University of Nairobi

School of Mathematics

ISSN: 2410-1397

Master Project in Mathematics

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August 2020

Submitted to the School of Mathematics in partial fulfillment for a degree in Master of Science in Applied Mathematics

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Submitted to: The Graduate School, University of Nairobi, Kenya

Abstract

The goal of this project is to study some properties of η-Ricci soliton on LP-Sasakian manifolds.
In this work, η-Ricci solitons on LP-Sasakian manifolds which satisfy the semi-symmetric conditions $R(X,Y)\cdot R(Z,U)=0$[28],also given by U.C. De [34] are considered.
Particularly, we based our study on
$R(\xi,X).W_3(Y,Z)=0$ and $R(\xi,X).W_5(Y,Z)=0$
The study is motivated by the results that are obtained in η-Ricci solitons on Para-kenmotsu manifolds. This led us to study or simply to investigate the η-Ricci solitons on W_3 and W_5 semi-symmetric LP-Sasakian manifolds satisfying the same conditions and see if a similar results are obtained.In addition, we prove that W_3 and W_5 semi-symmetric LP-Sasakian manifolds satisfying the semi-symmetric conditions $R(\xi,X).W_3(Y,Z)=0$ and $R(\xi,X).W_5(Y,Z)=0$, and having the η-Ricci soliton structure are Einstein according to the value of λ and a further conditions
$W_3(\xi,X)\cdot R(Y,Z)=0$ and $W_5(\xi,X)\cdot R(Y,Z)=0$
where dot denote the derivative of algebra at every point of a tangent space,say $T_p(U)$.

Master Thesis in Mathematics at the University of Nairobi, Kenya.
ISSN 2410-1397: Research Report in Mathematics

DISTRIBUTOR: School of Mathematics, University of Nairobi, Kenya

Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

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In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

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Dedication

This project is dedicated to my late mother, Rose C.A Pambo, the bravest woman ever known to me. To my Brother Kennedy, for his wisdom, support and direction. Bro Dave, Omondi and to my sisters Winny, Tausi, Zainab, Ephy and Saline. And finally, to my best friend, Naomi, who reminds me of all the beautiful things in this world. God bless you.

Contents

Acknowledgments

First, I want to thank The Lord God Almighty for the far He has walked with me. I also take this opportunity to express my deepest gratitude to my supervisors Prof S.K Moindi and Prof B.M Nzimbi for introducing me to this topic,spending their time to guide me through the project and making it see the light. Dr Nyandwi, thank you for your words of encouragements.

I can't afford to forget my friend Muthama. Your tireless effort to take me through LaTex is immensely appreciated.
Special Thanks to the University of Nairobi, through the School of Mathematics for accepting me to be one of their student.

My classmates, I am grateful. Without you people, this paper would be impossible to write. ....

Pambo Samwel

Nairobi, 2020.

1 Introduction

Differential Geometry has gone through a series of development especially in the 19^{th} century. Many branches emerged in this field trying to address many abstract concepts in mathematics and bringing them into appreciable practical order. As a branch, Riemannian Geometry then studied Riemannian manifolds as well as smooth manifolds with Riemannian metric.
Since the emergence of Riemannian Geometry, the study of geometry of surfaces, geometry of curves as well the behavior of geodesic on those surfaces has been made possible. Riemannian Geometry made the development of algebraic and differential topology much easier, actually, the idea of smooth manifolds admitting Riemannian metric helped solve the problems of differential topology. An understanding of Einstein's general relativity theory also was made possible, even generating results on group theory became clear.
It was also in this century, 19^{th}, that the concept of metric tensors was clearly understood by people like Tullio Levi-Civita and Gregorio Ricci-Curbastro. Ricci-flow is an evolution equation for metric on a Riemannian manifold. It defines a kind of non-linear diffusion equation similar to that of heat equation for metric under Ricci-flow.
The Ricci-flow equation is given by

$$\frac{\partial g}{\partial t} = -2S \tag{1.1}$$

on a compact Riemannian manifold M with metric g.
Ricci-soliton is a similar solution to the Ricci-flow, but only if it moves by a one parameter family of diffeomorphism and scaling. The Ricci-soliton has its equation given by

$$L_v g + 2S + 2\lambda g = 0$$

Where , L_v is Lie derivative in the V direction, S is Ricci curvature tensor , g is a Riemannian metric , V is a vector field and λ is a scalar.
η-Ricci soliton is a more general notion of the Ricci-flow. This idea was put forward by J.J Cho and Makoto Kimura[08], and they gave its equation by

$$L_\xi g + 2S = -2\lambda g - 2\mu \eta \otimes \eta \tag{1.2}$$

λ and μ are constants.

1.0.1 Notations and Definitions

Definition 1.0.1. *Consider an n-dimensional manifold M. if we let ρ be a point on the manifold, then V_ρ is the set of all vector field defined at ρ*
Therefore, V_ρ is also an n-dimensional vector space.

Definition 1.0.2. *A 1-form vector $\vec{r}$ defined at ρ is a linear scalar operator acting on a vector space V_ρ to real numbers $\Re$. This then means,*
(1) $\vec{r} : V_\rho \longrightarrow \Re$;
(2) For any $\underset{\sim}{u}\underset{\sim}{v} \in V_\rho$ and
if $a\ ,b \in \Re$
$\Rightarrow \vec{r}(a\underset{\sim}{u} + b\underset{\sim}{v}) = a\vec{r}(\underset{\sim}{u}) + b\vec{r}(\underset{\sim}{v})$.
The set of all 1-forms defined at ρ is called a co-vector or a dual space of V_ρ, and it is denoted by V_ρ^*. This is also an n-dimensional vector space.

Definition 1.0.3. *For any vector $\underset{\sim}{u} \in V_\rho$ can be associated with a linear scalar operator acting on 1-form $\underset{\sim}{u} \in V_\rho$ to $\Re$*
i.e. $\underset{\sim}{u}\vec{r} \neq \vec{r}\underset{\sim}{u} : V_\rho^* \longrightarrow \Re$.
We note that, tensors therefore are a generalization of vectors and 1-forms (co-vectors).

Definition 1.0.4. *Let M be a smooth manifold, a tangent vector at a point $\rho \in M$ is a map $X_\rho : C^\infty(M) \longrightarrow \Re$ which satisfies*

1. $X_\rho(f+g) = X_\rho(f) + X_\rho(g)$
2. $X_\rho = 0$ *(for constant map)*
3. $X_\rho(fg) = f(\rho)X_g + g(\rho)X_\rho f$

$\forall f, g \in C^\infty(M)$ on their common domain.
The set of all tangent vectors to an n-dimensional manifold M at a point $P \in M$ forms an n-dimensional vector space which is called the tangent space, denoted by $T_\rho M$.

Definition 1.0.5. *Let M be a smooth manifold, then by a Riemannian metric tensor g on M, we have a smooth assignments of an inner product to each tangent space of M.*
This then means, for each $\rho \in M$, $g_\rho : T_\rho M \times T_\rho M \longrightarrow \Re$, is symmetric, positive definite and bi-linear map. That is, for any smooth vector fields X and Y on M,
$P \longmapsto g_\rho(X_\rho, Y_\rho)$ is a smooth function.
It is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$-tensor, $g \in T_0^2(M)$
In a coordinate system, we may write
$g = g_{ij} dX^i \otimes dX^j$
Then the pair (M,g) will be called Riemannian manifold.

Definition 1.0.6. *By S and R, where S denote Ricci Tensor and R, Riemannian curvature tensor of an n-dimensional Riemannian manifold (M,g), then S can be defined as*

$$S(X,Y) = \sum_{i=1}^{n} g\big(R(e_i,X)Y,e_i\big)$$

Where $\{e_1,e_2...e_n\}$ are orthonormal basis vector fields in TM, and
X,Y,Z $\in$ TM.

Definition 1.0.7. *Let M be a smooth manifold, an Affine connection (Levi-Civita) ∇ on M is a differential operator, sending smooth vector fields X and Y to a smooth vector field $\nabla_X Y$, which then satisfies the following conditions*

1. $\nabla_{X+Y}Z = \nabla_X Z + \nabla_Y Z$
2. $\nabla(Y+Z) = \nabla_X Y + \nabla_X Z$
3. $\nabla_{fX}Y = f\nabla_X Y$
4. $\nabla_X(fY) = X(f)Y + f(\nabla_X Y)$.

$\forall$ *vector fields X,Y and Z, and real valued function f on M.*
The vector field $\nabla_X Y$ is known as the covariant derivative of the vector field Y along X with respect to ∇

Definition 1.0.8. *A curve $\gamma(s)$ is a geodesic if its tangent vector $\dot{\gamma}(s)$ at each point are parallel.*

Definition 1.0.9. *A homeomorphism $f : X \longrightarrow Y$ is continuous bijection whose inverse $f^{-1} : Y \longrightarrow X$ is also continuous.*

Definition 1.0.10. *Let M be an n-dimensional contact manifold with contact form η, that is, $\eta\Lambda(d\lambda)^n \neq 0$, then, a contact manifold admits a vector field ξ called characteristic vector such that $\eta(\xi) = 1$ for any field $X \in \chi(M)$.*
Furthermore, if M admits a Riemannian metric g, and a tensor filed ϕ of type (1,1), such that,
$\phi^2 X = X - \eta(X)\xi$
$g(X,\xi) = \eta(X)$
$g(X,\phi Y) = d\eta(X,Y)$
Then we say that (ϕ,η,ξ,g) is a contact metric structure.

Definition 1.0.11. *A contact metric manifold is said to be sasakian if*
$(\nabla_X\phi)Y = g(X,Y)\xi - \eta(Y)X$.
where,
$\nabla_X\xi = -\phi X$
$R(X,Y)\xi = \eta(Y)X - \eta(X)Y$
For all vector fields X,Y $\in M$.

Definition 1.0.12. *An n-dimensional differentible manifold M is said to admit an almost para-contact Riemannian structure* (ϕ,η,ξ,g) *such that*
$\phi^2 X = X - \eta(X)\xi$
$\phi\xi = 0$, $\eta(\xi) = 1$, $\eta(\phi X) = 0$,
$g(X,\xi) = \eta(X)$,
$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y)$
$\forall$ *vector fields X,T on M.*
If (ϕ,η,ξ,g) *satisfy the equation*
$d\eta = 0$, $\nabla_X \xi = \phi X$
$(\nabla_X \phi) = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi$
Then M is called Para-sasakian manifold.
If M admits 1-form η, *such that*
$(\nabla_x \eta)Y = -g(X,Y) + \eta(X)\eta(Y)$, *for all* $X,Y \in M$,
Then para-sasakian manifold is said to be a special manifold.

Definition 1.0.13. *An n-dimensional differentiable manifold* M^n *is Lorentzian Para-Sasakian manifold if it admits a (1,1)-tensor field* ϕ, *vector field* ξ , *1-form* η *and a Lorentzian metric g which satisfies*
$\phi^2 X = X + \eta(X)\xi$
$\phi\xi = 0$, $\eta(\xi) = -1$, $\eta(\phi X) = 0$,
$g(X,\xi) = \eta(X)$,
$g(\phi X,\phi Y) = g(X,Y) + \eta(X)\eta(Y)$
$(\nabla_x \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi$,
$\nabla_X \xi = \phi X$
Where X and Y are arbitrary vector fields, ∇_X *denote covariant differentiation in the direction of X with respect to g.*

2 Preliminaries

In this chapter, We will be discussing in summery some concepts that we will extensively employ in this dissertation. Specifically, we are going to define tensors, we will discuss manifolds, connections, Sasakian manifolds, solitons and eta-Ricci-solitons and conditions satisfied by these metrics.

2.1 Terminologies and Definitions

2.1.1 Tensor defined

Ricci-flow is an evolution equation of heat or wave-type equation of the metric on Riemannian manifold, which is defined by

$$\frac{\partial}{\partial t} g_{ij} = 2S_{ij} \tag{2.1}$$

2.1.2 Vector and 1-Form

Definition 2.1.1. *An* $\begin{pmatrix} k \\ l \end{pmatrix}$ *-type tensor defined at point* p *is a linear scalar operator with* l *slots for 1-forms from* V_p^* *and* k *slots from* V_p*. Such tensor can also be called as as* l*-times contravariant and* k*-times covariant. The total number of slots,* $r = l + k$*, is called the rank of the tensor.*
Thus,

1. *Any vector is a* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$*-type tensor*

2. *Any 1-form is a* $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$*-type tensor*

Remark 1. *Tensors thus are a generalization of vectors and 1-forms.*
A tensor of type (k,l) *at* p *is a multi-linear map which takes* k *vectors and* l *covectors and gives a real number.*
A tensor T *of type* (k,l) *is denoted with* k *superscripts and* l *subscripts* (T_l^k)*and is said to be of rank* $k+l$*.*

Definition 2.1.2. *The number of position of indices of tensor components reveal all the general information about tensors as operators. For example, if a tensor* T *has components* T^{ik}_{jl}.
This then tells us that

1. T *is a* 4^{th}*rank tensor*
2. T *is* $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$*-type tensor*
3. *Its* 1^{st} *and* 3^{rd} *slots are for 1-form whereas the* 2^{nd} *and* 4^{th} *slots are for vectors.*

2.1.3 Manifolds

A (real) n-dimensional manifold is a topological space M for which every point X $\in$ M has a neighborhood homeomorphism to Euclidean space R^n.

Definition 2.1.3. *Let* M *be a topological space and* $U \subseteq M$ *an open set.*
Let $V \subseteq R^n$ *be open. A homeomorphism* ϕ*:* $U \longrightarrow V$*,*
$\phi(u) = (x_1(u), ...x_n(u))$ *is called a coordinate system on U, and the function* $x_1...x_n$ *the coordinate function.*
The pair (U,ϕ*) is called a chart on M.*
Also the inverse map ϕ^{-1} *is a parametization of U.*

Definition 2.1.4. *An atlas on M is a collection of charts* (U_α, ϕ_α) *such that* U_α *covers M. The homeomorphism* $\phi_\beta \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \bigcap U_\beta) \longrightarrow \phi_\beta(U_\alpha \bigcap U_\beta)$ *are the transition maps or coordinate transformation.*

Definition 2.1.5. *A second countable, Hausdorff topological space M is an n-dimensional topological manifold if it admits an atlas*
$\{U_\alpha, \phi_\beta\}$, $\phi_\alpha : U_\alpha \longrightarrow R^n$, $n \in N$
It is a smooth manifold if all transition maps are C^∞ *diffeomorphism, that is, all partial derivatives exists and are continuous.*

Definition 2.1.6. *We define an n-dimensional manifold with boundary M as in (definition 1.1.1), however, here we allow the image of each chart to be open subset of Euclidean space* R^n*, or open subset of the upper half-space* $R^n_+ = \{(x_1, ..., x_n) | x_n \geq 0\}$.
The pre-image of points $(x_1, x_2, ...x_{n-1}, 0) \in R^n_+$ *are the boundary... is the interior of M.*
A manifold with boundary is smooth if the transition maps are smooth. For an arbitrary subset $X \subseteq R^m$*, a function f:X*$\longrightarrow R^n$ *is called smooth if every point in X has some neighborhood where f can be extended to a smooth function.*

2.1.4 Differentiable Manifolds

Definition 2.1.7. *An n-dimensional topological manifold M together with a complete atlas is called n-dimensional C^∞ or smooth differentiable manifold.*

2.1.5 Sub-Manifolds

Definition 2.1.8. *A sub-manifold of a manifold M is a subset S which itself has the structure of a manifold, and for which the inclusion map $S \longrightarrow M$ satisfies certain properties such as connection properties.*

2.1.6 Connection

The connections are some notions developed from parallelism, for instance, from vector method from where it is extended to Calculus.
Let M be a C^∞-Manifold, a connection, infinitely small connection, or a covariant differentiation on M is an operator D_y that assigns to each pair of C^∞ vector X and Y with domain A, a C^∞ fixed D_x Y with domain A.
If Z is a C^∞ vector field on A, and f is a C^∞ real valued function on A, then

$$\begin{aligned} D_x +_y Z &= D_x Z + D_y Z \\ D_{fx} Y &= f D_x Y \\ D_x(Y+Z) &= D_x Y + D_x Z \\ D_x(fY) &= (Xf)Y + f D_x Y \end{aligned}$$

Where the first two relations shows linearity properties and the last two shows parallelism property.

2.1.7 Riemannian Manifolds

A manifold on which one has singled out a specific symmetric and positively defined 2-covariant referred to as Riemannian manifold. Metric tensor allows one to define length, angles and distances. Let M be a Riemannian manifold with metric tensor <,>. Let X and Y be in M. Now we can define the length of X as

$$|X| = \sqrt{< X, X >}$$

The angle θ between the two points X, Y in M can then be obtained by

$$< X, Y >= |X||Y|.$$

To obtain the length of a curve, we integrate the length of its tangent vector field.
For instance,let θ from a to b, denoted by
$|\theta|_a^b$ is defined by

$$|\theta|_a^b = \int_a^b \sqrt{< T(t), T(t) >} dt \tag{2.2}$$

Since the integral is continuous, then it exist.

2.1.8 Riemannian Connection

A connection D on a Riemannian manifold M is called Riemannian connection on M if it satisfies the following properties

1. $D_x Y - D_y X = [X, Y]$
2. $Z < X, Y > = < D_z X, Y > + < X, D_z Y >$

For any field X,Y and Z with common domain.

3. $D_{fx} Y = f D_x Y$
4. $D_x(fY) = (Xf)Y + f D_x$

2.1.9 An Affine Connection

Definition 2.1.9. *Let M be a smooth manifold. An affine connection ∇ on M is a differential operator, sending smooth vector fields X and Y to a smooth vector field $\nabla_X Y$, which satisfy the following conditions:*

$$\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z,\ \nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z,$$
$$\nabla_{fX} Y = f \nabla_X Y,\ \nabla_X(fY) = X(f)Y + f(\nabla_X Y.$$

for all smooth vector fields X,Y and Z and real valued functions f on M. A vector field $\nabla_X Y$ is known as the covariant derivative of the vector field Y along X with respect to affine connection ∇

2.1.10 Lie bracket

Definition 2.1.10. *Let X and Y be vector fields on space M. We define the Lie bracket (at times known as The Jacobi-Lie bracket, or commutator)[X,Y] to be operator*

[X,Y]=XY-YX.

As it turns out, the bracket of two vector fields is again a vector field, meaning it is a first order differential operator. In components,
letting

$$X = X^i \frac{\partial}{\partial x^i} \text{ and } Y = Y^j \frac{\partial}{\partial x^j}$$

$$\begin{aligned}[X,Y] =& X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} - Y^i \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial x^j} \\ =& X(Y^j) \frac{\partial}{\partial x^j} - Y(X^j) \frac{\partial}{\partial x^j} \\ =& XY - YX\end{aligned}$$

Thus [X,Y] is is the vector field.

2.1.11 A Contact Metric Manifold

Definition 2.1.11. *Let (M, ϕ, ξ, η, g) be an $n = (2m+1)$-dimensional almost contact metric manifold consisting of a (1,1) tensor ϕ, a vector field ξ, a 1-form η and a Riemannian metric g.*
Let χ(M) be the lie algebra of vector in M. Then consider X, Y, Z, V, W $\in$(M). If M_n is a k-contact Riemannian manifold, then

$\nabla_x \xi$ = -$\alpha \phi X + \beta(X - \eta(X)\xi)$, $\nabla_x \xi$ = -ϕX
$(\nabla_x \eta) Y$ = -$g(\phi X, Y)$
$S(X, \xi)$= $(n-1)\eta(X)$
$\eta(R(X,Y)Z)$= $g(Y,Z)\eta(X) - g(X,Z)\eta(Y)$

2.1.12 A Sasakian Manifold

Definition 2.1.12. *Let (M, ϕ, ξ, η, g) be an $n = (2m+1)$-dimensional almost contact metric manifold consisting of a (1,1) tensor ϕ, a vector field ξ, a 1-form η and a Riemannian metric g.*
Let κ(M) be the lie algebra of vector in M. Now considering X, Y, Z, V, W $\in$(M), we have

ϕ^2= $-1 + \eta(X)\xi$, $\eta(\xi) = 1$,$\eta \circ \phi = 0$,$\phi \xi = 0$

$g(X, \xi) = g(X, Y) - \eta(X)\eta(Y),$
$g(X, \xi) = \eta(X)$
$(X\phi)Y = g(X, Y)\xi - \eta(Y)X,$
$(x_\xi)Y = -\phi X$

Thus, M is Sasakian Manifold.
Also,

$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y$
$R(X,Y)\xi = \eta(Y)X - \eta(X)Y$
$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X$
$R(,)\xi = \eta(X)\xi - X$
$S(X,\xi) = (n-1)\eta(X)$
$\phi\xi = (n-1)\xi$

2.2 Statement of the problem

This project aims at studying the W_3 and W_5 Semi symmetric LP-Sasakian structures. This study is motivated by P.G Pokhariyal's and Blaga's findings about the semi-symmetric conditions of these tensors. we generate some new ideas and emphasize on new geometric results.

2.3 Objectives

The project aims to give a detailed study of the η-Ricci solitons for the given LP-Sasakian manifolds satisfying the semi-symmetric conditions The study then focus on generating some new ideas to produce new geometric results with physical meaning.
Also we investigate the basic principles of Riemannian and LP-sasakian spaces and finally investigate the results obtained and use them to put forward some new ideas.

3 Literature Review

Pokhariyal and Mishra [18] and also, Pokhariyal (1979), defined Weyl tensor to define the relativistic significance of curvature tensor.
The Weyl's projective curvature tensor was then defined on the basis of geodesic correspondence due to a particular type of distribution of vector field found in it.
This tensor was then given by the equation,

$$W(X,Y,X,T) = R\big(X,Y,Z,T\big) + \frac{1}{n-1}\Big[g(X,Y)Ric(Y,T) - g(X,T)Ric(Y,Z)\Big] \tag{3.1}$$

The relativistic significance of Weyl's projective curvature tensor was also studied by Singh *et al*, (1965). On τ-Curvature tensor in k-contact and Sasakian manifolds by Mukut Mani, Tripath and Gupta (2011),studied properties of quasi-τ-flat, $\xi-\tau$-flat and Sasakian manifolds.These authors gave necessary and sufficient conditions for the contact manifold to be $\varphi-\tau$-flat under same algebraic conditions. They also proved that a compact $\xi-\tau$-flat k-contact manifold with regular contact vector field, under the algebraic condition is a principal S^1 bundle over an almost Kaehler space of constant holomorphic section curvature.
De and Bismas,(2006) also studied the ξ-conformally flat contact metric manifold with ξ $\in$ N(k), they too, proved that k-contact metric manifold with $\xi \in$ N(k) is ξ-conformally flat if and only if it is in η-Einstein Manifold. Dwivedi and Kim(2010), did prove that a Sasakian Manifold is ξ-conharmonically flat iff it is in η-Einstein Manifold.
A semi-Riemannian manifold M is said to be semi-symmetric by Szabo[28],if it satisfy the semi-symmetric condition $R(X,Y)\cdot R = 0$, where $R(X,Y)$ acts as a derivative on R.
A semi-Riemannian manifold is said to be recurrent by Walker [02],if it satisfy the tensor recurrent condition $\nabla R = \alpha(X)R$, where α is a 1-Form.In 1972, Takagi[03] , gave an example of Riemannian manifold satisfying $R(X,Y)\cdot R = 0$,but not $\bigtriangledown R = 0$. Several other authors have also studied symmetric manifolds, the like of: K.Sekigawa [10], Sekigawa and Tanno[06] etc. These authores, among others, also proved that a projectively semi-symmetric k-contact mani fold is Sasakian.
In 1970, Pokhariyal and Mishra defined some of the tensors which included

$$W_1(X,Y,Z,T) = R\big(X,Y,Z,T\big) + \frac{1}{n-1}\Big[g(X,T)Ric(Y,Z) - g(Y,T)Ric(X,Z)\Big] \tag{3.2}$$

$$W_2(X,Y,Z,T) = R\big(X,Y,Z,T\big) + \frac{1}{n-1}\Big[g(X,Z)Ric(Y,T) - g(Y,Z)Ric(X,T)\Big] \tag{3.3}$$

$$W_3(X,Y,Z,T) = R\big(X,Y,Z,T\big) + \frac{1}{n-1}\Big[g(Y,Z)Ric(X,T) - g(Y,T)Ric(X,Z)\Big] \tag{3.4}$$

$$W_4(X,Y,Z,T) = R\big(X,Y,Z,T\big) + \frac{1}{n-1}\Big[g(X,Z)Ric(Y,T) - g(X,Y)Ric(Z,T)\Big] \tag{3.5}$$

$$W_5(X,Y,Z,T) = R\big(X,Y,Z,T\big) + \frac{1}{n-1}\Big[g(X,Z)Ric(Y,T) - g(Y,T)Ric(X,Z)\Big] \tag{3.6}$$

$$W_6(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\Big[g(X,Z)Y - S(Y,Z)X\Big] \tag{3.7}$$

Later, Pokhariyal G.P [20] defined a new tensor field W* on a Riemannian manifold as

$$W^*(X,Y,Z,U) =' R\big(X,Y,Z,U\big) - \frac{1}{2(n-1)}\Big[S(Y,Z)g(X,U) - S(X,Z)g(Y,U)\Big] \tag{3.8}$$

Where,

$$W^*(X,Y,Z,U) = g\big(W^*(X,Y)Z,U\big)$$

and

$$'R(X,Y,Z,U) = g\big('R(X,Y)Z,U\big)$$

Such a tensor field W* is known as m-projective curvature tensor.
In the same year(1982), Pokhariyal also gave the definition of W_5 Curvature Tensor, where he gave its equation as,

$$W_5(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\Big[g(X,Z)\phi Y - S(X,Z)Y\Big] \tag{3.9}$$

Which came from earlier definition, see (eqn 2.6)
W_7 ,W_8 and W_9 he also did define during the same period, [21]. Their definitions he gave as

$$W_7(X,Y,Z,T) = R\big(X,Y,Z,T\big) + \frac{1}{n-1}\Big[g(Y,Z)Ric(X,T) - g(X,T)Ric(Y,Z)\Big] \tag{3.10}$$

and,

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\Big[S(X,Y)Z - S(Y,Z)X\Big] \tag{3.11}$$

Where

$$\begin{aligned} S(X,Y) =& g(QX,Y) = (n-1)g(X,Y) \\ =& R(X,Y), \end{aligned}$$

and Q is the Ricci operator , i.e. the linear endomorphism of a tangent space at each of its points,
or equivalently,

$$W^{'}(X,Y,Z,U) = R^{'}\big(X,Y,Z,U\big) - \frac{1}{(n-1)}\Big[R(X,Y)g(Z,U) - R(Y,Z)g(X,U)\Big] \tag{3.12}$$

$$W_9(X,Y,Z,T) = R\big(X,Y,Z,T\big) + \frac{1}{n-1}\Big[g(Z,Y)Ric(X,Y) - g(Y,Z)Ric(X,T)\Big] \quad \textbf{(3.13)}$$

In 2016, Prakasha D,G.,Vasant C and Kakasab established that a $\phi - W_5$-flat generalized Sasakian Space-form as comformally flat and is $\phi - W_5$-flat semi-symmetric $\Leftrightarrow$ it is W_5-flat.
W_5-Curveture tensor has been a field of interest for many authors, and a rich results for application in geometric modeling have been obtained.
In 1992, Deszcz [24] introduced the notion of pseudosymmetric manifolds. Sometimes later, in 1986, Ojha [16],studies the properties of m-projective curvature tensor in Sasakian and Kähler manifolds. He argued that this tensors act as a bridge between conharmonic curvature tensor, conformal curvature tensors, concircular curvature tensors on one side, and the H-projective curvature tensors on another side.The idea of Lorentzian para-Sasakian was however introduced later in 1988 by Matsumoto K. [12],and later (1992), Mihai I. and Rosca R defined LP-Sasakian and obtained several results. Other authors have also studied LP-Sasakian manifolds, the like of U.C.De,(1999) and Shaikh A.A.,(2004)
Ahmet Yildiz and U.C.De,[31], Bagewadi C.S, and Kumar K.T, (2011) studied tensor fields respectively in Kenmotsu and LP-Sasakian manifolds.
Pokhariyal,[22] studied W_2-curvature tensor and its associated symmetric and skew-symmetric tensors in an Einstein Sasakian manifolds. U.C.De and Sarkar A.(2009),made a detailed study on Para-Sasakian admitting W_2-symmetric tensor and established that W_2-symmetric para-Sasakian manifold is of constant curvature,hence it is an LP-Sasakian Manifold, just as with W_2 recurrent P-Sasakian.
Moindi S.K, Pokhariyal P.G and Njori P.W., [14],also gave their contributions on W_2-Curveture Tensor and E-Curveture tensor and proved the theorem on W_2 recurrent P-Sasakian manifold.[12]
Mohit Kumar Dwivedi,[12] also made their contribution on Lorentzian α-sasakian manifolds satisfying certain conditions on the W_2-Curveture tensor.

3.0.1 Ricci-Flow

R.Hamilton, in (1982) introduced the concept of Ricci-flow which explain the difficulties of geometry of manifolds,for instance, if there exist singular points, they can be minimized under Ricci-flow.
Ricci-solitons then moved under Ricci-flow simply by diffeomorphism of the initial metric,that is, they are stationary points of the Ricci-flow. Hamilton then gave the Ricci-flow equation as

$$\frac{\partial g}{\partial t} = -2Ric(g) \quad \textbf{(3.14)}$$

Where Ric$\equiv$ S
In the space of metric on M.

3.0.2 Ricci Soliton

Definition 3.0.1. *Hamilton [28],then defined Ricci-Soliton as follows:*
A Ricci-Soliton (g,V,λ) on a Riemannian manifold M is given by,

$$L_v g + 2S + 2\lambda g = 0 \tag{3.15}$$

Where S-is Ricci tensor, L_v is Lie derivative operator on M in V direction and λ is a scalar.

The equation for Ricci-Soliton (2.11) is said to be either shrinking (when λ <0),steady (when λ =0) or Expanding (when λ>0).
Here,the metric g(t) is the pull-back of the initial metric g(0) by a one parameter family of diffeomorphism which is generated by vector field V on a Manifold M. Hamilton noted that a compact Ricci-solitons are the points of the Ricci-flow

$$\frac{\partial g}{\partial t} = -Ric(g)$$

which is projected from the space of metric onto its its quotient modulo, diffeomorphism snd scallings, which often arise as blow-up limit for the Ricci-flow on compact manifold. In 1923, Einstein proved that if a positive definite Riemannian manifold (M,g) admits a second order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then the manifold is reducible. Levi [07], obtained the necessary and sufficient condition for the existence of such tensors. Later,[17],[16], [15], Sharma then generalized Levi's results by showing that a second order parallel,(not necessarily symmetric and non-singular) tensor on an n-dimensional (n>0) space of a constant curvature, is a constant multiple of the metric tensor.
He did prove also that, in a Sasakian manifold, there is a non-zero parallel 2-form. Sharma found out while studying Ricci-solitons on k-contact manifolds, that,where the structure field ξ is a killing vector[15], a congition in which the first real derivative vanishes, this condition distorts the condition of Ricci solitons. Sharma also proved that a complete k-contact gradient solution is compact Einstein and Sasakian.

Ricci solitons have been studied and discussed on different manifolds such as Kähler manifolds[13], on para-sasakian manifolds [25], on Kenmotsu and f-Kenmotsu manifolds and many more others by different authors.
Kimura M. and J.T. Cho [08], introduced a general notion of η-Ricci solitons, which have

been studied on Hopf-hyper-surfaces in complex spaces by C.C'alin [06]
We also noted that, Blaga [01] obtained some results on Ricci-solitons satisfying the conditions$(\xi,.)_s.R = 0$ and $(\xi,.)_R.S = 0$ on the same manifold.
This then became the motivation for us to have interest in studying η-Ricci Solitons on LP-Sasakian manifolds.

4 η- Ricci Soliton on W_3 Semi-Symmetric LP Sasakian Manifolfds

4.1 introduction

In this chapter, we discuss the η -Ricci soliton defined by use of W_3 curvature tensor.

preliminaries

A Sasakian manifold is a k-contact, but the converse is only true if the dimension $(n) = 3$ However, a contact metric tensor is Sasakian if and only if

$$R(X,Y)T = g(Y)X - g(X)Y \tag{4.1}$$

In a Sasakian manifold $(M,\phi,\eta,\xi,\lambda,g)$, we can easily see,

$$R(T,X)Y = g(X,Y)T - g(Y)X \tag{4.2}$$

Generally, in $n = (2m-1)$-dimensional Sasakian Manifold with the structure (ϕ,η,ξ,g), we have

$$\begin{aligned} R^{'}(X,Y,Z,U) =& g\big(R(X,Y)Z,U\big) \\ =& g\Bigg(\big\{g(Y,Z)X - g(X,Z)Y\big\}\Bigg) \\ =& g(Y,Z)g(X,U) - g(X,Z)g(Y,U) \end{aligned} \tag{4.3}$$

Where R is the Riemannian curvature tensor of $rank(r) = n-1$

Moreover, the data (g,ξ,λ,μ), If it satisfy equation (0.2), then it is called a η-Ricci soliton on the manifold M[02].
More particularly, if $\mu = 0$, then (g,ξ,λ) is a Ricci soliton , from R.S Hamilton[28]. And therefore,the equation (0.2) it is then said to be is Shrinking, steady or expanding according to the value of λ[02], i.e; ($\lambda < 0, \lambda = 0$ and $\lambda > 0$ respectively).

4.2 Generalised Lorentzian Para-Sasakian Manifolds

In this chapter, we discuss the η -Ricci soliton defined by use of W_3 curvature tensor.

Let M be an n-dimensional smooth manifold, φ a tensor field of (1,1)-type, ξ a vector field, η a one form and g a Lorentzian metric on M,

Definition 4.2.1. *We say that, (φ, ξ, η, g) is a Lorentzian Para-Sasakian structure of M [12] if:*

i $\varphi\xi = 0$, $\eta \circ \varphi = 0$

ii $\eta(\xi) = -1$, $\varphi^2 = 1 + \eta \otimes \xi$

iii $g(\varphi^{\cdot}, \varphi^{\cdot}) = g + \eta \otimes \eta$

iv $(\nabla_X Y) = g(X,Y)\xi + 2\eta(X)\eta(Y) + \eta(Y)X$

for any X,Y $\in \mathfrak{X}(M)$,

From the definition, it follows that η is the g-dual of ξ, that is,
$\eta(X) = g(X, \xi)$, for any X $\in \mathfrak{X}(M)$,
ξ then satisfies

$$g(\xi, \xi) = -1 \tag{4.4}$$

Here, φ is a g-symmetric operator, i.e;
$g(\varphi X, Y) = g(X, \varphi Y)$
for any X,Y $\in \mathfrak{X}(M)$,
These structures, (from equation i-iv) have their properties given below.

In [03], and [12], different authors have proved that, On a Lorentzian Para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$,for any for any X,Y,Z $\in \mathfrak{X}(M)$, the following relations holds:

1. $\nabla_X \xi = \varphi X$
2. $\eta(\nabla_X = 0$, $\nabla_\xi \xi = 0$

3. $R(X,Y)\xi = -\eta(X)Y + \eta(Y)X$

4. $\eta(R(X,Y)Z) = \eta(X)g(Y,Z) - \eta(Y)g(X,Z)$, $\eta(R(X,Y)\xi) = 0$

5. $(\nabla_X)Y = (\nabla_Y)X = g(\varphi X,Y)$, $\nabla_\xi \eta = 0$

And
$L_\xi \varphi = 0, L_\xi \eta = 0, L_\xi g = 2g(\varphi^{\cdot},^{\cdot})$

Where R is the Riemannian Curveture tensor field, and ∇, the Levi Ci-vita associated to g. The proofs of these properties are given by Adara [01]

4.3 η- Ricci Soliton on W_3-Semi Symmetric LP Sasakian Manifolfds

Definition 4.3.1. *U.C De and N.Guha [34], gave the definition of Semi symmetric condition as* $W_3(X,Y)\cdot R(Z,U) = 0$

Definition 4.3.2. *On the same line, we can also give the* W_3*-semi symmetric condition as* $R(X,Y)\cdot W_3(Z,U) = 0$

Theorem 4.3.3. *If* (φ,ξ,η,g) *is a Lorentzian Para-sasakian structure on the manifold* M_n*, and if* (g,ξ,λ,μ) *is a* η*- Ricci soliton on* M_n*, and* $R(\xi,X).W_3(Y,Z) = 0$*, then if* $\lambda = 1$*, then* $\mu = n$

Proof. The condition that must be satisfied by W_3 is

$$W_3\big(R(\xi,X)Y,Z\big) + W_3\big(Y,R(\xi,X)Z\big) = 0 \tag{4.5}$$

Using the definition of

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y \tag{4.6}$$

in LP Sasakian,
η-Ricci Soliton equation is given by[01],

$$\begin{aligned} R(\xi,X) &= g(X,Y)\xi - g(\xi,Y)X \\ &= g(X,Y)\xi - \eta(Y)X \end{aligned} \tag{4.7}$$

and

$$R(\xi,X)Z = g(X,Z)\xi - \eta(Z)X$$

Hence,

$$W_3(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\Big[g(Y,Z)\phi X - Ric(X,Z)Y\Big] \tag{4.8}$$

Where

$$\phi X = (n-1)X$$

Which then implies that,

$$\begin{aligned} S(X,Y) =& g(\phi X,Y) \\ =& (n-1)g(X,Y) \end{aligned} \tag{4.9}$$

Now computing each term in equation (4.1) separately after taking twice inner product with respect to U and T respectively,and using equation (1.1) and (4.5), we obtained

$$\begin{aligned} W_3\Big(R(\xi,X)Y,Z,U,T\Big) =& R\Big(R(\xi,X)Y,Z,U,T\Big)+ \\ & \frac{1}{n-1}\Big[g(Z,U)Ric(R(\xi,X)Y,T) - Ric\Big(R(\xi,X)Y,U\Big)g(Z,T)\Big] \\ =& g(Z,U)g\Big(R(\xi,X)Y,T\Big) - g(Z,T)g\Big(R(\xi,X)Y,U\Big)+ \\ & \frac{1}{n-1}\Big[g(Z,U)Ric(R(\xi,X)Y,T) - Ric\big(R(\xi,X)Y,U\big)g(Z,T)\Big] \\ =& g(Z,U)g\Big(g(X,Y)\xi - \eta(Y)X,T\Big) - g(Z,T)g\Big(g(X,Y)\xi - \eta(Y)X,U\Big)+ \\ & \frac{1}{n-1}\Big[g(Z,U)Ric(R(\xi,X)Y,T) - Ric\big(R(\xi,X)Y,U\big)g(Z,T)\Big] \end{aligned} \tag{4.10}$$

Putting $Z=U=T=\xi$ in equation (4.6) above, and using equation (1.1), we obtained,

$$\begin{aligned}
W_3\Big(R(\xi,X)Y,Z,U,T\Big) =& -\Big\{g(X,Y)\eta(\xi)-\eta(X)\eta(Y)\Big\}-\Big\{g(X,Y)\eta(\xi)-\eta(X)\eta(Y)\Big\}+\\
&\frac{1}{n-1}\Big[g(Z,U)Ric(R(\xi,X)Y,T)-Ric\big(R(\xi,X)Y,U\big)g(Z,T)\Big]\\
=&+g(\varphi X,\varphi Y)-g(\varphi X,\varphi Y)+\\
&\frac{1}{n-1}\Big[g(Z,U)Ric(R(\xi,X)Y,T)-Ric\big(R(\xi,X)Y,U\big)g(Z,T)\Big]\\
=&\frac{1}{n-1}\Big[-Ric\big(R(\xi,X)Y,\xi\big)+Ric\big(R(\xi,X)Y,\xi\big)\Big]\\
=&0 \qquad \textbf{(4.11)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
W_3\Big(Y,R(\xi,X)Z,U,T\Big) =&R\Big(Y,R(\xi,X)Z,U,T\Big)+\\
&\frac{1}{n-1}\Big[g\Big(R(\xi,X)Z,U\Big)Ric(Y,T)-g\Big(R(\xi,X)Z,T\Big)Ric(Y,U)\Big]\\
=&g(Y,T)g\Big(R(\xi,X)Z,U\Big)-g(Y,U)g\Big(R\xi,X)Z,T\Big)+\\
&\frac{1}{n-1}\Big[g\Big(R(\xi,X)Z,U\Big)Ric(Y,T)-g\Big(R(\xi,X)Z,T\Big)Ric(Y,U)\Big]\\
=&g(Y,T)g\Big(g(X,Z)\xi-g(\xi,Z)X,U\Big)-g(Y,U)g\Big(g(X,Z)\xi-g(\xi,Z)X,T\Big)+\\
&\frac{1}{n-1}\Big[g\Big(R(\xi,X)Z,U\Big)Ric(Y,T)-g\Big(R(\xi,X)Z,T\Big)Ric(Y,U)\Big]
\end{aligned}$$

(4.12)

Now, putting the similar conditions of $Z=U=T=\xi$ in equation (4.8) above, we have,

$$\begin{aligned}
W_3\Big(Y,R(\xi,X)Z,U,T\Big) =&\eta(Y)\Big(-\eta(X)+\eta(Y)\Big)-\eta(Y)\Big(-\eta(X)+\eta(X)\Big)+\\
&\frac{1}{n-1}\Big[g\{\eta(X)\xi+X,\xi\}Ric(Y,\xi)-g\{\eta(X)\xi+X,\xi\}Ric(Y,\xi)\Big]\\
=&Ric(Y,\xi)+\frac{1}{n-1}\Big[-\eta(X)+\eta(X)-\big(-\eta(X)+\eta(X)\big)\Big]
\end{aligned}$$

(4.13)

From equation (4.9),

$$\Rightarrow Ric(Y,\xi)(0) = 0 \tag{4.14}$$

But we know $Ric(Y,\xi) \neq 0$
From LP-Sasakian manifold, we have

$$Ric(Y,\xi) = (n-1)\eta(Y) \tag{4.15}$$

But in η-Ricci soliton in LP-Sasakian manifold, we have

$$\begin{aligned} Ric(X,Y) =& S(X,Y) \\ =& g(\varphi Y,X) - \lambda g(X,Y) - \mu\eta(X)\eta(Y) \end{aligned} \tag{4.16}$$

Putting $X = \xi$, we have

$$\begin{aligned} Ric(\xi,Y) =& -\lambda\eta(Y) + \mu\eta(Y) \\ =& (\mu-\lambda)\eta(Y) \end{aligned} \tag{4.17}$$

Which when compared to the $Ric(\xi,Y)$, equation(4.11),
We have
$\mu - \lambda = n-1$
We observe that if $\mu = n$, then $\lambda = 1$
Hence the theorem.

□

Corollary 4.3.4. *If (φ,ξ,η,g) is a Lorentzian para-Sasakian structure on the Manifold M_n, then (g,ξ,λ,μ) is a η-Ricci Soliton on M_n, and if $R(\xi,X).W_3(Y,Z) = 0$, then (M_n,g) is Einstein Manifold.*

Theorem 4.3.5. *If (μ,ξ,η,g) is a Lorentzian para-Sasakian structure on the Manifold M_n, (g,ξ,λ,μ) is a η-Ricci Soliton on M_n, and $S(\xi,X).W_3(Y,Z) = 0$, Then,*
$\lambda - \mu = n-1$

`Proof.` The condition that must be satisfied by S is

$$\begin{aligned} &S\Big(X,W_3(Y,Z)U\Big)\xi - S\Big(\xi,W_3(Y,X)U\Big)X + S(X,Y)W_3(\xi,Z)U - S(\xi,Y)W_3(X,Z)U \\ &+ S(X,Z)W_3(Y,\xi)U - S(\xi,Z)W_3(Y,X)U + S(X,U)W_3(Y,Z)\xi - S(\xi,U)W_3(Y,Z)X = 0 \end{aligned} \tag{4.18}$$

For any X,Y,Z,U $\in \mathfrak{X}(M_n)$ Taking the inner product with respect to ξ, the relation (12) becomes

$$-S\Big(X,W_3(Y,Z)U\Big)-S\Big(\xi,W_3(Y,Z)U\Big)\eta(X)+S(X,Y)\eta\Big(W_3(\xi,Z)U\Big)-S(\xi,Y)\eta\Big(W_3(X,Z)U\Big)$$
$$+S(X,Z)\eta(W_3(Y,\xi)U)-S(\xi,Z)\eta(W_3(Y,X)U)+S(X,U)\eta(W_3(Y,Z)\xi)-S(\xi,U)\eta(W_3(Y,Z)X)=0. \tag{4.19}$$

Next, we replace the expression of S using

$$S(X,Y)=-g(\varphi X,Y)-\lambda g(X,Y)-\mu\eta(X)\eta(Y)$$

Which then implies

$$\begin{aligned}S(\xi,\xi)=&0+\lambda-\mu\\=&\lambda-\mu.\end{aligned}$$

We observe that the relation (13), there are 8 terms. Computing each of these terms differently and subjecting them to equivalent condition gives,

$$S\Big(X,W_3(Y,Z)U\Big)=-g\Big(\varphi X,W_3(Y,Z)U\Big)-\lambda g\Big(X,W_3(Y,Z)U\Big)-\mu\eta(X)\eta\Big(W_3(Y,Z)U\Big) \tag{4.20}$$

But from the definition of W_3, Pokhariyal,[18]

$$W_3(Y,Z)U=R(Y,Z)U+\frac{1}{n-1}\Big[g(Z,U)\phi y-Ric(Y,U)Z\Big].$$

Now letting Z=U=ξ and substituting Z,U with ξ in the above relation,
we obtain,

$$W_3(Y,\xi)\xi=R(Y,\xi)\xi+\frac{1}{n-1}\Big[\eta(\xi)\phi Y-Ric(Y,\xi)\xi\Big] \tag{4.21}$$

Which now implies that

$$\begin{aligned}g\Big(X,W_3(Y,\xi)\xi\Big)=&g\Big(X,Ric(Y,\xi)\xi\Big)+\frac{1}{n-1}\Big[g(\xi,\xi)Ric(X,Y)-Ric(Y,\xi)g(X,\xi)\Big]\\=&g(X,\eta(\xi)Y-\eta(Y)\xi+\frac{1}{n-1}\Big[-Ric(X,Y)-\eta(X)Ric(y,\xi)\Big]\\=&-g(X,Y)-\eta(X)\eta(Y)-\frac{1}{n-1}\Big[Ric(X,Y)+\eta(X)Ric(Y,\xi)\Big]\end{aligned} \tag{4.22}$$

But,

$$\begin{aligned}\eta\Big(W_3(Y,\xi)\xi\Big)=&\eta(Ric(Y,\xi)\xi+\frac{1}{n-1}\Big[g(\xi,\xi)Ric(Y,\xi)-Ric(Y,\xi)g(\xi,\xi)\Big]\\=&\eta(g(\xi,\xi)Y-\eta(Y)\xi\\=&-\eta(Y)+\eta(Y)\\=&0\end{aligned} \tag{4.23}$$

And also

$$
\begin{aligned}
Ric(\xi,\xi) =& g(Y,\xi) - \lambda g(\xi,\xi) - \mu\eta(\xi)Y \\
=& \lambda - \mu
\end{aligned}
$$

Since $g(Y,\xi)$ is vanishing. Then we obtain from our equation (4.15)

$$
\begin{aligned}
g\left(\varphi X, W_3(Y,\xi)\xi\right) =& g\left(\varphi X, Ric(Y,\xi)\xi\right) + \frac{1}{n-1}\left[-g(\varphi X,Y) - Ric(Y,\xi)g(\varphi X,\xi)\right] \\
=& g\left(\varphi X, \eta(\xi)Y - \eta(Y)\xi\right) - \frac{1}{n-1}(\varphi X,Y) \\
=& -g(\varphi X,Y) - \frac{g(\varphi X,Y)}{n-1} \\
=& -g(\varphi X,Y)\big[1+\frac{1}{n-1}\big] + g(\varphi X,Y)\big[1+\frac{1}{n-1}\big] + \\
& \lambda g(X,Y) + \eta(X)\eta(Y) + \frac{1}{n-1}\left[Ric(X,Y) + (\lambda-\mu)\eta(X)\right] \qquad \textbf{(4.24)}
\end{aligned}
$$

computing the second term in the equation (4.13) yield the following results

$$
\begin{aligned}
S\left(\xi, W_3(Y,Z)U\right) =& -g\left(\varphi\xi, W_3(Y,Z)U\right) - \lambda g\left(\xi, W_3(Y,Z)U\right) - \mu\eta(\xi)\eta\left(W_3(Y,Z)U\right) \\
=& -\lambda\eta\left(W_3(Y,Z)U\right) + \mu\eta\left(W_3(Y,Z)U\right) \qquad \textbf{(4.25)}
\end{aligned}
$$

Here again, using the definition of W_3,
and still following with the condition Z=U=ξ,
we again obtain the results as before, i,e,

$$
\begin{aligned}
\eta\left(W_3(Y,\xi)\xi\right) =& \eta\left(R(Y,\xi)\xi\right) + \frac{1}{n-1}\left[g(\xi,\xi)Ric(Y,\xi) - Ric(Y,\xi)g(\xi,\xi)\right] \\
=& 0
\end{aligned}
$$

Observe that this is what we have from relation (4.17).

We now compute the third term from our equation (4.13).

$$\begin{aligned}
g\Big(\varphi X, W_3(Y,\xi)\xi\Big) =& g\Big(\varphi X, R(Y,\xi)\xi\Big) + \frac{1}{n-1}\Big[-g(\varphi X,\varphi Y) - Ric(Y,\xi)\eta(\varphi X)\Big] \\
=& g\Big(\varphi X, \eta(Y)\xi\Big) - \eta(Y)\xi) - \frac{1}{n-1}\Big[Ric(\varphi X,Y)\Big] \\
=& -g(\varphi X,Y) - \eta(Y)g(\varphi X,\xi) - Ric(\varphi X,Y) \\
=& -g(\varphi X,Y) - \frac{1}{n-1}\Big[-g(\varphi X,Y) - \lambda g(\varphi X,Y) - \mu\eta(\varphi X)\eta(Y)\Big] \\
=& -g(\varphi X,Y) + \frac{1}{n-1}\Big[g(\varphi^2 X,Y) + \lambda g(\varphi X,Y)\Big] \\
=& -g(\varphi X,Y) + \frac{1}{n-1}\Big[g(X+\eta(X)\xi,Y) + \lambda g(\varphi X,Y)\Big] \\
=& -g(\varphi X,Y) + \frac{1}{n-1}\Big[g(X,Y) + \eta(X)\eta(Y) + \lambda g(\varphi X,Y)\Big] \qquad \textbf{(4.26)}
\end{aligned}$$

Next, we combine the expressions (4.18), (4.19) and (4.20),
From $S(X, W_3(Y,\xi)\xi)$,
We get,

$$\begin{aligned}
S\Big(X, W_3(Y,\xi)\xi\Big) =& \Big\{g(\varphi X,Y) + \frac{1}{n-1}\Big[g(X,Y) + \eta(X)\eta(Y) + \lambda g(\varphi X,Y)\Big]\Big\} \\
& \lambda\Big[-g(X,Y) - \eta(X)\eta(Y)\Big] + \frac{1}{n-1}\Big[-Ric(X,Y) - Ric(Y,\xi)\eta(X)\Big] \\
& -\mu\eta(X)[\frac{1}{n-1}(0)] \\
=& -g(\varphi X,Y)[1+\frac{\lambda}{n-1}] + g(X,Y)[\lambda - \frac{1}{n-1}] + [\lambda - \frac{1}{n-1}]\eta(X)\eta(Y) + \\
& \frac{1}{n-1}\Big[-g(\varphi X,Y) - \lambda g(X,Y) - \mu\eta(X)\eta(Y)\eta(X)\Big] - \Big[g(\varphi Y,\xi) + \lambda g(Y,\xi) - \mu\eta(Y \\
=& -g(\varphi X,Y)[1+\frac{\lambda}{n-1}] + g(X,Y)[\lambda - \frac{1}{n-1}] + [\lambda - \frac{1}{n-1}]\eta(X)\eta(Y) + \\
& \frac{1}{n-1}[-g(\varphi X,Y) - \lambda g(X,Y) - \lambda\eta(X)\eta(Y)] \\
=& -g(\varphi X,Y)[1+\frac{\lambda}{n-1}] + \Big[[\lambda - \frac{1}{n-1}](g(X,Y) + \eta(X)\eta(Y)\Big]
\end{aligned}$$

(4.27)

Here, observe that some terms in (4.19) cancels those in (4.20).
We again compute the fourth term in equation (4.13).
Applying the same condition as in the other terms, we obtain results as bellow
With Z=U=ξ,

$\eta(W_3(X,Z)U)$
become,

$$\eta(W_3(X,\xi)\xi) = \frac{2}{n-1}(\lambda+\mu)\eta(X)$$

Also,

$$\begin{aligned} S(\xi,Y) =& Ric(\xi,Y) \\ =& -g(\varphi\xi,Y) - \lambda g(\xi,Y) - \mu\eta(\xi)\eta(Y) \\ =& -\lambda\eta(Y) + \mu\eta(Y) \\ =& (\mu-\lambda)\eta(Y) \end{aligned} \tag{4.28}$$

This then implies that,

$$S(\xi,Y)\eta(W_3(X,\xi)\xi) = \frac{2}{n-1}(\lambda+\eta)(\mu-\lambda)\eta(X)\eta(Y) \tag{4.29}$$

Computing the fifth term in our equation (4.13), we obtain,

$$\begin{aligned} S(\xi,Y) =& -g(\varphi\xi,Y) - \lambda g(\xi,Y) - \mu\eta(\xi)\eta(Y) \\ =& -\lambda\eta(Y) + \mu\eta(Y) \\ =& (\mu-\lambda)\eta(X) \end{aligned} \tag{4.30}$$

And so, With U=ξ,
$\eta(W_3(Y,\xi)U)$ become,

$$\begin{aligned} \eta\left(W_3(Y,\xi)\xi\right) &= \frac{2}{n-1}(\lambda+\mu)\eta(X) \\ \Rightarrow S(X,\xi)\eta\left(W_3(Y,\xi)\xi\right) &= \frac{2}{n-1}(\lambda+\eta)(\mu-\lambda)\eta(X)\eta(\xi) \end{aligned} \tag{4.31}$$

Clearly, (4.22) cancels (4.25) as did (4.19) and (4.20).

We again compute the sixth term. The results obtained are as follows Setting Z=U=ξ,
$S(\xi,Z)\eta(W_3(Y,X)U)$ become,

$$\begin{aligned} S(\xi,\xi) =& -g(\varphi\xi,\xi) - \lambda g(\xi,\xi) - \mu\eta(\xi)\eta(\xi) \\ =& \lambda-\mu. \end{aligned}$$

Again,

$$\eta\left(W_3(Y,X)U\right) = \eta\left(R(Y,X)U\right) + \frac{1}{n-1}\Big[g(X,U)Ric(Y,\xi) - Ric(Y,U)\eta(X)\Big] \tag{4.32}$$

putting U=ξ, and substituting, we obtain

$$\begin{aligned}
\eta\left(W_3(Y,X)\xi\right) =&\eta\left(R(Y,X)\xi\right)+\frac{1}{n-1}\left[\eta(X)Ric(Y,\xi)-\eta(X)Ric(Y,\xi\right] \\
=&\eta\left(R(Y,X)\xi\right) \\
=&\eta\left(g(X,\xi)Y-g(Y,\xi)X\right) \\
=&\eta(\eta(X)Y-\eta(Y)X) \\
=&\eta(X)\eta(Y)-\eta(Y)\eta(X) \\
=&0. \qquad (4.33)
\end{aligned}$$

$$\Rightarrow S(\xi,\xi)\eta(W_3(Y,X)\xi)=0$$

Computing the seventh term, we get,
From $S(X,U)\eta(W_3(Y,Z)\xi)$
With U=ξ, we get

$$\begin{aligned}
S(X,\xi) =&-g(\varphi X,\xi)-\lambda g(X,\xi)-\mu\eta(X)\eta(\xi) \\
=&-\lambda\eta(X)+\mu\eta(X) \\
=&(\mu-\lambda)\eta(X) \qquad (4.34)
\end{aligned}$$

and also, we see that

$$\eta\left(W_3(Y,Z)\xi\right)=\eta\left(R(Y,Z)\xi\right)+\frac{1}{n-1}\left[g(Z,\xi)Ric(Y,\xi)Ric(Y,\xi)-Ric(Y,\xi)\eta(Z)\right] \qquad (4.35)$$

Putting Z=ξ in equation (29)

$$\Rightarrow \eta\left(W_3(Y,\xi)\xi\right)=\eta\left(R(Y,\xi)\xi\right)+\frac{1}{n-1}\left[-Ric(Y,\xi)+Ric(Y,\xi)\right] \qquad (4.36)$$

Therefore

$$\begin{aligned}
\eta(\xi) =&g(\xi,\xi) \\
=&-1
\end{aligned}$$

Hence

$$\begin{aligned}\Rightarrow \eta\left(W_3(Y,\xi)\xi\right) =&\eta\left(R(Y,\xi)\xi\right)\\ =&\eta\left(g(\xi,\xi)Y-g(Y,\xi)\xi\right)\\ =&\eta\left(-Y-\eta(Y)\xi\right)\\ =&-\eta(Y)-\eta(Y)\eta(\xi)\\ =&0 \end{aligned} \tag{4.37}$$

Which therefore leads to the conclusion,

$$S(X,\xi)\eta\left(W_3(Y,\xi)\xi\right)=0$$

Now, we compute the last term.
we consider,
$S(\xi,U)\eta(W_3(Y,Z)X$,with Z=U=ξ,
We get,

$$\begin{aligned}S(X,\xi)=&-g(\varphi X,\xi)-\lambda g(X,\xi)-\mu\eta(X)\eta(\xi)\\ =&(-\lambda+\mu)\eta(X).\end{aligned}$$

Next,

$$\eta\left(W_3(Y,Z)X\right)=\eta\left(R(Y,Z)X\right)+\frac{1}{n-1}\left[g(Z,X)Ric(Y,\xi)-Ric(Y,Z)g(Z,\xi)\right]$$

Substituting Z=ξ, we have

$$\begin{aligned}\eta\left(W_3(Y,\xi)X\right) =&\eta\left(R(Y,\xi)X\right)+\frac{1}{n-1}\left[\eta(X)Ric(Y,\xi)+Ric(Y,X)\right]\\ =&\eta(\eta(X)Y-g(X,Y)\xi)+\frac{1}{n-1}\left[\eta(X)Ric(Y,\xi)+Ric(Y,X)\right]\end{aligned} \tag{4.38}$$

Also,

$$\begin{aligned}g\left(X,W_3(Y,\xi\right)\xi) =&g\left(x,R(Y,\xi)\xi\right)+\frac{1}{n-1}\left[\eta(\xi)Ric(Y,X)-Ric(Y,\xi)\eta(X)\right]\\ =&g\left(X,\eta(\xi)Y-\eta(Y)\xi\right)+\frac{1}{n-1}\left[-Ric(Y,X)-Ric(Y,\xi)\eta(X)\right]\\ =&-g(X,Y)-\eta(X)\eta(Y)+\frac{1}{n-1}\left[-Ric(Y,X)-Ric(Y,\xi)\eta(X)\right]\end{aligned} \tag{4.39}$$

$$\Rightarrow g\left(X, W_3(Y,\xi)\xi\right) = -\eta(Y) + \eta(Y) + \frac{1}{n-1}\left[-Ric(\xi,Y)_R ic(Y,\xi)\right]$$

It follows that,

$$g\left(\varphi X, W_3(Y,\xi)\xi\right) = g(\varphi X, Ric(Y,\xi)\xi) + \frac{1}{n-1}\left[\eta(\xi)g(\varphi X,\varphi Y) - Ric(Y,\xi)g(\varphi X,\xi)\right] \tag{4.40}$$

putting X=Z=U=ξ ,
$S(\xi,U)\eta(W_3(Y,Z)X)$ becomes,
$S(\xi,\xi)\eta(W_3(Y,\xi)\xi)$.
But

$$\begin{aligned} S(\xi,\xi) &= -g(\varphi\xi,\xi) - \lambda g(\xi,\xi) - \mu\eta(\xi)\eta(\xi) \\ &= \lambda - \mu \end{aligned}$$

Since $-g(\varphi\xi,\xi) = 0$
And also $S(Y,\xi) = (n-1)\eta(Y)$
But in LP-Sasakian,
$\lambda - \mu = n-1$. Also, for

$$\eta\left(W_3(Y,Z)X\right) = \eta\left(R(Y,Z)X\right) + \frac{1}{n-1}\left[g(Z,X)g(\varphi Y,\xi) - Ric(Y,Z0g(Z,\xi)\right]$$

Putting Z=U=X=ξ in the above equation,
We have

$$\begin{aligned} \eta\left(W_3(Y,\xi)\xi\right) &= \eta\left(R(Y,\xi)\xi\right) + \frac{1}{n-1}\left[-Ric(Y,\xi) + Ric(Y,\xi)\right] \\ &= \eta(\eta(\xi)Y - \eta(Y)\xi) = 0 \end{aligned} \tag{4.41}$$

$$\Rightarrow -S(\xi,\xi)\cdot 0 = 0 \tag{4.42}$$

but $S(\xi,\xi) \neq 0$

$$\begin{aligned} \Rightarrow S(\xi,\xi) &= -g(\varphi\xi,\xi) - \lambda g(\xi,\xi) - \mu\eta(\xi)\eta(\xi) \\ &= -\lambda\eta(\xi) - \mu \\ &= \lambda - \mu. \end{aligned} \tag{4.43}$$

Hence we conclude, from LP-Sasakian Manifold, we have
$S(Y,\xi) = (n-1)\eta(Y)$
$\Rightarrow \lambda - \mu = -(n-1)$
$\Rightarrow \lambda - \mu = -n+1$
Now by letting $\lambda = 1$,
It is easy to see that $\mu = n$
And that ends our proof of the Theorem. □

Corollary 4.3.6. *If (φ, ξ, η, g) is a Lorentzian para-Sasakian structure on the Manifold M_n, (g,ξ, λ, μ) is a η-Ricci Soliton on M_n, and if $S(\xi, X).W_3(y, Z) = 0$, then (M_n, g) is Einstein Manifold.*

5 A Study of η- Ricci Soliton on W_5 Semi-Symmetric LP Sasakian Manifolds

5.1 introduction

In this chapter, we discuss the η -Ricci soliton defined by use of W_5 curvature tensor.

preliminaries

A Sasakian manifold is a k-contact, but the converse is only true if the dimension $(n) = 3$ However, a contact metric tensor is Sasakian if and only if

$$R(X,Y)T = g(Y)X - g(X)Y \tag{5.1}$$

In a Sasakian manifold $(M, \phi, \eta, \xi, \lambda, g)$, we can easily see,

$$R(T,X)Y = g(X,Y)T - g(Y)X \tag{5.2}$$

Generally, in $n = (2m-1)$-dimensional Sasakian Manifold with the structure (ϕ, η, ξ, g), we have

$$\begin{aligned} R^{'}(X,Y,Z,U) =& g\big(R(X,Y)Z,U\big) \\ =& g\Bigg(\big\{g(Y,Z)X - g(X,Z)Y\big\}\Bigg) \\ =& g(Y,Z)g(X,U) - g(X,Z)g(Y,U) \end{aligned} \tag{5.3}$$

Where R is the Riemannian curvature tensor of $rank(r) = n-1$

We also observe that the data (g, ξ, λ, μ), If it sufficiently satisfy equation (0.2), then it is said to be a η-Ricci soliton on the manifold M[02].
More particularly, if we let $\mu = 0$, then (g, ξ, λ) i.e, a Ricci soliton according to R.S Hamilton[28]. And thus,the equation (0.2) it is said to be is Shrinking, steady or expanding

according to the value of λ[02]

Generalised Lorentzian Para-Sasakian Manifolds

Let M be an n-dimensional smooth manifold, φ a tensor field of (1,1)-type, ξ a vector field, η a one form and g a Lorentzian metric on M,

Definition 5.1.1. *We say that, (φ, ξ, η, g) is a Lorentzian Para-Sasakian structure of M,[12] if:*

i. $\varphi\xi = 0$, $\eta \circ \varphi = 0$

ii. $\eta(\xi) = -1$, $\varphi^2 = 1 + \eta \otimes \xi$

iii. $g(\varphi^{\cdot}, \varphi^{\cdot}) = g + \eta \otimes \eta$

iv. $(\nabla_X Y) = g(X,Y)\xi + 2\eta(X)\eta(Y) + \eta(Y)X$

for any X,Y $\in \mathfrak{X}(M)$,

From the definition, it follows that η is the g-dual of ξ, that is,
$\eta(X) = g(X, \xi)$, for any X $\in \mathfrak{X}(M)$,
ξ then satisfies

$$g(\xi, \xi) = -1 \tag{5.4}$$

Here, φ is a g-symmetric operator, i.e;
$g(\varphi X, Y) = g(X, \varphi Y)$
for any X,Y $\in \mathfrak{X}(M)$,
These structures, (from equation i-iv) have their properties given in the following remark.

Remark 5.1.2. *In [03], and [08], different authors have proved that, On a Lorentzian Para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$,for any for any X,Y,Z $\in \mathfrak{X}(M)$, the following relations holds:*

(i). $\nabla_X \xi = \varphi X$

(ii). $\eta(\nabla_X = 0$, $\nabla_\xi \xi = 0$

(iii). $R(X,Y)\xi = -\eta(X)Y + \eta(Y)X$

(iv). $\eta(R(X,Y)Z) = \eta(X)g(Y,Z) - \eta(Y)g(X,Z)$, $\eta(R(X,Y)\xi) = 0$

(v). $(\nabla_X)Y = (\nabla_Y)X = g(\varphi X,Y)$, $\nabla_\xi \eta = 0$
And

$L_\xi \varphi = 0$, $L_\xi \eta = 0$, $L_\xi g = 2g(\varphi^{\cdot},^{\cdot})$

Where R is the Riemannian Curveture tensor field, and ∇, the Levi Ci-vita associated to g. The proofs of these properties are given by Adara[01]

5.2 η- Ricci Soliton on W_5-Semi Symmetric LP Sasakian Manifolfds

Definition 5.2.1. *U.C De and N.Guha[34], gave the definition of Semi symmetric condition as* $R(X,Y)\cdot R(Z,U) = 0$

Definition 5.2.2. *On the same line, we can also give the* W_5*-semi symmetric condition as* $R(X,Y)\cdot W_5(Z,U) = 0$

Theorem 5.2.3. *If (φ,ξ,η,g) is a Lorentzian Para-sasakian structure on the manifold M_n,and if (g,ξ,λ,μ) is a η- Ricci soliton on M_n, and*
$R(\xi,X).W_5 = 0$*, then* $\lambda = 1$*, and* $\mu = n$

`Proof.` The condition that must be satisfied by W_5 is

$$W_5(R(\xi,X)Y,Z) + W_5(Y,R(\xi,X)Z) = 0 \tag{5.5}$$

For any X,Y,Z,U $\in \mathfrak{X}(M)$.
Pokhariyal and Mishra[19], gave the definition of W_5 as

$$W_5(X,Y,Z,U) = R(X,Y,Z,U) + \frac{1}{n-1}\Big[g(X,Z)Ric(Y,U) - g(Y,U)Ric(X,Z)\Big] \tag{5.6}$$

Now we compute the two terms on the L.H.S of the equation (5.1) as follows:
The first term,

$$W_5\left(R(\xi,X)Y,Z,U,\xi\right) = R\left(R(\xi,X)Y,Z,U,\xi\right)+ \frac{1}{n-1}\left[g(R(\xi,X)Y,U)Ric(Z,\xi) - g(R(Z,\xi)Ric(R(\xi,X)Y,U)\right]. \quad \textbf{(5.7)}$$

Upon expanding the three terms on the R.H.S of equation (5.3), we obtain,
From the first,

$$\begin{aligned} R\left(R(\xi,X)Y,Z,U,\xi\right) =& R\left(g(X,Y)\xi - \eta(Y)X,Z,U,\xi\right) \\ =& g(X,Y)R(\xi,Z,U,\xi) - \eta(Y)R(X,Z,U,\xi) \\ =& g(X,Y)\left\{g(\xi,\xi)g(X,U) - g(Z,\xi)g(\xi,U)\right\} - \\ & \eta(Y)\left\{g(X,\xi)g(Z,U) - g(Z,\xi)g(X,U)\right\}. \\ =& -g(X,Y)g(\varphi Z,\varphi U) - \eta(Y)\left\{\eta(X)g(Z,U) - \eta(Z)g(X,U)\right\} \\ =& 0 \end{aligned} \quad \textbf{(5.8)}$$

when we put X = Y =U = ξ.
Then the second term,

$$\begin{aligned} g\left(R(\xi,X)Y,U\right)Ric(Z,\xi) =& g\left(g(X,Y)\xi - g(\xi,Y)X,U\right)Ric(Z,\xi) \\ =& g(X,Y)g(\xi,U)Ric(Z,\xi) - g(\xi,Y)g(X,U)Ric(Z,\xi) \\ =& \eta(U)g(X,Y)Ric(Z,\xi) - \eta(Y)g(X,U)Ric(Z,\xi) \\ =& Ric(Z,\xi)\{\eta(U)g(X,Y) - \eta(Y)g(X,U)\} \\ =& 0 \end{aligned} \quad \textbf{(5.9)}$$

For the same conditions X = Y =U = ξ.

And also, the third term vanishes with those same conditions as shown.

$$\begin{aligned} g(Z,\xi)Ric\left(R(\xi,X)Y,U\right) =& \eta(Z)Ric\left(g(X,Y)\xi - g(\xi,Y)X,U\right) \\ =& \eta(Z)g(X,Y)Ric(\xi,U) - \eta(Y)\eta(Z)Ric(X,U) \\ =& 0 \end{aligned} \quad \textbf{(5.10)}$$

Computing the second term of the equation (5.1) also led to the following results

$$W_5\left(Y,R(\xi,X)Z,U,\xi\right) = R\left(Y,R(\xi,X)Z,U,\xi\right) + \frac{1}{n-1}\left[g(Y,U)Ric\left(R(\xi,X)Z,\xi\right) - g\left(R(\xi,X)Z,\xi\right)Ric(Y,U)\right] \quad \textbf{(5.11)}$$

Again we observed that, equation (5.7) has three terms.
Next, we expand these terms independently and obtain,
From the first term, with the conditions that, U= Z = ξ.

$$\begin{aligned} R\left(Y,R(\xi,X)Z,U,\xi\right) =& g(Y,\xi)g\left(R(\xi,X)Z,U\right) - g\left(R(\xi,X)Z,\xi\right)g(Y,U) \\ =& \eta(Y)g(g(X,Z)\xi - \eta(Z)X,U) - g(Y,U)g\left(g(X,Z)\xi - \eta(Y)Z,\xi\right) \\ =& \eta(Y)\left\{g(X,Z)\eta(U) - \eta(Z)g(X,U)\right\} - g(Y,U)\left\{-g(X,Z) + \eta(X)\eta(Z)\right\} \\ =& \eta(Y)\left\{g(X,Z)\eta(U) - \eta(Z)g(X,U)\right\} + g(Y,U)\left\{g(X,Z) + \eta(X)\eta(Z)\right\} \\ =& 0 \end{aligned} \quad \textbf{(5.12)}$$

Expanding the second term,

$$\begin{aligned} g(Y,U)Ric(\xi,X)Z,\xi) =& g(Y,U)Ric(g(X,Z)\xi - \eta(Z)X,\xi) \\ =& g(Y,U)\left\{g(X,Z)Ric(\xi,\xi) - \eta(Z)Ric(X,\xi)\right\} \\ =& g(Y,U)\left\{-(n-1)g(X,Z) - (n-1)\eta(X)\eta(Z)\right\} \\ =& -(n-1)g(Y,U)\left\{g(X,Z) + \eta(X)\eta(Z)\right\} \\ =& 0 \end{aligned} \quad \textbf{(5.13)}$$

when we put U= Z = ξ.
Lastly, we expand the third term and obtain,

$$\begin{aligned} -Ric(Y,U)g\left(R(\xi,X)Z,\xi\right) =& -Ric(Y,U)\left\{g(g(X,Z)\xi - \eta(Z)X,\xi)\right\} \\ =& -Ric(Y,U)\left\{g(\xi,\xi)g(X,Z) - \eta(Z)\eta(X)\right\} \\ =& Ric(Y,U)\left\{g(X,Z) + \eta(X)\eta(Z)\right\} \end{aligned} \quad \textbf{(5.14)}$$

We clearly see that our expansions with the necessary conditions, leave only equation (5.10) not vanishing. With simplification and substitution into equation (5.1), and noting

that,
if we put X =Y = ξ into equation (5.10)
We have,
$Ric(\xi,U)\{\eta(Z)-\eta(Z)\}$
$\Rightarrow Ric(\xi,U)\neq 0$
But in LP-Sasakian,

$$Ric(\xi,U)=(n-1)\eta(U) \tag{5.15}$$

We also have from η-Ricci soliton,

$$\begin{aligned} S(X,Y) =& Ric(X,Y) \\ =& g(\varphi X,Y)-\lambda g(X,Y)-\mu\eta(X)\eta(Y) \end{aligned} \tag{5.16}$$

Setting $Y=U$ and $X=\xi$

$$\begin{aligned} Ric(\xi,U) =& -0-\lambda\eta(U)+\mu\eta(U) \\ =& (-\lambda+\mu)\eta(U) \end{aligned} \tag{5.17}$$

But from equation (5.11)

$$Ric(\xi,U)=(n-1)\eta(U)$$

$$\Rightarrow \mu-\lambda=n-1$$

Again, if $\mu=n$, then $\lambda=1$
Hence the Proof. □

Corollary 5.2.4. *If (φ,ξ,η,g) is a Lorentzian para-Sasakian structure on the Manifold M_n, then (g,ξ,λ,μ) is a η-Ricci Soliton on M_n, and if $R(\xi,X).W_5(Y,Z)=0$, then (M_n,g) is Einstein Manifold.*

Theorem 5.2.5. *If (φ,ξ,η,g) is a Lorentzian Para-sasakian structure on the manifold M_n,and if (g,ξ,λ,μ) is a η- Ricci soliton on M_n, and*
$W_5(\xi,X).R=0$, then $\lambda=1$, and $\mu=n$

Proof. The condition that W_5 must satisfy is given by the equation

$$W_5\Big(X(R(Y,Z)U\Big)\xi-W_5\Big(\xi,R(Y,Z)U\Big)X+W_5(X,Y)R(\xi,Z)U-W_5(\xi,Y)R(X,Z)U+W_5(X,Z)R(Y,\xi)U- \\ W_5(\xi,Z)R(Y,X)U+W_5(X,U)R(Y,X)\xi-W_5(\xi,U)R(Y,Z)X=0 \tag{5.18}$$

Observe that equation (5.14) has eight terms.
On expanding each of the terms independently, and taking inner product with respect to

ξ, we have, from the first term,

$$\begin{aligned}
W_5\Big(X,R(X,Z)U,\xi,T\Big) =&R\Big(X,R(Y,Z)U,\xi,T\Big)+\\
&\frac{1}{n-1}\Big[g(X,\xi)Ric\Big(R(Y,Z)U,T\Big)-g\Big(R(Y,Z)U,T\Big)Ric(X,\xi)\Big]\\
=&g(X,T)g\Big(R(X,Z)U,\xi\Big)-g\Big(R(Y,Z)U,T\Big)g(X,\xi)+\\
&\frac{1}{n-1}\Big[g(X,\xi)Ric\Big(R(Y,Z)U,T\Big)-g\Big(R(Y,Z)U,T\Big)Ric(X,\xi)\Big]\\
=&g(X,T)g\Big(g(Z,U)X-g(X,U)Z,T\Big)-\eta(X)g\Big(g(Z,U)Y-g(Y,U)Z,T\Big)+\\
&\frac{1}{n-1}\Big[g(X,\xi)Ric\Big(R(Y,Z)U,T\Big)-g\Big(R(Y,Z)U,T\Big)Ric(X,\xi)\Big]\\
=&g(X,T)\Big\{g(Z,U)g(Y,T)-g(Y,U)g(Z,T)\Big\}-\\
&\eta(X)\Big[g(Z,U)g(Y,T)-g(Y,U)g(Z,T)\Big]+\\
&\frac{1}{n-1}\Big[g(X,\xi)Ric\Big(R(Y,Z)U,T\Big)-g\Big(R(Y,Z)U,T\Big)Ric(X,\xi)\Big]
\end{aligned} \tag{5.19}$$

Now substituting $Y=\xi$

$$\begin{aligned}
W_5\Big(X,R(Y,Z)U,\xi,\xi\Big) =&\eta(X)\Big\{g(Z,U)\eta(Y)-\eta(Z)g(Y,U)\Big\}-\\
&\eta(X)\Big\{g(Z,U)\eta(Y)-g(Y,U)\eta(Z)\Big\}+\\
&\frac{1}{n-1}\Big[\eta(X)Ric(R(Y,Z)U,\xi)-g(R(Y,Z)U,\xi)Ric(X,\xi)\Big]\\
=&\frac{n-1}{n-1}\Big[\eta(X)\eta(Y,Z)U)-\eta(X)\eta(R(Y,Z)U)\Big]\\
=&0
\end{aligned} \tag{5.20}$$

Also, computing the second term,

$$
\begin{aligned}
W_5\Big(\xi(R(Y,Z)U,X,T\Big) =& R\Big(\xi,R(Y,Z)U,X,T\Big)+\\
&\frac{1}{n-1}\Big[g(\xi,X)Ric(R(Y,Z)U,T)-g(R(Y,Z)U,T)Ric(\xi,X)\Big]\\
=&g(\xi,T)g(R(Y,Z)U,X)-g(R(Y,Z)U,T)g(\xi,X)+\\
&\frac{1}{n-1}\Big[g(\xi,X)Ric(R(Y,Z)U,T)-g(R(Y,Z)U,T)Ric(\xi,X)\Big]\\
=&\eta(T)\Big\{g\big(g(Z,U)Y-g(Y,U)Z,X\big)\Big\}-\eta(X)\Big\{g\big(g(Z,U)Y-g(Y,U)Z,T\big)\Big\}+\\
&\frac{1}{n-1}\Big[g(\xi,X)Ric(R(Y,Z)U,T)-g(R(Y,Z)U,T)Ric(\xi,X)\Big]\\
=&\eta(T)\Big\{g(Z,U)g(Y,X)-g(Y,U)g(X,Z)\Big\}-\\
&\eta(X)\Big\{g(Z,U)g(Y,T)-g(Y,U)g(Z,T)\Big\}+\\
&\frac{1}{n-1}\Big[g(\xi,X)Ric(R(Y,Z)U,T)-g(R(Y,Z)U,T)Ric(\xi,X)\Big]
\end{aligned} \tag{5.21}
$$

Now, putting $X=Y=T=\xi$, and substituting into equation (5.17), we see that,

$$
\begin{aligned}
W_5\Big(\xi,R(\xi,Z)U,\xi,\xi\Big) =&\Big\{-\Big(-g(Z,U)-\eta(U)\eta(Z)\Big)+\Big(-g(Z,U)-\eta(U)\eta(Z)\Big)\Big\}+\\
&\frac{1}{n-1}\Big[-(n-1)g\big(R(\xi,Z)U,\xi\big)+(n-1)g\big(R(\xi,Z)U,\xi\big)\Big]\\
=&0
\end{aligned} \tag{5.22}
$$

$$\Rightarrow W_5\big(\xi,R(\xi,Z)U,X,\xi\big)=0$$

Computing the third term

$$
\begin{aligned}
W_5\Big(X,Y,R(\xi,Z)U,T\Big) =&R\Big(X,Y,R(\xi,Z)U,T\Big)+\\
&\frac{1}{n-1}\Big[g(X,T)Ric(R(\xi,Z)U,Y)-g(R(\xi,Z)U,Y)Ric(X,T)\Big]\\
=&g(X,T)g\Big(Y,R(\xi,Z)U\Big)-g(Y,T)g\Big(X,R(\xi,Z)U\Big)+\\
&\frac{1}{n-1}\Big[g(X,T)Ric(R(\xi,Z)U,Y)-g(R(\xi,Z)U,Y)Ric(X,T)\Big]
\end{aligned} \tag{5.23}
$$

putting $X = Y = T = \xi$, and substituting into equation (5.19), we obtain

$$
\begin{aligned}
W_5\Big(\xi,\xi,R(\xi,Z)U,\xi\Big) =&\Big\{-\Big(g(\xi,g(Z,U)\xi-g(U,\xi)Z\Big)+\Big(g(\xi,g(Z,U)\xi-g(\xi,U)Z\Big)\Big\}+\\
&\frac{1}{n-1}\Big[-(n-1)g\big(R(\xi,Z)U,\xi\big)+(n-1)g\big(R(\xi,Z)U,\xi\big)\Big]\\
=&0
\end{aligned}
\tag{5.24}
$$

Computing the fourth term

$$
\begin{aligned}
W_5\Big(\xi,Y,R(X,Z)U,T\Big) =&R\Big(\xi,Y,R(X,Z)U,T\Big)+\\
&\frac{1}{n-1}\Big[g\big(\xi,R(X,Z)U\big)Ric(Y,T)-g(Y,T)Ric\big(\xi,R(X,Z)U\big)\Big]\\
=&\eta(T)g\Big(R(X,Z)U,Y\Big)-g(Y,T)g\Big(\xi,R(X,Z)U\Big)+\\
&\frac{1}{n-1}\Big[g\big(\xi,R(X,Z)U\big)Ric(Y,T)-g(Y,T)Ric\big(\xi,R(X,Z)U\big)\Big]\\
=&\eta(T)\Big[g\Big(g(Z,U)X-g(X,U)Z,Y\Big)\Big]-g(Y,T)\Big[g\Big(\xi,g(Z,U)X-g(X,U)Z\Big)\Big]\\
&\frac{1}{n-1}\Big[g\big(\xi,R(X,Z)U\big)Ric(Y,T)-g(Y,T)Ric\big(\xi,R(X,Z)U\big)\Big]\\
=&\eta(T)\Big(g(Z,U)g(X,Y)-g(X,U)g(Z,Y)\Big)-\\
&g(Y,T)\Big(g(Z,U)\eta(X)-g(X,U)\eta(Z)\Big)+\\
&\frac{1}{n-1}\Big[g\big(\xi,R(X,Z)U\big)Ric(Y,T)-g(Y,T)Ric\big(\xi,R(X,Z)U\big)\Big]
\end{aligned}
\tag{5.25}
$$

again, putting $X = Y = T = \xi$, and substituting into equation (5.21), the computations yield,

$$
\begin{aligned}
\Rightarrow W_5\Big(\xi,\xi,R(\xi,Z)U,\xi\Big) =&-\Big(-g(Z,U)-\eta(U)\eta(Z)\Big)+\Big(-g(Z,U)-\eta(U)\eta(Z)\Big)\\
&\frac{1}{n-1}\Big[g\big(\xi,R(X,Z)U\big)Ric(Y,T)-g(Y,T)Ric\big(\xi,R(X,Z)U\big)\Big]\\
=&0-\frac{1}{n-1}\Big[-(n-1)\eta\big(R(\xi,Z)U\big)+(n-1)\eta\big(R(\xi,Z)U\big)\Big]\\
=&0.
\end{aligned}
\tag{5.26}
$$

Computation of the fifth term also gave the following results,

$$\begin{aligned}
W_5\Big(X,Y,R(Y,\xi)U,T\Big) =& R\Big(X,Z,R(Y,\xi)U,T)+\\
&\frac{1}{n-1}\Big[g\Big(X,R(Y,\xi)U\Big)Ric(Z,T)-g(Z,T)Ric\Big(X,R(Y,\xi)U\Big)\Big]\\
=&g(X,T)g\Big(Z,R(Y,\xi)U\Big)-g(Z,T)g\Big(X,R(Y,\xi)U\Big)+\\
&\frac{1}{n-1}\Big[g\Big(X,R(Y,\xi)U\Big)Ric(Z,T)-g(Z,T)Ric\Big(X,R(Y,\xi)U\Big)\Big]\\
=&g(X,T)\Big\{g\Big(Z,g(\xi,U)Y-g(Y,U)\xi\Big)\Big\}-\\
&g(X,T)\Big\{g\Big(X,g(\xi,U)Y-g(Y,U)\xi\Big)\Big\}+\\
&\frac{1}{n-1}\Big[g\Big(X,R(Y,\xi)U\Big)Ric(Z,T)-g(Z,T)Ric\Big(X,R(Y,\xi)U\Big)\Big]\\
=&g(X,T)\Big(\eta(U)g(Z,Y)-\eta(Z)g(Y,U)\Big)-\\
&g(Z,T)\Big(\eta(U)g(X,Y)-\eta(X)g(Y,U)\Big)+\\
&\frac{1}{n-1}\Big[g\Big(X,R(Y,\xi)U\Big)Ric(Z,T)-g(Z,T)Ric\Big(X,R(Y,\xi)U\Big)\Big]
\end{aligned} \tag{5.27}$$

putting $X=Y=T=\xi$, and substituting into equation (5.23)

$$\begin{aligned}
W_5\big(X,Z,R(Y,\xi)U,T\big) =& -\Big(\eta(U)\eta(Z)-\eta(Z)\eta(U)\Big)-\eta(Z)\Big(-\eta(U)+\eta(U)\Big)+\\
&\frac{1}{n-1}\Big[g\Big(X,R(Y,\xi)U\Big)Ric(Z,T)-g(Z,T)Ric\Big(X,R(Y,\xi)U\Big)\Big]\\
=&\frac{1}{n-1}\Big[(n-1)\eta(Z)\eta\Big(R(\xi,\xi)U\Big)+(n-1)\eta(Z)\eta\Big(R(\xi,\xi)U\Big)\Big]+\\
&\frac{1}{n-1}\Big[g\Big(X,R(Y,\xi)U\Big)Ric(Z,T)-g(Z,T)Ric\Big(X,R(Y,\xi)U\Big)\Big]\\
=&2\eta(Z)\eta\Big(R(\xi,\xi)U\Big)
\end{aligned} \tag{5.28}$$

but we know, from the definition of Ricci curvature tensor,

$$R(Y,\xi)U=\eta(U)Y-g(Y,U)\xi \tag{5.29}$$

Now, putting $Y = \xi$

$$\Rightarrow R(\xi,\xi)U = \big(\eta(U)\xi - \eta(U)\xi\big)$$
$$= 0 \tag{5.30}$$

We proceed to the Computation of the sixth term and obtain,

$$\begin{aligned}
W_5\Big(\xi, Y, R(Y,X)U, T\Big) =& R\Big(\xi, Z, R(Y,X)U, T\Big) + \\
& \frac{1}{n-1}\Big[g\Big(\xi, R(Y,X)U\Big)Ric(Z,T) - g(Z,T)Ric\Big(\xi, R(Y,X)U\Big)\Big] \\
=& g(\xi,T)g\Big(Z, R(Y,X)U\Big) - g(Z,T)g\Big(\xi, R(Y,X)U\Big) + \\
& \frac{1}{n-1}\Big[g\Big(\xi, R(Y,X)U\Big)Ric(Z,T) - g(Z,T)Ric\Big(\xi, R(Y,X)U\Big)\Big] \\
=& g(\xi,T)\Big\{g\Big(Z, g(X,U)Y - g(Y,U)X\Big)\Big\} - g(\xi,T)\Big\{g\Big(\xi, g(X,U)Y - g(Y,U)X \\
& \frac{1}{n-1}\Big[g\Big(\xi, R(Y,X)U\Big)Ric(Z,T) - g(Z,T)Ric\Big(\xi, R(Y,X)U\Big)\Big] \\
=& g(\xi,T)\Big(\eta(U)g(Z,Y) - \eta(Z)g(Y,U)\Big) - g(Z,T)\Big(\eta(U)g(\xi,Y) - \eta(\xi)g(Y,U) \\
& \frac{1}{n-1}\Big[g\Big(\xi, R(Y,X)U\Big)Ric(Z,T) - g(Z,T)Ric\Big(\xi, R(Y,X)U\Big)\Big]
\end{aligned} \tag{5.31}$$

putting $X = Y = T = \xi$, and substituting into equation (5.27)

$$\begin{aligned}
W_5\big(\xi, Z, R(Y,\xi)U, T\big) =& -\Big(\eta(U)\eta(Z) - \eta(Z)\eta(U)\Big) - \eta(Z)\Big(-\eta(U) + \eta(U)\Big) + \\
& \frac{1}{n-1}\Big[g\Big(\xi, R(Y,\xi)U\Big)Ric(Z,T) - g(Z,T)Ric\Big(\xi, R(Y,\xi)U\Big)\Big] \\
=& \frac{1}{n-1}\Big[(n-1)\eta(Z)\eta\big(R(\xi,\xi)U\big) + (n-1)\eta(Z)\eta\big(R(\xi,\xi)U\big)\Big] + \\
& \frac{1}{n-1}\Big[g\Big(\xi, R(Y,\xi)U\Big)Ric(Z,T) - g(Z,T)Ric\Big(\xi, R(Y,\xi)U\Big)\Big] \\
=& 2\eta(Z)\eta\Big(R(\xi,\xi)U\Big)
\end{aligned} \tag{5.32}$$

but we know, from the definition of Ricci curvature tensor,

$$R(Y,\xi)U = \eta(U)Y - g(Y,U)\xi \tag{5.33}$$

Now, putting $Y=\xi$

$$\begin{aligned}\Rightarrow R(\xi,\xi)U =&\big(\eta(U)\xi-\eta(U)\xi\big)\\ =&0\end{aligned} \tag{5.34}$$

Hence we can easily conclude that, even

$$W_5\big(\xi,Z,R(Y,\xi)U,T\big)=0$$

Computing the seventh term,

$$\begin{aligned}W_5\Big(X,U,R(Y,Z)\xi,T\Big) =&R\Big(X,U,R(Y,Z)\xi,T\Big)+\\ &\frac{1}{n-1}\Big[g\Big(X,R(Y,Z)\xi\Big)Ric(U,T)-g(U,T)Ric\Big(X,R(Y,Z)\xi\Big)\Big]\\ =&g(X,T)g\big(U,R(Y,X)\xi\big)-g(U,T)g\big(X,R(Y,Z)\xi\big)+\\ &\frac{1}{n-1}\Big[g\Big(X,R(Y,Z)\xi\Big)Ric(U,T)-g(U,T)Ric\Big(X,R(Y,Z)\xi\Big)\Big]\\ =&g(X,T)\Big(g\big(U,\eta(Z)Y-\eta(Y)Z\big)\Big)-g(U,T)\Big(g\big(X,\eta(Z)Y-\eta(Y)Z\big)\Big)+\\ &\frac{1}{n-1}\Big[g\Big(X,R(Y,Z)\xi\Big)Ric(U,T)-g(U,T)Ric\Big(X,R(Y,Z)\xi\Big)\Big]\\ =&g(X,T)\Big(\eta(Z)g(U,Y)-\eta(Y)g(U,Z)\Big)-g(U,T)\Big(\eta(Z)g(X,Y)-\eta(Y)g(X,Z)\Big)+\\ &\frac{1}{n-1}\Big[g\Big(X,R(Y,Z)\xi\Big)Ric(U,T)-g(U,T)Ric\Big(X,R(Y,Z)\xi\Big)\Big]\end{aligned} \tag{5.35}$$

Now put $X=Y=T=\xi$, in equation (5.31),

$$\begin{aligned}W_5\Big(X,U,R(Y,Z)\xi,T\Big) =&-\Big(\eta(Z)\eta(U)+g(U,Z)-\eta(U)\Big(-\eta(Z)+\eta(Z)\Big)+\\ &\frac{1}{n-1}\Big[\eta\big(R(\xi,Z)\xi\big)(n-1)\eta(U)-\eta(U)(n-1)\eta\big(R(\xi,Z)\xi\big)\Big]\\ =&-g(U,Z)+\eta(U)\eta(Z)\end{aligned} \tag{5.36}$$

Computing the last term of this equation (5.14) we obtain,

$$
\begin{aligned}
W_5\Big(\xi,U,R(Y,Z)X,T\Big) =& R\Big(\xi,U,R(Y,Z)X,T\Big)+\\
&\frac{1}{n-1}\Big[g\Big(\xi,R(Y,Z)X\Big)Ric(U,T)-g(U,T)Ric\Big(\xi,R(Y,Z)X\Big)\Big]\\
=& g(\xi,T)g\Big(U,R(Y,Z)X\Big)-g(U,T)g\Big(\xi,R(Y,Z)X\Big)+\\
&\frac{1}{n-1}\Big[g\Big(\xi,R(Y,Z)X\Big)Ric(U,T)-g(U,T)Ric\Big(\xi,R(Y,Z)X\Big)\Big]\\
=& \eta(T)\Big\{g\Big(U,g(Z,X)Y-g(Y,X)Z\Big)\Big\}-\\
&g(U,T)\Big\{g\Big(\xi,g(Z,X)Y-g(Y,X)Z\Big)\Big\}+\\
&\frac{1}{n-1}\Big[g\Big(\xi,R(Y,Z)X\Big)Ric(U,T)-g(U,T)Ric\Big(\xi,R(Y,Z)X\Big)\Big]\\
=& \eta(T)\Big(g(Z,X)g(U,Y)-g(Y,X)g(U,Z)\Big)-\\
&g(U,T)\Big(\eta(Y)g(Z,X)-\eta(Z)g(Y,X)\Big)+\\
&\frac{1}{n-1}\Big[g\Big(\xi,R(Y,Z)X\Big)Ric(U,T)-g(U,T)Ric\Big(\xi,R(Y,Z)X\Big)\Big]\\
=& -\Big(\eta(Z)\eta(U)+g(U,Z)\Big)-\eta(U)\Big(-\eta(Z)+\eta(Z)\Big)+\frac{1}{n-1}
\end{aligned}
\tag{5.37}
$$

Now,we see that only the 7^{th} and the 8^{th} terms don't vanish.Summing them up and putting, $X=U=T=\xi$,

$$
-\frac{1}{n-1}\Big[(n-1)\eta(U)g\Big(\xi,R(Y,Z)X\Big)-\eta(U)Ric\Big(\xi,R(Y,Z)X\Big)\Big]=0 \tag{5.38}
$$

Since $\eta(U)=\eta(T)=\eta(\xi)=1$
as we already know that, $\eta(\xi)=-1$
Then,

$$
-(n-1)g\Big(\xi,R(Y,Z)X\Big)+Ric\Big(\xi,R(Y,Z)X\Big)=0 \tag{5.39}
$$

$$
\Rightarrow Ric\Big(\xi,R(Y,Z)X\Big)=(n-1)g\Big(\xi,R(Y,Z)X\Big)
$$

Or as from the definition of Ricci-soliton,

$$
Ric(\xi,U)=(n-1)g(\xi,U) \tag{5.40}
$$

But η-Ricci soliton in LP-Sasakian is given by, (as in equation 5.12)

$$\begin{aligned} S(X,Y) =& Ric(X,Y) \\ =& -g(\varphi X,Y) - \lambda g(X,Y) - \mu\eta(X)\eta(Y) \end{aligned}$$

Putting $X = \xi$

$$\begin{aligned} S(\xi,Y) =& -\lambda\eta(Y) + \mu\eta(Y) \\ =& (\lambda - \mu)\eta(Y) \end{aligned} \tag{5.41}$$

Solving (5.40) and(5.41) simultaneously

$$\Rightarrow \lambda - \mu = n - 1$$

Therefore,
if $\lambda = 1$, then, $\mu = n$
Hence the Theorem. □

Corollary 5.2.6. *If (φ,ξ,η,g) is a Lorentzian Para-Sasakian structure on the Manifold M_n, (g,ξ,λ,μ) is a η-Ricci Soliton on M_n, and if $W_5(\xi,X).R(Y,Z) = 0$, then (M_n,g) is Einstein Manifold.*

6 Discussions, Conclusion and Way-forward

In chapter 4 and chapter 5, we were able to achieve same results when using W_3 and W_5 on η-Ricci soliton on LP-Sasakian manifolds.
That means, the linear combination of W_3 and W_5 will yield the same results for a new tensor, say,

$$\tau(X,Y)Z = C_1W_3 + C_2W_5$$

where C_1 and C_2 are arbitrary constants.
Now, we may think of W_3 and W_5 as the compatible tensors in the sense that we can use either of them to describe the same phenomenon in LP-Sasakian manifold.
The same results could be lifted to other manifolds to study η-Ricci soliton in Sasakian manifolds, in Para-contact manifolds, in Para-kemontsu manifolds and in almost-Kenmotsu manifolds.
The same tensor can be used by physicist to study the pseudo-metrics of the η-Ricci metrics, i.e, physical aspects of W_3 and W_5 curvature tensors. Note that for us we dealt with the geometrical aspects in general theory of relativity.
From the general symmetry of W_3 and W_5, we can study

$$W_3(\xi,X)\cdot W_5(Y,Z)U = 0$$

and

$$W_5(\xi,X)\cdot W_3(Y,Z)U = 0$$

Where X,Y,Z,U $\in \mathfrak{X}(M)$.
to achieve the same results for Ricci solitons with the given LP structure.

6.1 Way Forward

Many Authors have actually studied W_2 η-Ricci soliton in LP-Sasakian, others in......
So, even our study can also be extended to Sasakian, almost Sasakian, with the exemption of k-contact manifolds due to the Lie-Derivative vanishing, i.e, becomes zero, $[L_\xi g = 0]$. This destroys the condition for the η-Ricci soliton. ξ being the killing vector.

6.2 Future Research

The aim of this dissertation was to give a detailed study of the η-Ricci solitons for the given LP-structure on the given LP-Sasakian manifolds satisfying the semi-symmetric conditions. We then focused on W_3 and W_5 models and applied the semi-symmetric conditions and compared the results obtained. In future, we may wish to extend this work to W_8 and even W_9 with higher dimensions and see if those models will yield equally, geometrically similar results even in those higher dimensions. Say dimensions > 3.

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