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Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Mathematical Statistics

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Master Thesis

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Abstract

This project considers optimum stepwise group screening designs .Numerical methods are used to obtain the optimum group sizes.Newton's method as a numerical method , is used as a mathematical tool,statistically, in the minimization of Expected Total number of runs(tests) in the Stepwise group screening design for selecting and separating the defective factors from a population entailing both the defective and non-defective factors(observations).

The minimization of the expected total number of runs is obtained or ascertained when the optimum sizes of the group factors are obtained as required from the performance of the several iterations in Newton's method, for different stages, i.e, stage one,(one group-size, (x)), stage two, (two group sizes,(x, y)) and stage three, (three group-sizes,(x, y, z)).At each particular stage, comparisons are made with the calculus method used in obtaining the optimum group sizes in the procedures used by other researchers.All these two procedures are afterwards confirmed with the results from the computer search.

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Declaration and Approval

I, the undersigned, declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Matilda Awuor Owino
Reg No. I56/11896/2018

Signature　　　　　　　　　　Date

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

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Dedication

This project is dedicated to;

- My daughter, Blessing.
- My dad, Lucas Owino Nyadhing' and my late mum, Roseline Awino Owino.

Contents

Acknowledgments

First and foremost, I express my gratitude to the Almighty Father, God in heaven, for seeing me through this far.I also thank my six year old daughter, who, through thick and thin, has always been so understanding while I am studying and giving me ample time to study.My dad has always been very supportive in terms of advice and believing in me and I am grateful for his contribution.

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Matilda Owino

Nairobi, 2020.

1 Introduction

1.1 Basic concepts in Stepwise group screening design

Group screening design is a rich area of research that researchers have come up with different developments,and they have also solved problems in this area of study. The main objective of group screening design procedures is to minimize the expected total number of runs (tests) with the aim of minimizing the inspection costs.

The Stepwise group-screening designs, consists of the initial step, where the population under investigation (the population of observations/factors) is divided into different groups, depending on the population size, called, the first order group factors .The first order group factors are then tested for their effects or significance in a factorial experiment and, the first order group factors found to be defective are set aside.

In the second step, we start with any of the defective first order group factors and test the factors within it one by one until a defective observation (factor) is obtained and kept a side.The observations found to be non-defective during the testing process are also kept separate. The remaining factors are re-grouped, pooled together, and one test done on the group.This procedure terminates upon the test result obtained being negative, otherwise, the factors within that obtained group are tested one by one until another positive test is obtained from a factor.This procedure of tests is done progressively and the test procedure is terminated after the result obtained from testing a pooled group-factor tests negative, or until a group of size one is obtained.

The above tests procedures are done for all the defective first order group factors detected in the initial step.The test procedures performed in the initial step and in the second step is repeatedly done in the other subsequent steps successively until the analysis is ended with a test on a non-defective group factor or with group factor of size one.
After all these processes have been done, the calculus method is used in the minimization of expected total number of runs where the equation for the expected total number of runs is approximated first, then differentiated once with respect to the group size(s), accordingly, and then equated to zero. The values of the optimum group-sizes obtained after solving the particular equation is fixed into the original equation and then solved.The result obtained is said to be the minimized value of the equation of expected total number

of runs.All these are done for specified values of p, the probability of a factor being defective.

The above procedure is only for one type of search steps.For two type search steps,each of the defective first order group factors found in the initial step undergo another division into another set of group factors called the second order group factors.The second order group factors are tested using the same procedure that was used above to detect the defective factors, (but in this case, factors are replaced with second order group factors).This is done until all the defective second order group factors are obtained.
All the defective factors in the defective second order group factors are obtained using the same procedure used in type one search steps.

In type three search steps, each of the defective second order group factors found above undergo another division into groups called third order group factors.The third order group factors are tested using the same procedure used to obtain the defective second order group factors, only this time, replace second order group factor with the third order group factor.This is done until all the defective third order group factors are obtained .All the defective factors in all the defective third order group factors are also obtained using the same procedure used in type one search steps.

In this study, Newton's method is used as the principal method for obtaining the group sizes in all the different types, type one, type two and type three.It is used as a tool to obtain the optimum group-sizes such that when the obtained group-sizes are optimized, by the method of Newton, their values are placed in the equation of the expected total number of runs and this yields a result.

1.2 Background

The idea of getting observations or factors tested in groups and carrying out tests on individual factors or observations only on the account that a group is tested and found to be positive was first introduced by Robert,[**dorfman1943detection**], when he was approached to determine the best way possible in which defective individuals could be identified from a population where there was less probability of a person being defective or getting infected.
The method he was to use was to identify the individuals, all of them, from the large population, but minimizing the costs for the identification as much as possible.
Dorfman suggested a cost effective way of taking the individuals from a targeted population through testing the blood samples of the volunteers of the army for the purpose of detecting the defective cases that are infected.
Dorfman,suggested that instead of carrying out tests on each individual blood sample, small portions of each of samples of interest would be pooled together and the pooled sample to undergo a test initially.The next step would depend on the result obtained from the pooled group test.If the pooled blood sample was non-defective, then, the individuals whose blood samples had been tested are separated from the others as they were free of infection.However, if the result was obtained to be positive, then all the individuals whose blood sample had been tested on that pooled sample would undergo individual tests until all the positive individuals had been found in that pooled blood sample.On the account that the rate of infection was less, then the expected total costs incurred for the identification of those positive individuals would be lessened, as a result of expected total number of tests being less.
Various authors have modified the initial problem solved by Dorfman and came up with various versions of the group-testing method.

1.3 Literature Review

The concept of tests done on factors in groups as opposed to individually was and carrying out tests on the individuals from a group that hasbeen tested and proven tobe positive, was first introduced by [**dorfman1943detection**], on one infection(syphilis detection among soldiers).The method was to be economical and cost effective in that the blood samples of the army volunteers was proposed to be tested in a group instead of testing individual blood samples where portions of each of the blood samples was to be pooled and tested as a pool first.If the pooled sample was non-defective, then the tests would stop and the volunteers considered free of the infection.However, the remaining portion of every blood samples would undergo tests individually.The expected total number of tests, hence, the overall costs would be reduced if the infection prevalence rate of infection were less.

More studies were done on the work of Dorfman by [**sterrett1957detection**], where, in his case, he suggested that individual factors would be tested one after the other,individually until a defective factor was obtained from the already tested defective group.The rest of the factors in the defective pooled sample were tested again in a pool.If a negative result is obtained, then the pooled sample would be put separate and no further tests done on it.Otherwise, testing individual factors was done until another positive case recorded .The remaining tests were again done ia pool.The process was continued upto a point, all the positive factors were obtained. Sterrett improved Dorfman's work in that, Dorfman's plan was okay for low prevalence rates of a factor being defective and this meant, there were high chances for one defective factor in a pooled sample strong enough to authorize a pooled test in the event that a positive factor has been obtained.The work Sterrett did made it possible to reduce the expected total number of tests got by Dorfman by eight percent for a prevalence rate of five percent.

[**connordevelopments**], was the first researcher to tackle the group-tests challenge from the designs of experiments approach.This was followed afterwards by [**watson1961study**], where the group factors were studied using a Plackett and Burman design in runs.This was after the group factors had been subdivided from the population of factors into group factors.He studied two stage group screening designs with and without errors in observations using equal size groups.Where there were errors in observations, he got expressions for the power of tests in the two stages.He assumed continuous variations in groupsizes, and got the optimum group sizes by minimizing the expected total number of runs (tests) with with respect to the group sizes using ordinary calculus techniques.He also did expressions for the expected number of positive factors declared negative and for the expected number of negative factors declared positive.

[**patel1963note**], emphasized that it is hard to be sure that an increase in α increases , but the lower bound certainly increases. Patel's main intention was to remove some kind of doubt left in the first part of that statement.This, he did by proving that the expected number of positive factors declared positive was not a reducing function of the level of significance in stage one.This ended up eliminating the doubts which Watson had.

[**li1962sequential**], came up with a multi-stage designs for screening experimental variables and got results similar to those got by Patel,[**patel1963note**].

[**hunter1964catalyst**] , made use of the group screening design to do selection of the best catalyst from a list of possible catalysts for the oxidation of methane.They mentioned that, by arranging catalysts of interest for a reaction in a logical groups and carrying out tests of individual group in one run, the less active catalysts could be separated from the rest and the runs, in totality, be decreased.

[**finucan1964blood**], researched on a case of a multistage group screeningdesign without errorsin observations and in which all factors were positive with the same probability of being defective.Fanucan suggested that the method of finite differences would be used in solving for optimizing group-sizes in a two stage groupscreening design.

[**garey1974isolating**] isolated one defective in a finite set of n observations having at least one defective as a procedure of obtaining the optimal group testing.They used a binary testing tree after considering situations when the probability of each observation to be defective as known.

[**patel1984two**], researched more on the two stage group screening designs with equal prior probabilities of items to be defective and with no errors in observations and took a case when there was discontinuous variation in the group sizes of the items.Their method of obtaining the optimum group sizes was the method of finite differences and later compared their results with the case when the assumption of continuous variation in group sizes is used, from Watson's results

[**patel1984optimum**], still, got the optimum two stage groupscreening designs with errorsin observations.They took care of both the expected total number of runs and the expected total number of incorrect decisions.The optimum group sizes were got by performing the minimization of the expected total number of tests for a fixed value of

theexpectedtotal number ofincorrect decisions.
[trocine2000finding], did some findings oneffective and efficient screening of large number of variables.This, she did by developing a simulation model and employing designed experiments to efficiently optimize the system. The firststep is to screen for important independent variables. Several screening methods, like the stepwise group screening designs, one of the classical methods,are compared and contrasted in terms of efficiency, effectiveness and robustness.new, more novel designs including sequential bifurcation (SB) and iterated fractional factorial designs (IFFD) are some methods to be used.

[manene2002multi] did some work in multi type stepwise group screening designs.They showed how the expression for the minimization of the expected total number of runs came about.They further went a head and showed how the optimum group sizes were obtained for the different cases, for two group sizes, and for three group sizes.They used the calculus techniques, to find these optimum group sizes.

[kambo1984mathematical], discussed , in one of his topics, and explained the multi dimensional unconstrained minimization techniques.He explained how minimization could be done using different techniques like, Newton's method.Optimization of the variables of interest could be used and afterwards, minimization done.

[patel1987step], showed in the work how the expression for the expected totalnumber of tests was obtained.They also showed how the optimum group size was optimized using the calculus technique.

[manene2005multi],still in another paper, discussed the optimization of group sizes, but in a case where the factors have unequal probability of being defective.

1.4 Statement of The Problem

The overall main objective of group screening design is to get the minimum expected total number of tests when identifying the defective factors from a population.For the minimum expected total number of tests to be obtained, the optimum group size(s) have to be obtained first, which is used in the expression of the expected total number of tests.
The optimum group size(s) have been obtained by other researchers using different methods including the Calculus method which was used after approximations were done to the original expression of the expected total number of tests.
This created an interest of obtaining the optimum group sizes using a better, and different approach, that would minimize the expected total number of tests using the expression directly without undertaking any approximations.
In Newton's method, the equation of the expected total number of runs or tests is intended to be used as it is, without any approximations, hence, more convenient.

1.5 Methodology

Main Objective

The main objective of this study is to Improve on the optimization of group sizes for the Minimization of expected total number of runs(tests or observations)

Specific Objective

(i) To obtain the optimal group size(s) using Newton's method.

(ii) To minimize the expected total number of runs,tests or observations using the Newton's method.

(iii) To Compare the results obtained by the Calculus method with the Newton's method in tables.

In this study, the main objective is to improve on the optimization of the group size(s),in order to minimize the expected total number of runs, tests or observations.One of the most appropriate numerical method that would be recommended in this case is the use of Newton's method, from the Numerical Analysis book, [**kambo1984mathematical**],as one of tools for optimization of the group size(s), hence minimization of the expected total number of runs.
The Newton's method is chosen as the equation is also non-linear.

There are some iterative methods for solving the equation,(problem),where by , an iterative method, meaning , a systematic procedure for obtaining a sequence of successive approximations, $x_1, x_2, x_3, \ldots$ to a solution to Unconstrained Minimization Problem, the approximations being generated by the recurrence relation

$$x_{(k+1)} = x_k + \alpha_k \mathbf{d}_k \qquad (k = 1, 2, \ldots)$$

Some of the algorithms used in finding the solution to the Unconstrained Minimization Problem, (UMP) are;-

1. Steepest descent Method
2. Quasi-Newtons Method, also, the Newton's method
3. Davidon-Fletcher-Powell Method

In this study, the focus is on the Newton's method as one of the algorithms for solving the Unconstrained Minimization Challenges. Use of the Newton's method,as a tool, is one of the numerical methods that can be used instead of the normal calculus way of differentiation.

Detailed description of Newton's method;

The Newton's method is taken as a tool that uses the search direction $\mathbf{d}_k$ at x_k which is determined by getting the minimization of Taylor's linear approximation to $f^{'}(x)$ about x_j.That is, if

$$f(x) \approxeq f(x_j) + (x - x_j)^{\mathbf{T}} \triangledown f(x_k) = f(x_j) + \mathbf{d}^{\top} \triangledown f(x_j), \tag{1}$$

then $\mathbf{d_j} = -\triangledown f(x_j)$ is the solution to the problem:

$$\textbf{Minimize}\mathbf{d}^{\top} \triangledown f(x_j)$$

subject to

$$\mathbf{d}^{\top}\mathbf{d} = 1.$$

Under the examination of of Taylor's quadratic approximation to $f(x)$ about x_j.This approximation is given by

$$f(x) \approxeq y(x) = f(x_j) + (x - x_j)^{\mathbf{T}} \triangledown f(x_j) + \frac{1}{2}(x - x_j)^{\mathbf{T}} \mathbf{H}(\mathbf{x_j})(x - x_j), \tag{2}$$

where $H(x_j)$ is the **Hessian Matrix** of the function evaluated at x_j.Suppose $y(x)$ has a minimum at x_{k+1}.Then $\triangledown y(x_{k+1}) = 0$ yields

$$\triangledown f(x_j) + H(x_j)(x_{j+1} - x_j) = 0 \tag{3}$$

which implies

$$x_{j+1} = x_j - \mathbf{H}^{-1}(\mathbf{x_j}) \triangledown f(x_j) \tag{4}$$

provided the matrix $H(x_j)$ is non- singular.Equation (4) yields Newton's Iterative method for $j = 1, 2, 3, \ldots$.In this case, the direction of search (called Newton's direction), at x_j is $\mathbf{d_j} = -\mathbf{H}^{-1}(\mathbf{x_j}) \triangledown f(x_j)$ which is a descent direction at x_j if and only if,

$$-\triangledown f(x_j)^{\top} \mathbf{H}^{-1}(\mathbf{x_j}) \triangledown f(x_j) < 0$$

i.e,

$$\triangledown f(x_j)^{\top} \mathbf{H}^{-1}(\mathbf{x_j}) \triangledown f(x_j) > 0 \tag{5}$$

The equation 5 is without a doubt satisfied for all points x_j for which $\nabla f(x_j) \neq 0$ and $\mathbf{H}(\mathbf{x_j})$ is a positive definite matrix, $[y(x)$ has the minimum value at x_{j+1} under this condition].It is worth noting that the step length $\alpha_{\mathbf{j}}$ in equation 4 is 1.Equation (4) can modified to get the iterative process.

$$x_{j+1} = x_j - \alpha_{\mathbf{j}} \mathbf{H}^{-\mathbf{1}}(\mathbf{x_j}) \nabla f(x_j), \qquad j = 1, 2, 3, \ldots, \tag{6}$$

where $\alpha_{\mathbf{j}}$ is chosen such that $f(x_{j+1}) < f(x_j)$.If $\alpha_{\mathbf{j}} = \alpha$ is fixed, then the limited step Newton method is obtained.Occasionally,in some situations, $\alpha_{\mathbf{j}}$ may be chosen to be the value $\alpha_{\mathbf{j}}{}^*$ which minimizes $f(x_j - \alpha_{\mathbf{j}} \mathbf{H}^{-\mathbf{1}}(\mathbf{x_j}) \nabla f(x_j))$.The optimal choice, $\alpha_{\mathbf{j}} = \alpha_{\mathbf{j}}{}^*$, results into Optimal step Newton method.

Suppose Newton's method is applied to minimize the quadratic function;

$$f(x) = a + c^{\top} x + \frac{1}{2} x^{\top} \mathbf{G_x}, \tag{7}$$

where $\mathbf{G}$ is an $n \times n$ positive definite symmetric matrix.Hence , here , the Hessian matrix of the function is $\mathbf{G}$.Then starting with any $x_1 \in \mathbf{R^n}$, from equation (1.4)

$$x_2 = x_1 - G^{-1} \nabla f(x_1) = x_1 - G^{-1}(c + Gx_1) = -\mathbf{G^{-1}c}$$

Note that is to be taken is that x_2 is the minimum of the function since $\nabla f(x) = 0$ yields $x = -G^{-1}c = x_2$ and the matrix $\mathbf{G}$ is the positive definite.Hence, the Newton's method has the quadratic termination property and it converges to minimum of a positive definite quadratic function in just one step.This implies that an expectation of a faster convergence is hopefully realized when the Newton's method is applied to the general functions.Nevertheless, the sequence $\left\{x_k\right\}$ of Newton's iterates may converge to a saddle point or a maximum of the function since the equation 5, which is a condition, may not be satisfied at each iteration.Also, The algorithm needs more calculations for the function and it's derivative evaluation and also the matrix invasion process.The calculations involved take a lot of time and are resource consuming when done manually hence , in practice, the common computer programming software, called the R programming language is to be used to generate solutions to the equation 5.

Some of the commonly used ways of termination of the newton's method iterations are;-, one stops the iteration processes $x_1, x_2, x_3, \ldots,$ on an instant that,

$$\left| f(x_{j+1} - f(x_j)) \right| < \epsilon_1 \tag{8}$$

or

$$\left| x_{j+1} - x_j \right| < \epsilon_2 \tag{9}$$

or

$$\left|\nabla f(x_j)\right| < \epsilon_3 \tag{10}$$

where $\epsilon_i > 0$ are previously provided or known numerics. An interpretation for Newton's iterate x_{j+1} is as below,
If the function that one is dealing with is differentiable, then both the maxima and the minima are the solutions to the system

$$\nabla f(x_j) = 0. \tag{11}$$

Hence, the Newton's method in optimization of the function is got when the system, obtained by applying the Taylor's linear approximation to the components of $\nabla f(x_j)$ about x_j, is worked on.The obtained working is the next point x_{j+1} given by equation5.This implies that x_{j+1} is an approximation to a stationary point of the function.

A convergence theorem has (ref.Ortega 1972, for proof)to be stated which help in showing that Newton's method has quadratic convergence under some occasional conditions.

Theorem 1.5.1. *Under the assumptions that the function is differentiable twice-continuously, each at point of neighborhood of a stationary point x^* of the function and that the Hessian matrix $\mathbf{H}(\mathbf{x}^*)$ of the function evaluated at x^* is nonsingular.Then, there exists a neighborhood $\mathbf{N}_\delta(\mathbf{x}^*)$ of x^* such that, for each $x \in \mathbf{N}_\delta(\mathbf{x}^*)$, the sequence of Newton's iterates $\left|x_j\right|$ obtained from equations (5) converges to x^* and the convergence is super linear because*

$$\lim_{j\to\infty} \frac{\left|x_{j+1} - x^*\right|}{\left|x_j - x^*\right|} = 0 \tag{12}$$

Nevertheless, if the function is differentiable thrice in some neighborhood of x^*, then there is a positive integer, j_1 that exists depending on x_1 and a constant $c < +\infty$ such that

$$\left|x_{j+1} - x^*\right| \le c\left|x_j - x^*\right|^2 \quad for \quad all j \ge j_1. \tag{13}$$

Expected number of Tests

The equation for the expected total number of runs or number of tests for two group sizes. This is one of the equations that need to be minimized.This is basically done using the Newtons method by getting the most suitable values of the two variables, x and y.

$$E(B) = 1 + f + fp + \frac{2fq^y}{x} - \frac{f}{x}\left(1-q^y\right)^{-1}\left(1-q^{x+y}\right) + \frac{f}{y} - \frac{fq^y}{y} + \frac{2fq}{y} - \frac{f}{yp}\left(1-q^{y+1}\right) \tag{14}$$

where p is the probability of a factor or an observation being defective, and q,being, $(q = 1 - p)$, and x and y which are the group sizes.

1.6 Materials and Methods

These are mainly two:

(a) Newton's method of Minimization and

(b) The use of the R-programming language in the implementation of the Newton's method.

Starting off with the use of the R-programming language;

1.6.1 The Use of R-programming language to implement the Newtons method

The use of the R-programming language in this study is really important in the production of the results of the Newton's method, where one gets to undertake both the first and second differentiations of the equation, get the answers to the differentials for each particular case and proceeds to finding the points after a series of iterations to finally put in the equation to be minimized.
All these is made possible and efficient using the R-programming language.

1.6.2 Newton's method of minimization

The method used basically in this study is Newton's method and the description of the method of how it works to produce the results in the chapter of results is as clearly illustrated below.

The equation (14) above can be simplified further into the below equations. The first and second derivatives with respect to,x, the group size in the first order group factor.

$$E(B) = 1 + f + fp + \frac{2fq^y}{x} - \frac{f}{x}\frac{\left(1-q^{x+y}\right)}{\left(1-q^y\right)} + \frac{f}{y} - \frac{fq^y}{y} + \frac{2fq}{y} - \frac{f}{yp}\left(1-q^{y+1}\right)$$

$$\mathbf{a} = \frac{\partial E(B)}{\partial x} = \frac{-2fq^y}{x^2} + \frac{f}{x^2\left(1-q^y\right)} + \frac{fq^{x+y}\ln q}{x\left(1-q^y\right)} - \frac{fq^{x+y}}{x^2\left(1-q^y\right)} \tag{15}$$

$$\mathbf{b} = \frac{\partial^2 E(B)}{\partial x^2} = \frac{-2(-2)fq^y}{x^3} + \frac{-2f}{x^3\left(1-q^y\right)} + \frac{fq^y \ln q}{\left(1-q^y\right)}\left[\frac{q^x \ln q}{x} - \frac{q^x}{x^2}\right] - \frac{fq^y}{\left(1-q^y\right)}\left[\frac{q^x \ln q}{x^2} - \frac{2q^x}{x^3}\right]$$

$$\mathbf{b} = \frac{\partial^2 E(B)}{\partial x^2} = \frac{4fq^y}{x^3} - \frac{2f}{x^3\left(1-q^y\right)} + \frac{fq^{x+y}\left(\ln q\right)^2}{x\left(1-q^y\right)} - \frac{fq^{x+y}\ln q}{x^2\left(1-q^y\right)} - \frac{fq^{x+y}\ln q}{x^2\left(1-q^y\right)} + \frac{2fq^{x+y}}{x^3\left(1-q^y\right)}$$

$$\mathbf{b} = \frac{\partial^2 E(B)}{\partial x^2} = \frac{4fq^y}{x^3} - \frac{2f}{x^3\left(1-q^y\right)} + \frac{fq^{x+y}\left(\ln q\right)^2}{x\left(1-q^y\right)} - \frac{2fq^{x+y}\ln q}{x^2\left(1-q^y\right)} + \frac{2fq^{x+y}}{x^3\left(1-q^y\right)} \tag{16}$$

The first and second derivatives with respect to, y, the group size in the second order group-factor.

$$\mathbf{c} = \frac{\partial E(B)}{\partial y} = \frac{2fq^y \ln q}{x} - \frac{f}{x}\left[\frac{q^y \ln q}{\left(1-q^y\right)^2}\right] + \frac{fq^x}{x}\left[\frac{q^y \ln q}{\left(1-q^y\right)^2}\right] - \frac{f}{y^2} - f\left[\frac{q^y \ln q}{y} - \frac{q^y}{y^2}\right]$$
$$- \frac{2fq}{y^2} + \frac{f}{y^2 p} + \frac{fq}{p}\left[\frac{q^y \ln q}{y} - \frac{q^y}{y^2}\right]$$

$$\mathbf{c} = \frac{\partial E(B)}{\partial y} = \frac{2fq^y \ln q}{x} - \frac{fq^y \ln q}{x\left(1-q^y\right)^2} + \frac{fq^{x+y}\ln q}{x\left(1-q^y\right)^2} - \frac{f}{y^2} - \frac{fq^y \ln q}{y}$$

$$+ \frac{fq^y}{y^2} - \frac{2fq}{y^2} + \frac{f}{y^2 p} + \frac{fq^{y+1}\ln q}{yp} - \frac{fq^{y+1}}{y^2 p} \tag{17}$$

$$\mathbf{d} = \frac{\partial^2 E(B)}{\partial y^2} = \frac{2fq^y\left(\ln q\right)^2}{x} - \frac{f \ln q}{x}\left[\frac{q^y \ln q}{\left[\left(1-q^y\right)^2\right]^2} - \frac{q^{3y}\ln q}{\left[\left(1-q^y\right)^2\right]^2}\right]$$
$$+ \frac{fq^x \ln q}{x}\left[\frac{q^y \ln q}{\left[\left(1-q^y\right)^2\right]^2} - \frac{q^{3y}\ln q}{\left[\left(1-q^y\right)^2\right]^2}\right] + \frac{2f}{y^3} - f\ln q\left[\frac{q^y \ln q}{y} - \frac{q^y}{y^2}\right] +$$
$$f\left[\frac{q^y \ln q}{y^2} - \frac{2q^y}{y^3}\right] + \frac{4fq}{y^3} - \frac{2f}{y^3 p} + \frac{fq\ln q}{p}\left[\frac{q^y \ln q}{y^2} - \frac{2q^y}{y^3}\right] - \frac{fq}{p}\left[\frac{q^y \ln q}{y^2} - \frac{2q^y}{y^3}\right]$$

$$\mathbf{d} = \frac{\partial^2 E(B)}{\partial y^2} = \frac{2fq^y\left(\ln q\right)^2}{x} - \frac{fq^y\left(\ln q\right)^2}{x\left[\left(1-q^y\right)^2\right]^2} + \frac{fq^{3y}\left(\ln q\right)^2}{x\left[\left(1-q^y\right)^2\right]^2} + \frac{fq^{x+y}\left(\ln q\right)^2}{x\left[\left(1-q^y\right)^2\right]^2}$$

$$- \frac{fq^{x+3y}\left(\ln q\right)^2}{x\left[\left(1-q^y\right)^2\right]^2} + \frac{2f}{y^3} - \frac{fq^y\left(\ln q\right)^2}{y} + \frac{fq^y \ln q}{y^2} + \frac{fq^y \ln q}{y^2} - \frac{2fq^y}{y^3} + \frac{4fq}{y^3}$$

$$- \frac{2f}{y^3 p} + \frac{fq^{y+1}\left(\ln q\right)^2}{y^2 p} - \frac{2fq^{y+1}\ln q}{y^3 p} - \frac{fq^{y+1}\ln q}{y^2 p} + \frac{2fq^{y+1}}{y^3 p} \tag{18}$$

Getting the second derivatives of the first derivatives of the function with respect to the different variables, i.e;

Theorem 1.6.1.

$$\mathbf{e} = \frac{\partial^2 E(B)}{\partial x \partial y} = \frac{\partial^2 E(R)}{\partial y \partial x} \tag{19}$$

PROOF. Equation (15) can be simplified into the following;

$$\frac{\partial E(B)}{\partial x} = \frac{-2f}{x^2}\left[q^y\right] + \frac{f}{x^2}\left[\frac{1}{(1-q^y)}\right] + \frac{fq^x \ln q}{x}\left[\frac{q^y}{(1-q^y)}\right] - \frac{fq^x}{x^2}\left[\frac{q^y}{(1-q^y)}\right]$$

$$\mathbf{e} = \frac{\partial^2 E(B)}{\partial x \partial y} = \frac{f}{x^2}\left[\frac{q^y \ln q}{\left(1-q^y\right)^2}\right] + \frac{fq^x \ln q}{x}\left[\frac{q^y \ln q}{\left(1-q^y\right)^2}\right] - \frac{fq^x}{x^2}\left[\frac{q^y \ln q}{\left(1-q^y\right)^2}\right] - \frac{2fq^y \ln q}{x^2}$$

$$= \frac{fq^y \ln q}{x^2\left(1-q^y\right)^2} + \frac{fq^{x+y}\left(\ln q\right)^2}{x\left(1-q^y\right)^2} - \frac{fq^{x+y}\ln q}{x^2\left(1-q^y\right)^2} - \frac{2fq^y \ln q}{x^2}$$

$$\mathbf{e} = \frac{\partial^2 E(B)}{\partial x \partial y} = \frac{-2fq^y \ln q}{x^2} + \frac{fq^y \ln q}{x^2\left(1-q^y\right)^2} + \frac{fq^{x+y}\left(\ln q\right)^2}{x\left(1-q^y\right)^2} - \frac{fq^{x+y}\ln q}{x^2\left(1-q^y\right)^2} \tag{20}$$

Getting the same answer after differentiation, using the second approach, equation above can be simplified further, as,

$$\frac{\partial E(B)}{\partial y} = 2f \ln q q^y \left[\frac{1}{x}\right] - \frac{fq^y \ln q}{\left(1-q^y\right)^2}\left[\frac{1}{x}\right] + \frac{fq^y \ln q}{\left(1-q^y\right)^2}\left[\frac{q^x}{x}\right] - \frac{f}{y^2} - \frac{fq^y \ln q}{y} + \frac{fq^y}{y^2}$$

$$- \frac{2fq}{y^2} + \frac{f}{y^2 p} + \frac{fq^{y+1}\ln q}{yp} - \frac{fq^{y+1}}{y^2 p}$$

The equation now becomes;-

$$\mathbf{e} = \frac{\partial^2 E(B)}{\partial y \partial x} = \frac{-2f\ln q q^y}{x^2} + \frac{fq^y \ln q}{x^2\left(1-q^y\right)^2} + \frac{fq^y \ln q}{\left(1-q^y\right)^2}\left[\frac{q^x \ln q}{x} - \frac{q^x}{x^2}\right]$$

$$\mathbf{e} = \frac{\partial^2 E(B)}{\partial y \partial x} = \frac{-2f\ln q q^y}{x^2} + \frac{fq^y \ln q}{x^2\left(1-q^y\right)^2} + \frac{fq^{x+y}\left(\ln q\right)^2}{x\left(1-q^y\right)^2} - \frac{fq^{x+y}\ln q}{x^2\left(1-q^y\right)^2} \tag{21}$$

Hence this completes the proof of the above theorem , as it is clearly proven that the equations, are one and the same.

$$\mathrm{e} = \frac{\partial^2 E(B)}{\partial x \partial y} = \frac{\partial^2 E(B)}{\partial y \partial x}$$

□

Implementation of the Newton's method to this case study

Use of the Newton's method to minimize the function, in equation from the derivations, the function is given as,

$$E(B) = 1 + f + fp + \frac{2fq^y}{x} - \frac{f}{x}\frac{\left(1-q^{x+y}\right)}{\left(1-q^y\right)} + \frac{f}{y} - \frac{fq^y}{y} + \frac{2fq}{y} - \frac{f}{yp}\left(1-q^{y+1}\right)$$

from the above equation, the following were obtained as illustrated previously in equations (15), (16), (17), (18) and (19).All these equations are needed for the minimization of the expression in equation (14)

$$\mathbf{a} = \frac{\partial E(B)}{\partial x} = \frac{-2fq^y}{x^2} + \frac{f}{x^2\left(1-q^y\right)} + \frac{fq^{x+y}\ln q}{x\left(1-q^y\right)} - \frac{fq^{x+y}}{x^2\left(1-q^y\right)}$$

,

$$\mathbf{b} = \frac{\partial^2 E(B)}{\partial x^2} = \frac{4fq^y}{x^3} - \frac{2f}{x^3\left(1-q^y\right)} + \frac{fq^{x+y}\left(\ln q\right)^2}{x\left(1-q^y\right)} - \frac{fq^{x+y}\ln q}{x^2\left(1-q^y\right)} - \frac{fq^{x+y}\ln q}{x^2\left(1-q^y\right)} + \frac{2fq^{x+y}}{x^3\left(1-q^y\right)}$$

$$\mathbf{c} = \frac{\partial E(B)}{\partial y} = \frac{2fq^y \ln q}{x} - \frac{fq^y \ln q}{x\left(1-q^y\right)^2} + \frac{fq^{x+y}\ln q}{x\left(1-q^y\right)^2} - \frac{f}{y^2} - \frac{fq^y \ln q}{y}$$

$$+\frac{fq^{y}}{y^{2}}-\frac{2fq}{y^{2}}+\frac{f}{y^{2}p}+\frac{fq^{y+1}\ln q}{yp}-\frac{fq^{y+1}}{y^{2}p}$$

$$\mathbf{d}=\frac{\partial^{2}E(B)}{\partial y^{2}}=\frac{2fq^{y}\left(\ln q\right)^{2}}{x}-\frac{fq^{y}\left(\ln q\right)^{2}}{x\left[\left(1-q^{y}\right)^{2}\right]^{2}}+\frac{fq^{3y}\left(\ln q\right)^{2}}{x\left[\left(1-q^{y}\right)^{2}\right]^{2}}+\frac{fq^{x+y}\left(\ln q\right)^{2}}{x\left[\left(1-q^{y}\right)^{2}\right]^{2}}$$

$$-\frac{fq^{x+3y}\left(\ln q\right)^{2}}{x\left[\left(1-q^{y}\right)^{2}\right]^{2}}+\frac{2f}{y^{3}}-\frac{fq^{y}\left(\ln q\right)^{2}}{y}+\frac{fq^{y}\ln q}{y^{2}}+\frac{fq^{y}\ln q}{y^{2}}-\frac{2fq^{y}}{y^{3}}+\frac{4fq}{y^{3}}$$

$$-\frac{2f}{y^{3}p}+\frac{fq^{y+1}\left(\ln q\right)^{2}}{yp}-\frac{fq^{y+1}\ln q}{y^{2}p}-\frac{fq^{y+1}\ln q}{y^{2}p}+\frac{2fq^{y+1}}{y^{3}p}$$

$$\mathbf{e}=\frac{\partial^{2}E(B)}{\partial y\partial x}=\frac{-2f\ln q q^{y}}{x^{2}}+\frac{fq^{y}\ln q}{x^{2}\left(1-q^{y}\right)^{2}}+\frac{fq^{x+y}\left(\ln q\right)^{2}}{x\left(1-q^{y}\right)^{2}}-\frac{fq^{x+y}\ln q}{x^{2}\left(1-q^{y}\right)^{2}}$$

With the above findings,the minimization process is as below in the part of results.

2 The group size from Newton's method in One-type stepwise group screening design without errors

In this chapter, the focus is on the involvement of the Newton's method to obtain, x, only one optimum groupsize, in such a way that the value of the groupsize obtained,is fixed into the equation for the minimization of the expected total number of runs. First, the derivation of the expression for the total expected number of runs is to be done.
Quoting the work done by [**manene2002multi**], If one let f be the number of factors under investigation, the problem statement is to get only the defective factors with the least or minimum number of runs (observations).Working with this objective in mind, first divide the f factors into g first order group factors in step one.If each group factor has x factors, then

$$f = xg \tag{22}$$

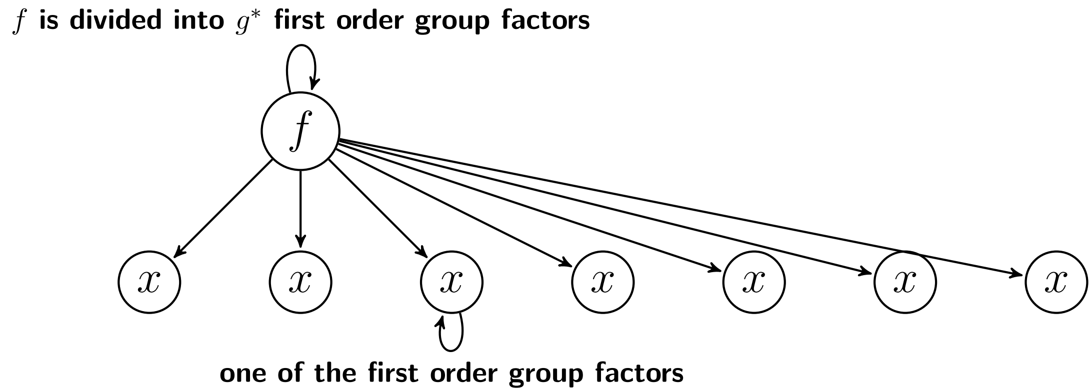
The group-factors are then tested for their significance by a factorial experiment consisting of $(g+1)$ observations.The ones found to be non-defective are set aside.In the second step, any defective first order group factor is started with and the factors within it are tested one by one till a defective factor is obtained.The factors found to be non-defective are set aside, while keeping the defective factor separate.The factors that are remaining are grouped into one group factor.In the third step, the group-factor obtained is tested after the second step is performed.If the group factor is non-defective, the test procedure is terminated.If at all the whole group factor is defective, step four is proceeded with.In the fourth step, factors within the first order group factor found to be defective in the third step are tested one by one, till a defective factor is found.Factors found to be non-defective are once more set aside keeping the defective factor separate.The remaining factors are grouped into a group-factor.In the fifth step, the group factor obtained in step four is tested.The test procedure is repeatedly done until the analysis is terminated when a group factor is tested negative,i.e, when a group factor is non-defective. The procedure surely terminates in a finite number of steps.If the probability of a factor to be defective is small, the probability of exactly one defective factor of a defective group factor is high enough to warrant a group analysis once a defective factor is obtained.Steps two onwards are carried out for all the first order group factors found to be defective in step one.This procedure plainly differs from the procedure first introduced by [**sterrett1957detection**] in that in the first step, the g first order group-factors are tested in a factorial experiment with $(g+1)$ runs.

Alternatively, if at all the control run used in the first step is used in the subsequent steps, then the steps two onwards could be performed in a series of experiments as follows:- In step two, one factor is taken from each first order group factors found to be defective in step one.The factors are then tested for their effects by an experiment.If no defective factor is obtained, another set of factors is taken one from each group-factor and test their significance.This procedure is repeated till at least one defective factor is observed.The non-defective factors are set aside, keeping the defective factor or factors, separate.The remaining factors from a group-factor that contained a defective factor are set aside and grouped into one group-factor.This process is constantly repeated till one defective factor from each group factor found to be defective in the initial step has been isolated.

In step three, the group factors set aside in step two are tested in an experiment using the control test used in step one.Again, the group factors found to be non-defective in the third step are set aside. In step four, a series of experiments are proceeded with as in the second step until one defective factor is isolated from each of the group factors found to be defective in the third step.The remaining factors from each of the group factors found to be defective in the third step are grouped into one group factor after the third step is performed.Again, the group factors set aside in the fourth step are tested in an experiment in the fifth step.This procedure is repeatedly done until the analysis terminates with all non-defective group-factors when all the defective factors have been isolated.Both these test procedures are equivalent when there are no errors in observations, but when errors in the observations are allowed, it is convenient to use an alternative procedure to derive theoretical results.In a simple way, the test procedure consists of testing the group-factors and the factors within the group factors found to be defective, one by one until a defective factor is detected by several steps alternatively.

2.1 Illustration of test in Initial step, one type stepwise search types

Figure 1. Division into g^* first order group factors



$$f = x \times g^*$$

The effects of the first order group factors tested in a factorial experiment, the non-defective first order group factors set aside.
Take a defective group factor, say, $x = 10$, having two defective factors within it.

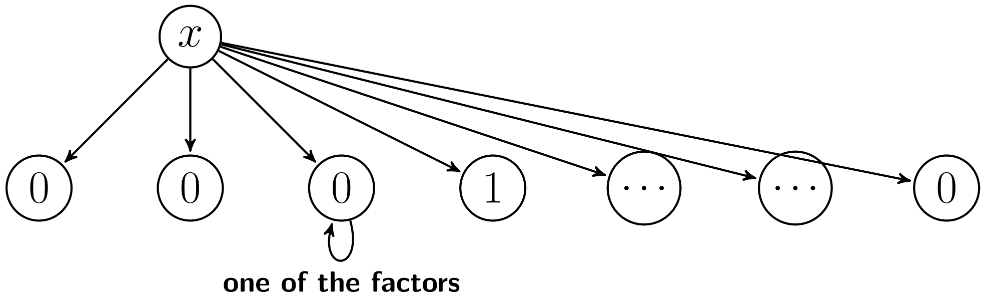
$$0 \quad 0 \quad 0 \quad 1 \,\big|\, 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0$$

With the above arrangement, tests are done to individual factors one by one until the first defective is found, here,it is the fourth factor.The first three are kept separate, and the first defective factor put away.
The remaining six factors are pooled together into a group, one test carried out on them.If the test result is non-defective, the procedure is terminated.If the result is defective, test all the six factors one by one, until another defective factor is obtained, which is the 7^{th} one in this case.The remaining three factors are grouped together again and tested in a pooled sample.The test result becomes negative and the procedure is terminated, Otherwise, if the result was defective, the process was continued with until a test is administered to a non-defective group factor or a test on a pooled group of size one.The below is an illustration of the same.

2.1.1 Step one type one search steps

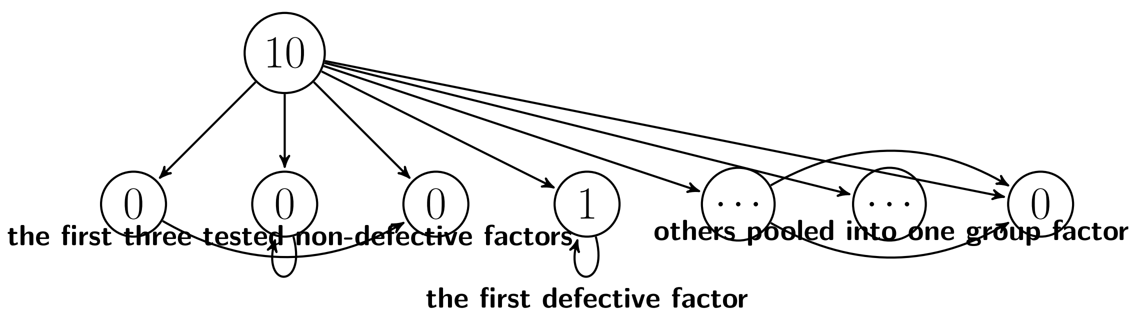
Any defective first order group-factor taken randomly and the factors within it tested one by one until a defective factor obtained, i.e,

Figure 2. Arrangement of factors within a defective first order group factor of size, $x = 10$



2.1.2 Step two type one search steps

Figure 3. Tests on factors within a defective first order group factor, $x = 10$



The first three factor found to be defective, kept separate.The fourth factor, tested to be the first positive factor. The remaining factors, after the fourth, found defective factor, the others pooled together into one group factor and tested for their effects.
The test procedures carried out in steps one and two of type one search steps, successively redone in the subsequent type one search steps until the analysis is concluded with a negative result on a group factor or with analysis on a group factor of size one.
Testing using type one search steps performed on all first order group factors found to be defective in the initial step.This is for a case of one group size, x.

2.2 The expected number of runs

By letting p be the probability that a factor is defective, a group factor is defective if it contains at least defective factor.Let p' be the probability that a group-factor in the first step is defective.If d is the number of defective factors in such a group, then

$$p' = \sum_{d=1}^{x} \binom{x}{d} p^d q^{x-d}$$

$$= 1 - q^x \tag{23}$$

where

$$q = 1 - p \tag{24}$$

In the initial step, all the g group-factors are tested for significance. Thus the number of runs(tests) needed in the initial step is given by,

$$B_1 = g + 1 \tag{25}$$

where the one extra test is the control test.This control test is used as a control test for the subsequent steps.By letting b be the number of defective group-factors in the first step, the probability distribution of b is given by

$$f(b) = \begin{cases} \binom{g}{b} (p')^b (1-p')^{g-b} & b = 0, 1, 2, \ldots, g \\ 0 & \text{Otherwise} \end{cases} \tag{26}$$

Thus,

$$E(b) = gp'$$

$$= \frac{f}{x}(1 - q^x) \tag{27}$$

In the subsequent steps, the analysis of the b group-factors found to be defective in the initial step is continued as described in the earlier part of this chapter .By letting $P_x(d)$ to denote the probability that a group-factor of size x contains exactly d defective factors if it is known to contain at least one defective factor, then,

$$P_x(d) = (1-q^x)^{-1}\binom{x}{d}p^d(1-p)^{x-d} \qquad d=1,2,\ldots,x \tag{28}$$

By letting $E_x(B_d)$ be the expected number of runs needed to analyze a group-factor, i.e, classifying as defective or non-defective all the factors all the factors within a group-factor of size x which is known to be defective if it contains exactly d defective factors.In order to obtain the expression for $E_x(B_d)$, considering a sequence of lemmas is started.

Lemma 2.2.1.

$$E_x(B_1) = \frac{x}{2}+1+\frac{1}{2}-\frac{2}{x}$$

Proof. Since it is known that it is equally likely that the defective factor is found at any trial.Consequently, the probability that it is found on any one trial is $\frac{1}{x}$.Given that the defective factor is found at the h^{th} trial; $h=1,2,\ldots,x-1$, then $h-$tests or runs are required to obtain it.The other test needed is the group test on one group-factor consisting of $(x-1)$ factors.This group-factor is non-defective if $d=1$.If the first $(x-1)$ factors tested are non-defective,then the x^{th} factor is considered to be the defective one.This x^{th} factor need not to be tested since the initial group-factor of size x is known to contain at least one defective factor.Thus

$$E_x(B_1) = \frac{1}{x}\sum_{h=1}^{x-1}(h+1)+\frac{1}{x}(x-1) \tag{29}$$

Upon simplification of equation (29),sequentially,

Since it is summation,the summation part of the equation is solved arithmetically as follows;
From arithmetic series, it is known that;

$$S_{x-1} = \frac{n}{2}(a+l) = \frac{x-1}{2}(a+l)$$

where a is the first term of the series, l is the last term of the series, both in arithmetic series, and n is total number of terms specifically in the arithmetic series.In this study, the total number of terms is $x-1$, hence, with the above reasoning,

$$
\begin{aligned}
a &= (h+1)\\
\text{but initially}\quad h &= 1\\
&\text{hence}\\
a &= (1+1)\\
a &= 2\\
&\text{Also,}\\
l &= ((x-1)+1)\\
l &= (x+(1-1))\\
l &= x\\
&\text{this then simplifies to}\\
\sum_{h=1}^{x-1}(h+1) &= \frac{x-1}{2}(a+l)\\
\sum_{h=1}^{x-1}(h+1) &= \frac{x-1}{2}(2+x)\\
\sum_{h=1}^{x-1}(h+1) &= \frac{(x-1)(2+x)}{2}\\
\sum_{h=1}^{x-1}(h+1) &= \frac{(x-1)(x+2)}{2}
\end{aligned}
$$

hence, going back to equation (28), it becomes,

$$\begin{aligned}
E_x(B_1) &= \frac{1}{x}\sum_{h=1}^{x-1}(h+1)+\frac{1}{x}(x-1)\\
E_x(B_1) &= \frac{1}{x}\frac{(x-1)(x+2)}{2}+\frac{1}{x}(x-1)\\
E_x(B_1) &= \frac{1}{x}\left\{\frac{(x-1)(x+2)+2(x-1)}{2}\right\}\\
E_x(B_1) &= \frac{1}{x}\left\{\frac{(x-1)(x+2)+2(x-1)}{2}\right\}\\
E_x(B_1) &= \frac{1}{x}\left\{\frac{(x-1)\Big((x+2)+2\Big)}{2}\right\}\\
E_x(B_1) &= \frac{1}{x}\left\{\frac{(x-1)(x+4)}{2}\right\}\\
E_x(B_1) &= \frac{1}{x}\left\{\frac{(x-1)(x+4)}{2}\right\}\\
E_x(B_1) &= \frac{1}{x}\left\{\frac{x(x+4)-1(x+4)}{2}\right\}\\
E_x(B_1) &= \frac{1}{x}\left\{\frac{x^2+4x-x-4}{2}\right\}\\
E_x(B_1) &= \frac{1}{x}\left\{\frac{x^2+3x-4}{2}\right\}\\
E_x(B_1) &= \left(\frac{x^2+3x-4}{2x}\right)\\
E_x(B_1) &= \frac{x^2}{2x}+\frac{3x}{2x}-\frac{4}{2x}\\
E_x(B_1) &= \frac{x}{2}+\frac{3}{2}-\frac{2}{x}
\end{aligned}$$

$$E_x(B_1) = \frac{x}{2}+1+\frac{1}{2}-\frac{2}{x} \tag{30}$$

This completes the proof of the lemma. □

Lemma 2.2.2.

$$E_x(B_2)=\frac{2x}{3}+2+\frac{2}{3}-\frac{4}{x}$$

PROOF. In the above case, the intension was to obtain the very first defective factor and thus reduce the problem to the case in which the group-factor has one defective factor only.The situation of a group-factor having only one defective factor previously considered in the above lemma 2.2.1.

The probability that the first factor that was tested was positive or defective was $\frac{2}{x}$.If the first factor tested was defective, then, averagely, $\Big\{1+1+E_{x-1}(B_1)\Big\}$ tests were

needed to complete the test procedure.In a case where h is the position of the first case of detection of the first defective case, hence $h = 1,2,3,\dots,x-2$, the probability that the $(h+1)^{st}$ factor tested is the first defective factor to be found was gotten to be $\prod_{v=1}^{h}\left(\frac{x-(v+1)}{x-(v-1)}\right)\frac{2}{x-h}$ and averagely , the number of runs or tests needed to complete the test procedure in this case was found to be $\left\{(h+1)+1+E_{x-(h+1)}(B_1)\right\}$.
This implies that

$$E_x(B_2) \;= \frac{2}{x}\left\{1+1+E_{x-1}(B_1)\right\}$$

$$+\sum_{h=1}^{x-2}\left[\prod_{v=1}^{h}\left(\frac{x-(v+1)}{x-(v-1)}\right)\frac{2}{x-h}\left\{(h+1)+1+E_{x-(h+1)}(B_1)\right\}\right] \tag{31}$$

□

Equation 30 was used and equation 31 rewritten to follow the below that;

$$\begin{aligned}
E_x(B_2) \;&= \frac{2}{x}\frac{x-1}{x-1}\left\{2+\tfrac{x-1}{2}+\tfrac{3}{2}-\tfrac{2}{x-1}\right\}\\
&+\frac{x-2}{x}\frac{2}{x-1}\left\{3+\tfrac{x-2}{2}+\tfrac{3}{2}-\tfrac{2}{x-2}\right\}\\
&+\frac{x-3}{x}\frac{2}{x-1}\left\{4+\tfrac{x-3}{x}+\tfrac{3}{2}-\tfrac{2}{x-3}\right\}\\
&+\dots\dots\dots\dots\dots\dots\dots\dots\\
&+\frac{2}{x}\frac{2}{x-1}\left\{(x-1)+\tfrac{2}{2}+\tfrac{3}{2}-\tfrac{2}{2}\right\}\\
&+\frac{1}{x}\frac{2}{x-1}\left\{x+\tfrac{1}{2}+\tfrac{3}{2}-\tfrac{2}{1}\right\}\\
&=\frac{2}{x(x-1)}\sum_{t=1}^{x-1}(t+1)(x-t)+\frac{1}{x(x-1)}\sum_{t=1}^{x-1}(x-t)^2\\
&+\frac{3}{x(x-1)}\sum_{t=1}^{x-1}(x-t)-\frac{4(x-1)}{x(x-1)}\\
&=\frac{2}{x(x-1)}\left[\tfrac{(x+1)x(x-1)}{6}+\tfrac{x(x-1)}{2}\right]+\frac{1}{x(x-1)}\left[\tfrac{x(x-1)(2x-1)}{6}\right]\\
&+\frac{3}{x(x-1)}\left[\tfrac{x(x-1)}{2}\right]-\frac{4}{x}
\end{aligned}$$

i.e.,

$$E_x(B_2) \;= \frac{2x}{3}+2+\frac{2}{3}-\frac{4}{x} \tag{32}$$

The above equation (32) proved the lemma.

Lemma 2.2.3.

$$E_x(B_3) = \frac{3x}{4} + 3 + \frac{3}{4} - \frac{6}{4}$$

PROOF. After one defective factor was found, the problem reduced to that considered in lemma (2.2.2).The probability that the first factor tested was defective was $\frac{3}{x}$ and the probability that for $h = 1, 2, \ldots, x-3$ the $(h+1)^{st}$ factor tested was the first defective was $\prod_{v=1}^{h}\left(\frac{x-(v+2)}{x-(v-1)}\right)\frac{3}{x-h}$.If the first factor tested was defective, then averagely, $\left\{1+1+E_{x-1}(B_2)\right\}$ tests were needed to complete the test procedure.However, if for $h = 1, 2, \ldots, x-3$, the $(h+1)^{st}$ factor tested was the first defective, then averagely, $\left\{(h+1)+1+E_{x-(h+1)}(B_2)\right\}$ runs was to be needed to complete the test procedure. Thus

$$E_x(B_3) \;= \frac{3}{x}\left\{1+1+E_{x-1}(B_2)\right\}$$

$$+\sum_{h=1}^{x-1}\left[\prod_{v=1}^{h}\left(\frac{x-(v+2)}{x-(v-1)}\right)\frac{3}{x-1}\left\{(h+1)+1+E_{x-(h+1)}(B_2)\right\}\right] \qquad \textbf{(33)}$$

Using equation (32), the following was found,

$$\begin{aligned}
E_x(B_3) \;&= \frac{3}{x}\frac{x-1}{x-1}\frac{x-2}{x-2}\left\{2+\tfrac{2(x-1)}{3}+\tfrac{8}{3}-\tfrac{4}{x-1}\right\}\\
&+\frac{3}{x}\frac{x-3}{x-1}\frac{x-2}{x-2}\left\{3+\tfrac{2(x-2)}{3}+\tfrac{8}{3}-\tfrac{4}{x-2}\right\}\\
&+\frac{3}{x}\frac{x-3}{x-1}\frac{x-4}{x-2}\left\{4+\tfrac{2(x-3)}{3}+\tfrac{8}{3}-\tfrac{4}{x-3}\right\}\\
&+\ldots\ldots\ldots\ldots\ldots\\
&+\frac{3}{x}\frac{2}{x-1}\frac{3}{x-2}\left\{(x-2)+\tfrac{2(3)}{3}+\tfrac{8}{3}-\tfrac{4}{3}\right\}\\
&+\frac{3}{x}\frac{1}{x-1}\frac{2}{x-2}\left\{(x-1)+\tfrac{2(2)}{3}+\tfrac{8}{3}-\tfrac{4}{3}\right\}.\\
&= \frac{3}{x(x-1)(x-2)}\sum_{t=1}^{x-2}(t+1)(x-t)(x-t-1)\\
&+\frac{2}{x(x-1)(x-2)}\sum_{t=1}^{x-2}(x-t)^2(x-t-1)\\
&+\frac{8}{x(x-1)(x-2)}\sum_{t=1}^{x-2}(x-t)(x-t-1)
\end{aligned}$$

$$-\frac{12}{x(x-1)(x-2)}\sum_{t=1}^{x-2}(x-t-1) \qquad \textbf{(34)}$$

The summations

$$\sum_{t=1}^{x-2} t(x-t)(x-t-1) \;= \frac{(x+1)x(x-1)(x-2)}{12}$$

$$\sum_{t=1}^{x-2} (x-t)^2(x-t-1) \;= \frac{(x+1)x(x-1)(x-2)}{12} \quad \textbf{(35)}$$

and

$$\sum_{t=1}^{x-2}(x-t)(x-t-1) \;= \frac{x(x-1)(x-2)}{3}$$

had been found by [**sterrett1957detection**].
Using the above summations in equations (34) the following was found,

$$E_x(B_3) \;= \frac{x+1}{4} + 1 + \frac{3x-1}{6} + \frac{8}{3} - \frac{6}{x}$$

$$= \frac{3x}{4} + 3 + \frac{3}{4} - \frac{6}{x} \quad \textbf{(36)}$$

This completed the proof. □

The general result , for generally total expected number runs for a large number of defectives, could be proven.

Theorem 2.2.4. *The average number of tests needed to analyze a defective group-factor of size x assuming that the group-factor contains exactly j defective factors was given by*

$$E_x(B_j) = \frac{jx}{j+1} + j + \frac{j}{j+1} - \frac{2j}{x} \qquad (j = 1, 2, \ldots, x)$$

Proof. The proof followed the mathematical induction.The validity of the theorem had been shown for $j = 1$.It was assumed that the theorem was true for $j = k-1, (1 \leq k-1 \leq x)$ that is,

$$E_x(B_{k-1}) = \frac{(k-1)x}{k} + (k-1) - \frac{2(k-1)}{x} \quad \textbf{(37)}$$

It was shown that the theorem was true for $j = k$.It followed that for $j = k$,

$$E_x(B_k) \;= \frac{k}{x}\left\{1 + 1 + E_{x-1}(B_{k-1})\right\}$$

$$+\sum_{h=1}^{x-k}\left[\prod_{v=1}^{h}\left(\frac{x-(v+k-1)}{x-(v-1)}\right)\frac{k}{x-h}\left\{(h+1)+1+E_{x-(h+1)}(B_{k-1})\right\}\right] \quad (38)$$

The probability that the first factor tested is defective was given to be the factor $\frac{k}{x}$ in the first term.The following $\left\{1+1+E_{x-1}(B_{k-1})\right\}$ was got to be the average number of runs needed to perform the analysis if the first factor is defective.The value, $\prod_{v=1}^{h}\left(\frac{x-(v+k-1)}{x-(v-1)}\right)$ was got to be the probability that the first h factors tested were non-defective;the probability that $(h+1)^{st}$ factor tested was defective was $\frac{k}{x-1}$.The term $\left\{(h+1)+1+E_{x-(h+1)}(B_{k-1})\right\}$ consisted of the number of tests needed to obtain the first defective factor on the $(h+1)^{st}$ trial, the group test on $x-(h+1)$ factors and the average number of tests that were needed to complete the analysis with $(k-1)$ defective factors in $x-(h+1)$ factors.
Upon substitution in equation (38) the values that were given in equation 37, the following were obtained

$$\begin{aligned}
E_x(B_k) &= \frac{k}{x}\frac{x-1}{x-1}\cdots\frac{x-(k-1)}{x-(k-1)}\left[2+\frac{k-1}{k}(x-1)+(k-1)+\frac{k-1}{k}-\frac{2(k-1)}{x-1}\right]\\
&+\frac{x-k}{x}\frac{k}{x-1}\frac{x-2}{x-2}\cdots\frac{x-(k-1)}{x-(k-1)}\left[3+\frac{k-1}{k}(x-2)+(k-1)+\frac{k-1}{k}-\frac{2(k-1)}{x-2}\right]\\
&+\frac{x-k}{x}\frac{x-k-1}{x-1}\frac{k}{x-2}\frac{x-3}{x-3}\cdots\frac{x-(k-1)}{x-(k-1)}\left[4+\frac{k-1}{k}(x-3)+(k-1)+\frac{k-1}{k}-\frac{2(k-1)}{x-3}\right]\\
&+\cdots\cdots\cdots\cdots\cdots\cdots\\
&+\frac{x-k}{x}\frac{x-k-1}{x-1}\cdots\frac{x-(x-2)}{x-(x-k-2)}\frac{k}{x-(x-k-1)}\frac{x-(x-k)}{x-(x-k)}\cdots\frac{x-(k-1)}{x-(k-1)}\\
&\times\left[(x-k+1)+\frac{k-1}{k}(k)+(k-1)+\frac{k-1}{k}-\frac{2(k-1)}{k}\right]\\
&+\frac{x-k}{x}\frac{x-k-1}{x-1}\cdots\frac{x-(x-1)}{x-(x-k-1)}\frac{k}{x-(x-k)}\frac{x-(x-k+1)}{x-(x-k+1)}\cdots\frac{x-(n-1)}{x-(k-1)}
\end{aligned}$$

$$\times\left[(x-k+2)+\frac{k-1}{k}(k-1)+(k-1)+\frac{k-1}{k}-\frac{2(k-1)}{k-1}\right] \quad (39)$$

By the re-arrangement and taking of appropriate summations, equations (39) became,

$$\begin{aligned}
E_x(B_k) &= \frac{k}{x^pk}\sum_{t=1}^{x-k+1}(t+1)(x-t)(x-t-1)\ldots(x-t-k+2)\\
&+\frac{k-1}{x^pk}\sum_{t=1}^{x-k+1}(x-t)^2(x-t-1)\ldots(x-t-k+2)\\
&+\frac{k^2-1}{x^Pk}\sum_{t=1}^{x-t+1}(x-t)(x-t-1)\ldots(x-t-k+2)
\end{aligned}$$

$$-\frac{2k(k-1)}{x^Pk}\sum_{t=1}^{x-k+1}(x-t-1)(x-t-2)\dots(x-t-k+2) \tag{40}$$

where

$$x^Pk \ = x(x-1)(x-2)\dots(x-k+1) \tag{41}$$

The summations

$$\sum_{t=1}^{x-k+1} t(x-t)(x-t-1)\dots(x-t-k+2) = \frac{x+1^P(k+1)}{k(k+1)}$$

$$\sum_{t=1}^{x-k+1}(x-t)^2(x-t-1)(x-t-2)\dots(x-t-k+2) = \frac{x+1^P(k+1)}{k(k+1)} - \frac{x^Pk}{k}$$

$$\sum_{t=1}^{x-k+1}(x-t)(x-t-1)\dots(x-t-k+2) = \frac{x^Pk}{k} \tag{42}$$

and

$$\sum_{t=1}^{x-k+1}(x-t-1)(x-t-2)\dots(x-t-k+2) = \frac{(x-1)^P(k-1)}{(k-1)}$$

had been determined by [**sterrett1957detection**].
When these summations were Used in equation (40),the following was obtained;

$$E_x(B_k) \ = \frac{x+1}{k+1} + 1 + \frac{(k-1)(x+1)}{k+1} - \frac{k-1}{k} + \frac{k^2-1}{k} - \frac{2k(k-1)}{(k-1)x}$$

$$= \frac{kx}{k+1} + k + \frac{k}{k+1} - \frac{2k}{x} \tag{43}$$

The above was the exact value $E_x(B_j)$ for $j=k$.Thus if the theorem was to be true for $j=k-1 (0<k-1<x)$, it was also true for $j=k$.
But the theorem was true for $j=1$, previously shown.Consequently,it was true also for $j=2$ and generally for any other $j \quad (j=1,2,\dots,x)$.This concluded the theorem.
If ${B^0}_L$ was let to denote the number of tests needed to analyze and classify all the factors within a group-factor of size x that is known to be defective as defective or non-defective.a group-factor.Then,

$$E({B^0}_L) \ = \sum_{j=1}^{x} E_x(B_j)P_x(j) \tag{44}$$

where $P_x(j)$ was as the definition in equation 28.
When equation (28) and theorem (2.2.1) in equation (44) was used to obtain,

$$E(B^0{}_L) = \sum_{j=1}^{x} \left\{ \tfrac{j(x+1)}{j+1} + j - \tfrac{2j}{x} \right\} \frac{1}{1-q^x} (x$$

$$j) P^j q^{x-j}$$

$$= \frac{1}{1-q^x} \Big\{ (x+1)(1-q^x) + xP - 2P \Big\} - \frac{x+1}{1-q^x} \sum_{j=1}^{x} \begin{pmatrix} x \\ j \end{pmatrix} P^j q^{x-j} \qquad \textbf{(45)}$$

Next

$$(x+1) \sum_{j=1}^{x} \frac{1}{j+1} (x$$

$$j) P^j q^{x-j}$$

$$= (x+1) \left\{ \tfrac{x}{2} P q^{x-1} + \tfrac{x(x-1)}{3\times 2} P^2 q^{x-2} + \ldots + \tfrac{1}{x+1} P^x \right\}$$

$$= \frac{1}{P} \left\{ \tfrac{(x+1)x}{2!} P^2 q^{x-1} + \tfrac{(x+1)x(x-1)}{3!} P^3 q^{x-2} + \ldots + P^{x+1} \right\}$$

$$= \frac{1}{P} \Big\{ 1 - q^{x+1} - (x+1) P q^x \Big\} \qquad \textbf{(46)}$$

equation (46) was used in equation (45) to find

$$E(B^0{}_L) = \frac{1}{1-q^x} \Big[(x+1)(1-q^x) + xP - 2P - \tfrac{1}{P} \Big\{ 1 - q^{x+1} - (x+1) P q^x \Big\} \Big]$$

$$= \frac{1}{1-q^x} \Big[(x+1) + xP - 2P - \tfrac{1}{P} \Big\{ 1 - q^{x+1} \Big\} \Big] \qquad \textbf{(47)}$$

B_L was let to denote the number of tests (runs) needed to do the analysis of all factors in the c group factors found to be defective in the first step.
Then,

$$B_L = cE(B^0{}_L)$$

$$= \frac{c}{1-q^x} \Big[(x+1) + xP - 2P - \tfrac{1}{P} \Big\{ 1 - q^{x+1} \Big\} \Big] \qquad \textbf{(48)}$$

Furthermore,B was let to be the total expected number of runs needed to investigate the f factors.Then,

$$B = B_I + B_L \tag{49}$$

□

Quoting the below theorem from [**patel1987step**],

Theorem 2.2.5. *B was let to denote the total number of runs needed to screen out the defective factors from among the f factors under investigation in a step-wise group screening experiment where P was the probability of any factor being defective and x was the size of the group-factor in the initial step, then,*

$$E(B) = 1 + fP + \frac{2fq}{x} + f - \frac{f}{xP}\left[1 - q^{x+1}\right]$$

where $q = 1 - q$

PROOF. In the first step, $g* = \frac{f}{x}$ group-factors were to be tested.This implied that the number of runs needed in step one was

$$B_I = g* + 1 \tag{50}$$

the one extra test being put as the control test.The number of tests in the subsequent steps was to

$$B_L = \frac{c}{1-q^x}\left[(x+1) + xP - 2P - \tfrac{1}{P}\left\{1 - q^{x+1}\right\}\right] \tag{51}$$

where c was the number of group-factors found to be defective in step one.Then

$$E(B_L) = \left[(x+1) + xP - 2p - \tfrac{1}{P}\left\{1 - q^{x+1}\right\}\right]\frac{E(c)}{1-q^x}$$

$$= \left[(x+1) + xP - 2p - \tfrac{1}{P}\left\{1 - q^{x+1}\right\}\right]\frac{f}{x} \tag{52}$$

The equation (27) was used,
The expected total number of runs was given by

$$E(B) = B_I + E(B_L) \tag{53}$$

equations (50) and (51) was used to get the below,

$$E(B) = 1 + \frac{f}{x} + f + \frac{f}{x} + fP - \frac{2fq}{x} - \frac{f}{xP}(1 - q^x)$$

$$1 + fP + \frac{2fq}{x} + f - \frac{f}{xP}\left\{1 - q^{x+1}\right\} \tag{54}$$

This was the conclusion of the proof of the theorem. □

The two equations, (55) and (54) are similar, hence this brings the derivation of the total expected number of runs to a conclusion.

$$E(B) = 1 + f + fp + \frac{2fq}{x} - \frac{f}{xp}\left(1 - q^{x+1}\right) \tag{55}$$

2.3 Comparison of the two methods for obtaining optimum group size, x

In this section, the study is aimed at getting the difference between the previously used method to obtain the optimum group size as compared to this method, the Newton's method to obtain the optimum group size of group factor in the initial step.

2.3.1 The Optimum size of the group factor using the Calculus method

In this method, the assumption was, the probability of a factor being defective is small,hence, the expression for the expected number of runs given the probability of a factor being defective is small was approximated as;

$$E(B) \approx 1+\frac{3fp}{2}+\frac{f}{x}-\frac{2fp}{x}+\frac{fxp}{2}$$

and proven as follows:

$$E(B) = 1+f+fp+\frac{2fq}{x}-\frac{f}{xp}\left(1-q^{x+1}\right)$$

$$\begin{aligned}\left(1-q^{x+1}\right) &\approx \frac{f(x+1)}{x}-\frac{f(x+1)p}{2}\\ &\approx \frac{fx}{x}+\frac{f}{x}-\frac{fxp}{2}-\frac{fp}{2}\\ &\approx f+\frac{f}{x}-\frac{fxp}{2}-\frac{fp}{2}\end{aligned}$$

replacing $\left(1-q^{x+1}\right)$, above with $f+\frac{f}{x}-\frac{fxp}{2}-\frac{fp}{2}$, equation (55) becomes,

$$\begin{aligned}
E(B) &= 1+f+fp+\frac{2fq}{x}-\frac{f}{xp}-\left(f+\tfrac{f}{x}-\tfrac{fxp}{2}-\tfrac{fp}{2}\right)\\
E(B) &= 1+f+fp+\frac{2fq}{x}-f-\frac{f}{x}+\frac{fxp}{2}+\frac{fp}{2}\\
E(B) &= 1+f-f+\frac{2fq}{x}-\frac{f}{x}+\frac{fxp}{2}+fp+\frac{fp}{2}\\
E(B) &= 1+\frac{2fq}{x}-\frac{f}{x}+\frac{fxp}{2}+\frac{3fp}{2}\\
E(B) &= 1+\frac{2f}{x}(1-p)-\frac{f}{x}+\frac{fxp}{2}+\frac{3fp}{2}\\
E(B) &= 1+\frac{2f}{x}-\frac{2fp}{x}-\frac{f}{x}+\frac{fxp}{2}+\frac{3fp}{2}\\
E(B) &= 1+\frac{2f}{x}-\frac{f}{x}-\frac{2fp}{x}+\frac{fxp}{2}+\frac{3fp}{2}\\
E(B) &= 1+\frac{f}{x}-\frac{2fp}{x}+\frac{fxp}{2}+\frac{3fp}{2}\\
E(B) &= 1+\frac{3fp}{2}+\frac{f}{x}-\frac{2fp}{x}+\frac{fxp}{2}
\end{aligned}$$

which completes the proof, and hence or otherwise, the optimum group size is dependent on the probability function, p.This was done through the following theorem which was proven given the condition that the p is small, hence, the expression for the expected total number of runs for a small value of p, probability of being defective, was used.

Theorem 2.3.1.

$$min \quad E(B) \approx 1+\frac{3fp}{2}+f\left(2p\right)^{\frac{1}{2}}\left(1-2p\right)^{\frac{1}{2}}$$

which was the corresponding minimum expected total number of runs obtained after the group-factor size, x, minimizing the expected total number of runs in a step-wise group screening design was given by

$$x = \left(\tfrac{2-4p}{p}\right)^{\frac{1}{2}}$$

p, the probability of a factor being defective,was assumed to be small. This condition was that p is small such that $p<\frac{1}{2}$.

The above writings were proven as below.

PROOF. x was assumed to have a continuous variation, the optimum group size was found by solving the equation below;

$$\frac{d}{dx}E(B) = 0$$

where

$$E(B) = 1 + f + fp + \frac{2fq}{x} - \frac{f}{xp}\left(1 - q^{x+1}\right)$$

Since the assumption of the value of p being small still was taken into consideration,the expression for the expected total number of runs for a small value of p, probability of being defective, given that $p < \frac{1}{2}$, was the one taken into consideration as follows;

$$E(B) = 1 + \frac{3fp}{2} + \frac{f}{x} - \frac{2fp}{x} + \frac{fxp}{2}$$

hence the following was obtained

$$\begin{aligned}
\frac{d}{dx}E(B) &= -\frac{f}{x^2} + \frac{2fp}{x^2} + \frac{fp}{2} = 0 \\
\frac{-2f + 4fp + fpx^2}{2x^2} &= 0 \\
-2f + 4fp + fpx^2 &= 0 \\
fpx^2 &= 2f - 4fp \\
x^2 &= \frac{2f - 4fp}{fp} \\
x^2 &= \frac{2 - 4p}{p}\left(\frac{f}{f}\right) \\
x^2 &= \frac{2 - 4p}{p}
\end{aligned}$$

$$x = \left(\frac{2-4p}{p}\right)^{\frac{1}{2}} \quad \textbf{(56)}$$

Mathematically, an equation is considered to be optimum if it satisfies the condition that the second derivative of the equation is either, greater than zero, for minimization, or the second derivative is less than zero if it is a maximum, i.e

$$\frac{d^2}{dx^2}E(B) > 0$$

for a minimum and

$$\frac{d^2}{dx^2}E(B) < 0$$

for a maximum

In this case, the focus is on minimization, hence $\frac{d^2}{dx^2}E(B) > 0$ is used.This implies that

$$\begin{aligned}
\frac{d^2}{dx^2}E(B) &= \frac{d}{dx}\left(\tfrac{d}{dx}E(B)\right) > 0\\
\frac{d^2}{dx^2}E(B) &= \frac{d}{dx}\left(-\tfrac{f}{x^2} + \tfrac{2fp}{x^2} + \tfrac{fp}{2}\right) > 0\\
\frac{d^2}{dx^2}E(B) &= \frac{2f}{x^3} - \frac{4fp}{x^3} > 0\\
2f - 4fp &> 0\\
2 - 4p &> 0\\
1 - 2p &> 0
\end{aligned}$$

This is true for $p = \frac{1}{2}$. □

This clearly shows that the value of x obtained and being worked with is within the jurisdiction of the minimum of $E(B)$ and hence used to obtain the minimum of $E(B)$, which consequently is replaced in the equation $1 + \frac{3fp}{2} + \frac{f}{x} - \frac{2fp}{x} + \frac{fxp}{2}$, hence the following;

$$\begin{aligned}
E(B) &= 1 + \frac{3fp}{2} + \frac{f}{x} - \frac{2fp}{x} + \frac{fxp}{2}\\
&= 1 + \frac{3fp}{2} + \frac{f}{\left(\frac{2-4p}{p}\right)^{\frac{1}{2}}} - \frac{2fp}{\left(\frac{2-4p}{p}\right)^{\frac{1}{2}}} + \frac{f\left(\frac{2-4p}{p}\right)^{\frac{1}{2}}p}{2}\\
&= 1 + \frac{3fp}{2} + f\left(\tfrac{p}{2-4p}\right)^{\frac{1}{2}} - 2fp\left(\tfrac{p}{2-4p}\right)^{\frac{1}{2}} + \frac{f}{2}\left(2-4p\right)^{\frac{1}{2}}p^{\frac{1}{2}}
\end{aligned}$$

Focusing on the part of the equation that has the value, x,before focusing on the whole equation,

$$
\begin{aligned}
&= f\left(\tfrac{p}{2-4p}\right)^{\frac{1}{2}} - 2fp\left(\tfrac{p}{2-4p}\right)^{\frac{1}{2}} + \frac{f}{2}\left(2-4p\right)^{\frac{1}{2}}p^{\frac{1}{2}} \\
&= f\left(\tfrac{p}{2-4p}\right)^{\frac{1}{2}}\left\{1-2p\right\} + \frac{f}{2}\left(1-2p\right)^{\frac{1}{2}}p^{\frac{1}{2}}2^{\frac{1}{2}} \\
&= \frac{fp^{\frac{1}{2}}}{2^{\frac{1}{2}}\left(1-2p\right)^{\frac{1}{2}}}\left\{1-2p\right\} + \frac{f}{2}\left(1-2p\right)^{\frac{1}{2}}\left(2p\right)^{\frac{1}{2}} \\
&= fp^{\frac{1}{2}}2^{\frac{-1}{2}}\left(1-2p\right)^{\frac{-1}{2}}\left\{1-2p\right\}^{\frac{2}{2}} + \frac{f}{2}\left(1-2p\right)^{\frac{1}{2}}\left(2p\right)^{\frac{1}{2}} \\
&= fp^{\frac{1}{2}}2^{\frac{-1}{2}}\left(1-2p\right)^{\frac{1}{2}} + \frac{f}{2}\left(1-2p\right)^{\frac{1}{2}}\left(2p\right)^{\frac{1}{2}} \\
&= fp^{\frac{1}{2}}2^{\frac{-1}{2}}\left(1-2p\right)^{\frac{1}{2}} + f2^{\frac{-2}{2}}2^{\frac{1}{2}}p^{\frac{1}{2}}\left(1-2p\right)^{\frac{1}{2}} \\
&= fp^{\frac{1}{2}}2^{\frac{-1}{2}}\left(1-2p\right)^{\frac{1}{2}} + fp^{\frac{1}{2}}2^{\frac{-1}{2}}\left(1-2p\right)^{\frac{1}{2}} \\
&= 2fp^{\frac{1}{2}}2^{\frac{-1}{2}}\left(1-2p\right)^{\frac{1}{2}} \\
&= 2^{\frac{2}{2}}2^{\frac{-1}{2}}fp^{\frac{1}{2}}\left(1-2p\right)^{\frac{1}{2}} \\
&= 2^{\frac{1}{2}}fp^{\frac{1}{2}}\left(1-2p\right)^{\frac{1}{2}} \\
&= f2^{\frac{1}{2}}p^{\frac{1}{2}}\left(1-2p\right)^{\frac{1}{2}} \\
&= f\left(2p\right)^{\frac{1}{2}}\left(1-2p\right)^{\frac{1}{2}}
\end{aligned}
$$

Going back to the whole equation above,

$$E(B) \;= 1 + \frac{3fp}{2} + f\left(2p\right)^{\frac{1}{2}}\left(1-2p\right)^{\frac{1}{2}} \tag{57}$$

This puts the proof into conclusion.
The values of x that minimizes the expected total number of runs, $E(B)$, for arbitrary values of p can also be found. Still borrowing from mathematics, minimization of the general equation (53) is done as follows,

$$E(B) \;= 1 + f + fp + \frac{2fq}{x} - \frac{f}{xp}\left(1-q^{x+1}\right)$$

The x values that gives the minimized form is the solution of the below equation,

$$\frac{d}{dx}E(B) = 0$$

that is,

$$\begin{aligned}
\frac{d}{dx}E(B) &= \frac{d}{dx}\left\{1+f+fp+\tfrac{2fq}{x}-\tfrac{f}{xp}\left(1-q^{x+1}\right)\right\}=0 \\
&= \frac{-2fq}{x^2}+\frac{f}{x^2p}+\frac{fq}{p}\left[\tfrac{q^x\ln q}{x}-\tfrac{q^x}{x^2}\right] \\
&= \frac{-2fq}{x^2}+\frac{f}{x^2p}+\frac{f\ln q q^{x+1}}{xp}-\frac{fq^{x+1}}{x^2p}=0 \\
&= \left(\tfrac{-2fq}{x^2}+\tfrac{f}{x^2p}+\tfrac{f\ln q q^{x+1}}{xp}-\tfrac{fq^{x+1}}{x^2p}\right)x^2p = 0x^2p \\
&= -2fpq+f+fx\ln q q^{x+1}-fq^{x+1}=0 \\
&= \left(-2pq+1+x\ln q q^{x+1}-q^{x+1}\right)f=0 \\
&= \left(-2pq+1+x\ln q q^{x+1}-q^{x+1}\right)\frac{f}{f}=\frac{0}{f} \\
&= \left(-2pq+1+x\ln q q^{x+1}-q^{x+1}\right)=0
\end{aligned}$$

$$= 1-q^{x+1}-2pq+xq^{x+1}\ln q = 0 \tag{58}$$

The equation (58) is a non-linear equation and the newton's method for minimization is one of the best method to solve the equation in such a way that one gets the optimized group size x such that if the obtained values, x, are put back into the equation, the equation is minimized.In [**sterrett1957detection**], Sterrett's work, , equation (56) was used to obtain the initial approximated values of x given a specified values of p in each case.
In this study, Newton's method is used to get a better approximation of optimum values of x, for specified values of p.

2.3.2 The optimum group size, x, using the Newton's method

Given the above derived equation of the total expected number of runs,Newton's method is performed on the equation to determine the minimum of the total expected number of runs , i.e the minimum of the equation.This is done by getting the optimized values of the group sizes in such a way that when the said values are put back into the original equation, the minimized values of the equation is obtained, as follows.
The formula for the minimization of an equation using the Newton's method, as discussed previously, in methodology is;

$$x_{0+1} = x_0 + t$$

where

$$t = -\frac{\frac{d}{dx}}{\frac{d^2}{dx^2}}$$

OR

if one lets

$$\frac{d}{dx} = f^{'}$$

and let

$$\frac{d^2}{dx^2} = f^{"}$$

hence the newton's method formula becomes,

$$x_{0+1} = x_0 + t$$

where

$$t = -\frac{f^{'}}{f^{"}}$$

With the above information,performing Newton's method of minimization is done where a series of iterations are done, given that one is having a set of initial point, until the last point obtained after iteration is the same as the second last point obtained after iteration or very close to each other.The finally obtained point is replaced in the equation (55).The results obtained from the above procedure results into the optimized values of group sizes x, such that when replaced into the equation, the minimization of the total expected number of runs is obtained.This is done as follows;

$$E(B) = 1+f+fp+\frac{2fq}{x}-\frac{f}{xp}\left(1-q^{x+1}\right)$$

Getting the first derivative of the equation (55), with respect to the group size, x, the following is obtained

$$\begin{aligned}
\frac{dE(B)}{dx} &= -\frac{2fq}{x^2}\frac{d}{dx}\left\{-\tfrac{f}{xp}+\tfrac{fq^{x+1}}{xp}\right\} \\
\frac{dE(B)}{dx} &= \left(-\tfrac{2fq}{x^2}\tfrac{d}{dx}\left\{-\tfrac{f}{xp}+\tfrac{fq}{p}\left(\tfrac{q^x}{x}\right)\right\}\right) \\
&= \frac{-2fq}{x^2}+\frac{f}{x^2p}+\frac{fq}{p}\left[\tfrac{q^x\ln q}{x}-\tfrac{q^x}{x^2}\right] \\
&= \frac{-2fq}{x^2}+\frac{f}{x^2p}+\frac{fq^{x+1}\ln q}{xp}-\frac{fq^{x+1}}{x^2p} \\
&= \frac{-2fq}{x^2}+\frac{f}{x^2p}+\frac{fq\ln q}{p}\left(\tfrac{q^x}{x}\right)-\frac{fq}{p}\left(\tfrac{q^x}{x^2}\right) \\
&= \frac{-2fq}{x^2}+\frac{f}{x^2p}+\frac{fq^{x+1}\ln q}{xp}-\frac{fq^{x+1}}{x^2p}
\end{aligned}$$

Consequently, the second derivative of the equation (55), the total expected number of runs, with respect to the group size,x, is as sequentially obtained below.

$$\begin{aligned}
\frac{d^2E(B)}{dx^2} &= \frac{4fq}{x^3}-\frac{2f}{x^3p}+\frac{d}{dx}\left\{\tfrac{fq\ln q}{p}\left(\tfrac{q^x}{x}\right)-\tfrac{fq}{p}\left(\tfrac{q^x}{x^2}\right)\right\} \\
&= \frac{4fq}{x^3}-\frac{2f}{x^3p}+\frac{fq\ln q}{p}\left[\tfrac{q^x\ln q}{x}-\tfrac{q^x}{x^2}\right]-\frac{fq}{p}\left[\tfrac{q^x\ln q}{x^2}-\tfrac{2q^x}{x^3}\right] \\
&= \frac{4fq}{x^3}-\frac{2f}{x^3p}+\frac{fq^{x+1}(\ln q)^2}{xp}-\frac{fq^{x+1}\ln q}{x^2p}-\frac{fq^{x+1}\ln q}{x^2p}+\frac{2fq^{x+1}}{x^3p} \\
\frac{d^2E(B)}{dx^2} &= \frac{4fq}{x^3}-\frac{2f}{x^3p}+\frac{fq^{x+1}(\ln q)^2}{xp}-\frac{2fq^{x+1}\ln q}{x^2p}+\frac{2fq^{x+1}}{x^3p}
\end{aligned}$$

3 Group size from Newton's method in Two-Type Step-Wise Group Screening Design without errors

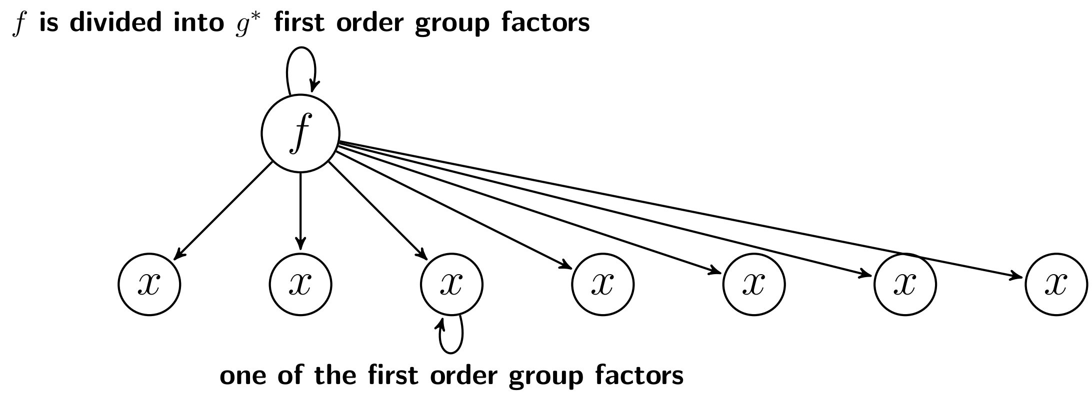
In this chapter, quoted from [**manene2002multi**], screening with two types of search steps was done by first dividing the factors, f, into groups, say g^* first order group factors, each of them having size, x, i.e. $(f = x \times g^*)$.In the first and initial step, the first order group factor were tested for their effects in a factorial experiment . m was let to be the number of first order group factors got to be defective in the initial step.Each of the m defective first order group factors was furthermore divided into g_1^* second order group-factors each having y factors , i.e. $(x = g_1^* \times y)$.In the first step of type one search steps, any of the defective first order group factors was started with, where, the particularly chosen defective first order group factor, from any of the defective first order group factors,is taken, and then the second order group factors within it is tested one by one until a defective second order group factor is obtained.The obtained defective second order group factor is kept away and the non defective ones put separate.In the second step, i.e, step two , of type one search steps,the remaining second order group factors are tested in a pooled group factor.Whenever a test was found to be negative, the test procedure was completed.However, in the third step, type one search steps, the testing of the remaining second order group factors is continued with one by one until another defective second order group-factor is put aside.The steps two and three of type one search steps are done repeatedly, successively in the other coming type one search steps until the analysis is completed with a test on a non-defective pooled group factor consisting of the second order group-factors or with a second order group factor of size one.The mentioned test procedure was performed to all the m first order group factors obtained to be defective in the initial step.

Suppose that m_1 second order group factors were obtained to be defective at the end of type one search steps.In the type two search steps, defective factors within the m_1 defective second order group factors are separated using the same method that was used in type one search steps, just that,in this particular case, factors were tested instead of the second order group factors.

3.1 Brief illustration of two type of search steps

g^* , taken to be the total number of the first order group factors.

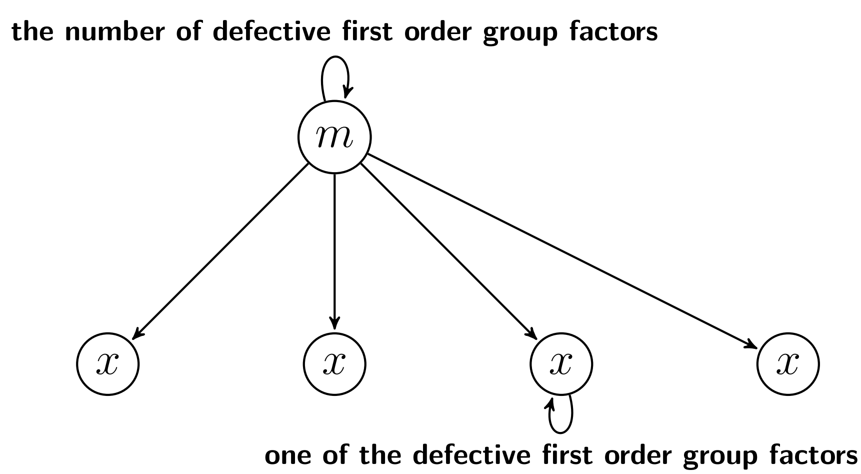
Figure 4. Division of f factors into g^* first order group factors



$$f = x \times g^*$$

where f is the total population under investigation,
x is the group size of each of the first order group factors,
In the initial step, all the first order group factors are tested for their effects in a factorial experiment,all the first order group factors found to be defective, say, m, in the initial step, are put separate.The non-defective first order group factors are also kept aside.Hence, $m-$ defective first order group factors,
i.e. $\left(g^* - m\right) =$ the number of non-defective first order group factors.i.e,illustrated as below;

Figure 5. the separated defective first order group factors,m



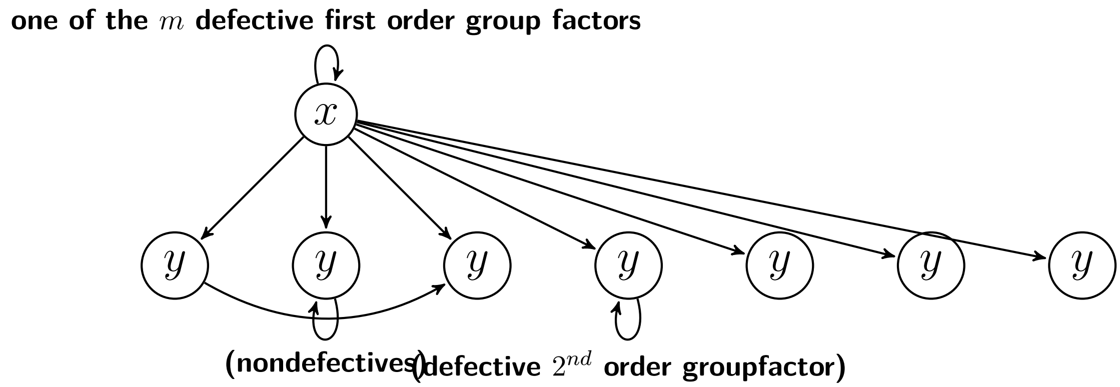
Each of the m first order group factors are taken and then further divided into ${g^*}_1$ second order group factors, each containing factors, y, i.e,

$$x = \left(g^*{}_1 \times y\right)$$

3.1.1 Step one type one search steps

Among the m defective first order group factors, any one of the first order group factors was started with.

Figure 6. step one type one search steps

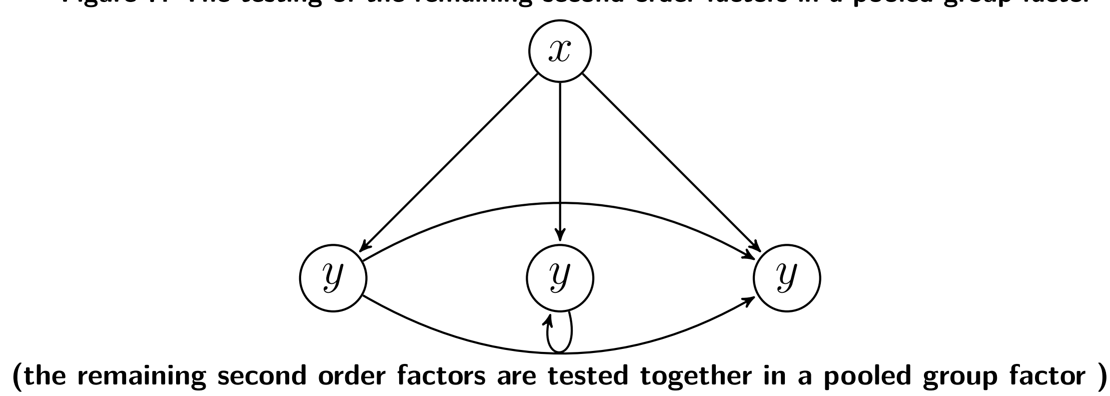


As illustrated in the above figure (6), the first three second order group factors were tested and found to be non defective, hence put separate.The fourth second order group factor is the first second order group factor found to be defective, and is also kept a side.

3.1.2 Step two of type one search steps

The remaining second order group factors were tested together in a pooled group-factor.

Figure 7. The testing of the remaining second order factors in a pooled group factor



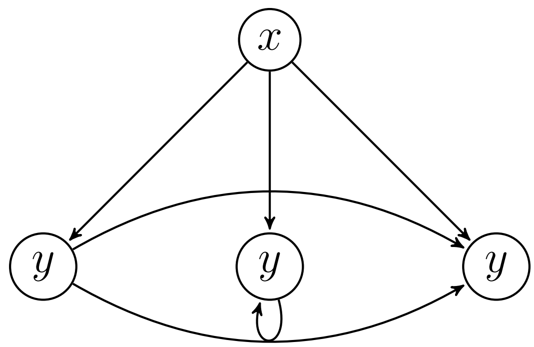
Whenever the test is negative, the procedure is concluded.

3.1.3 Step three of type one search steps

However,if the pooled group factors is tested to be positive, in step three of type one search steps, the remaining second order group factors were continued to be tested one by one until another defective second order group factor was found and set aside.

$$x = {g^*}_1 \times y$$

Figure 8. The remaining second order group factors,tested one by one until another defective second order group factor obtained and separated



(tests to each second order group factors one by one until another positive one obtained and set aside)

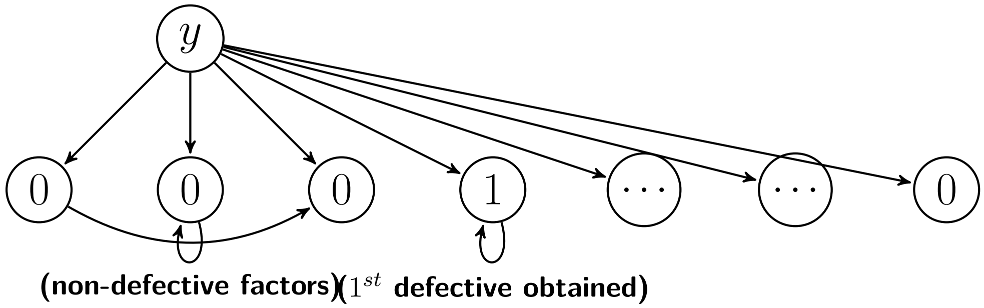
Steps two and three of the type one search steps repeated successively in the subsequent type one search steps until the analysis is is completed with a test on a non-defective pooled group-factor consisting of second order group-factors or with a test on a single group-factor.This test procedure done for all the m first order group-factors got to be defective in the initial step.

3.1.4 Type two search steps

Suppose that (m_1), second order group-factor, found to be defective at the end of type one search steps.In type two search steps, defective factors within the (m_1) defective second order group-factors isolated using the same procedure as used in type one search steps , however, in this case, the isolation done on the factors, instead of the second order group-factors.Say , $y = 10$, with the same arrangement as for the above already discussed case for $x = 10$.

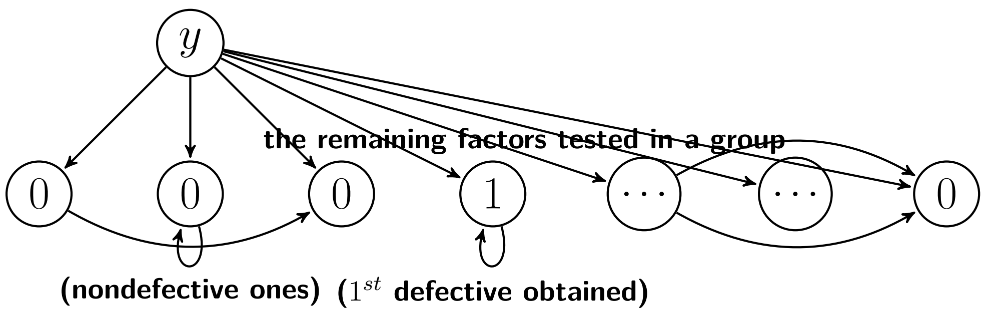
tests done on the factors within the defective second order group-factor, one by one until the first defective factor obtained, as illustrated in the above figure (9), where the first three factors are tested and proven to be non-defective, the fourth test, the first defective, or positive factor is obtained,

Figure 9. arrangements and tests of factors in a defective second order group factor, type two search steps



The tests from a pooled group

Figure 10. type two search steps



the rest of the factors tested together in a pooled group factor.The procedure ends with a negative test on the pooled group factor, or a negative test on group factor of size one. However, testing individually is continued with until the next defective factor is obtained, the remaining factors tested again in pooled group factor, if the result of the test is negative, the procedure is terminated .This was continued until a non defective test was found on a pooled group-factor, or found on a test of group-size of one.This was done to obtain all the defective factors in the population f, with two group sizes involved

3.2 The Expected total number of runs, (Tests)

The equation for obtaining the expected total number of runs was derived as below, before the optimization of the group sizes was done.The derivation procedures were performed as illustrated below,quoted from the [**manene2002multi**].$^*p_{1x}$ was let to be the probability that a first order group factor was defective, and $^*p_{2y}$, be the probability that a second order group factor was defective.Then,

$$^*p_{1x} = 1 - q^x \tag{59}$$

and

$$^*p_{2y} = 1 - q^y \tag{60}$$

In the initial step, the following number of tests were needed

$$T_x = 1 + g_{1x} \tag{61}$$

to test the g_{1x} first order group-factors orthogonally.The probability function of m, the number of defective first order group factors in the initial step, was

$$f(m) = P(m = m) \ = \begin{cases} \binom{g_{1x}}{m} {p_{1x}}^m \left(1 -^* p_{1x}\right)^{g_{1x}-m}, & m = 0, 1, 2, \ldots, g_{1x} \\ 0, & \text{otherwise.} \end{cases} \tag{62}$$

Thus

$$E(m) \ = g_{1x}^* P_{1x} = \frac{f}{x}\left(1 - q_x\right) \tag{63}$$

$^*p_{1y/1x}$ was let to denote the probability that a second order group-factor was defective given that it was within a defective first order group-factor.Then,

$$^*p_{2y/1x} \ = \frac{^*p_{2y}}{^*p_{1x}} \tag{64}$$

The conditional probability function of m_1, the number of second order group factor found to be defective at the end of type one search steps for given m, was as given below

$$f(m_1/m) \ = p(m_1 = m_1/m = m) \quad = \begin{cases} \binom{mg_{2y}}{m_1} P_{2y/1x}^{m_1} \left(1 -^* P_{2y/1x}^{m_1}\right),^{mg_{2y}-m_1} \\ \qquad\qquad\qquad m_1 = 0, 1, 2, \ldots, mg_{2y} \\ 0, \qquad Otherwise \end{cases} \tag{65}$$

Thus

$$E(m_1) = \underset{m m_1}{E\,E}(m_1 = m_1/m = m) = \frac{f}{y}(1-q^y) \tag{66}$$

By letting $p_{g_{2y}}(j_{1x})$ to be the probability that a defective first order group factor contains exactly j_{1x} defective second order group factors and $p_y(j_{2y})$ to be the probability that a defective second order group factor contains exactly j_{2y} defective factors, then, the following was held true,

$$P_{g_{2y}}(j_{1x}) = \Big(1-q^x\Big)^{-1}\binom{g_{2y}}{j_{1x}} {}^{*}p_{2y}{}^{j_{1x}}\Big(1-{}^{*}p_{2y}\Big)^{g_{2y}-j_{1x}}, \qquad j_{1x} = 1,2,\ldots,g_{2y} \tag{67}$$

and

$$p_y(j_{2y}) = \Big(1-q^y\Big)^{-1}\binom{y}{j_{2y}} p^{j_{2y}}\Big(1-p\Big)^{y-j_y}, \qquad j_y = 1,2,\ldots,y \tag{68}$$

And also, by letting $E_{g_{2y}}(B_{j_{1x}})\Big(j_{1x} = 1,2,\ldots,g_{2y}\Big)$ to be the expected number of runs required to classify as defective or non-defective all the g_{2y} second order group-factors within a first order group-factor which was known to be defective if it contains exactly j_{1x} defective second order group-factors.Further, $E_x(B_{j_{2y}})$ was let to denote the expected number of runs or tests required to classify as defective or non-defective all the factors within a defective second order group-factor if it contains exactly j_{2y} defective factors.Then, the following was also proven to be true,

$$E_{g_{2y}}(B_{j_{1x}}) = \frac{j_{1x}g_{2y}}{j_{1x}+1} + j_{1x} + \frac{j_{1x}}{j_{1x}+1} - \frac{2j_{1x}}{g_{2y}} \tag{69}$$

and

$$E_y(B_{j_{2y}}) = \frac{j_{2y}y}{j_{2y}+1} + j_{2y} + \frac{j_{2y}}{j_{2y}+1} - \frac{2j_{2y}}{gy} \tag{70}$$

The number of runs needed to separate and classify all the g_{2y} second order group-factors within a defective first order group factor as defective or non-defective by $T^0_{t_{1x}}$, Then,

$$E(B^0_{t_{1x}}) = \sum_{j_{2y}=1}^{g_{2y}} E_{g_{2y}}(B_{j_{1x}})P_{g_{2y}}(j_{1x}) \tag{71}$$

Equations (67) and (69) were used in equation (71) as follows;

$$E(B^0_{t_{1x}}) = \sum_{j_{2y}=1}^{g_{2y}} \times$$
$$\left\{ \frac{j_{1x}g_{2y}}{j_{1x}+1} + j_{1x} + \frac{j_{1x}}{j_{1x}+1} - \frac{2j_{1x}}{g_{2y}} \right\} \times$$
$$\left\{ \left(1-q^x\right)^{-1} \binom{g_{2y}}{j_{1x}} {}^*p_{2y}{}^{j_{1x}} \left(1-{}^*p_{2y}\right)^{g_{2y}-j_{1x}} \right\}$$

given that $j_{1x} = 1, 2, \ldots, g_{2y}$

and simplified , hence the following were obtained.

$$E(B^0_{t_{1x}}) = \left(1-q^x\right)^{-1} \left[(g_{2y}+1) + g_{2y}{}^*p_{2y} - 2^*p_{2y} - \frac{1}{{}^*p_{2y}} \left(1 - q^{*g_{2y}+1}\right) \right] \quad \textbf{(72)}$$

It was noted that $q^*{}_{2y} = 1 - p^*{}_{2y}$.
$T^0_{t_{1x}}$ was let to be the number of tests needed to classify all the mg_{2y} second order group-factors within the m defective first order group-factor as defective or non-defective.The following was then,

$$B_{t_{1x}} = mE(B^0_{t_{1x}}) \quad \textbf{(73)}$$

Suppose that $T^0_{t_{2y}}$ denoted the number of tests needed for all the y factors within a defective second order group-factor to be classified as defective or non-defective.Then

$$E(B^0_{t_{2y}}) = \sum_{j_{2y}=1}^{y} E_y(B_{j_{2y}}) p_y(j_{2y}) \quad \textbf{(74)}$$

the expressions for $p_y(j_{2y})$ and $E_y(B_{j_{2y}})$ given in equation(68) and (70) respectively, were substituted in equation (74) and simplified where the following were obtained.

$$E(T^0_{t_{2y}}) = \left(1-q^y\right)^{-1}][(y+1) + yp - 2p - \frac{1}{p}(1+q^{y+1}) \quad \textbf{(75)}$$

When the number of runs needed to classify all the $m_1 y$ factors within the m_1 second order group-factors found to be be defective at the end of type one search steps as defective or non defective was denoted by $B_{t_{2y}}$, then ,

$$B_{t_{2y}} = m_1 E(B^0_{t_{2y}}) \quad \textbf{(76)}$$

The below theorem was written to provide the expression for the expected total number of runs or tests needed to obtain all the defective factors.

Theorem 3.2.1. *B was let to denote the total number of runs or tests needed to do the selection of the defective factors from the whole specified population of interest, f, in a two-type step-wise group-screening design experiment.The following was the expression for the equation ;*

$$E(B) = 1+f+fp+\frac{2fq^y}{x}-\frac{f}{x}\left(1-q^y\right)^{-1}\left(1-q^{x+y}\right)+\frac{f}{y}-\frac{fq^y}{y}+\frac{2fq}{y}-\frac{f}{yp}\left(1-q^{y+1}\right),$$

where p was the probability of a factor being defective $\left(q=1-p\right)$, and x and y were the sizes of the first order group-factors and the sizes of the second order group-factors respectively.

PROOF. The expected total number of tests or runs was given by;

$$E(B) = B_x+E(B_{t_{1x}})+E(B_{t_{2y}}) = B_x+E\left\{mE(B^0_{t_{1x}})\right\}+E\left\{m_1E(B^0_{t_{2y}})\right\} \quad (77)$$

The above equations from equation 59 up to equation 76 were used to obtain the equation

$$E(B) = 1+f+fp+\frac{2fq^y}{x}-\frac{f}{x}\left(1-q^y\right)^{-1}\left(1-q^{x+y}\right)+\frac{f}{y}-\frac{fq^y}{y}+\frac{2fq}{y}-\frac{f}{yp}\left(1-q^{y+1}\right) \quad (78)$$

This completed the proof. □

The equation proven above is the equation for the determination of the expected total number of runs or tests in x, the group size in the first order group factor and y, the group size in the second order group factor.
For the minimized form of the equation to be obtained, optimum values of x, and y are obtained using the different methods like the method of calculus for minimization, used by other researchers like [**manene2002multi**]. Below is an illustration of what [**manene2002multi**] did in the optimization of the group sizes, x and y, hence the minimization of the expected total number of runs.

3.3 Comparison of the methods for obtaining the group sizes, x and y

Just as in the previous section of the previous chapter, this section also has the objective of obtaining the difference between the calculus method and the Newton's method, but this time, two variables, that is, two group sizes x and y, are involved.

3.3.1 Optimization of the groupsizes,x and y, from the calculus, First derivative approach

There are a series of assumptions that were made optimum group sizes to be obtained like;

p, the probability of a factor being defective , the values were assumed to be small.

Continuous variation in the optimum group sizes, x and y

Assuming p,(the probability of a factor being defective) to be small, the approximated sizes of the group , x and y, that does the minimization of the expected total number of runs(tests) in a two type stepwise group screening design without errors in observations are got from the following below explanations.
Assuming there is a continuous variation in the group size x and group size y,the optimum group sizes were obtained as follows which were later to be used to do the minimization of the expected total number of runs.
Given the equation for the expected total number of runs,or tests,

$$\begin{aligned} E(B) &= 1+f+fp+\frac{f}{x}-\frac{f}{y}+\frac{2fq}{y}-f-\frac{f}{y}+\frac{fpy}{2}+fp+ \\ &\frac{2f}{y}-\frac{f}{x}-\frac{f}{y}+fp+\frac{2f}{x}-\frac{2fpy}{x}-\frac{f}{x}-\frac{f}{y}+\frac{fp}{2}+\frac{fpx}{2y} \\ &= 1+\frac{7}{2}fp-\frac{2f}{y}+\frac{2fq}{y}+\frac{fpy}{2}-\frac{2fpy}{x}+\frac{fpx}{2y} \end{aligned} \tag{79}$$

With the equation above, from the calculus approach, the minimization of an the equation is done by getting the first derivatives of the equation, with respect to the first group sizes, x, and y, and consequently equating the differentiated equations to the zero value.
The optimum values of the group sizes are obtained by making the respective group sizes to be the subject of the formula.The obtained solutions are further taken back to the original equation of the expected total number of runs, and hence the minimization of the expected total number of runs.

The following below supports the above explanation.

$$\frac{\partial E(B)}{\partial x} = \frac{2fpy}{x^2} + \frac{fp}{2y} = 0$$

$$\frac{2fpy}{x^2} = -\frac{fp}{2y}$$

since the equation is strictly on positive integers, as the number of tests is strictly a positive digit,

$$\frac{2fpy}{x^2} = \frac{fp}{2y}$$

$$\frac{2fpy \times 2x^2y}{x^2} = \frac{fp}{2y} \times (2yx^2)$$

$$\frac{4fpx^2y^2}{x^2} = \frac{fp2yx^2}{2y}$$

$$x^2 = \frac{4fpy^2}{fp}$$

$$x = \left\{\frac{4fpy^2}{fp}\right\}^{\frac{1}{2}}$$

Also,

$$\begin{aligned}
\frac{\partial E(B)}{\partial y} &= \frac{2f}{y^2} - \frac{2fq}{y^2} + \frac{fp}{2} - \frac{2fp}{x} - \frac{fpx}{2y^2} = 0 \\
&= \frac{2f}{y^2} - \frac{2f}{y^2} + \frac{2fp}{y^2} + \frac{fp}{2} - \frac{2fp}{x} - \frac{fpx}{2y^2} \\
&= \frac{2fp}{y^2} + \frac{fp}{2} - \frac{2fp}{x} - \frac{fpx}{2y^2} \\
\frac{fp}{2} - \frac{2fp}{x} &= \frac{fpx}{2y^2} - \frac{2fp}{y^2} \\
\left(\frac{fpx-4fp}{2x}\right) &= \left(\frac{fpx-4fp}{2y^2}\right) \\
\left(\frac{y^2}{fpx-4fp}\right) &= \left(\frac{x}{fpx-4fp}\right) \\
y^2 &= x \\
y &= x^{\frac{1}{2}}
\end{aligned}$$

When the above expressions for the first group size and the second group size are used, given specified values of p the probability of a factor being defective, the optimum values should be obtained.This is illustrated using one of the tables in the chapter of results.

3.3.2 Optimization of the group sizes, x and y, from the Newton's method approach

The other method to be considered is the Newton's method for minimization, where a series of iterations are done on the group sizes, with continuous variations in the x and y values.The iterations are done to get the optimized values of the group sizes, values of the x's and y's , so that when these obtained values are taken back into the original equation, the equation in question is minimized.All these procedures are made possible and applicable as shown below.

$$E(B) = 1 + f + fp + \frac{2fq^y}{x} - \frac{f}{x}\frac{\left(1-q^{x+y}\right)}{\left(1-q^y\right)} + \frac{f}{y} - \frac{fq^y}{y} + \frac{2fq}{y} - \frac{f}{yp}\left(1-q^{y+1}\right)$$

3.3.3 The first, the second and the mixed variable, second derivatives

Using the above results from the previous chapter, the various derivatives are as follows.

$$\mathbf{a} = \frac{\partial E(B)}{\partial x}$$

$$\mathbf{b} = \frac{\partial^2 E(B)}{\partial x^2}$$

$$\mathbf{c} = \frac{\partial E(B)}{\partial y}$$

$$\mathbf{d} = \frac{\partial^2 E(B)}{\partial y^2}$$

and

$$\mathrm{e} = \frac{\partial^2 E(B)}{\partial x \partial y} = \frac{\partial^2 E(B)}{\partial y \partial x}$$

Given that the initial points are $x_0 = 76.31$ and $y_0 = 8.71$ as seen from the table in the part of results below, Then, the following are the results of the first derivatives when the initial points are substituted;

$$\mathbf{a_0} = \frac{\partial E(B)}{\partial x_0} = -0.001653995$$

and

$$\mathbf{c_0} = \frac{\partial E(B)}{\partial y_0} = 0.006942767$$

$$\nabla E(B)(X_0) = \begin{pmatrix} a_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} -0.001653995 \\ 0.006942767 \end{pmatrix}$$

3.3.4 The Hessian Matrix, $H(X_k)$

where

$$\begin{pmatrix} X_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

and consequently

$$\begin{pmatrix} X_k \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix}$$

$$\mathbf{b_0} = \frac{\partial^2 E(B)}{\partial {x_0}^2} = 0.0003980863$$

,

$$\mathbf{d_0} = \frac{\partial^2 E(B)}{\partial {y_0}^2} = 0.03010448$$

and

$$\mathbf{e_0} = \frac{\partial^2 E(B)}{\partial x_0 \partial y_0} = \frac{\partial^2 E(R)}{\partial y_0 \partial x_0} = -0.001601403$$

$$H(X_0) = H\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} b_0 & e_0 \\ e_0 & d_0 \end{pmatrix} = \begin{pmatrix} 0.0003980863 & -0.001601403 \\ -0.001601403 & 0.03010448 \end{pmatrix}$$

This now implies that the inverse of the Hessian Matrix, $\mathbf{H^{-1}}(\mathbf{X_0})$ is;

$$\mathbf{H^{-1}}(\mathbf{X_0}) = \begin{pmatrix} 3195.9097 & 170.00590 \\ 170.0059 & 42.26108 \end{pmatrix}$$

$$\mathbf{H^{-1}}(\mathbf{X_0}) \triangledown E(R)(X_0) = \begin{pmatrix} -4.10570706 \\ 0.01221994 \end{pmatrix}$$

3.3.5 Application of the Newton's method to the equation of the expected total number of runs

Therefore,

$$X_1 = X_{0+1} = X_0 - \mathbf{H^{-1}}(\mathbf{X_0}) \triangledown E(B)(X_0)$$

This implies that, the next point, X_1;

$$X_1 = X_{0+1} = \left\{ \begin{pmatrix} 76.31 \\ 8.71 \end{pmatrix} - \begin{pmatrix} -4.10570706 \\ 0.01221994 \end{pmatrix} \right\}$$

Hence

$$X_1 = X_{0+1} = \begin{pmatrix} 80.41571 \\ 8.69778 \end{pmatrix}$$

and so,

$$X_1 = \begin{pmatrix} 80.41571 \\ 8.69778 \end{pmatrix}$$

These newly obtained points in the first iteration,in Newton's method, act as the initial points in the second iteration process.This means that, for one to find the next points, put the newly obtained points in the original equation that is to be minimized, while maintaining all the conditions, like $p = 0.003, f = 100$ and $q = 0.997$.This is repeatedly done all through upto a point where the lastly iterated points are equivalent or equal to the previously or second last points found after iteration.When this point is reached, the final points are substituted in the original equation and hence therefore, the answer obtained is the minimized form of the equation.All this is done, keeping in mind that this is just the iteration done for only $p = 0.003$.

Now,Therefore,

$$X_2 = X_{1+1} = X_1 - \mathbf{H^{-1}}(\mathbf{X_1}) \nabla E(B)(X_1)$$

Given that the points in the second iteration are $x_1 = 80.41571$ and $y_1 = 8.69778$ as previously found from the last iteration, Then, the following are the results of the first derivatives when the respective points are substituted;

$$\mathbf{a_1} = \frac{\partial E(B)}{\partial x_1} = -0.0001314003$$

and

$$\mathbf{c_1} = \frac{\partial E(B)}{\partial y_1} = -1.332816e-05$$

$$\nabla E(B)(X_1) = \begin{pmatrix} a_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} -0.0001314003 \\ -1.332816e-05 \end{pmatrix}$$

The Hessian Matrix, $H(X_k)$

$$\mathbf{b_1} = \frac{\partial^2 E(B)}{\partial {x_1}^2} = 0.0003362398$$

,

$$\mathbf{d_1} = \frac{\partial^2 E(B)}{\partial {y_1}^2} = 0.0318364$$

and

$$\mathbf{e_1} = \frac{\partial^2 E(B)}{\partial x_1 \partial y_1} = \frac{\partial^2 E(B)}{\partial y_1 \partial x_1} = -0.001602488$$

$$H(X_1) = H \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} b_1 & e_1 \\ e_1 & d_1 \end{pmatrix} = \begin{pmatrix} 0.0003362398 & -0.001602488 \\ -0.001602488 & 0.0318364 \end{pmatrix}$$

The inverse of the Hessian Matrix, $\mathbf{H^{-1}}(\mathbf{X_1})$ is;

$$\mathbf{H^{-1}}(\mathbf{X_1}) = \begin{pmatrix} 3912.6927 & 196.94572 \\ 196.9457 & 41.32387 \end{pmatrix}$$

$$\mathbf{H^{-1}}(\mathbf{X_1}) \triangledown E(R)(B_1) = \begin{pmatrix} -0.51675374 \\ -0.02642949 \end{pmatrix}$$

Hence,

$$X_2 = X_{1+1} = X_1 - \mathbf{H^{-1}}(\mathbf{X_1}) \triangledown E(B)(X_1)$$

The next point,X_2;

$$X_2 = X_{1+1} = \left\{ \begin{pmatrix} 80.41571 \\ 8.69778 \end{pmatrix} - \begin{pmatrix} -0.51675374 \\ -0.02642949 \end{pmatrix} \right\}$$

As a result,

$$X_2 = X_{1+1} = \begin{pmatrix} 80.932464 \\ 8.724209 \end{pmatrix}$$

and so,

$$X_2 = \begin{pmatrix} 80.932464 \\ 8.724209 \end{pmatrix}$$

The next points of iteration is

$$X_3 = X_{2+1} = X_2 - \mathbf{H^{-1}}(\mathbf{X_2})\nabla E(B)(X_2)$$

This iteration process, while holding the initial conditions constant, $p = 0.003$ and $f = 100$, continues until it reaches a point where the final iterated point is equal to the second last iterated point.When this happens, the procedure is terminated.These are the final points in the iteration that when substituted in the original equation, that is, $E(B)$, the equation is said to be minimized. These iterations that have been done previously in this study, showcase only about iteration to obtain one pair of points, hence one row result on the below found table of results.However, the same procedure are needed to obtain the results on the whole table, where there is different probabilities of a factor being defective that produces a different result of the minimized equation (14), all throughout the table.Under normal circumstances where one would obtain the results using the manual way, that is, applying the Newton's method of minimization directly and doing the iterations one by one until one gets the answers , is possible, however, the procedure is very tedious,time consuming and this encourages one to use one of the software in Mathematics in the computations of such long procedures.Hence in my study, the famously known program, a great mathematical tool that was used is the R- programming language that was sufficiently coded and efficiently produced the results obtained in of the tables that are in chapter of results.

4 Group size from Newton's method in Multi-Type,(Three-Type) Step-Wise Group-Screening Design without errors

Generally, in the multi -type case of the step-wise group-screening design, also there is the initial step,and the many type search steps.In this case study, the many type search steps will be up to the third type,i.e, three-types of of search steps.

The whole population, f, of concern is first and fore-most divided into g_{1x} first-order group factors of x observations,(runs), each.The first order group-factors are consequently tested for their effects by test done initially.Given that m, first order group-factors are obtained as defective, each of the m defective first order group-factors is further partitioned into g_{2y}, second order group-factors each of size y,$(x = y * g_{2y})$.

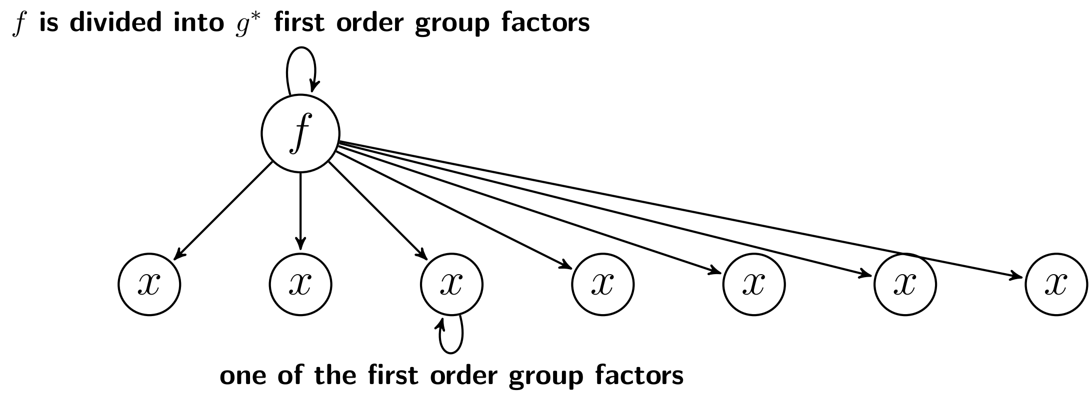
In step-one of type one search steps, any defective first order group-factor is started with and the second order group-factors are tested one by one until a defective second order group factor is obtained.Both the non-defective second order group factors and the defective second order group factor are differently and distinctly put away noting the difference between the two groups.In the second step- step of type one search steps, the remaining second order group-factors are tested in a pooled sample group-factor, and the tests are concluded with a negative result on the test.Otherwise, in the third step, the testing of the remaining second order group factors are continuously tested individually until upon the detection of another defective second order group factor.The steps of the second and the third type one search steps are performed repeatedly in the other upcoming type one search steps until the procedure is concluded after an observation of negative result from a test on non-defective pooled group-factor, or with one second order group-factor.The test procedure outlined is done on all the m first order group factors obtained to be defective in the initial step.

Supposedly that m_1 second order group-factors are the number obtained to be defective upon termination of type one search steps.Every one of the m_1 defective second order group-factors is progressively partitioned into g_{3z} third order group factors of z factors each , $y = z * g_{3z}$.Type two search steps are then used to classify the third order group-factors as defective and non defective, similarly to the case of type one search steps were used to classify the second order group-factors as defective or non defective.The research in this study, the multi-type, will be up to the three-type, and hence the expected total number of runs for this case is obtained before the minimization of the expected total number of runs. On a general, and final note, on focusing on the three-type stepwise group screening design, suppose that there are third order group factors that are found to be defective at the end of type two search steps.In the first step, of the type three search steps, any one of the defective third order group-factors is taken and the factors within it tested, one by one until a defective factor is obtained.The non- defective factors are isolated, putting the defective factors away from the population.In the second step of type three search steps, the remaining factors are tested in a pooled group-factor.This guidelines, are concluded upon negative result obtained on a group-factor test.Otherwise, the steps, one and two of type three search steps are repeatedly done subsequently in the type three search steps until the termination of the analysis is realized after the final test on a non-defective pooled group-factor or with a final test on a single factor.

4.1 Brief illustration of three type of search steps

g^* , taken to be the total number of the first order group factors.

Figure 11. Division of f factors into g^* first order group factors

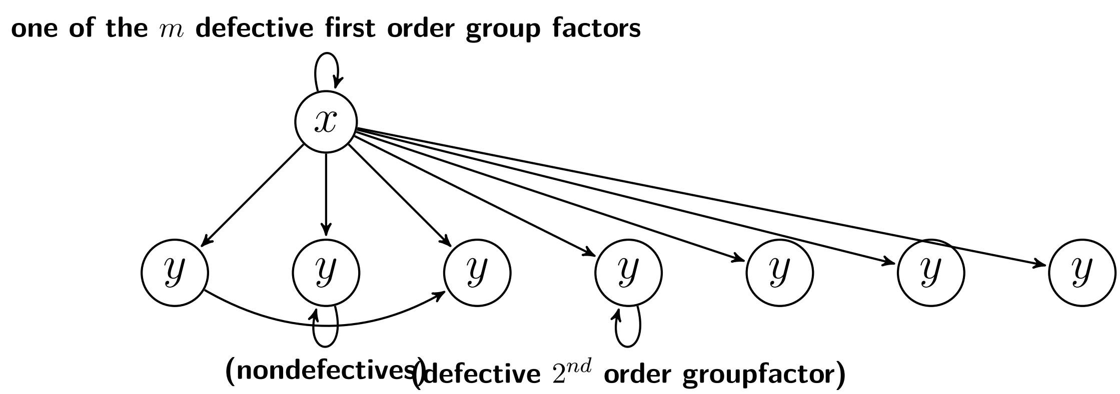


The defective first order group factors obtained using the factorial experiment.

4.1.1 Step one type one search steps

Among the m defective first order group factors, any one of the first order group factors was started with.

Figure 12. step one type one search steps

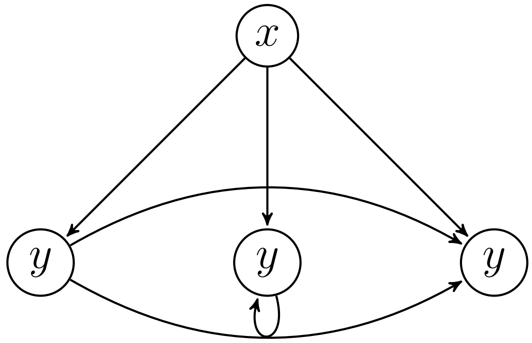


As illustrated in the above figure (12), the first three second order group factors were tested and found to be non defective, hence put separate.The fourth second order group factor is the first second order group factor found to be defective, and is also kept a side.

4.1.2 Step two of type one search steps

The remaining second order group factors were tested together in a pooled group-factor. Whenever the test is negative, the procedure is concluded.

Figure 13. The testing of the remaining second order factors in a pooled group factor



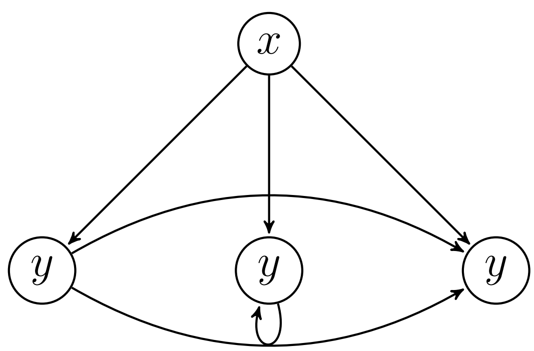
(the remaining second order factors are tested together in a pooled group factor)

4.1.3 Step three of type one search steps

However,if the pooled group factors is tested to be positive, in step three of type one search steps, the remaining second order group factors were continued to be tested one by one until another defective second order group factor was found and set aside.

$$x = {g^*}_1 \times y$$

Figure 14. The remaining second order group factors,tested one by one until another defective second order group factor obtained and separated



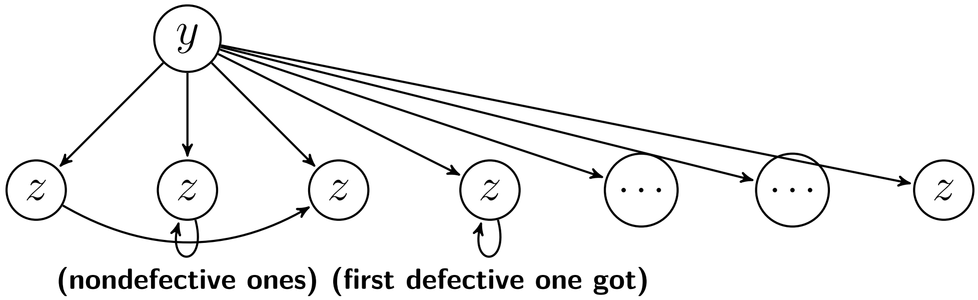
(tests to each second order group factors one by one until another positive one got and separated)

Steps two and three of the type one search steps repeated successively in the subsequent type one search steps until the analysis is completed with a test on a non-defective pooled group-factor consisting of second order group-factors or with a test on a single group-factor.This test procedure done for all the m first order group-factors got to be defective in the initial step.This is done until all the second order group factors are obtained.

4.1.4 Type two search steps

Suppose that (m_1), second order group-factors, found to be defective at the end of type one search steps.

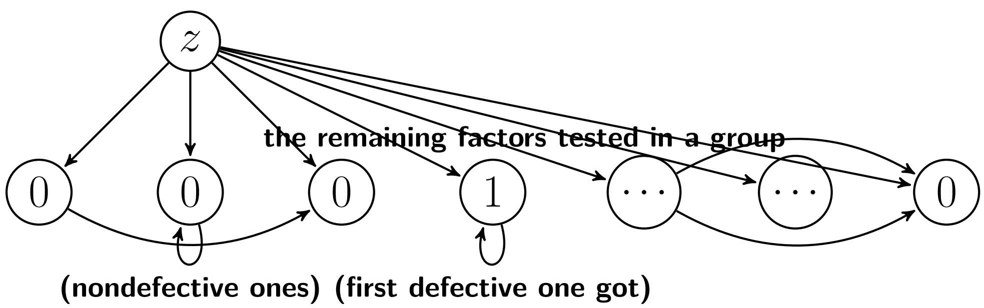
Figure 15. arrangements and tests of factors in a defective second order group factor, type two search steps



Taking any one of the obtained defective second order group factors,
In type two search steps, defective third order group factors within the (m_1) defective second order group-factors are isolated using the same procedure as used in type one search steps , however, in this case, the isolation done on the third order group factors, instead of the second order group-factors.This is done until all the defective third order group factors are obtained and isolated.
Taking any one of the defective third order group factors,say $z = 10$ factors within it arranged as below,

The tests on the factors within a defective third order group factor

Figure 16. type three search steps



Tests done on each of the factors until a defective factor is obtained, put aside,the defective one separated.The rest of the factors tested together in a pooled group factor.The procedure ends with a negative test on the pooled group factor, or a negative test on group factor of size one.
However, testing individually is continued with until the next defective factor is obtained, the remaining factors tested again in pooled group factor, if the result of the test is negative, the procedure is terminated .This was continued until a non defective test was found on a pooled group-factor, or found on a test of group-size of one.This was done to obtain all the defective factors in the population f, with three group sizes involved.

4.2 The Expected Total number of runs (Tests)

Quoted from the work done researchers in [**manene2002multi**], by B, denoting the total number of runs needed to highlight and determine the defective factors from the f factors being considered as the whole population of interest, in stepwise group-screening experiment considering the three types of search steps.Then, the following was the equation for finding the expected total number of runs or tests.

$$\begin{aligned} E(B) &= 1+f+fp+\frac{f}{x}-\frac{f}{z}+\frac{2fq}{z}-\frac{f}{zp}\left(1-q^{z+1}\right)+ \\ &\frac{2f}{y}-\frac{f}{x}-\frac{fq^y}{y}+\frac{2fq^y}{x}-\frac{f}{x}\frac{\left(1-q^{x+y}\right)}{\left(1-q^y\right)}+\frac{2f}{z}-\frac{f}{y}-\frac{fq^z}{z}+\frac{2fq^z}{y}-\frac{f}{y}\frac{\left(1-q^{y+z}\right)}{\left(1-q^z\right)} \end{aligned}$$

Given the assumption that the p, value was small, the following approximations were made in the calculus method,

$$\begin{aligned} \frac{f}{zp}\left(1-q^{z+1}\right) &\approx f+\frac{f}{z}-\frac{fpz}{2}-\frac{fp}{2}, \\ \frac{f}{x}\frac{\left(1-q^{x+y}\right)}{\left(1-q^y\right)} &\approx \frac{f}{x}+\frac{f}{y}-\frac{fp}{2}-\frac{fpy}{2z} \\ \frac{2fq^y}{x} &\approx \frac{2f}{x}-\frac{2fpy}{x} \\ \frac{2fq^z}{y} &\approx \frac{2f}{y}-\frac{2fpz}{y} \\ \frac{fq^y}{y} &\approx \frac{f}{y}-fp \\ \frac{fq^z}{z} &\approx \frac{f}{z}-fp \end{aligned}$$

Having all the above, replace the required expressions into the original equation of the expected total number tests by their equivalent expressions. This consequently yields the following;

$$
\begin{aligned}
E(B) &= 1+f+fp+\frac{f}{x}-\frac{f}{z}+\frac{2fq}{z}-\left(f+\tfrac{fpz}{2}-\tfrac{fp}{2}\right)+\\
&\quad \frac{2f}{y}-\frac{f}{x}-\frac{fq^{y}}{y}+\frac{2f}{x}-\frac{2fpy}{x}-\left(\tfrac{f}{x}+\tfrac{f}{y}-\tfrac{fp}{2}-\tfrac{fpx}{2y}\right)+\\
&\quad \frac{2y}{z}-\frac{f}{y}-\frac{fq^{z}}{z}+\frac{2f}{y}-\frac{2fpz}{y}-\left(\tfrac{f}{y}+\tfrac{f}{z}-\tfrac{fp}{2}-\tfrac{fpy}{2z}\right)\\
&= 1+f+fp+\frac{f}{x}-\frac{f}{z}+\frac{2fq}{z}-f-\frac{f}{z}+\frac{fpz}{2}+\frac{fp}{2}+\\
&\quad \frac{2f}{y}-\frac{f}{x}-\frac{fq^{y}}{y}+\frac{2f}{x}-\frac{2fpy}{x}-\frac{f}{x}-\frac{f}{y}+\frac{fp}{2}+\frac{fpx}{2y}+\\
&\quad \frac{2f}{z}-\frac{f}{y}-\frac{fq^{z}}{z}+\frac{2f}{y}-\frac{2fpz}{y}-\frac{f}{y}-\frac{f}{z}+\frac{fp}{2}+\frac{fpy}{2z}\\
E(B) &= 1+fp+\frac{fp}{2}+fp+\frac{fp}{2}+fp+\frac{fp}{2}+\frac{2fq}{z}-\frac{f}{z}+\\
&\quad \frac{fpz}{2}-\frac{2fpy}{x}+\frac{fpx}{2y}-\frac{2fpz}{y}-\frac{f}{z}+\frac{fpy}{2z}
\end{aligned}
$$

$$
E(B) = 1+\frac{9}{2}fp+\frac{2fq}{z}-\frac{2f}{z}+\frac{fpz}{z}-\frac{2fpy}{x}+\frac{fpx}{2y}-\frac{2fpz}{y}+\frac{fpy}{2z}+\frac{f}{x} \qquad \textbf{(80)}
$$

The above equation is the equation for obtaining the expected total number of tests or runs.

4.3 Comparison of the methods for obtaining group sizes

The optimum values of the group sizes are to be obtained in such a way that, when taken back into the original equation of the expected total number of runs, the equation is minimized, hence the obtaining the minimum expected total number of runs, which is the main objective.This was done previously, using the below method of calculus.

4.3.1 Optimum size of the group factor using the Calculus method,the first derivative

From the above, more simplified equation of the expected total number of runs or tests, the calculus method for minimization was easily obtained.This was done by getting the first derivations of the equation with respect to each of the group sizes, x, y and z.All the first derivatives were equated to the zero value and then solved consequently as shown below.
For the first group size, x.

$$\frac{\partial E(B)}{\partial x} = \frac{2fpy}{x^2} + \frac{fp}{2y} - \frac{f}{x^2} = 0$$

this then implies,

$$\begin{aligned}
\frac{fp}{2y} &= \frac{f}{x^2} - \frac{2fpy}{x^2} \\
\frac{-f(2x^2 \cdot 2y)}{x^2} + \frac{fp}{2y}(2x^2 \cdot 2y) + \frac{2fpy(2x^2 \cdot 2y)}{x^2} &= 0 \\
-4fy + 2fpx^2 + 8fpy^2 &= 0 \\
2fpx^2 &= 4fy - 8fpy^2 \\
fpx^2 &= 2fy - 4fpy^2 \\
x^2 &= \frac{2fy - 4fpy^2}{fp} \\
x &= \left(\tfrac{2fy - 4fpy^2}{fp}\right)^{\frac{1}{2}}
\end{aligned}$$

For the second group size, y.

$$
\begin{aligned}
\frac{\partial E(B)}{\partial y} &= \frac{-2fp}{x} - \frac{fpx}{2y^2} + \frac{2fpz}{y^2} + \frac{fp}{2z} = 0 \\
&\text{this then implies,} \\
\frac{fp}{2z} - \frac{2fp}{x} &= \frac{fpx}{2y^2} - \frac{2fpz}{y^2} \\
\left(\tfrac{fp}{2z} - \tfrac{2fp}{x}\right)\left(2y^2 \cdot 2z \cdot x\right) &= \left(\tfrac{fpx}{2y^2} - \tfrac{2fpz}{y^2}\right)\left(2y^2 \cdot 2z \cdot x\right) \\
fp \cdot 2y^2 \cdot x - 8fpy^2z &= 2fpx^2z - 8fpz^2x \\
y^2\left(2fpx - 8fpz\right) &= 2fpx^2z - 8fpz^2x \\
y^2 &= \left(\tfrac{2fpx^2z - 8fpz^2x}{2fpx - 8fpz}\right) \\
y^2 &= \frac{2fp}{2fp}\left(\tfrac{x^2z - 4z^2x}{x - 4z}\right) \\
y^2 &= \left(\tfrac{x^2z - 4z^2x}{x - 4z}\right) \\
&\text{taking the below into consideration,} \\
x^2z - 4z^2x &= xz\left(x - 4z\right) \\
&\text{then the following workings hold,} \\
y^2 &= \frac{xz\left(x - 4z\right)}{\left(x - 4z\right)} \\
y^2 &= xz \\
y &= \left(xz\right)^{\frac{1}{2}}
\end{aligned}
$$

This concludes the first differentiation of the equation with respect to the second group size .

Also,the first derivative of the equation with respect to the third group-size is obtained as below.

$$
\begin{aligned}
\frac{\partial E(B)}{\partial z} &= \frac{-2fq}{z^2} + \frac{f}{z^2} + \frac{fp}{2} - \frac{2fp}{y} - \frac{fpy}{2z^2} = 0 \\
&= \frac{-2f}{z^2}(1-p) + \frac{2f}{z^2} + \frac{f}{z^2} + \frac{fp}{2} - \frac{2fp}{y} - \frac{fpy}{2z^2} = 0 \\
&= \frac{-2f}{z^2} + \frac{2f}{z^2} + \frac{2fp}{z^2} + \frac{f}{z^2} + \frac{fp}{2} - \frac{2fp}{y} - \frac{fpy}{2z^2} = 0 \\
&= \frac{-2f}{z^2} + \frac{2f}{z^2} + \frac{2fp}{z^2} + \frac{f}{z^2} + \frac{fp}{2} - \frac{2fp}{y} - \frac{fpy}{2z^2} = 0 \\
&= \left(\tfrac{-2f}{z^2} + \tfrac{2f}{z^2} + \tfrac{2fp}{z^2} + \tfrac{f}{z^2} + \tfrac{fp}{2} - \tfrac{2fp}{y} - \tfrac{fpy}{2z^2}\right) \times \left(2z^2 \cdot y\right) = 0\left(2z^2 \cdot y\right) \\
&= 4fpy + fpz^2y - 4fpz^2 - fpy^2 = 0 \\
z^2\left(fpy - 4fp\right) &= y\left(fpy - 4fp\right) \\
z^2 &= y\frac{\left(fpy - 4fp\right)}{\left(fpy - 4fp\right)} \\
z^2 &= y \\
z &= y^{\frac{1}{2}}
\end{aligned}
$$

This concludes the first differentiation of the equation with respect to the third group size, hence concluding the calculus method for the minimization of the equation, given the assumptions stated earlier.The results that were obtained by these have been indicated in the chapter of results, with reference from [**manene2002multi**]. However, due to the assumptions that is in the accompaniments of the calculus method, which leads to a lot of approximations, the below method, the Newton method is used to improve the workings of optimization of the group sizes such that the minimization of the expected total number of runs or tests is obtained, but only this time, there are no approximations done in the original equation, as the optimum group sizes are obtained using the Newton's method.The Newton's method for minimization is as illustrated below.

4.3.2 Optimum size of the group factor using the Newton's method

Newton's method is based mainly of the Hessian matrix, which consists of the derivatives, hence the respective derivatives have to be obtained, as follows.

The derivatives of the equation with respect to the group sizes

Similarly, as in the case of the calculus method in the previous subsection, the main aim is to minimize the expected Total number of runs.In this case,the equation of the expected total number of runs is used just as it is and this is to the advantage in that, the much approximations are not done and this improves the values of the optimized group sizes hence even better results are obtained in the minimization of the expected total number of runs or tests.

The equation of the expected total number of runs or tests that is to be minimized is the following;

$$E(B) = 1+f+fp+\frac{f}{x}-\frac{f}{z}+\frac{2fq}{z}-\frac{f(1-q^{z+1})}{zp}+\frac{2f}{y}-\frac{f}{x}-\frac{fq^y}{y}+\frac{2fq^y}{x}-$$
$$\frac{f}{x}\frac{\left(1-q^{x+y}\right)}{\left(1-q^y\right)}+\frac{2f}{z}-\frac{f}{y}-\frac{fq^z}{z}+\frac{2fq^z}{y}-\frac{f\left(1-q^{y+z}\right)}{y\left(1-q^z\right)}$$

$$E(B) = 1+f+fp+\frac{f}{z}+\frac{2fq}{z}-\frac{f\left(1-q^{z+1}\right)}{zp}+\frac{f}{y}+\frac{2fq^z}{y}-\frac{fq^y}{y}+\frac{2fq^y}{x}-$$
$$\frac{f\left(1-q^{x+y}\right)}{x\left(1-q^y\right)}-\frac{fq^z}{z}-\frac{f\left(1-q^{y+z}\right)}{y\left(1-q^z\right)}$$

With the above equation, the derivatives of the equation can be found with respect to the group sizes, as below;

The derivative of the equation of the expected total number of runs, with respect to the first group size.

$$\frac{\partial E(B)}{\partial x} = \frac{-2fq^y}{x^2}-\frac{\partial}{\partial x}\left(\frac{f}{1-q^y}\right)+\frac{fq^y}{\left(1-q^y\right)}\left(\frac{q^x}{x}\right)$$
$$= \frac{-2fq^y}{x^2}+\frac{f}{x^2(1-q^y)}+\frac{fq^y}{(1-q^y)}\left(\frac{q^x\log q}{x}-\frac{q^x}{x}\right)$$
$$\frac{\partial E(B)}{\partial x} = \frac{-2fq^y}{x^2}+\frac{f}{x^2(1-q^y)}+\frac{fq^{x+y}\log q}{x(1-q^y)}-\frac{fq^{x+y}}{x^2(1-q^y)}$$

The conclusion of derivation of the equation with respect to the first group size, x.

The derivative of the second group size with respect to the first group size,

$$\begin{aligned}
\frac{\partial E(B)}{\partial x} &= \frac{-2f}{x^2}(q^y)+\frac{f}{x^2}\frac{1}{(1-q^y)}+\frac{fq^x\ln q}{x}\left(\frac{q^y}{1-q^y}\right)-\frac{fq^x}{x^2}\left(\frac{q^y}{1-q^y}\right)\\
\frac{\partial^2 E(B)}{\partial x\partial y} &= \frac{-2f}{x^2}(q^y\ln q)+\frac{f}{x^2}\left(\frac{q^y\ln q}{(1-q^y)^2}\right)+\frac{fq^x\ln q}{x}\left(\frac{q^y\ln q}{(1-q^y)^2}\right)-\frac{fq^x}{x^2}\left(\frac{q^y\ln q}{(1-q^y)^2}\right)\\
\frac{\partial^2 E(B)}{\partial x\partial y} &= \frac{-2fq^y\ln q}{x^2}+\frac{fq^y\ln q}{x^2(1-q^y)^2}+\frac{fq^{x+y}(\ln q)^2}{x(1-q^y)^2}-\frac{fq^{x+y}\ln q}{x^2(1-q^y)^2}
\end{aligned}$$

Also, **The second derivative of the equation with respect to the first group size,**

$$\begin{aligned}
\frac{\partial E(B)}{\partial x^2} &= \frac{4fq^y}{x^3}-\frac{2f}{x^3(1-q^y)}+\frac{fq^y\ln q}{(1-q^y)}\left[\frac{q^x\ln q}{x}-\frac{q^x}{x^2}\right]-\frac{fq^y}{(1-q^y)}\left[\frac{q^x\ln q}{x^2}-\frac{2q^x}{x^3}\right]\\
&= \frac{4fq^y}{x^3}-\frac{2f}{x^3(1-q^y)}+\frac{fq^{x+y}(\ln q)^2}{x(1-q^y)}-\frac{fq^{x+y}\ln q}{x^2(1-q^y)}-\frac{fq^{x+y}\ln q}{x^2(1-q^y)}+\frac{2fq^{x+y}}{x^3(1-q^y)}\\
\frac{\partial E(B)}{\partial x^2} &= \frac{4fq^y}{x^3}-\frac{2f}{x^3(1-q^y)}+\frac{fq^{x+y}(\ln q)^2}{x(1-q^y)}-\frac{2fq^{x+y}\ln q}{x^2(1-q^y)}+\frac{2fq^{x+y}}{x^3(1-q^y)}
\end{aligned}$$

The first derivative of the equation with respect to the second group size,

$$\begin{aligned}
\frac{\partial E(B)}{\partial y} &= \frac{2fq^y\ln q}{x}-\frac{f}{x}\left(\frac{q^y\ln q}{(1-q^y)^2}\right)+\frac{fq^x}{x}\left(\frac{q^y\ln q}{(1-q^y)^2}\right)-\frac{f}{y^2}-\frac{2fq^z}{y^2}-f\left(\frac{q^y\ln q}{y}-\frac{q^y}{y^2}\right)+\\
&\quad \frac{f}{y^2(1-q^z)}+\frac{fq^z}{(1-q^z)}\left(\frac{q^y\ln q}{y}-\frac{q^y}{y^2}\right)\\
\frac{\partial E(B)}{\partial y} &= \frac{2fq^y\ln q}{x}-\frac{fq^y\ln q}{x(1-q^y)^2}+\frac{fq^{x+y}\ln q}{x(1-q^y)^2}-\frac{f}{y^2}-\frac{2fq^z}{y^2}-\frac{fq^y\ln q}{y}+\frac{fq^y}{y^2}+\\
&\quad \frac{f}{y^2(1-q^z)}+\frac{fq^{y+z}\ln q}{y(1-q^z)}-\frac{fq^{y+z}}{y^2(1-q^z)}
\end{aligned}$$

Also,

The derivative of the second group size of the equation with respect to the third group size

$$\begin{aligned}
\frac{\partial^2 E(B)}{\partial y\partial z} &= \frac{-2fq^z\ln q}{y^2}+\frac{f}{y^2}\left(\frac{q^z\ln q}{(1-q^z)^2}\right)+\frac{fq^y\ln q}{y}\left(\frac{q^z\ln q}{(1-q^z)^2}\right)-\frac{fq^y}{y^2}\left(\frac{q^z\ln q}{(1-q^z)^2}\right)\\
\frac{\partial^2 E(B)}{\partial y\partial z} &= \frac{-2fq^z\ln q}{y^2}+\frac{fq^z\ln q}{y^2(1-q^z)^2}+\frac{fq^{y+z}(\ln q)^2}{y(1-q^z)^2}-\frac{fq^{y+z}\ln q}{y^2(1-q^z)^2}
\end{aligned}$$

The second derivative with respect to the second group size,

$$\begin{aligned}
\frac{\partial^2 E(B)}{\partial y^2} &= \frac{2fq^y(\ln q)^2}{x} - \frac{f\ln q}{x}\left(\frac{q^y\ln q - q^{3y}\ln q}{\left[(1-q^y)^2\right]^2}\right) + \frac{fq^x\ln q}{x}\left(\frac{q^y\ln q - q^{3y}\ln q}{\left[(1-q^y)^2\right]^2}\right) + \frac{2f}{y^3} + \frac{4fq^z}{y^3} - \\
&\quad f\ln q\left(\frac{q^y\ln q}{y} - \frac{q^y}{y^2}\right) + f\left(\frac{q^y\ln q}{y^2} - \frac{2q^y}{y^3}\right) - \frac{2f}{y^3(1-q^z)} + \frac{fq^z\ln q}{(1-q^z)}\left(\frac{q^y\ln q}{y} - \frac{q^y}{y^2}\right) - \\
&\quad \frac{fq^z}{(1-q^z)}\left(\frac{q^y\ln q}{y^2} - \frac{2q^y}{y^3}\right) \\
\frac{\partial^2 E(B)}{\partial y^2} &= \frac{2fq^y(\ln q)^2}{x} - \frac{fq^y(\ln q)^2}{x\left[(1-q^y)^2\right]^2} + \frac{fq^{3y}(\ln q)^2}{x\left[(1-q^y)^2\right]^2} + \frac{fq^{x+y}(\ln q)^2}{x\left[(1-q^y)^2\right]^2} - \frac{fq^{x+3y}(\ln q)^2}{x\left[(1-q^y)^2\right]^2} + \\
&\quad \frac{2f}{y^3} + \frac{4fq^z}{y^3} - \frac{fq^y(\ln q)^2}{y} + \frac{fq^y\ln q}{y^2} + \frac{fq^y\ln q}{y^2} - \frac{2fq^y}{y^3} - \frac{2f}{y^3(1-q^z)} + \\
&\quad \frac{fq^{y+z}(\ln q)^2}{y(1-q^z)} - \frac{fq^{y+z}\ln q}{y^2(1-q^z)} - \frac{fq^{y+z}\ln q}{y^2(1-q^z)} + \frac{2fq^{y+z}}{y^3(1-q^z)} \\
\frac{\partial^2 E(B)}{\partial y^2} &= \frac{2fq^y(\ln q)^2}{x} - \frac{fq^y(\ln q)^2}{x\left[(1-q^y)^2\right]^2} + \frac{fq^{3y}(\ln q)^2}{x\left[(1-q^y)^2\right]^2} + \frac{fq^{x+y}(\ln q)^2}{x\left[(1-q^y)^2\right]^2} - \frac{fq^{x+3y}(\ln q)^2}{x\left[(1-q^y)^2\right]^2} + \\
&\quad \frac{2f}{y^3} + \frac{4fq^z}{y^3} - \frac{fq^y(\ln q)^2}{y} + \frac{2fq^y(\ln q)^2}{y^2} - \frac{2fq^y}{y^3} - \frac{2f}{y^3(1-q^z)} + \frac{fq^{y+z}(\ln q)^2}{y(1-q^z)} - \\
&\quad \frac{2fq^{y+z}\ln q}{y^2(1-q^z)} + \frac{2fq^{y+z}}{y^3(1-q^z)}
\end{aligned}$$

This concludes the differentiation of the equation with respect to the second group size.

Finally,

Differentiation of the equation with respect to the third group size,

$$\begin{aligned}\frac{\partial E(B)}{\partial z} &= \frac{2fq^z\ln q}{y} - \frac{f}{y}\left(\tfrac{q^z\ln q}{(1-q^z)^2}\right) + \frac{fq^y}{y}\left(\tfrac{q^z\ln q}{(1-q^z)^2}\right) - \frac{f}{z^2} - \frac{2fq}{z^2} - f\left[\tfrac{q^z\ln q}{z} - \tfrac{q^z}{z^2}\right] + \frac{f}{z^2p} + \\ &\quad \frac{fq}{p}\left[\tfrac{q^z\ln q}{z} - \tfrac{q^z}{z^2}\right] \\ \frac{\partial E(B)}{\partial z} &= \frac{2fq^z\ln q}{y} - \frac{fq^z\ln q}{y(1-q^z)^2} + \frac{fq^{y+z}\ln q}{y(1-q^z)^2} - \frac{f}{z^2} - \frac{2fq}{z^2} - \frac{fq^z\ln q}{z} + \frac{fq^z}{z^2} + \frac{f}{z^2p} + \\ &\quad \frac{fq^{z+1}\ln q}{zp} - \frac{fq^{z+1}}{z^2p}\end{aligned}$$

The second derivative of the equation differentiated with respect to the third group size, with respect to the first group size

$$\frac{\partial E(B)}{\partial z\partial x} = 0$$

Also,it is known that,

$$\begin{aligned}\frac{\partial^2 E(B)}{\partial z\partial x} &= \frac{\partial^2 E(B)}{\partial x\partial z} \\ \frac{\partial^2 E(B)}{\partial z\partial y} &= \frac{\partial^2 E(B)}{\partial y\partial z} \\ \frac{\partial^2 E(B)}{\partial y\partial x} &= \frac{\partial E(B)}{\partial x\partial y}\end{aligned}$$

The second derivative of the equation with respect to the third group size

$$\begin{aligned}
\frac{\partial^2 E(B)}{\partial z^2} &= \frac{2fq^z(\ln q)^2}{y} - \frac{f\ln q}{y}\left(\frac{q^z\ln q - q^{3y}\ln q}{\left[(1-q^z)^2\right]}^2\right) + \frac{fq^y\ln q}{y}\left(\frac{q^z\ln q - q^{3y}\ln q}{\left[(1-q^z)^2\right]}^2\right) + \frac{2f}{z^3} + \\
&\quad \frac{4fq}{z^3} - f\ln q\left(\frac{q^z\ln q}{z} - \frac{q^z}{z^2}\right) + f\left(\frac{q^z\ln q}{z^2} - \frac{2q^z}{z^3}\right) - \frac{2f}{z^3p} + \frac{fq\ln q}{p}\left(\frac{q^z\ln q}{z} - \frac{q^z}{z^2}\right) - \\
&\quad \frac{fq}{p}\left(\frac{q^z\ln q}{z^2} - \frac{2q^z}{z^3}\right) \\
\frac{\partial^2 E(B)}{\partial z^2} &= \frac{2fq^z(\ln q)^2}{y} - \frac{fq^z(\ln q)^2}{y\left[(1-q^z)^2\right]^2} + \frac{fq^{3z}(\ln q)^2}{y\left[(1-q^z)^2\right]^2} + \frac{fq^{y+z}(\ln q)^2}{y\left[(1-q^z)^2\right]^2} - \frac{fq^{y+3z}(\ln q)^2}{y\left[(1-q^z)^2\right]^2} + \\
&\quad \frac{2f}{z^3} + \frac{4fq}{z^3} - \frac{fq^z(1-q^z)^2}{z} + \frac{fq^z\ln q}{z^2} + \frac{fq^z\ln q}{z^2} - \frac{2fq^z}{z^3} - \frac{2f}{z^3p} + \\
&\quad \frac{fq^{z+1}(\ln q)^2}{zp} - \frac{fq^{z+1}\ln q}{z^2p} + \frac{2fq^{z+1}}{z^3p} \\
\frac{\partial^2 E(B)}{\partial z^2} &= \frac{2fq^z(\ln q)^2}{y} - \frac{fq^z(\ln q)^2}{y\left[(1-q^z)^2\right]^2} + \frac{fq^{3z}(\ln q)^2}{y\left[(1-q^z)^2\right]^2} + \frac{fq^{y+z}(\ln q)^2}{y\left[(1-q^z)^2\right]^2} - \frac{fq^{y+3z}(\ln q)^2}{y\left[(1-q^z)^2\right]^2} + \\
&\quad \frac{2f}{z^3} + \frac{4fq}{z^3} - \frac{fq^z(\ln q)^2}{z} + \frac{2fq^z(\ln q)^2}{z^2} - \frac{2fq^z}{z^3} - \frac{2f}{z^3p} + \frac{fq^{z+1}(\ln q)^2}{zp} - \\
&\quad \frac{2fq^{z+1}\ln q}{z^2p} + \frac{2fq^{z+1}}{z^3p}
\end{aligned}$$

The next step, after obtaining the differentials with respect to the equation of the expected total number of runs, is to write the expression for the Hessian matrix, as part of the Newton method to perform the minimization of the expected number of runs.

4.3.3 The Hessian Matrix

The expression for the Hessian matrix, in the three variables, i.e.,three group sizes, is as follows;

$$H = \begin{bmatrix} \frac{\partial^2 E(B)}{\partial x^2} & \frac{\partial^2 E(B)}{\partial x \partial y} & \frac{\partial^2 E(B)}{\partial x \partial z} \\ \frac{\partial^2 E(B)}{\partial y \partial x} & \frac{\partial^2 E(B)}{\partial y^2} & \frac{\partial^2 E(B)}{\partial y \partial z} \\ \frac{\partial^2 E(B)}{\partial z \partial x} & \frac{\partial^2 E(B)}{\partial z \partial y} & \frac{\partial^2 E(B)}{\partial z^2} \end{bmatrix}$$

When the Hessian matrix, above is obtained, and consequently, its inverse, is also obtained, the Newton method proceeds by the following below;

Given a set of initial points, for all the three group sizes, i.e.

$$\textbf{Initial points of the group sizes} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

The given set of initial points, as discussed in the earlier chapter, should be given in such a manner that the Hessian matrix is not a singular matrix, i.e the determinant of the Hessian matrix should not be zero.

Upon the satisfaction of the above, given that the expressions for the first differentials with regards to the equation of the expected total number of runs was already obtained earlier, the initial points of the group sizes is put into the first derivatives with respect to all the group sizes, some set of values are obtained.The value of the inverse of the Hessian matrix is multiplied by these obtained values of the first derivative of the equation with respect to all group sizes.i.e.

$$\textbf{Inverse of Hessian matrix} \times \begin{bmatrix} \frac{\partial E(B)}{\partial x} \\ \frac{\partial E(B)}{\partial y} \\ \frac{\partial E(B)}{\partial z} \end{bmatrix}$$

The procedure gives out some values.

When these calculated values are subtracted from the earlier given set of initial points of the group sizes, then, the values got would make the first set of the iteration, and these obtained points counts for the initial points in the next iteration step.i.e

$$\textbf{Next points of the group sizes} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} - \left\{ \begin{matrix} \textbf{Inverse of} \\ \textbf{Hessian} \\ \textbf{matrix} \end{matrix} \times \begin{bmatrix} \frac{\partial E(B)}{\partial x} \\ \frac{\partial E(B)}{\partial y} \\ \frac{\partial E(B)}{\partial z} \end{bmatrix} \right\}$$

This procedure is continued systematically and while noting the consistency until the iteration is terminated when the set of lastly iterated points are equal to the second lastly iterated points, the values of the group sizes.This is the point when the following is zero.i.e.

$$\left\{ \begin{matrix} \textbf{Inverse of} \\ \textbf{Hessian} \\ \textbf{matrix} \end{matrix} \times \begin{bmatrix} \frac{\partial E(B)}{\partial x} \\ \\ \frac{\partial E(B)}{\partial y} \\ \\ \frac{\partial E(B)}{\partial z} \end{bmatrix} \right\} = 0$$

Or is some value that is almost, approximately zero.
When this happens, then one can conclude that the obtained points are the optimized values of the group sizes,such that when these values are taken back to the equation of the expected total number of tests,the equation is minimized, with specified p, the probability of a factor being defective.
In this case, given the set of conditions that were set earlier about initial points, and given that optimization terminates much faster depending on the wise choice of the initial points, the set of points from the tables obtained by some researchers, in [**manene2002multi**], were taken as the set of initial points.With all these, the above procedure can be demonstrated with some steps as below.

$$
\begin{aligned}
f &= 100 \\
p &= 0.005 \\
q &= 1-p \\
x_0 &= 89.44 \\
y_0 &= 20.00 \\
z_0 &= 4.47
\end{aligned}
$$

with all the above conditions set into place in the R-programming language, the values are obtained as below

$$
\begin{aligned}
E(B) &= 1+f+fp+\frac{f}{z}+\frac{2fq}{z}-\frac{f\left(1-q^{z+1}\right)}{zp}+\frac{f}{y}+\frac{2fq^{z}}{y}-\frac{fq^{y}}{y}+\frac{2fq^{y}}{x}- \\
&\quad \frac{f\left(1-q^{x+y}\right)}{x\left(1-q^{y}\right)}-\frac{fq^{z}}{z}-\frac{f\left(1-q^{y+z}\right)}{y\left(1-q^{z}\right)} \\
E(B) &= 6.783254 \\
\frac{\partial E(B)}{\partial x} &= -0.001229677 \\
\frac{\partial^2 E(B)}{\partial x^2} &= 0.000197651 \\
\frac{\partial^2 E(B)}{\partial x \partial y} &= -0.0003535865 \\
\frac{\partial^2 E(B)}{\partial x \partial z} &= 0 \\
\frac{\partial E(B)}{\partial y} &= 0.00294206 \\
\frac{\partial^2 E(B)}{\partial y^2} &= 0.005906151 \\
\frac{\partial^2 E(B)}{\partial y \partial x} &= \frac{\partial^2 E(B)}{\partial x \partial y} = -0.0003535865 \\
\frac{\partial^2 E(B)}{\partial y \partial z} &= -0.009284474
\end{aligned}
$$

$$\frac{\partial E(B)}{\partial z} = 0.003203008$$
$$\frac{\partial^2 E(B)}{\partial z^2} = 0.134929$$
$$\frac{\partial^2 E(B)}{\partial z \partial x} = \frac{\partial^2 E(B)}{\partial x \partial z} = 0$$
$$\frac{\partial^2 E(B)}{\partial z \partial y} = \frac{\partial^2 E(B)}{\partial y \partial z} = -0.009284474$$

The Hessian matrix then becomes,

$$H = \begin{bmatrix} 0.000197651 & -0.0003535865 & 0 \\ -0.0003535865 & 0.005906151 & -0.009284474 \\ 0 & -0.009284474 & 0.134929 \end{bmatrix}$$

with the above values of the Hessian matrix, the inverse of the same can be obtained as below;

$$H^{-1} = \begin{bmatrix} 5749.9298 & 385.98573 & 26.559703 \\ 385.9857 & 215.76183 & 14.846585 \\ 26.5597 & 14.84659 & 8.432899 \end{bmatrix}$$

After the values of the inverse of the Hessian matrix is obtained, the values of the first derivatives of the equation with respect to all the group sizes is obtained as follows;

$$\frac{\partial E(B)}{\partial x} = -0.001229677$$
$$\frac{\partial E(B)}{\partial y} = 0.00294206$$
$$\frac{\partial E(B)}{\partial z} = 0.003203008$$

The next step is to multiply the inverse of the Hessian matrix with the values of the first derivatives of the equation with respect to all the group sizes.This yields the following below outcomes;

$$\therefore H^{-1} \times \begin{bmatrix} \frac{\partial E(B)}{\partial x} \\ \frac{\partial E(B)}{\partial y} \\ \frac{\partial E(B)}{\partial z} \end{bmatrix} =$$

implies

$$= \begin{bmatrix} 5749.9298 & 385.98573 & 26.559703 \\ 385.9857 & 215.76183 & 14.846585 \\ 26.5597 & 14.84659 & 8.432899 \end{bmatrix} \times \begin{bmatrix} -0.001229677 \\ 0.00294206 \\ 0.003203008 \end{bmatrix}$$

The below are the obtained outcomes.

$$H^{-1} \times \begin{bmatrix} \frac{\partial E(B)}{\partial x} \\ \frac{\partial E(B)}{\partial y} \\ \frac{\partial E(B)}{\partial z} \end{bmatrix} = \begin{bmatrix} -5.84989054 \\ 0.20770039 \\ 0.03803034 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} - \begin{bmatrix} \textbf{Inverse of} & \\ \textbf{Hessian} & \times \begin{bmatrix} \frac{\partial E(B)}{\partial x} \\ \frac{\partial E(B)}{\partial y} \\ \frac{\partial E(B)}{\partial z} \end{bmatrix} \\ \textbf{matrix} & \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 89.44 \\ 20.00 \\ 4.47 \end{bmatrix} - \begin{bmatrix} -5.84989054 \\ 0.20770039 \\ 0.03803034 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 95.28989 \\ 19.79230 \\ 4.43197 \end{bmatrix}$$

The above obtained points are subtracted from the initial set of points, giving out the points which are used as the initial points in the next iteration procedure.
These iteration processes are continuously performed until one comes to the termination with equal solutions from the last and second last iterations. These then, are the optimum group sizes that when put back into the equation of the expected total number of runs, the equation is minimized.
The results for the above procedure for different values of the p, the probability being defective, is produced in a table in the below chapter of results.

5 Results And Discussion

Results The following tables below are the results for the group sizes, x, in one stage, two group sizes, x and y, in two stages, and three group sizes, x, y and z, in three stages, for all the two comparative methods, Calculus method and Newton's method

Table 1.

The Results for One Group Size

Table of group size and the minimum of $E(B)'$, from [manene2002multi],the Calculus Approach

	1-type	swgds results
p	x	min E(R)
0.001	44.72	5.584702
0.002	31.62	7.546292
0.003	25.81	9.074705
0.004	22.36	10.3781
0.005	20.00	11.53743
0.006	18.26	12.59405
0.008	15.81	14.48954
0.010	14.14	16.17897
0.020	10.00	22.96568
0.030	8.16	28.3199
0.035	7.56	30.68915
0.040	6.90	32.94944
0.045	6.67	35.00754
0.050	6.32	37.00267
0.055	6.03	38.90481
0.060	5.77	40.72972
0.070	5.35	44.17189
0.090	4.71	50.41398
0.100	4.47	53.27326
0.130	3.95	61.05036
0.140	3.78	63.43124
0.200	3.16	75.9421
0.300	2.58	92.0984
0.390	1.50	100.0717

Table 2. Table of group size and the minimum of $E(B)$, from Newton's method

	1-type	Newton's method
p	x	min E(R)
0.001	45.35625	5.584269
0.002	32.24404	7.545137
0.003	26.4305	9.072637
0.004	22.96217	10.37514
0.005	20.59334	11.53345
0.006	18.8433	12.58906
0.008	16.38379	14.48223
0.010	14.7024	16.1693
0.020	10.5101	22.94456
0.030	8.633392	28.28898
0.035	8.008529	30.65396
0.040	7.501856	32.87152
0.045	7.079571	34.96645
0.050	6.720023	36.95753
0.055	6.408612	38.859
0.060	6.135088	40.68196
0.070	5.673842	44.1261
0.090	4.978194	50.37068
0.100	4.704362	53.23555
0.130	4.058404	61.03933
0.140	3.883239	63.42022
0.200	3.061879	75.92689
0.300	2.087468	91.47275
0.390	1.327092	99.87957

Table 3. Table of group size and minimum of $E(B)$ from Computer search

	1-type	Computer search
p	x	min E(B)
0.001	45	5.584404
0.002	32	7.545311
0.003	26	9.073625
0.004	23	10.37515
0.005	21	11.53523
0.006	19	12.5894
0.008	16	14.48546
0.010	15	16.17185
0.020	11	22.96215
0.030	9	28.30523
0.035	8	30.65397
0.040	8	32.91687
0.045	6 7	34.96792
0.050	7	36.97727
0.055	6	38.91266
0.060	6	40.68822
0.070	6	44.16687
0.090	5	50.37094
0.100	5	53.2882
0.130	4	61.04248
0.140	4	63.4334
0.200	3	75.93333
0.300	2	91.50
0.390	1	100

Table 4. Comparison of tables of group sizes and minimum of $E(B)$, from Calculus method, the new Newton's method and Computer search

	1-type	swgds results	Newton's method	results	Computer search	results
p	x	min E(B)	x	min E(B)	x	min E(B)
0.001	44.72	5.584702	45.35625	5.584269	45	5.584404
0.002	31.62	7.546292	32.24404	7.545137	32	7.545311
0.003	25.81	9.074705	26.4305	9.072637	26	9.073625
0.004	22.36	10.3781	22.96217	10.37514	23	10.37515
0.005	20.00	11.53743	20.59334	11.53345	21	11.53523
0.006	18.26	12.59405	18.8433	12.58906	19	12.5894
0.008	15.81	14.48954	16.38379	14.48223	16	14.48546
0.010	14.14	16.17897	14.7024	16.1693	15	16.17185
0.020	10.00	22.96568	10.5101	22.94456	11	22.96215
0.030	8.16	28.3199	8.633392	28.28898	9	28.30523
0.035	7.56	30.68915	8.008529	30.65396	8	30.65397
0.040	6.90	32.94944	7.501856	32.87152	8	32.91687
0.045	6.67	35.00754	7.079571	34.96645	7	34.96792
0.050	6.32	37.00267	6.720023	36.95753	7	36.97727
0.055	6.03	38.90481	6.408612	38.859	6	38.91266
0.060	5.77	40.72972	6.135088	40.68196	6	40.68822
0.070	5.35	44.17189	5.673842	44.1261	6	44.16687
0.090	4.71	50.41398	4.978194	50.37068	5	50.37094
0.100	4.47	53.27326	4.704362	53.23555	5	53.2882
0.130	3.95	61.05036	4.058404	61.03933	4	61.04248
0.140	3.78	63.43124	3.883239	63.42022	4	63.4334
0.200	3.16	75.9421	3.061879	75.92689	3	75.93333
0.300	2.58	92.0984	2.087468	91.47275	2	91.50
0.390	1.50	100.0717	1.327092	99.87957	1	101

Table 5.

The Results for the two group sizes

Relative performance of two-type step-wise designs assuming continuous variation in x and y for $f = 100$ and for specific values of p, using Newton's minimization technique

	2-type swgs		
p	x	y	minE(B)
0.003	80.938730	8.724474	5.566827
0.004	67.052862	7.923704	6.588446
0.005	57.935053	7.352909	7.536420
0.006	51.404227	6.916687	8.429295
0.008	42.542155	6.279305	10.09200
0.010	36.711912	5.824516	11.63225
0.020	23.078892	4.603204	18.24997
0.030	17.448976	4.002685	23.8273
0.035	15.647967	3.793028	26.36976
0.045	13.044342	3.470921	31.09307
0.050	12.061841	3.342673	33.30581
0.055	11.223021	3.229941	35.43449
0.060	10.493023	3.129687	37.48759
0.070	9.292236	2.958290	41.39354
0.090	7.536313	2.697576	48.53655
0.100	6.868085	2.597662	51.8272
0.140	4.385241	1.348095	63.69427

Table 6. Relative performance of two-type step-wise designs assuming continuous variation in x and y for $f = 100$ and for specific values of p, using the Calculus approach,first derivative

	2-type swgs		
p	x	y	minE(B)
0.003	76.31	8.71	5.570482
0.004	63.00	7.94	6.593604
0.005	54.29	7.37	7.542812
0.006	48.08	6.93	8.436744
0.008	39.69	6.30	10.10161
0.010	34.20	5.85	11.64373
0.020	21.54	4.64	18.26635
0.030	16.44	4.05	23.84331
0.035	14.84	3.85	26.38454
0.045	12.55	3.54	31.10407
0.050	11.70	3.42	33.31489
0.055	10.98	3.31	35.44136
0.060	10.36	3.22	37.49332
0.070	9.35	3.06	41.39792
0.090	7.90	2.81	48.54949
0.100	7.37	2.71	51.85386
0.140	5.89	2.43	63.67094
0.200	4.64	2.15	78.28644
0.250	4.00	2.00	88.40234
0.300	3.54	1.88	97.0673

Table 7. Relative performance of two-type step-wise designs assuming continuous variation in x and y for $f = 100$ and for specific values of p, using Computer Search

	2-type swgs		
p	x	y	minE(B)
0.003	82	9	5.57
0.004	67	8	6.59
0.005	56	7	7.54
0.006	52	7	8.43
0.008	42	6	10.10
0.010	37	6	11.63
0.020	24	5	18.27
0.030	17	4	23.83
0.035	16	4	26.38
0.040	15	4	28.82
0.045	14	4	31.17
0.050	12	3	33.36
0.055	11	3	35.45
0.060	10	3	37.50
0.070	9	3	41.40
0.090	8	3	48.58
0.100	7	3	51.89
0.130	5	2	60.85
0.140	4	4	63.43*

Table 8. Relative performance of two-type step-wise designs assuming continuous variation in x and y for $f = 100$ and for specific values of p, results for both the approach used in Calculus method, Newton's minimization technique and method of Computer Search

	2-type swgs			2-type swgs			2-type swgs		
p	x	y	minE(B)	x	y	minE(B)	x	y	min
0.003	76.31	8.71	5.570482	80.938730	8.724474	5.566827	82	9	5
0.004	63.00	7.94	6.593604	67.052862	7.923704	6.588446	67	8	6
0.005	54.29	7.37	7.542812	57.935053	7.352909	7.53642	56	7	7
0.006	48.08	6.93	8.436744	51.404227	6.916687	8.429295	52	7	8
0.008	39.69	6.30	10.10161	42.542155	6.279305	10.09201	42	6	10
0.010	34.20	5.85	11.64373	36.711912	5.824516	11.63225	37	6	11
0.020	21.54	4.64	18.26635	23.078892	4.603204	18.24997	24	5	18
0.030	16.44	4.05	23.84331	17.448976	4.002685	23.8273	17	4	23
0.035	14.84	3.85	26.38454	15.647967	3.793028	26.36976	16	4	26
0.045	12.55	3.54	31.10407	13.044342	3.470921	31.09307	14	4	31
0.050	11.70	3.42	33.31489	12.061841	3.342673	33.30581	12	3	33
0.055	10.98	3.31	35.44136	11.223021	3.229941	35.43449	11	3	35
0.060	10.36	3.22	37.49332	10.496023	3.129687	37.48759	10	3	37
0.070	9.35	3.06	41.39792	9.292236	2.958290	41.39354	9	3	41
0.090	7.90	2.81	48.54949	7.536313	2.6975576	48.53655	8	3	48
0.100	7.37	2.71	51.85386	6.868085	2.597662	51.8272	7	3	51
0.140	5.89	2.43	63.67094	4.385241	1.348095	63.69427	4	4	63

Table 9.

Results for the Three group sizes

Performance of 3-type stepwise group screening design under the assumption of continuous variation in group sizes, x,y and z for $f = 100$ and for specified values of p, from the calculus method, first derivative

p	x	y	z	min E(B)
0.005	89.44	20.00	4.47	6.778895
0.006	78.01	18.26	4.27	7.66
0.008	62.87	15.81	3.98	9.30
0.010	53.18	14.14	3.76	10.85
0.020	31.62	10.00	3.16	17.66
0.030	23.33	8.16	2.86	23.41728
0.035	20.78	7.56	2.75	26.07776
0.045	17.21	6.67	2.58	31.03866
0.050	15.91	6.32	2.51	33.36919
0.055	14.81	6.03	2.46	35.61395
0.060	13.87	5.77	2.40	37.7814

Table 10. Performance of 3-type stepwise group screening design under the assumption of continuous variation in group sizes, x,y and z for $f = 100$ and for specified values of p, from the Newton's method

p	x	y	z	min E(B)
0.005	95.671992	19.643863	4.394413	6.778895
0.006	83.242768	17.896832	4.192884	7.652503
0.008	66.706277	15.442385	3.892769	9.299821
0.010	56.067391	13.765516	3.674256	10.84471
0.020	32.162881	9.598217	3.069063	17.61455
0.030	22.848103	7.761596	2.765695	23.41481
0.035	19.969030	7.166771	2.661847	26.07378
0.045	15.917648	6.336470	2.513488	31.02579
0.050	14.425568	6.066234	2.465330	33.34778
0.055	13.159185	5.921829	2.440499	35.58076
0.060				

Table 11. Performance of 3-type stepwise group screening design under the assumption of continuous variation in group sizes, x,y and z for $f = 100$ and for specified values of p, from the Computer search

p	x	y	z	min E(B)
0.005	92	18	4	6.78
0.006	81	17	4	7.65
0.008	68	16	4	10.85
0.010	58	15	4	9.30
0.020	31	9	3	17.62
0.030	23	8	3	23.42
0.035	20	8	3	26.10
0.040	18	7	3	28.64
0.045	16	7	3	31.07
0.050	12	5	3	33.36

Table 12. Performance of 3-type stepwise group screening design under the assumption of continuous variation in group sizes, x,y and z for $f = 100$ and for specified values of p, from the, Calculus method,first derivatives, Newton method and Computer search

p	x	y	z	min E(B)	x	y	z	min E(B)	x	y
0.005	89.44	20.00	4.47	6.778895	95.671992	19.643863	4.394413	6.778895	92	18
0.006	78.01	18.26	4.27	7.66	83.242768	17.896832	4.192884	7.652503	81	17
0.008	62.87	15.81	3.98	9.30	66.706277	15.442385	3.892769	9.299821	68	16
0.010	53.18	14.14	3.76	10.85	56.067391	13.765516	3.674256	10.84471	58	15
0.020	31.62	10.00	3.16	17.66	32.162881	9.598217	3.069063	17.61455	31	9
0.030	23.33	8.16	2.86	23.41728	22.848103	7.761596	2.765695	23.41481	23	8
0.035	20.78	7.56	2.75	26.07776	19.969030	7.166771	2.661847	26.07378	20	8
0.045	17.21	6.67	2.58	31.03866	15.917648	6.336470	2.513488	31.02579	16	7
0.050	15.91	6.32	2.51	33.36919	14.425568	6.066234	2.465330	33.34778	12	5

Discussion

The main aim of group screening design is to do the selection and separation of the defective and non-defective items, factors or observations from the population of interest.The selection is done, but with the aim of performing the actions with a lot of accuracy, while minimizing all the costs of doing the selection.This is done by minimizing the expected total number of tests.

The Newton's method of minimization is a better method as one uses the equation of the expected total number of tests and performs the Newton's iterations until one reaches a point where one cannot do iterations any longer due to the last iteration done being similar to the second last iteration done.This Newton method is proven to be a better method as the results obtained are more accurate and when the results are compared to the results of the computer search,they are better than the results obtained by [**manene2002multi**].

The study is mainly focused on minimization of the expected total number of observations, (runs or tests) using a different method that other researchers have not dwelt on.Other researchers in the area of group screening design mainly used the calculus way of differentiation by obtaining only first derivatives, which due to the complications of the equation of the expected total number of runs(tests),and consequently doing some approximations to the equation of the expected total number of observations (runs or tests), lead them to forcefully take into considerations some assumptions which in turn lead to the results being obtained having some approximations.In my research, however,I have focused on using the main equation of expected total number of runs in totality, as a whole which has taken into consideration, the aspect of approximations, hence , basically,the results that are obtained from the iterations produced more precise and better results without having any approximations to the equation, since the equation has not been approximated in any way.

In this study, the Newton's method, as one of the tools for optimization of the group sizes has been used in order to get a more precise optimized results for the group sizes so as to improve the values that would be able to produce the most minimum values of the expected total number of runs or tests that are needed.
As Newton's method being the main center of focus, the calculus method that had been used by Manene in his research [**manene2002multi**], was used as the main source of

comparison with the Newton's method.Newton's method has been used in this study, as a method to improve the results that [**manene2002multi**] did find in their study.

In [**manene2002multi**], the expression for the expected total number of runs has been approximated first.The optimum group sizes are thereafter obtained by getting the first derivative and equating the expression to zero.Newton's method came in as a method to obtain the optimum group sizes, without any approximations whatsoever, as it has been illustrated in the above chapters.The differentials for obtaining the Hessian matrix have been obtained from the original equation without change.

The above results are tables obtained from the Newton's method of minimization,the Calculus method and the method of Computer Search as indicated for each table. In this case, the minimization of the expected total number of runs, $E(B)$, is obtained using all the three methods.
These tables are compared to other two tables, the first one that [**manene2002multi**] got his results, the calculus method, while the second one is the table showing results on the performance of the computer search in obtaining the most accurate results of the first group size, in the first four tables,(1), (2), (3), and (4),the first and the second group sizes, in the second four tables, (5), (6), (7), and (8), and finally the first, the second and the third group sizes, in the third four tables,(9), (10), (11) and (12) respectively.
This improvements can be seen vividly from the tables in chapter five of the study.
First, it should be taken into consideration, the different ways in which the two methods obtained their optimum group sizes for each case.This is because, the calculus method used a lot of approximations, which were as a result of the assumptions that were put into considerations initially as explained in the earlier chapters.Newton's method is used as a tool to improve on the gaps caused by these assumptions, hence taking care of the approximations.These results, from both, the calculus and Newton, methods were both compared with the results obtained from the computer search.It has been clearly shown, from the results in the different tables that the Newton's method used here is giving out better results, that are almost equal to the results obtained from the computer search.This is a significant improvement in the field of statistics which should be recognized and more research done on the same.

Discussions of the results obtained after doing the iterations using the Newton's method as compared to what results Manene had. In the article,[**manene2002multi**]

5.1 Results from tables, $1, 2, 3$ and 4

Comparison of the results obtained from the calculus method and the Newton's method, both the two methods compared to the results form the computer search for the one group size x.

5.1.1 Results for one type type step-wise group screening design

Upon taking observation of the first three tables, $(1), (2)$ and (3) respectively,
First, one realizes that there is an improvement of the choice of the group sizes, given specified probability of defective, p.This is realized as the values of the group size, x, is seen to be more close to the values of the group sizes obtained from the computer search.Taking an example in the case of p value being 0.004,in table (4), the value of the group size in Newton's method is closer to the group size obtained using the computer search than the group size obtained using the calculus method.These results are observed to be true, along the table.

The next observed and important part to note is the values of the expected total number of runs or tests which have also been improved.It should be noted that the values of the expected total number of tests obtained are strictly dependent on the choice of the group size one chooses to operate with, which is consequently dependent on the specified values of p, the probability of a factor being defective.

5.2 Results from tables, $5, 6, 7$ and 8

Comparison of the results obtained from the calculus method and the Newton's method, both the two methods compared to the results form the computer search for the two group sizes, x and y.

5.2.1 Results for two-type step-wise group screening design

It is observed , from the second four tables from the above chapter,that the improvement made by the Newton's method is realized.
This can be seen , especially in the values of the first group size, x,.This clearly means that the choice of the first group size from the population, has been improved greatly by the Newton's method, as it is clearly seen how the values of the first group size is closer to the results of the group sizes from the computer search, as compared to how close the values of the group size obtained by the calculus method results are to the computer search results.This can be seen from the observation across the table (8), specifically focusing on the p value of 0.030, where the value of the first group size, 17.448976, in Newton's method is closer to the value of the first group size in the computer search, 17, as compared to how close the value of the first group size in the calculus method, 16.44, is close to the one obtained from the computer search.

This consequently has a systematic change in the values of the minimum values of the expected number of runs or tests, all through, along the table.The values of the second group sizes have also been improved along the table, but not the extent of the improvement of the first group size.

5.3 Results from tables, $9, 10, 11$ and 12

Comparison of the results obtained from the calculus method and the Newton's method, both the two methods compared to the results form the computer search for the three group sizes, x, y and z.

5.3.1 Tables obtained from the three type stepwise group screening design

In the case of three group sizes, the case is not any different as the other cases, for two and one group sizes.This can be seen from the illustration from the last four tables of the three group sizes,Newton's method is seen to have made an improvement to the calculus method as the values of the optimized group sizes in the Newton's method are closer to the values obtained from the computer search more than how the same values are towards the calculus method.This is evidenced especially in the first and second group sizes,along the table, in table (12).

An example of the illustration is in table, (12), a case when the probability of a factor being defective is 0.006,where the values of the first, 83.242768 and the second group sizes, 17.896832 obtained from Newton's method are closer to the computer search method,$x = 81$, and $y = 17$ more than the case of calculus method, where x, the values of first group size is 78.01, and the second group size, 18.26.The same case applies to all the cases along the table.
The values obtained for the third group size is also showing an improvement from the Calculus method, even though slightly.

This study has actually shown that the study of Newton's method as a tool to improve the optimization of the group sizes, such that there is minimization of the expected total number of runs or tests is really possible. This has been demonstrated greatly by above tables and the discussion from the said tables.
This analysis of the tables have also shown that Newton's method has improved on the choice of selection of the group sizes, to bring out the best and more precise results among all, during the minimization of the expected total number of tests.This was demonstrated during the several iterations done in the r-programming language,which could show optimization of the group sizes was done.

5.4 Discussions on group screening plans for different types of group screening designs with their relative performance

Here, the numerical values tabulated above for group screening plans minimizes the expected total number of runs.This is the main aim of group screening designs.In this study, the focus was on the Step-wise group screening design, where Newton method was used as a tool in the improvement of what **[manene2002multi]** did in their research work, as they used calculus method to obtain the optimum group sizes which consequently led to minimization of the expected total number of runs or tests.This was done under the numerous assumptions, hence, thereafter, the approximations due to the earlier made assumptions.This gap was addressed by the use of Newton's method in the optimization of the group sizes, hence the minimization of the expected total number of tests, where there were no approximations in the equation of the expected total number of runs, as it was used as it is, to perform the iterations, hence, optimum group sizes, and conclusively, the minimization of the expected number of tests.

This study is focused on the step-wise group screening design, even though, there are researchers who have used other methods, in the group screening design, as tool for calculating the optimum group sizes, hence the minimization of the expected total number of tests.

Some researchers in the same area have used the finite difference method to do the optimization , even though in different stages of the group screening design.

6 Conclusions and Recommendations

6.1 Conclusions

- Calculus method was used by other previous researchers to obtain the optimum group sizes.
- The original expression for the expected total number of tests was approximated.The approximated expression was the one differentiated with respect to the various group sizes, and equated to zero.Thus, optimizing the group sizes for each case.
- This was a gap that I addressed by using the Newton's method.
- Newton's method, I used the original expression for the expected total number of tests as it is to get the derivatives, did numerous iterations, thereafter getting the optimum group sizes without any approximations of the original expressions of the total number of tests.
- The tabulated results show that the Newton method produces better results, hence fulfilling my main objective of improving the optimal group sizes for the minimization of expected total number of tests.

6.2 Recommendation

In this study, the focus was majorly on obtaining the optimum group sizes Step-wise group screening design without errors, using the Newton's method which was basically suggested to be used by [**patel1987step**].

(i) More study could further be done on other researchers who have dwelt on the same topic of group screening design.Newton's method for optimization of group sizes for the minimization of the expected total number of runs or tests could be used as a better and more precise method in these other studies.

(ii) As indicated in the earlier chapters of the project, there are other methods that could be used to obtain the optimized group sizes for the minimization of the equation of the expected total number of runs, which is the main objective in group screening design, apart from the Newton's method.This can be seen from the referenced book , [**kambo1984mathematical**].

(iii) Newton's method could be used, better still, in a case of group screening problem with errors in observations, as an improvement to what [**watson1961study**] did, as he used the ordinary calculus techniques like [**manene2002multi**] did.

(iv) Some more investigations could be done on the comparison of the stepwise screening and S-stage group screening design, using the Newton's method.

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