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Numerical Solution of One-Dimensional Incompressible Steady Flow Burgers' equation

Research Report in Mathematics, Number 04, 2018

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August 2018

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Applied Mathematics

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Master of Science Project

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Abstract

The aim of this project is to find the numerical solution of one dimensional ,steady incompressible Burgers' equation by using the Runge-Kutta method. We shall solve the equation by first converting the non-linear Navier Stokes equation into the non-linear viscous burgers equation by using the Orlowski and Sobczyk transformation(OST).After solving we will represent the solutions graphically.

In chapter one,we look at the historical background of Burgers'Equations(BE) and its applications, ways in which fluid motion is described how fluid motion is classified.We will also consider the concept of Dimensional Analysis with the main focus on similarity and dimensionless numbers.We shall derive some common non-dimensional numbers as well. In this chapter ,we consider the equations that govern fluid flow,the momentum equations and the Navier -Stokes equations (NSE) in the various coordinate systems and their derivations.

In chapter two, we will look at some literature review on Burgers'Equations.

We shall look at the methodology in chapter three with emphasis on the Orlowski and Sobczyk transformation(OST)as a method of transforming the Navier-Stokes equation to Burgers' Equation.We will also discuss some types of Runge-Kutta methods.

Finally chapter four we will solve the one-dimensional Burgers'Equation for steady incompressible flow numerically using fourth order Runge-Kutta method(RK4) and represent the solutions graphically.

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Declaration and Approval

I the undersigned declare that this project report is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

__________________ __________________

Signature Date

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Reg No. I56/81308/2015

In our capacity as supervisors of the candidate,we certify that this report has our approval for submission.

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Dedication

This project is dedicated to my dear husband Jason Orina, my children Candice, Newton, Keith-Gabry and Felister.

Contents

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I would like to first thank the Almighty God for this far that He has brought and enabled me to work on this project. I would also wish to thank my supervisor Prof.C.B.Singh who patiently guided me throughout the period i have been working on this work.Special appreciation goes to Prof.G.P.Pokhariyal who came in at handy at the eleventh hour to supervise my work after Prof.Singh was taken ill.May God richly bless them. I also acknowledge and appreciate all my post-graduate lecturers namely; Prof.Ogana,Dr.Moindi,Dr.Were and Dr.Nyandwi together with all the staff in the School of Mathematics.

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Orina Pamela Omangi

Nairobi, 2017.

1 INTRODUCTION

The Burgers' equations are an example of non-linear partial differential equations which are obtained by reducing the Navier- stokes Equations by using the Cole-Hopf transformation.They are model equations that describe the interaction of convection and diffusion.They are non-linear in nature since they possess the advection and diffusion terms.There are no general methods of solving these equations analytically.

These equations arise from the Navier- Stokes equations without the pressure term.They are rare equations whose solutions can be obtained analytically or numerically.Analytical solutions are important since they verify the accuracies of numerical solutions.
In this paper we shall solve one- dimensional incompressible steady flow Burgers' equation numerically.

1.1 Historical Background

The Burgers' equation is a special form of the Navier-Stokes and the continuity equations which was introduced by Bateman who first studied it in his article $1939-1965$, where he used it as a mathematical model on the study of turbulence theory. This equation is obtained by simplifying the Navier-Stokes (NSE),by dropping the pressure term.It is a fundamental partial differential equation in fluid mechanics. The Burgers' Equation (BE) is also referred to as the non-linear diffusion equation after J.M Burgers. The full solution of one dimensional Burgers' Equation was found by Cole and Hopf. The general form of the BE is

$$u_t + uu_x = \nu \nabla^2 u_{xx},$$

where u(x,t)represents the velocity the subscripts x and t denote partial derivatives and ν is the coefficient of kinematic viscosity. The viscid BE represents advective non-linearlity and a Reynolds number defined from the diffusion term. This can be written as;

$$u_t + uu_x = \frac{1}{Re} u_{xx}.$$

Whereas the inviscid BE is;

$$u_t + uu_x = 0.$$

1.1.1 Applications of Burgers'Equation

The solution of Burgers' equations are used in various fields of pure mathematics, applied mathematics and physical science. In cosmology,it is used to approximate and understand the formation and distribution of matter on large scales. In hydrodynamics,it's used as a standard model of turbulence used to study propagation of nonlinear waves and shock formation. The other uses are in describing processes in gas dynamics,nonlinear accostics,heat conduction,plasma physics and as a model for physical events like elastic and hydrodynamic waves. In order to solve the one dimensional Burgers' equations which are non-linear in nature,A.Orlowski and K.Sobczyk presented a transformation of inhomogeneous Burgers' equation to homogeneous form,in 1989. We shall use the Orlowski and Sobczyk transformations (OST) to reduce the non-linear NSE equation to non -linear BE and then solve it.

1.2 Ways of Describing Fluid Motion

There are two ways of describing fluid motion:

(i) Lagrangian: In this method the position and the velocity of individual particles is tracked.This motion is based on Newton's laws of motion.This method is difficulty to use practically since;

a. the fluid consists of very many molecules,

b. interaction between molecules are hard to describe or model.

(ii) Eulerian: This method gives the description of the flow field in terms of the velocity,acceleration,pressure ,temperature and e.t.c as functions of position and time . Its attention is based on a fixed point in space. It's useful since we consider the flow in a particular region and not based on individual particles.

1.2.1 Streamlines,pathlines and streaklines

These are lines which are used to describe fluid motion. Streamlines or lines of flow are curves drawn in a fluid such that the tangent at each point at any time is in the direction of fluid velocity at that point. Pathlines are curves drawn in a fluid such that individual fluid particle travels along this curve. Streaklines are lines formed by all particles passing a given point in the flow.

1.3 Classification of Fluid Flow

There are various classifications of fluid flow namely;

- **Kinematic and Dynamic flow fields**
  In kinematic flow field,we consider the velocity field alone whereas in dynamic flow field we consider the forces acting on the particles.
- **Uniform and Non-uniform Flow**
  A flow is said to be uniform if the flow velocity is constant i.e if the velocity is the same in magnitude and direction at every point or is constant with space whereas it's non-uniform if the flow velocity changes at a given instant and at every point or the flow changes over space.
- **Steady and Unsteady Flow**
  The flow is said to be steady if the fluid flow conditions e.g velocity,pressure,temperature,applied magnetic field and cross-sectional area are independent of time while it's said to be unsteady if the flow variables depend on time.
- **Steady Uniform Flow and non-uniform Flow**
  In steady uniform flow the conditions do not change with position in the stream or with time ,whereas in steady non-uniform flow the conditions change from point to point in the stream but they don't change with time.
- **Unsteady Uniform Flow and Unsteady Non-uniform flow**
  For unsteady uniform flow,at a given instant in time the conditions at every point are constant but they change with time while for unsteady non-uniform flow,the conditions vary from point to point and also they vary with time at every point.

- **Laminar and Turbulent flow**
  Laminar flow refers to the motion of particles in an orderly manner.In this flow,the fluid particles move in a straight line parallel to the boundary walls and the fluid particles do not encounter a disturbance along their path. Turbulence flow refers to a disorderly manner of flow of particles with different velocities and energies.This occurs if the fluid particle encounter a disturbance suddenly while flowing.

- **Internal and External Flows**
  Internal flows are flows which are completely bounded by solid surfaces.They include flows through pipes,ducts,nozzles,diffusers,sudden expansions and contractions,valves and fittings. Open channel flow is the internal flow of liquids in which the duct does not fully flow i.e where there is a free surface subject to a constant pressure.Examples of such flows are flows in rivers,irrigation ditches and aqueducts. External flows are flows over bodies immersed in an unbounded fluid.They include flows over spheres and streamlined bodies(e.g airfoils,auto mobiles and airplanes)

1.3.1 Other Classifications of Fluid Flow

Fluid flow can also be classified in terms of a nondimensional parameter called the the Mach(Ma) number,which is given as;

$$Ma = \frac{V}{c},$$

where, V is the representative velocity and c is the speed of sound in a fluid. If;

- $Ma < .3$ the flow is incompressible
- $Ma > .3$ the flow is compressible
- $Ma = 1$ the flow is sonic
- $Ma > 1$ the flow is supersonic

1.4 DIMENSIONAL ANALYSIS

Dimensional Analysis offers a method which can be used to reduce a complex physical problem to a simplest and most economical form,that is, it involves simplification of physical problems by appealing to dimensional homogeneity to reduce the number of relevant variables. It's principle use is to reduce from a study of the dimensions of variables in any physical system certain limitations on the form of any possible relationship between close variables. At it's heart is the concept of similarity. Dimensional analysis is useful for the following purposes:

- presenting and interpreting experimental data,
- checking equations,
- physical modelling,
- establishing the relative importance of a particular physical phenomena,
- attacking problems not amenable to a direct theoretical solution and
- reducing the number of appropriate parameters for the problem in a question by neglecting some in order to simplify it hence easing it's solution.

1.4.1 Non-dimensionalization

This is the process of writing differential equations in a non-dimensional form by using dimensionless variables which are obtained through proper use of characteristic scales.

1.4.2 Non-dimensionalization procedure

The dimensional variable is transformed into a non-dimensional one by dividing it by a quantity composing of one or more physical properties having the same dimensions as the original one. For example, spatial coordinates are divided by a characteristic length;velocity by a characteristic velocity;pressure by a reference dynamic pressure and time by the ratio of the characteristic length to a reference velocity.

Types of Similarity

There are three types of similarity,namely;

(i)*Geometrical Similarity*:
Two systems are said to be geometrically similar if the ratio of corresponding lengths in the two systems is constant so that one is a scale model of the other.

(ii) *Dynamical similarity*:
Two systems are said to be dynamically similar if the ratio of the several forces acting on corresponding fluid elements are the same in both systems.

(iii) *Kinematic similarity*:
Systems are kinematically similarity if the ratio of all the corresponding velocities are the same.

1.4.3 Non-Dimensional Numbers

These are quantities describing certain physical systems with no units attached to them since they are ratios. They don't change regardless of the type of unit of measurement used.

1.4.4 Derivation Of common Non-Dimensional Numbers

The derivation of these numbers is based on dimensional analysis as applied to fluid dynamical problems and by considering forces on a small volume of fluid,given by the ratio of inertial force to resisting force.

Inertial Force

To derive this force ,let L be the characteristic length in the system to be considered and t be the typical time. Then,

$$\text{mass of the element}, m = \rho L^3$$
$$\text{acceleration}, a = \frac{L}{t^2}$$
$$\text{Inertial force} = mass \times acceleratio$$
$$= \rho L^3 \times \frac{L}{t^2}$$
$$= \rho L^2 \times (\frac{L^2}{t})^2$$
$$\text{But velocity}, v = \frac{L}{t}.$$

Therefore,

$$\text{inertial force} = \rho L^2 V^2.$$

Reynolds Number ,Re

If the motion is controlled by the viscous resistance ,then the ratio of the inertial force to the viscous force is the same i.e.

$$\text{viscous force} = \text{viscous shear stress} \times area$$

But

$$\text{velocity gradient} = \frac{v}{L}$$

Therefore,

$$\text{viscous force} = \mu \frac{v}{L} \times L^2$$
$$= \mu v L,$$

Thus,

$$Re = \frac{\text{inertial force}}{\text{viscous force}} = \frac{\rho L^2 v^2}{\mu v L},$$

where ρ is the density of the fluid and μ is the coefficient of dynamic viscosity, $Re = \frac{LV}{\vartheta}$ where $\vartheta = \frac{\mu}{\rho}$ is the coefficient of kinematic friction L is the characteristic length and v is the mean fluid velocity. The Reynolds number can be used to determine whether the flow is lamina or turbulent. Laminar flow occurs when $Re < 500000$ while turbulent flow occurs when $Re > 500000$.

Froude Number,Fr

This occurs in the case of free-surface flows. It's a measure of the resistance of partially immersed objects moving through fluids. It's given as the ratio of the inertial force to the gravitational force. The resisting force in this case is due to gravity.

$$\text{Gravitational force} = mass \times \text{ acceleration due to gravity} = \rho L^3 \times g,$$

therefore,

$$Fr = \frac{\text{inertial force}}{\text{gravitational force}} = \frac{\rho L^2 v^2}{\rho L^3 g} = \frac{v}{\sqrt{Lg}}$$

High froude number implies higher fluid resistance.For free surface flow ; the nature of the flow is determined by the value of the Froude number i.e;
$Fr > 1$-flow is supercritical,
$Fr < 1$-flow is subcritical,
$Fr = 1$ -flow is critical.
The flow at the interface between two regions is hydraulic jump.

Mach number,Ma

This occurs in the case of compressible flows and is given as the ratio of inertial force to the compressibility force or elastic force. For elastic compression of a fluid,the elastic force depends on the bulk modulus K of the fluid, i.e.

$$\text{elastic force} = KL^2$$

Thus,

$$\begin{aligned} Ma &= \frac{\text{inertial force}}{\text{compressibility force}} \\ &= \frac{\rho V^2 L^2}{KL^2} \\ &= \frac{v}{\sqrt{\frac{K}{\rho}}}. \end{aligned}$$

Alternatively,the Mach number can be given as the ratio of the object's speed to the speed of sound in that medium,when it travels through any medium.
This concept is used mostly to describe the speed of an aircraft.

$$Ma = \frac{\text{velocity of flow}}{\text{velocity of sound}}$$

.

Flights are classified using mach numbers as ;

(i) sonic if $Ma = 1$

(ii) subsonic if $Ma < 1$

(iii)supersonic if $Ma > 1$

(iv)transonic if $0.8 < Ma < 1.3$

(v)hypersonic if $Ma > 5$

Weber Number,We

This occurs in cases of surface tension effects and is defined as the ratio of the inertial force to that of surface tension.

$$\text{surface tension} = \sigma L,$$

where σ is the surface tension per unit length. Thus

$$\begin{aligned} We &= \frac{\text{inertial force}}{\text{surface tension force}} \\ &= \frac{\rho V^2 L^2}{\sigma L} \\ &= \frac{\rho V^2 L}{\sigma}. \end{aligned}$$

This number is useful in droplet breakup and in thin film flow.It's also useful in analyzing multi phase flows involving interface between two different fluid flows.

Peclet number,Pe

In case of heat transfer it's applied in forced convection and is given as the ratio of heat transfer by convection to conduction.

$$\begin{aligned} Pe &= \frac{\rho c_p V L}{k} \\ &= RePr. \end{aligned}$$

or as the product of the Reynold number and Prandtl number.At low Pe, heat transfer dominates by conduction ,as Pe increases convection gains until it dominates at $Pe = 1000$. In case of mass transfer ,the Peclet number is given by the ratio of bulk mass transfer to diffusive mass transfer.

$$\begin{aligned} Pe &= \frac{VL}{D} \\ &= ReSc \end{aligned}$$

or as the product of the Reynolds number and the Schmidt number. Large Re shows low dependence of the flow on downstream locations and high dependence on upstream locations.

1.4.5 Equations Governing Fluid Flow

The equations governing fluid flow are;

- the equation of continuity,
- the equation of momentum,
- the energy equations.

However,in this paper we shall consider the first two equations.

The equation of continuity (equation of mass conservation)

This equation relates the flow field variables at a point in terms of the fluid density ρ and fluid velocity vector,**q**

1.4.6 Deriving equation of continuity in cartesian coordinate system

Consider an infinitesmal element as shown in figure 8 below,as a small volume in the xy-plane with a depth dz. Let us assume that the flow is in the xy-plane and not in the z direction.

Then ,

$$\begin{aligned} \text{change in mass inside the element} &= \text{mass flow into the element} - \\ &\quad \text{mass flow out of the element} \end{aligned}$$

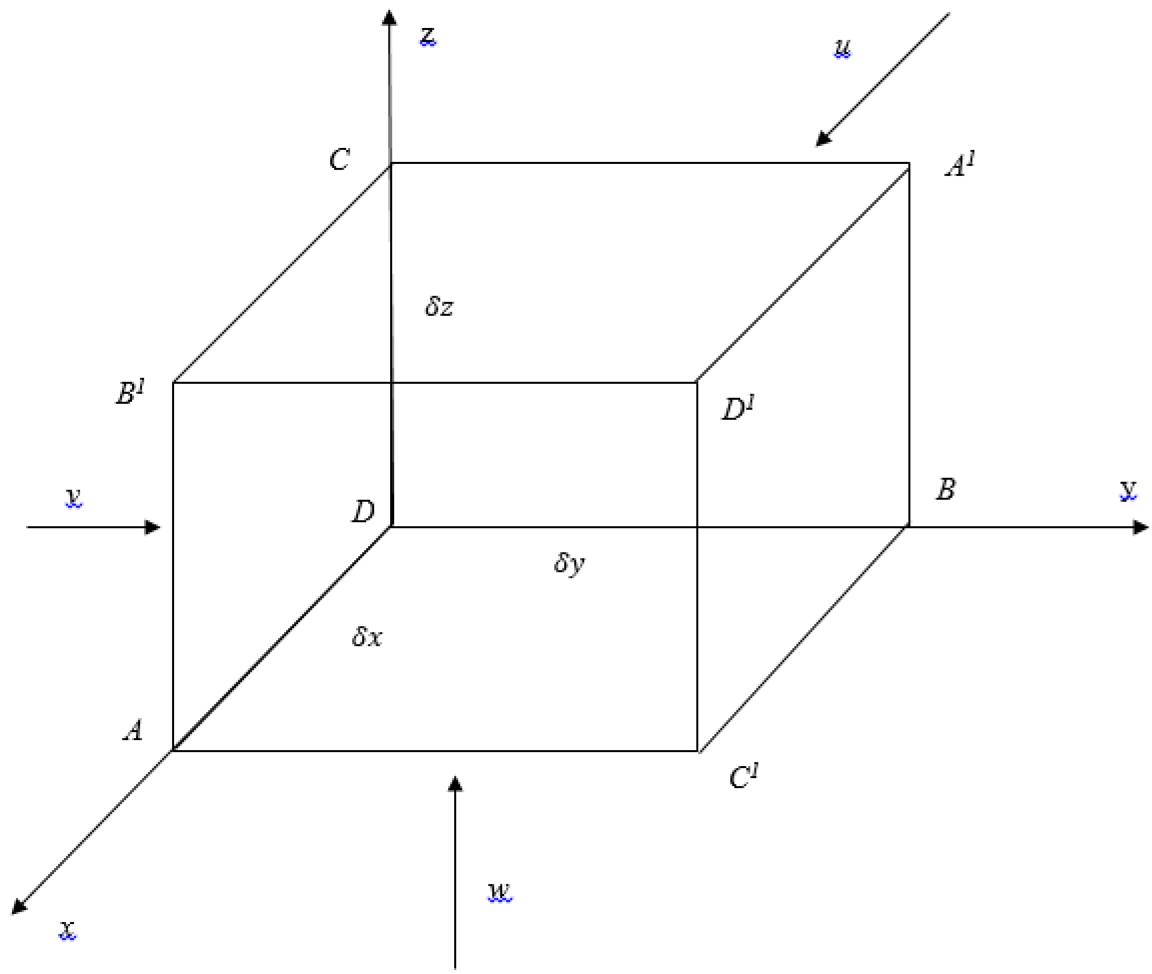

Figure 1. Fluid Flow within a Parallelopiped

Let ρ be the density of the fluid, then the mass flow into the face AB^1CD

$$= \rho v dx dz, \tag{1}$$

mass flow out of face $BC^1D^1A^1$

$$= \rho v + \frac{\partial \rho v}{\partial y} dy dz, \tag{2}$$

Net mass flow through the two faces

$$= \rho v dx dz - (\rho v + \frac{\partial (\rho v)}{\partial y} dy dz). \tag{3}$$

Similarly mass flow into the face DCA^1B

$$= \rho u dy dz \tag{4}$$

mass flow out of the face $AB^1D^1C^1$

$$= (\rho u + \frac{\partial(\rho u)}{\partial x})dydz, \tag{5}$$

net mass flow through the two faces

$$= \rho udydz - (\rho u + \frac{\partial(\rho u)}{\partial x})dydz, \tag{6}$$

change in mass inside the element

$$= \frac{\partial(\rho dxdydz)}{\partial t}. \tag{7}$$

Therefore combining (3), (6) and (7) we get

$$\rho udydz - (\rho u + \frac{\partial(\rho u)}{\partial x}dydz + \rho vdxdz - (\rho v + \frac{\partial(\rho v)}{\partial y})dxdz = \frac{\partial(\rho dxdydz)}{\partial t} \tag{8}$$

(8)simplifies to

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial \rho v}{\partial y} = -\frac{\partial \rho}{\partial t}. \tag{9}$$

Differentiating (9) and including the z-direction we get

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0. \tag{10}$$

Where u,v and w are the velocity components in the x,y and z directions respectively.
Equation(10) is the equation of continuity in three dimensions and in cartesian or rectangular coordinate system. The equation is valid for all types of fluid flow i.e for compressible,incompressible,steady and unsteady. In vector form equation (3.10) is given as;

$$\frac{D\rho}{Dt} + \rho\nabla.\mathbf{q} = 0, \tag{11}$$

or as

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{q}) = 0 \quad (12)$$

(11) is the non-conservative form of mass conservation while (12) is the conservative form of mass conservation where,

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$$
$$\mathbf{q} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}$$
$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \nabla.(\mathbf{q}\rho)$$
$$= 0 \text{ is the material derivative}$$
$$\frac{\partial \rho}{\partial t} \text{ is the partial or local derivative}$$
$$\mathbf{q}.\nabla\rho \text{is the convective derivative}$$

For steady flow,that is,when motion is independent of time (10) becomes;

$$u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0 \quad (13)$$

or in vector form

$$\nabla.(\rho \mathbf{q}) = 0. \quad (14)$$

For incompressible flow ,that is, ρ is a constant.the equations reduce to;

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (15)$$

or

$$\nabla.\mathbf{q} = 0$$

Other coordinate systems

(a)Cylindrical(r,θ ,z)

$$\frac{1}{r}\frac{\partial rq_r}{\partial r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z} = 0$$

where;

$$x = rcos\theta$$
$$y = rsin\theta$$
$$z = z$$

and q_r,q_θ,$q_z$are the velocity components in the r,θ,and z directions respectively.

(b)spherical (r,θ,Φ)

$$\frac{1}{r^2}\frac{\partial r^2 q_r}{\partial r} + \frac{1}{rsin\theta}\frac{\partial q_\theta sin\theta}{\partial \theta} + \frac{1}{rsin\theta}\frac{\partial q_\phi}{\partial \phi} = 0$$

where

$$x = rsin\phi cos\theta$$
$$y = rsin\phi sin\theta$$
$$z = rcos\phi$$

and q_r,q_θ,$q_\phi$are the velocity components in the r,θ,ϕ directions respectively.
The equations stated above apply in the case of incompressible fluid flow.
For compressible flow the equations are;

(c) cylindrical

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial \rho r q_r}{\partial r} + \frac{1}{r}\frac{\partial \rho q_\theta}{\partial \theta} + \frac{\partial \rho q_z}{\partial z} = 0$$

(d) spherical

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (\rho r^2 q_r)}{\partial r} + \frac{1}{rsin\theta}\frac{\partial \rho q_\theta sin\theta}{\partial \theta} + \frac{1}{r^2 sin\theta}\frac{\partial \rho q_\phi}{\partial \phi} = 0$$

1.5 The momentum equations(Navier-Stokes equations)

These equations are based on the Newton's second law of motion or law of conservation of linear momentum for newtonian fluids. The equations are commonly used in describing motion of fluids in models relating to ocean currents,flow of water in pipes and turbulent flows. These equations are also developed in software packages used in designing mechanical machines e.g airplanes,boats,cars and bicycles. They are also used in research fields like in geophysics and in industries e.g chemical,biomedical,aeronautical and chemical. These equations are a mathematical model describing the behavior of fluids and consist of a set of second order non-linear partial differential equations in four independent variables,momentum equations and the continuity equations, describing the flow of viscous incompressible fluids.

1.5.1 Derivation of Navier Stokes Equation in Vector form

The Navier Stokes equations are derived from the law of conservation of mass and momentum conservation. Consider a viscous fluid occupying a certain region such that if V is the volume enclosed by a surface S,which moves with the fluid and contains the same fluid particles at all times as shown in figure 9, below.

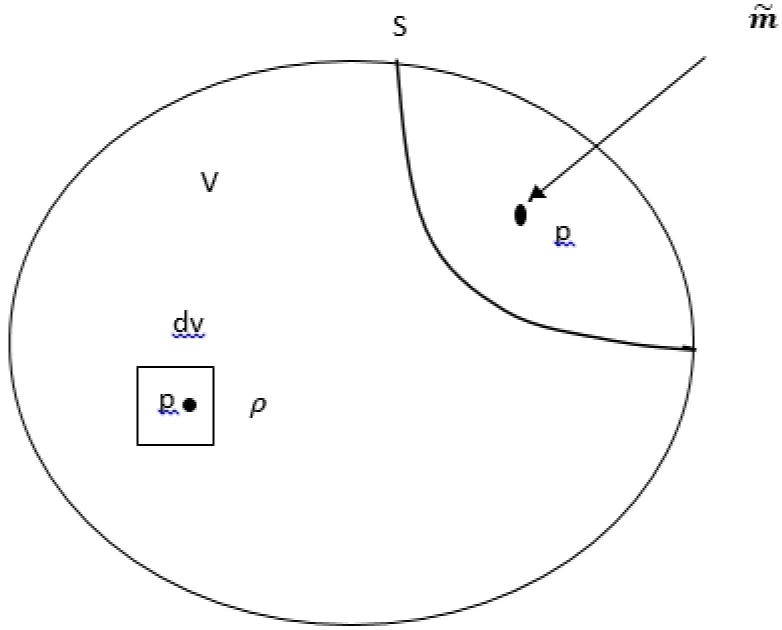

Figure 2. Fluid Flow through a closed surface,S

Let dV be the volume element surrounding the fluid particle P,ρ be the density of the fluid within the surface S. Then,

$$\text{m}asso f the element, m = \rho dV. \quad (16)$$

This mass remains constant throughout the flow.
Let **q** be the velocity of the fluid particl Then momentum M of the volume V is given by;

$$M = \int \rho \mathbf{q} dv, \quad (17)$$

with the integral carried over the entire volume.
If p is the normal pressure force with outward unit normal $\hat{u}$,then the surface force due to p is given by;

$$-\int p\hat{u} ds = -\int_v \nabla p dv \quad (18)$$

Frictional force acting on the volume V is given by;

$$\int_v \nabla^2 \vec{\mathbf{q}} dv \quad (19)$$

The external force per unit mass acting on the fluid is $\vec{\mathbf{F}}$.
Therefore the total force acting on the fluid with space S at any time is;

$$\int_v \vec{\mathbf{F}}\rho dv \tag{20}$$

Adding 18,19 and 20,we get the total force acting on volume V as;

$$\int_v \vec{\mathbf{F}}\rho dv - \int_v \nabla p dv + \int_v \mu \nabla^2 \vec{\mathbf{q}} dv \tag{21}$$

By Newton's second law of motion,which states that the rate of change of linear momentum is equal to the total force acting on the mass of the fluid ,we get;

$$\begin{aligned} \frac{DM}{Dt} &= \int_v \vec{\mathbf{F}}\rho dv - \int_v^{\nabla p dv} + \int_v \Delta p dv + \int_v \mu \nabla^2 \vec{\mathbf{q}} dv \\ \frac{DM}{Dt} &= \int_v (\vec{\mathbf{F}}\rho - \nabla p + \mu \nabla^2) dv \end{aligned} \tag{22}$$

But from (17),

$$\begin{aligned} M &= \int_v \rho \vec{\mathbf{q}} dv \\ \frac{DM}{Dt} &= \int_v \frac{D\vec{\mathbf{q}}}{Dt} \rho dv + \int v \vec{\mathbf{q}} \frac{D}{Dt}(\rho dv) \end{aligned}$$

Therefore (22) becomes ;

$$\int_v \frac{D\vec{\mathbf{q}}}{Dt} . \rho dv + \int_v \vec{\mathbf{q}} \frac{D}{Dt}(\rho dv) = \int_v (\vec{\mathbf{F}}\rho - \nabla denp + \mu \nabla^2 \vec{\mathbf{q}}) dv \tag{23}$$

But from (16) $\rho dv = constant$.Therefore $\frac{D}{Dt}(\rho dv) = 0$ Taking volume V as arbitrary in the region we have considered;

$$\rho \frac{D\vec{\mathbf{q}}}{Dt} = \rho \vec{\mathbf{F}} - \nabla p + \mu \nabla^2 \vec{\mathbf{q}}$$

or

$$\frac{D\vec{\mathbf{q}}}{Dt} = \vec{\mathbf{F}} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{\mathbf{q}} \tag{24}$$

This is the Navier Stokes equation in vector form.

1.5.2 Navier Stokes equations in various coordinate systems

For a newtonian fluid flow and in vector form the Navier Stokes equation is given by;

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{F} + \nu\nabla^2\mathbf{v}$$

or

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{F}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity
ρ is the density of the fluid
$\mathbf{F}$ is the external body forces
$\mathbf{v}.\nabla\mathbf{v}$ is the advection term
$\frac{\partial \mathbf{v}}{\partial t}$ is the acceleration term
$\frac{1}{\rho}\nabla p$ is the pressure term
$\nu\nabla^2\mathbf{v}$ is the velocity diffusion term
The advection and pressure terms are non-linear,the reason why Navier Stokes equations are non-linear.
Also,

$$\mathbf{v} = u\vec{i} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}$$

where u,v,w are the velocity components in the x, y and z directions respectively.

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

From the expressions above we can resolve the NSE in cartesian form by substituting them in the vector form equation to get;
x-component:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) + F_x$$

The y-component:

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + F_y$$

The z-component:

$$\frac{\partial w}{\partial t}+\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}+\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})+F_z$$

Where F_x,F_y and F_z are the components of force in the x ,y and z-directions respectively.

cylindrical coordinate system-(r,θ,z)

For an incompressible fluid with constant kinematic ν and constant density ρ;

r-component:

$$\frac{Dv_r}{Dt}-\frac{v_\theta^2}{r}=-\frac{1}{\rho}\frac{\partial \rho}{\partial r}+F_r+\nu[\nabla^2 v_r-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}]$$

θ component:

$$\frac{Dv_\theta}{Dt}+\frac{v_\theta v_r}{r}=-\frac{1}{r\rho}\frac{\partial p}{\partial \theta}+F_\theta+\nu[\nabla^2 v_\theta-\frac{v_\theta}{r^2}\frac{\partial v_r}{\partial \theta}]$$

z-component:

$$\frac{Dv_z}{t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+F_z+\nu\nabla^2 v_z$$

where v_r ,v_θ and v_z are the velocity components in the $r,\theta,andz$ directions respectively.

p is the pressure and

F_r,F_θ,F_z are the force components in the $r,\theta,andz$directions respectively.

Also

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+v_r\frac{\partial}{\partial r}+\frac{v_\theta}{r}\frac{\partial}{\partial \theta}+v_z\frac{\partial}{\partial z}$$

is the material or the Lagrangian derivative.

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}$$

is the Laplacian operator.

Spherical coordinate system (r,θ,ϕ)

r-component:

$$\frac{Dv_r}{t}+\frac{v_\theta^2+v\phi^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+F_r+\nu[\nabla^2 v_r-\frac{2v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{2v_\theta cot\theta}{r^2}-\frac{2}{r^2 sin\theta}\frac{\partial \phi}{\partial \phi}]$$

θ-component:

$$\frac{Dv_\theta}{t}+\frac{v_\theta v_r}{r}-\frac{v_\theta^2 cot\theta}{r}=-\frac{1}{r\rho}\frac{\partial p}{\partial\theta}+F_\theta+\nu[\nabla^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2 sin^2\theta}-\frac{2cot\theta}{r^2 sin\theta}\frac{\partial v_\phi}{\partial\phi}]$$

ϕ-component:

$$\frac{Dv_\phi}{t}+\frac{v_\phi v_r}{r}+\frac{V_\theta V\phi cot\theta}{r}=-\frac{1}{rsin\theta}\frac{\partial p}{\partial\phi}+F\phi+\nu[\nabla^2 v_\phi+\frac{2}{r^2 sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{v\phi}{r^2 sin\theta}+\frac{2cot\theta}{r^2 sin\theta}\frac{\partial}{\partial}$$

where v_r,v_θ,v_ϕ are the velocity components in the r,θ,ϕ directions respectively,p is the pressure,ρ is the density and F_r,F_θ,F_ϕ are the body forces in the r ,θ,and ϕ directions respectively.
The Lagrangian/material derivative is:

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+v_r\frac{\partial}{\partial r}+\frac{v_\theta}{r}\frac{\partial}{\partial\theta}+\frac{v_\phi}{rsin\theta}\frac{\partial}{\partial\phi}$$

The Laplacian operator is :

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}+\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Note:
For a non-viscous fluid the Navier Stokes equations in vector form reduce to:

$$\frac{Du}{Dt}=F_x-\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{Dv}{Dt}=F_y-\frac{1}{\rho}\frac{\partial p}{\partial y}$$
$$\frac{Dw}{Dt}=F_z-\frac{1}{\rho}\frac{\partial p}{\partial z}$$

These are the Euler's equations of motion in cartesian form.

Non-dimensionalization of the Navier- Stokes equations

To transform a dimensional variable into a dimensionless one,we divide that variable with a quantity consisting of one or more physical properties which have the same dimension as the original variable. Consider the

Navier Stokes equations without body forces:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} \tag{25}$$

and

$$\nabla.\mathbf{v} = 0$$

To non-dimensionalize these equations ,we divide the special coordinates by a characteristic length ,L,the velocity divided by a characteristic velocity,U and time divided by the ratio of characteristic length to a reverence velocity.The dimensional variables are;

$$\mathbf{v}^* = \frac{\mathbf{v}}{U}$$
$$x^* = \frac{x}{L}$$
$$y^* = \frac{y}{L}$$
$$z^* = \frac{z}{L}$$
$$t^* = \frac{tU}{L}$$
$$p^* = \frac{p}{\rho U^2}$$
$$\nabla^* = L\nabla$$
$$(\nabla^*)^2 = L^2\nabla^2$$

Substituting for dimensionless variables in the equations above we obtain;
Mass conservation:

$$\nabla.\mathbf{v} = 0$$
$$\frac{1}{L}\nabla^*.(U.\mathbf{v}^*) = 0$$
$$\frac{U}{L}\nabla^*.\mathbf{v}^* = 0$$

Dividing the above equation by $\frac{U}{L}$ we get:

$$\nabla^*.\mathbf{v}^* = 0$$

which is dimensionless mass conservation equation.
For the conservation of momentum equation:

$$\frac{U^2}{L}(\frac{\partial \mathbf{v}^*}{\partial t^*} + \mathbf{v}^*.\nabla^*\mathbf{v}^*) = -\frac{U^2}{L}\nabla^* p^* + \frac{\nu U}{L^2}(\nabla^*)^2\mathbf{v}^* \tag{26}$$

Dividing the equation above by $\frac{U^2}{L}$we get;

$$\frac{\partial \mathbf{v}^*}{\partial t^*} + \mathbf{v}^*.\nabla^*\mathbf{v}^* = -\nabla^* p^* + \frac{\nu}{LU}(\nabla^*)^2\mathbf{v}^* \tag{27}$$

But

$$\frac{\nu}{LU} = \frac{1}{Re}$$

Therefore the dimensionless momentum equation is ;

$$\frac{\partial \mathbf{v}^*}{\partial t^*} + \mathbf{v}^*.\nabla^*\mathbf{v}^* = -\nabla^* p^* + \frac{1}{Re}(\nabla^*)^2\mathbf{v}^* \tag{28}$$

This is also the Navier-Stokes equation in non-dimensionalized form.

1.6 Problem Statement

The Burgers' Equations,though non-linear in nature have been solved both analytically and numerically by various scholars.Most solutions that have been obtained are for unsteady fluid flows.This prompted us to find a solution for the steady fluid flow.
In this project we converted the Navier- Stokes equation to Burgers' Equation by using the Orlowski and Soczyk transformation together with the Reynolds number(dimensionless)to make the equation non-dimensional.
We solved the Burgers' Equation numerically by using the fourth order Runge-Kutta method and represented the solutions graphically.

1.6.1 Objectives of This Project

The Main objective of this paper was to: solve the one dimensional Burgers' equation for steady flow numerically and represent the solution graphically.

The specific objectives were to;

(i) explain Dimensional Analysis and derive some common non-dimensional numbers.

(ii) discuss the equations governing fluid flow and their representation in various coordinate systems.

(iii) convert the Navier- Stokes equation to Burgers' equation

2 Literature Review

The Burgers' Equations are important equations that govern fluid flow.Since these equations are non-linear in nature,we don't have a general method of solving them analytically.However various scholars have solved the Burgers' equations both analytically and numerically.These have been done mostly for one dimensional unsteady fluid flow. Various studies on one dimensional Navier-Stokes equations coupled by one dimensional viscous Burgers' equations have been carried out in order to solve Burgers' equation.

Neijib Smaoui,**"Analyzing the dynamics of forced Burgers' equation"**, numerically studied the long-time dynamic of a system of reaction-diffusion equation arising from the viscous Burgers' equation which is one dimensional Navier Stokes Equation without the pressure term.

Young,McDonough," **Exact solution to Burgers equation exhibiting erratic turbulent-like behaviour"**,studied on exact solution to a 1-D Burgers equation which exhibited erratic turbulent-like behaviour. They also mentioned that the governing equation is analogous to 1-D Navier Stokes Equation and proposed a model that provides a proper tool of testing numerical algorithm.

Kurt,Cenesiz,Tasbozer,(2016),**"Exact Solution for the conformable Burgers' Equation by the Cole-Hopf transform"** ,used the Cole-Hopf transform to solve conformable Burgers' Equation.

Azad , Andallah,(2014),**"Generating Exact Solutions of the 2-D incompressible Navier-Stokes equation"**,studied on analytical solution of 2-D NSE by using the method of separation of variables after converting the NSE to Burgers' equation ,followed by heat equation which is linear and can be solved.

Aminikhah,(2013),**"An Analytical Approximation for coupled viscous Burgers' Equation"** ,used the laplace transform and homotopy pertubation method to obtain approximate solutions of homogeneous and nonhomogeneous coupled Burgers equations.

Srivastata,Tamsir,(2013),”**Generating Exact solution of 3D coupled unsteady nonlinear generalized viscous Burgers’ Equation”**, derived a general analytical solution of the 3D homogeneous coupled unsteady nonlinear viscous Burgers’ equation via Cole-Hopf transform and separation of variables.
Dehghan,Hamidi,Shakourifar,(2007), ”**The Solution of coupled Burgers’ equation using Adomian-Pade technique”**,studied on analytical solution to 1D coupled Burgers’ Equation using Adomain decomposition method.
Yan,Yue,(2003),”**Variable Separable Solutions for the (2+1)-dimensional Burgers Equation”**,generated variable solution for the (2+1)-dimensional Burgers’ Equation.
Hon,Mao,(1998)”**An efficient numerical scheme for Burgers’ equation”**,used variety of numerical techniques based on finite-difference,finite-element and boundary element methods to solve Burgers’ equation fo small values of the kinematic viscosity,v,which correspond to steep fronts in the propagation of dynamic waveforms.
Debnath,1(1997),” **Partial Differential Equations for Scientists and Engineers”**,studied Burgers’ Equation as a mathematical model in various areas like gas dynamics ,heat conduction and elasticity theory.
Orlowski ,Soczyk,(1989),” **Rep.Math.Phys”**, presented a transformation of inhomogeneous Burgers’ equation to homogeneous form.
Fletcher,(1983),”**Generating exact solution of the two dimensional Burgers’ equation”**,generated exact solution of the two dimensional Burgers’ equation.
Caldwell, Wanless, Cook,(1981),”**A Finite Element approach to Burgers’ Equation”** ,used the finite element method to solve numerically Burgers’ equation for small values of kinematic viscosity.
Jeng and Meecham,(1972),” **Solution of forced Burgers’equation”**,studied the linearlization of the non-linear Burgers’ equation by the Cole-Hopf transformation.
Cole,(1951),” **A Quasilinear Parabolic equations occurring in aerodynamics,”**,independently showed that the Burgers’ Equation can be transformed into the linear diffusion equation and be solved analytically for initial conditions.

Bateman,(1915),**”Some recent researches on the motion of fluids”**,introduced the one dimensional quasi-linear parabolic partial differential equation and gave it’s steady solutions.

3 Methodology

To obtain the solution of Burgers' Equation,we first convert the Navier-Stokes Equation to Burgers' Equation by using the Orlowski and Sobczyk transformation(OST) then using the fourth order Runge-Kutta method we get its numerical solution.

3.0.1 The Orlowski and Sobczyk Transformation-OST

The OST is defined as;

$$x' = x - \phi(t)$$
$$t' = t$$
$$u'(x',t') = u(x,t) - W(t)$$

where

$$W(t) = \int_0^t f(\tau)d\tau = \int_0^t W(\tau)d\tau$$

and

$$\phi(t) = \int_o^t W(\tau)d\tau$$

The transformed derivatives are substituted in (28) to get the transformed Navier Stokes equation which is analogous to non-linear dimensionless form of Burgers equation. The process is: NSE$\xrightarrow{OST} BE$

3.0.2 Runge-Kutta Methods

These are numerical methods which are used to solve differential equations by involving successive approximations.They are a family of implicit and explicit iterative methods used to obtain approximate solutions to ordinary differential equations(ODE).
They were developed by a German mathematician C.Runge and M.W.Kutta in 1900.
The following is the list of Runge-Kutta methods:

- Explicit Runge -Kutta methods

- Adaptative Runge- Kutta methods
- Implicit Runge -Kutta methods
- Nonconfluent Runge -Kutta methods

Explicit Runge-Kutta Method

The common one is the $''RK4''$ which is also reffered to as classical Runge-Kutta method or the Runge-Kutta method. The formular for this method is given as

$$v' = f(x,v)$$

$$v(x_0) = v_0$$

where v is an unknown function(scalar or vector)of the space x,which we want to approximate. Taking $h > 0$ as the step size then;

$$v_{n+1} = v_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

for $n = 0, 1, 2, 3, ...$ where

$$k_1 = hf(x_n, v_n)$$

$$k_2 = hf(x_n + \frac{h}{2}, v_n + \frac{k_1}{2})$$

$$k_3 = hf(x_n + \frac{h}{2}, v_n + \frac{k_2}{2})$$

$$k_4 = hf(x_n + h, v_n + k_3)$$

Here v_{n+1} is the RK4 approximation of $v_{x_{n+1}}$. We obtain the next value v_{n+1} by the sum of the current value v_n and the weighted average of four increments with each increment being the product of the size of the interval ,h. Also,

- k_1 is the increment based on the slope at the beginning of the interval ,using v (Euler's method)
- k_2 is the increment based on the slope at the midpoint of the interval ,using v and k_1

- k_3 is the increment based on the slope at the midpoint ,using v and k_2
- k_4 is the increment based on the slope at the end of the interval,using v and k_3

The RK4 is a fourth order method which implies that the local truncation error is of order $O(h^5)$ while the total accumulated error is of order $O(h^4)$

4 Solution of One-dimensional Steady Incompressible Burgers' Equation

4.1 Numerical Solution of One-dimensional steady Incompressible Burgers' Equation

In this section,we are going to solve numerically one dimensional steady Burgers' Equation (1D BE)of the form:

$$vv_x = \frac{1}{Re} v_{xx}, \tag{29}$$

where v,represents the velocity,subscripts x represent derivatives with respect to x .

4.1.1 Governing Equations

The governing equations of fluid flow are the equations of continuity and momentum.
In dimensionalized form these equations are given as;
The equation of continuity

$$\frac{\partial v^*}{\partial x^*} = 0, \tag{30}$$

The equation of momentum

$$\frac{\partial v^*}{\partial t^*} + v^* \frac{\partial v^*}{\partial x^*} = -\frac{1}{\rho} \frac{\partial p^*}{\partial x^*} + \nu \frac{\partial^2 v^*}{\partial x^{*2}}, \tag{31}$$

where ρ is the density of the fluid which is constant and ν is the kinematic viscosity.
Using the dimensionless definitions,

$v^* = \frac{v}{U}, x^* = \frac{x}{L}, t^* = \frac{tU}{L}, p^* = \frac{p}{\rho U^2}$
where L is the characteristic length and U is the characteristic velocity.
We convert the dimensionalized governing equations into non-dimensional ID NSE as;

$$v_x = 0$$

,

$$v_t + vv_x = -p_x + \frac{1}{Re} v_{xx}, \quad (32)$$

where $\frac{1}{Re} = \frac{\nu}{UL}$ is the Reynolds number.
Letting $-p_x = f(t)$ then equation (6.4)can be written as

$$v_t + vv_x = f(t) + \frac{1}{Re} v_{xx}, \quad (33)$$

The Burgers' Equation for steady flow and without the pressure term is;

$$vv_x = \frac{1}{Re} v_{xx}. \quad (34)$$

4.1.2 To Obtain Numerical solution of 1D Burgers' Equation

To obtain numerical solution of (29) we first reduce the NSE equation to Burgers' equation by using Orlowski and Sobczyk Transformation(OST). The OST is defined as

$$x' = x - \phi(t) \quad (35)$$

$$t' = t \quad (36)$$

$$v'(x', t') = v(x, t) - W(t), \quad (37)$$

where

$$W(t) = \int_0^t f(\tau) d\tau$$

$$\phi(t) = \int_0^t W(\tau) d\tau$$

. But

$$W(\tau) = \int_0^t f(\tau) d\tau$$
$$= [F(\tau)]_0^t$$
$$= F(t) - F(0)$$

. And

$$\phi(t) = \int_0^t W(\tau) d\tau$$
$$= \int_0^t F(\tau) - F(0) d\tau$$
$$= [G(\tau) - \tau F(0)$$
$$= G(t) - tF(0) - G(0).$$

Substituting these in (30),(31),(32) we get

$$x' = x - G(t) + tF(0) + G(0)$$
$$t' = t$$
$$v'(x', t') = v(x, t) - F'(t) + F'(0).$$

Again,

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}[v' + F(t) - F(0)]$$
$$= \frac{\partial v'}{\partial t} + f(t)$$
$$= \frac{\partial v'}{\partial x'} . \frac{\partial x'}{\partial t} + \frac{\partial v'}{\partial t'} . \frac{\partial t'}{\partial t} + f(t).$$
$$= \frac{\partial v'}{\partial x'} . \frac{\partial}{\partial t}[x - G(t) + tF(0) + G(0)] + \frac{\partial v'}{\partial t} . \frac{\partial t}{\partial t} + f(t)$$
$$= \frac{\partial v'}{\partial x'} . [-F(t) + F(0)] + \frac{\partial v'}{\partial t'} + f(t).$$

Also

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}[v' + F(t) - F(0)]$$
$$= \frac{\partial v'}{\partial x}$$

$$= \frac{\partial v'}{\partial x'} . \frac{\partial x'}{\partial x} + \frac{\partial v'}{\partial t'} . \frac{\partial t'}{\partial x}$$

$$= \frac{\partial v'}{\partial x'} . \frac{\partial}{\partial x} [x - G(t) + tF(0) + G(0)] + \frac{\partial v'}{\partial t'} . \frac{\partial t}{\partial x},$$

$$= \frac{\partial v'}{\partial x'}.$$

Again

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x} [\frac{\partial v}{\partial x}]$$

$$= \frac{\partial}{\partial x} [\frac{\partial v'}{\partial x'}]$$

$$= \frac{\partial}{\partial x'} [\frac{\partial v'}{\partial x'}] . [\frac{\partial x'}{\partial x}] + \frac{\partial}{\partial t'} [\frac{\partial v'}{\partial x'}] [\frac{\partial t'}{\partial x}]$$

$$= \frac{\partial^2 v'}{\partial x'^2} . \frac{\partial}{\partial x} [x - G(t) + tF(t) + G(0)] + \frac{\partial^2 v'}{\partial x' \partial y} [\frac{\partial t}{\partial x}]$$

$$= \frac{\partial^2 v'}{\partial x'^2},$$

substituting the transformed derivatives in (28),we get

$$v'_{x'} [-F(t) + F(0)] + v'_{t'} + [v' + F(t) - F(0)] v'_{x'} = f(t) + \frac{1}{Re} (v'_{x'x'}$$

$$\Rightarrow v'_{t'} + v' v'_{x'} = \frac{1}{Re} (v'_{x'x'}). \tag{38}$$

From (27) ,we get

$$v'_{x'} = 0.$$

Therefore ,(33) is the transformed 1D NSE on application of OST,and is analogous to the non-dimensional form of the Burgers' equation which is ;

$$\frac{\partial v^*}{\partial t^*} + v^* \frac{\partial v*}{\partial x*} = \frac{1}{Re} \frac{\partial^2 v^*}{\partial x^{*2}}.$$

For steady flow,the above reduces to;

$$v^* \frac{\partial v*}{\partial x*} = \frac{1}{Re} \frac{\partial^2 v^*}{\partial x^{*2}}.$$

This equation can also be written as:

$$vv^{'} = \frac{1}{Re}v^{''}. \tag{39}$$

Next we solve equation (34) with initial conditions

$$v_0(x) = 0$$
$$v_o^{'}(x) = 1$$
$$v(x) = sin(\pi x)$$

Since equation(39) is a second order ordinary differential equation it cannot be solved directly,hence we convert it to first order differential equation by; let

$$v' = u \tag{40}$$

then this equation becomes;

$$vu = \frac{1}{Re}u'$$

This implies that;

$$u' = Reuv \tag{41}$$

we then solve equations (40)and (41),the set of first order ordinary differential equations for;

$$0 \leq x \leq 2$$

by using the fourth order Runge-Kutta (RK4) method , taking different step sizes,h,for a particular Reynold number and varying the Reynold numbers for a specific step size we will solve the equation.
In our case we will take two Reynold numbers i.e $Re = 5$ and $Re = 10$,and two step sizes i.e $h = 0.2$ and $h = 0.15$.
We will use the Matlab to compute the results, the computation is attached at the end of this project.

Solving for h=0.2,Re=5

$$x_0 = 0$$

$k_1 = hf(x_0, v_0, u_0) = 0.2$
$l_1 = hg(x_0, v_0, u_0) = 0$
$k_2 = hf(x_0 + \frac{h}{2}, v_0 + \frac{k_1}{2}, u_0 + \frac{l_1}{2}) = 0.2$
$l_2 = hg(x_0 + \frac{h}{2}, v_0 + \frac{k_1}{2}, u_0 + \frac{l_1}{2}) = 0.31415926535898$
$k_3 = hf(x_0 + \frac{h}{2}, v_0 + \frac{k_2}{2}, u_0 + \frac{l_2}{2}) = 0.23141592653590$
$l_3 = hg(x_0 + \frac{h}{2}, v_0 + \frac{k_2}{2}, u_0 + \frac{l_2}{2}) = 0.35755727034308$
$k_4 = hf(x_0 + h, v_0 + k_3, u_0 + l_3) = 0.27151145406862$
$l_4 = hg(x_0 + h, v_0 + k_3, u_0 + l_3) = 0.79795214265009$
$v_1 = v_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.22239055119007$
$u_1 = u_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 1.35689753567570$

$$x_1 = 0.2$$

$k_1 = hf(x_1, v_1, u_1) = 0.27137950713514$
$l_1 = hg(x_1, v_1, u_1) = 0.79756436034218$
$k_2 = hf(x_1 + \frac{h}{2}, v_1 + \frac{k_1}{2}, u_1 + \frac{l_1}{2}) = 0.35113594316936$
$l_2 = hg(x_1 + \frac{h}{2}, v_1 + \frac{k_1}{2}, u_1 + \frac{l_1}{2}) = 1.42037472679943$
$k_3 = hf(x_1 + \frac{h}{2}, v_1 + \frac{k_2}{2}, u_1 + \frac{l_2}{2}) = 0.41341697981508$
$l_3 = hg(x_1 + \frac{h}{2}, v_1 + \frac{k_2}{2}, u_1 + \frac{l_2}{2}) = 1.67230681216784$
$k_4 = hf(x_1 + h, v_1 + k_3, u_1 + l_3) = 0.60584086956871$
$l_4 = hg(x_1 + h, v_1 + k_3, u_1 + l_3) = 2.88094453420621$
$v_2 = v_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.62344492163552$
$u_2 = u_1 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 3.00087619775619$

$$x_2 = 0.4$$

$$
\begin{aligned}
k_1 &= hf(x_2, v_2, u_2) = 0.60017523955124 \\
l_1 &= hg(x_2, v_2, u_2) = 2.85400286247105 \\
k_2 &= hf(x_2 + \tfrac{h}{2}, v_2 + \tfrac{k_1}{2}, u_2 + \tfrac{l_1}{2}) = 0.88557552579834 \\
l_2 &= hg(x_2 + \tfrac{h}{2}, v_2 + \tfrac{k_1}{2}, u_2 + \tfrac{l_1}{2}) = 4.42787762899172 \\
k_3 &= hf(x_2 + \tfrac{h}{2}, v_2 + \tfrac{k_2}{2}, u_2 + \tfrac{l_2}{2}) = 1.04296300245041 \\
l_3 &= hg(x_2 + \tfrac{h}{2}, v_2 + \tfrac{k_2}{2}, u_2 + \tfrac{l_2}{2}) = 5.21481501225205 \\
k_4 &= hf(x_2 + h, v_2 + k_3, u_2 + l_3) = 1.64313824200165 \\
l_4 &= hg(x_2 + h, v_2 + k_3, u_2 + l_3) = 7.81358666114715 \\
v_3 &= v_2 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.64017667797726 \\
u_3 &= u_2 + \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 7.99303866544048
\end{aligned}
$$

$$x_3 = 0.6$$

$$
\begin{aligned}
k_1 &= hf(x_3, v_3, u_3) = 1.59860773308810 \\
l_1 &= hg(x_3, v_3, u_3) = 7.60183150776629 \\
k_2 &= hf(x_3 + \tfrac{h}{2}, v_3 + \tfrac{k_1}{2}, u_3 + \tfrac{l_1}{2}) = 2.35879088386472 \\
k_3 &= hf(x_3 + \tfrac{h}{2}, v_3 + \tfrac{k_2}{2}, u_3 + \tfrac{l_2}{2}) = 2.55275868869973 \\
l_3 &= hg(x_3 + \tfrac{h}{2}, v_3 + \tfrac{k_2}{2}, u_3 + \tfrac{l_2}{2}) = 10.32612580848193 \\
k_4 &= hf(x_3 + h, v_3 + k_3, u_3 + l_3) = 3.66383289478448 \\
l_4 &= hg(x_3 + h, v_3 + k_3, u_3 + l_3) = 10.76773471209180 \\
v_4 &= v_3 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 4.15443330681084 \\
u_4 &= u_3 + \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 17.67717815694958
\end{aligned}
$$

$$x_4 = 0.8$$

$k_1 = hf(x_4, v_4, u_4) = 3.53543563138992$
$l_1 = hg(x_4, v_4, u_4) = 10.39038462280161$
$k_2 = hf(x_4 + \frac{h}{2}, v_4 + \frac{k_1}{2}, u_4 + \frac{l_1}{2}) = 4.57447409367008$
$l_2 = hg(x_4 + \frac{h}{2}, v_4 + \frac{k_1}{2}, u_4 + \frac{l_1}{2}) = 7.06795117635995$
$k_3 = hf(x_4 + \frac{h}{2}, v_4 + \frac{k_2}{2}, u_4 + \frac{l_2}{2}) = 4.24223074902591$
$l_3 = hg(x_4 + \frac{h}{2}, v_4 + \frac{k_2}{2}, u_4 + \frac{l_2}{2}) = 6.55460697754485$
$k_4 = hf(x_4 + h, v_4 + k_{3,u_4+l_3}) = 4.84635702689889$
$l_4 = hg(x_4 + h, v_4 + k_3, u_4 + l_3) = 0$
$v_5 = v_4 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 8.49030036409097$
$u_5 = u_4 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 23.94976164538478$

$$x_5 = 1.0$$

$k_1 = hf(x_5, v_5, u_5) = 4.78995232907696$
$l_1 = hg(x_5, v_5, u_5) = 0$
$k_2 = hf(x_5 + \frac{h}{2}, v_5 + \frac{k_1}{2}, u_5 + \frac{l_1}{2}) = 4.78995232907696$
$l_2 = hg(x_5 + \frac{h}{2}, v_5 + \frac{k_1}{2}, u_5 + \frac{l_1}{2}) = -7.40088335965321$
$k_3 = hf(x_5 + \frac{h}{2}, v_5 + \frac{k_2}{2}, u_5 + \frac{l_2}{2}) = 4.04986399311163$
$l_3 = hg(x_5 + \frac{h}{2}, v_5 + \frac{k_2}{2}, u_0 + \frac{l_2}{2}) = -6.25738399389341$
$k_4 = hf(x_5 + h, v_5 + k_3, u_5 + l_3) = 3.53847553029827$
$l_4 = hg(x_5 + h, v_5 + k_3, u_4 + l_3) = -10.39931866153557$
$v_6 = v_5 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 12.82497711471637$
$u_6 = u_5 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 17.66378608394665$

$$x_6 = 1.2$$

$k_1 = hf(x_6, v_6, u_6) = 3.53275721678933$
$l_1 = hg(x_6, v_6, u_6) = -10.38251295979285$
$k_2 = hf(x_6 + \frac{h}{2}, v_6 + \frac{k_1}{2}, u_6 + \frac{l_1}{2}) = 2.49450592081004$
$l_2 = hg(x_6 + \frac{h}{2}, v_6 + \frac{k_1}{2}, u_6 + \frac{l_1}{2}) = -10.09048841252126$
$k_3 = hf(x_6 + \frac{h}{2}, v_6 + \frac{k_2}{2}, u_6 + \frac{l_2}{2}) = 2.52370837553720$
$l_3 = hg(x_6 + \frac{h}{2}, v_6 + \frac{k_2}{2}, u_0 + \frac{l_2}{2}) = -10.20861482327995$
$k_4 = hf(x_6 + h, v_6 + k_3, u_6 + l_3) = 1.49103425213334$
$l_4 = hg(x_6 + h, v_6 + k_3, u_6 + l_3) = -7.09028920755342$
$v_7 = v_7 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 15.33501379165256$
$u_7 = u_7 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 7.98528464412187$

$$x_7 = 1.8$$

$k_1 = hf(x_7, v_7, u_7) = 1.59705692882437$
$l_1 = hg(x_7, v_7, u_7) = -7.59445699526373$
$k_2 = hf(x_7 + \frac{h}{2}, v_7 + \frac{k_1}{2}, u_7 + \frac{l_1}{2}) = 0.83761122929800$
$l_2 = hg(x_7 + \frac{h}{2}, v_7 + \frac{k_1}{2}, u_7 + \frac{l_1}{2}) = -4.18805614649000$
$k_3 = hf(x_7 + \frac{h}{2}, v_7 + \frac{k_2}{2}, u_7 + \frac{l_2}{2}) = 1.17825131417537$
$l_3 = hg(x_7 + \frac{h}{2}, v_7 + \frac{k_2}{2}, u_7 + \frac{l_2}{2}) = -589125657087686$
$k_4 = hf(x_6 + h, v_6 + k_3, u_6 + l_3) = 0.41880561464900$
$l_4 = hg(x_7 + h, v_7 + k_3, u_7 + l_3) = -1.99153904436464$
$v_8 = v_7 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 16.34294506338925$
$u_7 = u_7 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 3.02784773172818$

$$x_8 = 1.6$$

$$k_1 = hf(x_8, v_8, u_8) = 0.60556954634564$$
$$l_1 = hg(x_8, v_8, u_8) = -2.87965431560959$$
$$k_2 = hf(x_8 + \frac{h}{2}, v_8 + \frac{k_1}{2}, u_8 + \frac{l_1}{2}) = 0.31760411478468$$
$$l_2 = hg(x_8 + \frac{h}{2}, v_8 + \frac{k_1}{2}, u_8 + \frac{l_1}{2}) = -1.28473563172108$$
$$k_3 = hf(x_8 + \frac{h}{2}, v_8 + \frac{k_2}{2}, u_8 + \frac{l_2}{2}) = 0.47709598317353$$
$$l_3 = hg(x_8 + \frac{h}{2}, v_8 + \frac{k_2}{2}, u_8 + \frac{l_2}{2}) = -1.92989379167704$$
$$k_4 = hf(x_8 + h, v_8 + k_3, u_8 + l_3) = 0.21959078801023$$
$$l_4 = hg(x_8 + h, v_8 + k_3, u_8 + l_3) = -0.64536113365847$$
$$v_8 = v_7 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 16.74537181843463$$
$$u_8 = u_7 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 1.36880201571747$$

$$x_9 = 1.8$$

$$k_1 = hf(x_9, v_9, u_9) = 0.27376040314349$$
$$l_1 = hg(x_9, v_9, u_9) = -0.80456163814694$$
$$k_2 = hf(x_9 + \frac{h}{2}, v_9 + \frac{k_1}{2}, u_9 + \frac{l_1}{2}) = 0.19330423932880$$
$$l_2 = hg(x_9 + \frac{h}{2}, v_9 + \frac{k_1}{2}, u_9 + \frac{l_1}{2}) = -0.29867147518661$$
$$k_3 = hf(x_9 + \frac{h}{2}, v_9 + \frac{k_2}{2}, u_9 + \frac{l_2}{2}) = 0.24389325562483$$
$$l_3 = hg(x_9 + \frac{h}{2}, v_9 + \frac{k_2}{2}, u_9 + \frac{l_2}{2}) = -0.37683580400753$$
$$k_4 = hf(x_9 + h, v_9 + k_3, u_9 + l_3) = 0.19839324234199$$
$$l_4 = hg(x_9 + h, v_9 + k_3, u_9 + l_3) = 0.0$$
$$v_{10} = v_9 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 16.96979659100009$$
$$u_{10} = u_9 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 1.00953931629493$$

n	x_n	v_n
0	0.0	0.0
1	0.2	0.222391
2	0.4	0.623445
3	0.6	1.640177
4	0.8	4.154433
5	1.0	8.490300
6	1.2	12.824977
7	1.4	15.335014
8	1.6	16.342945
9	1.8	16.745372
10	2.0	16.969796

Table 5.5: Numerical values for 1D Burgers' equation, RK4 method, h=0.2, Re=5.

Solving for h=0.2,Re=10

$$x_0 = 0$$

$$k_1 = hf(x_0, v_0, u_0) = 0.2$$
$$l_1 = hg(x_0, v_0, u_0) = 0$$
$$k_2 = hf(x_0 + \tfrac{h}{2}, v_0 + \tfrac{k_1}{2}, u_0 + \tfrac{l_1}{2}) = 0.2$$
$$l_2 = hg(x_0 + \tfrac{h}{2}, v_0 + \tfrac{k_1}{2}, u_0 + \tfrac{l_1}{2}) = 0.61803398874989$$
$$k_3 = hf(x_0 + \tfrac{h}{2}, v_0 + \tfrac{k_2}{2}, u_0 + \tfrac{l_2}{2}) = 0.26180339887499$$
$$l_3 = hg(x_0 + \tfrac{h}{2}, v_0 + \tfrac{k_2}{2}, u_0 + \tfrac{l_2}{2}) = 0.80901699437495$$
$$k_4 = hf(x_0 + h, v_0 + k_3, u_0 + l_3) = 0.36180339887499$$
$$l_4 = hg(x_0 + h, v_0 + k_3, u_0 + l_3) = 2.12662702088010$$
$$v_1 = v_0 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.24756836610416$$
$$u_1 = u_0 + \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 1.83012149785496$$

$$x_1 = 0.2$$

$k_1 = hf(x_1, v_1, u_1) = 0.36602429957099$
$l_1 = hg(x_1, v_1, u_1) = 2.15143685268512$
$k_2 = hf(x_1 + \frac{h}{2}, v_1 + \frac{k_1}{2}, u_1 + \frac{l_1}{2}) = 0.58116798483950$
$l_2 = hg(x_1 + \frac{h}{2}, v_1 + \frac{k_1}{2}, u_1 + \frac{l_1}{2}) = 4.70174776321801$
$k_3 = hf(x_1 + \frac{h}{2}, v_1 + \frac{k_2}{2}, u_1 + \frac{l_2}{2}) = 0.83619907589279$
$l_3 = hg(x_1 + \frac{h}{2}, v_1 + \frac{k_2}{2}, u_1 + \frac{l_2}{2}) = 6.76499263077897$
$k_4 = hf(x_1 + h, v_1 + k_3, u_1 + l_3) = 1.71902282572679$
$l_4 = hg(x_1 + h, v_1 + k_3, u_1 + l_3) = 16.34887860067568$
$v_2 = v_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.06753190723122$
$u_2 = u_1 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 8.73575420474742$

$$x_2 = 0.4$$

$$k_1 = hf(x_2, v_2, u_2) = 1.74715084094948$$
$$l_1 = hg(x_2, v_2, u_2) = 16.61639192235565$$
$$k_2 = hf(x_2 + \tfrac{h}{2}, v_2 + \tfrac{k_1}{2}, u_2 + \tfrac{l_1}{2}) = 3.408790033185049$$
$$l_2 = hg(x_2 + \tfrac{h}{2}, v_2 + \tfrac{k_1}{2}, u_2 + \tfrac{l_1}{2}) = 34.08790033185049$$
$$k_3 = hf(x_2 + \tfrac{h}{2}, v_2 + \tfrac{k_2}{2}, u_2 + \tfrac{l_2}{2}) = 5.15594087413453$$
$$l_3 = hg(x_2 + \tfrac{h}{2}, v_2 + \tfrac{k_2}{2}, u_2 + \tfrac{l_2}{2}) = 51.55940874134534$$
$$k_4 = hf(x_2 + h, v_2 + k_3, u_2 + l_3) = 12.05903258921855$$
$$l_4 = hg(x_2 + h, v_2 + k_3, u_2 + l_3) = 114.6882152419192$$
$$v_3 = v_2 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 6.22347278136576$$
$$u_3 = u_2 + \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 59.16895842319184$$

$$x_3 = 0.6$$

$$k_1 = hf(x_3, v_3, u_3) = 11.83379168463837$$
$$l_1 = hg(x_3, v_3, u_3) = 112.5460469415472$$
$$k_2 = hf(x_3 + \tfrac{h}{2}, v_3 + \tfrac{k_1}{2}, u_3 + \tfrac{l_1}{2}) = 23.08839637879309$$
$$l_2 = hg(x_3 + \tfrac{h}{2}, v_3 + \tfrac{k_1}{2}, u_3 + \tfrac{l_1}{2}) = 186.7890504330861$$
$$k_3 = hf(x_3 + \tfrac{h}{2}, v_3 + \tfrac{k_2}{2}, u_3 + \tfrac{l_2}{2}) = 61.20437207887429$$
$$l_3 = hg(x_3 + \tfrac{h}{2}, v_3 + \tfrac{k_2}{2}, u_3 + \tfrac{l_2}{2}) = 246.8529019711795$$
$$k_4 = hf(x_3 + h, v_3 + k_3, u_3 + l_3) = 61.20437207887429$$
$$l_4 = hg(x_3 + h, v_3 + k_3, u_3 + l_3) = 282.4323291878441$$
$$v_4 = v_3 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 36.26353111086456$$
$$u_4 = u_3 + \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 282.4323291878441$$

$$x_4 = 0.8$$

$k_1 = hf(x_4, v_4, u_4) = 56.48646583756882$
$l_1 = hg(x_4, v_4, u_4) = 332.0191157344556$
$k_2 = hf(x_4 + \frac{h}{2}, v_4 + \frac{k_1}{2}, u_4 + \frac{l_1}{2}) = 89.68837741101437$
$l_2 = hg(x_4 + \frac{h}{2}, v_4 + \frac{k_1}{2}, u_4 + \frac{l_1}{2}) = 277.1523281791760$
$k_3 = hf(x_4 + \frac{h}{2}, v_4 + \frac{k_2}{2}, u_4 + \frac{l_2}{2}) = 84.20169865548643$
$l_3 = hg(x_4 + \frac{h}{2}, v_4 + \frac{k_2}{2}, u_4 + \frac{l_2}{2}) = 260.1975583978347$
$k_4 = hf(x_4 + h, v_4 + k_3, u_4 + l_3) = 108.5259775171357$
$l_4 = hg(x_4 + h, v_4 + k_3, u_4 + l_3) = 0$
$v_5 = v_4 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 121.7289636921489$
$u_5 = u_4 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 516.8854773359236$

$$x_5 = 1.0$$

$k_1 = hf(x_5, v_5, u_5) = 103.3770954671847$
$l_1 = hg(x_5, v_5, u_5) = 0$
$k_2 = hf(x_5 + \frac{h}{2}, v_5 + \frac{k_1}{2}, u_5 + \frac{l_1}{2}) = 103.3770954671847$
$l_2 = hg(x_5 + \frac{h}{2}, v_5 + \frac{k_1}{2}, u_5 + \frac{l_1}{2}) = -319.4527932848146$
$k_3 = hf(x_5 + \frac{h}{2}, v_5 + \frac{k_2}{2}, u_5 + \frac{l_2}{2}) = 71.43181613870327$
$l_3 = hg(x_5 + \frac{h}{2}, v_5 + \frac{k_2}{2}, u_5 + \frac{l_2}{2}) = -220.7364512592597$
$k_4 = hf(x_5 + h, v_5 + k_3, u_5 + l_3) = 59.22980521533280$
$l_4 = hg(x_5 + h, v_5 + k_3, u_5 + l_3) = -348.1440600172842$
$v_6 = v_5 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 207.0997510078645$
$u_6 = u_5 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 278.7983858183516$

$$x_6 = 1.2$$

$$k_1 = hf(x_6, v_6, u_6) = 55.75967716367032$$
$$l_1 = hg(x_6, v_6, u_6) = -327.7471590939480$$
$$k_2 = hf(x_6 + \frac{h}{2}, v_6 + \frac{k_1}{2}, u_6 + \frac{l_1}{2}) = 22.98496125427551$$
$$l_2 = hg(x_6 + \frac{h}{2}, v_6 + \frac{k_1}{2}, u_6 + \frac{l_1}{2}) = -185.9522426975859$$
$$k_3 = hf(x_6 + \frac{h}{2}, v_6 + \frac{k_2}{2}, u_6 + \frac{l_2}{2}) = 37.16445289391172$$
$$l_3 = hg(x_6 + \frac{h}{2}, v_6 + \frac{k_2}{2}, u_6 + \frac{l_2}{2}) = -300.6667397782177$$
$$k_4 = hf(x_6 + h, v_6 + k_3, u_6 + l_3) = -4.37367079197323$$
$$l_4 = hg(x_6 + h, v_6 + k_3, u_6 + l_3) = 41.59608106835929$$
$$v_7 = v_6 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 235.7138901192097$$
$$u_7 = u_6 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 68.90021198881888$$

$$x_7 = 1.4$$

$$k_1 = hf(x_7, v_7, u_7) = 13.78004239776378$$
$$l_1 = hg(x_7, v_7, u_7) = -131.0559911721673$$
$$k_2 = hf(x_7 + \frac{h}{2}, v_7 + \frac{k_1}{2}, u_7 + \frac{l_1}{2}) = 0.67444328054704$$
$$l_2 = hg(x_7 + \frac{h}{2}, v_7 + \frac{k_1}{2}, u_7 + \frac{l_1}{2}) = -6.74443280547044$$
$$k_3 = hf(x_7 + \frac{h}{2}, v_7 + \frac{k_2}{2}, u_7 + \frac{l_2}{2}) = 13.10559911721673$$
$$l_3 = hg(x_7 + \frac{h}{2}, v_7 + \frac{k_2}{2}, u_7 + \frac{l_2}{2}) = -131.0559911721673$$
$$k_4 = hf(x_7 + h, v_7 + k_3, u_7 + l_3) = -12.43115583666969$$
$$l_4 = hg(x_7 + h, v_7 + k_3, u_7 + l_3) = 118.2273176354524$$
$$v_8 = v_7 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 240.5320520119800$$
$$u_8 = u_7 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 20.82862507348715$$

$$x_8 = 1.6$$

$$k_1 = hf(x_8, v_8, u_8) = 4.16572501469743$$
$$l_1 = hg(x_8, v_8, u_8) = -39.61839920321715$$
$$k_2 = hf(x_8 + \tfrac{h}{2}, v_8 + \tfrac{k_1}{2}, u_8 + \tfrac{l_1}{2}) = 0.20388509437571$$
$$l_2 = hg(x_8 + \tfrac{h}{2}, v_8 + \tfrac{k_1}{2}, u_8 + \tfrac{l_1}{2}) = -1.64946506249693$$
$$k_3 = hf(x_8 + \tfrac{h}{2}, v_8 + \tfrac{k_2}{2}, u_8 + \tfrac{l_2}{2}) = 4.00077850844774$$
$$l_3 = hg(x_8 + \tfrac{h}{2}, v_8 + \tfrac{k_2}{2}, u_8 + \tfrac{l_2}{2}) = -32.36697804064271$$
$$k_4 = hf(x_8 + h, v_8 + k_3, u_8 + l_3) = -2.30767059343111$$
$$l_4 = hg(x_8 + h, v_8 + k_3, u_8 + l_3) = 13.56414741967829$$
$$v_9 = v_8 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 242.243822831322$$
$$u_9 = u_8 + \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 5.14743540851746$$

$$x_9 = 1.8$$

$$k_1 = hf(x_9, v_9, u_9) = 1.02948708170349$$
$$l_1 = hg(x_9, v_9, u_9) = -6.05117324050929$$
$$k_2 = hf(x_9 + \tfrac{h}{2}, v_9 + \tfrac{k_1}{2}, u_9 + \tfrac{l_1}{2}) = 0.42436975765256$$
$$l_2 = hg(x_9 + \tfrac{h}{2}, v_9 + \tfrac{k_1}{2}, u_9 + \tfrac{l_1}{2}) = -1.31137467013420$$
$$k_3 = hf(x_9 + \tfrac{h}{2}, v_9 + \tfrac{k_2}{2}, u_9 + \tfrac{l_2}{2}) = 0.89834961469007$$
$$l_3 = hg(x_9 + \tfrac{h}{2}, v_9 + \tfrac{k_2}{2}, u_9 + \tfrac{l_2}{2}) = -2.77605297829419$$
$$k_4 = hf(x_9 + h, v_9 + k_3, u_9 + l_3) = 0.47427648604465$$
$$l_4 = hg(x_9 + h, v_9 + k_3, u_9 + l_3) = 0$$
$$v_{10} = v_9 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 242.9348160018711$$
$$u_{10} = u_9 + \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 2.77643065228978$$

n	x_n	v_n
0	0.0	0.0
1	0.2	0.247568
2	0.4	1.067532
3	0.6	6.223473
4	0.8	36.263531
5	1.0	121.728964
6	1.2	207.099751
7	1.4	235.713890
8	1.6	240.532052
9	1.8	242.243823
10	2.0	242.9348160

Table 5.6: Numerical values for 1D Burgers' equation, RK4 method, h=0.2, Re=10.

Solving for h=0.15,Re=5

$$x_0 = 0$$

$k_1 = hf(x_0, v_0, u_0) = 0.15$
$l_1 = hg(x_0, v_0, u_0) = 0$
$k_2 = hf(x_0 + \frac{h}{2}, v_0 + \frac{k_1}{2}, u_0 + \frac{l_1}{2}) = 0.15$
$l_2 = hg(x_0 + \frac{h}{2}, v_0 + \frac{k_1}{2}, u_0 + \frac{l_1}{2}) = 0.17508402289193$
$k_3 = hf(x_0 + \frac{h}{2}, v_0 + \frac{k_2}{2}, u_0 + \frac{l_2}{2}) = 0.16313130171689$
$l_3 = hg(x_0 + \frac{h}{2}, v_0 + \frac{k_2}{2}, u_0 + \frac{l_2}{2}) = 0.19041123042794$
$k_4 = hf(x_0 + h, v_0 + k_3, u_0 + l_3) = 0.17856168456419$
$l_4 = hg(x_0 + h, v_0 + k_3, u_0 + l_3) = 0.40532654204816$
$v_1 = v_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.15913738133300$
$u_1 = u_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 1.18938617478132$

$$x_1 = 0.15$$

$k_1 = hf(x_1, v_1, u_1) = 0.17840792621720$
$l_1 = hg(x_1, v_1, u_1) = 0.40497751790421$
$k_2 = hf(x_1 + \frac{h}{2}, v_1 + \frac{k_1}{2}, u_1 + \frac{l_1}{2}) = 0.20878124006001$
$l_2 = hg(x_1 + \frac{h}{2}, v_1 + \frac{k_1}{2}, u_1 + \frac{l_1}{2}) = 0.67796284442466$
$k_3 = hf(x_1 + \frac{h}{2}, v_1 + \frac{k_2}{2}, u_1 + \frac{l_2}{2}) = 0.22925513954905$
$l_3 = hg(x_1 + \frac{h}{2}, v_1 + \frac{k_2}{2}, u_1 + \frac{l_2}{2}) = 0.74444651474896$
$k_4 = hf(x_1 + h, v_1 + k_3, u_1 + l_3) = 0.29007490342954$
$l_4 = hg(x_1 + h, v_1 + k_3, u_1 + l_3) = 1.17337763258085$
$v_2 = v_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.38322997947714$

$u_2 = u_1 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 1.92658181958670$

$$x_2 = 0.3$$

$$
\begin{aligned}
k_1 &= hf(x_2, v_2, u_2) = 0.28898727293800\\
l_1 &= hg(x_2, v_2, u_2) = 1.33494713311838\\
k_2 &= hf(x_2 + \tfrac{h}{2}, v_2 + \tfrac{k_1}{2}, u_2 + \tfrac{l_1}{2}) = 0.38910830792188\\
l_2 &= hg(x_2 + \tfrac{h}{2}, v_2 + \tfrac{k_1}{2}, u_2 + \tfrac{l_1}{2}) = 1.7944600809564\\
k_3 &= hf(x_2 + \tfrac{h}{2}, v_2 + \tfrac{k_2}{2}, u_2 + \tfrac{l_2}{2}) = 0.42379572354518\\
l_3 &= hg(x_2 + \tfrac{h}{2}, v_2 + \tfrac{k_2}{2}, u_2 + \tfrac{l_2}{2}) = 1.95768097474601\\
k_4 &= hf(x_2 + h, v_2 + k_3, u_2 + l_3) = 0.58263941914991\\
l_4 &= hg(x_2 + h, v_2 + k_3, u_2 + l_3) = 2.87733080532743\\
v_3 &= v_2 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.79946910531415\\
u_3 &= u_2 + \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 3.88033713694155
\end{aligned}
$$

$$x_3 = 0.45$$

$$
\begin{aligned}
k_1 &= hf(x_3, v_3, u_3) = 0.58205057054123\\
l_1 &= hg(x_3, v_3, u_3) = 2.87442281080161\\
k_2 &= hf(x_3 + \tfrac{h}{2}, v_3 + \tfrac{k_1}{2}, u_3 + \tfrac{l_1}{2}) = 0.79763228135135\\
l_2 &= hg(x_3 + \tfrac{h}{2}, v_3 + \tfrac{k_1}{2}, u_3 + \tfrac{l_1}{2}) = 3.97586723612132\\
k_3 &= hf(x_3 + \tfrac{h}{2}, v_3 + \tfrac{k_2}{2}, u_3 + \tfrac{l_2}{2}) = 0.88024061325033\\
l_3 &= hg(x_3 + \tfrac{h}{2}, v_3 + \tfrac{k_2}{2}, u_3 + \tfrac{l_2}{2}) = 4.38763562602567\\
k_4 &= hf(x_3 + h, v_3 + k_3, u_3 + l_3) = 1.24019591444508\\
l_4 &= hg(x_3 + h, v_3 + k_3, u_3 + l_3) = 5.89748202957811\\
v_4 &= v_3 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.66246781767909\\
u_4 &= u_3 + \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 8.13015556438716
\end{aligned}
$$

$$x_4 = 0.6$$

$$k_1 = hf(x_4, v_4, u_4) = 1.21952333465807$$
$$l_1 = hg(x_4, v_4, u_4) = 5.19907288248256$$
$$k_2 = hf(x_4 + \frac{h}{2}, v_4 + \frac{k_1}{2}, u_4 + \frac{l_1}{2}) = 1.60945380084427$$
$$l_2 = hg(x_4 + \frac{h}{2}, v_4 + \frac{k_1}{2}, u_4 + \frac{l_1}{2}) = 6.86142476636087$$
$$k_3 = hf(x_4 + \frac{h}{2}, v_4 + \frac{k_2}{2}, u_4 + \frac{l_2}{2}) = 1.73413019213514$$
$$l_3 = hg(x_4 + \frac{h}{2}, v_4 + \frac{k_2}{2}, u_4 + \frac{l_2}{2}) = 7.39294526016749$$
$$k_4 = hf(x_4 + h, v_4 + k_3, u_4 + l_3) = 2.32846512368320$$
$$l_4 = hg(x_4 + h, v_4 + k_3, u_4 + l_3) = 8.23236739356381$$
$$v_5 = v_4 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 3.36832722506244$$
$$u_5 = u_4 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 15.12018561923768$$

$$x_5 = 0.75$$

$$k_1 = hf(x_5, v_5, u_5) = 2.26802784288565$$
$$l_1 = hg(x_5, v_5, u_5) = 8.01868933812171$$
$$k_2 = hf(x_5 + \frac{h}{2}, v_5 + \frac{k_1}{2}, u_5 + \frac{l_1}{2}) = 0.71390170035913$$
$$l_2 = hg(x_5 + \frac{h}{2}, v_5 + \frac{k_1}{2}, u_4 + \frac{l_1}{2}) = 1.86506306892960$$
$$k_3 = hf(x_5 + \frac{h}{2}, v_5 + \frac{k_2}{2}, u_5 + \frac{l_2}{2}) = 2.40790757305537$$
$$l_3 = hg(x_5 + \frac{h}{2}, v_5 + \frac{k_2}{2}, u_5 + \frac{l_2}{2}) = 6.2906415445048$$
$$k_4 = hf(x_5 + h, v_5 + k_3, u_5 + l_3) = 3.21162403105322$$
$$l_4 = hg(x_5 + h, v_5 + k_5, u_5 + l_3) = 4.96223202569210$$
$$v_6 = v_5 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 5.32220562852375$$
$$u_6 = u_5 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 20.00224062100001$$

$$x_6 = 0.9$$

$$k_1 = hf(x_6, v_6, u_6) = 3.00033609315000$$
$$l_1 = hg(x_6, v_6, u_6) = 4.63577420809943$$
$$k_2 = hf(x_6 + \frac{h}{2}, v_6 + \frac{k_1}{2}, u_6 + \frac{l_1}{2}) = 3.34801915875746$$
$$l_2 = hg(x_6 + \frac{h}{2}, v_6 + \frac{k_1}{2}, u_6 + \frac{l_1}{2}) = 1.31341277837805$$
$$k_3 = hf(x_6 + \frac{h}{2}, v_6 + \frac{k_2}{2}, u_6 + \frac{l_2}{2}) = 3.09884205152835$$
$$l_3 = hg(x_6 + \frac{h}{2}, v_6 + \frac{k_2}{2}, u_6 + \frac{l_2}{2}) = 1.21566535202475$$
$$k_4 = hf(x_6 + h, v_6 + k_3, u_6 + l_3) = 3.18268535202475$$
$$l_4 = hg(x_6 + h, v_6 + k_6, u_6 + l_3) = -2.48940840217685$$
$$v_7 = v_6 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 8.50166293948148$$
$$u_7 = u_6 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 21.20299309005701$$

$$x_7 = 1.05$$

$$k_1 = hf(x_7, v_7, u_7) = 3.18044896350855$$
$$l_1 = hg(x_7, v_7, u_7) = -2.48765916097108$$
$$k_2 = hf(x_7 + \frac{h}{2}, v_7 + \frac{k_1}{2}, u_7 + \frac{l_1}{2}) = 2.99387452643572$$
$$l_2 = hg(x_7 + \frac{h}{2}, v_7 + \frac{k_1}{2}, u_7 + \frac{l_1}{2}) = -5.72853089923414$$
$$k_3 = hf(x_7 + \frac{h}{2}, v_7 + \frac{k_2}{2}, u_7 + \frac{l_2}{2}) = 2.75080914606599$$
$$l_3 = hg(x_6 + \frac{h}{2}, v_6 + \frac{k_2}{2}, u_6 + \frac{l_2}{2}) = -5.26344542898907$$
$$k_4 = hf(x_7 + h, v_7 + k_3, u_7 + l_3) = 2.39093214916019$$
$$l_4 = hg(x_7 + h, v_7 + k_6, u_7 + l_3) = -7.02677328254154$$
$$v_8 = v_7 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 11.3451210156017$$
$$u_8 = u_7 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 15.95326224006383$$

$$x_8 = 1.2$$

$$k_1 = hf(x_8, v_8, u_8) = 2.39298933600957$$
$$l_1 = hg(x_8, v_8, u_8) = -7.03281920299793$$
$$k_2 = hf(x_8 + \frac{h}{2}, v_8 + \frac{k_1}{2}, u_8 + \frac{l_1}{2}) = 1.86552789578473$$
$$l_2 = hg(x_8 + \frac{h}{2}, v_8 + \frac{k_1}{2}, u_8 + \frac{l_1}{2}) = -7.09279270473991$$
$$k_3 = hf(x_8 + \frac{h}{2}, v_8 + \frac{k_2}{2}, u_8 + \frac{l_2}{2}) = 1.86102988315408$$
$$l_3 = hg(x_8 + \frac{h}{2}, v_8 + \frac{k_2}{2}, u_8 + \frac{l_2}{2}) = -7.07569112655146$$
$$k_4 = hf(x_8 + h, v_8 + k_3, u_8 + l_3) = 1.33163566702686$$
$$l_4 = hg(x_8 + h, v_8 + k_8, u_8 + l_3) = -5.93248033581429$$
$$v_9 = v_8 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 13.29631014526706$$
$$u_9 = u_8 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 9.06955103983134$$

$$x_9 = 1.35$$

$$k_1 = hf(x_9, v_9, u_9) = 1.36043265597470$$
$$l_1 = hg(x_9, v_9, u_9) = -6.06077186096184$$
$$k_2 = hf(x_9 + \frac{h}{2}, v_9 + \frac{k_1}{2}, u_9 + \frac{l_1}{2}) = 0.9058746640256$$
$$l_2 = hg(x_9 + \frac{h}{2}, v_9 + \frac{k_1}{2}, u_9 + \frac{l_1}{2}) = -4.40422687248562$$
$$k_3 = hf(x_9 + \frac{h}{2}, v_9 + \frac{k_2}{2}, u_9 + \frac{l_2}{2}) = 1.03011564053828$$
$$l_3 = hg(x_9 + \frac{h}{2}, v_9 + \frac{k_2}{2}, u_9 + \frac{l_2}{2}) = -5.00826731695304$$
$$k_4 = hf(x_9 + h, v_9 + k_3, u_9 + l_3) = 0.60919255843174$$
$$l_4 = hg(x_9 + h, v_9 + k_9, u_9 + l_3) = -3.04596279215872$$
$$v_{10} = v_9 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 14.26991114998175$$
$$u_{10} = u_9 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 4.41426386783169$$

$$x_{10} = 1.5$$

$$k_1 = hf(x_{10}, v_{10}, u_{10}) = 0.66213958017475$$
$$l_1 = hg(x_{10}, v_{10}, u_{10}) = -0.75$$
$$k_2 = hf(x_{10} + \frac{h}{2}, v_{10} + \frac{k_1}{2}, u_{10} + \frac{l_1}{2}) = 0.60588958017475$$
$$l_2 = hg(x_{10} + \frac{h}{2}, v_{10} + \frac{k_1}{2}, u_{10} + \frac{l_1}{2}) = -2.94574401422154$$
$$k_3 = hf(x_{10} + \frac{h}{2}, v_{10} + \frac{k_2}{2}, u_{10} + \frac{l_2}{2}) = 0.44120877910814$$
$$l_3 = hg(x_{10} + \frac{h}{2}, v_{10} + \frac{k_2}{2}, u_{10} + \frac{l_2}{2}) = -2.14509072710068$$
$$k_4 = hf(x_{10} + h, v_{10} + k_3, u_{10} + l_3) = 0.34037597110965$$
$$l_4 = hg(x_{10} + h, v_{10} + k_{10}, u_{10} + l_3) = -1.51638605467825$$
$$v_{11} = v_{10} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 14.78602986162344$$
$$u_{11} = u_{10} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 2.33958794494458$$

$$x_{11} = 1.65$$

$$k_1 = hf(x_{11}, v_{11}, u_{11}) = 0.3509381914169$$
$$l_1 = hg(x_8, v_8, u_8) = -1.56344109214356$$
$$k_2 = hf(x_{11} + \frac{h}{2}, v_{11} + \frac{k_1}{2}, u_{11} + \frac{l_1}{2}) = 0.23368010983092$$
$$l_2 = hg(x_{11} + \frac{h}{2}, v_{11} + \frac{k_1}{2}, u_{11} + \frac{l_1}{2}) = -0.88845874778751$$
$$k_3 = hf(x_{11} + \frac{h}{2}, v_{11} + \frac{k_2}{2}, u_{11} + \frac{l_2}{2}) = 0.28430378565762$$
$$l_3 = hg(x_{11} + \frac{h}{2}, v_{11} + \frac{k_2}{2}, u_{11} + \frac{l_2}{2}) = -1.08093147328365$$
$$k_4 = hf(x_{11} + h, v_{11} + k_3, u_{11} + l_3) = 0.18879847074914$$
$$l_4 = hg(x_8 + h, v_8 + k_8, u_8 + l_3) = -0.55486478380858$$
$$v_{12} = v_{11} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 15.04864727053476$$
$$u_{12} = u_{11} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 1.33007355859550$$

$$x_{12} = 1.8$$

$$k_1 = hf(x_{12}, v_{12}, u_{12}) = 0.19951103378933$$
$$l_1 = hg(x_{12}, v_{12}, u_{12}) = -0.58634821665495$$
$$k_2 = hf(x_{12} + \frac{h}{2}, v_{12} + \frac{k_1}{2}, u_{12} + \frac{l_1}{2}) = 0.15553491754020$$
$$l_2 = hg(x_{12} + \frac{h}{2}, v_{12} + \frac{k_1}{2}, u_{12} + \frac{l_1}{2}) = -0.29760318048453$$
$$k_3 = hf(x_{12} + \frac{h}{2}, v_{12} + \frac{k_2}{2}, u_{12} + \frac{l_2}{2}) = 0.17719079525299$$
$$l_3 = hg(x_{12} + \frac{h}{2}, v_{12} + \frac{k_2}{2}, u_{12} + \frac{l_2}{2}) = -0.33903990855456$$
$$k_4 = hf(x_{12} + h, v_{12} + k_3, u_{12} + l_3) = 0.14865504750614$$
$$l_4 = hg(x_8 + h, v_8 + k_8, u_8 + l_3) = -0.11627386416077$$
$$v_{13} = v_{12} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 15.21758352168174$$
$$u_{13} = u_{12h} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 1.0007551544652$$

n	x_n	v_n
0	0.0	0.0
1	0.15	0.159137
2	0.3	0.383230
3	0.45	0.799470
4	0.6	1.662468
5	0.75	3.368327
6	0.9	5.322206
7	1.05	8.501663
8	1.2	11.345121
9	1.35	13.296310
10	1.5	14.269911
11	1.65	14.786030
12	1.8	15.048647
13	1.95	15.217584

Table 5.6: Numerical values for 1D Burgers' equation, RK4 method, h=0.15, Re=

Solving for h=0.15,Re=10

$$x_0 = 0$$

$k_1 = hf(x_0, v_0, u_0) = 0.15$
$l_1 = hg(x_0, v_0, u_0) = 0$
$k_2 = hf(x_0 + \frac{h}{2}, v_0 + \frac{k_1}{2}, u_0 + \frac{l_1}{2}) = 0.15$
$l_2 = hg(x_0 + \frac{h}{2}, v_0 + \frac{k_1}{2}, u_0 + \frac{l_1}{2}) = 0.35016804578386$
$k_3 = hf(x_0 + \frac{h}{2}, v_0 + \frac{k_2}{2}, u_0 + \frac{l_2}{2}) = 0.17626260343379$
$l_3 = hg(x_0 + \frac{h}{2}, v_0 + \frac{k_2}{2}, u_0 + \frac{l_2}{2}) = 0.41147687592790$
$k_4 = hf(x_0 + h, v_0 + k_3, u_0 + l_3) = 0.21172153138919$
$l_4 = hg(x_0 + h, v_0 + k_3, u_0 + l_3) = 0.96119563840998$
$v_1 = v_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.16904112304279$
$u_1 = u_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 1.41408091363892$

$$x_1 = 0.15$$

$k_1 = hf(x_1, v_1, u_1) = 0.21211213704584$
$l_1 = hg(x_1, v_1, u_1) = 0.96296895098263$
$k_2 = hf(x_1 + \frac{h}{2}, v_1 + \frac{k_1}{2}, u_1 + \frac{l_1}{2}) = 0.28433480836953$
$l_2 = hg(x_1 + \frac{h}{2}, v_1 + \frac{k_1}{2}, u_1 + \frac{l_1}{2}) = 1.84660686367931$
$k_3 = hf(x_1 + \frac{h}{2}, v_1 + \frac{k_2}{2}, u_1 + \frac{l_2}{2}) = 0.35060765182179$
$l_3 = hg(x_1 + \frac{h}{2}, v_1 + \frac{k_2}{2}, u_1 + \frac{l_2}{2}) = 2.27701455205287$
$k_4 = hf(x_1 + h, v_1 + k_3, u_1 + l_3) = 0.55366431985377$
$l_4 = hg(x_1 + h, v_1 + k_3, u_1 + l_3) = 4.47923843940745$
$v_2 = v_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.50831801925650$
$u_2 = u_1 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 3.69565595061466$

$$x_2 = 0.3$$

$$k_1 = hf(x_2, v_2, u_2) = 0.55434839259220$$
$$l_1 = hg(x_2, v_2, u_2) = 4.85477270411524$$
$$k_2 = hf(x_2 + \frac{h}{2}, v_2 + \frac{k_1}{2}, u_2 + \frac{l_1}{2}) = 0.89070634540084$$
$$l_2 = hg(x_2 + \frac{h}{2}, v_2 + \frac{k_1}{2}, u_2 + \frac{l_1}{2}) = 8.22905361993766$$
$$k_3 = hf(x_2 + \frac{h}{2}, v_2 + \frac{k_2}{2}, u_2 + \frac{l_2}{2}) = 1.17152741408752$$
$$l_3 = hg(x_2 + \frac{h}{2}, v_2 + \frac{k_2}{2}, u_2 + \frac{l_2}{2}) = 10.82350199651338$$
$$k_4 = hf(x_2 + h, v_2 + k_3, u_2 + l_3) = 2.17787369206921$$
$$l_4 = hg(x_2 + h, v_2 + k_3, u_2 + l_3) = 21.51060452945639$$
$$v_3 = v_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.65109961986286$$
$$u_3 = u_2 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 14.37907069502694$$

$$x_3 = 0.45$$

$$k_1 = hf(x_3, v_3, u_3) = 2.15686060425404$$
$$l_1 = hg(x_0, v_0, u_0) = 21.30306071110701$$
$$k_2 = hf(x_3 + \frac{h}{2}, v_3 + \frac{k_1}{2}, u_3 + \frac{l_1}{2}) = 3.75459015758707$$
$$l_2 = hg(x_3 + \frac{h}{2}, v_3 + \frac{k_1}{2}, u_3 + \frac{l_1}{2}) = 37.43016009162344$$
$$k_3 = hf(x_3 + \frac{h}{2}, v_3 + \frac{k_2}{2}, u_3 + \frac{l_2}{2}) = 4.96412261112580$$
$$l_3 = hg(x_3 + \frac{h}{2}, v_3 + \frac{k_2}{2}, u_3 + \frac{l_2}{2}) = 49.48819877807864$$
$$k_4 = hf(x_3 + h, v_3 + k_3, u_3 + l_3) = 9.58009042096584$$
$$l_4 = hg(x_3 + h, v_3 + k_3, u_3 + l_3) = 91.11207421556341$$
$$v_4 = v_3 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 6.51349571363712$$
$$u_4 = u_3 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 62.08771280603936$$

$$x_4 = 0.6$$

$$k_1 = hf(x_4, v_4, u_4) = 9.31315692090590$$
$$l_1 = hg(x_4, v_4, u_4) = 88.57338576906869$$
$$k_2 = hf(x_4 + \frac{h}{2}, v_4 + \frac{k_1}{2}, u_4 + \frac{l_1}{2}) = 15.95616085368606$$
$$l_2 = hg(x_4 + \frac{h}{2}, v_4 + \frac{k_1}{2}, u_4 + \frac{l_1}{2}) = 136.0486361266195$$
$$k_3 = hf(x_4 + \frac{h}{2}, v_4 + \frac{k_2}{2}, u_4 + \frac{l_2}{2}) = 19.51680463040237$$
$$l_3 = hg(x_4 + \frac{h}{2}, v_4 + \frac{k_2}{2}, u_4 + \frac{l_2}{2}) = 166.4081150773298$$
$$k_4 = hf(x_4 + h, v_4 + k_3, u_4 + l_3) = 34.27437418250538$$
$$l_4 = hg(x_4 + h, v_4 + k_3, u_4 + l_3) = 242.3564240537469$$
$$v_5 = v_4 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 25.60240605886848$$
$$u_5 = u_4 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 218.0615981778250$$

$$x_5 = 0.75$$

$$k_1 = hf(x_5, v_5, u_5) = 32.70923972667375$$
$$l_1 = hg(x_5, v_5, u_5) = 231.2892521818743$$
$$k_2 = hf(x_5 + \frac{h}{2}, v_5 + \frac{k_1}{2}, u_5 + \frac{l_1}{2}) = 50.0559336403133$$
$$l_2 = hg(x_5 + \frac{h}{2}, v_5 + \frac{k_1}{2}, u_5 + \frac{l_1}{2}) = 261.5415348258102$$
$$k_3 = hf(x_5 + \frac{h}{2}, v_5 + \frac{k_2}{2}, u_5 + \frac{l_2}{2}) = 52.32485483860951$$
$$l_3 = hg(x_5 + \frac{h}{2}, v_5 + \frac{k_2}{2}, u_5 + \frac{l_2}{2}) = 273.3966155214384$$
$$k_4 = hf(x_5 + h, v_5 + k_3, u_5 + l_3) = 73.71873205488951$$
$$l_4 = hg(x_5 + h, v_5 + k_3, u_5 + l_3) = 227.8034100873406$$
$$v_6 = v_5 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 77.4673308487030$$
$$u_6 = u_5 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 472.8897586717770$$

$$x_6 = 0.9$$

$$k_1 = hf(x_6, v_6, u_6) = 70.93346380076655$$
$$l_1 = hg(x_6, v_6, u_6) = 219.1964578431702$$
$$k_2 = hf(x_6 + \frac{h}{2}, v_6 + \frac{k_1}{2}, u_6 + \frac{l_1}{2}) = 87.37319813900432$$
$$l_2 = hg(x_6 + \frac{h}{2}, v_6 + \frac{k_1}{2}, u_6 + \frac{l_1}{2}) = 68.55222116836116$$
$$k_3 = hf(x_6 + \frac{h}{2}, v_6 + \frac{k_2}{2}, u_6 + \frac{l_2}{2}) = 76.07488038839364$$
$$l_3 = hg(x_6 + \frac{h}{2}, v_6 + \frac{k_2}{2}, u_6 + \frac{l_2}{2}) = 59.68766322877340$$
$$k_4 = hf(x_6 + h, v_6 + k_3, u_6 + l_3) = 79.88661328508256$$
$$l_4 = hg(x_3 + h, v_3 + k_3, u_3 + l_3) = -124.9701961312768$$
$$v_7 = v_6 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 157.0867032055445$$
$$u_7 = u_6 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 531.3407637561375$$

$$x_7 = 1.05$$

$$k_1 = hf(x_7, v_7, u_7) = 79.70111456342062$$
$$l_1 = hg(x_7, v_7, u_7) = -124.6800121983885$$
$$k_2 = hf(x_7 + \frac{h}{2}, v_7 + \frac{k_1}{2}, u_7 + \frac{l_1}{2}) = 70.35011364854148$$
$$l_2 = hg(x_7 + \frac{h}{2}, v_7 + \frac{k_1}{2}, u_7 + \frac{l_1}{2}) = -269.2182295829800$$
$$k_3 = hf(x_7 + \frac{h}{2}, v_7 + \frac{k_2}{2}, u_7 + \frac{l_2}{2}) = 59.50974734469711$$
$$l_3 = hg(x_7 + \frac{h}{2}, v_7 + \frac{k_2}{2}, u_7 + \frac{l_2}{2}) = -227.7339437304798$$
$$k_4 = hf(x_7 + h, v_7 + k_3, u_7 + l_3) = 45.54102300384866$$
$$l_4 = hg(x_7 + h, v_7 + k_3, u_7 + l_3) = -267.6834169597451$$
$$v_8 = v_7 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 221.2470131311689$$
$$u_8 = u_7 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 300.2961344586287$$

$$x_8 = 1.2$$

$$k_1 = hf(x_8, v_8, u_8) = 45.04442016879430$$
$$l_1 = hg(x_8, v_8, u_8) = -264.7644587328292$$
$$k_2 = hf(x_8 + \frac{h}{2}, v_8 + \frac{k_1}{2}, u_8 + \frac{l_1}{2}) = 25.18708576383211$$
$$l_2 = hg(x_8 + \frac{h}{2}, v_8 + \frac{k_1}{2}, u_8 + \frac{l_1}{2}) = -191.5241027089754$$
$$k_3 = hf(x_8 + \frac{h}{2}, v_8 + \frac{k_2}{2}, u_8 + \frac{l_2}{2}) = 30.68011246562114$$
$$l_3 = hg(x_8 + \frac{h}{2}, v_8 + \frac{k_2}{2}, u_8 + \frac{l_2}{2}) = 282.0006200561406$$
$$k_4 = hf(x_8 + h, v_8 + k_3, u_8 + l_3) = 87.34451317721538$$
$$l_4 = hg(x_8 + h, v_8 + k_3, u_8 + l_3) = -778.2453109295576$$
$$v_9 = v_8 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 261.9342347653216$$
$$u_9 = u_8 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 156.6200119639526$$

$$x_9 = 1.35$$

$$k_1 = hf(x_9, v_9, u_9) = 23.49300179459289$$
$$l_1 = hg(x_9, v_9, u_9) = -209.3241787175130$$
$$k_2 = hf(x_9 + \frac{h}{2}, v_9 + \frac{k_1}{2}, u_9 + \frac{l_1}{2}) = 7.79368839077942$$
$$l_2 = hg(x_9 + \frac{h}{2}, v_9 + \frac{k_1}{2}, u_9 + \frac{l_1}{2}) = -75.78348160146479$$
$$k_3 = hf(x_9 + \frac{h}{2}, v_9 + \frac{k_2}{2}, u_9 + \frac{l_2}{2}) = 17.80924067448303$$
$$l_3 = hg(x_9 + \frac{h}{2}, v_9 + \frac{k_2}{2}, u_9 + \frac{l_2}{2}) = -173.1716993699013$$
$$k_4 = hf(x_9 + h, v_9 + k_3, u_9 + l_3) = -2.48275311089231$$
$$l_4 = hg(x_9 + h, v_9 + k_3, u_9 + l_3) = 24.82753110892307$$
$$v_{10} = v_9 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 273.97025256769259$$
$$u_{10} = u_9 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 42.88551037206558$$

$$x_{10} = 1.5$$

$k_1 = hf(x_{10}, v_{10}, u_{10}) = 6.43282655580984$
$l_1 = hg(x_{10}, v_{10}, u_{10}) = -64.32826555809837$
$k_2 = hf(x_{10} + \frac{h}{2}, v_{10} + \frac{k_1}{2}, u_{10} + \frac{l_1}{2}) = 1.60820663895246$
$l_2 = hg(x_{10} + \frac{h}{2}, v_{10} + \frac{k_1}{2}, u_{10} + \frac{l_1}{2}) = -15.63771761501219$
$k_3 = hf(x_{10} + \frac{h}{2}, v_{10} + \frac{k_2}{2}, u_{10} + \frac{l_2}{2}) = 5.25999773468392$
$l_3 = hg(x_{10} + \frac{h}{2}, v_{10} + \frac{k_2}{2}, u_{10} + \frac{l_2}{2}) = -51.14663578566566$
$k_4 = hf(x_{10} + h, v_{10} + k_3, u_{10} + l_3) = -1.23916881204001$
$l_4 = hg(x_7 + h, v_7 + k_3, u_7 + l_3) = 11.04107496098400$
$v_{11} = v_{10} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 277.1252636495329$
$u_{11} = u_{10} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 11.74286080565391$

$$x_{11} = 1.65$$

$k_1 = hf(x_{11}, v_{11}, u_{11}) = 1.76142912084809$
$l_1 = hg(x_{11}, v_{11}, u_{11}) = -15.69444838571026$
$k_2 = hf(x_{11} + \frac{h}{2}, v_{11} + \frac{k_1}{2}, u_{11} + \frac{l_1}{2}) = 0.58434549191982$
$l_2 = hg(x_{11} + \frac{h}{2}, v_{11} + \frac{k_1}{2}, u_{11} + \frac{l_1}{2}) = -4.44339798027313$
$k_3 = hf(x_{11} + \frac{h}{2}, v_{11} + \frac{k_2}{2}, u_{11} + \frac{l_2}{2}) = 1.42817427232760$
$l_3 = hg(x_{11} + \frac{h}{2}, v_{11} + \frac{k_2}{2}, u_{11} + \frac{l_2}{2}) = -10.85992236594391$
$k_4 = hf(x_{11} + h, v_{11} + k_3, u_{11} + l_3) = 0.1324407659560$
$l_4 = hg(x_{11} + h, v_{11} + k_3, u_{11} + l_3) = -0.77846729031549$
$v_{12} = v_{11} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 278.1117485520828$
$u_{12} = u_{11} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 3.896268077577267$

$$x_{12} = 1.8$$

$$k_1 = hf(x_{12}, v_{12}, u_{12}) = 0.58444021163659$$
$$l_1 = hg(x_{12}, v_{12}, u_{12}) = -3.43525337246679$$
$$k_2 = hf(x_{12} + \frac{h}{2}, v_{12} + \frac{k_1}{2}, u_{12} + \frac{l_1}{2}) = 0.32679620870158$$
$$l_2 = hg(x_{12} + \frac{h}{2}, v_{12} + \frac{k_1}{2}, u_{12} + \frac{l_1}{2}) = -1.25059494829819$$
$$k_3 = hf(x_{12} + \frac{h}{2}, v_{12} + \frac{k_2}{2}, u_{12} + \frac{l_2}{2}) = 0.49064559051423$$
$$l_3 = hg(x_{12} + \frac{h}{2}, v_{12} + \frac{k_2}{2}, u_{12} + \frac{l_2}{2}) = -1.87761938652780$$
$$k_4 = hf(x_{12} + h, v_{12} + k_3, u_{12} + l_3) = 0.30279730365742$$
$$l_4 = hg(x_{12} + h, v_{12} + k_3, u_{12} + l_3) = -0.47367934213273$$
$$v_{13} = v_{12} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 278.5321020710370$$
$$u_{13} = u_{12} + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 2.20204118020201$$

n	x_n	v_n
0	0.0	0.0
1	0.15	0.169041
2	0.3	0.508318
3	0.45	1.651020
4	0.6	6.513496
5	0.75	25.602406
6	0.9	77.467331
7	1.05	157.086703
8	1.2	221.247013
9	1.35	261.934235
10	1.5	273.970253
11	1.65	277.125263
12	1.8	278.111749
13	1.95	278.532102

Table 5.6: Numerical values for 1D Burgers' equation, RK4 method, h=0.15, Re=10

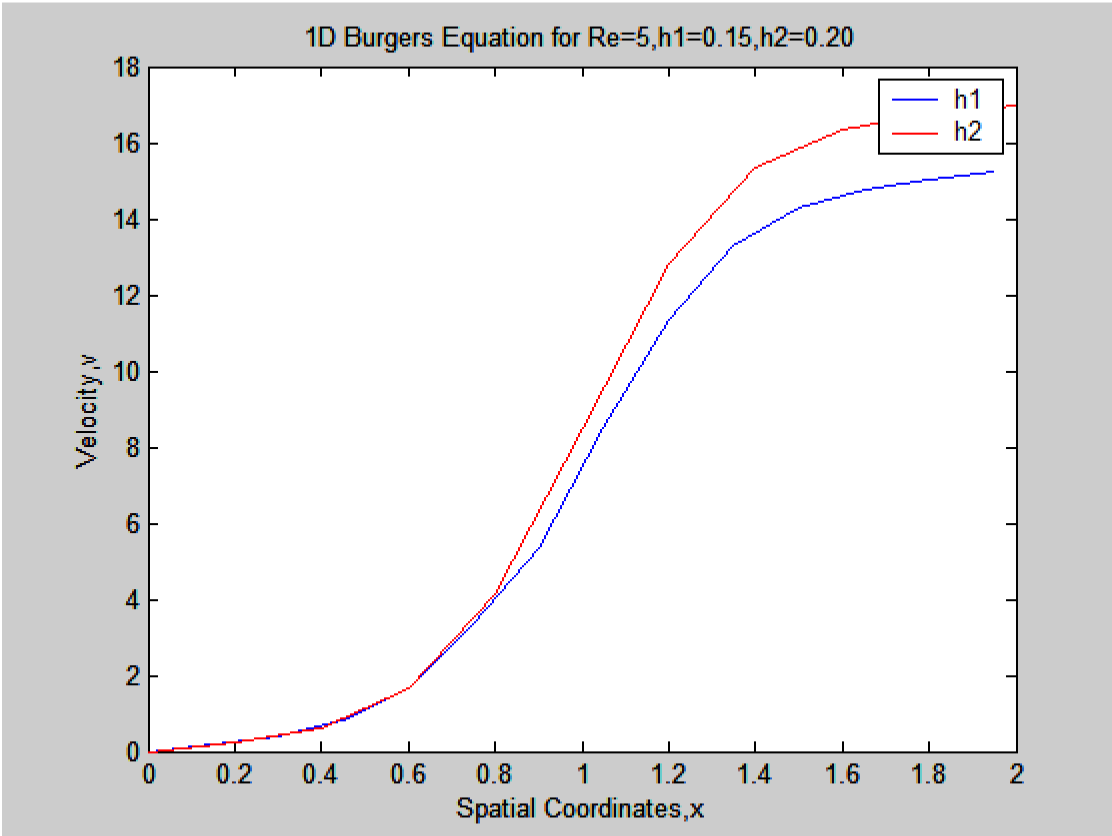

Figure 3. Numerical solution of 1D Burgers' equation taking h=0.15, h=0.2 and Re=5

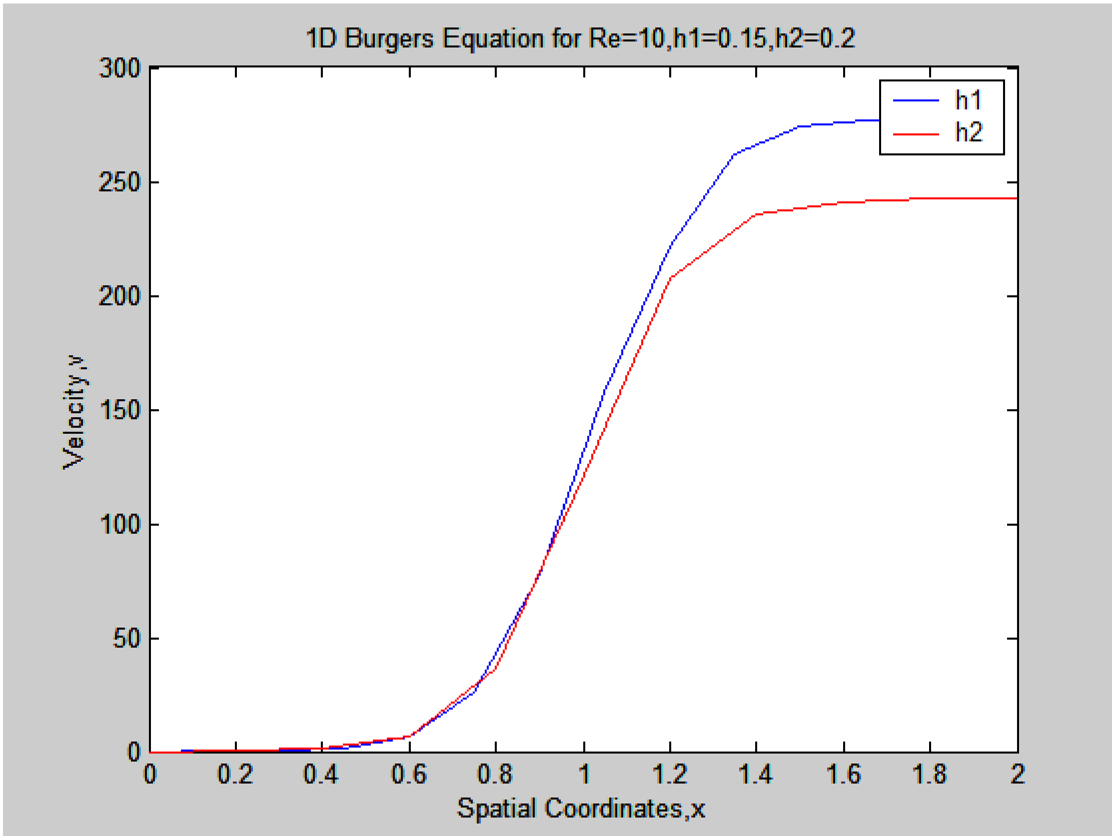

Figure 4. Numerical solution of 1D Burgers' equation taking h=0.15, h=0.2 and Re=10

4.2 DISCUSSIONS AND CONCLUSIONS

We have solved the equation numerically by using the Runge-Kutta method of order four.
We have drawn figures (3) and (4) for Re=5,Re=10 with space widths h=0.15 and h=0.20 respectively.
We notice that the velocity component increases exponentially with the spatial coordinates,which is a new development.
We also notice that the graphs become smoother with reduced step size .
This implies that the error is reduced with decreased step size.
We finally notice that the graphs become more smooth with reduced Reynold number.
We would not compare the findings with other authors since the problem had not been solved by them.However, we recommend that the solution of the same problem be done with same method but valid Reynold numbers and valid step sizes.
The other recommendation is to solve the same problem analytically and also by using different numerical methods.Once these are done, then the solutions can be compared and the degree of the accuracy determined for each method in order to draw a conclusion on which method is better as far as the degree of accuracy is concerned.

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