University of Nairobi

School of Mathematics

ISSN: 2410-1397

Master Project in Applied Mathematics

A STUDY OF W_6-CURVATURE TENSORS ON LORENTZIAN PARA-SASAKIAN MANIFOLD.

Research Report in Mathematics, Number 08 , 2018

WILSON KAMAMI WANJIRU

June 2018

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Applied Mathematics

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WILSON KAMAMI WANJIRU

School of Mathematics
College of Biological and Physical sciences
Chiromo, off Riverside Drive
30197-00100 Nairobi, Kenya

Master Thesis

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Applied Mathematics

Submitted to: The Graduate School, University of Nairobi, Kenya

Abstract

The properties of W_6-curvature tensor are studied in lp- sasakian manifold and following theorem proved
A W_6 skew symmetric lp-sasakian manifold
A W_6 symmetric lp-sasakian maifold
A study of properties of w_6,P and Q curvature tensors in lp-sasakian manifold.

Master Thesis in Mathematics at the University of Nairobi, Kenya.
ISSN 2410-1397: Research Report in Mathematics

DISTRIBUTOR: School of Mathematics, University of Nairobi, Kenya

Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

WILSON KAMAMI WANJIRU
Reg No. I56/89387/2016

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

Signature Date

Prof G.P POkhariyal
School of Mathematics,
University of Nairobi,
Box 30197, 00100 Nairobi, Kenya.
E-mail: gpokhariyal@uonbi.ac.ke

Signature Date

Prof.Stephen Moindi
School of Mathematics
University of Nairobi,
Box 30197, 00100 Nairobi, Kenya.
E-mail: moindi@uonbi.ac.ke

Dedication

This project is dedicated to my grandmother and my mother Mrs Elizabeth Mwangi and Mrs Purity Mwangi for your unconditional support
My family: Thanks for believing in me and allowing me to further my studies.
My daughter Shiru: Your smile every morning gave me hope and encouragement..

Contents

Acknowledgments

First, I thank my supervisors Prof G.P Pokhariyal and Prof Moindi Stephen.It has been my pleasure to interact with them both in and outside the classroom. I must acknoledge school of mathematics,my classmates and family members not forgetting my friends who helped me both emotionally and financially. ....

Wilson Kamami Wanjiru

Nairobi, 2018.

1 Introduction

Riemannian geometry was first put forward in generality by Bernhard Riemann in the nineteenth century. It deals with a broad range of geometries whose metric properties vary from point to point, including the standard types of Non-Euclidean geometry.

Any smooth manifold admits a Riemannian metric, which often helps to solve problems of differential topology. It also serves as an entry level for the more complicated structure of pseudo-Riemannian manifolds, which (in four dimensions) are the main objects of the theory of general relativity. Other generalizations of Riemannian geometry include Finsler geometry.There exists a close analogy of differential geometry with the mathematical structure of defects in regular crystals. Dislocations and Discrimination produce torsion's and curvature.

1.1 Definition

1.1.1 Differentiable manifold

The basic idea that leads to differentiable manifold is to try to select a family or sub collection of neighbourhood so that the change of cordinates is always given by differentiable functions. As to definitions of differentiable manifold we first look at n-dimensional serial space R^n as product space of R. where R is set of real numbers. R^n is obtained taking n-copies of R

Example

R^n = R× R×R×R.........×R;

R n-times where n is any integer greater than zero in R^n each element can be represented by $n-tuples$ so that for every $X\varepsilon R^n$

$x=(x_1,x_2,....,x_n)$, where $x_i\varepsilon$R

and i=1,2,3.....n

let us take two arbitrary points in R^n then for every such pair we can define metric on R^n by d(x,y)=$(\sum_{i=1}^{n}(x_i-y_i)^2)^{\frac{1}{2}}$

then R^n becomes metric space with metric topology as defined above for future discussion R^n is been considered as topological space with M being open subset. The definition of topological manifold M of dimensional n is a Hausdorff space with countable basis of open sets and with further property that each point has a neighborhood homomorphic to open subset of R^n.

Differentiable manifold

Let v_n be non empty para compact Hausdorff space.Then v_n is said to be n-dimensional topological manifold if every point $x\varepsilon\ v_n$ has open neighborhood U in v_n which is homomorphic to an open subspace of the n-dimensional euclidean space R^n

1.1.2 Differentiable structure

Concept of differentiable structure is studied in this section which form basis of differentiable manifold.First we look at element of differentiable structure namely chart and atlases.

charts

For chart X we mean imbeddingΦ:U $\rightarrow R^n$ of open subspace U of X into R^n such that Φ(U) in open subspace of R^n. where U domain of chart

let Pi: $R^n \rightarrow R$ such that $i = 1, 2, ...n$ denote natural projection defined

$Pi(t_1, t_2, ...t_n) = t_i$ for all $t_1, t_2, t_3.....t_n \varepsilon R^n$. Then for every chart Φ:U $\rightarrow R^n$ on X. The Φ=p_i0Φ:U$\rightarrow$R is known as i^th coordinate function in U with respect to the chart Φ and for every xεU the real number $t_1 = \Phi_i(x)$ is the i^th cordinate of point x with respect to chart Φ.

The chart Φ:U $\rightarrow R^n$ is called local coordinate system u eor every xε U then real numbers $(t_1, t_2......t_n)$=$(\Phi_1......., \Phi_n) = \Phi(x) \varepsilon R^n$ are said to be coordinates of point x with respect to Φ.

Let f:W$\rightarrow$R denotes function of non-empty space W of R^n .Then f is said to be

1. (a) i. of class c^k,k=1,2,3.......... if and only if f has partial continous derivative of all order r$\leq$k

ii. of all class c^0 if and only if its continous

iii. of class c^∞or smooth if its of class c^k for every for every positive interger.

iv. of class c^w if it is analytic function.

A function f:w$\rightarrow R^n$ for an open subspace w of R^n into R^n is said to be of class c^k if and only if for every i=1,2,3..........n, the composed function $f_i = p_i : w \rightarrow R^n$ of class C^k

Atlas

It is a collection α of charts of X satisfying following condition

i. The domain of the chart in α cover the n-manifold

ii. For any two chart Φ:U $\rightarrow R^n$ and φ:V $\rightarrow R^n$ in α with U n V $\neq \Phi$ the function

   f(Φ, φ):U n V $\rightarrow R^n$

   defined by f(Φ, φ)(t)=$\varphi[\Phi^-1(t)]$, for every point t$\varepsilon\Phi(UnV)$ is of class c^k.

Function f(Φ, φ)is know as connecting function of 2 charts Φ and φ.for every xεUnV we have

f(Φ, φ)$[\Phi(x)] = \varphi(x)$, hence f(Φ, φ) is ussually called the transformation for change of local cordinate system from Φ

to φ.Thus, we have enough concept to define differentiable structure.

Definition 1

Let c^k be set of all atlases on X of class c^k .if K $\neq$0, this set maybe empty.The relation of X defined by α β if and only if αUβ is an atlas in c^k(x) for any two atlases $\alpha, \beta\varepsilon c^k$(x). This is an equivalence relation in c^k(x)into disjoint equivalence classes. Each of equivalence classes is called is called differentiable structure of class c^k in the given n-manifold X. Two atlases α and β are known to be compatible if their union is an atlas.

Definition 2

A differentiable or c^{∞} (or smooth) structure on topological manifold M is a family $U = (U_{\alpha}, \varphi_{\alpha})$

of coordinates neighborhood such that the following are satisfied

i. The u_{α} cover U

ii. for any α, β the neighborhood $((U_{\alpha}, \varphi_{\alpha})$ and $(U_{\beta}, \varphi_{\beta})$ are c^{∞} compatible

iii. Any coordinate neighborhood $(U_{\alpha}, \varphi_{\alpha})$ compatible with every conditions $(U_{\beta}, \varphi_{\beta})\varepsilon$U is itself U.

Differentiable n-manifold

An n-manifold X together with given differentiable structure A of class c^{k} on X is called differentiable n manifold .

Diffeomorphism

Let X and Y be differentiable n-manifold of class c^{k}. Let also h≤k.if the function f:X→Y is homeomorphism and both f and $f^{-}1$(its inverse) are function of class c^{h} then f is called diffeomorphism

1.1.3 Tangent vector and tangent spaces

Definition 1

Let p be an element of $v-n$ and let c^{∞}(p) be set of real valued function that are c^{∞} on some neighborhood U of P. A vector X at p is said to be a tangent vector at p if it satisfy the following properties

i. xεv_{n} fεC(p) then Xfε C(p)

ii. x(f+g)= xf + Xg :f,g εC(p)

iii. X(fg)=fXg + gXf

iv. X(af) =axf aεR

The system consist of

i. The set T_p containing all tangent vectors at P

ii. The binary operation + satisfying (X+Y)f=Xf+Yf

iii. An operation scalar multiplication fXε T_p and (aX)f=aXf where aε R is a vector space called **Tangent space** to v_n at P. t_p approximates to V_n at P and is n dimensional.

Definition 2

let m_n be n- dimensional c^∞ manifold. if pε M_n and X be c^∞ real valued function of some neighborhood of P and satisfies

X$(a_1 f_1 + a_2 f_2) = a_1(X f_1) + a_2(X f_2)$

and

$X(f_1 f_2) = (X f_1) + f_1(X f_2)$

where$a_1 a_2$ εR and $f_1 f_2$ εc^∞ are real valued function at P.Then x is called **tangent vector at point P**.

The set of all tangent vector at point P with operation of addition (+) and multiplication (.) given

(X+Y)f_1=$X f_1 + Y f_1$

and

$(f_1 X) f_2 = f_1(X f_2)$

is a vector space and is called tangent space to M_n at P and is denoted by $T_(p)M$ or $t_(p)$.

1.1.4 Vector field

A vector field X on set B is a mapping that assign a to each p in B a vector x_p in the tangent space T_p.A vector field x on B is c^∞ if

i. B is open

ii. function xf at P is c^∞ on AnB , f is being a c^∞ real valued function on A in v_n

1.2 Tensor Analysis

1.2.1 Tensor Algebra

In this section tensors are defined as element of a vector space.The classical notation in definition is used but on most of the work index free notation is used. Let v' be an n-dimensional space and let e_i and $\bar{e_i}$ be two basis of v' then each vector of set $[\bar{e_i}]$ is linear combination of elements of the set $[e_i]$ i=1,2,3,4........,n and vice versa.

let us take

i. $\bar{e^i} = P_i^j e_j$: where $p_i^j, q_i^j \varepsilon$ F

ii. $e_i = q_i^j \bar{e_j}$: where f is scalar field

putting 2 in 1 above we shall get the following equation

$e_i = p_i^k q_k^j \bar{e_j}$, since$[\bar{e_j}]$ is linear independent we have
$p_i^j q_j^k \bar{e_j} = \delta_i^k$, consequently

((p))((q))= I_n

i.e ((p)) and ((q)) are inverse to each other for any vector X ε V_n.we have

X=$\bar{X^k}$ $\bar{e_k} = X^i e_j$,where $\bar{X_k}$ and x^i are component of X respect to $\bar{e_1}$ and e_i

from i and ii we have

i. $\bar{X^k} = q_i^k \bar{X^i}$

ii. $\bar{X}^k = p_i^k \bar{X}^i$

which are equation of laws of transformations of vector X.The vector x or any vector in v_n is called contravariant vector of order 1 or tensor type (1,0)

Tensors

A linear scalar function or form of V' is linear mapping such that A(X), XεV' is a scalar and A(fX+gY)=fA(X)+g(A,Y) ;f,g εF and X,Yε V'

Dual spaces

Consider V' consisting of

i. a set V of all linear scalar function on V'

ii. a binary operation "+" satisfying

   (A+B)(x)=A(x)+B(x)

   A,BεV* ;XεV' then V' is vector space called dual of V'.

A bi-linear scalar function T over V×W is a mapping T:V×W→F. i.e T(X,A) are Xε V and AεW is scalar such that

T(fX+gY,hA+kB)=fhT(X,A)+fkT(X,B)+ghT(X,A)+gkT(Y,B)

where A,BεW; X,YεV and f,g,h,k εF.Consider a system denoted by V'×V' or v^2 consisting of

i. a set v^{*2} of all bilinear scalar function of $V_1 \times V_1$.

ii. A binary operation say "+" satisfying

   (T+S)(A,B)=T(A,B)+S(A,B) ;T,S ε V^{*2} ;A,Bε v_1

iii. an operationof scalar multiplication satisfying
   (fT)(A,B)=fT(A,B) ; fεF ;A,Bε v_1 then v^2 is vector space called the tensor product of V' with itself.

Higher order tensors

We can define mixed tensor as $A^{t_1 t_2 \ldots . t_s}_{q^1 q^2 \ldots \ldots . q^p}$

This tensor is then called mixed tensor of contravariant order s and covariant order p.If by interchanging two indices the sign of tensors remain same then we say tensor is symmetric in those indices.
If sign changes then is skew-symmetric with respect to two indices.The properties on symmetry and skew-symmetric are independent of the cordinate system. A significant result from transformation laws of tensors is that "if component of a tensor are zero in one coordinate system ,then they are zero in any coordinate system".It is this property of tensor that is useful in physical application and when tensor is defined at all points of a curve in space v_n then we say consist of a tensor field.

Properties of tensors

i. **outer product**
   The outer product of two tensor is equal to tensor whose rank is sum of rank of given tensor and it also involves multiplication of components of the tensor.

ii. **Contraction**
   If we set one covariant index of tensor equal to one contravariant index then the resulting tensor will be of rank two less than original tensor.This process is contraction.

iii. **Inner multiplication**
   The outer multiplication of two tensor followed by contraction will result to a tensor known as inner product of given tensor

iv. Addition and subtraction of tensors of same rank and type result in tensor of same rank and type.

   NB: Two operation are defined only for tensor of same rank and type.

For us to verify whether functions would form components of tensor, we can use transformation laws of which they can be cumbersome so instead we can use the quotient law which is more convenient.

Quotient law

If an inner product of any quantity X with arbitrary tensor is also a tensor then X is also a tensor.
A tensor Q of type (r,0) is said to be symmetric in h^{th} and k^{th} places if
$S_{h,k}(Q) = Q$
and skew symmetric if
$S_{h,k}(Q) = -Q$ where $1 \leq h < k \leq r$ and $S_{h,k}$ is a linear mapping which interchanges vector at h^{th} and k^{th} places

Note it is also applies to a tensor of type (0,1).

1.2.2 Connexion

A connexion Δ is type preserving mapping assign to each pair of c^{∞} field (X,P), a c^{∞} vector field $\Delta_x P$ such that if X,Y,Z are c^{∞} vector field and f is a c^{∞} function then

i. $\Delta_x f = xf$

ii. $\Delta_x(fY) = xfY + f\Delta_x Y$

iii. $\Delta_{x+y} Z = \Delta_x Z + \Delta_y Z$

iv. $\Delta_{fx} Z = f\Delta_x Z$

and also

$\Delta_x(Y+Z) = \Delta_x Y + \Delta_x Z.$

1.2.3 Affine connexion

an affine connexion Δ on manifold m is map T(M)$\times$T(M)........ $>$ T(M),(X,Y).........$>$ $\Delta_x Y$ such that for all $x_i, y_i \varepsilon$T(M),i=1,2 we have

i. $\Delta_{x_1+x_2}(Y) = \Delta_{x_1} Y + \Delta_{x_2} Y$

ii. $\Delta_x(Y_1 + Y_2) = \Delta_x Y_1 + \Delta_x Y_2$

iii. $\Delta_x(fY) = (xf)y + f\Delta_x Y$

iv. $\Delta_{fx}(Y) = f\Delta_x Y$

where f is a c^∞ real valued function on M.

Definition 1

A c^∞ vector field X is said to be parallel along smooth curve $\gamma : t....... > \gamma(t)$ on M (with respect to Δ) if

$\Delta_T X = 0$

along γ where T=d(γ(t))/dt so if

$\Delta_T Y = 0$ everywhere along γ then X is parallel along γ

Definition 2

A Riemannian structure on M is covariant tensor field of order 2(degree) called Riemannian metric with the following properties

i. g(x,y)=g(y,x) for x,y εT(m)

ii. $g_x : T_x(m) \times T_x(m)$.............>R:x εM

   where g_x is a non-degenerate bilinear form on $T_x(m) \times T_x$ i.e an inner product on $T_x(m), g_x(y)$

iii. $g_x(y,x) = 0$; for all x$\varepsilon T_x(m)$ if and only if y=0

iv. g(x,y)$\geq$ 0 for all T(m):g(x,x)=0 which implies y=0

Definition 3

A connection Δ is compatible with Riemannian metric g if a parallel transformation a long any smooth curve γ in m preserves the inner product. i.e whenever x(t) and y(t) are parallel along γ then <x(t),y(t)> is independent to t.

1.2.4 Lie algebra

Let M be the set of all e infinity vector field on A the brackets [] is defined by mapping

[]:M $\times$ M $\rightarrow$ M such that for x,y in M
and
$[x,y]f = xyf - yxf$

where f is smooth function for x,y,z in M we have

i. $[X,Y] = -[Y,X]$
skew commutative(symmetric)

ii. $[X+Y,Z] = [X,Z]+[Y,Z]$

iii. $[fX,gY] = fg[X,Y]+f(XgY)-g(Yf)X$

iv. $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y] = o$

The last equation is known as **Jacobs identity**

Example

Let $M_n(R)$ denote the algebra of n$\times$n matrices over R with X,Y denoting the usual matrix product of X and Y.Then
$[X,Y] = XY - YX$
the "commuter" of X and Y defines a lie algebra structure on M_n(R) as easily verified.If f is c^∞ on any open set UcM then so is (XY-YZ)f and therefore Z is a c^∞ vector field on M as said.

We may define a product on T(m) using the fact ; namely ,define the product of X and Y by $[X,Y] = XY - YX$

Let us consider the following theorem;

Theorem 1.2.4.1

T(M) with the product $[X,Y]$ is a lie algebra.

Proof

If $\alpha,\beta\varepsilon$R and X_1,X_2,Y are c^{∞} vector field then it is straight forward to verify that
$[\alpha X_1+\beta X_2,Y]f=\alpha[X,Y]f+\beta[X_2,Y]f$

Thus $[X,Y]$ is linear in the first variable. Since the skew commutative $[X,Y]=-[Y,X]$ is clear from definition.We see linearity in the first variable implies linearity in the second variable.Therefore $[X,Y]$ is bilinear and skew commutative.There remain Jacobi identity which follows if we evaluate

$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]$

apply to c^{∞} function f.we obtain

$[X,[Y,Z]]f=X(([Y,Z])f)-[Y,Z](Xf)$

$=X(Y(Zf))-X(Z(Yf))-Y(Z(Xf))+Z(Y(Xf))$

permutting cyclically and adding establishes the identity.

1.2.5 Lie bracket and covariant Derivatives

let X,Y,Z be c^{∞} vector field on m_n.Then lie brackets is mapping

$[\,]:M_n\times M_n........>M_n$

such that

$[XY]f=X(Yf)-Y(Xf)$ f being c^{∞} function.

This satisfies the following properties

i. $[X,Y](f_1+f_2)=[X,Y]f_1+[X,Y]f_2$

ii. $[X,Y](f_1.f_2)=f_1[X,Y]f_2+f_2[X,Y]f_1$

iii. $[X,Y]+[Y,X]=0$

iv. $[X+Y,Z]=[X,Z]+[Y,Z]$

v. $[f_1X, f_2Y] = f_1f_2[X,Y] + f_1(Xf_2)Y - f_2(Yf_1)X$

and further it satisfy Jacobi identity

i.e

$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

The covariant derivative Δ is a mapping $\Delta : T^r_s .... > T^r_{s+1}$ such that
$\Delta_p(a_1, ...a_r, X_1.........X_{s+1})$=$(\Delta_{s+1}p)(a_1, ..., a_r, X_1, ..., X_s)$

where p εT^r_s: $a_,a_2, ...a_r\varepsilon T_{(p)}$ and $X_1, X_2.........X_s\varepsilon T^*_{(p)}$

1.2.6 Lie bracket and Exterior Derivatives

let X be c^∞ vector field on an open set A.lie derivative via X is a type preserving mapping

$L_x : T^r_s........... > T^r_s$ such that

i. $L_xf = xf$, where f is c^∞

ii. $L_xa = 0$, aε R

$L_xY = [X,Y], Y\varepsilon T_(p)$

$(L_xA)(Y) = X(A(Y)) - A([X,Y)$

where AεT^*_p and $(L_xp)(A_1.......A_r, X_1...........X_s) = X(p(A_1, A_2......, X_s), ...p(A_1, ..., [X,X_s])$

where pεT^r_s. Let v_p be c^∞. p forms an open set A. Then the mapping

$d : V_p........... > V_{p+1}$ given by
(df)(x)=xf, where x$\varepsilon T_(p)$ and f is c^∞ function on A thus from above its clear now we can define the following as

(dA)$(X_1, ...X_{p+1}) =$

$X_1(A(X_2, ..., X_{p+1})) + X_2(A(X_1, X_3, ...X_{p+1})) + ... + ...$

$+X_{p+1}(A(X_1,X_2......X_p))-A([X_1,X_2]X_3........X_{p+1})-A([X_1,X_3],X_2,X_4.............X_{p+1})$

$-A([X_2,X_3],X_1,X_4..........Xp+1)............$

for all arbitrary c^∞ fields XεV and A$in\ V_p$ is called **exterior derivative**

1.2.7 Torsion tensor of a connexion

The torsion tensor of a connexion D is defined as a vector valued bilinear function T which assigns to each pair of c^∞ vector X and Y with domain A, a c^∞ vector field T(X,Y) with domain A and is given by
T(X,Y)=$D_XY-D_YX-[X,Y]$.

A connexion is said to be symmetric if torsion tensor vanishes and a connexion D is said to be Riemannian if

i. T(X,Y)=0
   and
ii. $D_xg=0$

1.2.8 Curvature Tensor

Consider a connexion D then the operator K_{XY} defined by

$K_{XY}=[D_X,D_Y]-D_{[X,Y]}$ is called the curvature operator.
Then curvature K of the connexion D is defined as

K(X,Y,Z)=$K_{XY}Z$
which can be written as
K(X,Y,Z)=$[D_X,D_Y]Z-D_{X,Y}Z$

$=D_XD_YZ-D_YD_XZ-D_{X,Y}Z$

where k is vector valued function.The curvature tensor K sastisfy two identities

i. K(X,Y,Z)+K(Y,Z,X)+K(Z,X,Y)=0
   and

ii. $(D_X K)(Y,Z,W)+(D_Y K)(Z,X,W)+(D_z K)(X,Y,W)=0$

which are called Bianch first and second identities respectively.

Proof

Let D be a symmetric connexion then

K(X,Y,Z)+K(Y,Z,X)+K(Z,X,Y)

=$D_X D_Y Z - D_Y D_X Z - D[X,Y]Z + D_Y D_Z X - D_Z D_Y X - D[Z,Y]X + D_Z D_X Y - D_X D_Z Y$

$-D[X,Z]Y$

=$D_X[Y,Z] - D_{[Y,Z]}X + D_Y[X,Z] - D_{[Z,X]}Y + D_Z[X,Y] - D_{[X,Y]}Z$

=$[[X,Y],Z]+[[Y,X],Z]+[-[Z,X],Y]$=0 by Jacobi identities.Thus we have

K(X,Y,Z)+K(Y,Z,X)+K(Z,X,Y)=0

Similarly we also get

$(D_X K)(Y,Z,W)+(D_Y K)(Z,X,W)+(D_Z K)(X,Y,W)=0$
Let us put K'(X,Y,Z,W)=g(K(X,Y,Z),W).
It can be noted that K' satisfy the following conditions

A. is skew symmetric in the first two slot as well as in the last two slot

B. satisfy first and the second banachi identity

C. symetric in two pair of slot

   i.e (XY) and (Z,W)

Difference tensor of two connexion

Consider a smooth manifold M and let D and $\bar{D}$ be two connexion on M for two field X and Y on M .We define difference tensor by

B(X,Y)=$\bar{D}_X$Y-$D_X Y$

Linearlity of B slot is trivial result from properties of connexion.let us consider slot 2 and Let f be c^∞ on domain X and Y then

B(X,fY)=(Xf)Y+f$D_X Y - (Xf)Y - f\bar{D}_X Y$=fB(X,Y)

If we decomposed B(X,Y) into symmetric and skew symmetric pieces we have

Lets B(X,Y)=S(X,Y)+Z(X,Y)
where
S(X,Y)=1/2[B(X,Y)-B(Y,X)]
symmetric part

and

A(X,Y)=1/2[B(X,Y)-B(Y,X)]
skew symmetric part

Then we can express A in terms of torsion tensors T and $\bar{T}$ of connexion D and $\bar{D}$ respectively as for

2A(X,Y)=B(X,Y)-B(Y,X)

=$\bar{D}_X Y - D_X Y - \bar{D}_Y X - D_Y X$

=$\bar{T}(X,Y) - T(X,Y) + [X,Y] - [X,Y]$

=$\bar{T}(X,Y) - T(X,Y)$

Let the two connexion D and $\bar{D}$ be related in V_n by

$\bar{D}_X Y = D_X Y + A(X)Y + A(Y)X$

where A is a 1-form and X and Y are vector field in V_n then D and $\bar{D}$ are said to be projectively related.

1.2.9 Ricci Tensor

The tensor defined by Ric(Y,Z)=(C',K)(Y,Z) is called tensor of type(0,2), where C' denote contraction.Its symmetric tensor

Ric(X,Y)=Ric(Y,X), the ricci tensor of type (1,1) is defined by

g(R(X),Y)=Ric(X,Y), the scalar curvature r is defined by

$C_1'R = def$ r

1.2.10 The weyl projective curvature tensor

This is defined by

$W(X,Y,Z) =$

$K(X,Y,Z)+\frac{1}{n+1}[L(X,Y)-L(Y,X)]Z+\frac{n}{n^2-1}[L(X,Y)Y-L(Y,Z)Y]+\frac{1}{n^2-1}[L(Z,X)Y-L(Z,Y)X]$

It can be shown that symmetric connexion which are projectively related have the same curvature tensor.

The weyl's projective curvature tensor w satisfies the following properties:-

i. W(X,Y,Z)= -W(Y,X,Z)
ii. $(trW)(X,Y) = (C_3'W)(X,Y) = 0$
iii. W(X,Y,Z)+W(Y,Z,X)+W(Z,X,Y)=0

1.3 Literature review

Set of new curvature tensors was defined on the line of Weyl tensor by Pokhariyal and Mishra (1970), and Pokhariyal (1979); to study Relativistic signicance of curvature tensors. The Weyl's projective curvature tensor was defined on the basis of geodesic correspondence due to a particular type of distribution of vector fields contained in it.

These new tensors were not necessary due to its in variance in two spaces V_n and $\bar{v_n}$, but showed that the "distribution" (order in which the vectors in question are arranged before being acted upon by the tensor in question), of vector field over the metric potentials and matter tensors plays an important role in shaping the various physical and geometrical properties of a tensor, viz the formulation of gravitational waves, reduction of electromagnetic field to a purely electric field, vanishing of the contracted tensor in an Einstein space and the cyclic property.The relativistic significance of Weyl's projective curvature tensor has also been explored by Singh(1965).

The concept of curvature is very common in Differential Geometry. In this article we try to show its evolution along history, as well as some of its applications. This survey is limited both in number of topics dealt with and the extent with which they are treated. Some of them, like minimal submanifolds,Kahler manifolds or Morse Theory are completely omitted. Though in an implicit way, the curvature is already present in the Fifth Euclid's Postulate.

However it does not emerge explicitly in Mathematics until the appearance of the theory of curves and surfaces in the euclidean space. Taking basically the work of Gauss's as a starting point, Riemann defines the curvature tensor in an abstract and rigorous way.

The introduction of multilinear algebra in the second half of the XIX century allowed a better analytic formulation and its further development. It is worth stressing its fundamental role in the development of the Theory of Relativity. Besides, the curvature is present, not only in riemannian manifolds, but also in many other geometric structures, like homogeneous and symmetric spaces, the theory of connections, characteristic classes, etc. Having in mind that the physical world cannot be explained in a linear way,the curvature also arises in the theories of Mathematical Physics. Likewise, it seems interesting to note its presence in applied sciences, like Estereology.

The world we live in, and the mathematical models describing the geometrical and physical objects, cannot be properly explained with only linear constructions. In order to obtain an adequate description of Nature,it is necessary to introduce models in which the relations between parameters go beyond the linear ones.That is why the concept of curvature appears in a natural way.

According to Osserman, the notion of curvature is one of the main concepts of differential geometry; it could be argued that it is indeed the central one, by distinguishing the geometrical core of the subject from those aspects that are analytic, algebraic or topological. According to Berger, the curvature is the most important invariant of Riemann's Geometry, and the most natural one. In Gromov writes: "the curvature tensor of a Riemann manifold is a little monster of multilinear algebra whose complete geometrical meaning remains obscure".

Thus, for Riemannian manifolds without additional structures, the curvature is a complicated magnitude. Its properties in the simplest manifolds were the first to be studied. Later, the situation in a more general manifold could be compared to that in the simplest ones.

The latter are often called “model spaces”.The curvature also plays a fundamental role in Physics and other experimental sciences.

For example,the force required to move an object at a constant speed is, according to Newton’s laws, a constant multiple of the curvature of its trajectory; and the movement of a body in a gravitational field is determined, according to Einstein, by the curvature of the space-time.

2 Riemannian and Complex manifold

2.1 Riemannian manifold

2.1.1 Riemannian manifold

let T be tangent space as at point P of differentiable manifold V_n.Let us single out in V_n a real valued bilinear symmetric and positive definite function g on the ordered pair of tangent vectors at each point P on V_n.Then V_n is called Riemannian manifold and g is called the metric tensor of V_n.

We thus have two vector X,Y of T at P

i. g(X,Y)εR

ii. g(X,Y)=g(Y,X); g is symmetric

iii. g(aX+bY,Z)=ag(X,Z)+bg(Y,Z)

iv. g(X,X)$>$0

v. if X,Y are c^{∞} fields with domain A then g(X,Y) at P is a c^{∞} function on A.Let (G(X)(Y))=g(X,Y)then G is non singular and Let ${}^{-1}G$ be the inverse map then ${}^{-1}GOG = GO^{-1}G = I_n$

The angle θ between two vectors is defined by

$||X||||Y||cos\theta = g(X,Y)$

where

$||X|| = g(X,Y)$

Thus two vectors X and Y are perpendicular if g(X,Y)=0

A connexion D is said to be Riemannian if it satisfies

i. D is symmetric

$D_X Y - D_Y X = [X, X]$

ii. g is covariant constant with respect to D whicg gives

$D_X g = 0$
and
$g(D_X Y, Z) + g(Y, D_X Z) = X(g(Y, Z))$

An affine connexion D is said to be metric if $D_X g = 0$.

The riemannian manifold is said to be Einsteinian manifold if

$Ric(X, Y) = \frac{r}{n} g(X, Y)$

A Riemannian manifold is said to be flat if

K(X,Y,Z)=0

The torsion tensor Tor is vector valued linear dunction and is defined by

$Tor(X, Y) = D_X Y - D_Y X - [X, Y]$

if torsion vanishes then connexion is said to be torsion free or symmetric

2.1.2 Riemannian curvature tensor

The curvature tensor with respect to the Riemannian connexion is called the Riemannian curvature tensor.

Let K be Riemannian curvature tensor

$K(X, Y, Z) = (D_X D_Y - D_Y D_X - D_{X,Y})Z$

2.1.3 Riemannian connexion

Let X and W be vectors as P in R_n.Let Y and Zbe c^∞ field about P and let f be a c^∞real valued function about P.then we have

i. $\bar{D}_X(Y+Z)=\bar{D}_XZ+\bar{D}_YZ$

ii. $\bar{D}_{X+W}(Y)=\bar{D}_XY+\bar{D}_WY$

iii. $\bar{D}_{f(p)X}Y=f\bar{D}_XY$

iv. $\bar{D}_X(fY)=(Xf)Y_p+f_(p)\bar{D}_XY$

(2.1.3.1)

Using $\bar{D}$ we can define parallel vector field along a curve and geodesics.let r be a c^∞ curve on R_n with tangent T and let Y be an R_n vector field that is parallel along r if $\bar{D}_rY$=0 along r

The curve γ is geodesic if $\bar{D}_rT$=0 i.e if its tangent T is parallel along γ.Thus generalization of a definition of covariant differentiation or connexion on c^∞ manifold is clear i.e We merely need the existence of operator D which satisfies all four condition of above properties (2.1.3.1) listed for $\bar{D}$ and assigns to c^∞ vectors field X and Ywith domain A,a c^∞ field D_XY on A.

NB: connexion can be more than one on a given manifold.

Let us denote dot or inner product of X and Y tangent to R_n by

$<X,Y>=\sum_{i=1}^{n}X_iY_i$

if X and Y are c^∞ field then $<X,Y>$ is also c^∞ field and if A is the domain of X,Y and X Y are c^∞ fields then one easily check that

$\bar{D}_YZ-\bar{D}_ZY=[Y,Z]$ on A

(2.1.3.2)

and

$X_p<Y,Z>=<\bar{D}_XY,Z>P+<Y,\bar{D}_XZ>P$
(2.1.3.3)
for every X at p in A. From above we can generalize and fix some terms.

A Riemannian manifold is a c^{∞} manifold M on which one has singled out a c^{∞} real valued ,bilinear ,symmetric and positive define function$<,>$ on ordered pair of tangent vector at each point.Thus if X,Y and Z are in M_P then X,Y is real number and $<,>$ satisfies the following properties

i. $<X,Y>=<Y,X>$ symmetric

ii. $<X+Y,Z>=<X,Z>+<Y,Z>$ bilinear

   $<aX,Y>=a<X,Y>$ for all aεr

iii. $<X,X>$ >0 for all X$\neq 0$

iv. If X and Y are c^{∞} fields with domain A then $<X,Y>p=<X_p,Y_p>$ is ac^{∞} function on A when (3) is placed by (3*) (non singular)$<X,Y>=0$ for all X implies Y=0 then M is semi-Riemannian (or pseudo Riemannian) manifold.In either case the function is inner product,metric tensor,the Riemannian metric or infinite semi metric on M not the topological metric function.

   If D is c^{∞} connexion in semi-Riemannian manifold M then D is Riemannian connexion if it satisfies (2.1.3.2) and (2.1.3.3)

2.1.4 Properties of Riemannian curvature tensor

The Riemannian curvature tensors is linear over the ring of smooth function are coefficient on the right hand side and satisfy the following properties

i. K(X,Y,Z)= -K(Y,X,Z)

   and if f is smooth function then

ii. K(fX,Y,Z)= -fK(Y,X,Z) where D is Riemannian connexion.

   Let us define
   'K(X,Y,Z,W)=g(K(X,Y,Z),W)
   then 'K is skew symmetric in the first two slots and the last two slots.The Riemannian curvature tensor K satisfies Binanchi's first identity and Bianchi's second identity.

Curvature Tensors

In a Riemannian manifold the weyl projective tensor reduces to

$$W(X,Y,Z) = K(X,Y,Z) + \frac{1}{n-1}[Ric(X,Z)Y - Ric(Y,Z)X]$$

Conformal curvature tensor

The tensor V defined by

$$V(X,Y,Z) =$$

$$K(X,Y,Z) + \frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)RY + g(Y,Z)RX] +$$

$$\frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]$$

is same for manifolds in conformal correspondence.This tensor is called the conformal curvature tensor.

A manifold whose conformal curvature tensor vanishes at every point is said to be conformaly flat.A conformal curvature V satisfies Bianchi's first identity

$$V(X,Y,Z) + V(Y,Z,X) + V(Z,X,Y) = 0$$

Concircular curvature tensor

The concircular curvature tensor is defined by

$$C(X,Y,Z) = K(X,Y,Z) - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]$$

Conharmonic curvature tensor

The conharmonic curvature tensor is defined by

$$L(X,Y,Z) = K(X,Y,Z) - \frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)RX - g(X,Z)RY]$$

Riemannian curvature

Let X and Y be unit tangent vector at a point P of Riemannian manifold V_n ,these vectors determine a pensil of direction at P if the unit vectors along that direction are U then

U=fX+gY,where f,gεF

and

$f^2 + g^2 = 1$

the geodesic of V_n whose unit tangent vector are U,generate a two dimensional sub manifold of the tangent manifold T at P.

The gaussian curvature K(X,Y) at P of this two dimensional sub manifold was defined by Riemannian as sectional curvature at P of V_n in direction of X and Y.Thus

$K = -K(X,Y,X,Y)/||X||^2||Y||^2[1 - cos^2\theta]$;where θ is angle between X and Y.

A necessary and sufficient condition on V_n to be locally flat in the neighbourhood U of a point P is that Riemannian curvature of V_n at P vanishes.

If the Riemannian curvature K of $V - n$ at P of the direction X and Y then

K(X,Y,Z)=K[g(Y,Z)X-g(X,z)Y]............(*)

contracting we get

i. Ric=K(n-1)g

ii. R=[K(n-1)]n................(**)

contracting (ii) we get
R=Kn(n-1)
hence a Riemannian manifold of constant curvature is an Einstein manifold.

Shur's theorem

If a Riemannian curvature K of V_n at every point of neighborhood U of V_n is independent of the direction choosen then K is constant throughout the neighborhood U provided n>2 putting (*) and (**) together we get W=0

Conversely, if W=0

K(X,Y,Z)=$\frac{1}{n-1}[g(Y,Z)RX-g(X,Z)RY]$

contracting equation we get

$Ric(Y,Z)=\frac{r}{n}g(Y,Z)$

which sometimes expressed as RX=$\frac{r}{n}X$ and putting the two equation into the first one we get

$K(X,Y,Z)=\frac{r}{n(n-1)}[g(Y,Z)X-g(X,Z)Y]$

which shows that a manifold is constant Riemaniian curvature.Hence a necessary and sufficient condition for the manifold V_n to be of constant Riemannian curvature is not the weyl projective curvature tensor vanishes identically throughout V_n.
Similarly the conformal curvature tensor vanishes from manifold with constant Riemannian curvature.

2.1.5 Difference tensor of two connexions

Consider a smooth manifold M and let D and $\bar{D}$ be two connexion on M for two field X and Y on M .we define difference tensor by

B(X,Y)=$\bar{D}_X$Y-D_XY

Linearlity of B slot is trivial result from properties of connexion and let us consider slot 2.

Let f be c^∞ on domain X and Y then

B(X,fY)=(Xf)Y+f$D_XY-(Xf)Y-f\bar{D}_XY$=fB(X,Y)

Lf we decomposed B(X,Y) into symmetric and skew symmetric pieces we have;

Lets B(X,Y)=S(X,Y)+Z(X,Y)

where S(X,Y)=1/2[B(X,Y)-B(Y,X)] (symmetric part)
and
A(X,Y)=1/2[B(X,Y)-B(Y,X)](skew symmetric part)

Then we can express A in terms of torsion tensors T and $\bar{T}$ of connexion D and $\bar{D}$ respectively as for

2A(X,Y)=B(X,Y)-B(Y,X)

=$\bar{D}_X Y - D_X Y - \bar{D}_Y X - D_Y X$

=$\bar{T}(X,Y) - T(X,Y) + [X,Y] - [X,Y]$

=$\bar{T}(X,Y) - T(X,Y)$

Theorem

The following statements are equivalent

i. The connexion D and $\bar{D}$ have same geodesic
ii. B(X,X)=0 for all vector X
iii. S=0
iv. B=A
   proof omitted

Theorem

The connexion D and $\bar{D}$ are equal if they have the same geodesic and the same torsion tensors.

Proof

That the first part implies the second is trival.conversely ,if the geodesic are the same then S=0 and if the torsion tensors are equal then A=0;hence B=0 and D=$\bar{D}$

2.1.6 Riemannian curvature tensor

The curvature tensor of connexion D is a linear transformation valued tensor R that assigns to each pair of vector X and Y at linear transformation R(X,Y) of M_n into itself.we define R(X,Y)Z by imbedding X,Y and Z in c^∞ field about M and setting R(X,Y)Z=$(D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z)_m$
(2.1.5.1)

Hence we notice that R(X,Y)= -R(Y,X) and if f is c^{∞} then

$$R(fX,Y)Z = fD_XD_YZ - (Yf)D_XZ - fD_YD_XZ + (Yf)D_XZ - fD_{X,Y}Z =$$

$$= fR(X,Y)Z$$

(2.1.5.2)

also

R(X,Y)(fZ)=$D_X(Yf)X + fD_YZ - D_Y((Xf)Z - fD_XZ) - ([X,Y]f)Z - fD_{[X,Y]}Z$

$$= (XY)(fZ) + (Yf)D_XZ + (Xf)D_YZ + fD_XD_YZ - (YX)(fZ) - (Xf)D_YZ - (Yf)D_XZ$$

$$- fD_YD_XZ - ([X,Y]f)Z - fD_{[X,Y]}Z$$

$$= fR(X,Y)Z$$

(2.1.5.3)

The linearlity of R(X,Y)Z with respect to addition (in each slot)is trivial to check.The curvature of symmetric linear connexion on M satisfies Bianchi identities

R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0

(2.1.5.4)

for all vector X,Y,Z in M for which the left hand side is defined to prove this,we recall that for symmetric connexion

$$D_AB - D_BA = [a,b]$$

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y =$$

$$D_X[Y,Z] + D_Y[Z,X] + D_Z[X,Y] - D_{[Y,Z]}X - D_{[Z,X]}Y - D_{[X,Y]}Z$$

$$= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

By Jacobi identity.

if we define

$Z < X, Y > = < D_Z X, Y > + < X, D_Z Y >$

(2.1.5.5)

for all vector X,Y,Z with common domain,then using about definition we can define a 4 rank covariant tensor called Rimann-Christoffel curvature tensor as

$K(X, Y, Z, W) = < X, R(Z, W) Y >$

(2.1.5.6)

for all X,Y,Z and W is same domain.

Thus from the above definition the following result arises

i. K(X,Y,Z,W)= -K(Y,X,Z,W)

ii. K(X,Y,Z,W)= -K(X,Y,W,Z)

iii. K(X,Y,Z,W)= K(Z,W,X,Y)

(2.1.5.7)

Theorem

Let M be differential i.e Riemannian n-maniford.then there is unique torsion free connexion D such that D on M satisfies

i. D is symmetric

ii. $D_X g = 0$ for all X εT(M).

parallel translation preserves inner products,this connexion is called the Riemannian or Levi-civita connexion.

Proof

Uniqueness from proposition (2.1.5.3) we obtain

$Xg(Y,Z)-g(D_XY,Z)-g(Y,D_XZ)=0$

using D is torsion free this yields

i. $Xg(Y,Z)=g(D_XY,Z)$

$=g([X,Y],Z)+g(Y,D_XZ)$

cyclically permuting X,Y and Z we get

ii. $Yg(Z,X)=g(D_XY,X)+g([Y,Z],X)+g(Z,D_YX)$

iii. $Zg(X,Y)=g(D_XZ,Y)+g([Z,X],Y)+g(X,D_ZY)$

substituting (i) from (ii)+(iii) we get

$2g(D_ZY,X)=-X<Y,Z>+Y<Z,X>+Z<X,Y>-<[Z,X],Y>-<$
$[Y,Z],X>$

$+<[X,Y],Z>$

the right hand of this last expression does not involve D,so we have a formula for $g(D_ZY)$ on X.As $<,>$ is non singular i.e

The map $T(M)$........$T^*(m)$ induced by g being an isomorphism and X is arbitrary, D_ZY is uniquely determined so D is unique.
If we define D_ZY by using the expression 2g above then D is a connexion and we find condition (i) and (ii) of the theorem satisfied.
Q.E.D

2.2 Complex Manifold

2.2.1 Complex Manifold

An even dimensional differentiable manifold V_n;n=2m which can be endowed by a system of complex coordinate neighborhood (U,α) in such a way that in the intersection UnU' of two complex cordinate patches (U,α),(U',α'),α' are complex analytic function of α is called a complex manifold.

2.2.2 Almost complex manifold

If on an even dimensional differentiable manifold V_n;n=2m of differentiability class C^{r+1} there exist a vector valued real linear function f of differentiability class C^r satisfying

i. $F^2 + I_n = 0$

   which implies

ii. $\bar{X} + X = 0$ where $\bar{X} = FX$

then V_n is said toi be an almost complex manifold and f is said to be an almost complex structure V_n.We shall apply the following notation

i. The operation of premultiplying a vector by F will be known as barring the vector.

ii. we shall denote T(V_n) the set of c^∞ vector field of V_n.

iii. in this and what follows the equation containing X,Y,Z......hold for arbitrary vectors fields X,Y,Z.......$\varepsilon T(V_n)$ unless explicitly stated otherwise.

3 Sasakian manifold

3.1 The w_6 curvature tensor

Weyl's projective curvature tensoris given by

$$W(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(X,U)Ric(Y,Z)$$

The other tensors have defined by (Pokhariyal and mishra)(1970,1982) are given by

$$W_1(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,U)Ric(Y,Z) - g(Y,U)Ric(X,Z)]$$

$$W_2(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(Y,Z)Ric(X,U)]$$

$$W_3(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(Y,Z)Ric(X,U) - g(Y,U)Ric(X,Z)]$$

$$W_4(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(X,Y)Ric(Z,U)]$$

$$W_5(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(Y,U)Ric(X,Z)]$$

For W_6 which is studied in this project is given by

$$W_6(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)Y - XRic(Y,Z)]$$

$$W_6(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)]$$

Which is from the following definition

Definition

In a (2n+1) dimensional Riemannian manifold M the τ-curvature tensor is given by Tripathi and Gupta(2011)

$$T(X,Y)Z = a_0R(X,Y)Z + a_1S(Y,Z)X + a_2S(X,Z)Y + a_3S(X,Y)Z + a_4g(Y,Z)QX + a_5g(X,Z)QY + a_6g(X,Y)QZ + a_7r(g(Y,Z)X - g(X,Z)Y) \text{.....................(**)}$$

where R is curvature tensor,S is ricci tensor,Q is Ricci operator and r is scalar curvature

(1.12) $'W_6(X,Y,Z,U) =' R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)]$.

we thus break the equation into symmetric P and skew symmetric Q parts in X and Y.

We start with symmetric part P

$'P(X,Y,Z,U) = \frac{1}{2}['W_6(X,Y,Z,U) +' W_6(Y,X,Z,U)]$

$= \frac{1}{2}['R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)] +' R(Y,X,Z,U) + \frac{1}{n-1}[g(Y,X)Ric(Z,U)$
$- g(Y,U)Ric(X,Z)]]$

$= \frac{1}{2}{}'R(X,Y,Z,U) + \frac{1}{2}{}'R(Y,X,Z,U) + \frac{1}{2(n-1)}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z) + g(Y,X)Ric(Z,U)$
$- g(Y,U)Ric(X,Z)]$

$= \frac{1}{2(n-1)}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z) + g(Y,X)Ric(Z,U) - g(Y,U)Ric(X,Z)]$.
or
(1.13)$'P(X,Y,Z,U) = \frac{1}{2(n-1)}[2g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z) - g(Y,U)Ric(X,Z)]$.

now we take a look at skew-symmetric part Q

$'Q(X,Y,Z,U) = \frac{1}{2}['W_6(X,Y,Z,U) -' W_6(Y,X,Z,U)]$

$= \frac{1}{2}['R(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)] -' R(Y,X,Z,U) - \frac{1}{n-1}[g(Y,X)Ric(Z,U)$
$- g(Y,U)Ric(X,Z)]]$

$= \frac{1}{2}{}'R(X,Y,Z,U) - \frac{1}{2}{}'R(Y,X,Z,U) + \frac{1}{2(n-1)}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z) - g(Y,X)Ric(Z,U)$
$+ g(Y,U)Ric(X,Z)]$

$=' R(X,Y,Z,U) + \frac{1}{2(n-1)}[g(X,Y)Ric(Z,U) - g(X,U)Ric(Y,Z)] - g(Y,X)Ric(Z,U) + g(Y,U)Ric(X,Z)]$
OR
(1.14)$'Q(X,Y,Z,U) =' R(X,Y,Z,U) - \frac{1}{2(n-1)}[+g(X,U)Ric(Y,Z)] - g(Y,U)Ric(X,Z)]$
.

2.LP-Sasakian manifold

In this section we study properties of W_6,P,q curvature tensors in LP-sasakian manifold.

Theorem 2.1
In an n-dimensional LP-Sasakian manifold we have

i. $'W_6(T,Y,Z,T) = -g(Y,Z) + \frac{1}{n-1}Ric(Y,Z)$

ii. $W_6(X,Y,T) = YA(X)\frac{2-n}{n-1}$

iii. $W_6(T,Y,T) = Y\frac{n-2}{n-1}$

Proof (2.1)i
Putting U=T in (1.12) we get

$$'W_6(X,Y,Z,T) =' R(X,Y,Z,T) + \frac{1}{n-1}[g(X,Y)Ric(Z,T) - g(X,T)Ric(Y,Z)]$$

Using (1.4) we get

$$'W_6(X,Y,Z,T) =' R(X,Y,Z,T) + \frac{1}{n-1}[g(X,Y)Ric(Z,T) - A(X))Ric(Y,Z)]$$

Using (1.9) we get

$$'W_6(X,Y,Z,T) = g(Y,Z)A(X) - g(X,Z)A(Y)) + \frac{1}{n-1}[g(X,Y)Ric(Z,T) - A(X))Ric(Y,Z)]$$

Using (1.10) we get

$$'W_6(X,Y,Z,T) = g(Y,Z)A(X) - g(X,Z)A(Y)) + \frac{1}{n-1}[g(X,Y)(n-1)A(Z) - A(X))Ric(Y,Z)]$$

$= g(Y,Z)A(X) - g(X,Z)A(Y)) + g(X,Y)A(Z) - A(X)\frac{1}{n-1}Ric(Y,Z)$.........(2.2)

Putting X=T in(2.2) we get

$$'W_6(T,Y,Z,T) = g(Y,Z)A(T) - g(T,Z)A(Y)) + g(T,Y)A(Z) - A(T)\frac{1}{n-1}Ric(Y,Z)$$

Using (1.1),we get

$$'W_6(T,Y,Z,T) = -g(Y,Z) - g(T,Z)A(Y)) + g(T,Y)A(Z) + \frac{1}{n-1}Ric(Y,Z)$$

Again using (1.4),we get

$'W_6(T,Y,Z,T) = -g(Y,Z) - A(Z)A(Y)) + A(Y)A(Z) + \frac{1}{n-1}Ric(Y,Z)$

$'W_6(T,Y,Z,T) = -g(Y,Z) + \frac{1}{n-1}Ric(Y,Z)$.....Hence proved

Proof (2.1)ii
$'W_6(X,Y,Z,U) = g(W_6(X,Y,Z),U)$ and (1.12) we have

$W_6(X,Y,Z) = R(X,Y,Z) + \frac{1}{n-1}[g(X,Z)Y - XRic(Y,Z)]$

Putting T=Z

$W_6(X,Y,T) = R(X,Y,T) + \frac{1}{n-1}[g(X,T)Y - XRic(Y,T)]$

Using XA(Y)-YA(X) and (1.4)(1.10) we get

$W_6(X,Y,T) = XA(Y) - YA(X) + \frac{1}{n-1}[A(X)Y - X(n-1)A(Y)]$

$W_6(X,Y,T) = XA(Y) - YA(X) + \frac{1}{n-1}A(X)Y - XA(Y)$

$W_6(X,Y,T) = -YA(X) + \frac{1}{n-1}A(X)Y)$

$W_6(X,Y,T) = YA(X)\frac{2-n}{n-1}$..............................Hence proved

Proof (2.1)iii
Putting X=T in (2.1)ii we get

$W_6(T,Y,T) = YA(T)\frac{2-n}{n-1}$

Using (1.1) we get

$W_6(T,Y,T) = -Y\frac{2-n}{n-1}$

$W_6(T,Y,T) = Y\frac{n-2}{n-1}$...............................Hence proved

Theorem 2.2
In an n-dimensional LP-Sasakian manifold P tensor field satisfies

i. $'P(X,Y,Z,T) = g(X,Y)A(Z) - \frac{1}{2(n-1)}[A(X)Ric(Y,Z) + A(Y)Ric(X,Z)]$

ii. $'P(T,Y,Z,U) = -\frac{1}{2}g(Y,U)A(Z) + \frac{1}{2(n-1)}[2A(Y)Ric(Z,U) - A(U)Ric(Y,Z)]$

iii. $'P(T,Y,Z,T) = \frac{1}{2}[A(Y)A(Z) + \frac{1}{n-1}Ric(Y,Z)]$

Proof (2.2)i
Using (1.13) and putting T=U we have

$'P(X,Y,Z,T) = \frac{1}{2(n-1)}[g(X,Y)Ric(Z,T) - g(X,T)Ric(Y,Z)] + g(Y,X)Ric(Z,T) - g(Y,T)Ric(X,Z)]$

Using (1.4) and (1.10) we get

$= \frac{1}{2(n-1)}[g(X,Y)(n-1)A(Z) - A(X)Ric(Y,Z)] + g(Y,X)(n-1)A(Z) - A(Y)Ric(X,Z)]$

$= \frac{1}{2}[g(Y,X)A(Z) + g(X,Y)A(Z)] - \frac{1}{2(n-1)}[A(X)Ric(Y,Z)] - A(Y)Ric(X,Z)]$

$= g(X,Y)A(Z) - \frac{1}{2(n-1)}[A(X)Ric(Y,Z) + A(Y)Ric(X,Z)]$......Hence proved

Proof (2.2)ii
Using (1.13) and putting T=X we have

$'P(T,Y,Z,U) = \frac{1}{2(n-1)}[2g(T,Y)Ric(Z,U) - g(T,U)Ric(Y,Z) - g(Y,U)Ric(T,Z)]$

Using (1.4) and (1.10) we have

$'P(T,Y,Z,U) = \frac{1}{2(n-1)}[2A(Y)Ric(Z,U) - A(U)Ric(Y,Z) - g(Y,U)(n-1)A(Z)]$

$'P(T,Y,Z,U) = -\frac{1}{2}g(Y,U)A(Z) + \frac{1}{2(n-1)}[2A(Y)Ric(Z,U) - A(U)Ric(Y,Z)]$..Hence proved

Proof (2.2)iii
Putting X=T in (2.2)i we get

$'P(T,Y,Z,T) = g(T,Y)A(Z) - \frac{1}{2(n-1)}[A(T)Ric(Y,Z) + A(Y)Ric(T,Z)]$

Using (1.1),(1.4) and (1.10) we have

$'P(T,Y,Z,T) = A(Y)A(Z) - \frac{1}{2(n-1)}[-Ric(Y,Z) - A(Y)(n-1)A(Z)]$

$'P(T,Y,Z,T) = A(Y)A(Z) - \frac{1}{2}A(Y)A(Z) + \frac{1}{2(n-1)}Ric(Y,Z)$

$'P(T,Y,Z,T) = \frac{1}{2}[A(Y)A(Z) + \frac{1}{n-1}Ric(Y,Z)]$.........Hence proved

Theorem 2.3
In an n-dimensional LP-Sasakian manifold Q tensor field satisfies

i. $'Q(X,Y,Z,T) = A(X)[g(Y,Z) - \frac{1}{2(n-1)}Ric(Y,Z)] - A(Y)[g(X,Z) - \frac{1}{2(n-1)}Ric(X,Z)]$

ii. $'Q(T,Y,Z,U) = A(U)[g(Y,Z) - \frac{1}{2(n-1)}Ric(Y,Z)] - \frac{1}{2}g(Y,U)A(Z)$

iii. $'Q(T,Y,Z,T) = -g(Y,Z) + \frac{1}{2}[\frac{1}{n-1}Ric(Y,Z) - A(Y)A(Z)]$

Proof (2.3)i
Using (1.14) and putting T=U we have

$'Q(X,Y,Z,T) =' R(X,Y,Z,T) - \frac{1}{2(n-1)}[+g(X,T)Ric(Y,Z)] - g(Y,T)Ric(X,Z)]$

Using (1.4),(1.9) we have

$'Q(X,Y,Z,T) = g(Y,Z)A(X) - g(X,Z)A(Y) - \frac{1}{2(n-1)}[+A(X)Ric(Y,Z)] - A(Y)Ric(X,Z)]$

$'Q(X,Y,Z,T) = A(X)[g(Y,Z) - \frac{1}{2(n-1)}Ric(Y,Z)] - A(Y)[g(X,Z) - \frac{1}{2(n-1)}Ric(X,Z)]$....Hence proved

Proof (2.3)ii
Using (1.14) and putting T=X we have

$'Q(T,Y,Z,U) =' R(T,Y,Z,U) - \frac{1}{2(n-1)}[g(T,U)Ric(Y,Z)] - g(Y,U)Ric(T,Z)]$

Using (1.4),(1.9),(1.10) we have

$'Q(T,Y,Z,U) = g(Y,Z)A(U) - g(Y,U)A(Z) - \frac{1}{2(n-1)}[A(U)Ric(Y,Z)] - g(Y,U)(n-1)A(Z)]$

$'Q(T,Y,Z,U) = A(U)[g(Y,Z) - \frac{1}{2(n-1)}Ric(Y,Z)] - \frac{1}{2}g(Y,U)A(Z)$....Hence proved

Proof (2.3)iii
Let X=T in (2.3)i

$$'Q(T,Y,Z,T) = A(T)[g(Y,Z) - \frac{1}{2(n-1)}Ric(Y,Z)] - A(Y)[g(T,Z) - \frac{1}{2(n-1)}Ric(T,Z)]$$

Using (1.1),(1.9),(1.10) we have

$$'Q(T,Y,Z,T) = -[g(Y,Z) - \frac{1}{2(n-1)}Ric(Y,Z)] - A(Y)[A(Z) - \frac{1}{2(n-1)}(n-1)A(Z)]$$

$$'Q(T,Y,Z,T) = -g(Y,Z) + \frac{1}{2(n-1)}Ric(Y,Z) - A(Y)[A(Z) + \frac{1}{2}A(Y)A(Z)$$

$'Q(T,Y,Z,T) = -g(Y,Z) + \frac{1}{2}[\frac{1}{n-1}Ric(Y,Z) - A(Y)A(Z)]$........Hence proved

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