ON NUMERICAL RANGES OF SOME OPERATORS IN HILBERT SPACES

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BY

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Declaration

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

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In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

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Dedication

I dedicate this project to my mum Getrude Otae and sisters Nancy, Quintor and Lilian.

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List of Abbreviations

- T^*: The adjoint of an operator T.
- $\mathcal{H}$: Hilbert space over the complex number $\mathbb{C}$.
- $\|T\|$: The operator norm of T.
- $\|x\|$: The norm of vector x.
- $< x, y >$: The inner product of x and y on the Hilbert space $\mathcal{H}$.
- $w(T)$: Numerical radius of an operator T.
- $W(T)$: Numerical range of an operator T.
- $ker(T)$: Kernel of an operator T.
- $Ran(T)$: Range of an operator T.
- $\sigma(T)$: The spectrum of an operator T.
- $\rho(T)$: The resolvent set of an operator T.
- $B(\mathcal{H})$: Banach algebra of bounded linear operator on the Hilbert space $\mathcal{H}$.
- $\sigma_p(T)$: The point spectrum of an operator T.
- $\sigma_c(T)$: The continuous spectrum of an operator T.
- $\sigma_r(T)$: The residual spectrum of an operator T.
- $r(T)$: The Duggal transform of the operator T.
- $\tilde{T}$: The Aluthge transform of the operator T.
- $Hol(\sigma(T))$: The algebra of all complex valued functions which are analytic on some neighborhood of $\sigma(T)$.

- $T = U|T|$: The polar decomposition of the operator T.
- $\Gamma(\sigma(T)) = U^*\Delta(T)U$: The polar decomposition of the Duggal transform.
- $\{\Delta^n(T)\}_{n=0}^{\infty}$: The Aluthge sequence.

Abstract

In this project, we investigate the numerical ranges of some basic operators.
We develop the study of the Numerical ranges of these operators from the study of the resolvent sets, the spectrum(the classical classification of the spectrum) and the numerical radii of some of these operators under study.
We then cap it off by an insightful look at the Numerical ranges of some selected operators.
We will also get to study pseudo-spectrum and essential spectrum as part of the wider classification of the spectrum of some operators.

Contents

Chapter 1

Preliminaries

This chapter summarizes the necessary tools required to successfully execute the study of the numerical ranges of some operators. We outline this chapter in the following folds; Introduction, Notations and Terminologies and finally some Properties of Bounded Linear operators and Banach spaces, that we will use in the sequel.

1.1 Introduction

Functional analysis has its origin in the the theory of ordinary and partial differentials used to solve several physical problems. Joseph Fourier [1768-1830], on his work The Theory of Heat, triggered the development of trigonometric series which required the implicit definition of a function and convergence and also the Lebesque integrals. This also triggered the development of transforms such as the Aluthge transform by A. Aluthge[1900], Fourier transform, among other transforms. During this process, the spectral theory, which is the focal point of functional analysis was developed. David Hilbert[1836-1943], published a number of papers on integral equations in which he started transforming the integral equations to a finite system of equations under the restriction that the kernel function is symmetric. In the process of his study, he classified operators in terms of their spectral properties on a Hilbert space, which he referred to as infinite dimensional complete normed linear spaces.
Going forward, we are going to base our deliberations on linear operators. A linear operator is a linear transformation from a vector space to itself. We can therefore confirm that a linear operator is a transformation which maps linear subspaces to linear subspaces.
In studying the numerical ranges of operators, we are going to view our operators as matrices which is the model for operator theory. Toeplitz [1909] found out that every linear operator can be represented by a matrix for easier operations on these operators.
Also, Cauchy[1826] discovered eigenvalues and a generalization of a square matrix. Cauchy[1826] proved the spectral theory for self adjoint matrices, i.e, every real symmetric matrix is

diagonalizable, and this theory was later generalized into spectral theory for normal operators[Neumann, 1942].

1.2 Notations and Terminologies

Throughout this project, Hilbert spaces are non zero complex and separable. They may be finite or infinite dimensional. We are going to use upper case letters $\mathcal{H}$ and $\mathcal{K}$ e.t.c to, denote Hilbert spaces or subspaces of Hilbert spaces.
$B(\mathcal{H})$ will denote the Banach algebra for bounded linear operators on $\mathcal{H}$. $B(\mathcal{H}, \mathcal{K})$ will denote the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ equipped with the norm.
By an operator, we mean a bounded linear transformation.
If T is the matrix representation for a given vector x in $\mathcal{H}$ is equivalent to multiplying such an x by a constant say λ , that is $Tx = \lambda x$; then x is called the eigenvector of T corresponding to an eigenvalue λ.
The spectrum of a linear operator on a finite dimensional Hilbert space is the set of all its eigenvalues. The set of all such λ such that $\lambda \mathrm{I} - T$ has a densely defined continuous inverse is the resolvent set of T, denoted by $\rho(T)$. The complement of $\rho(T)$ denoted by $\sigma(T)$ is the spectrum of T.
For an operator T, we denote $\sigma(T), W(T), w(T), r(T), \sigma_p(T), \sigma_c(T), \sigma_r(T)$ the spectrum, the numerical range, the numerical radius, the spectral radius, the point spectrum, the continuous spectrum and the residual spectrum of T respectively.
An operator P on a Hilbert space $\mathcal{H}$ is said to be idempotent if $P = P^2$. If P is idempotent, then $Ran(P) = ker(I - P)$ so that $Ran(P)$ is a subspace of $\mathcal{H}$.
$T \in B[\mathcal{H}, \mathcal{K}]$ is invertible if it has an inverse and the $Ran(T) = \mathcal{K}$, and such an inverse must be bounded.
An operator $T \in B(\mathcal{H})$ is said to be:
Unitary if $T^*T = TT^* = \mathrm{I}$.
Normal if $T^*T = TT*$.
Self adjoint if $T^* = T$.
Idempotent if $T^2 = T$.
Isometry if $T^*T = \mathrm{I}$.
The kernel of an operator T defined by $Ker(T) = N(t) = \{x \in \mathcal{H} : Tx = 0\}$ is a subspace of $\mathcal{H}$ containing all the elements that have been mapped to the identity by the operator T.
$Conv(T)$ is the convex hull of T is the smallest convex set containing T.
A set Ω is said to be convex if for any two points $x, y \in \Omega$, we have that $z = tx + (1-t)y \in \Omega$, $\forall t \in [0, 1]$.
$Ran(T)$ is the range or image of T.

1.3 Some Properties of Bounded Linear Operators

Definition 1.3.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces over the complex plane $\mathbb{C}$. A function T which maps $\mathcal{H}_1$ into $\mathcal{H}_2$ i.e $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is called a linear operator if for all $x, y \in \mathcal{H}_1$ and $\alpha \in \mathbb{C}$;*
$T(x + y) = T(x) + T(y)$ and,
$T(\alpha x) = \alpha(T(x))$.

Definition 1.3.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces over the complex plane $\mathbb{C}$. A function $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is called bounded if $\sup_{\|x\|=1} \|Tx\| < \infty$ and the norm of T written $\|T\|$ is given by $\|T\| = \sup_{\|x\|=1} \|Tx\|$.*

Proposition 1.3.3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces over the complex plane $\mathbb{C}$. Let $T : H_1 \longrightarrow \mathcal{H}_2$ be a non zero linear operator. Then the following are equivalent;*

1. *$Ran(T)$ is a closed subspace of $\mathcal{H}$.*

2. *T is a bounded linear operator.*

3. *$Ker(T)$ is a closed subspace of $\mathcal{H}$.*

Definition 1.3.4. *If $T \in B(\mathcal{H})$ then its adjoint T^* is the unique operator in $B(\mathcal{H})$ that satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle, \forall x, y \in \mathcal{H}$.*

Theorem 1.3.5. *Let $T \in B(\mathcal{H})$, then the following results hold;*

1. *$Ran(T)$ is closed if and only if $Ran(T^*T)$ is closed.*
2. *T^*T is a positive self adjoint operator.*

Remark
An operator $T \in B(\mathcal{H})$ is said to be positive ($T \geq 0$) if it is self adjoint and $\langle Tx, x\rangle \geq 0$ $\forall x \in \mathcal{H}$.
Also, $T \in B(\mathcal{H})$ is said to be strictly positive ($T > 0$) if $\langle Tx, x\rangle > 0$ $\forall 0 \neq x \in \mathcal{H}$.

Example 1.3.6. *Let $\mathcal{H} = \mathbb{R}^2$ and $T : \mathcal{H} \longrightarrow \mathcal{H}$ defined by;*

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$. Clearly, T has a matrix representation $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. It is easy to check that T is positive.

Remark 1.3.7. *Many operators between Hilbert spaces in classical analysis and operator theory are in fact positive linear operators.*

Definition 1.3.8. *Let $\mathcal{H}$ be a Hilbert space and $x \in \mathcal{H}$. Let R be a relation in X. Then R is said to be ;*

1. *Reflexive if xRx, $\forall x \in X$.*

2. *Symmetric if $xRy \Longrightarrow y\ R\ x \forall x, y \in X$.*

3. *Antisymmetric if xRy and $yRx \Longrightarrow x = y$, $\forall x, y \in X$.*

4. *Transitive if xRy and $yRz \Longrightarrow xRz\ \forall x, y, z \in X$.*

If R is reflexive, symmetric and transitive, then R is called an equivalence relation on X and if R is reflexive, antisymmetric and transitive, then R is called a partial order on X. We have to note that an operator T is said to be *bounded* if there exists a constant C such that $\|Tx\|_y \leq C\|x\|_x$ for all $x \in X$ provided X is a vector space.

Proposition 1.3.9. *Let $T \in B(\mathcal{H})$ and X, Y vector space. Then the transformation $T : X \longrightarrow Y$ is continuous if and only if T is bounded.*

Proof. Suppose T is continuous, then there exists $\delta > 0$ such that $\|Tx - T0\|_y < 1$ whenever $\|x\|_x < \delta$.
Now take $\delta^{'} \in (0, \delta)$. Consider $\frac{\delta^{'}}{\|x\|_x}x$ so that $\|\frac{\delta^{'}}{\|x\|_x}x\| < \delta$ and so $\|T(\frac{\delta^{'}}{\|x\|_x}x)\| < 1$ which implies that $\|T_x\|_y \leq \frac{1}{\delta^{'}\|x\|_x}$.
Now let $\delta^{'}$ increase to δ, then $\|Tx\|_y \leq \frac{1}{\delta}\|x\|_x$.
Hence T is bounded.
Now assume that T is bounded. We want to show that T is continuous.
If T is bounded, then $\|Tx\| \leq C\|x\|$.
$\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \leq C\|x_1 - x_2\| < \epsilon$ provided that $\|x_1 - x_2\| \leq \frac{\epsilon}{C}$
Hence this shows that T is continuous □

Example 1.3.10. *Let X be a vector space and that $Y = \mathbb{C}$.*
An operator $\phi : X \longrightarrow \mathbb{F}$ is a linear functional on X. Then ϕ is continuous if and only if $ker\phi$ is closed in X (This holds when Y is finite dimensional).

Definition 1.3.11. $\|T\| = sup\{\|Tx\|_y : x \in X, \|x\| = 1\}$. *Then $\|T\|$ is the smallest constant such that $\|Tx\| \leq \|T\|\|x\|_x$ for all $x \in X$. $\|.\|$ is a norm on $B(X, Y)$, the space of all bounded linear operators $T : X \longrightarrow Y$.*

For instance, given that $S, T \in B(\mathcal{H})$ then $(S+T)x$ is a norm on X as seen below;

$$\|(S+T)x\| = \|Sx + Tx\|$$

$$\leq \|Sx\| + \|Tx\|$$

$$\leq \|S\|\|x\| + \|T\|\|x\|$$

$$= (\|S\| + \|T\|)\|x\|.$$

So $\|S+T\| \leq \|S\| + \|T\|$.
Also, let $T \in B(X,Y)$ and $S \in B(Y,Z)$. Then, the composition $ST \in B(X,Z)$ is a norm.

Proof.

$$\|(ST)x\| = \|S(Tx)\|_z$$

$$\leq \|S\|_{B(y,z)}\|Tx\|_y$$

$$\leq \|S\|_{B(y,z)}\|T\|_{B(x,y)}\|x\|_x.$$

So $\|ST\| \leq \|S\|_{B(y,z)}\|T\|_{B(x,y)}$.

□

1.4 Banach Spaces

Definition 1.4.1. *A space X is called Banach if every cauchy sequence $(x_n)_k \subset X$ converges to an element $x \in X$.*

A Banach space is a complete normed space.
If $|x_n - x| \longrightarrow 0$ as $n \longrightarrow \infty$ for some $x \in X$, then $\|x_k - x_n\| = \|x_k - x + x - x_n\| \leq \|x - x_k\| + \|x - x_n\| \longrightarrow 0$ as $n, k \longrightarrow \infty$.

Shift Operators
Let $X = l^2$ or $X = C_0$ where $C_0 = \{(x_j)_{j\geq 1} : lim_{j\longrightarrow\infty} x_j = 0\} \subseteq l^2$
C_0 takes $\|.\|_\infty$, $\|x\|_\infty = sup_{j\geq 1}|x_j|$ and thus C_0 is closed in l^∞.
Therefore C_0 is also a Banach space. $L, R \in B(\mathcal{H})$ where L and R represent Left and Right shift operators respectively, and for $x = (x_1, x_2, x_3, \cdots)$, then;
$L_x = (x_2, x_3, x_4 \cdots)$ and $R_x = (0, x_1, x_2, x_3, \cdots)$
In this case, $\|R_x\| = \|x\|$ and so R is an isometry from X to X, $R \in B(X)$, $\|R\| = 1$.
Also, $\|L_x\| \leq \|x\|$ with equality if $x_1 = 0$.

Hence $\|L\| = 1$.

Multiplication Operators

Let $X = l^2$ or C_0 and let $(a_j)_{j\geq 1}$ in $\mathbb{F}$. Then $M_x = (a_j x_j)_{j\geq 1}$ and $x = (x_j)$ where a_j is bounded.
Then $M \in B(X)$ and $\|M\|_{B(x)} = sup_{j\geq 1}|a_j|$.
Also let $C[0,1], h \in C[0,1]$.
Define $(M_h f)(t) = h(t)f(t)$.
Then $M_h \in B\Bigg(C[0,1]\Bigg)$ and $\|M_h\|_{B(C[0,1])} = \|h\|_\infty$.

Differentiation and Integration Operators

Define $D : C^{'}[0,1] \longrightarrow C[0,1]$ where $Df = f^{'}$.
If $C^{'}[0,1], C[0,1]$ have $\|.\|_\infty$, D is NOT bounded.
If $C^{'}[0,1]$ has a norm $\|f\|_\infty + \|f^{'\|_\infty}$ and C has $\|.\|_\infty$ then D is bounded.
Let $(Vf)(t) = \displaystyle\int_0^t ds$. The V, called the Voltera operator is bounded from $(C[0,1]), \|.\|_\infty$ to $(C[0,1]), \|.\|_\infty$ and $\|V\| = 1$.

Chapter 2

Spectrum of Some Operators

2.1 Introduction

In this chapter we are going to study the parts of the spectrum of an operator; the point spectrum, continuous spectrum, residual spectrum, continuous point spectrum, approximate point spectrum and essential spectrum.
We will also get to look at the fredholm operator and some of its properties as far as the essential spectrum is concerned.We will also get to look at the pseudo-spectrum.
Finally, we will take a look at the spectra of some selected classes of bounded linear operators.
The concept of eigenvalues of matrices plays a fundamental role in linear algebra and is the starting point in finding the canonical forms of matrices and developing functional calculus.
The similar theory can be developed on infinite dimensional spaces for compact operators, but for infinite dimensional, the situation is rather involved for general operators.
In particular, many important examples of operators have no eigenvalues at all.

Definition 2.1.1. *Let $T : D(T) \longrightarrow X$ be a linear transformation where X is a Hilbert space and $D(T)$, the domain of T, is a linear manifold on X.*
Let $\mathbb{F}$ be the complex field $\mathbb{C}$, and let I be the identity on X.
The resolvent set $\rho(T)$ of T is the set of all scalars $\lambda \in \mathbb{C}$ for which the linear transformation $\lambda I - T : D(T) \longrightarrow X$ has a densely defined continuous inverse.
That is, $\rho(T) = \{\lambda \in \mathbb{F} : (\lambda I - T)^{-1} \in B[\mathbb{R}(\lambda I - T), D(T)]$ and $R(\lambda I - T)^{-1} = X\}$.

The spectrum $\sigma(T)$ of T is the complement of the resolvent set $\rho(T)$ in $\mathbb{C}$.
Let $X = H$, a Hilbert space Throughout , $T : X \longrightarrow X$ will denote a bounded linear transformation of X into itself so that the domain of T i.e $D(T) = X$, where $X \neq 0$ is a complex Banach space.
We can therefore precisely state that the resolvent set $\rho(T)$ is the set of all complex numbers λ for which $(\lambda I - T) \in B[X]$ is invertible.

Therefore,
$\rho(T) = \{\lambda \in \mathbb{C} : (\lambda I - T) \in \mathcal{G}[X]\}$, (where $\mathcal{G}[X]$ is set of invertible operators on H,)
$= \{\lambda \in \mathbb{C} : (\lambda I - T)$ has an inverse in $B[X]\}$
$= \{\lambda \in \mathbb{C} : \mathcal{N}(\lambda I - T) = 0$ and $R(\lambda I - T) = X\}$
From the above, we can therefore implicitly define the spectrum of an operator as;

Definition 2.1.2. *The spectrum, $\sigma(T)$ of an operator T is given by;*
$\sigma(T) = \{\mathbb{C} \setminus \rho(T) = \{\lambda \in \mathbb{C} : (\lambda I - T)$ *has no inverse in* $B[X]\}$
$= \{\lambda \in \mathbb{C} : N(\lambda I - T) \neq 0$ *or* $R(\lambda I - T) \neq X\}$.

Theorem 2.1.3. *The resolvent set $\rho(T)$ is nonempty and open and the spectrum $\sigma(T)$ is compact.*

Proof. Let $T \in B[X]$. By the Neumann expansion,if $\|T\| < |\lambda|$, then $\lambda \in \rho(T)$.
Equivalently, since $\rho(T) = \mathbb{C} \setminus \rho(T)$, then $|\lambda| \leq \|T\|$, for every $\lambda \in \sigma(T)$.
Thus, $\lambda(T)$ is bounded and therefore we can assert that $\rho(T) \neq \emptyset$.
Now, we make a claim that if $\lambda \in \rho(T)$, then the open ball $B_\sigma(\lambda)$ with the center at λ and radius $\sigma = \|(\lambda I - T)^{-1}\|^{-1}$ is inclusive in $\rho(T)$.
For the proof of our claim, if $\lambda \in \rho(T)$, then$(\lambda I - T) \in \mathcal{G}[X]$ so that $(\lambda I - T)^{-1}$ is nonzero and bounded and hence $0 < \|(\lambda I - T)^{-1}\|^{-1} < \infty$.
Now set $\sigma = \|(\lambda I - T) - 1\|^{-1}$, and let $B_\delta(0)$ be the nonempty open ball of radius δ about the origin of the complex plane $\mathbb{C}$ and take $v \in B_\sigma(0)$.
Since $|v| < \|(\lambda I - T)^{-1}\|^{-1}$ and it follows that, $\|v(\lambda I - T)^{-1}\| < 1$.
Then, $[I - v(\lambda I - T)^{-1}] \in \mathcal{G}[X]$ and so, $(\lambda - v)I - T = (\lambda I - T)[I - v(\lambda I - T)^{-1}] \in \mathcal{G}[X]$.
Thus $\lambda - v \in \rho(T)$ so that $B_\sigma(\lambda) = B_\delta(0) + \lambda = \{v \in \mathbb{C} : v = v + \lambda$ for some $v \in B_\sigma(0)\} \subseteq \rho(T)$ which is the proof of our claim.
Since $\rho(T)$ includes nonempty open balls centered at each of its points, then we have that $\rho(T)$ is open.
Since $\sigma(T)$ is a complement of $\rho(T)$, we thus have that $\sigma(T)$ is closed which completes the proof. □

2.2 A Classical Partition of the Spectrum

The spectrum of a bounded linear operator T in a Hilbert space $\mathcal{H}$ is the set of all scalars $\lambda \in \mathbb{C}$ for which the operator $\lambda\mathrm{I} - T$ fails to be an invertible element of the Banach algebra $B[X]$.
We can therefore split the spectrum of an operator T into many disjoint parts.
A classical partition comprises three main parts but may also contain some overlapping parts that we will get to look at the tail end of the classical partitioning of the spectrum.

2.2.1 Point Spectrum

The point spectrum, denoted, $\sigma_p(T)$ of T is the set of all those λ for which $(\lambda\mathrm{I} - T)$ has no inverse.
In other words, $\sigma_p(T) = \{\lambda \in \mathbb{C} : \mathcal{N}(\lambda\mathrm{I} - T) \neq 0\}$.
A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of T if there exists a nonzero vector $x \in X$ such that $Tx = \lambda x$.
Equivalently, λ is an eigenvalue of T if $\mathcal{N}(\lambda\mathrm{I} - T) \neq 0$.
If λ is an eigenvalue of T then the nonzero vectors in $N(\lambda\mathrm{I} - T)$ are the eigenvectors of T and $N(\lambda I - T)$ is the eigenspace associated with the eigenvalue λ.
Precisely, the point spectrum, $\sigma_p(T)$, is the set of all eigenvalues of T for a finite dimensional vector space.

2.2.2 Continuous Spectrum

Continuous spectrum of T, denoted by $\sigma_c(T)$ is the set of all those $\lambda \in \mathbb{C}$ for which $\lambda\mathrm{I} - T$ has a densely defined but unbounded inverse on its range.
Symbolically, $\sigma_c(T) = \{\lambda \in \mathbb{C} : (\lambda\mathrm{I} - T) = \{0\}, \overline{R(\lambda\mathrm{I} - T)} = X$ and $R(\lambda\mathrm{I} - T \neq X\}$.

2.2.3 Residual Spectrum

The residual spectrum of T, denoted by $\sigma_r(T)$, is the set of all those $\lambda \in \mathbb{C}$ such that $\lambda\mathrm{I} - T$ has an inverse on its range that is not densely defined i.e, $\sigma_r(T) = \{\lambda \in \mathbb{C} : \mathcal{N}(\lambda\mathrm{I} - T) = \{0\}$ and $\overline{R(\lambda\mathrm{I} - T)} \neq X\}$.
The collection $\{\sigma_p(T), \sigma_c(T)$ and $\sigma_r(T)\}$ is a disjoint covering of the spectrum, $\sigma(T)$ to mean that they are pairwise disjoint and
$\sigma_p(T) \bigcup \sigma_c(T) \bigcup \sigma_r(T) = \sigma(T)$.

The following also form some overlapping parts of the spectrum;

2.2.4 Essential Spectrum

Fredholm Operator
Fredholm operator is a bounded linear operator between two Banach spaces, with finite dimensional kernel and cokernel, and with a closed range.
Essential spectrum, denoted by, $\sigma_e(T)$, is the set of all complex numbers such that $(T - \lambda I)$ is not **Fredholm**.
Alternatively, we can say that an operator $T : X \longrightarrow Y$ is Fredholm if it is invertible modulo a compact operator i.e, if there exists a bounded linear operator $S : Y \longrightarrow X$ such that $Id_x - ST$ and $Id_y - TS$ are compact operators on X and Y respectively.

The index of the Fredholm operator $T$is given by; Ind $(T) = dimkerT - codimranT$ or in other words,Ind $T = dimkerT - dimcokerT$.

Some Properties of the Fredholm Operator
The set of fredholm operators from X to Y is open in the Banach apace $\mathcal{L}(X, Y)$ of the bounded linear operator, equipped with the operator norm. More precisely, when T_0 is Fredholm, from X to Y, there exists an $\epsilon > 0$ such that every T in $\mathcal{L}(X, Y)$ with $\|T - T_0\| < \epsilon$ is Fredholm, with the same index as that of T_0.
When T is Fredholm from X to Y and V is Fredholm from Y to Z, then the composition VT is Fredholm from X to Z and the $ind(VT) = ind(V) + ind(T)$.
When T is Fredholm, the adjoint operator $T^{'}$ is Fredholm from $Y^{'}$ to $X^{'}$ and the $ind(T^{'}) = -ind(T)$ and when X and Y are Hilbert spaces, then the same conclusion holds for the self adjoint operator T^*.
When T is Fredholm and K a compact operator, then $T + K$ is Fredholm.
Example of Fredholm Operator
Let $\mathcal{H}$ be an Hilbert space with an orthonomal basis $\{e_n\}$ indexed by the nonnegative integers. Let S be the right shift operator on $\mathcal{H}$ defined by $S(e_n) = e_{n+1}, n \geq 0$. This operator S is injective and has a closed range of codimension 1, hence S is Fredholm with $ind(S) = -1$.
The powers $S^k, k \geq 0$ are Fredholm with index $-k$.
The adjoint S^* is the left shift operator and $S^*(e_0) = 0$ and $S^*(e_n) = e_{n-1}, n \geq 0$.
The left shift operator S^* is Fredholm with index 1.

2.2.5 Approximate point spectrum

The approximate point spectrum of an operator T, denoted by, $\sigma_{ap}(T)$ is defined or given by;
$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ is not bounded below$\}$.

2.2.6 Pseudospectrum

Introduction
The main question that we need to ask is whether the spectrum of an operator conclusively addresses all the pertinent issues of a bounded linear operator as far as structure, composition and behavior of such operators are concerned.
We also need to address the specific position or where really the non normal operator "lives" in the complex plane $\mathbb{C}$.
As a precondition, we ascertain that the spectrum does not conclusively tackle these questions as a result of what we refer to as, "The spectral pollution[[21]]."

We find that the spectrum is too small and the field of values, otherwise called the numerical range, is too huge to try to tackle some of these spectral issues.
The concept of the pseudospectrum tries to give solutions to these spectral questions and therefore we will, in passing, introduce this concept of pseudospectrum and leave out its application in solving the behavior of an operator for further research.

Now, consider a function, $\lambda \in \mathbb{C}$, the norm of the resolvent $(\lambda I - T)^{-1}$, where λ is an eigenvalue of the operator T, $\|(\lambda I - T)^{-1}\|$ can be thought of as infinite and otherwise finite.
If T is normal then, $\|(\lambda I - T)^{-1}\| = \frac{1}{dist(\lambda, \sigma(T))}$.
Thus, in a normal case, the surface $\|(\lambda I - T)^{-1}\|$ is determined entirely by the eigenvalues.
In the non normal case, $\|(\lambda I - T)^{-1}\| = \frac{1}{dist(\lambda, \sigma(T))}$ is simply a lower bound.

Definition 2.2.1. *[13] The pseudospectrum, denoted by, $\sigma_\epsilon(T)$ of T is defined as;*
$\sigma_\epsilon(T) = \{\lambda \in \mathbb{C} : \|(\lambda I - T)^{-1}\| \geq \epsilon^{-1}\}$.

We note that the pseudospectra of T are closed, strictly nested sets with $\sigma_0(T) = \sigma(T)$.
The norm, $\|(\lambda I - T)^{-1}\|$ is the largest singular value of the pseudospectrum, i.e the inverse of the smallest singular value of $(\lambda I - T)$.
Therefore, as an equivalent definition, we have that;
$\sigma_\epsilon(T) = \sigma(T) \bigcup \{\lambda \in \mathbb{C} \setminus \sigma(T) : \|R_A(\lambda)\| \geq \epsilon^{-1}\}$.
The following theorem gives some important aspects of the Pseudospectrum;

Theorem 2.2.2. *Let $T \in B(\mathcal{H})$ and $\epsilon > 0$. Then the following three statements are equivalent;*

1. *$\lambda \in \sigma_\epsilon(T)$.*

2. *There exists $U \in B(\mathcal{H})$ with $\|U\| < \epsilon$ such that $\lambda \in \sigma(T + U)$.*

3. *$\lambda \in \sigma(T)$ or there exists $v \in \mathcal{H}$ with $\|v\| = 1$ such that $\|(\lambda I - T)v\| < \epsilon$.*

2.3 Spectral Radius

The following definitions, due to [14] are well known.
Let $T \in B(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space.The spectral radius $r(T)$ is defined by;
$r(T) = sup\{|\lambda| : \lambda \in \sigma(T)\} = max\{|\lambda|\}, \forall \lambda \in \sigma(T)$.
Clearly,$r(T^n) = r(T)^n \leq \|T^n\| \leq \|T\|^n, n \geq 0$.

Theorem 2.3.1. *(Gelfand-Burling Formula)*
$r_\sigma(T^n) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ *for every* $n \geq 0$. *That is,* $r(T) = r_\sigma(T) = \sup_{\lambda\in\sigma(T)} |\lambda| = \max_{\lambda\in\sigma\ (T)} |\lambda| = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ *where* $r(T)$ *is the limit of the sequence*$\{\|T^n\|^{\frac{1}{n}}\}$.

Clearly, $r(T^*) = r(T)$.
For T^*T is non negative, it follows that , for any $T \in B(\mathcal{H})$, we have that;
$r(T^*T) = r(T^*T) = \|T^*T\| = \|TT^*\| = \|T\|^2 = \|T^*\|^2$.

2.4 Classes of some Bounded Linear Operators

In this section we do a sneak preview of some of the classes of bounded linear operators and get to look at their spectra where possible. We will get to look at the Unitary operators, the normal operators and some of the non normal operators.

2.4.1 Unitary Operators

Definition 2.4.1. *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be two Hilbert spaces of the same dimension. Then we say that* $\mathcal{H}_1$ *is equivalent to* $\mathcal{H}_2$ *or vice versa if there exists* $V \in B(\mathcal{H}_1, \mathcal{H}_2)$ *such that* V *is surjective and* $\|Vx\| = \|x\|, \forall x \in \mathcal{H}_1$.

An operator V with the property that $\|Vx\| = \|x\|$ for all x is referred to as an isometry. More clearly, we say that an isometry is that type of an operator that preserves distances in normed linear spaces, and that which preserves the distance in inner product spaces.

Theorem 2.4.2. *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be two Hilbert spaces and let* $V \in B(\mathcal{H}_1, \mathcal{H}_2)$. *Then the following assertions are equivalent;*

1. V *is an isometry.*
2. $V^*V = I_{\mathcal{H}1}$, *an identity operator in* $\mathcal{H}_1$.
3. $\langle Vx, Vy\rangle = \langle x, y\rangle, \forall x, y \in \mathcal{H}_1$.

Proof. (1) $\Longrightarrow$ (2)
For all $x \in \mathcal{H}_1$, we have that,
$\langle (V^*V - I_{\mathcal{H}1})x, x\rangle = \|Vx\|^2 - \|x\| = 0$
Hence $V^*V - I_{\mathcal{H}1} = 0$. Hence (1)$\Longrightarrow$ (2).
(2)$\Longrightarrow$ (3)
$\langle Vx, Vy\rangle = \langle V^*Vx, y\rangle = \langle x, y\rangle$. Hence (2) $\Longrightarrow$ (3).
(3) $\Longrightarrow$ (1).
$\|Vx\|^2 = \langle Vx, Vx\rangle = \langle x, x\rangle = \|x\|^2$. Hence (3) $\Longrightarrow$ (1). □

A good example of an isometry is the forward shift operator on ℓ^2.

Remark 2.4.3. *For $\mathcal{H}$ a Hilbert space, a linear isometry which maps $\mathcal{H}$ onto itself is called a unitary operator.*

Clearly, an operator $T \in B(\mathcal{H})$ is unitary if and only if it has an inverse and $T^{-1} = T^*$.
Also, we say that an operator $T \in B(\mathcal{H})$ is unitary if $T^*T = TT^* = I$.
We can also say that an operator $T \in B(\mathcal{H})$ is unitary if it is an invertible isometry.
Framed differently, we can assert that a unitary operator is an isometric isomorphism between Hilbert spaces.

Theorem 2.4.4. *The class of all unitary operators is a multiplicative group.*

Proof. Clearly, the class of unitary operators contains the identity I. It also contains the identity operator and the inverse V^{-1} of any element V.
Clearly, the class also contains the product V_1V_2 of any two elements V_1 and V_2 and the following is satisfied;
$\langle V_1V_2e, V_1V_2e^{'}\rangle = \langle V_2e, V_2e^{'}\rangle = \langle e, e^{'}\rangle$ and since multiplication of elements is associative, we have the result. $\square$

Example 2.4.5. *Given a bounded linear operator in $\mathcal{L}^2(\mathbb{R})$, then,*

1. *For $a \in \mathbb{R}$, the operator T_a, called the translation by a operator, defined by;*
   $(T_af)(x) = f(x-a), x \in \mathbb{R}$,
2. *For $b \in \mathbb{R}$, the operator E_b, called the modulation by b, defined by;*
   $(E_bf)(x) = e^{2\pi ibx}f(x), x \in \mathbb{R}$
3. *For $c > 0$, the operator D_c, called the dilation by c, defined by;*
   $(D_cf)(x) = \frac{1}{\sqrt{c}}f(\frac{x}{c}), x \in \mathbb{R}$.
   The operators T_a, E_b and D_c are unitary operators of $\mathcal{L}^2(\mathbb{R})$ onto $\mathcal{L}^2(\mathbb{R})$.

Theorem 2.4.6. *If $T \in B(\mathcal{H})$ is unitary, then $\sigma(T) \subset \{\lambda : |\lambda| = 1\}$.*
Or stated otherwise, if $T \in B(\mathcal{H})$ then $\sigma(T)$ lies inside the unit circle.

Proof. For $T \in B(\mathcal{H})$ unitary, we have that $TT^* = T^*T = I$ and so T is invertible with $T^{-1} = T^*$.
We also know that, for a unitary operator, if $\|Tx\| = \|x\|$, then $\|T\| = 1 = \|T^{-1}\|$ and therefore we must have that;
$\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\sigma(T^{-1}) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.
Now to show that $\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, it is sufficient to show that $\lambda I - T$ is invertible for every λ satisfying $|\lambda| < 1$.
Since T is already invertible, it suffices to show that $(\lambda I - T)$ is invertible for every λ satisfying the relation, $0 < |\lambda| < 1$.

From the relation $-\lambda T(\frac{1}{\lambda}I - T^{-1}) = (\lambda I - T)$, it is clear that for $\lambda \neq 0$, if $(\frac{1}{\lambda}I - T^{-1})$ is invertible, then $\lambda I - T$ trivially becomes invertible.
Now we have to note that if $0 < |\lambda| < 1$, then $|\frac{1}{\lambda}| > 1$ and so in this case, $(\frac{1}{\lambda}I - T^{-1})$ is invertible and as a result, $\lambda I - T$ is invertible.
Thus we have that $\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ which ends the proof □

2.4.2 Normal Operators

An operator $T \in B(\mathcal{H})$ is said to be normal if it commutes with its adjoint.
The following theorem gives a necessary and sufficient condition for an operator $T \in B(\mathcal{H})$ to be normal;

Theorem 2.4.7. *An operator $T \in B(\mathcal{H})$ is said to be normal if and only if ;*
$\|Tx\| = \|T^*\|, \forall x \in \mathcal{H}$.

If $T \in B(\mathcal{H})$ is normal, then,

1. $Ker(T) = Ker(T^{*)}$.
2. $Ran(T)$ is dense in $\mathcal{H}$ if and only if T is injective.
3. T is invertible if and only if $\exists\delta > 0$ such that $\|Tx\| \geq \delta\|x\|, \forall x \in \mathcal{H}$.
4. If $Tx = \alpha x$ for some $x \in \mathcal{H}$, $\alpha \in \mathbb{C}$, then $T^*x = \overline{\alpha}x$.
5. If α, β are distinct eigenvalues of T, then the corresponding eigenspaces are orthogonal to each other.

Theorem 2.4.8. *Let $T \in B(\mathcal{H})$. Then the following assertions are equivalent;*

1. *T is normal.*
2. *$\|T^*x\| = \|Tx\|$ for every $x \in \mathcal{H}$.*
3. *T^n is normal for every integer $n \geq 1$.*
4. *$\|T^{*n}x\| = \|T^nx\|$ for every $x \in \mathcal{H}$ and every integer $n \geq 1$.*

Proof. (1) $\Longrightarrow$ (2)
For any arbitrary operator, $T \in B(\mathcal{H})$, we have that $\|T^*x\|^2 - \|Tx\|^2 = \langle T^*x, T^*x\rangle - \langle Tx, Tx\rangle = \langle (TT^* - T^*T)x, x\rangle$ for every $x \in \mathcal{H}$, and since $(TT^* - T^*T)$ is self adjoint, it follows that T is normal if and only if $\|T^*x\| = \|Tx\|$ for every $x \in \mathcal{H}$.
Hence (1)$\Longrightarrow$ (2).
(3) $\Longrightarrow$ (4)
$Since$T$^{*n} = T^{n*}$ for each integer $n \geq 1$, the equivalence above assumes that T^n is normal

for every integer $n \geq 1$ if and only if $\|T^{*n}x\| = \|T^n x\|$ for every $x \in \mathcal{H}$ and every integer $n \geq 1$.
Hence (3) $\Longrightarrow$ (4).
(1) $\Longrightarrow$ (3)
Finally, we have by induction that T commutes with T^* if and only if T^n commutes with T^{*n} for every integer $n \geq 1$.
Since $T^{*n} = T^{n*}$ for every integer $n \geq 1$, we have that T is normal if and only if T^n is normal for every integer $n \geq 1$.
Hence (1) $\Longrightarrow$ (3). □

Remark 2.4.9. *From the above deliberations, we can assert that every self adjoint operator $T \in B(\mathcal{H})$ is normal.*
Clearly, we can also state that every unitary operator is also normal.

Proposition 2.4.10. *An operator $T \in B(\mathcal{H})$ is an orthogonal projection if and only if it is a normal projection.*

Proof. If T is an orthogonal projection, then it is hermitian and hence a normal projection.
Conversely, suppose T is normal, then $\|T^*x\| = \|Tx\|$ for every $x \in \mathcal{H}$ so that $Ker(T^*) = Ker(T)$.
If T is a projection then $Ran(T) = Ker(I - T)$ so that $Ran(T) = \overline{Ran(T)}$.
Therefore, if T is a normal projection ,then $Ker(T)^{\perp} = Ker(T^*)^{\perp} = \overline{Ran(T)} = Ran(T)$ so that $Ran(T) \perp Ker(T)$, and hence T is an orthogonal projection. □

Example 2.4.11. *We first have to underscore that any self adjoint operator $T \in B(\mathcal{H})$ is normal.*
We can also assert that any example of a unitary operator is trivially a normal operator.
Now let T be the operator on $\mathcal{L}^2(\mathbb{R})$ defined by,
$(Tf)(t) = e^{-|t|}f(t)$. *Then,*
$\langle Tf, g\rangle = \int_{-\infty}^{\infty} e^{-|t|}f(t)\overline{g(t)}dt$
$= \int_{-\infty}^{\infty} f(t)\overline{[e^{-|t|}g(t)]}dt$
$= \langle f, Tf\rangle$ *which is a self adjoint operator and hence a normal operator.*

We finally state the following theorem that holds for normal operators as far as the spectral partition is concerned;

Theorem 2.4.12. *[34] Let $T \in B(\mathcal{H})$ be a normal operator and let $\lambda \in \mathbb{C}$. Then we have the following;*

1. $\rho(T) = \{\lambda \in \mathbb{C} : \mathcal{R}(T_\lambda) = \mathcal{H}\}$.
2. $\sigma_p(T) = \{\lambda \in \mathbb{C} : \overline{\mathcal{R}(T_\lambda)} \neq \mathcal{H}\}$ *where* $\overline{R(T_\lambda)}$ *is the closure of* $\mathcal{R}(T_\lambda)$.
3. $\sigma_c(T) = \{\lambda \in \mathbb{C} : \overline{\mathcal{R}(T_\lambda)} = \mathcal{H}$ *and* $\mathcal{R}(T_\lambda) \neq \mathcal{H}\}$.
4. $\sigma_r(T)$ *is empty.*

2.4.3 Non Normal Operators

Non normal operators are those normal operators defined by weakening the requirements of the normal operators.
Under this class, we have the following sub-classes of operators:

1. **Quasinormal Operators** An operator $T \in B(\mathcal{H})$ is said to be quasinormal if it commutes with T^*T i.e $[T^*T, T] = 0, where [,] denotes the commutator.$
From the above definition, we can confirm that every isometry is quasinormal.

Theorem 2.4.13 ([20], Theorem 7.29)**.** *Let $T \in B(\mathcal{H}, K)$ with polar decomposition $T = U|T|$, where U is a partial isometry and $|T|$ is a positive operator, is quasinormal if and only if $U|T| = |T|U$.*

Proof. Let $T = U|T|$ and suppose $U|T| = |T|U$. Then U commutes with $|T|^2$. Since $|T|$ commutes with $|T|^2$, it follows that $T = U|T|$ commutes with T^*T which actually equals $|T^2|$.
Hence T is quasinormal.
Conversely, suppose T is quasinormal. Then T commutes with $|T|$.
Now, by Weierstrass Theorem on the approximation of continuous functions, we have that,
$(U|T| - |T|U)|T| = T|T| - |T|T = 0$, so that $U|T| - |T|U$ annihilates $Ran(|T|)^{\perp} = Ker(|T|) = Ker(U)$
Trivially, $|T| - |T|U$ annihilates $Ran(|T|)$, in other words, $(U|T| - |T|U = 0)$ on $Ran(|T|)$.[[33]]
Since $Ran(|T|)^{\perp} = Ker(|T|) = Ker(U)$, it is trivial that $U|T| - |T|U$ annihilates $Ran(|T|)$ also, and if $x \in Ran(|T|)^{\perp} = Ker(|T|) = Ker(U)$, and then by the definition of U, we have that $Ux = 0$, and it then follows that $U|T| = |T|U = 0$.
Hence $U|T| = |T|U$. □

2. **Hyponormal Operators**
An operator $T \in B(\mathcal{H})$ is said to be hyponormal if $T^*T \geq T^*$ or $T^*T - T^* \geq 0$.

Theorem 2.4.14. *Let $T \in B(\mathcal{H})$. Then, T is hyponormal if and only if $\|T^*x\| \leq \|Tx\|$, for every $x \in \mathcal{H}$.*

Proof. Let $T \in B(\mathcal{H})$. Then $T^* \leq T^*T$ if and only if $\langle T^*x, x\rangle \leq \langle T^*Tx, x\rangle$ i.e, $\|T^*x\|^2 \leq \|Tx\|^2$, for every $x \in \mathcal{H}$.
Equivalently, $\|T^*x\| \leq \|Tx\|$ hence the result. □

Definition 2.4.15. *An operator $T \in B(\mathcal{H})$ is said to be* ***co-hyponormal*** *if its adjoint T^* is hyponormal, i.e if $T^* \geq T^*T$ or $\|T^*x\| \geq \|Tx\|$.*

From the definitions of the hyponormal and the co-hyponormal, we can confirm therefore that an operator $T \in B(\mathcal{H})$ is normal if it is both hyponormal and co-hyponormal.

3. **Seminormal Operators**
   An operator $T \in B(\mathcal{H})$ is said to be seminormal if it is either hyponormal or co-hyponormal.
   Every hyponormal operator is seminormal but the converse is not true in general.
   A good example of a seminormal operator are the unilateral shifts.

Theorem 2.4.16. *If $T \in B(\mathcal{H})$ is hyponormal and λ, a , $b \in \mathbb{C}$, then;*

(a) $T + \lambda\mathrm{I}$ *is hyponormal.*

(b) $aT + bT$ *is hyponormal or seminormal.*

Chapter 3

Numerical Ranges of Some Bounded Linear Operators

3.1 Abstract

The sole purpose of this chapter is to introduce various properties of the numerical range that are very fundamental in the proof of many approximation properties.
Since we now have the notion that the spectrum of an operator lacks enough information to fully "solve" an operator, we will introduce the numerical ranges of some of these classes of bounded linear operators to try to answer to some of these questions that the spectra fail to address.
In this section, we will also demonstrate how the numerical range contains more information than the spectrum of bounded linear operators.
Since the numerical ranges are intimately connected to the numerical radii of bounded linear operators, we will therefore be drawing more information on the numerical radii of operators in this section.
We shall also demonstrate some examples of the numerical ranges that would be very vital in the proof of some of the theorems in this section, e.g the Hausdorff-Toeplitz theorem which states that the numerical range of a bounded linear operator is a convex subset of the complex plane [[11]].
The Hausdorff-Toeplitz theorem solely is enough indication that the numerical range says "something" about an operator.
For example, some bounded linear operators may have a singleton $\{\lambda\}$ as a spectrum, but it is easy to see that only λI can have this set as the numerical range.
Again, if an operator lies on the real line, then we lack enough information on the spectrum of the operator, but should the numerical range of the operator be real, then we will see that the operator is Hermitian.
We will also look at the containment of the spectrum in the closure of the numerical range

and the Hildebrandt's theorem which asserts that the intersection of the numerical ranges of all operators similar to a given one T is precisely the convex hull of the spectrum of T, and the works of Donogue/Hildebrandt asserting that the corner points of the numerical range are the eigenvalues.
In the very final stages of this chapter, we will get to look at some of the applications of the numerical ranges of some classes of operators in different applicable fields.

3.2 Numerical Range

Definition 3.2.1. *For a bounded linear operator T on a Hilbert space $\mathcal{H}$, the numerical range $W(T)$ is the image of the unit sphere of $\mathcal{H}$ under the quadratic form $x \longrightarrow \langle Tx, x\rangle$ associated with the operator(see [5]). More precisely, $W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$.*

The numerical range is also called the field of values of T.
Thus the numerical range of an operator, like the spectrum, is a subset of the complex plane whose geometrical properties should say 'something' about the operator. We will be comparing the properties of spectrum and the numerical range of some operators.
The image of the unit sphere is the union of all the closed segments that join the origin to the points of the numerical range, the entire range is the union of all the closed rays from the origin through points of the numerical range.
The geometrical properties of the numerical range of an operator often provides useful information about the algebraic and analytic properties of an operator.

Definition 3.2.2. *A set Ω is convex if for any two points $x, y \in \Omega$ we have that $z = tx + (1-t)y \in \Omega$, for all $t \in [0,1]$.*

We should note that changing the condition $t \in [0,1]$ to $t \in \mathbb{R}$ would result in z describing a straight line through the points x and y.
The empty set and the set containing a single point are regarded as convex. We also note that the intersection of any family of Ω_i (finite or infinite) of convex sets is convex.
Indeed, if $x, y \in \Omega_i$, they belong to each Ω_i, then so does the "line segment" z.

Example 3.2.3. *In $\mathbb{R}^2$, the set $\Omega = \{(x_1, x_2) : x_1, x_2 \geq 0\}$ is an example of a convex set.*

As a direct result of the convex set defined above, we introduce a very fundamental result, the Hausdorff-Toeplitz theorem. Before we can introduce the theorem, we need the following lemma that will be very important in the proof of the theorem;

Lemma 3.2.4. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a closed linear subspace of $\mathcal{H}$. Let $P_{\mathcal{L}}$ be the projection onto $\mathcal{L}$ and let $T \in B(\mathcal{H})$. Then, $W(P_{\mathcal{L}}T\restriction_{\mathcal{L}}) \subseteq W(T)$.*

Proof. Clearly, if $x \in \mathcal{L}$ is a unit vector, then $x \in H$ is also a unit vector and therefore;
$\langle (P_{\mathcal{L}}T\restriction_{\mathcal{L}})x, x\rangle_{\mathcal{L}} = \langle Tx, x\rangle_{\mathcal{H}} \in W(T)$.
Hence $W(P_{\mathcal{L}}T\restriction_{\mathcal{L}}) \subset W(T)$ as desired. □

With the proof of the above lemma in mind, we now introduce the Hausdorff- Toeplitz theorem;

Theorem 3.2.5 ([11]). *Let $T \in B(\mathcal{H})$. Then the numerical range, $W(T)$, is a convex subset of the complex plane $\mathbb{C}$.*

Proof. By 3.2.4 and [[15]], let $\alpha, \beta \in W(T)$ be distinct scalars. Then $\exists x, y \in \mathcal{H}$ unit vectors such that, $\langle Tx, x\rangle = \alpha$ and $\langle Ty, y\rangle = \beta$.
Also, let $x = \lambda y$ for some $\lambda \in \mathbb{C}$, then $|\lambda| = 1$ since $\|x\| = 1 = \|y\|$ and as a result, $\alpha = \langle Tx, x\rangle = \langle \lambda Ty, \lambda y\rangle = \beta$ which is a contradiction. Similarly, $y \neq \lambda x$ for any $\lambda \in \mathbb{C}$ so that x and y are linearly independent.
Now let $\mathcal{L} = span\{x, y\}$, a two dimensional subspace of $\mathcal{H}$. Since $x, y \in \mathcal{L}$, 3.2.4 proved above gives, $\alpha, \beta \in W(P_{\mathcal{L}}T\restriction_{\mathcal{L}}) \subseteq W(T)$.
But since $\mathcal{L}$ is two dimensional, $W(P_{\mathcal{L}}T\restriction_{\mathcal{L}})$ is convex.
Hence, $t\alpha + (1-t)\beta \in W(P_{\mathcal{L}}T\restriction_{\mathcal{L}}) \subseteq W(T)$ for $0 < t < 1$.
Hence, since $\alpha, \beta \in W(T)$, were arbitrarily picked, $W(T)$ is convex as desired. □

It is important to note that the Hausdorff-Toeplitz theorem produces some of the very wonderful results as far as the numerical range is concerned. At this stage, we introduce one very important result in form of a theorem as can be seen hereafter.

Theorem 3.2.6. *Let $T \in B(\mathcal{H})$ be such that $\lambda \in \partial W(T)$. If no closed disk of $W(T)$ contains λ, then λ is an eigenvalue of T.*

Proof. Let $\lambda \in \partial W(T)$ be such that no closed disk of $W(T)$ contains λ. Let $x \in \mathcal{H}$ be a unit vector such that $\langle Tx, x\rangle = \lambda$. We note that if x were an eigenvector of T with an associated eigenvalue α, then $\alpha = \langle \alpha x, x\rangle = \langle Tx, x\rangle = \alpha$.
Now suppose to the contrary that x is not an eigenvector of T. Then $\mathcal{L} = span\{x, Tx\}$ is a two dimensional subspace of $\mathcal{H}$. Let $A = P_{\mathcal{L}}T\restriction_{\mathcal{L}}$. Therefore since $x \in \mathcal{L}$, $\lambda \in W(A) \subseteq W(T)$.
Moreover, since $\lambda \in \partial W(T)$, it is also clear that $\lambda \in \partial W(T)$.
However, A is not a multiple of the identity or else $\lambda \in W(A)$ would imply that $A = \lambda \mathrm{I}_{\mathcal{L}}$ which may in turn imply that $Tx = Ax = \lambda x$ which is a contradiction. Hence every point of $W(T)$ is contained in a closed disk contained in $W(T)$. Since $\lambda \in W(T) \subseteq W(T)$, and as a consequence, $W(T)$ contains a closed disk containing λ which is also a contradiction. Hence x is an eigenvector with the corresponding eigenvalue λ as required. □

3.3 Numerical Radius

Associated with the numerical range is the numerical radius.

Definition 3.3.1. *Let $T \in B(\mathcal{H})$. The numerical radius of T, denoted by $w(T)$, is the number $w(T) = sup\{|\lambda| : \lambda \in W(T)\}$.*

In other words, the numerical radius of an operator, $T \in B(\mathcal{H})$ is the radius of the smallest circle in the complex plane centered at the origin that encloses the numerical range of the operator $W(T)$ of T. We can also say that the numerical radius , $w(T)$, of an operator T is the greatest distance between any point in the numerical range and the origin.
Obviously, $w(T^*) = w(T)$ for every $T \in B(\mathcal{H})$ and for every vector $x \in \mathcal{H}$, we have that $|\langle Tx, x\rangle| \leq w(T).\|x\|^2$.
The following result shows that the numerical radius of a self adjoint operator as well as a normal operator equals the norm of such operators;

Theorem 3.3.2. *Let $T \in B(\mathcal{H})$ be a self adjoint operator. Then $w(T) = \|T\|$.*

Proof. First, we have that,

$$w(T) \leq \{\|Tx\|\|x\| : x \in \mathcal{H}, \|x\| \leq 1\} \leq \|T\|. \tag{3.1}$$

Also, we have to recall that $\|T\| = \{|\langle Tx, y\rangle| : x, y \in \mathcal{H}, \|x\|, \|y\| \leq 1\}$.
Now we fix x and y with $\|x\| \leq 1$ and $\|y\| \leq 1$.
Let $\theta \in [0, 2\pi]$ such that $|\langle Tx, y\rangle| = e^{i\theta}\langle Tx, y\rangle$.
Let $y^{'} = e^{-i\theta}y \in \mathcal{H}$ so that $\|y^{'}\| \leq 1$ and $\langle Tx, y^{'}\rangle = |\langle Tx, y\rangle| \in \mathbb{R}$.
Recalling the polarization identity, we have that;
anges$\langle Tx, y^{'}\rangle = \frac{1}{4}\{\langle T(x+y^{'}), x+y^{'}\rangle - \langle T(x-y^{'}), x-y^{'}\rangle$ + i $\langle T(x+iy^{'}), x+iy^{'}\rangle - i\langle T(x-iy^{'}), x-iy^{'}\rangle\} \longrightarrow (*)$.
Since T is self adjoint, it follows therefore that each inner product in equation $(*)$ above is a real number and therefore , from our choice of $y^{'}$, $\langle Tx, y^{'}\rangle$ is real, we must have that the complex terms in equation $(*)$ sum to zero and thus we have therefore that;
$\langle Tx, y^{'}\rangle = \frac{1}{4}\{\langle T(x+y^{'}), x+y^{'}\rangle - \langle T(x-y^{'}), x-y^{'}\rangle\}$.
Now, from the definition of the numerical radius, we see that, $|\langle Tx, x\rangle| \leq \|x\|^2 w(T)$ for all $x \in \mathcal{H}$.
Hence, $|\langle Tx, y\rangle| \leq \frac{1}{4}\{|\langle T(x+y^{'}), x+y^{'}\rangle| + |\langle T(x-y^{'}), x-y^{'}\rangle| \leq \frac{1}{4}w(T)(\|x+y\|^2 + \|x-y\|^2)$ and by applying the parallelogram law, we have,

$$|\langle Tx, y\rangle| \leq \frac{1}{4}w(T)(2\|x\|^2 + 2\|y\|^2) \leq w(T). \tag{3.2}$$

Since $x, y \in \mathcal{H}$ were arbitrarily picked, with the norm at most one, the two inclusions yield the expression for $\|T\|$, i.e $\|T\| \leq w(T)$, i.e $\|T\| = w(T)$ which concludes the proof. □

3.4 Some Main Results of Numerical Ranges

We now want to look at some main results in terms of theorems that connect the numerical ranges of operators and some very important aspects of the operator theory.

3.4.1 Spectral Inclusion

One of the very important applications of the numerical range is to bound the spectrum. Here, we purpose to show that the numerical range bounds the spectrum. We do this by looking at the boundary of the spectrum.
We should note here that the boundary of the spectrum is contained in the approximate point spectrum, $\sigma_{ap}(T)$, of the operator T.
The approximate point spectrum consists of all those λ, complex, for which there exists a sequence of unit vectors x_n such that $\|(T-\lambda I)x_n\| \longrightarrow 0$.
Now we know that the numerical range $W(T)$ is convex, a result from the Hausdorff-Toeplitz theorem, it then suffices to show that the approximate point spectrum is contained in the closure of its numerical range, i.e, $\sigma_{ap} \subset \overline{W(T)}$.

Theorem 3.4.1. ***Spectral Inclusion****[29] The spectrum of an operator is contained in the closure of its numerical range, i.e,* $\sigma(T) \subseteq \overline{W(T)}$.

Proof. Let $\lambda \in \sigma_{ap}(T)$ and also let $\{x_n\}$ be a sequence of unit vectors such that $\|(T-\lambda I)x_n\| \longrightarrow 0$.
Borrowing from the Schwartz Inequality, and using Theorem 3.4.1 and also [[7], Theorem 2] we have that;
$|\langle (T-\lambda I)x_n, x_n\rangle| \leq \|(T-\lambda I)x_n\| \longrightarrow 0.$
Thus, $\langle Tx_n, x_n\rangle \longrightarrow \lambda$ and so $\lambda \in \overline{W(T)}$ which completes the proof. □

The spectral inclusion theory enables us to locate the spectrum of the sum of any two operators A and B in a Hilbert space H,, and we have that,
$\sigma(A+B) \subset \overline{W(A+B)} \subset W(A)+W(B)$.
Now, since $\overline{W(T)}$ is convex, then we can come to a conclusion that the convex hull, $conv(\sigma(T))$ is also contained in the closure of the numerical range, $\overline{W(T)}$.
We have to note that even though the numerical range is used to bound the spectrum of an operator, it is also clear that the spectrum can sometimes be much smaller.
For example, let $\mathcal{H} = \mathbb{C}^2$ and $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. If $x = (x_1, x_2)$ then we have that $Tx = (0, x_1), \langle Tx, x\rangle = x_1\overline{x_2}$ and $W(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{2}\}$.
However, $\sigma(T) = \{0\}$.
We also note that the Hermitian operators have their spectra sharply bounded by their numerical ranges.

Theorem 3.4.2. *An operator $T \in B(\mathcal{H})$ is self adjoint if and only if $W(T)$ is real [12].*

Proof. Suppose $T \in B(\mathcal{H})$ is self adjoint. Then we have that $\langle Tx, x\rangle = \langle x, T^*x\rangle = \overline{\langle Tx, x\rangle}$ for all $x \in \mathcal{H}$.

Hence we have that $W(T)$ is real.
Conversely, suppose $\langle Tx, x\rangle$ is real, for all $x \in \mathcal{H}$, then we have that;
$\langle Tx, x\rangle - \langle x, Tx\rangle = 0 = \langle (T - T^*)x, x\rangle$, and x is of unit length.
Thus, the operator $T - T^*$ has only $\{0\}$ in its numerical range, and such an operator has to be null.
This implies that $T - T^* = 0$ and hence $T = T^*$ hence the proof. □

We now want to show that the numerical radius of an operator, $T \in B(\mathcal{H})$ is an equivalent norm of the operator T.

Theorem 3.4.3. *Let $\mathcal{H}$ be a complex Hilbert space. For $T \in B(\mathcal{H})$, then $w(T) \leq \|T\| \leq 2w(T)$.*

Proof. Let $\lambda = \langle Tx, x\rangle$ such that $\|x\| = 1$. From Schwartz inequality, we have that;
$|\lambda| \leq |\langle Tx, x\rangle| \leq \|Tx\| \leq \|T\|$.
Now to show the other inequality, we employ the polarization principle, and we have that;
$4\langle Tx, y\rangle = \langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\langle T(x+iy), x+iy\rangle - i\langle T(x-iy), x-iy\rangle$.
Hence we have that;
$4|\langle Tx, x\rangle| \leq w(T)[\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2] = 4w(T)[\|x\|^2 + \|y\|^2]$.
Now let $\|x\| = \|y\| = 1$, then we have;
$4|\langle Tx, y\rangle| \leq 8w(T)$ which implies that, $\|T\| \leq 2w(T)$ which completes the proof. □

We need to note that in the real Hilbert spaces, this result does not hold as demonstrated by the example below.

Example 3.4.4. *Let $\mathcal{H} = \mathbb{R}^2$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For $x = (x_1, x_2), \|x\| = 1$, we have that $Tx = (-x_2, x_1)$ and $\langle Tx, x\rangle = 0$.*
However, $\|T\| = 1$.

Theorem 3.4.5. *If the numerical radius, $w(T) = \|T\|$, then the spectral radius, $r(T) = \|T\|$.*

Proof. Let $w(T) = \|T\| = 1$. Then there exists a sequence of unit vectors $\{x_n\}$ such that $\langle Tx_n, x_n\rangle \longrightarrow \lambda \in W(T), |\lambda| = 1$.
From the inequality, $|\langle Tx_n, x_n\rangle| \leq \|Tx_n\| \leq 1$, we have that $\|Tx_n\| \longrightarrow 1$.
Hence, $\|(T - \lambda I)x_n\|^2 = \|Tx_n\|^2 - \langle Tx_n, \lambda x_n\rangle - \langle \lambda x_n, Tx_n\rangle + \|x_n\|^2$.
Hence $\lambda \in \sigma_{ap}(T)$ and $r(T) = 1 = \|T\|$ and we have the result. □

Theorem 3.4.6. *If $\lambda \in W(T), |\lambda| = \|T\|$, then $\lambda \in \sigma_p(T)$.*

Theorem 3.4.7. *Let $T \in B(\mathcal{H})$. Then;*

1. *$W(T) = W(T^*) = W(T)^*$.*

2. *$W(T) = W(U^*TU)$ for any unitary operator U, i.e, invariance under unitary equivalence.*

3. *$W(\lambda T) = \lambda W(T)$, for any $\lambda \in \mathbb{C}$.*

4. *$W(\lambda I + T) = \lambda + W(T)$, for any $\lambda \in \mathbb{C}$. In particular, $W(T) = \{\mu\}$ if and only if $T = \mu I$.*

Proposition 3.4.8. *Let $T \in B(\mathcal{H})$.Then;*

1. *$W(T)$ lies in the closed disc of radius $\|T\|$ centered at the origin.*

2. *$\sigma_p(T) \subseteq W(T)$, i.e, contains all the eigenvalues of T.*

3. *$W(T^*) = \{\overline{\lambda} : \lambda \in W(T)\}$.*

4. *If $\mathcal{H}$ is finite dimensional, then $W(T)$ is closed and bounded.*

Corollary 3.4.9. *Let $T \in B(\mathcal{H})$. Then;*
$\sigma_p(T) \bigcup \sigma_r(T) \subseteq W(T)$.

Proof. $\lambda \in \sigma_p(T) \Longrightarrow \lambda \in W(T)$. Now, if $\lambda \in \sigma_r(T)$, then $\overline{\lambda} \in \sigma_p(T^*)$. Thus, $\overline{\lambda} \in W(T^*)$ so that $\lambda \in W(T)$.
Hence $\sigma_p(T) \bigcup \sigma_r(T) \subseteq W(T)$ which completes the proof. □

Proposition 3.4.10. *Let $T \in B(\mathcal{H})$. Then;*

1. *$\sigma(T) \subseteq \overline{W(T)}$.*

2. *$W(T \bigoplus S) = conv\{W(T) \bigcup W(S)\}$, where $conv(\Omega)$ denotes the convex hull of $\Omega \subseteq \mathbb{C}$.*

Proof. If $\lambda \in \sigma_{ap}(T)$ then there exists a sequence x_n of unit vectors in $\mathcal{H}$ such that;
$\|(\lambda I - T)x\| \longrightarrow 0$ and therefore,
$0 \leq |\lambda - \langle Tx, x\rangle| = |\langle(\lambda I - T)x_n, x_n\rangle|$.
$\leq \|(\lambda I - T)x_n\| \longrightarrow 0$ so that $\langle Tx_n, x_n\rangle \longrightarrow \lambda$.
Since each $\underline{\langle Tx_n, x_n\rangle}$ lies in $W(T)$, it follows, by the closed set theorem, that $\lambda \in W(T)$.
Hence, $\sigma_{ap}(T) \subseteq \overline{W(T)}$ and so, $\sigma(T) = \sigma_r(T) \bigcup \sigma_{ap} \subseteq W(T)$. □

Theorem 3.4.11 ([32], Theorem 3.1)**.** *Let $T, S \in B(\mathcal{H})$. If $W(T)$ is a line segment, then T is normal.*

Theorem 3.4.12. *[4] Let $T \in B(\mathcal{H})$. If T is normal, then $\overline{W(T)} = conv(\sigma(T))$.*

Remark 3.4.13. *$W(T)$ is completely determined by the $\sigma(T)$ if T is a normal operator.*

However, there are normal operators with the same spectrum but different numerical ranges.

Example 3.4.14. *Let $A = diag(1, \frac{1}{2}, \frac{1}{3}, \cdots)$ and $B = (0, 1, \frac{1}{2}, \frac{1}{3}, \cdots)$ be two diagonal operators acting on the Hilbert space $\mathcal{H} = \ell^2$. Then $W(A) = (0, 1] \neq W(B) = [0, 1]$ and $\sigma(A) = \sigma(B) = \left\{\frac{1}{n} : n \geq 1\right\} \bigcup \{0\}$.*

Proposition 3.4.15. *Let A and B ne two normal operators on a Hilbert space $\mathcal{H}$. Suppose that $\sigma(A) = \sigma(B)$. Then;*
$\overline{W(A)} = conv(\sigma(A)) = conv(\sigma(B)) = \overline{W(B)}$.

Thus, $W(A)$ and $W(B)$ can only have a difference in their boundaries $\partial W(A)$ and $\partial W(B)$. Therefore, to describe the numerical range of a normal operator T, it is important to determine which boundary points of $W(T)$ belongs to the $W(T)$.

Corollary 3.4.16. *If $T \in B(\mathcal{H})$ is normal, then $w(T) = \|T\|$.*

Proof. Clearly, $w(T) \leq \|T\|$ for any operator $T \in B(\mathcal{H})$. Since T is normal, we have that $\overline{W(T)} = conv(\sigma(T))$. □

Definition 3.4.17. *Let $T \in B(\mathcal{H})$. The Crawford Number of the operator T is the number.*

Theorem 3.4.18. *(Bendixson-Hirsch Theorem)*
If $T = A + iB$, is the cartesian decomposition of $T \in B(\mathcal{H})$ with A, B self adjoint, then $W(T) \subseteq \overline{W(A)} + i\overline{W(B)}$.

3.5 Numerical Ranges of Commuting Operators

Here,we present some results concerning $W(AB)$ and $w(AB)$ whenever $AB = BA$ for all $A, B \in B(\mathcal{H})$.
We need to note that we don't really expect much as far as $W(AB)$ is concerned, but under some special conditions, we can come up with something as far as $W(AB)$ is concerned.

Theorem 3.5.1. *[25] Let A be a nonnegative, self-adjoint operator and $AB = BA$. Then $W(AB) = W(A)W(B)$.*

Proof. We have that $\langle ABx, x\rangle = \langle BA^{\frac{1}{2}}x, A^{\frac{1}{2}}x\rangle$, where $A^{\frac{1}{2}}$ is the nonnegative square root of A.
Thus, $\langle ABx, x\rangle = \langle Bf, f\rangle \|A^{\frac{1}{2}}x\|^2 = \langle Bf, f\rangle\langle Ax, x\rangle$.
Where $f = \frac{A^{\frac{1}{2}}x}{\|A^{\frac{1}{2}}x\|}$ with $A^{\frac{1}{2}}x \neq 0$ and $\|f\| = 1$ which completes the proof. $\square$

We now turn to the numerical radius, $w(AB)$ of self adjoint operators. We first note that

$w(A)w(B)$ can be exceeded by $w(AB)$ as is imminent in the example below;
Let A act on C^4, with $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.
Then we have that $w(A) = cos(\frac{\pi}{5}) = 0.80901699$.
Also, we have that $w(A^2) = w(A^3) = 0.5$ so that;
$0.5 = w(A.A^2) > w(A).w(A^2) = 0.4045088$.

Theorem 3.5.2. *[6] $w(AB) \leq 4w(A)(B)$. When $AB = BA$ then it always holds that $w(AB) \leq 2w(A)(B)$.*

Corollary 3.5.3. *Let A be a normal operator commuting with B. Then $w(AB) \leq w(A)w(B)$.*

Now we consider the direct sum of operators. Let us consider the matrix case for an nn

matrices A and B, then the direct sum is the $2n \times 2n$ matrix;
$A \bigoplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ on $\mathcal{H} \bigoplus \mathcal{H}$. And by the direct sum property, we have that;
$W(A \bigoplus B) = conv(W(A) \bigcup W(B))$.

3.6 Some Basic Properties and Examples of Numerical Range of Operators

Here, we are going to develop the properties and examples of some basic numerical ranges of operators that we will be making a reference to most oftenly. We shall also demonstrate some clear examples that we will be using in the study.
Now, since the numerical range and the numerical radius are connected, we are therefore going to touch a bit on the the linkages of these two aspects in this section.
We will have an insightful look at the numerical range of a 2×2 matrices as far as their eigenvalues and eigenvectors are concerned and we will thereafter come to a conclusion

that the proof of the Hausdorff-Toeplitz Theorem is firmly anchored on the aspects of the eigenvalues and the eigenvectors of a 2×2 matrix.
Now, before going straight into some specific examples of the numerical ranges, let us have a look at the following fundamental results that would help us understand the subsequent deliberations.

Proposition 3.6.1. *Let $T, S \in B(\mathcal{H})$. Then the following hold;*

1. $W(T^*)=\overline{W(T)}$.
2. $W(T)$ *contains all the eigenvalues of* T.
3. $W(T)$ *is contained in the closed disc of radius* $\|T\|$ *and the origin.*
4. *If* $a, b \in \mathbb{C}$, *then* $W(aT + bI_H) = aW(T) + b$.
5. *If* $U \in B(\mathcal{H})$ *is unitary, then* $W(UTU^*) = W(T)$.
6. $W(T) \subseteq \mathbb{R}$ *if and only if* T *is self adjoint.*
7. *If* $\mathcal{H}$ *is finite dimensional,* $W(T)$ *is closed and thus compact.*
8. $W(T + S) \subseteq W(T) + W(S)$ *for all* $T, S \in B(\mathcal{H})$.

Proof. Part (1) follows from the fact that $\langle T^*x, x\rangle = \overline{\langle Tx, x\rangle}$ for all $x \in H$. Hence $W(T^*) = \overline{W(T)}$.
Part (2) follows from the fact that if α is an eigenvalue of T with non zero eigenvector x_0, then $x = \frac{1}{\|x_0\|}$ is a unit eigenvector for T with eigenvalue α and thus $\alpha = \langle Tx, x\rangle \in W(T)$ as required.
Part (3) is a trivial result from the fact that $|\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq \|T\|, \forall x \in H$ with $\|x\| = 1$.
Part (4) also follows from the fact that $\langle (aT + bI_{\mathcal{H}})x, x\rangle = a\langle Tx, x\rangle + b = aW(T) + b$ such that $\|x\| = 1$.
For part (5), we know that $\langle UTU^*x, x\rangle = \langle T(U^*x), U^*x\rangle$ for all $x \in \mathcal{H}$ and $\|x\| = 1$ if and only if U^*x has a norm one, i.e, $\|U^*x\| = 1$.
Part (6) follows from the fact that $T \in B(\mathcal{H})$ is self adjoint if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for every $x \in \mathcal{H}$ which is equivalent to $\langle Tx, x\rangle \in \mathbb{R}$ for every $x \in \mathcal{H}$ such that $\|x\| = 1$.
For part (7), if $W(T)$ is closed by part (3) above, then compactness immediately follows. Now suppose $(\lambda_n)_{n\geq 1}$ is a sequence in $W(T)$ that converges to $\lambda \in \mathbb{C}$. For each $n \in, \mathbb{N}$ choose $x_n \in \mathcal{H}$ with $\|x\| = 1$ such that $\lambda_n = \langle Tx_n, x_n\rangle$.
Since $\mathcal{H}$ is a finite dimensional Hilbert space, the unit ball of $\mathcal{H}$ is compact and thus there exists a sequence $(x_{nk})_{k\geq 1}$ that converges to a unit vector $x \in \mathcal{H}$.
This implies that

$$|\langle Tx, x\rangle| = \lim_{n\to\infty} \langle Tx_n, x_n\rangle = \lim_{n\to\infty} \lambda_n = \lambda$$

and thus $\lambda \in W(T)$ as required.
Finally, part (8) follows trivially from the definition of the numerical range. $\square$

3.7 Examples of Numerical Ranges

Let us first have a look at the unilateral backward shift.

Example 3.7.1. *Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{e_n\}_{n\geq 1}$. Let $T \in B(\mathcal{H})$ be the unilateral forward shift operator; that is $T(e_1) = 0$ and $T(e_n) = e_{n-1}$ for $n \geq 2$.*
Then $W(T)$ is the open unit disc centered at the origin.
Now, we notice that if $x \in \mathcal{H}$ has a norm one, then $|\langle Tx, x\rangle| \leq \|Tx\|\|x\| \leq 1$ with equality if and only if Tx and x are multiples of each other and $\|Tx\| = 1$. This implies that $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.
However, if $x = \sum_{n\geq 1} a_n e_n$, the equation $Tx = \lambda x$ implies that $\lambda_{a_n} = a_{n+1}, \forall n \in \mathbb{N}$. Thus, $|a_n| = |a_1|$ for all $n \in N$ which is an impossibility since

$$1 = \|x\|^2 = \sum_{n\geq 1} |a_n|^2$$

Thus $W(T)$ is a subset of the open unit disc.
Now, to see that $W(T)$ is actually the open unit disc itself, let $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$. Let

$$x_0 = \sum_{n\geq 1} \lambda^n e_n \in \mathcal{H}$$

which exists as

$$\sum_{n\geq 1} |\lambda^n|^2$$

converges. Thus, $Tx_0 = \lambda x_0$. Hence λ is an eigenvalue for T and thus $\lambda \in W(T)$. Hence $W(T)$ is the open unit disc.

Remark 3.7.2. *We have to notice that if T is the unilateral backward shift, then $W(T)$ is open and not closed.*
This also demonstrate that $w(T) = 1 = \|T\|$.

We now want to demonstrate an example of the numerical range of the diagonal operators.

Example 3.7.3. *Let $\mathcal{H}$ be a separable Hilbert space and let $T \in B(\mathcal{H})$ be a diagonal operator; i.e, there exists an orthonormal basis $\{e_n\}_{n\geq 1}$ of $\mathcal{H}$ and a bounded set $\{a_n\}_{n\geq 1}$*

of scalars such that $Te_n = a_n e_n$ *for all* $n \in N$.
Let

$$x = \sum_{n \geq 1} c_n e_n$$

be an arbitrary unit vector .
Thus

$$\sum_{n \geq 1} |c_n|^2 = 1$$

and

$$\langle Tx, x \rangle = \left\langle \sum_{n \geq 1} a_n e_n, \sum_{n \geq 1} c_n e_n \right\rangle$$

$$= \sum_{n \geq 1} a_n |c_n|^2$$

Hence

$$W(T) = \left\{ \sum_{n \geq 1} a_n b_n : b_n \geq 0, \sum_{n \geq 1} b_n = 1 \right\}$$

We claim that $W(T) = conv(\{a_n\}_{n \geq 1})$; the convex hull of $\{a_n\}_{n \geq 1}$.
And we have that $W(T)$ is convex and $conv(\{a_n\}_{n \geq 1}) \subseteq W(T)$.
Suppose $\lambda \in W(T)$. Then, either $\lambda \in conv(\{a_n\}_{n \geq 1})$ or there exists a closed half plane with λ on the boundary that contains the $conv(\{a_n\}_{n \geq 1})$.
Borrowing from , $W(aT + bI_{\mathcal{H}}) = aW(T) + b$ and [2], we have that $conv(a\{a_n\}_{n \geq 1} + b) = aconv(\{a_n\}_{n \geq 1}) + b$ for all $a, b \in \mathbb{C}$ and by translation and rotation, we can assume that $\lambda = 0$ and $conv(\{a_n\}_{n \geq 1})$ is contained in the closed upper half plane.
Since $\lambda \in W(T)$, there exists $\{b_n\}_{n \geq 1}$ such that $b_n \geq 0$ for all

$$n \in N, \sum_{n \geq 1} b_n = 1$$

and

$$0 = \sum_{n \geq 0} a_n b_n$$

Since each $b_n \geq 0$ and each a_n is contained in the closed upper half plane, $b_n = 0$ whenever a_n contains an imaginary point.
Therefore, since

$$\sum_{n \geq 1} b_n = 1$$

and

$$0 = \sum_{n\geq 1} a_n b_n$$

,

either $a_m = 0$ or $b_m = 0$ for some $m \in \mathbb{N}$ or there exists $m_1, m_2 \in \mathbb{N}$ such that $a_m 1 > 0$ and $a_m 2 < 0$.
It is then clear that $0 \in (\{a_n\}_{n\geq 1})$.
Hence $W(T) = conv(\{a - n\}_{n\geq 1})$ as desired.

The following remarks are direct consequences of the example /(3.7.2);

Remark 3.7.4. *Let $T \in B(\mathcal{H})$ be a diagonal self adjoint operator with spectrum $[0,1]$. Then, depending on whether 0 and 1 appear along the diagonal of T, then $W(T)$ is either $[0,1], (0,1], [0,1)$ or $(0,1)$ and any of this occurs for some self adjoint operators.*
This shows that the numerical range is not invariant under approximate unitary equivalence.
This also shows that the numerical range need not be closed.
Also, as a consequence of the above example, let $S \in B(\mathcal{H})$ be a normal operator, then it need not be the case that $\sigma(S) \subseteq W(S)$ nor $W(S) \subseteq \sigma(S)$.
We have seen that a self adjoint diagonal operator with spectrum $[0,1]$ can have $(0,1)$ as its numerical radius and so $\sigma(S) \subseteq W(S)$ may not be the case.
Furthermore, if S is a diagonal normal operator with diagonal entries $\{a_n)\}_{n\geq 1}$, then $\sigma(S) = \overline{\{a_n\}_{n\geq 1}}$ yet $W(S) = conv(\{a_n\}_{n\geq 1})$ and so for certain choices of $a_n, W(S)$ need not be a subset of $\sigma(S)$.

Example 3.7.5. *Consider* $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{C})$

Then $W(T)$ is the closed disc of radius $\frac{1}{2}$ centered at the origin.
To see this, it suffices to show that $x \in \mathbb{C}^2$ is a unit vector if and only if we can write $x = (cos(\alpha)e^{ei\theta_1}, sin(\alpha)e^{i\theta_2})$ for some $\theta_j \in [0, 2\pi)$ and $\alpha \in [0, \frac{\pi}{2}]$.
However, $\big\langle T(cos(\alpha)e^{i\theta_1}, sin(\alpha)e^{i\theta_2}), (cos(\alpha)e^{i\theta_1}, sin(\alpha)e^{i\theta_2})\big\rangle = cos(\alpha)sin(\alpha)e^{i(\theta_2-\theta_1)}$.
By ranging over all possible $\theta_j \in [0, 2\pi)$ and $\alpha \in [0, \frac{\pi}{2}]$ and using the fact that the range of $cos(\alpha)sin(\alpha) = \frac{1}{2}sin(2\alpha)$ over $\alpha = [0, \frac{\pi}{2}]$ is $[0, \frac{1}{2}]$, we see that $W(T)$ is precisely the closed disc of radius $\frac{1}{2}$ centered at the origin.

Remark 3.7.6. *We note that the example above shows that $w(T) = \frac{1}{2} \neq 1 = \|T\|$ thus showing that the numerical range and the operator norm are not equal norms.*

Now, we are going to demonstrate a very good consequence of our example and see how it describes the numerical ranges of elements of $\mathcal{M}_2(\mathbb{C})$.

This consequence also is a prime step in the proof of the Hausdorff- Toeplitz theorem that already we have seen its proof so far.

Theorem 3.7.7. *For $T \in \mathcal{M}_2(\mathbb{C})$, either;*

1. *If $T = \lambda I_2$, then $W(T) = \{\lambda\}$.*

2. *If the eigenvalues of T are equal and T is not a multiple of the identity, then $W(T)$ is a non trivial closed disc centered at the eigenvalues of T, or,*

3. *If the eigenvalues λ_1 and λ_2 of T are distinct, then $W(T)$ is a possibly degenerate ellipse with foci λ_1 and λ_2. Moreover, if x_i is any unit eigenvector for λ_i, then the eccentricity of $W(A)$ is $|\langle x_1, x_2\rangle|^{-1}$ and the length of the major axis is $\frac{|\lambda_1-\lambda_2|}{\sqrt{1-|\langle x_1,x_2\rangle|^2}}$.*

$c(T) = inf\{|\lambda| : \lambda \in W(T)\}$.

Proof. Part (1) clearly follows from part (4) of Proposition **3.4.1** above.
Now, to see that part (2) holds, suppose that the eigenvalues of T are equal and T is not a multiple of the identity.
Let $\lambda \in \mathbb{C}$ be the eigenvalue of T. Then there exists a unitary operator $U \in \mathcal{M}_2(\mathbb{C})$ such that $T = U(\lambda I_2 + aM)U^*$ where $a \in \mathbb{C}$ is non zero and M is the matrix, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
Therefore, we have;

$$\begin{aligned} W(T) &= W(U(\lambda I_2 + aM)U^*) \\ &= W(\lambda I_2 + aM) \\ &= \lambda + aW(M) \end{aligned}.$$

And so the result follows. □

Finally, suppose $T \in \mathcal{M}_2(\mathbb{C})$ has two distinct values. Since $W(T+aI_2) = W(T)+a, \forall a \in \mathbb{C}$, it is clear that we may assume there exists $\lambda \in \mathbb{C}$ such that the eigenvalues of T are λ and $-\lambda$.
Since the eigenvalues of T are $\pm\lambda$, it is easy to see that $tr(T) = 0$.
Now let x_1 be a unit eigenvector for λ and let x_2 be a unit eigenvector for $-\lambda$. If x_1 and x_2 are orthogonal, then $W(T)$ is the line segment connecting λ to $-\lambda$.
Since a line segment is an ellipse with foci at the end points , with infinite eccentricity, and with a major axis of length $2\lambda = \frac{|\lambda-(-\lambda)|}{\sqrt{1-|\langle x_1,x_2\rangle|^2}}$, we complete the proof.

Example 3.7.8 ([17]). *Let $T = \begin{pmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{pmatrix}$.*

Then the numerical range of T is ;

1. An ellipse with foci λ_1 and λ_2 having a minor axis of length $|\alpha|$, if $\lambda_1 \neq \lambda_2$.

2. A closed disc centered at λ_i if $\lambda_1 = \lambda_2$.

3. A line segment joining λ_1 and λ_2 if $\lambda_1, \lambda_2 \in \mathbb{R}$, are distinct and $\alpha = 0$.

Let $T = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$,
where $a, b \in \mathbb{C}$. Then $W(T)$ is an(possibly degenerate) ellipse with foci at $F = \pm\sqrt{ab}$.
To see this, suppose $a, b \in \mathbb{C}$ and writing in polar form, $a = |a|e^{i\alpha}$ and $b = |b|e^{i\beta}$, we

observe that if ;
$S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\alpha-\beta}{2}} \end{pmatrix}$, then S is unitary and so $STS^{-1} = e^{i\frac{\alpha+\beta}{2}} \begin{pmatrix} 0 & |a| \\ |b| & 0 \end{pmatrix}$ and we see that $W(T)$ is an ellipse with foci at $\pm\sqrt{|a||b|}e^{i\frac{\alpha+\beta}{2}} = \pm\sqrt{ab}$, which are the eigenvalues of T.

Theorem 3.7.9. ***(The Elliptic Range Theorem)***
If T is a linear operator on $\mathbb{C}^2$, then $W(T)$ is a (possibly degenerate) elliptic disc.

Theorem 3.7.10. *If T is a linear operator,with trace zero, then T is unitarily equivalent to a matrix operator with zero diagonal.*

Let $T = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $W(T)$ is the ellipse with foci $F = -1$ and $F = 1$ and minor axis 1 and a major axis 2.23 and $w(T) = 1.115$.

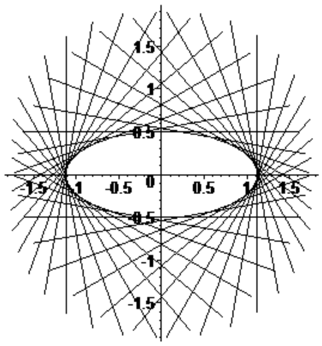

Figure 3.1: Numerical range of T

Example 3.7.11. *Let* $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$*. Then* $W(M)$ *is a circular disc centered at the origin with radius* $\frac{1}{2}$*.*
Clearly, the numerical radius , $w(M) = \frac{1}{2}$*.*

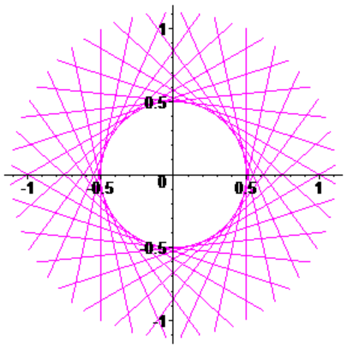

Figure 3.2: Numerical range of M

3.8 Numerical Ranges of 3×3 Matrices

3.8.1 Introduction

In this section, we are going to see a series of tests that will help us determine the shape of the numerical ranges, $W(T)$ for 3×3 matrices.
By now, we clearly understand that the numerical range of an operator T is a complex subset of the complex plane $\mathbb{C}$ which contains all the eigenvalues of the operator T and

therefore its convex hull, denoted by $conv(\sigma(T))$.
We should also be in a position to recall that for a normal operator $T, W(T) = conv(\sigma(T))$.
We have also had the chance to look at the numerical ranges of 2×2 matrices with coinciding eigenvalues and come to a generalization that their numerical range is the ellipse with foci at the eigenvalues λ_1 and λ_2 of T and a minor axis of length $S = (trace(T^*T) - |\lambda_i|^2 - |\lambda_2|^2)^{\frac{1}{2}}$.
When $S = 0$, for a normal operator, the ellipse becomes a line segment joining the two eigenvalues λ_1 and λ_2.
Now, the numerical ranges of 3×3 was a classification due to Kippenhahn. He argued that the numerical ranges of 3×3 matrices take the following forms [9]. It is either;

1. The convex hull of its eigenvalues.

2. The convex hull of an ellipse and a point (which reduces to an ellipse if the point is inside the ellipse).

3. A shape with a flat portion on the boundary, and

4. An ovular shape.

Let us now have an insight into each of the forms of the numerical ranges above;

3.8.2 When Is the Numerical Range, W(T) An Ellipse ?

Theorem 3.8.1. *[34]*
Let $T = \begin{pmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{pmatrix}$ *, an upper triangle matrix. Then its associated curve* $C(T)$ *consists of an ellipse(possibly degenerating to a disc)[22] and a point if and only if;*

1. $d = |x|^2 + |y|^2 + |z|^2 > 0$ *, and*

2. *The number* $\lambda = \frac{c|x|^2+b|y|^2+a|z|^2-x\overline{y}z}{d}$ *coincides with at least one of the eigenvalue* a, b, c.

Upon the satisfaction of these conditions, then, $C(T)$ is the union of λ with the ellipse having its foci at two other eigenvalues of T and minor axis of length $S = \sqrt{d}$. See [31]
With this theorem, we now have a tool to help us formulate a necessary and sufficient condition for a 3×3 matrix T to have an ellipse as its numerical range.

Theorem 3.8.2 ([19]). *Let T represent a 3×3 matrix with eigenvalues $\lambda_j, j = 1, 2, 3$. Then $W(T)$ is an ellipse if and only if the following three conditions are satisfied;*

1. $d = |x|^2 + |y|^2 + |z|^2 > 0$.

2. *The number*
$$\lambda = traceT + \left(\frac{1}{d}\right)\left(\sum_{j=1}^{3} |\lambda_j|^2 \lambda_2 - trace(T^*T)\right)$$
coincides with at least one of the eigenvalues λ_j , and,

3. $(|\lambda_1 - \lambda_3| + |\lambda_2 - \lambda_3|)^2 - \lambda_1 - |\lambda_2|^2 \leq d$, *where the eigenvalue that coincides with λ is λ_3.*

Proof. Conditions (1) and (2) are equivalent to stating that $C(T)$ is a union of the ellipse E, with foci at λ_1, λ_2 (and a minor axis of length $\sqrt{d}$) and the point λ_3.
The third condition implies that λ_3 lies right inside E and according to Kippenhahn's classification, this provides the only case where $W(T)$ is an ellipse. $\square$

After this result, we can go on to describe the 3×3 matrices for which the numerical range, $W(T)$ is a disc.

Corollary 3.8.3. *[3] $W(T)$ is a disc if;*

1. *T has multiple eigenvalue μ.*

2. $2\mu trace(T^*T) = trace(T^*T^2) + 2|\mu^2|\mu + (2\mu - \lambda)|\lambda|^2$.

3. $4|\mu - \lambda|^2 + 2|\mu|^2 = trace(T^*T)$.

Now let T be in the form, $\begin{pmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{pmatrix}$, then we may substitute conditions (2) and (3) above by;
($2'$) to be $x\overline{y}z = (\delta_c, \mu|x|^2 + \delta_b, \mu|y|^2 + \delta_a, \mu|z|^2)$, where δ is the usual Kronecker symbol and;
($3'$) to be $4|\mu - \lambda|^2 \leq |x|^2 + |y|^2 + |x|^2$.
Upon the satisfaction of these conditions, then we have that $W(T)$ is centered at μ with radius $\frac{1}{2}\sqrt{trace(T^*T) - 2|\mu|^2 - |\lambda|^2}$.

Proof. $W(T)$ is a disc if and only if it is an ellipse and in addition, the foci of this ellipse coincide. This is an implication that T has multiple eigenvalues, denoted by μ, and its third eigenvalue coincides with $\lambda = \frac{(c|x|^2 + b|y|^2 + a|z|^2 - x\overline{y}z)}{d}$ which completes the proof. $\square$

3.8.3 When is the Numerical Range, $W(T)$, a Flat Surface on its Boundary?

Throughout, we will hold the assumption that T is a 3×3 irreducible matrix in the form of $T = H + iK$ with H and K being self adjoint matrices.
We are also going to ignore the derivation of the canonical form an irreducible matrix with a flat portion on the boundary of its Numerical range.
According to Kippenhahn's classification, $W(T)$ has a flat portion on the boundary of the numerical range if and only if there exists a line $ux + vy + w = 0$, tangent to the associated curve $C(T)$ at two distinct points. The double tangent line corresponds to an eigenvalue $-w$ of $uH + vK$, with a multiplicity of 2.
Since $u, v \in \mathbb{R}$, it follows therefore that $uH + vK$ is self adjoint and $uH + vK + wI$ has the rank of 1.
Conversely, let $uH + vK + wI$ be of rank 1, then $-w$ becomes an eigenvalue of $uH + vK$ with multiplicity 2, and hence the double tangent.
Now, if T is irreducible,then $uH + vK$ fails to posses an eigenvalue of multiplicity 3, and if otherwise, then the self adjoint matrix $uH + vK$ would be a scalar, H and K would be commutative, and hence T would, as a result, be a normal matrix.
The following proposition gives a summary of the the above assertion, i.e, the numerical range being a flat portion on its boundary;

Proposition 3.8.4 ([27]). *Let $T = H + iK$ be irreducible. Then the following statements are equivalent;*

1. *$W(T)$ has a flat portion on the boundary.*

2. *$Rank(uH + vK + wI) = 1$, for some $u, v, w \in \mathbb{R}$.*

3. *for $u, v \in \mathbb{R}$ both not zero, $uH + vK$ has a multiple eigenvalue.*

Under these three conditions, the flat portion of the boundary lies on the line $ux+vy+w = 0$.
Also, we have from the Kippenhahn's classification that an irreducible 3×3 matrix can have at most a flat portion on the boundary of its numerical range [28].

Example 3.8.5. *Let* $T = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

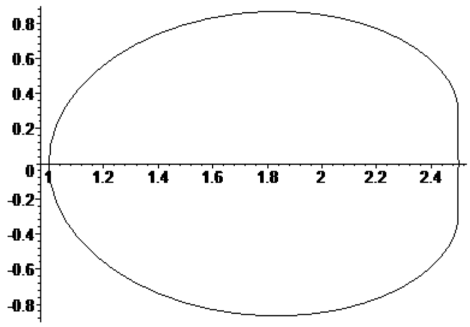

Figure 3.3: Numerical range of T

Example 3.8.6. *Let* $Q = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. *Then, we can see from the diagram below that the numerical range takes an ovular shape.*

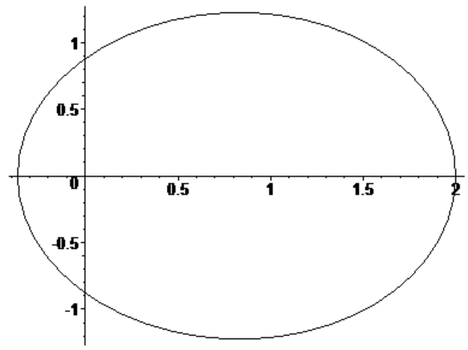

Figure 3.4: Numerical range of Q

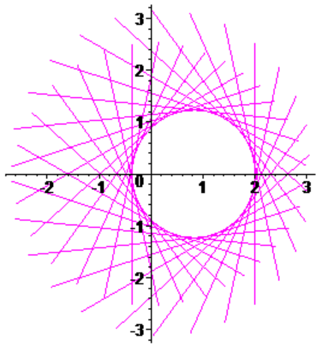

Figure 3.5: Numerical range of Q Constrained

Example 3.8.7. *Finally, let* $P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Then we notice from a previous statement that the numerical range is a disc centered at the eigenvalues.

3.9 Numerical Range of Operators of Higher Dimensions

We note that the higher the dimension, the stranger the numerical range.

For example, let $N = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ Clearly, we can see that N is a normal operator.

Therefore, $W(N)$ is the convex hull of its eigenvalues and we can see this numerical range in the figure below;

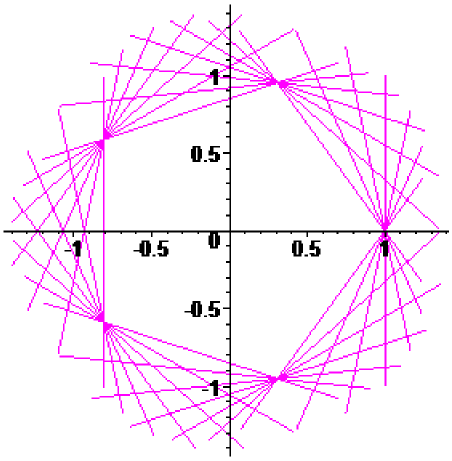

Figure 3.6: Numerical range of N

We can see from the figure that the corners are the eigenvalues of the operator above and indeed the eigenvalues are exactly five, one real and the rest being complex conjugates of each other.
From Maple, we find that $w(N) = 0.99999998$ and the norm of N, $\|N\| = 1$.
Therefore, $w(N) \leq \|N\|$ and the operators that exhibit these behaviors are referred to as normaloid.

Chapter 4

Numerical Range of Aluthge and Duggal Transforms of Some Operators

In this chapter, we look briefly on the spectral properties as well as the numerical ranges of the Aluthge and the Duggal transforms.

4.1 Aluthge Transformation of operators

Here, we associate with every operator $T \in B(\mathcal{H})$, its Aluthge transform denoted by $\tilde{T}$. We will study, in this section, the different connections there exist between the operator T and its Aluthge transform $\tilde{T}$ as far as their spectra, numerical ranges and lattices of invariant subspaces are concerned.

Definition 4.1.1. *L If $T = U|T|$ is any polar decomposition of T with U as a partial isometry and $|T| = (T^*T)^{\frac{1}{2}}$, then the Aluthge transform $\tilde{T}$ of T is the operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, i.e, $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, is called the Aluthge transform of T. See [24].*

Also, define $\Delta(T) = (\tilde{T}), \forall\ T \in B(\mathcal{H})$. Then for each nonnegative n, the n^{th} Aluthge transform denoted by $\Delta^n(T)$ of T is defined as;
$\Delta^n(T) = \Delta(\Delta^{n-1}(T)), \Delta^0(T) = T$. See [23].

In this definition, we call the operator sequence $\{\Delta^n(T)\}_{n=0}^{\infty}$ the Aluthge sequence of T. Then Δ is a map defined on $B(\mathcal{H})$.
We need to note that $\tilde{T}$ is purely independent of the choice of the partial isometry U in

the polar decomposition of T.
The partial isometry U is uniquely determined by the kernel condition $N(U) = N(T^*)$.
We show here, the existence of very intimate connections between any arbitrary $T \in B(\mathcal{H})$ and its associated Aluthge transform $\tilde{T}$.

Lemma 4.1.2. *[1] Let $T = U|T|$ be an arbitrary operator on $B(\mathcal{H})$ and let $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ be its Aluthge transform. Then we have the following;*

$$|T|^{\frac{1}{2}}T = \tilde{T}|T|^{\frac{1}{2}} \tag{4.1}$$

and,

$$T(U|T|^{\frac{1}{2}}) = (U|T|^{\frac{1}{2}}). \tag{4.2}$$

We say, in particular, that T is a quasiaffinity if and only if $|T|$ is a quasiaffinity and U is a unitary operator, and so, $\tilde{T}$ is also a quasiaffinity provided T is a quasiaffinity.
Moreover, for this case, T and $\tilde{T}$ are referred to as quasisimilar. Furthermore, T is invertible if and only if $\tilde{T}$ is invertible and for this case, we say that T and its Aluthge transform $\tilde{T}$ are similar.

4.1.1 Some Examples of Aluthge Transforms

1. Let T be a unilateral shift on $l^2(\mathbf{N})$ such that $T(x_1, x_2, ...) = (0, \lambda_1 x_1, \lambda_2 x_2, ...)$. Then $\Delta(T)(x_1, x_2, ...) = (0, \sqrt{\lambda_1\lambda_2 x_1}, \sqrt{\lambda_2\lambda_3 x_2, ...})$.

2. Let T be a bilateral weighted shift on $l^2(\mathbf{Z})$ such that ;
$T(..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...) = (...\lambda_{-3}x_{-3}, \lambda_{-2}x_{-2}, \lambda_{-1}x_{-1}, \lambda_0 x_0, \lambda_1 x_1, ...)$.
Then $\tilde{T}(...x_{-2}, x_{-1}, x_0, x_1, \cdot) = (...\sqrt{\lambda_{-3}\lambda_{-2}x_{-3}}, \sqrt{\lambda_{-2}\lambda_{-1}x_{-2}}, \sqrt{\lambda_{-1}, \lambda_0 x_{-1}}, \sqrt{\lambda_0\lambda_1 x_0}, ...)$.

3. Consider the Hilbert space $\mathcal{H} = L^2([0,1], \mu)$ where μ is a lebesgue measure and let $\{e_n\}_{n=1}^{\infty}$ be any orthonormal basis for $\mathcal{H}$ such that e_1 is the constant function 1. Let $U \in B(\mathcal{H})$ be defined by $Ue_n = e_{n+1}, n \in N$. Consider $T = U(M_x)^2$, where M_x is the multiplication by the position function. Then, $\tilde{T} = M_x U M_x$ is a quasiaffinity.

4.2 Spectral Picture of Aluthge Transform

We recall that if $T \in B(\mathcal{H})$ then the spectrum of $T, \sigma(T)$, is given by $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda\mathrm{I} - T$ is not invertible$\}$, and the resolvent set is the regular value of the operator T defined by, $\rho(T) = \{\lambda \in C : \lambda\mathrm{I} - T$ is invertible$\}$ [16].
So far, we have had the chance to look at the partitioning of the spectrum into point

spectrum, continuous spectrum, residual spectrum, approximate point spectrum and the essential spectrum of the bounded linear operators.
We now look at some of the results that touch on the partitioning of the spectrum;

Proposition 4.2.1. *Let $T \in B(\mathcal{H})$. Then $\sigma(T) = \sigma_p(T) \bigcup \sigma_c(T) \bigcup \sigma_r(T)$ holds where $\sigma_p(T), \sigma_c(T), \sigma_r(T)$ are mutually disjoint parts of the spectrum, $\sigma(T)$.*

Proposition 4.2.2. *[30] Let $T \in B(\mathcal{H})$ then $\sigma(T) = \sigma_{ap}(T) \bigcup \sigma_{cp}(T)$ where σ_{cp}, σ_{ap} denote continuous point spectrum and the approximate point spectrum respectively which are not necessarily disjoint parts of the spectrum $\sigma(T)$.*

Theorem 4.2.3. *Let $T = U|T|$, be the polar decomposition of the operator $T \in B(\mathcal{H})$ and let $\tilde{T}$ denote the Aluthge transform of T. Then the following statements hold;*

1. *The spectrum of T, $\sigma(T) = \sigma[\sigma(T)]$.*
2. *The point spectrum of T, $\sigma_p(T) = \sigma_p[\tilde{T}]$.*
3. *$\sigma_{ap} = \sigma_{ap}[\tilde{T}]$.*
4. *The essential spectrum of T, $\sigma_e(T) = \sigma_e[\sigma(T)]$.*
5. *$\|\tilde{T})\| \leq \|T^{\frac{1}{2}}\| \leq \|T\|$.*

4.3 Spectral Radius of the Aluthge Transform

Let $T \in B(\mathcal{H})$ be an invertible operator on the complex Hilbert space $\mathcal{H}$. For $0 < \lambda < 1$, we define the λ-Aluthge transform of the operator $T \in B(\mathcal{H})$, as;
$\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$ where $T = U|T|$ is the polar decomposition of T.
The spectral radius of T is therefore given by

$$r(T) = lim_{n \longrightarrow \infty} |||\Delta_\lambda^n(T)|||$$

where $|||.|||$ is the unitary norm such that $(B(\mathcal{H}), |||.|||)$ is a Banach algebra with $\|I\| = 1$.

Lemma 4.3.1. *[10] Let $B(\mathcal{H})$ be a Banach algebra associated with the norm $|||.|||$. Then for $T \in B(\mathcal{H})$, we have that;*

$$r(T) = lim_{n \longrightarrow \infty} |||T^n|||^{\frac{1}{n}} = inf_{n \in \boldsymbol{N}} |||T^n|||^{\frac{1}{n}}$$

.

Theorem 4.3.2. *Let $B(\mathcal{H})$ be the Banach algebra associated with the unitary invariant norm $|||.|||$ and $|||I||| = 1$. Let $T \in B(\mathcal{H})$ be invertible and $0 < \lambda < 1$. Then;*
$lim_{n \longrightarrow \infty} |||\Delta_\lambda^n||| = r(T)$.

Proof. We note that, for $n \in N$, the sequence $\{|||(T_k)^n|||^{\frac{1}{n}}\}_{k \in N}$ is non increasing and converges to $m = lim_{n \longrightarrow \infty} |||T_n|||$ for all $n, k \in N, m \leq |||(T_k)^n|||^{\frac{1}{n}}$.
Now suppose that $r(T) < m$, i.e, $r(T) < m$ for all k, then for a fixed $k \in N$, and for a sufficiently large n, we have that;
$|||(T_k)^n|||^{\frac{1}{n}} < m$ which is a contradiction and so $r(T) = m$.
It then follows that our theorem is true for $T \in B(\mathcal{H})$ invertible. □

4.4 The Numerical Range of Aluthge Transform

We recall that the numerical range of an operator $T \in B(\mathcal{H})$ is given by, $W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1\}$.
We also have to recall that $r(T) \leq w(T) \leq \|T\|$ and $\frac{1}{2}\|T\| \leq w(T) \leq \|T\|$ where $w(T)$ is the numerical radius of $T \in B(\mathcal{H})$.

Theorem 4.4.1. *For each $T \in B(\mathcal{H}), \bigcap_{n=1}^{\infty} \overline{W(\Delta^n(T))} = conv(\sigma(T))$.*

Theorem 4.4.2. *For each $T \in B(\mathcal{H})$, then $conv(\sigma(T)) = \overline{W(T)}$ is equivalent to;*
$\overline{(W(T))} = \overline{W(\sigma(T))}$.

With these classes of operators in mind, we now turn our attention to the convergence of the Aluthge sequence.
Aluthge sequence of weighted shift converges in the strong operator topology (SOT) if its weighted sequence $\{\lambda_n\}$ converges.
The following results give a further explanation on the convergence of the Aluthge sequence;

Theorem 4.4.3. *For any 2×2 matrix T, their exists a matrix N such that $lim_{n \longrightarrow \infty} \Delta^n(T) = N$ and $\sigma(T) = \sigma(N)$.*

We need to note that the Aluthge sequence converges for the general operators but not always as can be seen in the following result;

Theorem 4.4.4. *There exists an operator T such that the Aluthge sequence does not converge in the weak operator topology (WOT). Furthermore, there exists a hyponormal operator whose Aluthge sequence converges in SOT and not norm topology.*

Theorem 4.4.5. *Let T be the hyponormal bilateral shift operator on $l^2(Z)$ with a weighted sequence $\{\lambda_n\}$.*
Letm=sup$\{\lambda_n\}$ and $n = inf\{\lambda_n\}$. Then the Aluthge sequence converges to a quasinormal operator in the norm topology if and only if $m = n$.

Example 4.4.6. *Let T be a bilateral shift with weight sequence λ_n, where;*
$\lambda_n = \begin{cases} \frac{1}{2}, n < 0 \\ 1, n \geq 0 \end{cases}$

Then the Aluthge sequence does not converge to a quasinormal operator in the norm topology but converges in the SOT.
Therefore, we say that every Aluthge sequence of a hyponormal operator converges to a quasinormal operator in the SOT.

4.5 Some Main Results on the Aluthge Transform of Operators

We now want to show the relationships there exist between the bounded linear operators $T \in B(\mathcal{H})$ and their Aluthge transforms $\tilde{T}$.
We recall that, $\sigma(T), \sigma_p(T), \sigma_{ap}(T)$ denote the spectrum, the point spectrum and the approximate point spectrum respectively.

Theorem 4.5.1. *[23] For every $T = U|T|$ in $B(\mathcal{H}), \sigma(T) = \sigma(\tilde{T}), \sigma_{ap}(T) = \sigma_{ap}(T), \sigma_p(T) = \sigma_p(T), \sigma_{ap}(T^*) \setminus (0) = \sigma_{ap}((\tilde{T})^*) \setminus (0)$ and $\sigma_p(T^*) \setminus (0) = \sigma_p((\tilde{T})^*) \setminus (0)$.*

Proof. We have already shown that for $T \in B(\mathcal{H}), \sigma(T) = \sigma(\tilde{T})$.
Now suppose that $\lambda \in \lambda_{ap}(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that;

$$lim_{n\to\infty} \|U|T|_{x_n} - \lambda x_n\| = 0. \tag{4.3}$$

If $\lambda = 0$ then the sequence $\{|T|^{\frac{1}{2}} x_n\}$ tends to zero in the norm and so does $\{\tilde{T}x_n\}$. Thus $0 \in \sigma_{ap}(\tilde{T})$
If $\lambda \neq 0$ the the sequence $\{|T|^{\frac{1}{2}} x_n\}$ fails to tend to zero in the norm by (4.3).
Now, applying $|T|^{\frac{1}{2}}$ to (4.3), we obtain;

$$lim_{n\to\infty} \|\tilde{T}(|T|^{\frac{1}{2}} x_n) - \lambda(|T|^{\frac{1}{2}} x_n)\| = 0. \tag{4.4}$$

Thus $\lambda \in \sigma_{ap}(\tilde{T})$ and $\sigma_{ap}(T) \subset \sigma_{ap}(\tilde{T})$.
Now with the same argument, we see that with $\{x_n\}$ a constant sequence, we have that

$\sigma_p(T) \subset \sigma_p(\tilde{T})$.
Suppose now that $\lambda \in \sigma_{ap}(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that;

$$lim_{n\to\infty}\||T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}x_n - \lambda x_n\| = 0. \tag{4.5}$$

If $\lambda = 0$, then either $\{|T|^{\frac{1}{2}}x_n\}$ tends to zero in the norm, so that $\{Tx_n\}$ tends to zero in norm, or $\{|T|^{\frac{1}{2}}x_n\}$ does not converge to zero .
But since $|T|^{\frac{1}{2}}$, and as a consequence T, maps this last sequence to a null sequence, then $0 \in \sigma_{ap}(T)$.
Now let $\lambda \neq 0$, then the sequence $\{|T|^{\frac{1}{2}}x_n\}$ does not converge to zero by (4.5) and hence we have that $\{U|T|^{\frac{1}{2}}\}$ also fails to converge to zero.
Now, applying $U|T|^{\frac{1}{2}}$ to (4.5) we have that ;

$$lim_{n\to\infty}\|T(U|T|^{\frac{1}{2}}x_n) - \lambda(U|T|^{\frac{1}{2}}x_n)\| = 0. \tag{4.6}$$

which yields $\lambda \in \sigma_{ap}(T)$.
Thus $\lambda_{ap}(\tilde{T}) \subset \lambda_{ap}(T)$ and since $\lambda_{ap}(\tilde{T}) = \lambda_{ap}(T)$, and following the same argument, with a constant sequence $\{x_n\}$ we have that $\sigma_p(\tilde{T}) \subset \sigma_p(T)$ and thus $\sigma_p(T) = \sigma_p(\tilde{T})$.
Now suppose $0 \neq \lambda \in \sigma_{ap}(T^*)$. Then there exists a sequence $\{y_n\}$ of unit vectors in $\mathcal{H}$ such that;

$$lim_{n\to\infty}\||T|U^*y_n - \lambda y_n\| = 0. \tag{4.7}$$

Now applying $|T|^{\frac{1}{2}}U^*$ to (4.7), we have ;

$$lim_{n\to\infty}\|(\tilde{T})^*(|T|^{\frac{1}{2}}U^*y_n) - \lambda(|T|^{\frac{1}{2}})U^*y_n\| = 0. \tag{4.8}$$

Since $\lambda \neq 0$ we have from (4.7) that $\{|T|^{\frac{1}{2}}U^*y_n\}$ cannot converge to zero in norm , and thus from (4.8), we find that $\lambda \in \sigma_{ap}((\tilde{T})^*)$.
Thus, $\sigma_{ap}(T^*) \setminus \{0\} \subset \lambda_{ap}((\tilde{T})^*) \setminus \{0\}$ and following the same argument with the constant sequence y_n, we have that $\sigma_p(T^*) \setminus |0 \subset \sigma_p(T^*)^* \setminus \{0\}$.
Next, suppose that $0 \neq \lambda \in \sigma_{ap}((\tilde{T})^*)$. Then there exists a sequence z_n of unit vectors in $\mathcal{H}$ such that;

$$lim_{n\to\infty}\||T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}z_n - \lambda z_n\| = 0. \tag{4.9}$$

Since $\lambda \neq 0$, the sequence $\{|T|^{\frac{1}{2}}z_n\}$ fails to converge to zero and on using $|T|^{\frac{1}{2}}$ on (4.9), we have that;
$lim_{n\to\infty}\|T^*(|T|^{\frac{1}{2}}z_n) - \lambda(|T|^{\frac{1}{2}}z_n)\| = 0$ so that $\lambda \in \sigma_{ap}(T^*)$.
Thus, $\sigma_{ap}(T^*) \setminus \{0\} = \sigma_{ap}((\tilde{T})^*) \setminus \{0\}$, and following the same argument with a constant sequence z_n, we have that ;
$\sigma_{ap}(T^*) \setminus \{0\} = \sigma_p((\tilde{T})^*) \setminus \{0\}$ and hence we have the result. $\square$

Proposition 4.5.2. *[18] Let $T = U|T|$ be an arbitrary operation in $B(\mathcal{H})$. Then $W(\tilde{T}) \subset W(U)W(|T|)$. Moreover, if $T \in B(\mathcal{H})$, then $W(\tilde{T}) \subset W(T)$.*

Proposition 4.5.3. *[35] An operator $T = U|T|$ in $B(\mathcal{H})$ satisfies $\tilde{T} = T$ if and only if T is quasinormal.*

Proof. If T is quasinormal, then U commutes with $|T|$ and hence it also commutes with $|T|^{\frac{1}{2}}$.
Hence from the above relation, we have that $T = \tilde{T}$.
Conversely, suppose $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = U|T|$, then;
$(|T|^{\frac{1}{2}}U - U|T|^{\frac{1}{2}})|T|^{\frac{1}{2}} = 0$ and so, $|T|^{\frac{1}{2}}U - U|T|^{\frac{1}{2}}$ vanishes on the range of $|T|$.
But both U and $|T|^{\frac{1}{2}}$ vanish orthocomplement of the range of $|T|$.
Hence T is quasinormal as desired. □

Theorem 4.5.4. *[26] Let $T = U|T|$ (Polar decomposition) be an arbitrary quasiaffinity in $B(\mathcal{H})$. Then the mapping $\phi : N \longrightarrow (|T|^{\frac{1}{2}}N)^{-}, N \in Lat(T)$, maps Lat(T) into Lat($\tilde{T}$), and furthermore, if $(0) \neq \phi(N) = (|T|^{\frac{1}{2}}N)^{-} \neq \mathcal{H}$.*
Moreover, the mapping $\psi : M \longrightarrow (U|T|^{\frac{1}{2}}M)^{-} \neq \mathcal{H}$.
As a consequence, Lat(T) is nontrivial if and only if Lat($\tilde{T}$) is nontrivial.

Proof. Clearly, $\phi\{0\} = \{0\})$ and $\phi(\mathcal{H}) = \mathcal{H}$.
Now suppose $(0) \neq N \neq \mathcal{H}$ with $N \in Lat(T)$.
Then ;

$$U|T|^{\frac{1}{2}}(|T|^{\frac{1}{2}}N) \subset N. \tag{4.10}$$

and since $U|T|^{\frac{1}{2}}$ is a quasiaffinity, it maps dense sets in $\mathcal{H}$ to dense sets, and thus the linear manifold $|T|^{\frac{1}{2}}N$ cannot be dense in $\mathcal{H}$.
Therefore, $\phi(N) = (|T|^{\frac{1}{2}}N)^{-}$ is neither (0) nor $\mathcal{H}$.
Moreover, on applying $|T|^{\frac{1}{2}}$ to (4.10), we have, $\tilde{T}(|T|^{\frac{1}{2}}N) \subset |T|^{\frac{1}{2}}N$ and hence $\phi(N)$ is a nontrivial element $Lat(\tilde{T})$ which ends the proof. □

4.6 The Duggal Transform of Operators

4.6.1 Introduction

For $T \in B(\mathcal{H})$, we define a positive operator T, i.e, $T \geq 0$ if $\langle Tx, x\rangle \geq 0, \forall x \in \mathcal{H}$.
We recall that for every self-adjoint operator operator T, we write its polar decomposition as $T = U|T|$ of T, with a partial isometry U and $|T| = (TT^{*})^{\frac{1}{2}}$.
We say that U is uniquely determined by the kernel condition $ker(U) = ker(T)$.
Now, we define a transformation $\Gamma(T) = |T|U$ called the Duggal transformation of T.

For each nonnegative integer n, the n^{th} transformation, $\Gamma^n(T)$ can therefore be defined as;
$\Gamma^n(T) = \Gamma(\Gamma^{n-1}(T))$ and $\Gamma^0(T) = T$.

Definition 4.6.1. *Let $T \in B(\mathcal{H})$. Then T is said to be binormal if $[|T|, |T^*|] = 0$.*

Having defined the Duggal transformation, we now turn our attention to the polar decomposition of the Duggal transformation.
The polar decomposition of the Duggal transform is defined as, $\Gamma(T) = U|\Gamma(T)|$.

If T is binormal, then $\Gamma(T) = \Gamma(U)|\Gamma(T)|$ to be the polar decomposition of the Duggal transform $\Gamma(T)$.

4.7 Duggal Transformation of Binormal Operators

Lemma 4.7.1. *Let $T = U|T|$ be the polar decomposition of the operator T. Then the following assertions hold;*

1. *$r(T) = U^*TU$*
2. *If T is invertible, then $r(T)$ is invertible and therefore $r(T) = |T|T|T|^{-1}$.*

Theorem 4.7.2. *Let $T = U|T|$ be the polar decomposition of T. If T is binormal, then $\sigma(T) = U^*UU|\sigma(T)|$ is the polar decomposition of $\sigma(T)$.*

Theorem 4.7.3. *Let $T = U|T|$ be the polar decomposition of the operator T, and U be a co-isometry. Then T is binormal if and only if $r(T)$ is binormal.*

Proof. From [[8], Theorem 3.2.5], we have that since $r(T) = U^*TU$, then $(r(T))^*r(T) = U^*|T|^2U \geq 0$ and $|r(T)| = U^*|T|U$.
Also, $r(T)(r(T))^* = U^*|T^*|^2U \geq 0$ and $(r(T))^* = U^*|T^*|U$.
Hence T is binormal and it also follows immediately that $r(T)$ is also binormal.
Conversely, suppose $r(T)$ is binormal, then we have that $U^*|T^*||T|U = U^*|T||T^*|U$.
Now, multiplying by U and U^* on both sides, we have, $|T^*||T| = |T||T^*|$.
This implies that T is binormal. □

Theorem 4.7.4. *[2] Let $T = U|T|$ be the polar decomposition of the operator T. If T is binormal, then $\Gamma(\sigma(T)) = \Delta(r(T))$.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Since T is invertible, U is unitary, then $r(T) = U^*TU$.
Therefore, $\Delta(r(T)) = U^*\sigma(T)U$.
Since $\sigma(T) = U|\sigma(T)|$ is the polar decomposition of $\sigma(T)$, then $\Gamma(\sigma(T)) = U^*\sigma(T)U$ as desired. $\square$

4.8 Some Main Results on Duggal Transformation

We recall the following facts; Let $T \in B(\mathcal{H})$, then;

1. T is called hyponormal if $T^*T \geq TT^*$.
2. For $p > 0$, T is p-hyponormal if $(T^*T)^p \geq (TT^*)^p$.
3. If T is invertible, then T is called log-hyponormal if $logT^*T \geq logTT^*$.

Theorem 4.8.1. *[2] Let $T \in B(\mathcal{H})$, then;*

1. $\|\tilde{T}\| \leq \|T\|, \|r(T)\| \leq \|r(T)\|$.
2. *T is quasinormal if and only if $T = r(T)$.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Then $\|U\| = 1$.
Also, $\|T\| = \||T|^2\|^{\frac{1}{2}} = \||T|\| = \||T|^{\frac{1}{2}}\|^2$ and hence $\||T|^{\frac{1}{2}}\| = \|T\|^{\frac{1}{2}}$.
Now,

$$\begin{aligned} \|\tilde{T}\| &= \||T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\| \\ &\leq \||T|^{\frac{1}{2}}\|.\|U\|.\||T|^{\frac{1}{2}}\| \\ &= \||T|^{\frac{1}{2}}\|^2 \\ &= \|T\|. \end{aligned}.$$

Also,

$$\begin{aligned} \|r(T)\| &= \||T|U\| \\ &\leq \||T|^{\frac{1}{2}}\|.\|U\| \\ &= \||T|\| \\ &= \|T\|. \end{aligned}.$$

Furthermore, $T = r(T) \Longrightarrow T = |T|U \Longrightarrow U|T| = |T|U$
$\Longrightarrow |T|$ commutes with $U \Longrightarrow |T|^{\frac{1}{2}}$ commutes with U.

$\Longrightarrow |T|^{\frac{1}{2}}U = U|T|^{\frac{1}{2}}$.
$\Longrightarrow |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = U|T|$.
$\Longrightarrow \tilde{T} = T$.
On the other hand, $T = \tilde{T} \Longrightarrow T = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$.
$\Longrightarrow T|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}U|T| \Longrightarrow T|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}T$.
$\Longrightarrow |T|^{\frac{1}{2}}$ commutes with T $\Longrightarrow |T|$ commutes with T.
$\Longrightarrow T^*T$ commutes with T $\Longrightarrow T$ is quasinormal $\Longrightarrow U$ and $|T|$ commute.
$\Longrightarrow |T|U = U|T| \Longrightarrow r(T) = T$.
Also, T is quasinormal $\Longleftrightarrow$ U and $|T|$ commute.
$\Longleftrightarrow r(T) = T$.
Thus T is quasinormal $\Longleftrightarrow T = r(T)$.
$\Longleftrightarrow T = \tilde{T}$ which completes the proof. $\square$

Definition 4.8.2. *For $T \in B(\mathcal{H})$, let $Hol(\sigma(T))$ be the algebra of all complex valued functions which are analytic on some neighborhood of $\sigma(T)$, where the linear combinations and products in $Hol(\sigma(T))$ are trivially defined.*

The (Riesz-Dunford) algebra $A_T \subseteq B(\mathcal{H})$ is defined as;
$A_T = \{f(T) : f \in Hol(\sigma(T))\}$, where the operator $f(T) \in B(\mathcal{H})$.

Theorem 4.8.3. *For every $T \in B(\mathcal{H})$, with $\tilde{T}, r(T)$ and $Hol(\sigma(T))$, we have that;*

1. *The maps $\tilde{\Phi} : A_T \longrightarrow A_{\tilde{T}}$ and $\Gamma(\Phi) : A_T \longrightarrow A_{r(T)}$ defined by;*
   $\tilde{\Phi}(f(T)) = f(\tilde{T}), \Gamma(\Phi)(f(r(T))) = f(r(T)), f \in Hol(\sigma(T))$ are well defined contractive algebra homomorphisms. Thus,
   $$max\Big\{ \|f(\tilde{T})\|, \|f(r(T))\| \Big\} \leq \|f(T)\|, f \in Hol(\sigma(T)).$$

2. *More generally, the maps $\tilde{\Phi}, \Gamma(\Phi)$ are completely contractive, to mean, that for every $n \in \mathbb{N}$ and every $n \times n$ matrix f_{ij} with entries from $Hol(\sigma(T))$,*
   $$max\Big\{ \|(f_{ij}(r(T)))\|, \|(f_{ij}(\tilde{T}))\| \Big\} \leq \|(f_{ij}(T))\|.$$

3. *Every spectral set for T is a spectral set for both $\tilde{T}$ and $r(T)$. For fixed $k > 1$, every k-spectral set is a k-spectral set for both $\tilde{T}$ and $r(T)$.*

4. *If $W(S)$denotes the numerical range of an operator $S \in B(\mathcal{H})$, then we have that;*
   $\overline{W(f(r(S)))}U\overline{W(f(\tilde{(S)}))} \subset W(f(S)), f \in Hol(\sigma(S))$.

4.9 Some Applications of Numerical Ranges

Numerical ranges of operators are widely used as a tool in solving problems in different disciplines.
The following are some of the areas that we find the notion of numerical ranges of operators very useful;

1. Carrying out Experiments on the Accuracy of the *Chebyshev-Frobenius* Companion matrix method for finding the solutions of a truncated series of *Chebyshev Polynomials.*

   George Frobenius showed that the roots of a polynomial in the form $f_N(x) = \sum_{j=0}^{N} b_j x^j$ are actually the eigenvalues of the Frobenius Companion matrix of the polynomial.
   For example, when $N = 5$, the Frobenius Companion matrix becomes;
   $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ (-1)\frac{b_0}{b_5} & (-1)\frac{b_1}{b_5} & (-1)\frac{b_2}{b_5} & (-1)\frac{b_3}{b_5} & (-1)\frac{b_4}{b_5} \end{pmatrix}$ which can be generalised to any arbitrary N.

2. Finding the Solution of the Multichannel Schrödinger Equation.
   We apply the spectral theory in this area to determine the "scattering" and bounding states of the multichannel Schrödinger equation.
   Bound levels and scattering matrix elements are determined with spectral accuracy using relatively small number points, called the field of values.

3. The study of the Numerical ranges is also very important in determining the behavior of non-normal matrices and operators.

Chapter 5

Conclusion and Recommendations

5.1 Conclusion

The concept of the numerical range of bounded linear operators on Hilbert Spaces plays a central role in the study of the structure and behavior of these operators.
It tends to exploit the gaps that the study of the spectral properties of an operator has always failed to address.
For example, during our study, we have been able to realize that, for $T \in B(\mathcal{H}), T$ is self adjoint if and only if its numerical range lies on the real line, i.e, if and only if $W(T) \subseteq \mathbb{R}$.
We need also to recall that, the numerical ranges of bounded linear operators are not always closed, but this particular instance, i.e, $W(T) \subseteq \mathbb{R}$, the numerical range is closed since it is a line joining two points on the real line.

5.2 Recommendation

During our study of the numerical range of bounded linear operators, we have come across some obstacles, especially in the product of operators and their numerical ranges.
From this study, we have been able to confirm that, given $S, T \in B(\mathcal{H})$, we have that $W(ST) \subseteq W(S)W(T)$. However, we were unable to actually show that, for these operators, $S, T \in B(\mathcal{H})$, $W(ST) = W(S)W(T)$.
So on this difficulty, we recommend a further research that would would impose, on both the operators S and T, some conditions so that the equality is achieved.

Bibliography

[1] A. Aluthge, *On p-hyponormal operator*, Integ. Equat. Oper. Theory **1** (1998), 310–333.

[2] T. Ando, *Aluthge transformation and the convex hull of matrix*, Lin. Multln. Alg. **25** (1989), 275–287.

[3] Balantine, *Numerical range of a matrix: Some effective criteria*, Lin. Alg. Appl. **19** (1978), 117–188.

[4] G. D. E. Barra, *The boundary of the numerical range*, Glasgow Math J. **22(5)** (1981), 69–72.

[5] F. L. Bauer, *On a field of values subordinate to a norm*, Numer. Math. **4** (1962), 103–113.

[6] R. Bouldin, *A note on the numerical range*, Rev. Roum. Math. Pures ET. Appl **18(2)** (1973), 189–190.

[7] Richard Bouldin, *The numerical range of a product*, University of Georgia, Georgia, 1970.

[8] E. Ko C. Foias, I. B. Jung and C. Pearcy, *Complete contractivity of maps associated with the aluthge and duggal transforms*, Pacific Journ. Math. **209(2)** (2003), 249–259.

[9] I. Spitskovsky D. Keeler, L. Rodman, *The numerical range of* 3×3 *matrices*, Lin. Alg. Appl. **252** (1997), 115–139.

[10] Heinz D. Wang and Mclutosh, *Inequalities, aluthge transformation and spectral radius*, Math. Inequal. Appl. **6** (2003), 121–124.

[11] Ch. Davis, *The toeplitz- hausdorff theorem explained*, Canad. Math. Bull. **14** (1971), 245–246.

[12] Mao-Ting et al, *On product of operators and numerical range*, Numer. Math. **64(1)** (2015), 58–67.

[13] K. Fan, *On a theorem of weyl concerning eigenvalues of linear transformations*, Inter. Proc. Nat. Acad. Sci. **3** (1949), 625–655.

[14] H. Flanders, *On the norm and spectral radius*, Lin. Multilin. Alg. **2** (1974), 239–240.

[15] H. Fujita, *Proof that the numerical range of an operator is convex*, Sugaku **17** (1966), 232–233.

[16] T. Furuta, *On the polar decomposition of an operator*, Acta Science Math. **46** (1983), 261–268.

[17] Moshe Goldberg, *On certain dimensional numerical ranges and numerical radii*, Lin. Multlin. Alg. **7(4)** (2007), 329–342.

[18] A. D. Andrew W. L. Green, *Spectral theory of operators in hilbert spaces*, Springer, New York, 2002.

[19] Karl E. Gustafson, *The field of values of linear operators and matrices*, Springer-Verlag, New York, 1997.

[20] P.R. Halmos, *Hilbert space problem book*, Springer, New York, 1982.

[21] Anders C. Hansen, *On approximation of spectra of linear operators on hilbert spaces*, Journ. of Funct. Analysis **2** (2008), 68–75.

[22] Thomas Ryan Harris, *Numerical range of cube roots of identity*, Journ. of Funct. Analysis **2** (2011), 2639–2657.

[23] E. Ko I. B. Jung and C. Pearcy, *The iterated aluthge transformation of operators*, Integ. Equa. Oper. Theo. **45** (2003), 375–387.

[24] E. Ko C. Pearcy I. B. Jung, *Aluthge transform of operators; integral equations and operator theory*, Journ. Of Funct. Anal **37** (2000), 437–448.

[25] R. A. Horn C. R. Johnson, *Topics in matrix analysis*, Camb. Press, New York, 1991.

[26] Carlos S. Kubrusly, *Introduction to models and decomposition in operator theory*, Birkhauser, Boston, 1997.

[27] I. Spitskovsky L. Rodman, 3×3 *matrices with a flat portion on the boundary of the numerical range*, Lin. Alg. Appl. **397** (2005), 193–207.

[28] ______, *On generalised numerical ranges of quadratic operators, recent advances in matrix and operator theory*, Oper. Theory Adv. Appl **197** (2008), 241–256.

[29] E. Tadmor G. Zwas M. Goldberg, *The numerical radius of positive matrices*, Lnear Alg. Appl. **12** (1975), 209–214.

[30] T. Yamazaki M. Ito and M. Yanagida, *On polar decomposition of aluthge transform*, Reader Result Journ. Oper. Theory **51** (2004), 303–329.

[31] A. McIntosh, *The toeplitz- hausdorff theorem and ellipticity conditions*, Amer. Math. Monthly **85** (1978), 475–477.

[32] Ahmed M. Sabir, *The numerical range of* 6×6 *irreducible matrices*, Raf. J. of Comp. and Math's **4** (2007), 1–37.

[33] S. J. Bernau F. Smithes, *A note on normal operators*, Proc. Camb. Phil. Soc. **59** (1963), 727–729.

[34] Pei Yuan Wu, *Polygons and numerical ranges*, Amer. Math. Monthly **107(6)** (2000), 528–540.

[35] ______, *Numerical range of aluthge transform of operators*, Lin. Alg. Appl. **387** (2002), 295–298.