entire function

Furthermore since $\|R(\lambda)\| = 0$

As $|\lambda| \to \infty$

Hence before Liouville's theorem $R(\lambda)$ is constant and $R(\lambda) = 0$ for every λ.

But this is clearly impossible.

Remark 1.4.23

We are now ready to decompose the spectrum into various subsets noting cleary that it should be non empty .see[1].

Definition 1.4.24

Point spectrum is defined and denoted as follows;

$\sigma_p(A) = \{\lambda \in \mathbb{C}, : \ker(\lambda I-A) = (0)\}$

The elements of $\sigma_p(A)$ are eigenvalues.

Clearly;

$\sigma_p(A) \subset \sigma(A)$--(1)

Hence;

$\sigma(A) = \sigma_p(A)$ for every operator on a finite-dimensional space

Definition 1.4.25

Let approximate point spectrum be denoted by $\sigma_{ap}(A)$. Hence;

$\sigma_{ap}(A) = \{ \lambda \in \mathbb{C}$ the sequence (x_n) such $(\lambda I\text{-}A)\, x_n \rightarrow 0\}$

Sometimes (x_n) is called an approximate eigenvector with eigenvalues λ

Clearly $\sigma_p(A) \subset \sigma_{ap}(A)$--(2)

Remark 1.4.26

We now state a very important theorem as a result of (1) and (2) above

Theorem 1.4.27 [5]

Let $A \in B(H)$

$\sigma_p(A) \subset \sigma_{ap}(A) \subset \sigma(A)$

Proof

If $\lambda \in \sigma(A)$, then let x be an eigenvector corresponding to eigenvalue λ such that;

$||x|| = 1$, and let

$x_n = x$ for n = 1, 2, ------, then

$0 = ||A(x_n) - k\, x_n|| \rightarrow 0$,

So that

$\lambda \in \sigma_{ap}(A)$

On the other hand, if $\lambda \in \sigma(A)$, then for every $x \in X$ with $||x|| = 1$,

$1 = ||x|| = ||(A\text{-}\lambda I)^{-1}\, 0 (A\text{-}\lambda I)x|| \leq ||(A\text{-}\lambda I)^{-1}||\ ||(A(x)\text{-}\lambda(x)||$

So that $\lambda \notin \sigma(A)$

and thus;

$\sigma_p(A) \subset \sigma_{ap}(A) \subset \sigma(A)$ follows.

Definition 1.4.30

Continuous spectrum denoted by $\sigma_c(A)$ is the totality of a complex numbers λ for which $\sigma(A)$ has discontinuous inverse with domain being dense.

Thus;

$\sigma_c(A) = \sigma(T) \backslash \sigma_{com}(A) \cup \sigma_p(T) \rightarrow \sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$

Where $\sigma_r(A)$ is the residue spectrum defined as follows

Definition 1.4.31

The residue spectrum is defined and denoted by

$\sigma_r(A) = \sigma_{com}(A)\sigma_p(A)$

Note $\sigma(A) = \sigma_{ap}(A) \cup \sigma_{com}(A)$

Where $\sigma_{com}(A)$ is the compression spectrum which is the set of all λ such that λ belongs to $\sigma(A)$ - $\sigma_{ap}(A)$

Theorem 1.4.32 [5]

If x is a finite dimensional normed linear space and $A: x \rightarrow x$ is linear operator then

$\sigma_r(A) = \phi$, $\sigma_{com}(A) = \phi$ Thus $\sigma(A) = \sigma_p(A)$.

Proof

We first note that every ;linear transformation of a finite dimensional normed linear space is bounded and thus for all $\lambda \in \mathbb{C}$, $(\lambda I-A)^{-1}$ is bounded if it exists on;

$R(\lambda I-A)$-------------------------- (*)

Thus $\sigma_{com}(A) = \phi$ for its existence asserts that $(\lambda I-A)^{-1}$ is unbounded which contradicts the (*) above

Now for $\lambda \in \sigma_r(A)$, $(\lambda I-A)^{-1}$ exists and is bounded for x has finite dimensions. Therefore $(\lambda I-A)$ is one to one of x

Let $\{x_1, x_2, \text{---------} x_n\}$ be a basis of x.

Then since $(\lambda I-A)$ is one to one it follows that the set $\{(\lambda I-A)x_1, (\lambda I-A)x_2, \text{-----------}, (\lambda I-A)x_n\}$ which spans $R(\lambda I-A)$ is linearly independent. Thus $R(\lambda I-A) = x$

Therefore if $\lambda \in \sigma_r(A)$ then

$R(\lambda I-A) = x$. But this contradicts the definition of $\sigma_r(A)$. Thus there is no such λ.

Hence $\sigma_r(A)=\phi$

Theorem 1.4.33 [1]

Let $A \in B(H)$, then $\lambda \in \sigma_{ap}(A)$ if and only if $(\lambda I-A)$ does not have a bounded inverse on B(H).

Proof

Suppose $\lambda \in \sigma_{ap}(A)$, then for each $n \in J^+ \ni x_n \in D(A)$ where D(A) is the domain of A with $||x_n|| = 1$ such that;

$||(\lambda I-A)x_n|| < {}^1/_n$

Thus it is not possible to find k>0 such that $||(\lambda I-A)x|| \geq k||x||$

$\forall x \in D(A)$

ie (λI-A) is not bounded from below

Thus λI-A does not have a bounded inverse on B(H).

Conversely, let λI-A to have no bounded inverse. In this case k>0 satisfying.

$\|(\lambda I-A)x\| \geq k\|x\| \ \forall\ x\in D(A)$

does not exist. This means that for any $\in$>0 an $x\in D(A)$ with $\|x\| = 1$ can be found such that

$\|\lambda I-A)x\| < \in$

This implies that $\lambda\in \sigma_{ap}(A)$

Remark 1.4.34

From the earlier result already shown by definition 1.4.24., we conclude that $\sigma_{ap}(A) \subset \sigma(A)$

Here we need to recall the following can corollary also see thorem 1.4.27

Corollary 1.4.35 [1]

$\sigma_{ap}(A) \subset \sigma(A)$

Proof

Let $\lambda\in \sigma(A)$

Let denote are solvent set by $\rho(A)$

Then $\lambda\notin\sigma(A)\rightarrow \lambda\in\rho(A)$

$\rightarrow$ λI-A has a bounded inverse

$\rightarrow \lambda\notin \sigma_{ap}(A)$ (as above).

ie $\lambda\notin\sigma(A) \rightarrow \sigma_{ap}(A)$

ie $\lambda\in \sigma_{ap}(A) \rightarrow \lambda\in \sigma(A)$

ie $\sigma(A) \subseteq \sigma(A)$

Remark 1.4.36

In the introduction we stated that a numerical range as a set whose closure is known to contain spectrum, we now study the properties of spectrum and numerical range.

Definition 1.4.37

The numerical range of A is defined and denoted as follows

$W(A) =\{\lambda\in \mathbb{C} : \lambda = \langle A(x), x\rangle \text{ for some } x\in H \text{ with } \|x\| =1\}$

One of the most interesting and surprising facts about the numerical range of any bounded operator on a complex Hilbert space is that it is a convex set. The line segment joining any two points in it, is itself contained in it.

Theorem 1.4.38 [5]

Let A $\in$ B(H) where it is finite dimensional;

Then;

$\sigma(A) \subseteq W(A)$

Theorem 1.4.40 [14]

The eigenvalues of every operator A belong to W(A)

Proof

If $A_x = \lambda_x$ with $\|x\| = 1$, then $\langle Ax, x \rangle = \lambda$

If A is normal then;

$\|A\| = \text{Sup}\ \{|\lambda| : \lambda \in W(A)\}$

So that there always exists a λ in W(A) such that $|\lambda| = \|A\|$

It follows that if a normal operator has sufficiency many eigenvalues to approximate its norm, but does not have one whose module is as large as the norm, then its numerical range will not be closed.

Remark 1.4.41

Now we study the spectrum in various classes of bounded operators on a Hilbert space H. One of the striking features of the collection of bounded operators on H is that very few of them commute with each other; ie AB does not general equal BA for A, B $\in$ B(H). In case a abounded operator A commutes at least with its own adjoint A* it forms important classes of operators on H, eg normal, unitary, self – adjoint .

Its important to recall the following definition on classes of operators.

Definition 1.4.42

Let A $\in$ B(H)

A Recall is called normal if $A^*A = AA^*$

unitary if $A^*A = I = AA^*$ ie $A^* = A^{-1}$

Self adjoint if $A^* = A$

Hyponomal if $A^*A \geq AA^*$

Remark 1.4.43

We first give various results on normal operators in relation to the spectrum.

Lamma 1.4.44 [5]

Let $A \in B(H)$ be a normal operator. Then

a) If $\lambda \neq \mu$ ker $(\lambda I-A) \perp$ ker (UI-T)

b) For every $\lambda \in \not\subset$ ker $(\lambda I-A)$ and ker $((\lambda I-A)^{\perp}$ are invariant under both.

A and A*

Proof

a) If $Ax = \lambda x$ and $Ay = \mu y$ then

$A^*y = \bar{\mu}y$ because ker A = ker A*

Hence we have;

$y \in \ker(\mu I-A) = \ker(\bar{\mu}I-A^*)$

Therefore

$\lambda\langle x,y\rangle = \langle Ax,y\rangle = \langle x, T^*y\rangle$

$= \langle x, \bar{\mu}y\rangle = \mu\langle x,y\rangle$

$\lambda \neq \mu$

and so $\langle x,y\rangle=0$

b) as $\lambda I-A$ commutes with

A and A*

Ker $(\lambda I-A)$ is invariant under both A and A* for all

$y \in \ker(\lambda I-A)$ we have

$\langle Ax,y\rangle = \langle x, A^*y\rangle = 0$

Hence $Ax \in (\ker(\lambda I-A)^{\perp}$

Similarly

$\langle A^*x,y\rangle = \langle x, T_y\rangle = 0$

for every $y \in \ker(\lambda I-T)$ and so

$T^*x = \ker(\lambda I-T)^{\perp}$

Theorem 1.4.45 [1]

Let $A \in B(H)$ be normal

a) if λ is an eigenvalue of A and x is a corresponding eigenvector,then $\bar{\lambda}$ is an eignenvalue of A* and the same x is an eigenvector of A* corresponding to λ.

b) Eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

c) Every spectral value of A is an approximate eigenvalue of A.

Proof

a) Let $Ax = \lambda x$ for some $\lambda \in \sigma(A)$ and $0 \neq x \in H$. Then $||A(x)|| = ||A^*x||$ for all $x \in H$

then;

$||A^*x - \lambda x|| = ||(A-\lambda I)^*x||$

$= ||(A - \lambda I)x|| = 0$

Hence $A^*x = \lambda x$

This proves (a)

b) Let $Ax_1 = \lambda_1 x_1$ and

$Ax_2 = \lambda_2 x_2$ for some $\lambda_1 \neq \lambda_2$

In and $x_1, x_2 \in H$

Then by (a) above

$A^*x_2 = \overline{\lambda} x_2$ so that

$\lambda_1 \langle x_1, x_2 \rangle = \langle \lambda_1 x_1, x_2 \rangle$

$= \langle Ax_1, x_2 \rangle$

$= \langle x_1, A^*x_2 \rangle$

$= \langle x_1, \overline{\lambda}_2 x_2 \rangle$

$= \lambda_2 \langle x_1, x_2 \rangle$

Since $\lambda_1 \neq \lambda_2$ we see that

$\langle x_1, x_2 \rangle = 0$

This proves (b)

c) Let $\lambda \in \sigma(A)$

Then we know that

$\sigma(A) = \{\lambda: \lambda \in \sigma_p(A)\} \cup \{\overline{\lambda}: \lambda \in \sigma_p(A^*)\}$

$= \{\lambda: \lambda \in \sigma_{ap}(A)\} \cup \{\overline{\lambda}: \lambda \in \sigma_p(A^*)\}$

then either $\overline{\lambda} \in \sigma_p(A^*)$ or $\lambda \in \sigma_{ap}(A)$

if $\overline{\lambda} \in \sigma_p(A^*)$ then by (a)

above, $\lambda \in \sigma_{p(A)} \subset \sigma_{ap}(A)$

Thus in any case, λ is

An approximate eigenvalue of A.

Thus in any case, λ is an approximate eigenvalues of A.

Remark 1.4.46

We had considered $A \in (B)H$ being normal and so the prove of (c) above clearly shows that in any case, λ is an approximate eigenvalue of A, which is a very important result and so we state the following theorem.

Theorem 1.4.47 [1]

If $A \in B(H)$ is normal then $R\sigma(A) = \phi$

Proof

Now, $\lambda \in \mathbb{C}$,elongs to $R\sigma(A)$ if $(\lambda 1 - A)^{-1}$ exist as a map but $R(\lambda 1 - A) \neq H$

Let $\lambda \in \mathbb{C}$,be such that $R(\lambda I - A) \neq H$

We show that this condition implies that $\lambda \in \sigma_p(A)$ when A is normal Which in turn implies $R\sigma(A) = \phi$

Since $R(\lambda I - A) \neq H$ it follows that $R(\lambda I - A)^{\perp} \neq \{0\}$

$\Rightarrow N(\lambda I - A^*) \neq 0$ where N is the Null set. By use of $R(A)^{\perp} = N(A^*)$ i.e. $\exists x \in H$

such that $(\lambda I - A^*)x = 0$____(*)

Since A normal, so is $\lambda I - A$.

This is $\equiv$ to the condition that;

$||(\lambda I - A)y|| = ||(\lambda I - A^*)y||$ for all $y \in H$____(**)

By (*) and (**) we have that $||(\lambda I - A)x|| = 0$ for some $x \neq 0 \rightarrow (\lambda I - A)x = 0$ for some $x \neq 0$.

i.e. $Ax = \lambda x$ has non-trivial roots in x.

i.e. $\lambda \in \sigma_p(A)$

Hence $R\sigma(A) = \phi$ Q.E.D.

Remark 1.4.48

Let us now turn to exhibit some results on self-adjoint operators and so we state the following theorem; which is very important.

Theorem 1.4.49 [25]

Let $A \in B(H)$ be self – adjoint. Then $\sigma(A) \subseteq R$.

Proof

Let $\lambda \in \not\subset$ with im $\lambda \neq 0$. Then for any $x \in H$ such that $x \neq 0$, we have that;

$0 < |\lambda - \bar{\lambda}| \, ||x||^2$

$= |<(A-\lambda I) x, x> - <(A-\bar{\lambda} I) x, x>|$

$= |<A-\lambda I) x, x> - <x,(A-\lambda I)x>|$

$\leq 2||(A-\lambda I)x|| \, ||x||$

if $\lambda \in \sigma(A) = \sigma_{ap}(A)$ then there exists a sequence of vector (x_n)

with $||x_n|| = 1$ such that

$(A-\lambda I)x_n \rightarrow 0$ ____(a)

since $2||(A-\lambda I)x_n||$ must

be greater than or equal to

$|\lambda - \underline{\lambda}|$ we have that

Lim $||(A-\lambda I)x_n|| \geq |\lambda - \bar{\lambda}| > 0$ ____(b)

$n \rightarrow \infty$

if im $\lambda \neq 0$

Thus (a) and (b) are compatible

If λ is such that

$\lambda = \bar{\lambda}$

im $\lambda = 0$

Therefore for λ to belong to

$\sigma_{ap}(A)$ we have that

im $\lambda = 0$

$\therefore \sigma_{ap}(A) \subseteq R$ ____(*)

Or $\sigma_{ap}(A) \subseteq R$

Remark 1.4.50

We now exhibit results of unitary operators.

Theorem 1.4.51 [25]

A is unitary iff $||A(x)|| = ||x||$ for all $x \in H$ and A is onto

Proof

If A is unitary, then for $x \in H$;

$<Ax, Ax> = <A^*Ax, x> = <x, x> = ||x||^2$

Also, since A is invertible, the range of A is the whole of H. Conversely, assume that $||Ax|| = ||x||$ for all $x \in H$ and A is onto for $x \in H$

$<(A^*A-I)x, x)> = <Ax, Ax> - <x, x>$

$= ||Ax||^2 - ||x||^2 = 0$

Since $(A^*A-I) = 0$. To prove $AA^* = I$ we show that A is invertible in B(H) if $A(x) = 0$ then $||x|| = ||A(x)|| = 0$, so that A is onto, $A^{-1}: H \to H$;

Is a well defined map and it is linear.

Also for $y \in H$, if $A(x) = y$

then $||A^{-1}(y)|| = ||A^{-1}A(x)|| = ||y||$

Hence A^{-1} is bounded and belongs to B(H) ie A is invertible in B(H).

Now $AA^* = (AA^*)(AA^{-1}) = A(A^*A)^{-1} = I$

and hence A is unitary.

Theorem 1.4.52 [1]

The spectrum of unitary operator U lies entirely on the unit circle.

Proof.

i) Since $||U|| = 1$, it follows that

if $|\lambda| > ||A||$, the operator $\lambda I - A$

is invertible, then the spectrum

of U is confined to the closed unit disc, $\{\lambda | |\lambda| \leq 1\}$.

ii) Let $|\mu| <|$. Then for any non-zero vector g, we have

$||\mu g|| = || g|| > ||\mu g||$, and so

$(\mu I - U)g \neq 0$

Thus $\mu I - U$ is one– to-one.

iii) Assuming as in (ii), that $|\mu| < 1$,

we shall show that the range of (μI-U) is dense in H. If this were not so, there would exist a non-zero vector h such that <(μI-U)g, h>=0 for every vector g. Choosing g the vector $\cup^{*}$h, we would obtain,

$<\mu\cup^{*}h, h> = <\cup \cup^{*}h, h> = <h, h>$

and by the Schwartz inequality we obtain

$||h||^{2} \leq |\mu| . || U^{*}|| . ||h||^{2}$. Dividing by $||h||^{2}$ and recalling that $|| U^{*}|| = 1$,

We would obtain $|\mu| \geq 1$, contradicting our hypothesis.

iv) Continuing to assume that $|\mu| < 1$, we can extend the result (of) (iii) to show that the range of μI-U consists of all of H.

Hence, μI-U is invertible, and so the spectrum is confined entirely to the circumference of the unit disc.

Remark 1.4.53

Now with the concepts of the spectrum and its properties we can now extend the study to spectrum and similarity.

CHAPTER TWO

ON QUASI-SIMILARITY AND SPECTRUM

2.0 INTRODUCTION

The property of quasisimilarity is more relaxed condition on oprerator than the property of similarity. Khalagai[15]; It is well known that any two similar operators have equal spectra. However the quasi-similar operators requires that more conditions are imposed for them to have equal spectra. In this chapter we give a brief run down of such results from various authors.

2.1 QUASI-SIMILARITY OF OPERATORS

RECALL;

Definition 2.1.1

An operator $A\in B(H)$ is said to be similar to another operator $B\in B(H)$ if there exists an invertible operator $S\in B(H)$ such that;

$B = S^{-1}AS$

Remarks 2.1.2

i) A unitary operator according to Patel [24] is called cramped if and only if its spectrum is an area of the unit circles with central angle less than π

ii) We now state without proof the following lemma on boundedness of two similar operators.

Lemma 2.1.3[12]

If any operator A is similar to an operator B, then A is bounded below iff B is bounded below. In other words in A and B are similar then; $\sigma_{ap}(A) = \sigma_{ap}(B)$

Example 2.1.4

Let H be two – dimensional Hilbert space and T be an operator on H with the matrix

then $T^* = S^{-1}T^{-1}S$.

however, T cannot be similar to an isometry as its spectrum is not in the disc.

Remark 2.1.5

Its clear that if A is any operator such that $S^{-1}AS=A^*$,see definition 2.1.1;where $0\in W(A)$ then spectrum of A is real, then similar operators have equal approximate spectrum. See[12].

Theorem 2.1.6[14]

Similar operators have.

i) The same spectrum

ii) The same point spectrum

iii) The same approximate point spectrum, and

iv) The same compression spectrum.

Remark 2.1.7

We now state a weaker condition of similarity called **Quasi-similarity**

Definition 2.1.8

Suppose T_1 and T_2 are spectral operators with resolutions of the identity E_1 and E_2 respectively. We say T_2 is weakly similar to T_1 if there is a densely defined closed linear transformation A on H with densely defined inverse such that.

i) $(AT_2 A^{-1})x = T_2x$ for every x in the domain of A^{-1} and

ii) for every Borel set B, there is a constant M_B such that

$||(AE_1 (B)A^{-1})x|| \leq M_B||x||$ for each x

in the domain of A^{-1}

Remark 2.1.9

Now that we have shown properties of spectrum of the operator:theorem 2.1.7 We extend this study to a more weaker condition of similarity called quasisimality.

Theorem 2.1.10 [14]

Quasi – similar spectral operators are weakly similar.

Remark 2.1.11

It is a well known fact in operator Theory that if A and B are operators with at least one of them invertible then AB and BA are similar operators by Khalagai [15]. We now state the following theorem with respect to quasi-similarity without the loss of generality of the concept of invertibility of the operator.

Theorem 2.1.12 [16]

Let $A,B \in B(H)$ be quasi-Invertible.

Then AB and BA are quasisimilar.

Proof

We first note that in the equations:

$(AB)A = A(BA)$

and

(BA)B = B(AB)

We let T =AB and S = BA

Thus we have

TA = AS

and

SB = BT

Now A and B are quasi-invertible implies T and S are quasisimilar. Hence AB and BA are quasisimilar.

Corollary 2.1.13 [16]

Let A,B $\in$ B(H), be quasi-invertible.

Then σ (AB) = σ (BA)

Under any one of the following conditions:

(i) AB and BA are hyponormal

(ii) AB is dominant and (BA)* is M-hyponornal.

(iii)AB and BA are p-hyponormal with U and V unitary in the polar decomposition AB = U│AB│ and BA = V│BA│.

Theorem 2.1.14 [16]

Quasi – similar hyponormal operators have equal spectra

Proof

If A and B are quasi-similar hyponormal operators; then for any complex number λ, A-λI and B-λI are also quasi – similar and hyponormal, so by the corollary 2.1.11 they are both invertible. Thus the spectrum of A is the same as that of B. Q.E.D.

Remark 2.1.15

We have shown that if A and B are quasi-invertible then AB and BA are quasi-similar.We now extend the study next aspect of quasi-similarity and CI operators.

2.2 QUASI-SIMILARITY AND CI OPERATORS

If AB and BA are similar operators then σ(AB) = σ(BA). W. Gong and D. Han [11] proved among other results that an operator

B∈ B (H) is CI operator iff

σ(B*B) = σ(BB*)

We use this result to deduce a number of results on CI operators.

Corollary 2.2.1 [16]

Let B be quasi-invertible.

Then B is a CI operator.

Proof

We note from what was proved by W.Gong[11] ie.since B is quasi-inevertible we have that

σ(B*B) = σ(BB*)

Hence B is a CI operator. Q.E.D.

Corollary 2.2.2 [13]

Let B∈ B (H) be such that O∉ W(B). Then both B^* and B are CI operators.

Proof

We first note that if O ∉ W(B) then both B and B* are quasi-invertible.

Hence by corollary 4 above B and B* are CI operators. Q.E.D.

Theorem 2.2.3 [16]

If B is an M-hyponormal operator satisfying the equation

BX = XB*

Where X is quasi-invertible then B is a CI operator.

Proof

Since B is M-hypononormal

BX = XB* implies

B*X = XB

Taking adjoints we have:

BX* = X*B* and B*X* = X*B

Now using the equations above we have:

B*BX = B* X B* = XBB*

and

BB*X* = B X* B = X*B*B

i.e BB* and B*B are quasi-similar since X* is also quasi-invertible.

Thus σ(BB*) = σ(B*B) implying B is a CI operator.

Corollary 2.2.4 [16]

If an M-hyponormal operator B is quasi-similar to its adjoint B* then B is a CI operator.

Proof

In this case there exist quasi-invertible operators X and Y such that

BX = XB* and B*Y = YB

Thus the proof is immediate

The following result due to Duggal [7] is required in the proof of our next theorem. Q.E.D.

Theorem 2.2.5 [16]

Let A:$H_1 \to H_1$, B: $H_2 \to H_2$ and

X: $H_2 \to H_1$ be operators such that

AX = XB

Where H_1 and H_2 are Hilbert spaces.

If A is dominant and B* is M-hyponormal them

A*X = XB*

Theorem 2.2.6 [16]

Let A,B,X$\in$ B(H) be such that

BX = XA, where B is dominant, A* is M-hyponormal and X is quasi-invertible. If B is a CI operator, then A is also a CI operator.

Proof

In this case,

BX = XA implies B*X = XA*

Taking adjoints we also have:

A*X* = X*B*

and

AX* = X*B

Now using these equations we have

B*BX = B*XA = XA*A

and

A*AX* = A*X*B = X*B*B

i.e B*B and A* A are quasi-similar and hence

σ(B*B) = σ(A*A)

Similarly we have that

BB*X = BXA* = XAA*

and

AA*X* = A X*B* = X*BB*

i.e BB* and AA* are quasisimilar and hence

$\sigma(B$B*) = σ(AA*)

Now if B is a CI operator then we have that

σ(B*B) = σ(BB*) = σ(AA*) = σ(A*A)

Hence A is also a CI operator.

Corollary 2.2.7[16]

If a dominant operator B is quasisimilar to any operator A with A* M-hyponormal, then B is a CI operator implies A is also a CI operator.

Proof

In this case, there exist quasi-invertible operators X and Y such that

BX = XA and AY = YB Taking adjoints we also have:

A*X* = X*B*

and

AX* = X*B

And by theorem2.2.8 if B is quasisimilar to operator A , the we take the adjoint of A to p m-hyponomal implying tha BB* and AA* are quasisimilar , then B is CI operator.

Hence A is also a CI operator. Q.E.D.

Theorem 2.2.8 [16]

Let $A, B \in B(H)$ be quasi similar normal operators. Then A and B are unitarily equivalent and hence have equal spectra.

CHAPTER THREE

ON QUASI-SIMILARITY AND ESSENTIAL SPECTRUM

3.0 INTRODUCTION

The concept of quasi-similarity and equality of spectra for a given pair of operators has been considered by a number of authors, among them W.C. Clary [2] who showed that quasi-similar hyponormal operators have equal spectra. J.M. Khalagai and B. Nyamai [15] showed that if $A\ and\ B$ quasi-similar operators, with A dominant and B^* M.hyponormal then $A\ and\ B$ have equal spectra. B.P.Duggal [7] showed that if $A_{i\,.}\ i = 1{,}2$ are quasi-similar operators such that $U_{i\,.}$is unitary in the polar decomposition $A_{i\,.} = U_{i\,.}|A_{i\,.}|$ then A_1 and A_2 have equal spectra and also equal essential spectra. J.P Williams [27] showed that there are several cases which imply that two operators A and B have equal essential spectra under quasi-similarity. For example if A and B are both hyponormal or are both partial isometrics or are quasi-normal.

In this chapter we prove results on equality of spectra and essential spectra for classes of operators that satisfy the Putnam-Fuglede property.

3.1 QUASI-SIMILARITY AND SPECTRUM OF THE OPERATOR

Theorem 3.1.1 [27]

(Putnam-Fuglede property)

Let $A, B \in B(H)$ be normal operators . For any other operator X we have that $AX = XB$ implies $A^*X = XB^*$.

The following result by J.M. Khalagai and B.Nyamai [15] will be required.

Theorem 3.1.2 [15]

Let A,B and X be operators such that AX=XB implies $A^*X = XB^*$. If X is either one-one or has dense range then A and B are normal operators.

The following result by R.G. Douglas [5] will also be required for the proofs of our results.

ie theorem 2.2.8 above.

.

Theorem 3.1.3 [17]

Let $A, B, X \in B(H)$ be operators such that,$AX = XB$ where $A\ \ and\ B$ satisfy Putnam –Fuglede property and X is a quasi-affinity.

Then we have:

i. $\sigma(A) = \sigma(B)$

ii. $\sigma(AA^*) = \sigma(BB^*)$

iii. $\sigma(A^*A) = \sigma(B^*B)$

Proof

We note that $AX = XB$____(1)

implies $A^* X = XB^*$. On taking adjoints we have

$BX^* = X^*A$____(2).

Since X is a quasiaffinity it follows from (1) and (2) that $A\ and\ B$ are quasi-similar. It now follows from both theorem 2.2.8 and 3.1.2 above that A and B are unitarily equivalent normal operators. Hence $\sigma(A) = \sigma(B)$.

Also using the equations $AX = XB,\ A^*X = XB^*, BX^* = X^*A$ and $B^*X^* = X^*A^*$ we get

$A^*AX = A^*XB = XB^*B$____(3)

and

$B^*BX^* = B^*X^*A = X^*A^*A$____(4)

Also $AA^*X = A\,X\,B^* = XBB^*$____(5)

and

$BB^*X^* = BX^*A^* = X^*AA^*$____(6)

From (3)and (4) A^*A and B^*B are quasi similar positive operators. Hence $\sigma(A^*A) = \sigma(B^*B)$.

Also from (5) and (6) A^*A and B^*B are quasi-similar positive operators.

Hence $\sigma(AA^*) = \sigma(BB^*)$.

We note that from the results by J.P Williams [27] the following corollary is immediate.

Corollary 3.1.4 [17]

Let A,B, X $\in$ B(H) be operators such that

AX=XB

where X is quasi-affinity then

$\sigma(A) = \sigma(B), \sigma(A^*A) = \sigma(B^*B)$ and $\sigma(AA^*) = \sigma(BB^*)$ under any one of the following conditions:

i. A is dominant and B^* is m-hyponomal
ii. A is dominant and B^* is p-hyponomal
iii. A and B^* are p-hyponomal

Proof

We note that all the classes of operators stated above are known to satisfy Putnam-Fuglede property and exist quasi-affinites X and Y such that AX=XB and BY=YA. We also note that in case of part,(iii),if in addition we have that in the polar decomposition A=V$|A|$ and B=V$|B|$.

then $\sigma_e(A) = \sigma_e(B)$.

W.Gong and.Han [11]) proved among other results that an operator B$\in$B(H) is CI operator iff $\sigma(B^*B) = \sigma(BB^*)$.

From this result the following corollary is immediate. Q.E.D.

Corollary 3.1.5[17]

Let $A, B, X \in B(H)$ be operators such that:

$AX = XB$ where A and B satisfy Putnam –Fuglede property and X is a quasi-affinity.

Then we have:

i. $\sigma_e(A) = \sigma_e(B)$
ii. $\sigma_e(AA^*) = \sigma_e(BB^*)$
iii. $\sigma_e(A^*A) = \sigma_e(B^*B)$

Proof

(i) If A, B $\in$B(H) , then the essential spectrum are equal for positive operator and by Putnam-Fuglede property theorem 3.1.1, if $A, B, X \in B(H)$ we have $AX = XB$ which cleary implies $A^*X = XB^*$ and so if X is quasi affinity then; $\sigma(A) = \sigma(B)$ by corollary 3.1.4 above. And for positive operators hence the result for essential spectra ie; $\sigma_e(A) = \sigma_e(B)$ which proves part (i).

(ii) Part (i) implies part (ii) by theorem 3.1.3 also see[17]., then $\sigma(AA^*) = \sigma(BB^*)$ and hence by theorem 3.1.3 and corollary 3.14 then we have the result $\sigma_e(AA^*) = \sigma_e(BB^*)$, which proves part (ii).

(ii) It’s a well know fact that $\sigma(AA^*) = \sigma(BB^*)$see[17] and we also note $AA^*X = A X B^* = XBB^*$ where X is quasiaffinity ; similary $B^*BX^* = B^*X^*A = X^*A^*A$ and therefore we get that A^*A and B^*B are quasi-similar positive operators which implies that $\sigma(AA^*) = \sigma(BB^*)$ hence we conclude that $\sigma_e(A^*A) = \sigma_e(B^*B)$ which proves part (iii) see[17]. Q.E.D.

Corollary3.1.6 [17]

Let $A, B \in B(H)$ be quasi-similar operator which satisfy Putnam-Fuglede property. Then we have that B is a $C.I$ operator whenever A is and conversely.

Proof

We note that for such operators A and B.

$\sigma(AA^*) = \sigma(BB^*)$ and $\sigma(B^*B) = \sigma(A^*A)$.

Thus if A is CI operator then $\sigma(A^*A) = \sigma(AA^*) = \sigma(B^*B) = \sigma(BB^*)$. Hence A is a C.I operator.

In our next result we show that by imposing more stringent conditions on the intertwining we obtain equality of operators rather than equality of their spectra. In so doing we require the following result by I. H Sheth and J.M Khalagai [7]

Theorem 3.1.7[26]

Let $T, S, X \in B(H)$ be operators such that.

$TX = XS$ and $SX = XT$ with $O \notin W(X)$ where $S - T$ is normal.

Then $S = T$

We have the following result in this direction.

Theorem 3.1.8[17]

Let $A, B, X \in B(H)$ be operators such that A and B satisfy Putnam-Fuglede property and $AX = XB$ with X self adjoint and then $A^*A = B^*B$ and $AA^* = BB^*$

Proof

We first note that $AX = XB \Longrightarrow A^*X = XB^*$

Thus taking adjoints we also have $BX = XA$.

Thus using these equations we have: $A^*AX = A^*XB = XB^*B$ and $AA^*X = AXB^* = XBB^*$

Now letting $T = A^*A$ and $S = B^*B$

we have that $S - T = B^*B - A^*A$ is normal.

Hence by theorem 3.1.7 above

$B^*B = A^*A$

Similarly if $T = AA^*$ and $S = BB^*$

then $BB^* - AA^*$ is normal.

Hence $BB^* = AA^*$

Consequently $AA^*$and BB^* have equal spectra and essentialspectra.this is the same case with the operators $AA^*$and BB^*

REMARK 3.1.9

We note that $A = B$ implies $A^*A = B^*B$ and $AA^* = BB^*$by theorem 3.1.7 also see [26]

However the converse is not necessarily true.

CHAPTER FOUR

ON EQUALITY OF SPECTRA AND ESSENTIAL SPECTRA

4.0 INTRODUCATION

It is a well known fact in operator theory that for any operator A, the essential spectrum of A is contained in the spectrum of A [17]. In this chapter we show that quasisimilar pure dominant operators have their essential spectra equal to their spectra provided one of the interfering quasiaffinities is compact.

An operator $A \in B(H)$ is said to be a quasiaffinity if A is both one-one and has dense range. Two operators A and B are said to be similar if there is an invertible operator S such that $AS = SB,$ while A and B are said to be quasisimilar if there exist quasiaffinities X and Y such that $AX = XB$ and $BY = YA$.

4.1 EQUALITY OF SPECTRA

Here we show that quasisimilar pure dominant operators have their essential spectra equal to their spectra provided one of the interfering quasiaffinities is compact.We first state the following theorem.

Theorem 4.1.1 [28]

Suppose that T is a pure dominant operator, K is a compact operator having dense range and $KT = TK$. Then spectrum of T is equal to essential spectrum of T.

Remark4.1.2

It is at this point that we also pick up the quest of delving into this theory .

<u>Definition 4.1.3</u>

Let $x \in \mathcal{H}$ We define $\varrho_T(x)$ to be the set of complex numbers α for which there exists a neighbourhood V_α of α with μ analytic on having values in H such that $(zI - T)u(z) = x$ on $\forall_\alpha$.

We say that T has the single valued extension property (in short SVEP) if $(zI - T)\,u(z) = 0$ implies $u = 0$ for any analytic function u defined on any donation D of a complex plane with values in H.

An operator $T \in H$ is said to satisfy Dunford's property (c) if for each closed subset F of the complex plane the corresponding local spectrum subspace $H_T(F) - \{x \in H{:}\, \sigma(T, x) \subset F\}$ is closed.

Theorem 4.1.4 [28]

Suppose A and B are dominant operators satisfying Dunford's property (c) and are quasisimilar with at least one of the implementing quasiaffinities compact, then we have A and B are equal spectra and also equal essential spectra.

Note that Problem of looking for conditions under which the essential spectrum is equal to spectrum of a given operator has also been considered by a number of authors. In particular J. P. Williams [23] apart from showing that there are several cases under which quasisimilar operators A and B have equal essential spectra also proved the following result on equality of spectrum and essential spectrum for a given operator.

LEMMA 4.1.5 [4]

(i) If A is compact then so is A^*

(ii) If A is compact and B is bounded then AB and BA are also compact.

Remark 4.1.6

We not from the Lemma above that if A is compact and B is bounded then their products are also compact and we now state the following theorem if we take compact quasiaffinities.

Theorem 4.1.7 [18]

Let $A, B \in B(H)$ be quasisimilar pure dominant operators with at least one of the intertwining quasiaffinities compact. Then we have:

$\sigma_e(A) = \sigma(A)$

$\sigma_e(B) = \sigma(B)$

Proof: Since A and B are quasisimilar there exist two quasiaffinities X and Y such that

$AX = XB$ and $BY = YA$

We also have that either X or Y is compact implies XY and YX are compact operators each with dense range. It can also be verified easily that $[A, XY] = 0$ and $[B, YX] = 0$. Now from theorem A above we have $\sigma_e(A) = \sigma(A)$ and $\sigma_e(B) = \sigma(B)$

Hence the result. Q.E.D.

<u>**Corollary 4.1.8 [18]**</u>

Let $A, B \in B(H)$ be quasiinvertible operator's with either A and B compact. If AB and BA are pure dominant operators then we have $\sigma_e(AB) = \sigma(AB)$ and $\sigma_e(BA) = \sigma(BA)$

Proof: We first note that quasi invertibility is the same as quasiaffinity. Thus A and B are quasiaffinity and AB and BA are also compact quasiaffinity which are quasisimilar since we have:

$(AB)A = A(BA)$ and

$(BA)B = B(AB)$

Hence by theorem 1 above $\sigma_e(AB) = \sigma(AB)$ and $\sigma_e(BA) = \sigma(BA)$ Q.E.D.

Theorem 4.1.9 [18]

Let A be a pure dominant operator and B be such that $AX = XB$ implies $A^*X = XB^*$ where X is a compact quasiaffinity, then $\sigma_e(A) = \sigma(A)$.

Proof: Since $AX = XB$ implies $A^*X = XB^*$ it can easily be verified that

$[A, XX^*] = 0$ and

$[B, X^*X] = 0$

Where XX^* is compact with dense range. Hence by theorem 4.1.1 above:

$\sigma_e(A) = \sigma(A)$

Corollary 4.1.10 [18]

If A is a pure dominant operator such that

$AX = XA^*$ and $A^*X = XA$

Where X is a compact quasiaffinity then $\sigma_e(A) = \sigma(A)$

Proof: In this case $[A, XX^*] = 0$ where XX^* is compact with dense range, hence the result.

Corollary 4.1.11 [18]

Let $A, B \in B(H)$ be quasiinvertible operators with either A or B compact. If AB and BA are pure dominant operators satisfying Dunford's property (c) then we have

$\sigma_e(AB) = \sigma(AB) = \sigma(BA) = \sigma_e(BA)$

Corollary 4.1.12 [18]

Let A and B be quasisimilar pure dominant operators which satisfy Dunford's condition (c) with at least one of the implementing quasiaffinities compact. Then we have $\sigma_e(AB) = \sigma(A) = \sigma(B) = \sigma_e(B)$

Proof

Let A and B be quasi similar of pure dominant part then from theorem 4.1.9 then $\sigma_e(AB) = \sigma(AB)$____(1)

Similary; for the commuting operators the following condition also hold

$\sigma(BA) = \sigma_e(BA)$____(2)

From (1) and (2) hence we can conclude

$\sigma_e(AB) = \sigma(AB) = \sigma(BA) = \sigma_e(BA)$ Q.E.D.

4.2EQUALITY OF ESSENTIAL SPETRA OF QUASISIMILAR QUASINORMAL OPERATORS

Let T_1 and T_2 be quasisimilar hyponormal operators on infinite dimensional Hilbert spaces.

S. Clary proved in [2] that $\sigma(T_1) = \sigma(T_2)$. Duggal [9] showed that there are several cases which imply $\sigma_e(T_1) = \sigma_e(T_2)$. For example, if T_1 and T_2 are both biquasisimilar, if T_1 and T_2 are both weighted shifts (bilateral or unilateral), or if T_1 and T_2 are both partial isometries, then $\sigma_e(T_1) = \sigma_e(T_2)$.

The purpose here is to prove that if T_1 and T_2 are both quasinormal, then $\sigma_e(T_1) = \sigma_e(T_2)$. Suppose that T is an operator . Thus, in order to prove that two quasisimilar quasinormal operators T_1 and T_2 have equal essential spectra, it suffices to study the pure parts of T_1 and T_2. Hence we shall begin by considering the pure parts of quasinormal operators.

Denote the index of T. It is well-known that $\sigma_e(T) = \{\lambda \epsilon\ C: T - \lambda \text{ is not Fredholm}\}$. A *hole* in $\sigma_e(T)$ is a bounded component of $C/\sigma_e(T)$. It is also well-known that if H is a hole is $\sigma_e(T)$, then $i(T-\lambda)$ is constant on H.

We shall first prove the following Theorem.

THEOREM 4.2.1 [27].

Suppose that T_1 and T_2 are quasisimilar quasinormal operators an infinite dimensional Hilbert spaces Then $\sigma_e(T_1) = \sigma_e(T_2)$.

Proof. Note T_1 is unitarily equivalent to $N_i \oplus V_{\mathcal{H}_i} \hat{P}_i$ on $\mathcal{H}_i + \widehat{\mathcal{H}}_i$ where N_i is a normal operator on the Hilbert space $\mathcal{H}_i$ and P_i is a positive definite operator on $\mathcal{H}_i$, $(i = 1,2)$ implies that both $T_1$and T_2 are normal. Thus, in this case, T_1 and T_2 are unitarily equivalent and $\sigma_e(T_1) = \sigma_e(T_2)$. Hence we may assume that both $\mathcal{H}_1$ and $\mathcal{H}_2$ are nonzero. In order to complete the proof, it suffices to show that $\sigma_e(V_{\mathcal{H}_1}\hat{P}_1) = \sigma_e(V_{\mathcal{H}_2}\hat{P}_2)$. There exist quasiaffinites X and Y such that$X(N_1 \oplus V_{\mathcal{H}_1}\hat{P}_1) = (N_2 \oplus V_{\mathcal{H}_2}\hat{P}_2)X$ and $(N_1 \oplus V_{\mathcal{H}_1}\hat{P}_1) = Y(N_2 \oplus V_{\mathcal{H}_1}\hat{P}_2)$

EXAMPLE 4.2.2[27]

Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let V be the unilateral shift on $\mathcal{H}$ defined by $Ve_n = e_{n+1}, n = 1, 2, \dots.$ According to Lemma 9, there exist a quasiaffinity W and a positive operator R in $\mathcal{L}(\mathcal{H})$ by $Ue_n = \left({}^{1}\!/_{2}{}^{n}\right)_{e_n,}$ $n = 1, 2, \dots.$ The operators U is a quasiaffinity and $1/2VU = UV$.Let $T_1 = \hat{V} \oplus 1/2 \oplus 1/2\hat{R}$ on $\widehat{\mathcal{H}} \oplus \mathcal{H} \oplus \widehat{\mathcal{H}} \to \widehat{\mathcal{H}} \oplus \widehat{\mathcal{H}}$

by$X\left((x_1, x_2, x_3, \dots) \oplus x_0 \oplus (y_1, y_2, y_3, \dots.)\right) = (x_1, x_2, x_3, \dots) \oplus (Wx_0, x_1, x_2, x_3, \dots)$ and

$Y: \widehat{\mathcal{H}} \oplus \widehat{\mathcal{H}} \to \widehat{\mathcal{H}} \oplus \mathcal{H} \oplus \widehat{\mathcal{H}}$ b $Y\left((x_1, x_2, x_3, \dots) \oplus (y_1, y_2, y_3, \dots)\right) = (x_2, x_3, \dots) \oplus Ux_1 \oplus (y_1, y_2, y_3, \dots)$

It is clear that X and Y are quasiaffinites and a routine calculation shows that $XT_1 = T_2X$ and $T_1Y = YT_2$. Hence T_1 and T_2 are quasisimilar. Note that the pure part of T_1 is $\hat{V} \oplus 1/2V$ and the pure part of T_2 is $\hat{V}$. We shall show now that $\hat{V} \oplus 1/2V$ and $\hat{V}$ are not quasisimilar by using the same argument. Suppose that there exists a quasiaffinity $Z: \hat{\mathcal{H}} \oplus \mathcal{H} \to \hat{\mathcal{H}}$ such that $Z(\hat{V} \oplus 1/2V) = \hat{V}Z$. Define $W: \mathcal{H} \to \hat{\mathcal{H}}$ by $Wx = Z(0 \oplus x)$.

Let $m \to \mathcal{H}$ is injective. Since $\hat{V}$ is completely nonunitary, $\hat{V}|m$ is a nonunitary isometry thus for $1/2 < |\lambda| < 1$, λ is an eigenvalues of $(\hat{V}|m)^*$ and thus also of $(1/2)V^*$. The last statement is clearly a contradiction. Therefore, the pure parts of T_1 and T_2 are not quasisimilar J. Conway also proved in [3] that subnormal operators are similar if and only if their normal parts are unitarily equivalent and their pure parts are similar.Thus the equality of the essential spectra of quasisimilar quasinormal operators is not a result of similarity.\

REMARK4.2.3

The following example shows that two quasisimilar quasinormal operators need not be similar even if both operators are pure.

Example 4.2.4[27]

Let $\mathcal{H}$ be a Hilbert space with a orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let $c_1 = 1, d_1 = 1/2, d_{2n} = c_{2n} = 1/4$, $n = 1, 2, \dots.$, and $d_{2n+1} = c_{2n+1} = 1, n = 1, 2, \dots$. Define positive definite operators P_1 and P_2 on $\mathcal{H}$ by $P_1e_n = c_ne_n$ and $P_2e_n = d_ne_n, n = 1,2, \dots$ Define an operator Y_1 on $\mathcal{H}$ by the following: let $Y_1e_1 = e_2$ and for each positive integer n let $Y_1e_{2n} = e_{2n+2}$ and $Y_1e_{2n+1} = e_{2n-1}$.

Let $X_n = P_2^n(P_1^{-1})^n$ and $Y_{n+1} = P_1^nY_1(P_2^{-1})^n$, $n = 1,2, \dots$ Observe that, for each positive integer $3n$, we have $\|X_n\| = \|Y_n\| = 1, X_{n+1}P_1 = P_2X_n$, and $Y_{n+1}P_2 = P_1Y_n$

Let $X = \sum_{n=1}^{\infty} \oplus X_n$ and $X = \sum_{n=1}^{\infty} \oplus Y_n$. Then X and Y are quasiaffinities on $\hat{\mathcal{H}}$, $XV_{\mathcal{H}}\hat{P}_1 = V_{\mathcal{H}}\hat{P}_2X$, and $V_{\mathcal{H}}\hat{P}_1Y = YV_{\mathcal{H}}\hat{P}_2$. Hence $T_1 = V_{\mathcal{H}}\hat{P}_1$ and $T_2 = V_{\mathcal{H}}\hat{P}_2$ are quasinormal operators. We show next that T_1 and T_2 are not similar. The operator T_1 is unitarily equivalent to $\sum_{n=1}^{\infty} \oplus c_nV$ and T_2 is unitarily equivalent to $\sum_{n=1}^{\infty} \oplus d_nV$. It follows that $\|(T_1 - 1/2)x\| \geq 1/4\|x\|$ for each x in $\hat{\mathcal{H}}$. Thus $T_1 - 1/2$ has closed range. Since $1/2\,V$ is one of the direct summands of $\sum_{n=1}^{\infty} \oplus d_nV$ and $1/2\,V - 1/2$ does not have closed range, if follows that $T_1 - 1/2$ does not have closed range. Hence T_1 and T_2 are not similar.

Remark4.2.5

As mentioned before, J. Conway proved in [3] that the normal parts of quasisimilar subnormal operators are unitarily equivalent. In that paper he also provided an example4.2.2 which showed that the pure parts of quasisimilar subnormal operators need not be quasisimilar. Close scrutiny of his example will reveal that one of the two quasisimilar subnormal operators is not quasinormal. However, a slight modification of his example will show that the parts of quasisimilar quasinormal operators need not be quasisimilar.

Therefore, the pure parts of T_1 and T_2 are not quasisimilar J. Conway also proved in [3] that subnormal operators are similar if and only if their normal parts are unitarily equivalent and their pure parts are similar. Hence the two quasisimilar quasinormal operators are not similar by example 4.2.3. Thus the equality of the essential spectra of quasisimilar quasinormal operators is not a result of similarity.

CHAPTER FIVE

SUMMARY

5.0 Chapter wise summary

Chapter one is introduction.Where brief history of concepts was given. Thereafter literature review of the subject matter was outlined. Notations and terminologies that were used were also defined in this chapter.Also aspects of inclusions of classes of operators in Hilbert spaces were noted and towards the end of the chapter the concept of spectrum and its decomposition is introduced.

Chapter two we have looked at the role of quasi-invertibility in relationshipto quasi-similarity of opcerators.Indeed through theorem 2.1.15 and its colloraries we have results on some classes of operators which give equality of spectra for not only operators A and B but also their products AB and BA. This approach then takes us easily into theory of operators which are consistent in invertibility. Theorems 2.2.3, 2.2.4 and 2.2.5 together with their corollaries in the study constitute additional knowledge to what already exists on consistent in invertibility of operators ; which make asignificant contribution to knowledge in operator theory.

In chapter 3 we have strived to look at the role of essential spectrum in relationship to quasi-similarity of operators. We have shown that in most cases where two given operators have equal spectra it turns out that they also have equal essential spectra.Indeed theorem 3.1.3 together with its corollaries adds significant knowledge in operator theory.

In chapter 4 we have looked at conditions under which for agiven operator its spectrum is equal to its essential spectrum .We note that quasi-similarity of operator still plays an important role here.Through theorems 4.1.7 and 4.1.9 together with their corollaries the study have shown equality of spectra and essential spectra for given operators say A and B or their product AB or BA. How ever these results have been centered mainly on the class of pure dominant operators.

This study have shown under which conditions the operator can have equal spectra and essential spectra(theorem 4.1.9)

5.2 Conclusion

In this thesis, we have made several key contributions to the study of quasi-invertibility,quasi-similarity and equality of spectra with essential spectrum.These results could be used to give more insights into the problem of determining the structure of operators in some classes of operators in Hilbert spaces.Parallel results have been established:sometimes there was no need to specify the classes of operators.An attempt have been made to extend the results to infinite dimensional hence making conditions of finite dimensional conditions to be relaxed.

The results in this thesis have shown strong conditions of Quasi-similarity and and quasi-invertibilty of operators while focusing on essential spectrum.Finally the aspect of equality of spectra and essential spectrum was studied and results drawn. These results could be used to give more insight into problems of determining structure of operators which is a major content under study in operator theory.

It should be noted that many problems have their mathematical formulation as an operator equation in interpretation of quantum mechanical observables. Indeed the study have added knowledge and my assist in improving the formulation of quantum mechanics observables and solve problems in physics, engineering and physical chemistry.

5.2 FUTURE RESEARCH

In this last section we shall briefly present some problems which are of interest for possible future work.

- Study of weyl spectrum: investigating its properties and comparing them with essential spectrum.
- Study Browder spectrum its properties draw similaties of essential,weyl and Browder spectrums.

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APPENDIX 1

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