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USE OF HAAR WAVELETS TO SOLVE OPTIMAL CONTROL PROBLEM

Research Report in Mathematics, Number 15, 2020

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Master Thesis

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Abstract

Optimal control is an important branch in mathematics that has been widely applied in a number of fields including engineering, science and economics. We aimed at finding the performance indicator in optimal control problem for the best control by solving a nonlinear partial differential equation known as Hamilton Jacobi Bellman. In this project we established the efficiency, value addition and advantages of using Haar Wavelets in solving optimal control problem by looking at the fundamental of the optimal control theory then Hamilton Jacobi Bellman and finally application of Haar Wavelet method by solving some problems. Finally, we found out that with the Haar Wavelet function, we obtained very satisfactory exactness of the results even for a lower number of collocation points and that it was in deed of value addition in the computation of optimal control problems.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

EDWIN MURITHI NJERU
Reg No. I56/11046/2018

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

Signature Date

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Dedication

I dedicate this project to my daughters Shiphrah Mwende and Praise Mukeni for the overwhelming support and encouragement that they gave me.

Contents

Acknowledgments

I thank the Almighty God for his strength, kindness and favor and for this far he has brought me.

I wish to thank my supervisor Dr. James Okwoyo who devoted much of his time to supervise the writing of this project.

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Edwin Murithi Njeru

Nairobi, 2020.

Introduction

Optimal control problem without constraints can be solved successfully using most of the of the direct and indirect techniques. However, inequality constraints often generates both analytic and computational difficulties. Thus researchers aim at solving constrained optimal control problems with numerical methods.

Optimal control problems are divided into two categories, direct and indirect methods. The direct methods reduce an optimal control problem to a non-linear programming problems by parametrizing a discretizing the infinite dimensional optimal control problem into finite dimensional optimization problem. On the other hand the indirect method solve the Hamilton- Jacobi- Bellman equation, a first order necessary condition for optimality which are obtained from Pontryagin minimum principle. Parametrization methods are classified into three types: State, control and state control.

The optimality conditions usually lead to solution of two point boundary value problems which are difficult to solve. To overcome these difficulties various numerical methods such as shooting technique, non-linear programming, quasi-linearisation, e.t.c have been applied.

Both direct and indirect methods are important in solving optimal control problems, however, the difference between them is that, the indirect methods are believed to yield more accurate result whereas, the direct method tends to have better convergence properties.

The wavelet approach has some advantage compared with other numerical methods. Due to the sparsity of the transform matrices and small number of significant wavelet coefficients high accuracy of result is guaranteed already for the small number of grid points. The wavelet method are very convenient for solving boundary conditions since the boundary conditions are taken into account automatically

1 Chapter one

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1.1 Background for the Optimal Control

1.1.1 Performance Index:

We have to find the control history $u(t)$ that minimizes the function

$$I = \int_{t_0}^{T} F(t,x,u)dt \tag{1.1}$$

Here t, t_0, T are scalar, the state variables $x(t)$ and controls $u(t)$ are $n\ and\ r-dimensional$ vectors respectively. It is assumed that x, u have as many derivatives as are needed for the theory being developed.

1.1.2 State Variables

They are subjected to differential constraints

$$\dot{x} = f(t,x,u) \tag{1.2}$$

1.1.3 Adjoint Variables

If $\hat{H}$ denotes the Hamiltonian

$$\hat{H}(t,x,\psi) = -F + \psi f \tag{1.3}$$

then the adjoint varaibles (Langrange multiplier) are calculated from the equation

$$\dot{\psi} = -\frac{\partial \hat{H}}{\partial x} \tag{1.4}$$

Here $f\ \ and\ \ \psi$ are $n-dimensional$ vectors

1.1.4 Admissible Controls

If the vector u is unconstrained then optimal control are calculated from

$$\frac{\partial \hat{H}}{\partial u} = 0 \tag{1.5}$$

If $u \in U$, where U is a closed set, the maximum principle (Pontryagin's principle) holds.

$$\hat{H}(t, x(t), u(t), \psi(t) = \max_{u \in U} \hat{H}(t, x(t), u(t), \psi(t)) \tag{1.6}$$

For simplicity, we consider only the case where u is a scalar satisfying the inequality constraint $|u(t)| < u_0$

1.1.5 Boundary Conditions

We confine ourselves with case where the values $x_\alpha(t_0), \alpha = 1, 2 \ldots, r$ are prescribed and the remained values $x_\alpha(t_0), \beta = 1, 2, \ldots, n - r$ are free. It follows from the transversality condition that $\psi_\alpha(t_0)$ is free and $\psi_\beta(t_0) = 0$, similar results hold also for the final time $t = T$.

1.1.6 Free Final Time

If the final time T is not fixed then the complementary condition $\hat{H}|_{t=T} = 0$ hold.

If $F = 1$ we get a minimum time problem $T = min$

1.1.7 Integral Constraint

A scalar integral constrain has the form

$$\int_{t_0}^{T} G(x, u) dt = k \tag{1.7}$$

In this case, it is suitable to introduce a new state variable

$$\dot{x}_{n+1} = G(x, u), x_{n+1}(t_0 = 0) \; and \; x_{n+1}(t) = k \tag{1.8}$$

1.1.8 State Inequality Constraints

Consider the case , $g(t, x) \leq 0$

Where g is a prescribed function. It is assumed that the equality $g(t,x)=0$ holds for $t\in[t_1,t_2]$. In this case the adjoint system gets the form

$$\dot{\psi} = -\frac{\partial H}{\partial x} + \mu \bigtriangledown S, \quad t \in [t_1, t_2] \tag{1.9}$$

where $S(x,u)=\frac{\partial g}{\partial x} f, \;\; \bigtriangledown S = \frac{\partial S}{\partial x}$
Here $\mu(t)$ denotes the Langrange multiplier, it can be shown the following conditions hold;

1. $\psi(t_1+0) = \psi(t_1-0)$
2. $(t_2+0) = \psi(t_2-0) - \mu(t_2)\triangledown S(u(t_2), x(t_2))$

1.2 Control Theory

Control theory is a mathematical study of methods to steer the evolution of a dynamic system to achieve desired goals. For example stability on trading reference

Optimal control is a branch of control system that seeks to steer the evolution so as to optimize a specific objective functional. There is a close connection with calculus of variation. Mathematical study of control requires predictive model of the system evolution. Assume Markovian models everything relevant to future evolution of the system is captured in the current state.

1.2.1 Control Theory Objective

Choose input signal $u(\cdot) \in U \underline{\Delta} \{u : [0,\infty[- > u|u(\cdot)] \; \textit{is measurable}`\}$

To minimize the cost functional $J(x,u(\cdot))$ as $J(x,t,u(\cdot))$.

Many possible cost functional exist such as

Finite horizon: given horizon $T>0$ running cost ℓ and terminal cost g
$J(x,t),t,u(\cdot)\underline{\Delta}\int_t^T \mu(x(s),u(s))ds + g(x(T))$

minimum Time: given target, set $T \subset \mathbb{R}^{dx}$

$$J(x_0,u(\cdot)) \triangleq \begin{cases} \{min\{t|x(t)\in T\}\} \;\; if \;\; \{t|x(t)\in T \neq 0\} \\ +\infty \;\; otherwise. \end{cases} \tag{1.10}$$

Discounted infinite horizon: given discount factor $\lambda > 0$ and running cost Λ

$$J(x_0, u(\cdot)) \triangleq \int_0^\infty \ell(x(s), u(s)) e^{-\lambda s} ds \tag{1.11}$$

Alternatively maximize payoff functional " or " optimize objective functional"

1.3 Haar Wavelets

Wavelet analysis is a branch of mathematics and widely applied in signal analysis, image processing and numerical analysis. Wavelets method have proved to be very effective and efficient tool in solving problems of mathematical calculus. Alfred Haar introduced the notion of wavelet in 1910. His initial theory has been expanded into a wide variety of application but primarily it allows for the representation of various function by a combination of step functions and wavelet transform is one of the earliest example of what is known as a compact, dynamic, orthonormal wavelet transform.

The pioneering work in system analysis via Haar wavelets was led by Chen and Hsiao(1997), who first derived a Haar oparational matrix for the integral of the Haar function vector and paved the way for the Haar analysis of the dynamical system.

Haar wavelet was a system of square waves. The first curve was marked up as $h_0(t)$ and second curve marked up as $h_1(t)$, that is

$$h_0(t) = \begin{cases} 1, 0 \leq x \leq 1 \\ 0, otherwise \end{cases} \tag{1.12}$$

$$h_1(t) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{2} \\ -1, & \frac{1}{2} \leq x \leq 1 \\ 0, & otherwise \end{cases} \tag{1.13}$$

Where, $h_0(t)$ is scaling function, $h_1(t)$ is mother wavelet, for $x \in [0,1]$. Haar wavelet function is defined as follows $h_0(t) = \frac{1}{\sqrt{m}}$

$$h_i(x) = \frac{1}{\sqrt{m}} \begin{cases} 2^{\frac{j}{2}} & \frac{k-1}{2^j} \leq x < \frac{k-\frac{1}{2}}{2^j} \\ -2^{\frac{j}{2}} & \frac{k-\frac{1}{2}}{2^j} \leq x < \frac{k}{2^j} \\ 0 & otherwise \end{cases} \tag{1.14}$$

Integer $m = 2^j \; with \; (j = 0, 1, \ldots J)$ indicate the level of the wavelet $i = 0, 1, 2, \ldots m-1$ is the translation parameter. Maximal level of resolution is J. The index i is calculated according to the formula $i = m + k + 1$. In the case of minimal values $m = 1, \;\; k = 0$ we have $i = 2$, the maximal value of i is $i = 2m = 2^{j+1}$. It is assumed that the value $i = 1$ corresponds to the scaling function for which $h_1 = 1$ in $[0, 1]$.

1.4 Value Function

The value function specifies the best positive value of the cost functional starting from each state and possibly time.

$$V(x) = \inf_{u(\cdot) \in U} J(x, u(\cdot)) \;\; or \;\; V(x,t) = \inf_{u(\cdot) \in U} J(x, t, u(\cdot)) \tag{1.15}$$

Infimum may not be achievable and if infimum is attained then the possibly non-unique optimal input is often designated $u^*(\cdot)$ and sometimes the corresponding optimal trajectory is designated $x^*(\cdot)$

Intuitively to find the best trajectory from a point x, go to a neighbour $\hat{x}$ of x which minimizes the sum of the cost from x to $\hat{x}$ and the cost to go from $\hat{x}$.

This intuition is familiarized in the dynamic programming principle.

1.5 Dynamic Programming Principle (DPP)

The Hamilton Jacobi Bellman equation is a result of the dynamic programming principle of Bellman which allows us to split the value function.

Dynamic programming is an optimization approach that transforms a complex problem into a sequence of simpler problems; its essential characteristics is the multi-stage nature of the optimization procedure. More so than other optimization techniques, dynamic programming provides a general framework thus analyzing many problem types. Within this framework a variety of optimization techniques can be employed to solve particular aspects of a more general formulation. Usually creativity is required before one can recognize that a particular problem can be cast effectively as a dynamic program and often subtle insights are necessary to restructure the formulation so that it can be solved effectively.

1.6 Pontryasin's Minimum Principle

Pontryasin's minimum principle is closely related to the Hamilton Jacobi Bellman equation and provides condition that an optimal trajectory must satisfy . However that the minimum principle provides necessary conditions but not sufficient conditions for optimality in contrast the Hamilton Jacobi Bellman equation offered sufficient conditions. Using the minimum principle alone one is soften not able to conclude that a trajectory is optimal however in same cases it is quite useful for finding candidate optimal trajectories. Any trajectory that fails to satisfy the minimum principle cannot be optimal.

Following the method of Lagrange a control Hamiltonian function can be constructed by appending the state equation to the integrand L using the Lagrange multipliers $\lambda(t)$ as follows;

$$H(u(t),z(t),\lambda(t),t) = L(u(t),z(t),t) + \lambda^T(t)f(u(t),z(t),t) \tag{1.16}$$

where $z(t)$ is the optimal control and $u(t)$ is the corresponding optimal state. Then the Pontryagin minimum principle states that there exists a continuous function λ known as an **adjoint function** that is the solution of the **adjoint equation**.

$$\dot{\lambda(t)} = -H_u(u(t),z(t),\lambda(t),t) \tag{1.17}$$

along with the appropriate initial (or find) condition of λ. In equation above H_u denotes the differentiation of the Hamiltonian function with respect to the state. In particular the adjoint function is a Lagrange multiplier that brings the information of the state equation constraint to the optimization problem. According to the PMP the optimal control $z(t)$ and corresponding optimal state $u(t)$ and adjoint, $\lambda(t)$ must minimize the Hamiltonian so that

$$H(u(t),z(t),\lambda(t),t) \leq H(u(t),z^*(t),\lambda(t),t) \tag{1.18}$$

for all time and for all admissible (i.e feasible) trajectory control variables $z^*(t)$ while the adjoint equation above is satisfied. Admissible trajectories are defined as a set of variables that lay in the neighborhood of the minimal solution and satisfies all of the constraints.

1.7 Calculus of Variations

Calculus of variations is extremely useful in solving optimization problems. Queen Dido of Carthage was apparently the first person to attack a problem that can readily be solved by using variational calculus.

Dido having been promised all the land she could enclose with a bull's hide, cleverly she cut the hide into many lengths and tied the ends together. Having done this her problem was to find the closed curve with a fixed perimeter that encloses to maximum area.

The first problem in calculus of variation was the Branchistochrone problem formulated by J. Bernoulli in 1906, consider a bead sliding under gravity along a smooth wire joining two fixed point A *and* B(not on a vertical line). What is the shape of the wire in order that the bead when released from rest at a point A slides to B in minimum time?

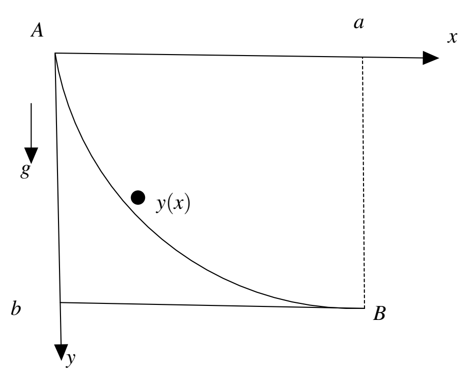


Figure 1

The figure shows the choice of axes with A taken to be the origin without loss of generality. Here we require to minimize

$$\int_A^B dt = \int_A^B \frac{ds}{v} \tag{1.19}$$

where s is the arc length along the wire and v is the instantaneous speed of the bead.

The theory of optimal control allows for the solution of a large class of non-linear control problem subject to complex state and control signal constraints. The theory is an extension of classical calculus of variation since it does not rely on the smoothness assumption, indeed in most cases the optimal control is highly discontinuous (bang- bang control, control along switching curves, sliding control). The formulation of the problem involves the minimization of cost function subject to initial and terminal constraints which is reminiscent of calculus of variation problems. The optimal control signal is typically obtained either as a function of time $u^*(t)$ or more interestingly as control application in feedback form as a function of state $u^*(x)$.

1.8 Performance Measures for Optimal Control Problems

The optimal control problem is to find a control $u^* \in U$ which causes the system

$$\dot{x}(t) = a(x(t), u(t), t) \tag{1.20}$$

to follow a trajectory $x^* \in X$ that minimizes the performance measure.

$$J = h(x(t_f), t_f) + \int_{t_o)}^{t_f} g(x(t), u(t), t)dt \tag{1.21}$$

1.8.1 Minimum Time Problems

Problem: To transfer a system from an arbitrary initial state $X(t_0) = X_0$ to a specified target set S in minimum time

The performance measure to be minimized is $J = t_f - t_0$

with t_f the first instant of time when $X(t)$ and S intersect. A typical examples are the interception of attacking aircraft and missiles and the skewing mode operation of a radar or a gun system.

1.8.2 Terminal Control Problem

Problem: To minimize the deviation of the final state of a system from its derived value $r(t_f)$.

A possible performance measure is

$$J = \sum_{i=1}^{n} [X_i(t_f) - r_i(t_f)]^2 \tag{1.22}$$

Since positive and negative derivations are equally undesirable, the error is squared.

1.8.3 Minimum-Control-Effort Problems

Problem: To transfer a system from an arbituary initial $X(t_0) = X_0$ to a specified target set S, with a minimum expenditure of control effort. The meaning of the term "minimum control effort" depends upon the particular physical application; therefore, the performance measure may assume various terms. For example consider a spacecraft on an interplanetary exploration. Let $u(t)$ be the thrust of the rocket engine and assume that the magnitude of thrust is proportional to the rate of diesel consumption. In order to minimize the total expenditure of fuel the performance measure

$$J = \int_{t_0}^{t_f} |u(t)| dt \tag{1.23}$$

would be selected. If there are several controls and the rate of expenditure of control effort of the i^{th} control is $C_i|u_i(t)|, i = 1, \cdots m$ (C_i is a constant of proportionality) then minimizing

$$J = \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} \beta_i |u_i(t)| \right] dt \tag{1.24}$$

would minimize the control effort expanded. The $\beta_i' s$ are non negative weighing factor.

As another example consider a voltage source driving a network containing non energy storage elements. Let $u(t)$ be the source voltage and suppose that the network is to be controlled with minimum source energy dissipation. The source current is directly proportional to $u(t)$ in this case so to minimize the energy dissipated, minimize the performance measure

$$J = \int_{t_0}^{t_f} u^2(t) dt \tag{1.25}$$

For several control inputs the general form of performance measure corresponding to equation above is

$$\begin{aligned} J = & \int_{t_0}^{t_f} [u^T(t) R u(t)] dt \\ = & \int_{t_0}^{t_f} ||u(t)||_R^2 dt \end{aligned} \tag{1.26}$$

where R is a real symmetric positive definite weighting matrix. The elements of R may be functions of time if it is derived to vary the weighting control effort expenditure during the interval $[t_0, t_f]$

1.8.4 Trading Problems

Problem: To maintain the system state $X(t)$ as close as possible to the desired state $r(t)$ in the interval $[t_0, t_f]$

As a performance measure we select

$$J = \int_{t_0}^{t_f} ||X(t) - r(t)||^2_{Q(t)} dt \tag{1.27}$$

A real symmetric matrix R is positive definite if $Z^T RZ > 0$ for all $Z \neq 0$

where $Q(t)$ is a real symmetric $n \times n$ matrix that is positive semi definite for all $t \in [t_0, t_f]$. The elements of the matrix Q are selected to weight the relative importance of the different components of the state vector and to normalize the numerical values of the derivations. If the set of admissible controls is bounded e.g $|u_i(t) \leq 1|, i = 1, 2, \cdots, m$ the equation above is a reasonable performance measure. However , if the controls are not bounded minimizing equation above results in controls with impulses and few derivatives. To avoid placing bounds on the admissible controls, or if control energy is to be conserved, we use the modified performance measure.

$$J = \int_{t_0}^{t_f} [||X(t) - r(t)||^2_{Q(t)} + ||u(t)||^2_{R(t)}] dt \tag{1.28}$$

$R(t)$ is a real symmetric positive definite $m \times m$ matrix for all $t \in [t_0, t_f]$.

It may be especially important that the rates be close to their desired values at the final time. In this case the performance measure

$$J = ||x(t_f) - r(t_f)||^2_H + \int_{t_0}^{t_f} [||X(t) - r(t)||^2_{Q(t)} + ||u(t)||^2_{R(t)}] dt \tag{1.29}$$

could be used. H is a real symmetric positive semi definite $n \times n$ matrix.

1.8.5 Regulator Problems

A regulator problem is the special case of a trading problem which results when the derived state values are zero $(r(t) = 0$ for all $t \in [t_0, t_f])$

1.9 Selecting Performance Measure

In selecting a performance measure the designer attempts to define a mathematical expression which when minimized indicates that the system is performing in the most desirable manner. Thus choosing a performance measure is a translation of the system is physical requirements into mathematical terms. In particular suppose that two admissible control histories which cause admissible state trajectories are specified and we are to select to better one.

To evaluate these controls, perform the test shown in the figure below.

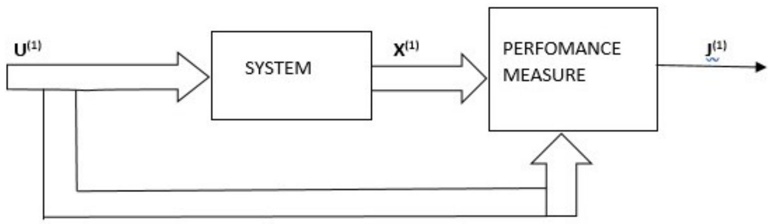


Figure (a)

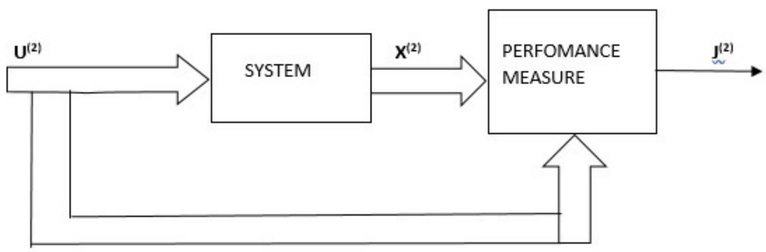


Figure (b)

:s)

First apply the control $u^{(1)}$ to the system and determine the value of the performance measure $J^{(1)}$ then repeat this procedure with $u^{(2)}$ applied to obtain $J^{(2)}$. If $J^{(1)} < J^{(2)}$ then we designate $u^{(1)}$ as better control. If $J^{(2)} < J^{(1)}$ $u^{(2)}$ is better. If $J^{(1)} = J^{(2)}$ the two controls are equally desirable.

An alternative test is to apply each control, record the rate trajectories and then subjectively decide which trajectory is better.

If the performance measure truly reflects desired system performance the trajectory selected by the designer as being more to his liking should yield the smaller value of J. If this is not the case the performance measure or the constraints should be modified.

1.10 Statement of the Problem

Haar Wavelet is a sequence, re-scaled square shaped function which together form a wavelet family or basis which allows a target function over an interval to be represent in terms of an orthonormal basis. It possesses useful properties such as orthogonality and has been applied to a wide range of application such as in system analysis (Chen and Hsiao, 1999) and numerical solution of nonlinear integral equation (Aziz and Siraj, 2013). In addition, the function has been used to solve optimal control problems and other applications. However, the problem is that, the real value addition of using Haar wavelet function in solving optimal control problem had not been assessed and therefore its full adoption and application has not been embraced. Therefore, we established the value addition of using Haar Wavelet function in solving optimal control problems.

1.11 Objective of the Study

In this project we aimed at:

1. To establish the value addition of using Haar wavelet function in solving optimal control problem.
2. To establish the properties of Haar wavelet function appropriate in solving optimal control problems.

1.12 Methodology of the Study

The main objective of this project was to establish the value addition of using Haar Wavelet function and to establish its feature. To achieve this the following approach was employed; first we reviewed the existing literature on the optimal control, Hamilton Jacobi Bellman and Haar Wavelet in order to find out what has already been done/ studied on this subject including understanding in details their development/evolution over time. Then three optimal control problems with different constraints were solved by introducing the Haar Wavelet and the results thereof compared to the exact solution in order to establish any similarity or otherwise of the solution in addition to assessing the complexity or simplicity of solving the problems.

2 Chapter 2

2.1 Literature Review

This chapter covers the literature review aspect in the area of optimal control Hamilton Jacobi Bellman and the Haar wavelet function. We looked at the various researches that were done in these areas and kind of conclusion that they came up with from their study. The section is outlined as follows : Introduction, the studies done on optimal control, then Haar wavelet equations and then a study on Hamilton Jacobi Bellman.

2.2 Introduction

Many researchers have studied the theoretical aspects of the inequality constraints of the trajectory. Mehva and Davis (1972) noted that the complications in conjugate gradient methods were caused by the exclusive use of control variables as independent variables in the search procedure. In response they presented the so called generalized gradient technique. Jaddu (1998) established some numerical methods based on a parametrization technique with Chebyshev polynomials to solve unconstrained and constrained optimal control problem using quasi linearlization method. Jaddu(2002) later extended this concept to non linear optimal control problems with terminal state and control inequality constraints as well as to simple boumds on state variables.

Infact Doi and Cochran (2009) converted optimal control problem into non linear programming (NLP) parameters at the collocation points using Haar wavelet technique. Han and Li (2011) also presented a numerical method to solve non linear optimal control problems with terminal state and state control inequality constraints. This method is based on quasilinearlization and Haar function. In addition Marzban and Razzaghi (2010) presented a numerical method to address constrained and non linear optimal control problems. Although their method was also based on Haar wavelets but it required a set of necessary conditions.

2.3 Studies on Optimal Control

In the study titled, "comparative analysis of numerical solution of optimal control problems" by Shangareera, Origoryev and Mustahna (2016) they developed a step by step algorithms of solving optimal control problem based on the method of successive approximation and the method of variations in the space of controls. They were seeking a numerical study and comparative analysis of the developed algorithms performed at different values of accuracy.

This was mainly driven by the larger variety of statements of problems of algorithms control and the specified difficulties causing there to be many various approaches for their numerical decision now.

The statement problem was the following optimal control that was considered

minimize

$$I(u) = G(x(T)) \tag{2.1}$$

subject to

$$\frac{dx_i}{dt} = f_i(t, x(t), u(t)), \;\; t \in [t_0, T], x(0) = x_0 \tag{2.2}$$

$$\phi(u) \leq 0 \tag{2.3}$$

where $u(t) \in R$ is the function characterizing the operating influence $x(t) \in R^n$ is function describing a condition of process and t is time.

In their conclusion, they had compared a method of variation with successive approximation method and that by comparison of the numerical results of the methods with exact solution the performance of the methods had been confirmed.

Vasilieva (2007) did a study on optimal control in the class of smooth and bounded functions. In his paper the objective was to develop the optimality conditions and optimization technique for a control problem with boundary condition whose class of admissible controls contains smooth continuously differentiable functions with inclusion or homogeneous inclusion constraints.

The investigation technique was the increment of the objective functional together with conjugate boundary value problem being considered as a certain type of control variations thus providing the admissibility of varied control under same adjustments of the parameters of variations. In contrast to the classic variation of lagrange and the needle-shaped variation of Boltyanskii. Statement of the problem was controllable process;

$$\{u, x\} = \{u(t) \in R^m \; x(t) \in R^n, \;\; t \in T = [t_0, t_1] \tag{2.4}$$

be defined by the conditions

$$J(u) = \phi_0(x(t_0), x(t)) + \int_T F(x,u,t)dt$$
$$\dot{x} = f(x,u,t) \quad \phi(x(t_0), x(t_1)) = 0 \tag{2.5}$$

Here admissible control $u(t), t \in T$ are smooth vector functions with fixed end points $u^0, u^1 \in R^m$ that is

$$u(\cdot) \in U = \{u \in C_1^m(T) : u(t_0) = U^0, u(t_1) = U^1\} \tag{2.6}$$

In conclusion Olga Vasilieva noted that the application of general methods for solving real world problems requires alot of computational effort and is rarely successful Algorithmic support to non-linear optimal control problems is provided by numerical methods of the gradient type linearlization methods and methods based on the maximum principle. The presence of control of constraints in many cases leads to the application of numerical techniques based on the maximum principle where the control trajectories are adjusted by mean of needle shaped or combined variation. The majority of such techniques rest on a compulsory assumption of explicit solvability of the maximum condition with respect to the maximal Hamiltonian function such as an assumption becomes completely excessive if the control trajectories are adjusted by means of interior variations.

Russu(2012), in his study titled, "the optimality of limit cycles in nature based tourism". He mentioned that virgin nature as well as historical and cultural monuments located in National parks all form part of national heritage. Then one of the objective of the administration of the governmental institution (National park) is to maximize profits from tourism and recreation. That is the difference from revenue and expenditure on recreation investments.

In the paper he tried to model some relevant aspects of these prey predator relations. The model was formulated in terms of optimal control theorem and transformed into an augmented dynamic system by means of optimal choice of control variables resulting from the application of Pointryagin maximum principle.

Formulation of the problem.

The model problem was formulated with two state variables

$x(t)$ vulnerable nature resources

$v(t)$ a number of persons visiting a national part for a period of 24 hours.

In his conclusion he noted that for reasonable parameter values the optimal trajectory exhibits a cyclical behaviour that is to mean large investments call many visitors this generates large revenues. The continuous cyclical pattern in fact allows the National park to regenerate.

2.4 Researches done on Haar Wavelet

In their study titled, " Haar wavelet method for numerical solution of telegraph equation " by Navesh,Dinesh and Parihar (2013),sought to modify the result given by Hariharan (2009) on the solution of fisher's equation based on the above topic. In their abstract, they stated that, "we are giving the solution of second order linear hyperbolic telegraph equation in one - space dimension. The telegraph equation is solved numerically by Haar wavelet method". They noted that telegraph equation appeared in many engineering field, such as modeling of anomalous diffusive and wave propagation phenomenon, modeling of anomalous diffusion and sub-diffusive systems continuous line random walk.

They used the second- order linear hyperbolic telegraph equation in one space dimension given by

$$\frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} + \beta^2 u = \frac{\partial^2 u}{\partial x^2} + f(x,t) \quad a \leq x \leq b, t \geq 0 \tag{2.7}$$

subject to initial conditions :

$$u(x,0) = f(x) \qquad a \leq x \leq b$$
$$\dot{u}(x,0) = f_1(x) \qquad a \leq x \leq b$$

and the Dirichlet boundary condition $u(a,t) = g_0(t), \quad u(b,t) = g_1(t), \qquad t \geq 0$

where α *and* β are known constant coefficients for $\alpha > 0, \ \ \beta = 0$ equation above represent a dumped wave equation and for $\alpha > \beta > 0$, u is called telegraph equation. They assumed that $f(x), f_1(x)$ and their derivation are continuous function of x and $g_0(t), \ g_1(t)$ and their derivatives are continuous function of t, both the electric voltage and the current in double conductor satisfy the telegraph equation where x is distance and t is time

In their conclusion they noted the following," Haar wavelet method is proposed for the numerical solution for the second order hyperbolic telegraph equation. Approximate solution of the telegraph equation obtain by MATLAB are compared with exact solution. This calculation demonstrate the accuracy of Haar wavelet solution.

The main advantage of their method is its simplicity and small computation cost, which is due to the sparsity of transform matrices and the small number of significant wavelet coefficient. It is worthy mentioning that Haar solution provide excellent result even for small values of $m(M = 16)$, for larger values of M we can obtain the result closer to the real values"

In the study by Siraj, Imaran, and Bozidar(2010), they looked at a concept of, " The numerical solution of second- order boundary value problems by collocation method with Haar wavelets ". They were seeking an efficient numerical method based on uniform Haar wavelets for the numerical solution of second order boundary value problem arising in the mathematical modeling of the deformation of beams and plate deflection theory, deflection of a cantilever beam under a concentrated load, obstacle problems and many other engineering applications.

They noted that Haar wavelets have gained popularity among reserchers for the useful properties such as simple, applicability, orthigonality and compact support. Compact support of Haar wavelets basis permits straight inclusion of the different types of boundary condition in the numerical algorithm.

The objective of their study was to construct a simple collocation method with the Haar basis function for the numerical solution of linear and non-linear second order BVP_s arising in the mathematical modeling of different engineering application. To test the applicability of the Haar wavelets they focused on the following types of boundary value problem defined in the interval $[a,b]$

$$y'' = \phi(x,y,y') \tag{2.8}$$

subject to the following six sets of boundary conditions that cope a reasonable spectrum of possible cases two different type of peridic boundary condition (PBCs)

(i) $y'(a) = \alpha_1 \qquad y'(b) = \beta_1$

(ii) $y(a) = \alpha_2 \qquad y(b) = \beta_2$

(iii) $y'(a) = \alpha_3 \qquad y(b) = \beta_3$

(iv) $y(a) = \alpha_4 \qquad y'(b) = \beta_4$

(v) $y(a) = y(b) \qquad y'(a) = y'(b)$ PBCs

(vi) $y(a) = \alpha_5 \qquad y(c) = y(b) \quad forc \in (a,b)$
where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta_1, \beta_2, \beta_3, \beta_4$ are real value constant and $a = 0, b = 1$

In their conclusion they noted that a simple and straightforward numerical technique based on the Haar wavelets was proposed in the numerical solution of different types of linear and non-linear second order ODEs. Minor modifications were needed to apply the same method to different sets of boundary conditions. The distinctive features was that it can be applied to initial and boundary value problems without transformation of BVPs and IVPs as needed for the Runga Kutta methods. The new method showed excellent performance for highly nonlinear BVPs, simple applicability and fast convergence of the Haar wavelets provide a solid foundation for using these function in the context of numerical approximation of integral equation, partial differential equations, and ordinary differential equation. They further noted that the only limitation of the approach in multi-dimensional problems is the increased computational cost due to inversion of $2M \times 2M$ sparse coefficient matrix.

Harpreet, Mittal and Mishra (2014), they looked at a topic on Haar wavelet solution on nonlinear oscillation equations. Their main objective was to present a numerical scheme using uniform Haar wavelet approximation and quasilinearization process for solving some nonlinear oscillator equations.

Nonlinear problem are of interest to many scientist and engineering because most of physical system in the real world are inherently nonlinear in nature. Many nonlinear differential equations arise in physical, chemical and biological contexts.

In their work, they considered a wave general nonlinear oscillator system of the form

$$\varepsilon u''(t) + \delta + \beta u^p(t))u'(t) - \mu u(t) + \alpha u^q(t) = g(F, \omega, t) \quad p, q \in \mathbb{N}) \tag{2.9}$$

with initial conditions

$$u(0) = y_0, \quad and \;\; u'(0) = y_1$$

depending on the parameter chosen equation above can take a number of special forms, where differentiation is with respect to independent time variable t and all parameters $\varepsilon, \delta, \beta, \mu$ *and* α are real constant Here ω is an angular frequency and $g(F, \omega, t)$. represent the periodic arising function of time with period $T = \frac{2p}{\omega}$

In their conclusion, they stated the following," The aim of the work was to represent a Haar wavelet method to solve well known nonlinear oscillator differential equations such as Duffing, Van der pol and Duffing -Van der pol with different parameters. To overcome the nonlinearities, quasilinearization was used. it was observed that the quasilinearization makes easier procedue for the Haar wavelets method to handle nonlinearity in shorter time. of computations. There was no need of iterations for achieving sufficient accuracy in numerical results. Therefore it was suggested that that quasilinearization can effectively be used to solve the nonlinear oscillator differential equations. In their method, they increased number of points $m = 2^j$ then coefficient matrix becomes ill-conditioned and it becomes difficult to find direct solutions. The Haar wavelet collocation method computes the solutions only at odd pints , however, results can be obtained at any point of the domain. The obtained numerical solutions are in a very good coincidence with those solution which are available in literature computed by other methods and indicates that the proposed method is feasible and convergent. Therefore they recommended the use of Haar wavelet to compute solution of nonlinear vibration problems.

In their study titled, "Haar wavelets matrices for the numerical solution of differential" by Sengeta, Singh and Kumar (2014), they implemented a computational scheme using Haar matrices to find the numerical solution of differential equations with known initial and boundary conditions. They also presented the exact solution, numerical solution and resultant absolute error. They noted that nowadays wavelets methods are extensively applied to the problem for numerical solution as wavelets method have several advantages are FEM and FDM. They further stated that wavelets analysis is new technique that can be performed in several ways, a continuous wavelets transform, a discretized continuous wavelet transform, and a true discrete wavelets transform.

In their conclusion, they noted that they had represented simple and straight forward numerical technique based on Haar wavelets for solving differential equations. They noted that simple the method was simple and has small computation cost and also very convenient for solving variety of boundary value problems. Their work presented numerical solution very close to the to the exact solution. So Haar wavelet method is very simple, fast and reliable. In addition, it was observed by them that Haar wavelet method can be extended for more collocation point.

2.5 Study on Hamilton Jacobi Bellman Equation

Peyrl, Herzoa and Geering (2005) did a study on numerical solution of the Hamilton Jacobi Bellman Equation for Stochastic Optimal control problems. In their work they wanted to provide a numerical solution of the Hamilton Jacobi Bellman equation for stochastic optimal control problems bearing in mind that there was a computation difficulty due to the nature of the HJB equation being a second order partial differential equation which was coupled with an optimization.

Problem formulation

They considered the $n-$ dimensional stochastic process x which is governed by the given stochastic differential equation (SDE)

$$dx = f(t,x,u)dt + g(t,x,u)dz \tag{2.10}$$

where dz denotes $k-$ dimensional uncorrelated standard Brownian motion defined on a fixed, filtered probability space $(\Omega, F\{F_t\}_{t\geq 0}, P)$. The vector u denotes the control variables constrained in some compact, convex set $U \subset R^m$ the drift term $f(t,x,u)$ and the diffusion $g(t,x,u)$ are given function

$$f : [0,T] \times G \times u \to R^n \tag{2.11}$$

$$f : [0,T] \times G \times u \to R^{n\times k} \tag{2.12}$$

for some open and bounded set $G \subset R^n$

The value functional of their study starting at an arbitrary time $t \in (0,T)$ and state $x \in G$ with respect to a fixed control law u is destined by

$$(t,x,u) = E\{\int_t^T L(s,x,u)ds + K(\tau, x(\tau))dt \tag{2.13}$$

where E denotes the expectation operator and L, K are scalar functions

$L : [0,T] \times G \times U \to R$
$K : [0,T] \times G \to R$

The final time of the problem denoted by τ is the time when the solution $x(t)$ leaves the open set

$Q = (0,T) \times G$
$\tau = inf\{s \geq t | (s, x(s)) \notin Q$

In conclusion they found out that by using a successive approximation algorithm the optimization gets separated from the boundary value problem. This makes the problem solvable by standard numerical methods. They also noted that for a problem of portfolio optimization where no analytical solution is known the numerical methods is applied.

That based on the literature review it very clear that Hamilton Jacobi Bellman equation provides the mathematical framework for optimization and that out of it the state equation is derived together with the control function. In this project we sought to find the value addition that Haar Wavelet adds in solving optimal control problem and we have already established that Haar Wavelet function possess special features/ properties that fit in very well in the optimization discussion. Therefore, we proceeded mainly focusing on the value addition and how Haar Wavelet features fits in solving optimal control problems.

3 Chapter 3

3.1 Hamilton Jacobi Bellman Equation

In dynamic programming one approximates continuously operating systems by discrete systems. However, Hamilton Jacobi Bellman (HJB) equations provides an alternative approach of approximating continuously operating systems which leads to a nonlinear partial differential equation. The derivation of the HJB equation starts by considering a process described by the state equation defined as follows;

$$\dot{x}(t) = a(x(t), u(t), t) \tag{3.1}$$

is to be controlled to minimize the performance measure

$$J = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(z), u(z), z) dz \tag{3.2}$$

where h and g are specified functions, t_0 and t_f are fixed and z are dummy variables of integration. Let us now use the impending principle to include this problem in a larger class of problems by considering the performance measure.

$$J(x(t), t, u(z))_{t \leq z \leq t_f} = h(x(t_f), t_f) + \int_{t}^{t_f} g(x(z), u(z), z) dz \tag{3.3}$$

where t can be any value less than or equal to t_f and $x(t)$ can be any admissible state value. Notice that the performance measure will depend on the numerical values for $x(t)$ and t and on the optimal control history in the interval $[t, t_f]$

Let us now attempt to determine the controls that minimize **[3.3]** for all admissible $x(t)$ and for all $t \leq t_f$. The minimum cost function is then:

$$J^*(x(t), t) = \min_{u_z, t \leq z \leq t_f} \left\{ \int_{t}^{t_f} g(x(z), u(z), z) dz + h(x(t_f), t_f) \right\} \tag{3.4}$$

By subdividing the interval we obtain

$$J^*(x(t),t) = \min_{u_z, t\leq z\leq t_f} \left\{ \int_t^{t+\delta t} g dz + \int_{t+\delta t}^{t_f} g dz + h(x(t_f),t_f) \right\} \tag{3.5}$$

The principle of optimality requires that

$$J^*(x(t),t) = \min_{u_z, t\leq z\leq t_f} \left\{ \int_t^{t+\delta t} g dz + J^*(x(t+\delta t), t+\delta t) \right\} \tag{3.6}$$

where $J^*(x(t+\delta t), t+\delta t)$ is the minimum cost of the process for the time interval $t+\delta t \leq z \leq t_f$ with initial state $x(t+\delta t)$

Assuming that the second partial derivative of $J*$ exists and are bounded, we can expand $J^*(x(t+\delta t), t+\delta t)$ in a Taylor series about the point $(x(t),t)$ to obtain

$$\begin{aligned} J^*(x(t),t) = \min_{u(z)} \Big\{ \textstyle\int_t^{t+\delta t} g dz + J^*(x(t),t) + \left[\frac{\partial J^*}{\partial t}(x(t),t)\right]\delta t \\ + \left[\frac{\partial J^*}{\partial x}(x(t),t)\right]^T \Big[x(t+\delta t) - x(t)\Big] \Big\} + \text{terms of higher derivatives} \end{aligned} \tag{3.7}$$

Now for small δt

$$\begin{aligned} J^*(x(\theta),t) = \min_{u(z)} \Big\{ g(x(t),u(t),t)\delta t + J^*(x(t),t) + J_t^*(x(t),t)\delta t \\ + J_x^{*T}(x(t),t)\Big[a(x(t),u(t),t)\Big]\delta t + 0(\delta t) \end{aligned} \tag{3.8}$$

where $0(\delta t)$ denotes the terms containing $[\delta t]^2$ and higher orders of δt that arise from the approximation of the integral and the function of the Taylor series expansion. Next removing the terms containing $J^*(x(t),t)$ and $J_t^*(x(t),t)$ from the minimization [since they do not depend on $u(t)$] we obtain

$$0 = J_t^*(x(t),t)\delta t + \min u(t) \left\{ g(x(t),u(t),t)\delta t + J_x^{*T}(x(t),t)\Big[a(x(t),u(t),t)\delta t\Big] + 0(\delta t) \right\} \tag{3.9}$$

Dividing by δt and taking the limit as $\delta t \to 0$ gives

$$0 = J_t^*(x(t),t)\delta t + \min u(t)\left\{ g(x(t),u(t),t)\delta t + J_x^{*T}(x(t),t)\left[a(x(t),u(t),t) \right] \right\} \tag{3.10}$$

To find the boundary value for this partial differential equation ,set $t = t_f$ from equation[**3.4**] it is apparent that

$$J^*(x(t_f),t_f) = h(x(t_f),t_f) \tag{3.11}$$

we define the Hamiltonian H as

$$H(x(t),u(t),J_x^*,t) = g(x(t),u(t),t) + J_x^{*T}(x(t),t)\left[a(x(t),u(t),t \right]) \tag{3.12}$$

and

$$H(x(t),u^*(x(t),J_x^*,t),J_x^*,t) = \min u(t) H(x(t),u(t),J_x^*,t) \tag{3.13}$$

since the minimizing control will depend on x, J_x^* and t. Using these definitions we obtain the Hamiltonian Jacobi Bellman equation as;

$$0 = J_t^*(x(t),t) + H(x(t),u^*(x(t),J_x^*,t),J_x^*,t \tag{3.14}$$

Example 3.1.1. *A first order system is described by the differential equation*

$$\dot{x}(t) = x(t) + u(t) \tag{3.15}$$

It is desired to find the control law that minimizes the performance

$$J = \frac{1}{4}x^2(T) + \int_0^T \frac{1}{4}u^2(t)dt \tag{3.16}$$

The final time T is specified and the admissible state and control values are not constrained by any boundaries.

Substituting $g = \frac{1}{4}u^2(t)$ and $a = x(t) + u(t)$ into equation **??** *we find that the Hamiltonian is omitting the arguments of J_x^*.*

$$H(x(t),u(t),J_x^*,t) = \frac{1}{4}u^2(t) + J_x^*[x(t)+u(t)] \quad \textbf{(3.17)}$$

and since control is unconstrained a necessary condition that the optimal control must satisfy is

$$\frac{\partial H}{\partial u} = \frac{1}{2}u(t) + J_x^*(x(t),t) = 0 \quad \textbf{(3.18)}$$

Observe that $\frac{\partial^2 H}{\partial u^2} = \frac{1}{2} > 0$

*Thus the control that satisfies equation **[3.18]** does minimize H from equation **[3.18]***

$$u^*(t) = -2J_x^*(x(t),t) \quad \textbf{(3.19)}$$

which when substituted in the Hamiltonian Jacobi Bellman equation gives

$$0 = J_t^* + \frac{1}{4}[-2J_x^*]^2 + [J_x^*]x(t) - 2[J_x^*]^2 = J_t^* - [J_x^*]^2 + [J_x^*]x(t) \quad \textbf{(3.20)}$$

*The boundary value is from **[3.16]***

$$J^*(x(T),T) = \frac{1}{4}x^2(T) \quad \textbf{(3.21)}$$

One way to solve the Hamiltonian Jacobi Bellman equation is to guess a form for the solution and see it can be made to satisfy the differential equation and the boundary condition. Let us assume a solution of the form:

$$J^*(x(t),t) = \frac{1}{2}k(t)x^2(t) \quad \textbf{(3.22)}$$

where $k(t)$ represents an unknown scalar function of t and that is to determine. Notice that :

$$J_x^*(x(t),t) = k(t)x(t) \quad \textbf{(3.23)}$$

*which together with equation **[3.19]** implies that*

$$u^*(t) = -2k(t)x(t) \quad \textbf{(3.24)}$$

*Thus if a function $k(t)$ can be found such that **[3.20]** and **[3.21]** are satisfied the optimal control is linear feedback of the state indeed this was the motivation of selecting the form **[3.22]** by making $k(T) = \frac{1}{2}$ the assumed solution matches the boundary condition specified by equation **[3.21]**.*

*Substituting **[3.23]** for J_x^* and*

$$J_t^*(x(t),t) = \frac{1}{2}k(t)x^2 \tag{3.25}$$

*into equation **[3.20]** gives*

$$0 = \frac{1}{2}k(t)x^2 - k^2(t)x^2(t) + k(t)x^2(t) \tag{3.26}$$

since this equation must be satisfied for all $x(t)$

$$\frac{1}{2}k(t) - k^2(t) + k(t) = 0 \tag{3.27}$$

$k(t)$ is a scalar function of t therefore the solution can be obtained by separation of variables with the result

$$k(t) = \frac{\varepsilon^{T-t}}{\varepsilon^{T-t} + \varepsilon^{-(T-t)}} \tag{3.28}$$

The optimal control law is then

$$u^*(t) = -2J_x^*(x(t),t) = -2k(t)x(t) \tag{3.29}$$

Notice that as $T \to \infty$ the linear time ranging feedback approaches constant feedback $(k(t) \to 1)$ and that the controlled system

$$\dot{x}(t) = x(t) - 2x(t) = -x(t) \tag{3.30}$$

is stable. If this were not the case the performance measure would be infinite.

3.2 Necessary condition for optimal control

To determine the necessary condition for optimal control, the problem is to find the admissible control u^* that causes the system

$$\dot{x}(t) = a(x(t), u(t), t) \tag{3.31}$$

to follow an admissible trajectory X^* that minimizes the performance measure

$$J(u) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t)dt \tag{3.32}$$

We shall initially assume that the admissible state and control regions are not bounded and that the initial condition $x(t_0) = x_0$ and the initial time to are specified. As usual x is the $n \times 1$ state vector and u is the $m \times 1$ vector of control inputs. Assume h is a differential function we can write

$$h(x(t_f), t_f) = \int_{t_o}^{t_f} \frac{d}{dt}[h(x(t), t)]dt + h(x(t_0), t_0) \tag{3.33}$$

so that the performance measure can be expressed as

$$J(h) = \int_{t_0}^{t_f} \left\{ g(x(t), u(t), t) + \frac{d}{dt}\left[h(x(t), t)\right] \right\} dt + h(x(t_0), t0) \tag{3.34}$$

since $x(t_0)$ and t_0 are fixed the minimization does not affect the $h(x(t_0), t_0)$ term so we need to consider only the functional.

$$J(u) = \int_{t_0}^{t_f} \left\{ g(x(t), u(t), t + \frac{d}{dt}[h(x(t), t)]) \right\} dt \tag{3.35}$$

using chain rule of differentiation, we find that this becomes

$$J(u) = \int_{t_0}^{t_f} \left\{ g(x(t), u(t), t) + [\frac{\partial h}{\partial x}(x(t), t)]^T \dot{x}((t) + \frac{\partial h}{\partial t}(x(t), t) \right\} dt \tag{3.36}$$

To include the differential equation constrains, we form the augmented functional.

$$J(h)=\int_{t_0}^{t_f}\left\{g(x(t),u(t),t)+\left[\frac{\partial h}{\partial x}(x(t),t)\right]^T\dot{x}(t)+\frac{\partial h}{\partial t}(x(t),t)+P^T(t)\left[a(x(t),u(t),t)-\dot{x}(t)\right]\right\}dt \tag{3.37}$$

by introducing the Lagrange multipliers $p_1(t),\cdots,p_n(t)$. Let us define

$$\begin{aligned} g_a(x(t),\dot{x}(t),u(t),p(t),t) &= g(x(t),u(t),t)+p^T(t)[a(x(t),u(t),t)-\dot{x}(t)] \\ &\quad +\left[\tfrac{\partial h}{\partial x}(x(t),t)\right]^T\dot{x}(t)+\tfrac{\partial h}{\partial t}(x(t),t) \end{aligned} \tag{3.38}$$

so that

$$J_a(u)=\int_{t_0}^{t_f}\{g_a(x(t),\dot{x}(t),u(t),p(t),t)\}dt \tag{3.39}$$

We shall assume that the end points at $t=t_f$ can be specified or free. To determine the variation of J_a we introduce the variations $\sigma x,\sigma\dot{x},\sigma u,\sigma p$ and σt_f. This gives

$$\begin{aligned} \sigma J_a(u^*) = 0 = &\left[\frac{\partial g_a}{\partial \dot{x}}(x^*(t_f),\dot{x}^*(t_f),u^*(t_f),p^*(t_f),t_f)\right]^T\sigma x_f \\ &+\Bigg[g_a(x^*(t_f),\dot{x}^*(t_f),u^*(t_f),p^*(t_f),t_f) \\ &-\left[\frac{\partial g_a}{\partial \dot{x}}(x^*(t_f),\dot{x}^*(t_f),u^*(t_f),p^*(t_f),t_f)\right]^T\dot{x}^*(t_f)\Bigg]\sigma t_f \\ &+\int_{t_0}^{t_f}\Bigg\{\Bigg[\left[\frac{\partial g_a}{\partial x}(x^*(t),\dot{x}^*(t),u^*(t),p^*(t),t)\right]^T \\ &-\frac{d}{dt}\left[\frac{\partial g_a}{\partial \dot{x}}(x^*(t),\dot{x}^*,u^*(t),p^*(t),t)\right]^T\Bigg]\sigma x(t) \\ &+\left[\frac{\partial g_a}{\partial u}(x^*(t),\dot{x}^*(t),u^*(t),p^*(t),t)\right]^T\sigma u(t) \\ &+\frac{\partial g_a}{\partial p}(x^*(t),\dot{x}^*(t),u^*(t),p^*(t),t)\Bigg]^T\sigma p(t)\Bigg\}dt \end{aligned} \tag{3.40}$$

Notice that the above result is obtained because $\dot{u}(t)$ and $\dot{p}(t)$ do not appear in g_a.

Next let us consider only those terms inside the integral which involve the function h these terms contain

$$\frac{\partial}{\partial x}\left[\left[\frac{\partial h}{\partial x}(\dot{x}^*(t),t)\right]^T\dot{x}^*(t)+\frac{\partial h}{\partial t}(x^*(t),t)\right]-\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{x}}\left[\left[\frac{\partial h}{\partial x}(x^*(t),t)\right]^T\dot{x}^*(t)\right]\right\} \tag{3.41}$$

writing out the indicated partial derivatives gives

$$\left[\frac{\partial^2}{\partial x^2}(x^*(t),t)\right]\dot{x}^*(t)+\left[\frac{\partial^2}{\partial t\partial x}(x^*(t),t)\right]-\frac{d}{dt}\left[\frac{\partial h}{\partial x}(x^*(t),t)\right] \tag{3.42}$$

or if we apply the chain rule to the last term

$$\left[\frac{\partial^2 h}{\partial x^2}(x^*(t),t)\right]\dot{x}^*(t)+\left[\frac{\partial^2 h}{\partial t\partial x}(x^*(t),t)-\left[\frac{\partial^2 h}{\partial x^2}(x^*(t),t)\right]\dot{x}^*(t)-\left[\frac{\partial^2 h}{\partial x\partial t}(x^*(t),t)\right]\right. \tag{3.43}$$

It is assumed that the second partial derivative are continuous, the order of differentiation can be interchanged and these terms add to zero. In the integral term we have, then

$$\begin{aligned}&\int_{t_0}^{t_f}\left\{\left[\frac{\partial y}{\partial x}(x^*(t),u^*(t),t)\right]^T+p^*(t)\left[\frac{\partial a}{\partial x}(x^*(t),u^*(t),t\right]\right.\\&-\frac{d}{dt}\left[-p^{*T}(t)\right]\sigma x(t)+\left[\left[\frac{\partial g}{\partial u}(x^*(t),u^*(t),t)\right]^T\right.\\&\left.+p^{*T}(t)\left[\frac{\partial a}{\partial u}(x^*(t),u^*(t),t)\right]\right]\sigma u(t)+\left[[a(x^*(t),u^*(t),t)-\dot{x}^*(t)]^T\right]\sigma p(t)\Big\}dt\end{aligned} \tag{3.44}$$

This integral must vanish on an extremal regardless of the boundary conditions. We first observe that the constraints

$$\dot{x}^*(t)=a(x^*(t),u^*(t),t) \tag{3.45}$$

must be satisfied by an extend so that the coefficient of $\sigma p(t)$ is zero. The lagrange multipliers are arbitrary so let us select them to make the coefficient of $\sigma x(t)$ equal to zero that is

$$\dot{p}^*(t)=\left[\frac{\partial a}{\partial x}(x^*(t),u^*(t),t)\right]^T p^*(t)-\frac{\partial g}{\partial x}(x^*(t),u^*(t),t) \tag{3.46}$$

we shall therefore call equation **[2.16]** the co-state equation and $p(t)$ is the co-state.

The remaining variation $\sigma u(t)$ is independent so its coefficient must be zero thus

$$0 = \frac{\partial g}{\partial u}(x^*(t), u^*(t), t) + \left[\frac{\partial a}{\partial u}(x^*(t), u^*(t), t)\right]^T p^*(t) \tag{3.47}$$

There are still terms outside the integral to deal with since the variation must be zero we have

$$\left[\frac{\partial h}{\partial x}(x^*(t_f), t_f) - p^*(t_f)\right]^T \sigma x_f + \left[g(x^*(t_f), u^*(t_f), t_f) + \frac{\partial h}{\partial t}(x^*(t_f), t_f) \right. \\ \left. + p^{*T}(t_f)\left[a(x^*(t_f), u^*(t_f), t_f)\right]\right] \sigma t_f = 0 \tag{3.48}$$

in writing equation **[3.48]** we have used the fact that

$$\dot{x}^*(t_f) = a(x^*(t_f), u^*(t_f), t_f) \tag{3.49}$$

Equations **[3.45, 3.46, 3.47]** are the necessary conditions we wish to determine. Notice that these necessary conditions consist of a set of $2n$ first order differential equations the state and co-state equations **[3.45** and **[3.46]** and a set of m algebraic relations **[3.47]** which must be satisfied throughout the interval $[t_0, t_f]$. The solution of the state and costate equations will contain $2n$ constants of integration. To evaluate these constants we use the equations $x^*(t_0) = x_0$ and an additional set of n or $n+1$ relationships depending on whether or not t_f is specified from **[3.48]**. Notice that as expected we are again confronted by a new point boundary value problem.

we can use the function H called Hamiltonian defined as

$$H(x(t), u(t), p(t), t) = g(x(t), u(t), t) + p^T(t)[a(x(t), u(t), t)] \tag{3.50}$$

Using this notation we can write the necessary conditions **[3.45]**, **[3.46]**,**[3.47]** using **[3.48]** as follows

$$\begin{aligned} \dot{x}^*(t) =& \frac{\partial H}{\partial p}(\dot{x}^*(t), u^*(t), p^*(t), t) \;\; \forall \, t \in [t_0, t_f] \\ \dot{p}^*(t) =& - \frac{\partial H}{\partial x}(\dot{x}^*(t), u^*(t), p^*(t), t) \;\; \forall \, t \in [t_0, t_f] \\ 0 =& \frac{\partial H}{\partial u}(\dot{x}^*(t), u^*(t), p^*(t), t) \;\; \forall \, t \in [t_0, t_f] \end{aligned} \tag{3.51}$$

and

$$\left[\frac{\partial h}{\partial x}(x^*(t_f), t_f) - p^*(t_f)\right]^T \sigma x_f + \left[H(x^(t_f), u^*(t_f), p^*(t_f), t_f) + \frac{\partial h}{\partial t}(x^*(t_f), t_f)\right] \sigma t_f = 0 \tag{3.52}$$

Equation **[3.52]** above is the boundary condition.

Let us now consider the boundary condition that may occur.

$CASE1$ Fixed final time and fixed final state.
In this case there will be boundary condition since $\sigma t_f = \sigma x_f = 0$

$CASEII$ Free final time and fixed final state.
In this case, $\sigma x_f = 0$ and therefore the boundary condition shall be

$$\left[H(\cdot) + \tfrac{\partial h(\cdot)}{\partial t}\right] \sigma t_f = 0$$

$CASEIII$ Fixed final time and free final state

In this case $\sigma_{t_f} = 0$. Therefore the boundary condition shall be

$$\left[\tfrac{\partial h}{\partial x}(\cdot) - p^*(t_f)\right]^T \sigma_{t_f} = 0$$

$CASEIV$ Free final time and free final state

The boundary condition shall be

$$\left[\tfrac{\partial h}{\partial x}(\cdot) - p^*(t_f)\right]^T \sigma_{x_f} + \left[H(\cdot) + \tfrac{\partial h}{\partial t}(\cdot)\right] \sigma_{t_f} = 0$$

In conclusion in all the four cases above equation **[2.21]** holds.

3.3 Viscosity Solution

3.3.1 Viscosity Solutions of Hamilto-Jacobi- Bellman equation

Here we shall see the link between the Hamilton Jacobi Bellman equation of an optimal control problem and viscosity solution. The value function of a deterministic optimal control problem is in fact a viscosity solution of the associated HJB equation. We restricted ourselves to the deterministic cases because the technicalities of the stochastic case require more involving analysis.

3.3.2 Deterministic Optimal Control

We consider here the finite time bounded domain deterministic problem. Let the state dynamics be given by

$$dx^{\alpha(\cdot)}(s) = b(x^{\alpha(\cdot)}, s, \alpha(s))ds \quad s \in (t,T) \quad x^{\alpha(\cdot)}(t) = x \tag{3.53}$$

The control set A is chosen to be

$A = \{\alpha : [0,T] \to \wedge | \alpha(\cdot)$ Lebesgue measurable

We will use the following lemma: Let XA be the indicator function of statements $A : XA = 1$ if $A, XA = 0$ if not $A.\tau$ is the time of first exit of $(x(s), s)$ from $\bar{u} \times [t,T]$

Consider Dynamic programming principle where for an $h > 0$ such that $t+h < T$ if $\bar{t} = \min(\tau, t+h)$ then

$$u(x,t) = \inf_{\alpha(\cdot)\in A}\left[\int_t^{\bar{t}} f(x(s), s, \alpha(s))ds + g(x(\bar{t}), \bar{t})x_{\tau<t+h} + u(x(\bar{t}), \bar{t})x_{t+h\leq\tau}\right] \tag{3.54}$$

Theorem 3.3.1. *(First order Hamiltonian Jacobi Bellman equation-viscosity sense)*

Provided that the value function u is uniformly continuous upto the boundary i.e $u \in c(\bar{o}), u$ is a viscosity solution of the HJB equation with no boundary data.

$-U_t + \sup_{\alpha\in\wedge}[-b^\alpha \cdot D_x u - f^\alpha = 0$ *on* o.

If furthermore $u = g$ on ∂O, then u is a viscosity solution of the HJB equation

$-U_t + \sup_{\alpha\in\wedge}[-b^\alpha \cdot D_x u - f^\alpha = 0$ *on* o

$u = g$ *on* ∂O,

3.4 Bellman Principle of Optimality

There are two approaches to dynamic optimization, the pontrjagin (Hamiltonian) approach and the Bellman approach. In this section we look at local conditions for having a finite optimum.

3.4.1 Discrete Time certainity

We start in discrete time and assume perfect foresight. The general problem we want to solve is

$$\begin{cases} \max_{(c_t)} \sum_{t=0}^{\infty} f(t, kt, c_t) \\ \text{s.t} k_{t+1} = g(t, k_t, c_t) \end{cases} \tag{3.55}$$

In addition we compose a budget constraint which for many examples is the restriction that k_t be eventually positive (i.e $\liminf_t k_t \geq 0$). This budget constraints excludes explosive solutions for c_t so that we can apply the Bellman method. The usual name for the variables involved is the control variable (Because it is under the control of the choice maker) and k_t is the state variable (because it describes the state of the system at the beginning of t when the agent makes the decision).

The equation $k_{t+1} = g(t, k_t, c_t)$ is called the state equation.

To get some intuition about the problem we think of k_t as capital available for production at time t and c_t as consumption at t. At time 0 for a starting level of capital k_0 the consumer checks the level of consumption c_0. This determines the level of capital available for the next period $k_1 = g(0, c_0, k_0)$. So at time 1 the consumer decides on the level of c_1 which together with k_1 determines k_2 and the cycle is repeated on and on. The infinite sum $\sum_{t=0}^{\infty} f(t_1, k_t, c_t)$ is to be thought of as the total utility of the consumer which the latter is supposed to maximize at time 0.

Bellman's idea for solving **[3.55]** is to define a value function V at each $t = 0, 1, 2, \cdots$

$$V(t, k_t) = max_{c_s} \sum_{c=t}^{\infty} f(s, k_s, c_s)$$

such that $k_{s+1} = g(s, k_s, c_s)$

which represent the common maximum "utility" given the initial level of k_t.

From Bellman's principle of optimality, for each $t = 0, 1, 2, \cdots$

$$V(t, k_t) = max_{c_t}\left[f(t, k_t, c_t) + V(t+1, g(t, k_t, c_t))\right] \tag{3.56}$$

This in principle reduces an infinite period of optimization problem to a two limit optimization problem. So now we see how we shall solve optimization problem **[3.55]** using the Bellman's equation **[3.56]**. We denote partial derivatives by using subscripts a star superscripts denotes the optimum. Then the first order condition from **[3.56]** reads

$$f_c(t, k_t, c_t^*) + V_k(t+1, g(t, k_t, c_t^*)) \cdot g_c(t, k_t, c_t^*) = 0 \tag{3.57}$$

Looking at this formula it is clear that we would like to be able to compute the derivative $V_k(t+1, k_{t+1})$ by trying to use formula **3.56]** since we are differentiating a maximum operator, we apply the envelope theorem and obtain

$$V_k(t, k_t) = f_k(t, k_t, c_t^*) + V_k(t+1, g(t, k_t, c_t^*)) \cdot g_k(t, k_t, c_t^*) \tag{3.58}$$

from **[3.57]** we can calculate $V_k(t+1, g(t_1, k_t, c_t^*))$ and substituting it in **??** we get

$$V_k(t, k_t) = (f_k - \frac{f_c}{g_c} \cdot g_k)(t, k_t, c_t^*) \tag{3.59}$$

Finally substitute this formula into **[3.57]** and obtain a condition which does not depend on the value function any more.

$$f_c(t, k_t, c_t^*) + g_c(t, k_t, c_t^*) \cdot (f_k - \frac{f_c}{g_c} \cdot g_k)(t+1, g(k_t, c_t^*), c_{t+1}^*) = 0 \tag{3.60}$$

Notice this formula is true for any k_t not necessary only for the optimal are upto that point. But in that case c_t^* and c_{t+1}^* are the optimal choices given k_t. In any case from now on we are only going to work at the optimum (t, k_t^*, c_t^*). The previous formula can be written as follows

$$f_k(t+1) - \frac{f_c(t+1)}{g_c(t+1)} \cdot g_k(t+1) = \frac{-f_c(t)}{g_c(t)} \tag{3.61}$$

This is the key equation that allows us to compute the optimum c_t^* using only the initial data (f_t)andg_t equation **??** is called Bellman- Euler equation

For purposes of comparison with continuous time version we write $g(t,k_t,c_t) = k_t + h(t,k_t,c_t)$. Denote $\triangle_t \phi = \phi$ Then we can rewrite **[3.61]** as:

$$\triangle_t \left(\frac{f_c(t)}{h_c(t)}\right) = f_k(t+1) - \frac{f_c(t+1)}{h_c(t+1)} \cdot h_k(t+1) \quad \textbf{(3.62)}$$

3.4.2 Discrete Time uncertainty

Now we assume everything to be stochastic and the agent solves the problem

$$\begin{cases} max_{(c_t)} E_0 \sum_{t=0}^{\infty} f(t,k_t,c_t) \\ \text{s.t} k_{t+1} = g(t,k_t,c_t) \end{cases}$$

As usual we denote by E_t the expectation given information available at time t. Then we can define the value function

$$v(t,k_t) = max_{(c_t)} E_t \sum f(s,k_s,c_s) \;\; such\ that \;\; k_{s+1} = g(s_1,k_s,c_s)$$

The Bellman principle of optimality **[3.56]** becomes

$$v(t,k_t) = max_{(c_t)} \left[f(t_1,k_t,c_t) + E_t v(t+1, g(t,k_t,c_t)) \right]$$

Now in order to derive the Euler equation with uncertainty, all we have to do is replace $v(t+1)$ in the formulas of the previous section by $E_t v(t+1)$ (using of course the fact that differentiation commutes with expectation) we arrive at the following Bellman Euler equation

$$E_t(f_k(t+1) - \frac{f_c(t+1)}{g_c(t+1)} \cdot g_k(t+1)) = -\frac{f_c(t)}{g_c(t)} \quad \textbf{(3.63)}$$

3.4.3 Continuous Time certainty

This is a bit trickier but same derivation as in discrete time can be used. The difference is that instead of the interval (t,t_{t+1}) we now look at $(t,t+dt)$.

The problem that the decision maker has to solve is;

$$\left\{ max_{c_t} \int_0^\infty f(t,k_t,c_t) \ \text{ s.t } \ \frac{d_{kt}}{d_t} = h(t_1,k_t,c_t) \right. \tag{3.64}$$

The constraint can be rewritten in differential notation

$$k_{t+dt} = k_t + h(t,k_t,c_t)dt \tag{3.65}$$

so we have a problem similar inform to **??** and we can solve it by an analogous method.

Define the value function

$$v(t,k_t) = max_{(c_s)} \int_t^\infty f(s,k_s,c_s)ds \ \mid \ \frac{dk_s}{ds} = h(s,k_s,c_s) \tag{3.66}$$

Using equation **[3.66]** the Bellman principle of optimality can be written as;

$$v(t,k_t) = max_{c_t} \left[\int_t^{t+dt} f(s,k_s,c_s)ds + v(t+dt,k_t + h(c,k_t,c_t)dt) \right] \tag{3.67}$$

we know that $\int_t^{t+dt} f(s,k_s,c_s)ds = f(t,k_t,c_t)dt$. The first order condition for minimum is

$$f_c(t)dt + v_k(t+dt,k_{t+dt}) \cdot h_c(t)dt = 0 \tag{3.68}$$

This is equivalent to

$$v_k(t+dt) = -\frac{f_c(t)}{h_c(t)} \tag{3.69}$$

As we apply the envelope theorem to derive

$$v_k(t) = f_k(t)dt + v_k(t+dt) \cdot (1 + h_k(t)dt) \tag{3.70}$$

substitute **[3.69]** into **[3.70]** to obtain

$$v_k(t) = -\frac{f_c(t)}{h_c(t)} + \left(f_k(t) - \frac{f_c(t)}{h_c(t)} \cdot h_k(t)\right)dt \tag{3.71}$$

If ϕ is any differential function, then $\phi(t+dt)dt = \phi(t)dt$. So we get the formula

$$v_k(t+dt) = -\frac{f_c(t+dt)}{h_c(t+dt)} + \left(f_k(t) - \frac{f_c(t)}{h_c(t)} \cdot h_k(t)\right)dt \tag{3.72}$$

putting equations **[3.69]** and **[3.72]** we get

$$\frac{f_c(t+dt)}{h_c(t+dt)} - \frac{f_c(t)}{h_c(t)} = \left(f_k(t) - \frac{f_c(t)}{h_c(t)} \cdot h_k(t)\right)dt \tag{3.73}$$

Using the formula $\phi(t+dt) - \phi(t) = \frac{d\phi}{dt}dt$ we can reunite the above formula as

$$\frac{d}{dt}\left(\frac{f_c(t)}{h_c(t)} = f_k(t) - \frac{f_c(t)}{h_c(t)} \cdot h_k(t)\right.$$

This is the Bellman-Euler equation in continuous time. It is pretty similar to our equation **[3.62]** in discrete time.

By $f_c(t)$ in the above formula what we really mean is $f_c(t, k_t, c_t)$ as usual calculated at the optimum. Then we calculate

$$\frac{d}{dt}(f_c) = f_{tc} + f_{kc} \cdot h + f_{cc} \cdot \frac{dc}{dt} \tag{3.74}$$

so we can rewrite the Bellman-Euler equation **??** is as follows

$$-(f_{tc} + f_{kc} \cdot h + f_{cc} \cdot \frac{dc}{dt}) = -\frac{f_c}{h_c}(h_{tc} + h_{kc} \cdot h + h_{cc} \cdot \frac{dc}{dt} - h_c \cdot h_k) - h_c \cdot f_k \tag{3.75}$$

In general in order to solve this, notice that we can rewrite **[3.74]** as $\frac{d_{ct}}{dt} = \lambda(t, k_t, c_t)$ so the optimum is given by the following system of ODE_s

$$\begin{cases} \frac{d_{ct}}{dt} = \lambda(t, k_t, c_t) \\ \frac{d_{kt}}{dt} = h(t, k_t, c_t) \end{cases} \tag{3.76}$$

3.4.4 Continuous Time uncertainty

First we assume that the uncertainty cases from the function h (for example, if h dependent as an uncertain endowment e_t). The problem is

$$\begin{cases} \max\limits_{c_t} E_0 \int_0^\infty f(t,k_t,c_t) \\ \text{s.t} \frac{dk_t}{dt} = h(t,k_t,c_t) \end{cases} \tag{3.77}$$

The value function takes the form

$$v(t,k_t) = max_{c_s} \int_t^\infty f(s,k_s,c_s)ds \ \mid \ \frac{dk_s}{ds} = h(s,k_s,c_s)$$

and the Bellman principle of optimality **[3.67]** becomes

$$v(t,k_t) = max_{c_t} \left[\int_t^{t+dt} f(s,k_s,c_s)ds + E_t v(t+dt, k_t + h(t,k_t,c_t)dt) \right] \tag{3.78}$$

we arrive at the following Bellman-Euler equation

$$E_t \frac{d}{dt}\left(\frac{f_c}{h_c}\right) = f_k - \frac{f_c}{h_c} \cdot h_k \tag{3.79}$$

3.5 Numerical Determination of Optimal Trajectories

Variational techniques have been used to derive the necessary conditions for optimal control. In problems with linear plant dynamics and quadratic performance criteria. It was found that it is possible to obtain the optimal control law by numerically integrating a matrix differential equation of the Ricatti type. Optimal control laws have also been determined for several other simple examples by applying Pontryagin minimum principle. In general however the variational approach leads to a nonlinear two point boundary value problem that cannot be solved analytically to obtain the minimal control law or even an optimal open loop control. To address this there are four iterative numerical techniques for determining optimal controls and trajectories. Three of the techniques steepest descent, variation of extremals and quasilinearlization are procedures for solving non linear two point boundary value problems. The fourth technique gradient projection does not make use of the necessary condition for optimality provided by the variational approach. Instead the optimization problem is solved by minimizing a function of several variables subject to various constraining relationships. A brief highlight of the three techniques is as follows:

3.5.1 The Method of Steepest Descent - Minimization of Functions by Steepest Descent

Consider f be a function of two independent variables y_1 and y_2 the value of the function at the point y_1, y_2 is denoted by $f(y_1, y_2)$. It is desired to find the point y_1^* and y_2^* where f assumes its minimum value $f(y_1^*, y_2^*)$.

If it is assumed that the variables y_1 and y_2 are not constrained by any boundaries, a necessary condition for y_1^* and y_2^* to be a point where f has a (relative) minimum is that the differential of f vanish at y_1^*, y_2^* that is

$$\begin{aligned} df(y_1^*, y_2^*) &= \left[\frac{\partial f}{\partial y_1}(y_1^*, y_2^*)\right] \triangle y_1 + \left[\frac{\partial f}{\partial y_2}(y_1^*, y_2^*)\right] \triangle y_2 \\ &= \left[\frac{\partial f}{\partial y}(y^*)\right]^T \triangle y = 0 \end{aligned} \tag{3.80}$$

$\partial f / \partial y$ is called the gradient of f with respect to y. Since y_1 and y_2 are independent, the components of $\triangle y$ are independent arbitrary and equation above implies

$$\frac{\partial f}{\partial y}(y^*) = 0$$

In other words for $f(y^*)$ to be a relative minimum it is necessary that the gradient of f be zero at the point y^*.

3.5.2 Variation of extremals

This method is called variation of extremals because every trajectory generated by the algorithm satisfies

$$\dot{x}^x(t) = \frac{\partial H}{\partial p} = a(x^*(t), u^*(t), t)$$

through

$$0 = \frac{\partial H}{\partial u} = \left[\frac{\partial a}{\partial u}(x^*(t), u^*(t), t)\right]^T p^*(t)$$

and hence is the extremal.

3.5.3 Quasilinearlization

Numerical integration can be used to solve non linear differential equation if a complete set of boundary conditions is specified at other initial time or the final time the method of variation of extremals consists of solving a sequence of such problems. In the method of quasilinearlization a sequence of linear two points boundary value problem is solved.

4 Chapter 4

4.1 Application of Haar Wavelet and Conclusion

In this section we focused on the Haar Wavelet function by looking at how it was developed, its inherent features and finally solving some optimal control problems using the function. Then we compared the results thereof of the problems with the numerical solutions (exact solutions) to establish if there was any similarity and efficiency in the process.

4.1.1 Overview of Haar Wavelets

Haar wavelets was first introduced by Alfred Haar in 1990, Haar wavelets is also referred to as Daubechies (I) wavelet.

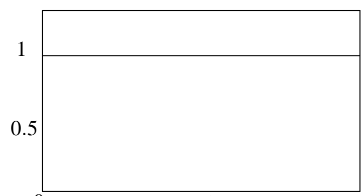
The scaling function $\phi(x)$ is defined as

$$\begin{aligned} \phi(x) &= 1, \qquad if \;\; x \in [0,1] \\ \phi(x) &= 0, \qquad if \;\; x \notin [0,1] \end{aligned}$$

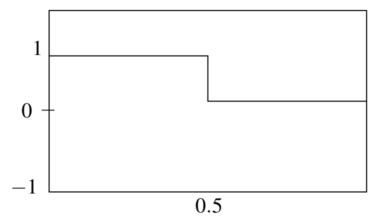
The wavelet function $\Psi(x)$ for this scaling function is defined as

$$\begin{aligned} \Psi(x) &= 1 \qquad if \;\; x \in [0,0.5] \\ \Psi(x) &= -1 \qquad if \;\; x \in [0.5,1] \\ \Psi(x) &= 0 \qquad if \;\; x \notin [0,1] \end{aligned}$$

The scaling function and wavelet for the Haar wavelet are shown in the figure below respectively.



(a) Scaling function



(b) wavelet function

4.1.2 Advantages / Features of Haar Wavelet Method

These were highlighted by Hariharan and Kannn(2013) and Radomir and Bogdan(2003).

1. High accuracy is obtained already for a small number of grid point.
2. The method is very convenient for solving boundary value problem since the boundary conditions are taken care of automatically.
3. Singularities can be treated as intermediate boundary conditions, this circumstance to a great extend simplifies the solution.
4. The obtained solution are mostly simpler compared with other known methods.
5. From the definition , it is obvious that the Haar functions are orthogonal function. Therefore
$$\int_0^1 har(m,\theta)har(n,\theta)d\theta = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

4.1.3 Haar Wavelet Method

Consider the interval $t \in [A,B]$ where A and B are given by constants. Define the quantity $m = 2^J$ where J is the maximal level of resolution . Distribute the interval $[A,B]$ in $2m$ submanifolds of equal length
$$\Delta t = (B-A)/2M$$

The other two parameters , the dilation parameter $j = 0, 1, \ldots, J$ and translation parameter $k = 0, 1 \ldots, m-1$ where $m = 2^J$. The wavelet number is defined as $i = m+k+1$.

We define the Haar wavelet as

$$h_i(t) = \begin{cases} 1, & for \;\; t \in [\xi_1(i), \xi_2(i)] \\ -1 & for \;\; t \in [\xi_2(i), \xi_3(i)] \\ 0, & elsewhere. \end{cases} \quad \textbf{(4.1)}$$

where

$$\xi_1(i) = A + 2k\mu\Delta t, \;\; \xi_2(i) = A + (2k+1)\mu\Delta t, \;\; \xi_3(i) = A + 2(k+1)\mu\Delta t \;\; and \;\; \mu = M/m$$

The case $i = 1$ corresponds to the scaling function $h_1(t) = 0$ for $t \in [A, B]$. In the following we need the integrals

$$P_i(t) = \int_A^t h_i(t)dt \quad \textbf{(4.2)}$$

In the view of **[1.1]** these integrals can be evaluated analytically and by doing this we find

$$P_i(t) = \begin{cases} 0 & for \;\; t \le \xi_1(i) \\ t - \xi_1(i) & for \;\; t \in [\xi_1(i), \xi_2(i)] \\ -t - \xi_1(i) + 2\xi_2(i) & for \;\; t \in [\xi_1(i), \xi_3(i)] \\ 0 & for \;\; t \ge \xi_3(i) \end{cases} \quad \textbf{(4.3)}$$

These formulas hold for $i > 1$. In the case $i = 1$ we have $\xi)_1 = A, \;\; \xi_2 = \xi_3 = B$ and

$$P_1(t) = t - A \quad \textbf{(4.4)}$$

The collocation points are

$$t_1 = 0.5(\hat{t}_{\ell-1} + \hat{t}_\ell) \quad \ell = 1, 2, \ldots, 2M \quad \textbf{(4.5)}$$

The symbol $\hat{t}_\ell$ denotes the ℓ^{th}grid point $\hat{t}_\ell = A + \ell\Delta t$

Equation **[4.1]** - **[4.4]** are discretized by replacing $t \to t\ell$. We introduce t he Haar matrices $H(i, \ell) = h_i(t_\ell)$:

$$P(i, \ell) = P_i(t\ell)$$

For solving the boundary value problem we need the values of P_i at $t = B$

In view of [4.3] and **[4.4]** we find

$$P_i(B) = \begin{cases} B - A & for\ i = 1 \\ 0 & for\ i \neq 1 \end{cases}$$

We introduce the matrix R with elements

$$R(i,\ell) = P(i,\ell) - P_i(B) \tag{4.6}$$

Next we introduce some formulas useful in solving some problems. First we introduce the vectors

$$E = [1,1,\ldots,,1], E_1 = [1,0,0,0,\ldots,0] \quad with\ \ t = (t_\ell),\ \ \hat{t} = t - AE$$

Then the following formula hold.

$$E_1 = E/H,\ \ `(E/H)P = E_1P = t - AE = \hat{t} \tag{4.7}$$

$$\frac{1}{\alpha!}(\hat{t}^{\alpha}/H)P = \frac{1}{(\alpha+1)!}\hat{t}^{\alpha+1} \tag{4.8}$$

If we have to integrate by the Haar wavelet method of equation

$$\dot{x} = f(t,x,\mu), x = (x_i) \tag{4.9}$$

Then the solution is sought in the form

$$\dot{x} = aH \tag{4.10}$$

and by integrating **[4.9]** we find

$$x = aP + c \tag{4.11}$$

where c stand for the vector of integration constants. These constants can be calculated from the initial boundary conditions. Replacing **[4.10]** and [**4.11**] into **[4.9]** we obtain we obtain a system of $2m$ equations for evaluating the wavelet coefficient $a = (a_i)$

4.2 Solved Examples

4.2.1 Case 1

Consider the problem

$$\int_0^2 xdt \to \int_0^2 \ddot{x}^2 dt = 1, \tag{4.12}$$
$$x(0) = \dot{x}(0) = 0, \ \ \ddot{x}(2) = 0$$

We interpret $\ddot{x}$ as control state variables

$$x_1 = x, \quad x_2 = \dot{x}$$
$$\dot{x}_3 = \ddot{x} = u, \quad x_4 = \int_0^t u^2 dt$$

State equations are

$$x_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u, \quad \dot{x}_4 = u^2 \tag{4.13}$$

with the boundary conditions;

$$x_1(0) = x_2(0) = x_4(0) = 0$$
$$x_3(2) = 0 \quad x_4(2) = 1$$

The Hamiltonian is given by

$$\hat{H} = -x_1 + \Psi_1 x_2 + \Psi_2 x_3 + \Psi_3 u + \Psi_4 u^2 \tag{4.14}$$

The adjoint system has the form:

$$\dot{\Psi}_1 = E, \quad \dot{\Psi}_2 = -\Psi_1, \quad \dot{\Psi}_3 = -\Psi_2, \quad \dot{\Psi}_4 = 0 \tag{4.15}$$

According to the transversality conditions

$$\Psi_1(2) = \Psi_2(2) = \Psi_3(2) = 0$$

To begin with we integrate **[4.15]** by assuming

$$\begin{aligned} \dot{\Psi}_1 &= b_1 H, & \Psi_1 &= b_1 R \\ \dot{\Psi}_2 &= b_2 H, & \Psi_2 &= b_2 R \\ \dot{\Psi}_3 &= b_3 H, & \Psi_3 &= b_3 R \\ \dot{\Psi}_4 &= 0, & \Psi_4 &= \lambda = constant \end{aligned} \tag{4.16}$$

The matrix R is calculated from **[4.6]** substituting **[4.16]** into **[4.15]** we obtain,

$$b_1 H = E, \quad b_1 R + b_2 H = 0, \quad b_2 R + b_3 H = 0 \tag{4.17}$$

From here we calculate the wavelet coefficient, b_1, b_2, b_3 and fro **[4.16]** the adjoint variables Ψ_1, Ψ_2, Ψ_3

Evaluating the optimal control from $\frac{\partial \hat{H}}{\partial u} = 0,$ we find

$$u = -\frac{\Psi_3}{2\Psi_4} = -\frac{\Psi_3}{2\lambda} \tag{4.18}$$

Since Ψ_3 is already known we can calculate the auxiliary variable

$$\hat{u} = \lambda u = -\frac{\Psi_3}{2} \tag{4.19}$$

Next the state variable are developed into the Haar series adopted from **[4.10]** and **[4.11]**

$$\begin{aligned} \dot{x}_1 = a_1 H, \quad x_1 = a_1 P, \quad \dot{x}_2 = a_2 H, \quad x_2 = a_2 P \\ \dot{x}_3 = a_3 H, \quad x_3 = a_3 P \quad \dot{x}_4 = a_4 H, \quad x_4 = a_4 P \end{aligned} \tag{4.20}$$

Replacing these results into **[4.13]** we get

$$\begin{aligned} a_1 H - a_2 P = 0, \quad a_2 H - a_3 P = 0 \\ a_3 H = \hat{u}/\lambda, \quad a_4 H = \hat{u}^2/\lambda^2 \end{aligned} \tag{4.21}$$

The langrange multiplier λ is calculated in the following way from **[4.21]**

$$\hat{a}_4 = \lambda^2 a_4 = \hat{u}^2/H \tag{4.22}$$

satisfying the boundary condition $x_4(2) = \phi_1$ we get from **[4.20]**

$$x_4|_{t=2} = a_4 P|_{t=2} = 2a_4(1) = 1 \tag{4.23}$$

From the two equations above we find

$$\lambda = \sqrt{2\hat{a}_4(1)} \tag{4.24}$$

The state variables x_1, x_2 are then calculated.

The exact solution is

$$x_{ex} = -\frac{1}{12\lambda}[\frac{1}{120}(t-2)^6 + \frac{4}{3}(t-2)^3 - \frac{72}{5}t + \frac{152}{15}]$$
$$u_{ex} = -\frac{1}{48\lambda}(t-2)[(t-2)^3 + 32], \quad \lambda_{ex} = 0,7559$$

For estimating the accuracy of our result, the error estimates below were introduced

$$\delta_x = \max_i |x_{ex}(t_i) - x(t_i)|$$
$$\delta_u = \max_i |u_{ex}(t_i) - u(t_i)|$$
$$\delta_\lambda = |\lambda_{ex} - \lambda|$$

Results of computer simulation were presented as

J	δ_x	δ_u	δ_λ
4	$6.5E-4$	$6.3E-4$	$3.8E-7$
5	$1.6E-4$	$1.6E-4$	$3.8E-7$
6	$4.0E-5$	$4.0E-5$	$3.8E-7$

Conclusion from above is that from the table its clear that already a small number of points ($J = 4; 32$ grid points) guarantees sufficient accuracy.

4.2.2 Case 2

This will be the case where constraints are state inequality Brysan and Ho (1975) discussed the problem below and used analytical methods. In this case we introduced Haar wavelet to see if the result would converge to the solution as obtained.

$$I = \frac{1}{2}\int_0^1 udt \rightarrow min, \quad x_1 \leq \ell \tag{4.25}$$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u, \quad x_1(0) = x_1(1) = 0, \quad x_2(0) = -x_2(1) = 1$$

where $\ell > 0$ is a given constant.

Consider the case where the equality $x_1(t) = \ell$ holds in the sub-interval $t \in [t_1, t_2]$. This subinterval cannot be near the boundaries $t = 0$ or $t = 1$ since in this case the prescribed boundary condition cannot be satisfied.

Due to symmetry we can solve the problem **[4.25]** only only for $t \in [0, 0.5]$. It is reasonable to assume that $x_1(t) < \ell$ for same interval $t \in [0, t_1]$ and $x_1(t) = \ell$ for $t \in [t, o.5]$

The Hamiltonian is given by;

$$\hat{H} = -\frac{1}{2}u^2 + \Psi_1 x_1 + \Psi_2 u \tag{4.26}$$

It follows from the fact that $\frac{\partial \hat{H}}{\partial u} = 0$ that $\Psi = u$

Next we put together the adjoint system below.

$$\dot{\Psi} = -\frac{\partial H}{\partial x} + \mu \bigtriangledown s \quad t \in [t_1, t_2] \tag{4.27}$$

Since in the present case $g = x_1 - \ell, \quad s = x_2, \ \bigtriangledown s = (0, 1)$

We find

$$\dot{\Psi}_1 = 0, \quad \dot{\Psi}_2 = -\Psi_1 + \mu \tag{4.28}$$

Whereas $\mu(t) = 0$ for $t \in [0, t_1]$ To begin with we consider the subinterval $t \in t_1, 0.5$ since $x_1 = \ell$ it follows that from the state equation that $x_2 = u = 0$ and consequently $\Psi_2 = 0$

Integrating **[4.28]** we find $\Psi_1 = c_1, \quad \mu(t) = c_1$ where c_1 is a constant of integration.

Now we rewrite **[4.25]** as ;

$$I = \frac{1}{2}\int_0^{t_1} u^2 dt \rightarrow min \tag{4.29}$$

$$\dot{x_1} = x_2, \quad \dot{x_2} = u, \quad x_1(0) = 0, \ x_2(0) = 1, \quad x_1(t_1) = \ell, \quad x_2(t_1) = 0$$

The wavelet solution is sought in the matrix form from equation **[4.10]** and **[4.11]**

$$\begin{array}{lllll} \dot{x_1} = a_1 H & x_1 = a_1 P, & \dot{x_2} = a_2 H & x_2 = a_2 P + E & \ldots(i) \\ \dot{\Psi}_1 = b_1 H, & \Psi_1 = b_1 P + c_1 E & \dot{\Psi}_2 = b_2 H & \Psi_2 = b_2 P + c_2 E & \ldots(ii) \end{array} \tag{4.30}$$

The matrix $H \ and \ P$ is calculated according to **[4.1]**,**[4.3]**,**[4.4]** assuming that $A = 0, \ B = t_1$

Replacing **[4.30]** into the state equation **[4.29]** and into the adjoint system **[4.28]**

$$
\begin{array}{cc}
a_1H = a_2P = E & a_2H = b_2P + c_2E \\
b_1H = 0 & b_2H = -b_1H - c_1E
\end{array} \tag{4.31}
$$

It follows from the third and fourth equation that

$$b_1 = 0, \quad b_2 = -c_1E/H = -c_1E_1, \quad c_2 = c_1t_1$$

Due to continuity $\Psi_2(t_1) = 0$, we obtain

$$\Psi_2 = c_1(t_1E - t) = u \tag{4.32}$$

Integrating of the second equation of **[4.30]** gives

$$
\begin{gathered}
a_2H = -c_1E_1P + c_2E = -c_1t + c_2E \\
x_2 = -c_1(E/H)P + (t/H)P + E
\end{gathered}
$$

In view of **[4.7]** and **[4.8]** this result can be put into form

$$x_2 = c_1t(t_1 - t/2) + 1 \tag{4.33}$$

Since

$$
\begin{aligned}
a_2P &= -c_1(t/H)P + c_1t_1(E/H)P \\
&= -c_1t^2/2 + c_1t_1t
\end{aligned}
$$

then

$$a_1 = (a_2P + E)/H = -\tfrac{1}{2}c_1t^2/H + c_1t_1t/H = E/H$$

and

$$x_1 = -\tfrac{1}{2}c_1(t^2/H)P + c_1t_1(t/H)P + (E/H)P$$

Then

$$x_1 = -c_1t^2(\tfrac{t}{6} - \tfrac{1}{2}t_1) + t$$

The constant $c_1,\ t_*$ are calculated from the boundary conditions

$$x_1(t_*) = \ell, \quad x_2(t_*) = 0$$

satisfying this condition, we find

$$\begin{gathered} t_1 = 3\ell, \quad c_1 = -2/t_1^2 \quad \text{and} \\ x_1 = \ell\zeta(\zeta^2 - 3\zeta + 3), \quad \zeta = t/t_1 \\ x_2 = (1-\zeta)^2, \quad u = -\tfrac{2}{3\ell}(1-\zeta) \end{gathered}$$

This is the same result obtained by Brysa and Ho and so we see that the Haar wavelet enables us to find the exact analytical solution of the problem

4.2.3 Case 3

This is the case where optimal control has a control inequality constraints.

We shall solve the problem

$$I = \int_0^1 (x_1^2 + x_2^2 + \alpha u^2)dt \to min \quad |u| \le u_0 \tag{4.34}$$

$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 + u, \quad x_1(0) = 0, \quad x_2(0) = -1$

It is assumed that the control is smooth and thus the function $u(t)$ must be continuous

Introducing the Hamiltonian

$$\hat{H} = -(x_1^2 + x_2^2 + \alpha u^2) + \Psi_1 x_2 + \Psi_2(-x_2 + u) \tag{4.35}$$

$\dot{\Psi} = -\frac{\partial \hat{H}}{\partial x_1}$

We get we put together the adjoint system

$$\dot{\Psi} = -\frac{\partial \hat{H}}{\partial x_1} = 2x_1, \quad \dot{\Psi}_2 = -\frac{\partial \hat{H}}{\partial x_2} = 2x_2 - \Psi_1 + \Psi_2 \tag{4.36}$$

According to the transversely conditions we have $\Psi_1(1) = \Psi_2(1) = 0$

In the region where $|u| < u_0$ it follows from the admissible control conditions that $\frac{\partial \hat{H}}{\partial u} = 0 \quad that \quad \Psi_2 = 2\alpha u$

Therefore, it is reasonable to assume that $u = u_0$ for $t \in [0,t]$ and $u < u_0$ for $t \in [t,1]$.

The value of t_1 is for the unknown and will be calculated. We assign some value to t_1 and integrate the state equation for $t \in [0,t_1]$.

According to the wavelet method we take (the matrices $H \quad and \quad P$ are calculated for $(a = 0, b = t_1)$ from equation **??** and equation **[4.11]**

$$\begin{aligned} \dot{x_1} &= a_1 H, & x_1 &= a_1 P \\ \dot{x_2} &= a_2 H & x_2 &= a_2 P - E \end{aligned} \tag{4.37}$$

Replacing the results into the state equations $\dot{x_1} = x_2 \quad and \quad \dot{x_2} = -x_2 + u_0$. We find;

$$\begin{aligned} a_1 H - a_2 P &= -E \\ a_2(H+P) &= (1+u_0)E \end{aligned} \tag{4.38}$$

Solving this system we evaluate the wavelet coefficient a_1, a_2 and calculate the function x_1, x_2 according to [4.37]

We need the values $x_1 = x_1(t_1)$ and $x_2 = x_2(t_1)$, μ follows from [4.3] that $P_1(t_1) = t_1$ and $P_i(t_1) = 0$ for $i \neq 1$

In view of the [4.37] we find

$$x_1^* = a_1(1)t_1 \qquad x_2^* = a_2(1)t_1 - 1 \tag{4.39}$$

We consider the subinterval $t \in [t_1, 1]$. Again we divide this interval into $2M$ equal parts and calculate the matrices $H, P \; and \; R$ from **[4.11]- [4.3]** and **[4.6]** assuming

$$A = t_1, \quad B = 1 - t_1$$

The solution is sought in the form

$$\begin{aligned} \dot{x_1} &= \hat{a}_1 H, & x_1 &= \hat{a}_1 P + x_1 E, & \dot{x_2} &= \hat{a}_2 H & x_2 &= \hat{a}_2 P + x_2^* E \\ \dot{\Psi} &= \hat{b_1} H & \Psi_1 &= \hat{b_1} R & \dot{\Psi}_2 &= \hat{b_2} H & \Psi_2 &= \hat{b_2} R \end{aligned} \tag{4.40}$$

Here $\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2$ denote wavelet coefficients for the sub-interval $t \in [t_1, 1]$. The matrix R is calculated according to **[4.6]**

Then substituting **[4.40]- [4.34]** and **[4.36]** and taking into account that

$$\Psi_2 = 2\alpha u \quad we \;\; get$$

$$\begin{array}{ll} \hat{a}_1 H - \hat{a}_2 P = x_2^* E & \hat{a}_2(H+P) = \frac{1}{2\alpha}\hat{b}_2 R - x_2^* E, \\ -2\hat{a}_1 P + \hat{b}_1 H = 2x_1^* E, & -2\hat{a}_2 P + \hat{b}_1 R + \hat{b}_2(H-R) = 2x_2^* E \end{array} \tag{4.41}$$

This was solved numerically.

The control $u(t)$ must be continuous at $t = t_*$. In the case of an arbitrary chosen value t_* the requirement of $u(t)$ being continuous is not fulfilled. This discrepancy can be estimated by the function.

$$\Delta = u(t_1 - 0) - u(t_1 + 0) \;\; since$$

$$u(t_1 - 0) = u_0$$

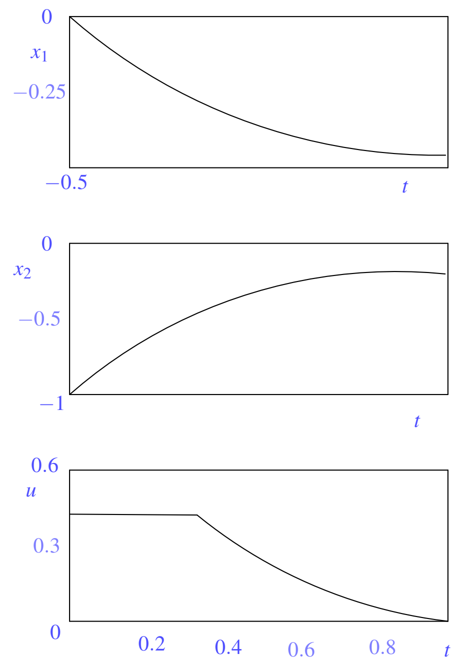
$$u(t_1 + 0) = \frac{1}{2\alpha}\Psi_2(t_1 + 0) = \frac{1}{2\alpha}\hat{b}_2 R|_{t=t_2} = -\frac{1}{2\alpha}\hat{b}_2(1)(1 - t_1)$$

we obtain

$$\Delta = \frac{1}{2\alpha}\hat{b}_2(1)(1 - t_1) + u_0 \tag{4.42}$$

We varied t_* until the condition $\Delta = 0$ was fulfilled with the necessary exactness.

Computer simulation was carried out for $u_0 = 0.5, \quad \alpha = 0.5$. The results are plotted in the figure below.



above solution of problem **[4.34]** for $u_0 = 0.5, \quad \alpha = 0.5$

The exactness of solution was estimated by calculating the values of $t_1, x_1(1), x_2(1)$ at different levels of resolution J. The results are presented in table below.

J	t_1	$x_1(1)$	$x_2(1)$
4	0.338569	−0.48679	−0.20527
5	0.338570	−0.48676	−0.20535
6	0.338574	−0.48675	−0,20536

Above parameter t_1 and boundary values of the problem **[4.34]**

In conclusion, this result of $x_1(1) \quad and \quad x_2(1)$ compare with the result simulated by computer as captured in figure 1 above proving that Haar wavelet efficiency is achieved.

4.2.4 Conclusion:

From the three cases we solved, we have shown that, by the introduction of Haar Wavelet function, we obtained very satisfactory exactness of the results even for a lower number of collocation points. The Haar Wavelet was therefore in deed of value add in computation of optimal control problems in providing a simple and straight forward approach in addition to providing virtually exact solution. This was in concurrence with Sengeta, Singh and Kumar (2014) who noted that they had represented simple and straight forward numerical technique based on Haar Wavelet in solving differential equations.

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