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Master Thesis
Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Applied Mathematics

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Abstract

Youth unemployment is on the rise globally and it poses huge economic, social and political challenges which results to unsustainable growth of the country's economy. The main aim of this thesis is to focus on youth unemployment in Kenya by developing a mathematical model consisting of first order systems of Ordinary Differential Equations (ODEs) and solve it numerically and to give results from mathematical perspective.Firstly, we give introduction to Unemployment problem, ODEs, meaning of a solution to an ODE and also conditions for a problem to satisfy for it to be referred as well posed. We then discuss in brief analytical solution and also the numerical approach of finding a solution to an ODE. A consideration of error analysis,convergence,consistency and stability of numerical methods in general is discussed.

We consider Runge-Kutta (RK) methods of different orders, derivation of Euler methods, second order,third order and RK method of fourth order are obtained and a brief formula for fifth order is given.Afterwards,error analysis for RK methods as well as the stability analysis are discussed. After describing the methods for a single equation, we focus on RK methods for solving systems of ODEs since it is our area of interest. We also analyze the stability for one-step and multi-step methods for ODEs.

Once we are fully equipped with the necessary tools,we now obtained the numerical solution the model using RK method of fourth order and discuss the results.

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Declaration and Approval

I hereby declare that this thesis is my original research work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

GIDEON KIPRUTO CHELULEI
Reg No. I56/13501/2018

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

Signature Date

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Dedication

This project is dedicated to my family. A special gratitude to my uttermost beloved parents, Philemon Chelulei and Esther Kiprono whose words of encouragement and push for tenacity ring in my ears. My siblings: Jonathan Chelulei, Rael Chelulei, Sylvia Chelulei and Ben Kimutai who have always been on my side at every step and are very special to me.

List of Abbreviations

IVP	Initial Value Problem
ODE	Ordinary Differential Equation
DE	Differential Equation
PDE	Partial Differential Equation
L.H.S	Left Hand side
R.H.S	Rift Hand side
w.r.t	with respect to
Fig	Figure
Tab	Table
Eq	Equation
Iff	If and only if
Lc	Lipschitz constant
BT	Butcher Tableau
RK	Runge-Kutta
RK4	Forth order Runge-Kutta method
ERK	Explicit Runge-Kutta
IKR	Implicit Runge-Kutta
SDIRK	Singly Diagonally Implicit Runge-Kutta
DIRK	Diagonally Implicit Runge-Kutta
ESDIRK	Explicit Singly Diagonally Implicit Runge-Kutta

List of symbols

$\mathbb{R}$	The set of real numbers
$\Re$	The real part of complex number
$\mathbb{C}$	The set of complex numbers
$\subset$	A subset
$\in$	Belongs to
$\forall$	For all
$\exists$	There exists
$\|x\|$	The absolute value of a real number,the modulus of a complex number
$\|\|x\|\|$	A norm in a vector space

Contents

Figures and Tables

Figures

Tables

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First and foremost, I would like to acknowledge my indebtedness and very special appreciation to my supervisor, Dr. Charles Nyandwi who made this work a reality. You've been a fantastic friend and a mentor to me. I would like to thank you for your recommendation, patience and allowing me to grow as a research scientist. Your guidance, encouragement and words of advice at every step of my thesis have been invaluable.

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Gideon Kipruto Chelulei

Nairobi, 2020.

1 Introduction

1.1 Background

The world currently, faces many serious problems like climate change, pollution, terrorism, diseases, poverty, inequality,corruption, earthquakes and many more. Among them is unemployment problem which has spread in the whole world and keeps on rising day by day.Unemployment interrelates to other problems like corruption, poverty, terrorism etc.

Unemployed people are those individuals seeking for a job either in government or private sector yet they are unable to find one. According to International labour Organization (ILO) adopted in 1982, unemployed person is a person of working age (15 years or above) who meets all the following three conditions:

(a) having no employment

(b) being available to take up a job within two weeks

(c) having actively looked for a job in the previous 30 days or having found one starting in the next three weeks.

The working age population can be grouped into three classes

(i) **Employed people**-are those paid by working for at least one hour or more in a week.

(ii) **Unemployed people**-are those who have no paying job(s) but are actively looking for one.

(iii) **Not in labour force**-are people not in the first two categories due to different reasons like, retirement, still studying, permanently unable to work etc.

Many factors contribute to unemployment and they vary depending from one country to another. Common factors are: slow economic growth, economic decline which makes businesses to struggle and therefore reducing the payroll expenses by laying off some of the employees thus causing unemployment, technological advancement like the use computers, robots, automation are efficient, accurate and less time consuming but on the other hand replaces the human resource leading to unemployment, unfavorable working environment for foreign investors who could have created many job opportunities for the locals also contribute to high levels of unemployment, increase in population which leads to a higher labour force which lowers the job supply and therefore leaving many unemployed.

The effects of unemployment are felt both economically and socially and not only affect an individual nor to the community but to the whole country as well. In a country where we have unemployed people, this causes a social problem where part of the population struggles in order to maintain a minimum welfare as well as the consumption level.

The rate of unemployment is the percentage of number of people who are unemployed over the sum of employed and the unemployed population. It is a commonly used indicator for determining how labour market works. Labour market is a term used by economist when referring to supply of labour and demand for labour. Unemployment rate also gives an insights into how the economy is performing more generally. Unemployment rate typically rises considerably during recessions then falls as the economy recovers[17], it indicates that the economy of a given country is struggling. High rates of unemployment requires rigorous measures by the government so as to provide social protection to its citizens. There are government policies and programs which have been laid down by each country on how to create job opportunities for its citizens.

Youths are very important in all countries across the globe. They are seen as enablers for achieving development goals.Youths are seen as people with energy, talents, creativity, open-minded, productivity and dynamical. They are future investors, leaders, workers, entrepreneurs, producers and customers.All the above mentioned qualities are what forms potential youths who are precious resource to the economies.A better world can be created by creative and dynamical youths.

For this reason, excluding youths from economies, societies and labour market is impossible. Youths must be part of economies and labour force so as to be able to achieve the development goals that we desire.

The higher rate of youth unemployment is not a local problem or only for the developing countries but it is a worldwide challenge which if not contained then, it can be out of control and therefore,it requires serious interventions so to mitigate this problem.

In Africa, 133 million young people are illiterate [ILO,2010]. Many youths lack or have little basic skills and thus are excluded from economies and labour market. Some who have little formal education have skills which are irrelevant to the current job demand and to make it worse, the current labour market require someone whose education and professional skills match the job that one is looking for and also the experience that one has acquired. For these reasons, many youths remain unemployed. The case of youth unemployment in Sub-Saharan Africa is projected to be at twenty percent[6].

Kenya currently faces unemployment problem that affects mainly the youths. As young people, they sop depending on other people's income and become independent[24].

We do not have a general definition of a youth, the several definitions depend majorly on the context and the purpose in which they are in use. The United Nations for example, gives a definition of a youth as an individual with a range of $15-24$ years of age which implies maturity, nurturing and building of skills and knowledge in preparedness for integration into economic, social and political domains of life. World Bank emphasis on the age range of $12-24$ as the time when pivotal foundations are set for learning and building acquirement. For Kenya, the definition include age range of $15-35$. The laws of Kenya define rights and responsibilities of an individual in accordance with the age of a person.

In the National Youth Council act of 2009 and the nation constitution of Kenya 2010 define youths as people age between 18 and 34. Also,the Age of Majority Act (Cap 33) states that "a person shall be of full age and cease to be under any disability by reason of age after attaining the age of 18 years. The unemployment Act of 2007 that depicts the laws that governs employment and protection of employees specifies that no person shall employ a child below 13 years whether gainfully or otherwise, while those between the age of $13-16$ may only get employed in nonlabour intensive functions.

In the past four decades,and in recognition of Kenya's unemployment and underemployment difficulties, successive government administrations have assumed and prioritized employment creation as a core policy (Republic of Kenya, $1969, 1983, 2008b, 2008c$)[24]. Recently, policies aiming at creating employment have also focused on youth.Some of these initiatives are: The sector plan for labour, Youth and Human Resource Development sector $(2008-2012)$, Kazi Kwa Vijana (KKV) program(2009) Launched in April 2009. The aim of KKV is to employ between $200,000$ and $300,000$ young people annually in both rural and urban areas in labour intensive public works project that are implemented by different government ministries. We also have a recent enrolled program called Kazi mtaani (2020).It is a national initiative that is designed to cushion the most vulnerable but able bodied citizens living in informal settlement from the effects and response strategies of the COVID-19 pandemic. The first phase kicked off in April 2020 employing over $26,000$ in 8 counties. The second phase will be expanded to cover over 34 counties and employ $200,000$ workers. The program targets resident of informal settlements who are over 18 years of age and are unable to find work due to disruption of economic activity. It has restriction of being open to youths only.

The government has also established training institutions which equipped learners with the required skills needed in the job market. Among them are colleges and Technical and Vocational Education and Training (TVETs)

Despite all government efforts to curb youth unemployment, youth unemployment is still on the rise. Currently, there are many unemployed youths and also many are graduating annually.

We are of the view that, the measures put in place does not directly address this issue of unemployment. We propose that if more allocation of funds is given to the the appropriate ministry and developing policies which ensures that more vacancies are created continuously,then the number of unemployed young people will significantly reduce .

This study is important as it informs the Kenyan government policy makers, its partners and other relevant bodies who would like to know the current situation of youth unemployment in Kenya and in addressing this issue by creating more vacancies for the young people. It will also be useful to institutions of learning for researchers who would like to do more research on unemployment.

1.2 Objectives

- To study and understand the concept of unemployment.
- To study and understand the concepts of error, convergence, consistency and stability of numerical methods.
- To study derivations of RK methods.
- To study and understand RK methods for systems of first order ODEs.

1.3 Problem Statement

This involves the study of numerical solution of ODEs in solving first order systems of nonlinear ODEs and we consider unemployment model as a real time problem.We solve the system using a fourth order RK method for systems. The solution to be obtained will help to comprehend and understand how the solution of unemployment problem has been done using the most efficient numerical method.

1.3.1 Why RK method?

- RK method is a widely used and is also an effective tool used in solving IVPs of ODEs.
- It can develop high order accurate numerical method without the need of high order derivatives of functions.
- It can be used to solve differential equations whose higher order derivatives are complicated and also the complicated stiff problems.
- In many instances, the method has been used to solve a single initial value problem and for this reason we focus our study on how to extend the method to the first order systems of ODEs.

1.3.2 Why Unemployment model?

Unemployment is a major problem not only for Kenyan government but is a global phenomenon which is on the rise and it has made many young university graduates demoralized. We would like to bring our contribution by giving a numerical solution to the problem which will inform the government agencies and its partners on the state of unemployment in the country and the measures to adopt.

1.4 Literature Review

1.4.1 A review on unemployment

In 2003 $Nikolopoulos$ and $Tzanetis$[16] in their work derived and analyzed a model considering housing allocation of homeless families due to natural disaster. Using some concept in this paper, in 2011 $Misra$ and $Singh$ [14] considered in their model three variables, the numbers of employed, temporarily employed and regularly employed persons and found out that the rate of movement of people from temporarily employed to regularly employed persons increases under certain conditions consequently,unemployment can be minimized.

From those models, in 2015 $Pathan$ and $Bhathawala$ [18]in their model found that the unemployment problem can be solved if the new job vacancies created by the government and private sectors are with a rate proportional to the number of unemployed people and also encouraged self-employment as another solution to the unemployed persons.

In 2016 $Pathan$ and $Bathawala$ [19]in their work developed model of unemployment focusing four variables namely: number of unemployed people, number of new migrants, number of employed people and number of newly created job vacancies by the government and private sector. They found that if the rate of new unemployed immigrants increased then the rate of unemployed people increased and therefore in order to tackle the situation, the government and the private sector needs to create job vacancies for the natives who are unemployed and also unemployed immigrants proportional to their number.

In 2017 *Pathan* and *Bathawala* [20] in their paper, discovered that in order to reduce the number of unemployed people, then the rate of unemployed people who joined employed class should be higher, they also observed that if present jobs varies, then the rate of self-employed should be higher in order to reduce unemployment.

In 2018 *Raneah*, *Ashi* and *Sarah* [2] in their unemployment model, they considered three variables namely; the number of unemployed persons, the number of employed and the number of available job vacancies. On varying some parameters, they found that in order to deal with the unemployment problem effectively, the government should consider creating and providing more job opportunities which will results to an increase in the number of employed and thus decreasing the number of unemployed people and therefore reducing the rate of unemployment.

In our text, we have considered the proposed unemployment model by *Raneah*, *Ashi* and *Sarah* for unemployment in Kenya. We obtain a numerical solution to the nonlinear system bt Runge-kutta method. We consider age bracket of $18-34$ years since this category have majority of the country's population according 2019 census results where there were 10.1 million youths between the age of $18-34$ years representing 21.2 percent of the country's population and 4.1 million aged $15-24$ years in the labor force and furthermore,are "fresh", energetic work force, well grounded with technological expertise which when properly utilized can bring significant contribution to the country's economic growth.

1.4.2 A review on Runge-Kutta methods

In 2010,*Ruitao*,*Chan*,*Fang* and *Xu* [13]in their paper solved nonlinear equations using a four-order Runge-Kutta method and they got a more stable and easily convergent results.

In 2013,[9] simruy in his paper considered a numerical solution of systems of differential equations using a fourth Runge-Kutta method and Euler method and concluded that a fourth order Runge-Kutta method is more accurate and gives consistent result.

In 2019,*Abhinadan*,*Clayton* and *Lemma*[5] presented a study on numerical solution of nonlinear differential equations using Runge-Kutta-Fehlberg method and found that it gives a more accurate approximations with relatively low cost compared to classical methods.

1.5 Thesis Outline

The outline of the thesis is as follows: In chapter2, the first section is about the conditions for an ordinary differential equation to have a solution, then we also discussed the analytical and numerical approach of finding solution to an ODE, in the second section we have some definition of terms, in the third section we discuss the concept of error analysis, convergence, consistency and stability for numerical methods. Chapter 3 involves the derivation of Runge-Kutta (RK) methods for second, third and fourth order then we give a formula for fifth order RK method, we obtained local truncation error for first and second order RK methods, we also discuss the stability regions for RK methods. Chapter 4, we consider system of ordinary differential equations and how RK methods works for the systems and also stability for a system. In chapter 5, we consider an example of a real world problem; unemployment model and then use RK method to solve and analyze the model by implementing the method using MATLAB software. In chapter 6, we give the numerical results as well as the discussion.Finally,chapter 7 involves conclusion of our work and the future research of the study.

2 Ordinary Differential Equations

2.1 Introduction

A differential equation is an equation, where the unknown is a function and both the function and its derivatives may appear in the equation. They are in two categories; ordinary differential equations (ODE) and partial differential equations (PDEs). An ODE is a differential equation containing one or more functions of one independent variables and the derivatives of those functions. While a PDE is a differential equation where the unknown function depends on two or more independent variables and their partial derivatives appear in the equations. These equations can be a single equation or a system of equations. An ordinary differential equation is an equation for a function $y(x)$, defined on an interval $I \subset \mathbb{R}$ and with values in the real or complex numbers or in the space $\mathbb{R}^m(\mathbb{C})$,which can be written as

$$\mathbf{F}\left(x, y(x), y'(x), y''(x), ..., y^m(x)\right) = 0 \tag{1}$$

Where $\mathbf{F}$ is an arbitrary function which contains the independent variable x,the dependent variable y and its derivatives with respect to (w.r.t) x.

Nonlinear differential equations is of more interest because of its wide applications. Nonlinear ODEs plays a great role in many branches of pure and applied mathematics and also their applications to analytical chemistry, biology, engineering, applied mechanics, quantum physic, economics and astronomy. From the last decade, researchers concentrated towards theoretical and numerical solutions to nonlinear ODEs.

Many nonlinear systems of ODEs cannot be solved explicitly and for this reason we need to have a better numerical method so as to accurately obtain an approximate solution.

Our aim is to obtain a numerical scheme for solving systems of ODEs which rapidly converges to the analytical solutions compared to other methods.

We will now discuss the conditions for an ODE to have a Solution and two approaches of finding a solution of an ODE.

2.1.1 Conditions for an ODE to have a Solution

A function $\Phi(x)$ is said to be a solution to a differential Eq. (1) if it satisfies the following requirements:

- The function $\Phi(x)$ is defined in the region $x \in I$.
- The function $\Phi(x)$ is differentiable hence the derivatives $\Phi(x), \Phi'(x), ..., \Phi^{(m)}(x)$ all exist in the region $x \in I$.
- If the variables $y, y', ..., y^{(m)}$ are replaced in the right-hand side (R.H.S) of Eq. (1) by the functions $\Phi(x), \Phi'(x), ..., \Phi^{(m)}(x)$, then Eq. (1) becomes zero for all $x \in I$.

A solution can be general if it consists of constant of integration or particular solution if the initial conditions are given.

An ODE is of order m if $y^{(m)}$ is derivative of the highest order present in the equation examples can be seen in Tab. 2.

ODE	Order
$xy'' - y' + 4x^3y = 0..$	2
$y'y = 4x$	1
$y''' = x^2y'' + y$	3

Table 2. Order of differential equations.

If dimension m of the variable y is greater than one,then we have a system of differential equations.

We would like to consider the methods for solving ordinary differential equations with only one independent variable. An explicit first order differential equation is an equation given by

$$\frac{dy}{dx} = f(x, y(x)) \tag{2}$$

Let consider Eq. (2) together with the initial condition which gives initial value problem (IVP) expressed as

$$\begin{aligned} y' = \frac{dy}{dx} &= f(x, y(x)) \\ y(0) &= y_0 \end{aligned} \tag{3}$$

for the interval $x_0 \leq x \leq x_w$.We have the independent variable being x, w as the number of points and f as a function of derivation. The objective is getting the unknown function $y(x)$ whose derivatives satisfies Eq.(2) with corresponding initial values. Equation (2) is of first order differential equation since the highest derivative present is of order one.

For a case of an IVP, we first discuss the conditions or requirements that a problem has to satisfy for it to have a solution.

2.1.2 A well posed problem

Before even looking for analytical or numerical solution to an ODE we first check what it means to say that a problem is well posed. To do this we consider the following definitions and theorems.

Definition 2.1.1. *A problem is [4] well formulated when an equation is associated with the correct number of initial or boundary conditions for its solution, while a problem is well posed when the solution does not only exists but is unique and continuously depends on the initial data.*

Definition 2.1.2. *A function f(x, y) satisfies a Lipschitz condition (Lc) in a region D of $\mathbb{R}^2$ if there exists a constant $L > 0$ such that*

$$||f(x,y) - f(x,z)|| \leq L||y - z|| \tag{4}$$

whenever (x,y) and (x,z) are in D. L is called the Lipschitz constant.

Theorem 2.1.3 (Existance). *If $f(x,y)$ is continuous in a set D*

$$D = \{(x,y)\|y - y_0\| \leq k, |y - y_0| \leq Y\}, \tag{5}$$

then there exists at least one solution of the IVP and it is of class C^1 for $|y - y_0| \leq Y$,where

$$Y = min\{\, Y, \frac{K}{M}\} and M = max\|f(x,y)\|, for (x,y) \in D$$

Theorem 2.1.4 (Uniqueness). *If in addition to continuity condition ot theorem of existence,then the [4]function f satisfies a Lc in a region D, then a solution $y(x)$ to the IVP is unique and*

$$\|y(x) - y_0\| \leq M\bar{Y}. \tag{6}$$

We now consider the problem of the continuous dependence on the initial data[4], we let $\bar{y}$ and $\tilde{y}$ be the two solutions of the two IVPs with the initial data $y(x_0) = \bar{y}_0$ and $y(x_0) = \tilde{y}_0$ respectively

Theorem 2.1.5 (Continuous dependence on initial data). *If f is continuous and satisfies the Lc, then*

$$\|\bar{y}(x) - \tilde{y}\| \leq \|\bar{y}_0 - \tilde{y}_0\| e^{L|y - y_0|} \tag{7}$$

[4].

Finally we now consider the analysis of continuous dependence on f, consider for $x \in [x_0, x_0 + Y]$ the initial value problems

$$\begin{array}{cc} \dfrac{dy}{dx} = f(x,y(x)) & \dfrac{dy}{dx} = \vec{y}(x,y(x)) \\ y(x_0) = y_0 & y(x_0) = \vec{y}_0 \end{array} \quad and \tag{8}$$

with f and $\vec{y}$ defined and continuous in a common domain D

Theorem 2.1.6 (Continuous dependence on a function f). *If one of the f or $\vec{y}$ satisfies a Lc with constant L and if*

$$\|f(x,y)-\vec{y}(x,y)\| \leq \varepsilon, \forall (x,y) \in D, \tag{9}$$

then

$$\|y(x)-\vec{y}(x)\| \leq \|y_0-\vec{y}_0\| e^{L|x-x_0|} + \frac{\varepsilon}{L}[e^{L|x-x_0|}-1] \tag{10}$$

where $y(x)$ and $\vec{y}(x)$ are solutions of the two initial value problems defined in Eq.(8).

If all the three conditions are fulfilled, then a given IVP is well posed[4]. We now extend to nonlinear system of ODEs. We first develop the Fundamental Existance-Uniqueness theorem.

Theorem 2.1.7. *[21][**The Fundamental Existance-Uniqueness theorem**] Let A be an open subset of $\mathbb{R}^n$ containing y_0 and assume that $\boldsymbol{f} \in C^1(A)$. Then, exists an $e > 0$ such that the IVP*

$$\frac{d\boldsymbol{y}}{d\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{y}) \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0 \tag{11}$$

has a unique solution $\boldsymbol{y}(\boldsymbol{x})$ on an interval $[-e,e]$

2.1.3 Approaches of finding a solution of an ODE

There are two ways of getting a solution to a differential equation (DE). One is by analytical approach and the other is through numerical approach.
The analytical solution of an ODE is the true solution of an equation $\frac{dy}{dx} = f(x,y(x))$ which is obtained by applying the appropriate method of integration.

2.1.4 Analytical methods for linear ODEs

Generally, we can obtain analytical solutions if we are dealing with linear systems of ODEs.
For any linear system

$$y' = B(x)y + C(x) \tag{12}$$

$B(x)$ is an $m \times m$ matrix of the coefficients $B_{ij}(x)$ and $C(x)$is the column vector with m components. With continuous coefficients on an interval [0,Y] satisfies the Lc with

$$L = \sum_{i,j=1}^{m} max|B_{ij}(x)|. \tag{13}$$

The IVP

$$\frac{dy}{dx} = B(x)y + C(x)$$
$$y(x_0) = y_0, \tag{14}$$

with $B_{ij}(x)$ and $C(x)$defined and continuous for $|x - x_0| \leq Y$, has unique solution for $|x - x_0| \leq Y$ by the theorems of uniqueness and existence.
If $y_1(x), y_2(x), ..., y_w(x)$ are w solutions to homogeneous linear system

$$\frac{dy}{dx} = A(x)y, y \in \mathbb{R}^m, \tag{15}$$

then any linear combination

$$\sum_{i=1}^{w} b_i y_i(x), b_1, b_2, ..., b_w \in \mathbb{R}, \tag{16}$$

is still a solution of Eq.(15).

Theorem 2.1.8 (:**Solutions of linear homogeneous equations**). *There exist m linearly independent solutions $y_1, ..., y_m$ of Eq.*(15) *such that any solution of Eq.*(15) *is written as*

$$y(x) = \sum_{i=1}^{m} \vec{b}_i y_i(x) \tag{17}$$

for suitable $\vec{b}_1, ..., \vec{b}_m \in \mathbb{R}$.

The IVP

$$y' = B(x)y$$
$$y(x_0) = y_0, \tag{18}$$

is then solved by imposing to $y(x)$ given by Eq.(17), the m initial conditions to find the values of constants $\vec{b}_1, ..., \vec{b}_m$.
Independent set of solutions for an homogeneous problem is known as a fundamental set [4] and is represented in matrix form

$$Y(x) = \begin{pmatrix} y_{1,1}(x) & \cdots & y_{1,n}(x) \\ \vdots & \ddots & \vdots \\ y_{m,1}(x) & \cdots & y_{m,m}(x) \end{pmatrix} \tag{19}$$

called the fundamental matrix.
The above result is used to obtain also exact solution of nonhomogeneous linear Eq.(12).

2.1.5 Numerical solution for IVPs

Generally, mathematical models are nonlinear and for this reason, the analysis of mathematical problems needs more sophisticated analytic methods and explicit solutions are almost impossible to get, therefore we apply computational schemes which are based on the concept of discretizing the variable $x \in [a,b]$ into a appropriate set of points:

$$I_x = \{x_0, x_1, x_2, ..., x_w\} \tag{20}$$

and approximate the solution along the discretizes points using a suitable numerical scheme. These methods can be categorized as single-step and multi-step methods. Single-step methods take the approach that to approximate the solution at x_{w+1} using y_w we obtain an approximate at intermediate steps in (x_w, x_{w+1}) and use these to get y_{w+1}. Examples of single-step methods include, Euler method, Runge-Kutta methods, Taylor's series method etc.

Multi-step methods uses approximate solution at previous points to obtain y_{w+1}. For example if we want to use y_{w-1} in addition to y_n to generate y_{w+1} as well as $f(x_{n+}, y_{n+})$and $f(x_{w-1}, y_{w-1})$we call this a two-step method. It would have a general form

$$y_{w+1} = c_1 y_w + c_2 y_{w-1} + h[d_1 f(x_w, y_w) + d_2 f(x_{w-1}, y_{w-1})] \tag{21}$$

where h is the step size,

$$h = \frac{x_w - x_0}{N}.$$

N is any positive real number representing the number of steps. This is an example of an explicit scheme since we have the unknown value y_{w+1} equals to all known values.

we can also have an implicit two-step scheme which is given by

$$y_{w+1} = c_1 y_w + c_2 y_{w-1} + h[d_0 f(x_{w+1}, y_{w+1}) + d_1 f(x_w, y_w) + d_2 f(x_{w-1}, y_{w-1})] \tag{22}$$

which is the general form of a two-step multi-step method and if $d_0 = 0$ then it is explicit, otherwise it is implicit. Examples of multi-step methods include Adams-Multon formulae, Adams-Bashforth formulae etc.

1. **Euler method**
   It is the simplest numerical method:

   $$y_{w+1} = y_w + hf(x_w, y_w), w = 0, 1, 2, ... \tag{23}$$

   It is an explicit single-step method for the case of one step.

2. **Adams-Bashforth**
   A simple multistep method is the two-step Adams-Bashforth method which is given by

   $$y_{w+2} = y_w + \frac{h}{2}[3f(x_{w+1}, y_{w+1}) - f(x_w, y_w)] \tag{24}$$

One of the difficulties with multi-step methods over single step method is the starting values. For single-step methods, one only needs the initial conditions while for the multi-step methods we need solutions at several values before implementing the method. For example in two-step method we need y_0 and y_1, for a four step we need y_0, y_1, y_2 and y_3 before implementing the formula. We can use a single-step method to calculate the starting values but we need to be sure we use a scheme with the same or better order of accuracy as our multi-step method.

2.1.6 Definition of terms

1. One-step numerical method is given by

$$y_{w+1} = y_w + h\theta(x_w, y_w, h), \tag{25}$$

   where $\theta(x_w, y_w, h)$ is the increment function. Here, the term"one-step method" means that the numerical solution y_{w+1} only depends on the current numerical approximation y_w and not on the previous approximations y_{w-1}, y_{w-2}, etc..

2. If $\phi(x_w, y_w, h) = 0$ as $h \to 0$ then we have a **zero stable** numerical method.

3. **An Explicit RK method** is a one step method with presentation

$$y_{w+1} = y_w + \sum_{q=1}^{l} P_q K_q,$$

   where

$$\begin{aligned} &K_1 = hf(x_w, y_w) \\ &\vdots \\ &K_l = hf(x_w + hc_l, y + \sum_{q=1}^{l-1} m_{lq} K_q), l = 2, 3, ..., \end{aligned}$$

   where l is the order of the method and if $l = q$,then the sum equals to zero. This forms the family of Explicit Runge-Kutta methods[10].

4. A Runge-Kutta method has absolute stability region when the region has

$$|\xi(\lambda h)| \leq 1,$$

   and a strict stability region when the region

$$|\xi(\lambda h)| < 1.$$

5. The **stability function** $\xi(x) = \xi(\lambda h)$ is the function generated by applying the one-step method
$$y_{w+1} = y_w + hf(x_w, y_w),$$
to a test problem
$$y'(x) = \lambda y(x). \tag{26}$$
Therefore,
$$y_{w+1} = \xi(\lambda h) y_w$$
and recursively,
$$y^{(w)} = \xi(\lambda h)^w y_w.$$
The **stability region** of a one-step method is the set
$$A = \{x \in \mathbb{C} : |\xi(x)| \leq 1\}$$

6. A numerical scheme is an **A-stable**, if the stability region of the method has the left half-plane of $\mathbb{C}$, hence
$$\{x \in \mathbb{C} : |\xi(x)| \leq 0\} \subset A.$$

7. A one-step method is called **B-stable** if for monotonic IVP $\frac{dy}{dx} = f(x)$ with arbitrary initial value y_0 and z_0 there holds:
$$|y_1 - z_1| \leq |y_0 - z_0|$$
independent of the step length h.

8. An A-stable,one-step method is **L-stable**, if:
$$\lim_{x \to \infty} |\xi(x)| = 0$$
holds. L-stable method is also said to be **strongly A-stable**.

9. The iteration
$$y_{w+1} = \xi(h\lambda) y_w$$
is **divergent** for $|\xi(h\lambda)| > 1$, converges otherwise.

The **absolute stability** region is the set
$$\{h\lambda \in \mathbb{C} : |\xi(h\lambda)| \leq 1|\}$$
which is the unit circle under ξ.

The **relative stability** region is the set
$$\{h\lambda \in \mathbb{C} : |\xi(h\lambda)| \leq |e^{h\lambda}|\}.$$

The relative stability compares the growth of the iteration to the growth of the exact solution.

10. **A DE** of first order is said to be **linear** if it involves on the dependent variable y and its first order derivatives, otherwise it is nonlinear.

11. A system of ODEs is said to be **autonomous** if the R.H.S of it is independent of time variable t. Otherwise it is said to be **non-autonomous**.

12. A system of ODEs is said to be **linear** if we can write as

$$\begin{aligned}\frac{dy_1}{dt} &= b_{11}(t)y_1(t) + b_{12}(t)y_2(t) + \dots + b_{1m}(t)y_m(t) + c_1(t),\\ \frac{dy_2}{dt} &= b_{21}(t)y_1(t) + b_{22}(t)y_2(t) + \dots + b_{2m}(t)y_m(t) + c_2(t),\\ &\vdots\\ \frac{dy_m}{dt} &= b_{m1}(t)y_1(t) + b_{m2}(t)y_2(t) + \dots + b_{mm}(t)y_m(t) + c_m(t)\end{aligned} \tag{27}$$

Or

$$\frac{d\mathbf{Y}}{dt} = \mathbf{B}(t)\mathbf{Y}(t) + \mathbf{C}(t) \tag{28}$$

where $B(t)$ is an $m \times m$ matrix of coefficients $b_{ij}(t)$ while C(t) is a column matrix with m elements. If $C(t) = 0$ then we have an homogeneous system otherwise it is inhomogeneous.

13. A system of first order DEs is written as

$$\begin{aligned}y_1' &= g_1(x, y_1, y_2, \dots, y_m),\\ y_2' &= g_2(x, y_1, y_2, \dots, y_m),\\ &\vdots\\ y_m' &= g_m(x, y_1, y_2, \dots, y_m)\end{aligned} \tag{29}$$

This system is said to be nonlinear if the terms of $g_{i'}s$ are nonlinear functions of the dependent variables $y_{i'}s$.

2.2 Analysis of error, convergence, consistency and stability of numerical methods

2.2.1 Error analysis

Errors arises as a result of using a numerical method to solve a problem. Error occurs in two categories namely, truncation error and round off error.

(i) **Truncation error**. This occurs when approximations are used to estimate some quantity. It is obtained by taking difference of the theoretical solution and the approximate solution. This error is of two types:

- The Local truncation error is given by T_{w+r} at a point $x_{(w+r)}$. Which is given by

$$T_{w+r} = y(x_{w+r}) - y(x_w) - h\phi(x_w, y_w, h), h > 0, \tag{30}$$

where $\phi(x_w, y_w, h)$ is the increment function of the numerical method. This error arises when when solving an IVP and accumulates at step.

$$T_{w+1} = y(x_{w+1}) - y_{w+1}. \tag{31}$$

In order to get the error,we subtract the R.H.S from the left hand side (L.H.S) of the numerical scheme with and aid of Taylors series expansion.Remainder term is what we call the local truncation error denoted by τ_w.Dividing the result by step length h we get

$$\tau_w = \frac{T_w}{h}. \tag{32}$$

- Global truncation:We represent by er_w which is expressed as

$$er_w = y(x_w) - y_w.$$

Here, $y(x_w)$ is the analytical result while y_w is the approximated result. The global truncation error results when local error accumulates in all of its iterations.

(ii) **Round off error**:It occurs due to the way in which computers represents numerical values.After calculating the approximations of methods,the result is dropped in specific location.It is denoted by R_{w+r} which is also committed at the w^{th} application of the methods[9].

2.2.2 Stability for Numerical Methods

When obtaining a numerical solution to the given Eq. (2) it is prudent to check if the solution obtained is not far much different from analytical solution to the problem. Therefore, the idea of nearness is what we called the numerical stability.

The stability for single-step methods

Definition 2.2.1 (Consistency [7]**).** *Let τ_w be the Local truncation error at the w^{th} step of the numerical scheme. The scheme is consistent with the DE it approximates if*

$$\lim_{h \to 0,\, 1 \le w \le k} \max |\tau_w| = 0,$$

where k represents the last iteration.

Definition 2.2.2 (Convergence[7] **).** *A numerical scheme is convergent w.r.t the DE if*

$$\lim_{h\to 0,\, 1\leq w\leq k} \max |y(x_w) - y_w| = 0.$$

Definition 2.2.3 (Stability[7]**).** *A numerical scheme is* ***stable*** *if slight disturbance in the initial conditions for the DE results correspondingly no more than small changes in the subsequent approximations. Let y_w be the approximation to the solution $y(x_w, x_0)$ where the last component indicates the initial data that generated solution. Let $\bar{y}_w$be the be approximation to $x(x_w, x_1)$for each $\varepsilon > 0$ there is a $\delta > 0$ sufficiently small so that $|y_w - \bar{y}_w| < \varepsilon$ whenever $|x_1 - x_0| < \delta$.*

Stability for Multi-step methods

A general multi-step method is expressed as

$$y_{w+1} = \sum_{i=0}^{r} c_i y_{w-i} + h \sum_{i=-1}^{r} d_i f(x_{w-i}, y_{w-i}), \tag{33}$$

for $h > 0$[22] and $c_0, c_1, ..., c_p, d_), d_1, ..., d_r$ being constants such that $|c_p| + |d_r| \neq 0$ as r is non-negative. This is called $(p+1)$ step method[9].
The general multi-step of our IVP satisfies

$$y(x_{w+1}) = \sum_{i=0}^{r} c_i y(x_{w-i}) + h \sum_{i=-1}^{r} d_i f(x_{w-i}, y(x_{w-i})) + T_{w+r,} \tag{34}$$

where T_{n+p} is the local truncation error[9] and

$$y_{w+1} = \sum_{i=0}^{r} c_i y_{w-i} + h \sum_{i=-1}^{r} d_i f(x_{w-i}, y_{w-i}) + R_{w+r}, \tag{35}$$

where y_w is approximation of multi-step method and R_{w+r} is the round off error. Now subtracting Eq. (35) from Eq.(34) and considering the global error $e_n = y(x_w) - y_w$, then we get

$$y_{w+1} - y(x_{w+1}) = e_{w+1} = \sum_{i} c_i e_{w-i} + h \sum_{i=-1}^{r} d_i \{f(x_{w-i}, y(x_{w-i})) - f(x_{w-i}, y_{w-i})\} + \varepsilon_{w+r}.$$

The difference between round-off error and the local truncation error gives ε is given by $\varepsilon = R_{w+r} - T_{w+r,}$. The truncation error for any differentiable function say $y(x)$ is expressed as

$$T_w(y) = y(x_{w+1}) - \{\sum_{i=0}^{r} c_i y(x_{w-i}) + h \sum_{i=-1}^{r} d_i y'(x_{w-i})\}, w \geq r. \tag{36}$$

We get this error for multi-step method by expanding $y(x_{w+1})$ by Taylor's series expansion and the error will be given by

$$T_w = O(h^{r+1}).$$

We consider Eq. (32) for consistency check which is expressed as

$$\tau_w(y) \to 0, h \to 0$$

rewriting the multi-step method using expression of Eq.(32) we get

$$y(x_{w+1}) = \sum_{i=0}^{r} c_i y(x_{w-i}) + h \sum_{i=-1}^{r} d_i y'(x_{w-i}) + h\tau_w(y).$$

We then introduce

$$\tau(h) \equiv \max_{x_0 < x < x_w} |\tau_w(y)|. \tag{37}$$

A multi-step method is **consistent[9]** if the function $y(x)$ is continuously differentiable on $[x_0, x_w]$ and satisfying the following condition

$$\tau(h) \to 0, h \to 0 \tag{38}$$

[9].The order of a numerical method plays a crucial role on how fast the method converges, if condition of Eq. (37) approaches zero very fast, then the approximation is closer to the theoretical solution[9]. Also, if the step size h chosen is very small, then intervals increases and this makes the truncation error closer to zero which is given by

$$\tau(h) = O(h^r). \tag{39}$$

Theorem 2.2.4. *[3]: Let $k \geq 1$ be a given integer. For Equation (37) [9]to hold for all continuously differentiable functions $y(x)$, i.e for multi-step method (33) to be consistent, then it is necessary and sufficient that*

$$\sum_{i=0}^{r} c_i = 1 \tag{40}$$

$$\sum_{i=0}^{r} d_i - \sum_{i=0}^{r} ici = 1 \tag{41}$$

[9]Further, for $\tau(h) = O(h^k)$ to be valid for all functions $y(x)$ that are $k+1$ times continuously differentiable, it is necessary and sufficient that Eq. (40) and Eq. (41)hold and that

$$\sum_{i=0}^{r} (-i)^j c_i + i \sum_{i=-1}^{r} (-i)^{j-1} d_i = 1, j = 2, 3, ..., k.$$

For the largest value of r, which (39) holds, is referred to as order of the convergence of the method . A multi-step method will converge if the approximate solution y_w tends to exact solution $y(x_w)$ so as the step size h approaches zero.
This gives convergence in form of a limit that is, as $h \to 0, w \to \infty, h = \frac{x_w - x_0}{w}$ remaining fixed where $x_0 \le x \le x_w$.This can be written as

$$\lim_{h\to 0} y_w - y(x_w).$$

For the convergence of initial value problem Eq. (2) and the general multi-step method in Equation Eq.(33), the following theorem will be important.

Theorem 2.2.5. *[9] Assume that the derivative function $f(x,y(x))$ is continuous and satisfies the Lipschitz condition*

$$||f(x,y_1) - f(x,y_2)|| \le L||y_1 - y_2|| \tag{42}$$

$\forall\, y_1, y_2 \in [-\infty, +\infty], x \in [x_0, x_w]$ *and for a positive constant L called the Lipschitz constant. Let the initial errors satisfy*

$$\mu(h) \equiv \max_{0\le t\le r} |y(x_t) - y_t| \to 0, h \to 0 \tag{43}$$

[9]. Assume that all the solution $y(x)$ is continuously differentiable and that the method is consistent, i.e it satisfies Eq. (38)*.[9]Finally, assume that the coefficients c_i are all non-negative,$c_i \ge 0, i = 0, i, ..., r$ then the multi-step method* (33)*is convergent and*

$$\max_{x_0\le x\le x_w} |y(x_w) = y_w| \le q_1\mu(h) + q_2\tau(h), \tag{44}$$

for suitable constants q_1, q_2.[9]If the solution $y(x)$ is $(k+1)$ times continuously differentiable, the method (33) *is of order k and the initial errors $\mu(h) = O(h^k)$, then the order of convergence of the method is k.*

Convergence of the numerical methods does not stand on its own,rather it majorly lies on the consistency and stability of the numerical scheme. Convergence and stability of a numerical scheme relates with the polynomial roots [9].

$$r(a) = a^{(r+1)} - \sum_{i=0}^{r} c_i a^{(r-i)}. \tag{45}$$

We observe that $r(1) = 0$ by considering the consistency condition (40). Let $a_0 = 1$ and $a_1, a_2, ..., a_r$ be the roots of the polynomial (45). The multi-step method (33) satisfies the root condition if

$$|a_i| \le 1, i = 0, 1, ..., r \tag{46}$$

$$|a_i| = 1 \implies \alpha'(a_i) = \delta(1) \neq 0. \tag{47}$$

The associated characteristic polynomial is given by

$$\rho(a, h\lambda) = \alpha(a) - h\lambda\gamma(a). \tag{48}$$

The general solution of Eq. (43) can be obtained if the roots $a_i(h\lambda)$ are all distinct

$$y_w = \sum_{i=0}^{r} \beta_i [a_i(h\lambda)]^w. \tag{49}$$

Definition 2.2.6. *[3]Assume the consistency conditions* (40) *and* (41) *then the multi-step method* (33) *is stable iff the root condition is satisfied.*

Stability is so important when finding the approximate solution,if we are certain that a given method is stable then we are satisfied that we will get a solution close to the analytical solution. The numerical errors generated during the approximation should not be propagated. If the initial condition y_0 has numerical errors, then the errors will increase at each step.
Now analyzing the stability of the numerical methods.We slightly modify the IVP (2) with $y_0 + \varepsilon$.The result is

$$\begin{aligned} y'_\varepsilon(x) &= f(x, y_\varepsilon(x)), \\ y_\varepsilon(x) &= y_0 + \varepsilon. \end{aligned} \tag{50}$$

It can be seen that $y(x)$ and $y_\varepsilon(x)$ do exist on a given interval for a small value of ε and besides that

$$||y_\varepsilon - y||_\infty \equiv \max_{x_0 < x < x_w} |y_\varepsilon(x) - y(x)| \leq r\varepsilon, r > 0. \tag{51}$$

The same analysis is used for multi-step method. We now state the solution of (31) with the initial values $y_0, y_1, ..., y_r$ for a DE $y' = f(x,y)$ which slightly modify to new values $l_0, l_2, ..., l_r$. Then,

$$\max_{x_0 < x < x_w} |y_w - l_w| \leq \varepsilon. \tag{52}$$

These initial values holds depending on h,contrary,the numerical solution $\{y_w | 0 \leq w \leq B(h)\}$ is stable if it does not depend on h and we do have a constant r $\forall$ sufficiently small ε which is

$$\max_{0 < w < B(h)} |y_w - l_w| \leq r\varepsilon, 0 \leq h \leq h_0,$$

and $b(h)$ is taken as the largest subscript B which $x_B \leq x_w$ which also satisfy the Lipschitz condition.
We say a numerical solution of a multi-step scheme is stable when all the approximate solution y_w is stable[9].When maximum error is not greater than the initial error, ε then the initial value problem is said to be well-conditioned, otherwise we say it is ill-conditioned. Therefore, small changes in initial data will cause a change to the solution of $y(x)$ of Eq.(2) for a small ε.

Theorem 2.2.7. *[9] A linear multi-step method is* ***zero stable*** *for any ODE of the Eq.*(2) *where f satisfies Lipschitz condition iff its first characteristic polynomial has zeros inside the unit disc with any which lies on the unit circle being simple.*

We consider two essential conditions for convergence by considering the theorem below.

Theorem 2.2.8. *[9]Consistency and zero stability are the basic conditions for convergence of linear multi-step.*

Definition 2.2.9. *[11]The linear multi-step method Eq.* (33) *is* ***relatively stable*** *for a given λh, the root a_i satisfy $|a_i(\lambda h)| \leq a_0(\lambda h), i = 1,2,...,m$ for some sufficiently small values that the step length h can have. If a linear multi-step method is not relatively stable then it is weakly stable.*

Definition 2.2.10. *[9]A numerical method is* ***absolutely stable*** *in a region $\Re$ of complex plane if $\forall\ \lambda h \in \Re$ all roots of the stability polynomial $\beta(a, \lambda h)$ associated with the method satisfy*

$$|a_i| < 1, i = 1,2,...,r.$$

Stability also entails the absolute stability. The parameters λ and h,the eigenvalues of the Jacobian matrix and the step size respectively. Absolute stability will depends on their product,λh. If it is analyze separately then it si insufficient. The region of absolute stability is considered to be on complex plane since it is assumed that λ is a negative real or complex part. When the numerical method is applied to the test equation in order to obtain the stability region, then the modulus of the pth step iteration should be < 1. For system of differential equations, the coefficient matrix forms the Jacobian matrix. The the the eigenvalues are obtained by obtaining the determinant of Jacobian matrix equated to zero..Solving the system, we obtain λ_m,m=1,2,3,... If all the eigenvalues are real, then the matrix is symmetric and it is possible to have complex eigenvalues. If the eigenvalues are not real then the product λh will also be complex. The numerical scheme will be absolutely stable if λh lies in the absolute stability region[9].

3 Runge-Kutta Methods

3.1 Introduction

Runge-Kutta one-step numerical methods are well known methods for solving ODEs. Runge-Kutta method was developed by German mathematicians duo *Carl Runge* (1856-1927) and *M.W.Kutta* (1867-1944)[5]. In (1901), Carl Runge developed numerical methods for solving the differential equations that arose in his study of atomic spectra. Runge-Kutta method gives better approximations compared to other numerical methods. It is also suitable in cases when the computations of higher order derivatives are complicated.

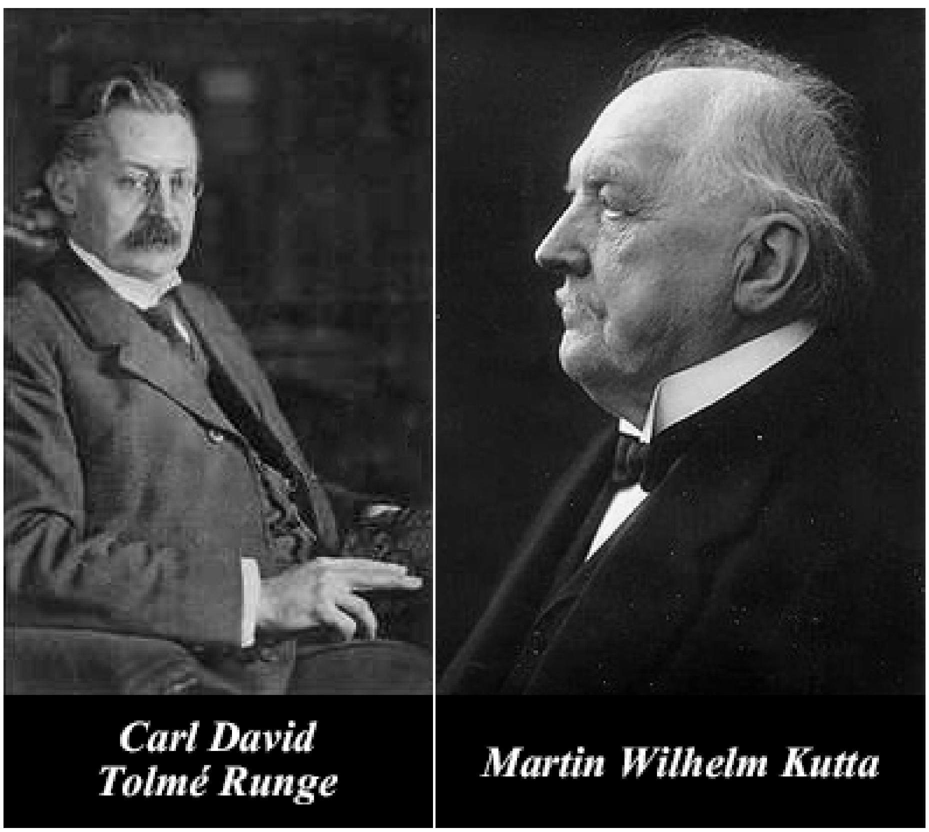


Figure 1. A photo of *Carl Runge* **and** *M.W.Kutta*.

A general l-stage Runge-Kutta method according to *Simruy* [9] is given by

$$y_{(w+1)} = y_w + h\phi(x_w, y_w, h), w = 0, 1, ..., N-1, \quad (53)$$

where $\phi(x_w, y_w, h)$ is called the increment function which is given by

$$\phi(x_w, y_w, h) = \sum_{q=1}^{l} P_q K_q,$$

where $P_q{}'s$ are constants and $k_q{}'s$ are given by

$$\begin{aligned}
K_1 &= hf(x_w, y_w) \\
K_2 &= hf(x_w + c_2 h, y_w + m_{21} K_1) \\
K_3 &= hf(x_w + hc_3, y_w + m_{31} K_1 + m_{32} K_2) \\
&\vdots \\
K_l &= hf\left(x_w + hc_l, y_w + \sum_{q=1}^{l-1} m_{lq} K_q\right), l = 2, 3, \ldots
\end{aligned}$$

where m_{lq}, c_l are constants and each of the function $k's$ represent the slope of the solution which are approximated to $y(x)$. The coefficients m_{lq}, c_l, p_l specify each method of Runge-Kutta and are represented in a scheme known as Butcher Tableau.

Definition 3.1.1. *A **Butcher Tableau (BT)** is a representation of coefficients of the increment function in the following matrix form:*

$$\begin{array}{c|c}
C & M \\
\hline
 & p^T
\end{array}$$

Table 3. A BT for Runge–Kutta method.

If the method is an implicit RK method then its butcher table is represented below If the

$$\begin{array}{c|ccccc}
c_1 & m_{11} & m_{12} & \cdots & m_{1l-1} & m_{1l} \\
c_2 & m_{21} & m_{22} & \cdots & m_{2l-1} & m_{2l} \\
c_3 & m_{31} & m_{32} & \cdots & m_{3l-1} & m_{3l} \\
\vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\
c_l & m_{l1} & m_{l2} & \cdots & m_{l,q-1} & m_{ll} \\
\hline
 & P_1 & P_2 & \cdots & P_{l-1} & P_l
\end{array}$$

Table 4. The BT for implicit Runge–Kutta method.

method is explicit RK then we have

$$\begin{array}{c|ccccc}
0 & & & & & \\
c_2 & m_{21} & & & & \\
c_3 & m_{31} & m_{32} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
c_l & m_{l1} & m_{l2} & \cdots & m_{l,q-1} & \\
\hline
 & P_1 & P_2 & \cdots & P_{l-1} & P_l
\end{array}$$

Table 5. A BT for an explicit Runge–Kutta method.

Where

$$c_q = \sum_{q=1}^{l} m_{ql}, l = 1,2,... \quad \textbf{(54)}$$

From the structures of matrix M, we can specify the sub classes of RK method. The most general case is the group of **Implicit** RK methods (IRK) also called **fully implicit** (FIRK).They usually have nonzero elements in the coefficient matrix M. The method is called **explicit** RK (ERK) if it is strictly a lower triangular matrix. If the diagonal elements nonzero, then we have **diagonally implicit** RK methods(DIRK). If all the diagonal elements are there then we have **singly diagonal implicit** RK (SDIRK). A special case of ERK methods is the **explicit singly diagonal implicit** RK methods (ESDIRK) which only have zeros on the first row of matrix M. If we have a four stage RK method, then the representation of matrices M is as follows:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ m_{21} & 0 & 0 & 0 \\ m_{31} & m_{32} & 0 & 0 \\ m_{41} & m_{42} & m_{43} & 0 \end{pmatrix}_{\textbf{ERK}} \qquad \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}_{\textbf{IRK}}$$

$$\begin{pmatrix} \beta & 0 & 0 & 0 \\ m_{21} & \beta & 0 & 0 \\ m_{31} & m_{32} & \beta & 0 \\ m_{41} & m_{42} & m_{43} & \beta \end{pmatrix}_{\textbf{SDIRK}} \qquad \begin{pmatrix} m_{11} & 0 & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ m_{31} & m_{32} & m_{33} & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}_{\textbf{DIRK}}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ m_{21} & \beta & 0 & 0 \\ m_{31} & m_{32} & \beta & 0 \\ m_{41} & m_{42} & m_{43} & \beta \end{pmatrix}_{\textbf{ESDIRK}}$$

Of these classes, ERK method is the simplest to solve however explicit methods have small area of stability which makes problems mainly in case of stiff equations.

3.1.1 Stiff systems of ODEs

Applications in mechanical engineering, chemical kinematics, numerical solutions in PDEs demonstrate the property of stiffness. However, it not easy to tell if a system is either stiff or not. Many definitions have been given concerning stiffness. According to $Saufianim$[1], the definition of stiffness w.r.t the linear system of first order equation is defined as

$$y' = Ay + \phi(x), y(x_0) = \eta, x_0 < x < x_w, \tag{55}$$

where $y^T = (y_1, y_2, y_3, ...y_m), \eta^T = \eta_1, \eta_2, \eta_3, ..., \eta_m$ and A is a $m \times m$ matrix having eigenvalues $\lambda_k, k = 1, 2, 3, ..., m$.

Definition 3.1.2. *[11] A linear method is said to be stiff if*

(a) $Re(\lambda_k) < 0. k = 1, 2, 3, ..., m$

(b) $max|Re(\lambda_k)| \ggg min|Re(\lambda_k)|$ *where* λ_k *are the eigenvalues of matrix* A *and the ratio*

$$R = \frac{max|Re(\lambda_k)|}{min|Re(\lambda_k)|}$$

is referred to as the stiffness ratio. If $R \ggg 1$*, then the system is said to be stiff.*

3.2 Derivation of Explicit Runge-Kutta methods

3.2.1 Euler method-First order Runge-Kutta method

Euler method can be derived from forward difference approximation of the first order derivative as

$$\frac{dy}{dx} \approx [y(x+h) - y(x)]/h = f(x,y)$$

applying it to Eq. (2) at $x = x_w$ we obtain

$$f(x_w, y(x_w)) \approx \frac{y(x_w+h) - y(x_w)}{h}$$
$$y(x_w+h) = y(x_w) + hf(x_w.y(x_w))$$
$$y(x_{w+1}) = y(x_w) + hf(x_w.y(x_w))$$

Now substituting the theoretical solution $y(x_w)$ with the approximate solution y_w we arrive at

$$y_{w+1} = y_w + hf(x_w, y_w). \tag{56}$$

Its BT is

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

Table 6. The BT for the explicit first order Runge–Kutta method.

Alternatively, we we can integrate the equation

$$y' = f(x,y)$$

w.r.t x from $x_w$to $x_w + h$ to get

$$y(x_w+h) - y(x_w) = \int_{x_w}^{x_w+h} f(x,y)dx. \tag{57}$$

Approximating the R.H.S using rectangle rule

$$\int_{x_w}^{x_w+h} f(x,y)dx \approx hf(x_w, y(x_w)).$$

Therefore Eq. (57) becomes

$$y(x_w+h) = y(x_w) + hf(x_w, y(x_w))$$

which eventually becomes (56). Another method of derivation is to expand the function $y(x)$ at a point x_w to get

$$y(x_w+h) = y(x_w) + hy'(x_w) + \frac{h^2}{2}y''(x_w) + O(h^3), \tag{58}$$

but from (2) we have $y' = f(x,y)$ and using this in (58) and omitting higher order terms we arrive at

$$y(x_w+h) = y(x_w) + hf(x_w, y(x_w)).$$

Now substituting the analytical solution $y(x_{w+1}), y(x_w)$ and the approximate solution y_{w+1}, y_w we get (56).

3.2.2 Second order Runge-Kutta method

The second Runge-Kutta method is given as

$$y_{w+1} = y_w + (P_1K_1 + P_2K_2), \tag{59}$$

where

$$\begin{aligned} K_1 &= hf(x_w, y_w) \\ K_2 &= hf(x_w + c_2h, y_w + m_{21}K_1). \end{aligned} \tag{60}$$

We first obtain Taylor's series expansion of $y(x)$ assuming that it has derivatives of all orders we,then compare the increment functions of Taylor's series and that of Runge-Kutta method of second order. Taylor's series expansion of fourth order is given by

$$y(x_{w+1}) = y(x_w+h) = y(x_w) + hy'(x_w) + \frac{h^2}{2}y''(x_w) + \frac{h^3}{3!}y'''(x_w) + \frac{h^4}{4!}y^{(iv)}(x_w) + \ldots$$

where

$$y'(x_w) = f(x_w, y_w), y_w = y(x_w).$$

We now obtain the derivatives of $y''(x_w)$,$y'''(x_w)$ and $y^{(iv)}(x_w)$. Since $y'(x_w) = f(x_w, y_w)$, without loss of generality, lets take $y'(x) = f(x, y(x))$,then

$$y'' = \frac{d}{dx}[y'(x)] = \frac{d}{dx}[f(x, y(x))]$$

$$= \frac{\partial f}{\partial x} + y'(x)\frac{\partial f}{\partial y} = f_x + ff_y$$

$$y'''(x) = \frac{d}{dx}[y''(x)] = \frac{d}{dx}[f_x + ff_y] = \frac{\partial(f_x + ff_y)}{\partial x} + f\frac{\partial(f_x + ff_y)}{\partial y}$$

$$= f_{xx} + 2ff_{xy} + f^2 f_{yy} + ff_y^2 + f_x f_y$$

$$= f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_y(f_x + ff_y).$$

Let use the convention that $A = f_x + ff_y, B = f_{xx} + 2ff_{xy} + f^2 f_{yy}$
Then

$$y^{(iv)}(x) = \frac{d}{dx}[y'''(x)] = \frac{d}{dx}[B + f_y A]$$

$$= \frac{\partial B}{\partial x} + f\frac{\partial B}{\partial y} + f_{xy}A + ff_{yy}A + f_y(\frac{\partial A}{\partial x} + f\frac{\partial A}{\partial y})$$

$= f_{xxx} + 2f_x f_{xy} + 2ff_{xxy} + 2ff_x f_{yy} + f^2 f_{xyy} + f[f_{xxy} + 2f_y f_{xy} + 2ff_{xyy} + 2ff_y f_{yy} + f^2 f_{yyy}] +$
$[f_{xy} + ff_{yy}](f_x + ff_y) + f_y[(f_{xx} + f_x f_y + ff_{xy}) + f(f_{xy} + f_y + ff_{yy})]$
$= f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy} + f_y(f_{xx} + 2ff_{xy} + f^2 f_{yy}) + f_x(3f_{xy} + 3ff_{yy})ffy(3f_{xy} +$
$3ff_{yy}) + f_y^2(f_x + ff_y)$
$= C + f_y B + 3A(f_{xy} + ff_{yy}) + f_y^2 A$
where $C = f_{fxxx} + 3ff_{xxxy} + 3f^2 f_{xyy} + f^3 f_{yyy}$.
The Taylor series of $y(x)$ is therefore

$$y(x+h) = y(x) + hf + \frac{h^2}{2}A + \frac{h^3}{6}(B + f_y A) + \frac{h^4}{24}(C + f_y B + 3(f_{xy} + ff_{yy})A + f_y^2 A) + \dots \quad \textbf{(61)}$$

We now obtain K_1, k_2 using Taylor's series expansion formula

$$K_n = h\sum_{n=0}^{\infty}\frac{1}{n!}\left(hc_2\frac{\partial}{\partial x} + m_{21}K_1\frac{\partial}{\partial y}\right)^n f(x,y) \quad \textbf{(62)}$$

Then

$$K_1 = hf(x,y) = hf$$

$$K_2 = h\sum_{n=0}^{\infty}\frac{1}{n!}(hc_2\frac{\partial}{\partial x} + m_{21}hf\frac{\partial}{\partial y})^n f(x,y)$$

$= hf + h(c_2 hf_x + m_{21}hff_y) + \frac{h}{2}(c^2h^2 f_{xx} + 2c_2 m_{21}h^2 ff_{xy} + m_{21}^2 h^2 f^2 f_{yy}) + \dots$

$$= hf + h^2(c_2 f_x + m_{21}ff_y) + \frac{h^3}{2}(c_2^2 f_{xx} + 2c_2 m_{21}ff_{xy} + m_{21}^2 f^2 f_{yy}) + \dots$$

Then putting these values of K_1, K_2 into Eq. (59) we get

$$y_{w+1} = y_{(x+h)} = y(x) + hf(P_1 + P_2) + h^2 P_2(c_2 f_x + m_{21} f f_y) + \frac{h^3}{2} P_2(c_2^2 f_{xx} + 2c_2 m_{21} f f_{xy} + m_{21}^2 f^2 f_{yy}) + \ldots \quad (63)$$

Where the function and its derivatives are evaluated at $(x_w, y(x_w))$.

Now comparing the coefficients of different powers of h in Eq. (61) and Eq. (63) we get

$$P_1 + P_2 = 1 \quad (64)$$

$$c_2 P_2 = \frac{1}{2} \quad (65)$$

$$m_{21} P_2 = \frac{1}{2}. \quad (66)$$

Here we have a system of three equations with four unknowns and therefore we won't be able to get a unique solution since the parameters P_1, P_2, c_2 and m_{21} have infinite choices. From Eq.n (65) and (66) $\Longrightarrow$ $c_2 = m_{21}$. Using (65) $P_2 = \frac{1}{2c_2}$,for $c_2 \neq 0$ then from (64) $P_1 = 1 - \frac{1}{2c_2}$.
If $c_2 = 1, m_{21} = 1\frac{1}{2}$ and $P_1 = P_2 = 1$.
Substituting for the values into Equation (59)and (60) we arrive at

$$y_{w+1} = y_w + (K_1 + K_2)/2, \quad (67)$$

where

$$\begin{aligned} K_1 &= hf(x_w, y_w) \\ K_2 &= hf(x_w + h, y_w + K_1). \end{aligned} \quad (68)$$

Which is the classical RK method of order 2 which is also known as the $Euler - Cauchy$ method or $Heun's$ method. If we choose $c_2 = \frac{1}{2} = m_{21}, P_1 = 0, P_2 = 1$, then we obtain a $modified\ Euler$ method of order 2 given as

$$y_{w+1} = y_w + hf(x_w + h/2, y_w + hf(x_w, y_w)/2).$$

Another possible choice is if we take $c_2 = \frac{2}{3}$, then we have $m_{21} = \frac{2}{3}, P_2 = \frac{3}{4}, P_1 = 1 - \frac{3}{4} = \frac{1}{4}$.

The corresponding BT for classical second order RK method is shown as

0		
1	1	
	$\frac{1}{2}$	$\frac{1}{2}$

Table 7. The BT for the explicit second order Runge–Kutta method.

3.2.3 Third Order RK Method

To derive third order RK method we apply the same construction used to construct the second order RK method. A general third order RK method is given by

$$y_{w+1} = y_w + P_1K_1 + P_2K_2 + P_3K_3, \tag{69}$$

where

$$\begin{aligned} K_1 &= hf \\ K_2 &= hf(x_w + c_2h, y_w + m_{21}K_1) \\ K_3 &= hf\left(x_w + c_3h, y_w + (m_{31}K_1 + m_{32}K_2)\right). \end{aligned} \tag{70}$$

We now obtain K_2, K_3 using Taylor's series expansion (62) to get

$$K_1 = hf$$

$$K_2 = h\sum_{n=0}^{\infty}\frac{1}{n!}\left(hc_2\frac{\partial}{\partial x} + m_{21}K_1\frac{\partial}{\partial y}\right)^n f(x,y)$$

$$= h[f + c_2hf_x + m_{21}K_1f_y + \frac{1}{2!}(c_2^2h^2f_{xx} + 2c_2hm_{21}K_1f_{xy} + m_{21}^2K_1^2f^2f_{yy}) + \ldots]$$

$$= hf + h^2(c_2f_x + m_{21}ff_y) + \frac{h^3}{2}(c_2^2f_{xx} + 2c_2m_{21}ff_{xy} + m_{21}^2f^2f_{yy}) + \ldots$$

$$K_3 = hf(hf(x + c_3h, y + (m_{31}K_1 + m_{32}K_2)))$$

$$= h\sum_{0}^{\infty}\frac{1}{n!}\left(c_3h\frac{\partial}{\partial x}+m_{31}K_1+m_{32}K_2\frac{\partial}{\partial y}\right)^n f(x,y)$$

$$= h[f+(c_3h\frac{\partial}{\partial x}+(m_{31}K_1+m_{32}K_2)\frac{\partial}{\partial y})f+\frac{1}{2!}(c_3h+\frac{\partial}{\partial x}+(m_{31}K_1+m_{32}K_2)\frac{\partial}{\partial y})^2 f+...+]$$

Substituting for the value of K_1, K_2 into the first order differentials we get
$K_3 = h[f+(c_3hf_x+m_{31}hff_y+m_{32}hff_y+m_{32}h^2(c_2f_x+m_{21}ff_y))f_y+...$
$+\frac{1}{2!}(c_3^2h^2f_{xx}+2c_2h(m_{31}K_1+m_{32}K_2)f_{xy}+(m_{31}K_1+m_{32}K_2)^2)f_{yy}+...]$

Now substituting the values of K_1, K_2 into the second order differentials we arrive at
$K_3 = hf+h^2[c_3f_x+(m_{31}+m_{32})ff_y]+h^3[m_{32}(c_2f_x+m_{21}ff_y)fy+\frac{1}{2}c_2^2f_{xx}+(c_3m_{31}+c_2m_{32})ff_{xy}+$
$\frac{1}{2}(m_{31}^2+2m_{31}m_{32}+m_{32}^2)f^2f_{yy}+...]$

Substituting for the values of K_1, K_2 and K_3 in Eq. (69) we get

$$y_{w+1} = y(x+h) = y(x)+h(P_1+P_2+P_3)f+h^2[P_2(c_2f_x+m_{21})ff_y+P_3(c_3f_x+(m_{31}+m_{32})ff_y)]+$$
$$h^3\{P_2[\frac{1}{2}c_2^2f_{xx}+c_2m_{21}ff_{xy}+\frac{1}{2}m_{21}^2f^2f_{yy}]+P_3[m_{32}(c_2f_x+m_{21ff_y})f_y+$$
$$\frac{1}{2}c_3^2f_{xx}+(c_3m_{31}+c_3m_{32})ff_{xy}+\frac{1}{2}(m_{31}+m_{32})^2f^2f_{yy}]\}+... \quad \textbf{(71)}$$

Where the function and its derivatives are evaluated at $(x_w, y(x_w))$. Now comparing the coefficients of different powers of h in Eq. (61) and Eq. (71) we get
Coefficients of h:
$P_1+P_2+P_3 = 1$
coefficient of h^2 :
$P_2(c_2f_x+m_{21})ff_y+P_3(c_3f_x+(m_{31}+m_{32})ff_y)$

$$= (c_2P_2+c_3P_3)f_x+(m_{21}P_2+(m_{31}+m_{32})P_3)ff_y = \frac{1}{2}A = \frac{1}{2}(f_x+ff_y)$$

$\implies c_2P_2+c_3P_3 = \frac{1}{2}$
$m_{21}P_2+(m_{31}+m_{32})P_3 = \frac{1}{2}$
Coefficients of h^3 :
$P_2[\frac{1}{2}c_2^2f_{xx}+c_2m_{21}ff_{xy}+\frac{1}{2}m_{21}^2f^2f_{yy}]+P_3[m_{32}(c_2f_x+m_{21ff_y})f_y+\frac{1}{2}c_3^2f_{xx}+(c_3m_{31}+c_3m_{32})ff_{xy}+$
$\frac{1}{2}(m_{31}+m_{32})^2f^2f_{yy} = \frac{1}{6}(B+f_yA)$

Collecting coefficients of $f_{xx}, ff_{xy}, f^2 f_{yy}, f_x f_y, ff_y^2$ we get
$(\frac{1}{2}c2_2P_2+\frac{1}{2}c_3^2)f_{xx}+(c_2m_{21}P_2+c_3(m_{31}+m_{32})P_3)f^2fxy+\frac{1}{2}(m_{21}^2P_2+(m_{31}+m_{32}^2)P_3)f^2f_{yy}+$
$c_2m_{32}P_3f_xf_y+m_{21}m_{31}P_3ff_y^2$=$\frac{1}{6}(f_{xx}+2ff_{xy}+f^2f_{yy}+f_xf_y+ff_y^2)=\frac{1}{6}(B+f_yA)$

comparing the R.H.S and L.H.S of the above equation we arrive at

$$\frac{1}{2}c_2^2P_2+\frac{1}{2}c_3^2P_3=\frac{1}{6}$$

$$c_2m_{21}P_2+c_3(m_{31}+m_{32})P_3=\frac{1}{3}$$

$$\frac{1}{2}(m_{21}^2P_2+(m_{31}+m_{32})^2P_3=\frac{1}{6}$$

$$c_2m_{32}P_3=\frac{1}{6}$$

$$m_{21}m_{32}P_3=\frac{1}{6}.$$

We arrive at the following system:

$$P_1+P_2+P_3=1 \tag{72}$$

$$c_2P_2+c_3P_3=\frac{1}{2} \tag{73}$$

$$m_{21}P_2+(m_{31}+m_{32})P_3=\frac{1}{2} \tag{74}$$

$$c_2^2P_2+c_3^2P_3=\frac{1}{3} \tag{75}$$

$$c_2m_{21}P_2+c_3(m_{31}+m_{32})P_3=\frac{1}{3} \tag{76}$$

$$m_{21}^2P_2+(m_{31}+m_{32})^2P_3=\frac{1}{3} \tag{77}$$

$$c_2m_{32}P_3=\frac{1}{6} \tag{78}$$

$$m_{21}m_{32}P_3=\frac{1}{6}. \tag{79}$$

We observe that, Eq. (78) and (79) $\implies c_2=m_{21}$ and substituting this value in (76) we observe that Eq. (75) and (76) are:

$$c_2^2P_2+c_3^2P_3=\frac{1}{3}$$

$$c_2^2P_2 + c_3(m_{31} + m_{32})P_3 = \frac{1}{3},$$

equating the two equations gives $c_3 = m_{31} + m_{32}$, for $c_3 \neq 0$. Then, the system of eight equations reduces to a system of six equations:

$$c_2 = m_{21} \quad (80)$$

$$m_{31} + m_{32} = c_3 \quad (81)$$

$$P_1 + P_2 + P_3 = 1 \quad (82)$$

$$c_2P_2 + c_3P_3 = \frac{1}{2} \quad (83)$$

$$c_2^2P_2 + c_3^2P_3 = \frac{1}{3} \quad (84)$$

$$c_2m_{32}P_3 = \frac{1}{6}. \quad (85)$$

(86)

We can write the last three equations in matrix form as:

$$\begin{pmatrix} c_2 & c_3 & \frac{1}{2} \\ c_2^2 & c_3^2 & -\frac{1}{3} \\ 0 & c_2m_{32} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} P_2 \\ P_3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The homogeneous system has unique solution of $P_2, P_3, 1$ iff the determinant of coefficients is not different zero that is

$$\begin{vmatrix} c_2 & c_3 & \frac{1}{2} \\ c_2^2 & c_3^2 & -\frac{1}{3} \\ 0 & c_2m_{32} & -\frac{1}{6} \end{vmatrix} = 0 = \begin{vmatrix} c_2 & c_3 & \frac{1}{2} \\ 0 & c_3^2 - c_2c_3 & -\frac{1}{3} + \frac{1}{2}c_2 \\ 0 & c_2m_{32} & -\frac{1}{6} \end{vmatrix} = c_2\left(-\frac{1}{6}c_3(c_3 - c_2) + c_2m_{32}(-\frac{1}{3} + \frac{1}{2}c_2)\right) = 0$$

if $c_2 \neq 0$, then on simplification we get

$$c_3 = \frac{c_3(c_3 - c_2)}{c_2(2 - 3c_2.)}.$$

If we choose $c_2 = \frac{1}{2}$ and $c_3 = 1$,we get $m_{21} = \frac{1}{2}, m_{32} = 2, m_{31} = -1$. Substituting these values in Eq. (72),(73) and (78) we get $P_3 = \frac{1}{6}, P_2 = \frac{2}{3}, P_1 = \frac{1}{6}$
Then substituting these values into Equation (69) and (70) we obtain a classical third order RK method given by

$$y_{w+1} = y_w + (K_1 + 4K_2 + K_3)/6$$

where

$$K_1 = hf(x_w, y_w)$$

$$K_2 = hf(x_w + h/2, y_w + K_1/2)$$

$$K_3 = hf(x_w + h, y_w - K_1 + 2K_2).$$

The corresponding BT is given by

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

Table 8. The BT for the explicit third order RK method.

3.2.4 Fourth Order RK Method

Fourth order RK method is given by

$$y_{w+1} = y_w + P_1K_1 + P_2K_2 + P_3K_3 + P_4K_4, \quad (87)$$

where

$$K_1 = hf(x_w, y_w)$$

$$K_2 = hf(x_w + c_2h, y_w + m_{21}K_1)$$

$$K_3 = hf(x_w + c_3, y_w + m_{31}K_1 + m_{32}K_2)$$

$$K_4 = hf(x_w + c_4, y_w + m_{41}K_1 + m_{42}K_2 + m_{43}K_3).$$

Here we have thirteen parameters which implies we should solve thirteen equations in order to get all the parameters, consequently we consider Eq. (87) where:

$$K_1 = hf(x_w, y_w)$$

$$K_2 = hf(x_w + c_2h, y_w + c_2K_1)$$

$$K_3 = hf(x_w + c_3h, y_w + c_3K_2)$$

$$K_4 = hf(x_w + c_4, y_w + c_4K_3)$$

We now obtain K_2, K_3, K_4 using using Taylor's series expansion given by Eq. (61)

$$K_1 = hf(x_w, y_w)$$

$$K_2 = hf(x_w, +c_2h, y_w + m_{21}K1)$$

$$= h\{f + (c_2h\frac{\partial}{\partial x} + c_2K_1\frac{\partial}{\partial y})f + \frac{1}{2}(c_2h\frac{\partial}{\partial x} + c_2K_1\frac{\partial}{\partial y})^2 f + \frac{1}{3!}(c_2h\frac{\partial}{\partial x} + c_2K_1\frac{\partial}{\partial y})^3 f + ...\}$$

$$= h\{f + c_2hA + \frac{1}{2}c_2^2h^2B + \frac{1}{6}c_2^3h^3C + ...\}$$

$$K_3 = hf(x_w + c_3h, y_w + c_3K_2)$$

$$= h\{f + (c_h\frac{\partial}{\partial x} + c_3K_2\frac{\partial}{\partial y})f + \frac{1}{2}(x_w + c_3h, y_w + c_3K_2)$$

$$= h\{f + (c_3h\frac{\partial}{\partial x} + c_3K_2\frac{\partial}{\partial y})^2 f + \frac{1}{3!}(c_3h\frac{\partial}{\partial x} + c_3K_2\frac{\partial}{\partial y}))^3 f + ...$$

Replacing for the values of K_1, K_2 we get
$= h\{f + (c_3hf_x + c_3hf_y(f + c_2hA + \frac{1}{2}c_2^2h^2B + ...)) + \frac{1}{2}(c_3^2h^2f_{xx} + 2c_3^2h^2(f + c_2hA + \frac{1}{2}c_2^2h^2B + ...)f_{xy} + c_3^2h^2(f + c_2hA + \frac{1}{2}c_2^2h^2B + ...)^2 f_{yy}) + \frac{1}{6}(c_3^3h^3f_{xxx} + 3c_3^3h^3(f + c_2hA + \frac{1}{2}c_2^2h^2B + ...)f_{xxy} + 3c_3^3h^3(f + c_2hA + \frac{1}{2}c_2^2h^2B + ...)^2)f_{xyy} + (c_3^3h^3(f + c_2hA + \frac{1}{2}c_2^2h^2B + ...)^3 f_{yyy}) + ...\}$

Therefore
$K_3 = h\{f + c_3hA + \frac{1}{2}h^2(c_3^2B + 2c_2c_3f_yA) + \frac{1}{6}h^3(c_3^3C + 3c_2^2c_3f_yB + 6c_2c_3^2(f_{xy} + ff_{yy})A + ...)\}$

Similarly, $K_4 = h\{f + c_4hA + \frac{1}{2}h^2(c_4^2 + 2c_3c_4f_yA) + \frac{1}{6}h^3(c_3^3C + 3c_2^2c_3f_yB + 6c_2c_3^2(f_{xy} + ff_{yy}A) + 6c_2c_3c_4f_y^2A) + \ldots\}$
The function and its derivatives are evaluated at $(x_w, y(x_w))$
Putting the values of K_1, K_2, K_3, K_4 into Equation (87)
we obtain $y_{w+1} = y_w + hf(P_1 + P_2 + P_3 + P_4) + h^2A(c_2P_2 + c_3P_3 + c_4P_4) + \frac{1}{2}(c_2^2P_2 + c_3^2P_3 + c_4^2P_4)h^3B + \frac{1}{6}(c_2^3P_2 + c_3^3P_3 + c_4^3P_4)h^4C + (c_2c_3P_3 + c_3c_4P_4)h^3f_yA + \frac{1}{2}(c_2^2c_3P_3 + c_3^2c_4P_4)h^4f_yB + (c_2c_3^2P_3 + c_3c_4^2P_4)h^4(f_{xy} + ff_{yy}) + c_2c_3c_4P_4f_y^2A + \ldots$

Now comparing (61) and the obtained equation we get

$$P_1 + P_2 + P_3 + P_4 = 1 \tag{88}$$
$$c_2P_2 + c_3P_3 + c_4P_4 = \frac{1}{2} \tag{89}$$
$$c_2^2P_2 + c_3^2P_3 + c_4^2P_4 = \frac{1}{3} \tag{90}$$
$$c_2^3P_2 + c_3^3P_3 + c_4^3P_4 = \frac{1}{4} \tag{91}$$
$$c_2c_3P_3 + c_3c_4P_4 = \frac{1}{6} \tag{92}$$
$$c_2c_3^2P_3 + c_3c_4^2P_4 = \frac{1}{8} \tag{93}$$
$$c_2^2c_3P_3 + c_3^2c_4P_4 = \frac{1}{12} \tag{94}$$
$$c_2c_3c_4P_4 = \frac{1}{24} \tag{95}$$

Solving the above system of eight equations with seven unknowns by choosing convenient values of parameters we get $c_2 =, c_3 = \frac{1}{2}, = c_4 = 1, P_1 = P_4 = \frac{1}{6}, P_2 = P_3 = \frac{1}{3}$.

Finally, substituting these values into Eq. (87) we obtain

$$y_{w+1} = y_w + (K_1 + 2K_2 + 2K_3 + K_4)/6$$

where

$$K_1 = hf(x_w, y_w)$$

$$K_2 = hf(x_w + h/2, y_w + K_1/2)$$

$$K_3 = hf(x_w + \frac{h}{2}, y_w + K_2/2)$$

$$K_4 = hf(x_w + h, y_w + K_3)$$

Which is popularly known as *a classical fourth order RK method.*Its BT is given as

0				
$\frac{1}{2}$	$\frac{1}{2}$			
$\frac{1}{2}$	0	$\frac{1}{2}$		
1	0	0	1	
	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

Table 9. The BT for the explicit fourth order RK method.

3.2.5 Fifth Order RK Method

A general fifth order RK method is given as

$$y_{w+1} = y_w + P_1K_1 + P_2K_2 + P_3K_3 + P_4K_4 + P_4K_5, \tag{96}$$

where

$$K_1 = fh(x_w, y_w)$$

$$K_2 = hf(x_w, +c_2h, y_w + m_{21}K_1)$$

$$K_3 = hf(x_w + c_3h, y_w + (m_{31}K_1 + m_{32}K_2))$$

$$K_4 = hf(x_w + c_4h, y_w + (m_{41}K_1 + m_{42}K_2 + m_{43}K_3))$$

$$K_5 = hf(x_w + c_5h, y_w + (m_{51}K_1 + m_{52}K_2 + m_{53}K_3 + m_{54}K_4).$$

Applying the same procedure used in deriving the second, third and fourth order RK methods and comparing it to fifth order Taylor's series, then choosing the parameters as $m_{52} = \frac{1}{2}, m_{53} = -1, m_{54} = 2m_{21} = c_2 = \frac{1}{4}, c_5 = 1, m_{51} = \frac{1}{2}, c_3 = \frac{1}{4}, c_4 = \frac{1}{2}, m_{41} = 0, m_{42} = \frac{1}{4}, m_{43} = \frac{1}{4}, m_{31} = -\frac{1}{4}, P_= P_5 = \frac{1}{6}, P_2 = \frac{1}{2}, P_3 = -\frac{1}{2}, P_4 = \frac{2}{3},$

Substituting these values we arrive at

$$y_{w+1} = y_w + \frac{1}{6}(K_1 + 3K_2 - 3K_3 + 4K_4 + K_5), \tag{97}$$

where

$$K_1 = hf(x_w, y_w)$$

$$K_2 = hf(x_w + \frac{h}{4}, y_w + \frac{K_1}{4})$$

$$K_3 = hf(x_w + h/4, y_w + \frac{1}{4}(-K_1 + 2K_2))$$

$$K_4 = hf(x_w + h/2, y_w + \frac{1}{4}(K_2 + K_3))$$

$$K_5 = hf(x_w + h, y_w + \frac{1}{2}(-K_1 + K_2 - 2K_3 + 4K_4)).$$

The corresponding Butcher's Tableau of parameters is

$$\begin{array}{c|ccccc}
0 & & & & & \\
\frac{1}{4} & \frac{1}{4} & & & & \\
\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & & \\
1 & -\frac{1}{2} & \frac{1}{2} & -1 & 2 & \\
\hline
 & \frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{2}{3} & \frac{1}{6}
\end{array}$$

Table 10. The BT for the explicit fifth order Runge–Kutta method.

3.2.6 Truncation Error

We now obtain the truncation error for first order RK method. From the exact solution which is the Taylor's series expansion (58) and now subtracting the approximate solution, Euler method (56) we obtain

$$T_{w+1} = |y(x_{w+1}) - y_{w+1}| = \frac{h^2}{2}y''(x_w) + O(h^3).$$

T_{w+1} is proportional to $O(h^2)$, thus the method is of first order. Global error occurs as a result of accumulation of the local truncation error after many steps of approximations.

We now estimate the error in second order RK formula then the errors for higher orders can be obtained by generalizing the computed error.

The second order RK method is given by

$$y_{w+1} = y_w + (K_1 + K_2)/2$$

where

$$K_1 = hf(x_w, y_w)$$

$$K_2 = hf(x_w + 2h, y_w + K_1)$$

Now, the truncation error is expressed as (31)

$$T_{w+1} = y(x_{w+1}) - y_{w+1}$$

Now, expanding the given formula by Taylor's series expansion we obtain

$$y(x_{w+1}) = y(x_w + h) = y(x_w) + hy'(x_w) + \frac{h^2}{2!}y''(x_w) + \frac{h^3}{3!}y'''(x_w) + \frac{h^4}{4!}y^{iv}(x_w) + \dots$$

or

$$y(x_{w+1}) = y_w + hy'_w + \frac{h^2}{2}y''_w + \frac{h^3}{3!}y'''_w + \frac{h^4}{4!}y^{iv}_w + \dots$$

Now substituting for the derivatives we arrive at

$$y(x_{w+1}) = y_w + hf + \frac{h^2}{2}(f_x + ff_y) + \frac{h^3}{3!}(f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + ff_y^2) + O(h^4). \quad \textbf{(98)}$$

On the other hand

$$y_{w+1} = y_w + (K_1 + K_2)/2$$

where

$$K_1 = hf(x_w, y_w) = hf$$

$$K_2 = hf(x_w + h, y_w + K_1) = hf(x_w + h, y_w + hf).$$

Which on expansion using the Taylor's series we get

$$K_2 = hf + h^2(f_x + ff_y) + \frac{h^3}{2!}(f_{xx} + 2ff_{xy} + f^2 f_{yy}) + \dots +$$

Therefore

$$y_{w+1} = y_w + hf + \frac{h^2}{2}(f_x + ff_y) + \frac{h^3}{4}(f_{xx} + 2ff_x + f^2 f_{yy}) + O(h^4). \quad (99)$$

Now subtracting Eq. (99) from (31) we obtain the truncation error

$$T_{w+1} = \frac{h^3}{12}|f_{xx} + 2ff_{xy} + f^2 f_{yy} - 2f_y f_x - 2ff_y^2|. \quad (100)$$

Here the truncation error is of order $O(h^3)$ this implies that the numerical method is of order 2 similarly, the truncation error for third and fourth order RK methods is $O(h^4), O(h^5)$ respectively, therefore in general if truncation error $T_{w+1} = O(h^{r+1})$, r defines the order of the numerical method.

3.2.7 Stability of RK methods

RK method is said to be **A-stable** if the modulus of the function is less than one for $Re(h\lambda) < 0$ for all complex $h\lambda$. In order to analyze the convergence of RK method, consistency and stability condition must hold. Let consider a one-step method of the form of Eq. (53) which is a numerical solution of (3)
Using the truncation error:

$$T_{w+1} = y(x_{w+1}) - y(x_w) - h\phi(x_w, y_w, h), \quad (101)$$

then the local truncation error is

$$\tau_w(y) = \frac{1}{h}T_{w+1}(y). \quad (102)$$

Now to show the convergence of solution(58), we require that

$$\tau_w(y) \to 0, h \to 0, \quad (103)$$

since,

$$\tau_w(y) = \frac{y(x_{w+1}) - y(x_w)}{h} - \phi(x_w, y_w, h)$$

we require that $\phi(x_w, y_w, h) \to y'(x) = f(x, y)$ as $h \to 0$
Accordingly, we define

$$\beta(h) = \sup_{\substack{x_0 < x < x_w \\ -\infty < y < \infty}} |f(x,y) - \phi(x,y,h)|$$

and assume that $\beta(h) \to 0, h \to 0$, then the consistency condition holds.

3.2.8 Absolute stability of RK methods

The exact solution of the IVP (3) is

$$y(x) = y_0 e^{x\lambda}$$

where λ is a non-positive real number. Here, the exact solution converges to zero on the R.H.S of the solution as x tends to positive infinity.

When the numerical scheme is applied to the test problem $y' = \lambda y$, the interval of line λh must agree with the condition of absolute stability and this shows that solution of RK method goes to zero as x tends to positive infinity[9].

We now analyze stability of RK methods using the test problem

$$y' = \lambda y. \tag{104}$$

First order RK method is given by

$$y_{w+1} = y_w + hf(x_w, y_w).$$

Now applying the model problem Eq.(104) into the formula we get

$$\begin{aligned} y_{w+1} &= y_w + h\lambda y_w \\ &= y_w(1 + h\lambda) \\ &= y_w \xi(\lambda h) \end{aligned}$$

where

$$\xi(\lambda h) = 1 + h\lambda.$$

The characteristic polynomial is

$$p(x) = x - \xi(\lambda h)$$

with a single root

$$x = \xi(\lambda h).$$

The method is absolutely stable when

$$|1 + h\lambda| \leq 1$$

or the region of stability is

$$-2 \leq h\lambda \leq 0. \tag{105}$$

We say the region of absolute stability as a region in complex z plane by taking an assumption that λ is complex and h as a real positive number. The region of absolute stability for Euler's method is a disk of a unit radius centered at -1[12]. We allow λ to be complex since we are usually solving a system of ODEs.
In the case of linear system,the eigenvalues of the matrix of coefficients that determines the stability. On the other hand, nonlinear case are typically linearize and consider the eigenvalues of the Jacobian matrix[12]. Hence,λ stands for eigenvalues values which could be real or complex even if the matrix is real.
For the **second order RK method** given by Eq. (67)

$$y_{w+1} = y_w + (K_1 + K_2)/2$$

where

$$K_1 = hf(x_w, y_w),$$

$$K_2 = hf(x_w + h, y_w + K_1).$$

Then,applying the model problem (104) into K_1, K_2 we get

$$K_1 = \lambda h y_w$$

$$K_2 = h\lambda(y_w + h\lambda y_w)$$

$$= y_w(\lambda h + (\lambda h)^2).$$

Therefore,

$$y_{w+1} = y_w + \frac{1}{2}(\lambda h y_w + y_w(\lambda h + (\lambda h)^2))$$

$$= y_w + y_w(\lambda h + \frac{(\lambda h)^2}{2})$$

$$= y_w[1 + \lambda h + \frac{(\lambda h)^2}{2}]$$

$$= y_w \xi(\lambda h)$$

where, $\xi(\lambda h) \approx e^{\lambda h}$ is the growth factor of the method. The method is absolutely stable if

$$|1+\lambda h+\frac{(\lambda h)^2}{2}| \leq 1 \tag{106}$$

and relatively stable if $|1+\lambda h+\frac{(\lambda h)^2}{2}| \leq e^{\lambda h}$. We now consider the stability of **third order RK method**, the formula is given by

$$y_{w+1} = y_w + (K_1 + 4K_2 + K_3)/6,$$

where

$$K_1 = hf(x_w, y_w)$$

$$K_2 = hf(x_w + h/2, y_w + K_1/2)$$

$$K_3 = hf(x_w + h, y_w - K_1 + 2K_2).$$

Now applying the model problem Eq.(104) into the method we get

$$K_1 = h\lambda y_w$$

$$K_2 = h\lambda(y_w + \frac{1}{2}h\lambda y_w)$$

$$= y_w(\lambda h + \frac{(\lambda h)^2}{2})$$

$$K_3 = \lambda h(y_w - \lambda h y_w + 2y_w(\lambda h + \frac{(\lambda h)^2}{2}))$$

$$= y_w(\lambda h + (\lambda h)^2 + (\lambda h)^3).$$

Therefore,

$$y_{w+1} = y_w + \frac{1}{6}(\lambda h y_w + 4y_w(\lambda h + \frac{(\lambda h)^2}{2}) + y_w(\lambda h + (\lambda h)^2 + (\lambda h)^3))$$

$$= y_w(1 + \lambda h + \frac{(\lambda h)^2}{2!} + \frac{(\lambda h)^3}{3!}).$$

This method is absolutely stable if the condition

$$|1+x+\frac{x^2}{2!}+\frac{x^3}{3!}| \leq 1 \tag{107}$$

is satisfied where $x=\lambda h, \forall\ \lambda h \in \xi(\lambda h)$.
Fourth order RK method is given by the formula

$$y_{w+1}=y_w+(K_1+2K_2+2K_3+K_4)/6,$$

where

$$K_1=hf(x_w,y_w)$$

$$K_2=hf(x_w+h/2,y_w+K_1/2)$$

$$K_3=hf(x_w+h/2,y_w+K_2/2)$$

$$K_4=hf(x_w+h,y_w+K_3)$$

Applying the model problem (104) to the formula we have

$$K_1=\lambda hy_w$$

$$K_2=\lambda h(y_w+\frac{\lambda hy_w}{2})$$

$$=y_w(\lambda h+\frac{(\lambda h)^2}{2})$$

$$K_3=\lambda h(y_w+\frac{K_2}{2})$$

$$=\lambda h(y_w+y_w(\lambda h+\frac{(\lambda h)^2}{2}))$$

$$=y_w(\lambda h+\frac{(\lambda h)^2}{2}+\frac{(\lambda h)^3}{4})$$

$$K_4 = \lambda h(y_w + K_3)$$

$$= \lambda h(y_w + y_w(\lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{4}))$$

$$= y_w(\lambda h + (\lambda h)^2 + \frac{(\lambda h)^3}{2} + \frac{(\lambda h)^4}{4})$$

Hence,

$$y_{w+1} = y_w + \frac{1}{6}(\lambda h y_w + 2y_w(\lambda h + \frac{(\lambda h)^2}{2}) + 2y_w(\lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{4}) + y_w(\lambda h + (\lambda h)^2 + \frac{(\lambda h)^3}{2} + \frac{(\lambda h)^4}{4}))$$

$$= y_w + \frac{1}{6}(6\lambda h + 3(\lambda h)^2 + (\lambda h)^3 + \frac{(\lambda h)^4}{4})y_w$$

$$= y_w(1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)}{3!} + \frac{(\lambda h)^4}{4!}).$$

Therefore, this method is absolutely stable if

$$|1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}| \leq 1 \quad \textbf{(108)}$$

where,$x = \lambda h$.

From the above analysis, the fifth order Runge-Kutta method is absolutely stable if

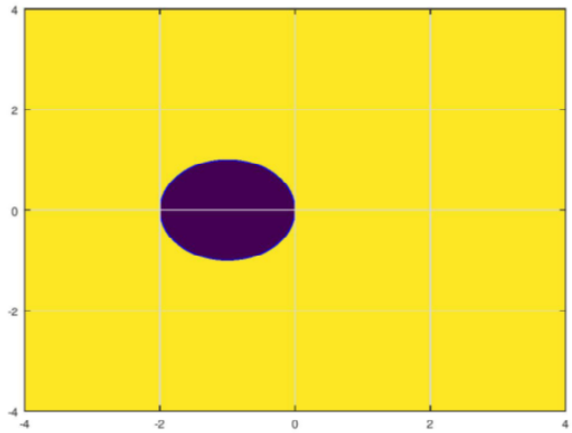
$$|1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)}{3!} + \frac{(\lambda h)^4}{4!} + \frac{(\lambda h)^5}{5!}| \leq 1. \quad \textbf{(109)}$$

In general,if RK method is of order l then its region of absolute stability is

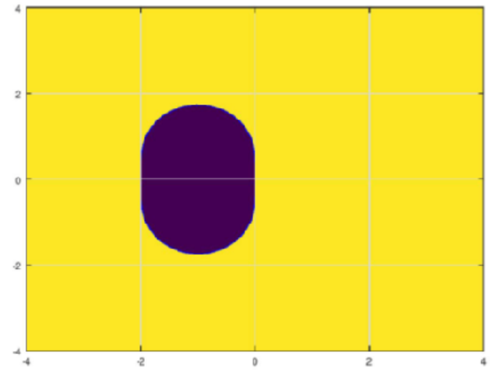
$$|1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)}{3!} + \frac{(\lambda h)^4}{4!} + \frac{(\lambda h)^5}{5!} + \ldots + \frac{(\lambda h)^L}{L!}| \leq 1. \quad \textbf{(110)}$$

3.2.9 The stability functions of one-step methods

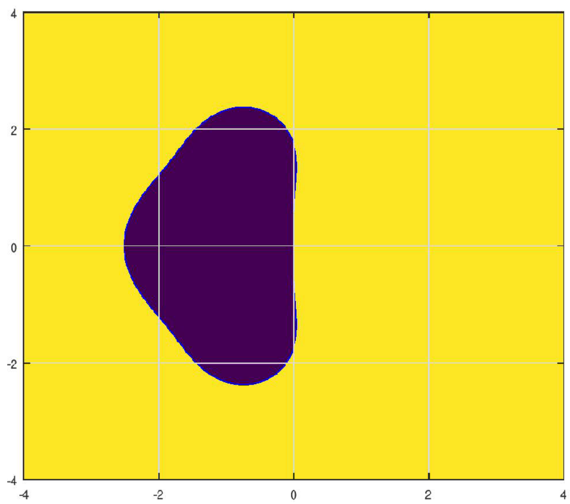
Seka and *Richard* in their article *"Order of the Runge-Kutta method and evolution of the stability region"* demonstrated through examples that the evolution size of the stability regions of RK methods for ODEs is independent of the order of the method[23]. Since you can have a higher order method which is not stable. A good example is the case of stability region of Runge-Kutta method of order 8 where, its stability region is much smaller compared that of lower orders that is of order $1, 2, 3, 4$ and 5. We demonstrate the stability regions of standard RK methods of order $1, 2, 3, 4, 5$ and 8. Using the stability regions of Eq. (105),(106),(107),(108),(109) and (110) for order 8 we obtain the following stability regions.



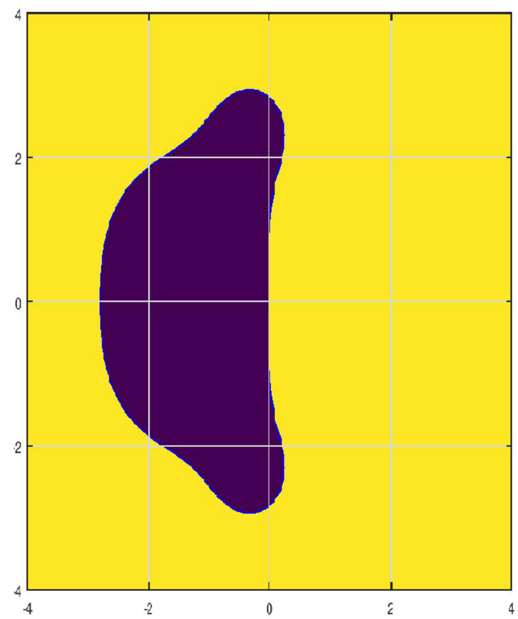
(a) Explicit Euler method



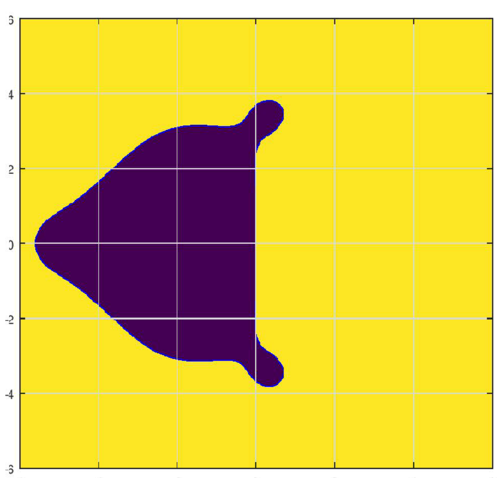
(b) Explicit second order RK method



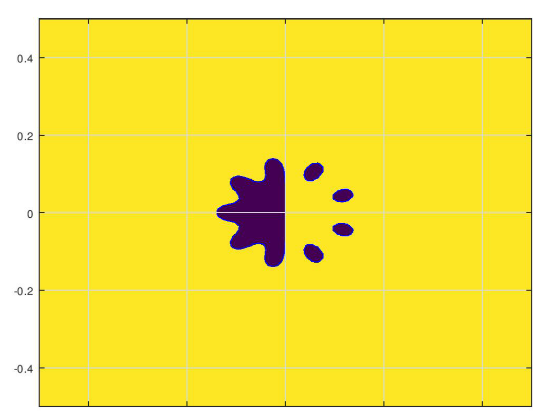
(c) Explicit third order RK method



(d) Explicit fourth order RK method



(e) Explicit fifth order RK method



(f) Explicit eight order RK method

Figure 2. Stability functions of explicit RK methods.

4 System of ODEs

4.1 Introduction

Some applications problems may occur as a system of several equations and the solution to the given system is needed in engineering, science, technology e.t.c which have complicated set up.

A system of n differential equations is expressed as

$$\begin{aligned} \frac{d\mathbf{Y}}{dx} &= \mathbf{F}(x, \mathbf{Y}) \\ \mathbf{Y}(x_0) &= \mathbf{Y}_0. \end{aligned} \tag{111}$$

Which can also be written as follows

$\frac{dy_1}{dx} = f_1(x, y_1(x), y_2(x), y_3(x), ..., y_n(x), y_1(x_0) = y_{1,0}$

$\frac{dy_2}{dx} = f_2(x, y_1(x), y_2(x), y_3(x), ..., y_n(x), y_2(x_0) = y_{2,0}$

$\vdots$

$\frac{dy_n}{dx} = f_n(x, y_1(x), y_2(x), y_3(x), ..., y_n(x), y_n(x_0) = y_{n,0}.$
With the given interval $x_0 \leq x \leq x_n$.

A general form of the above system can be written with the solution and the DE by a column vector as follows[9] $\mathbf{Y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ \vdots \\ y_n(x) \end{bmatrix}, \mathbf{Y}_0 = \begin{bmatrix} y_{1,0} \\ y_{2,0} \\ y_{3,0} \\ \vdots \\ y_{n,0} \end{bmatrix}$,

$$\mathbf{F}(x,\mathbf{Y}) = \begin{bmatrix} y'(x) = f_1(x, y_1(x), y_2(x), y_3(x), ..., y_w(x), y_1(x_0) = y_{1,0} \\ y_2'(x) = f_2(x, y_1(x), y_2(x), y_3(x), ..., y_w(x), y_2(x_0) = y_{2,0} \\ \vdots \\ y_n'(x) = f_n(x, y_1(x), y_2(x), y_3(x), ..., y_n(x), y_n(x_0) = y_{n,0} \end{bmatrix}$$

with $\mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$

4.2 Runge-Kutta methods for system of ODEs

We now extend the concept of RK methods to the systems of equations. As already mentioned, numerical schemes can easily be used to solve higher order differential equations or a system of ODEs. For the case of higher-order, we consider a second order differential equation

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx}) \tag{112}$$

we can rewrite this equation as system of two,first-order differential equations if we let

$$z = \frac{dy}{dx},$$

then

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

which on substitution we arrive at a system of two equations

$$\begin{aligned} \frac{dy}{dx} &= z \\ \frac{dz}{dx} &= f(x, y, z). \end{aligned} \tag{113}$$

For a system of two,first order ODEs together with the their initial conditions we have

$$\begin{aligned} \frac{dy}{dx} &= f(x,y,z), y(x_0) = y_0 \\ \frac{dz}{dx} &= g(x,y,z), z(x_0) = z_0 \end{aligned} \tag{114}$$

We then approximate the solution (y_w, z_w) at $x_w, w = 1, 2, ...,$ using the different formulas of RK methods.

The **second order RK method** is given as

$$\begin{aligned} y_{w+1} &= y_w + (K_1 + K_2)/2 \\ z_{w+1} &= z_w + (L_1 + L_2)/2 \end{aligned} \tag{115}$$

where

$$\begin{aligned} K_1 &= hf(x_w, y_w, z_w) \\ L_1 &= hg(x_w, y_w, z_w) \\ K_2 &= hf(x_w + h, y_w + K_1, z_w + L_1) \\ L_2 &= hg(x_w + h, y_w + K_1 + z_w + L_1) \end{aligned} \tag{116}$$

Third order RK method is given by

$$\begin{aligned} y_{w+1} &= y_w + (K_1 + 4K_2 + K_3)/6 \\ z_{w+1} &= z_w + (L_1 + 4L_2 + L_3)/6 \end{aligned} \tag{117}$$

where

$$\begin{aligned} K_1 &= hf(x_w, y_w, z_w) \\ L_1 &= hg(x_w, y_w, z_w) \\ K_2 &= hf(x_w + h/2, y_w + K_1/2, z_w + L_1/2) \\ L_2 &= hg(x_w + h/2, y_w + K_1/2, z_w + L_2/2) \\ K_3 &= hf(x_w + h, y_w - K_1 + 2K_2, z_w - L_1 + 2L_2) \\ L_3 &= hg(x_w + h, y_w - K_1 + 2K_2, z_w - L_1 + 2L_2) \end{aligned} \tag{118}$$

Fourth order RK method for a system of two equations.

$$\begin{aligned} y_{w+1} &= y_w + (K_1 + 2K_2 + 2K_3 + K_4)/6 \\ z_{w+1} &= z_w + (L_1 + 2L_2 + 2L_3 + L_4)/6 \end{aligned} \tag{119}$$

where

$$\begin{aligned}
K_1 &= hf(x_w, y_w, z_w)\\
L_1 &= hg(x_w, y_w, z_w)\\
K_2 &= hf(x_w + h/2, y_w + K_1/2, z_w + L_1/2)\\
L_2 &= hg(x_w + h/2, y_w + K_1/2, z_w + L_1/2)\\
K_3 &= hf(x_w + h/2, y_w + K_2/2, z_w + L_2/2)\\
L_3 &= hg(x_w + h/2, y_w + K_2/2, z_w + L_2/2)\\
K_4 &= hf(x_w + h, y_w + K_3, z_w + L_3)\\
L_4 &= hg(x_w + h, y_w + K_3, z_w + L_3)
\end{aligned} \tag{120}$$

We also extend this to the **fifth order Runge-Kutta method** and we obtain

$$\begin{aligned}
y_{w+1} &= y_w + \frac{1}{6}(K_1 + 3K_2 - 3K_3 + 4K_4 + K_5)\\
z_{w+1} &= z_w + \frac{1}{6}(L_1 + 3L_2 - 3L_3 + 4L_4 + L_5)
\end{aligned} \tag{121}$$

where

$$K_1 = hf(x_w, y_w, z_w)$$

$$L_1 = hg(x_w, y_w, z_w)$$

$$K_2 = hf(x_w + \frac{h}{4}, y_w + \frac{K_1}{4} + z_w + \frac{L_1}{4})$$

$$L_2 = hg(x_w + \frac{h}{4}, y_w + \frac{K_1}{4} + z_w + \frac{L_1}{4})$$

$$K_3 = hf(x_w + \frac{h}{4}, y_w + \frac{1}{4}(-K_1 + 2K_2) + z_w + \frac{1}{4}(-L_1 + 2L_2))$$

$$L_3 = hg(x_w + \frac{h}{4}, y_w + \frac{1}{4}(-K_1 + 2K_2) + z_w + \frac{1}{4}(-L_1 + 2L_2))$$

$$K_4 = hf(x_w + \frac{h}{2}, y_w + \frac{1}{4}(K_2 + K_3) + z_w + \frac{1}{4}(L_2 + L_3))$$

$$L_4 = hg(x_w + \frac{h}{2}, y_w + \frac{1}{4}(K_2 + K_3) + z_w + \frac{1}{4}(L_2 + L_3))$$

$$K_5 = hf(x_w + h, y_w + \frac{1}{2}(-K_1 + K_2 - 2K_3 + 4K_4) + z_w + \frac{1}{2}(-L_1 + L_2 - 2L_3 + 4L_4))$$

$$L_5 = hg(x_w + h, y_w + \frac{1}{2}(-K_1 + K_2 - 2K_3 + 4K_4) + z_w + \frac{1}{2}(-L_1 + L_2 - 2L_3 + 4L_4)).$$

For any l-stage RK methods for a system of two first order ODEs,the formula is given by:

$$
\begin{aligned}
y_{w+1} &= y_w + \sum_{q=1}^{l} P_q K_q, l = 2,3,4,\dots \\
z_{w+1} &= z_w + \sum_{q=1}^{l} P_q L_q, l = 2,3,4,\dots \\
&\textit{where} \\
K_1 &= hf(x_w, y_w, z_w) \\
L_1 &= hg(x_w, y_w, z_w) \\
K_2 &= hf(x_w + c_2 h, y_w + m_{21}K_1, z_w + o_{21}L_1) \\
L_2 &= hg(x_w + c_2 h, y_w + m_{21}K_1, z_w + o_{21}L_1) \\
K_3 &= hf(x_w + c_3 h, y_w + m_{31}K_1 + m_{32}K_2, z_w + o_{31}L_1 + o_{32}L_2) \\
&\vdots \\
L_3 &= hg(x_w + c_3 h, y_w + m_{31}K_1 + m_{32}K_2, z_w + o_{31}L_1 + o_{32}L_2) \\
K_l &= hf(x_w + c_l h, y_w + m_{l1}K_1 + m_{l2}K_2 + \dots + m_{ll_1}K_{l-1}, z_w + o_{l1}L_1 + o_{l2}L_2 + \dots + o_{ll_1}L_1) \\
L_l &= hg(x_w + c_l h, y_w + m_{l1}K_1 + m_{l2}K_2 + \dots + m_{ll_1}K_{l-1}, z_w + o_{l1}L_1 + o_{l2}L_2 + \dots + o_{ll_1}L_1).
\end{aligned}
\tag{122}
$$

4.3 Analysis on stability of one-step and multi-step method for a system of ODEs

A numerical method is said to be stable if for a small change to the IVP results to a small change in the solution. We consider numerical schemes which are numerically stable.In order to determine the stability for a system we keep in mind the initial condition (2) we then consider stability of numerical scheme for the test problem

$$
\begin{aligned}
y'(x) &= \lambda y(x) + k(x) \\
y(0) &= 1.
\end{aligned}
\tag{123}
$$

Now expanding $y' = f(x,y)$ to get

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0(x-x_0)) + f_y(x_0,y_0)(y-y_0).$$

Therefore,

$$
\begin{aligned}
y' &= f(x,y) \\
&= \lambda(y-y_0) + k(x)
\end{aligned}
\tag{124}
$$

where

$$
\begin{aligned}
k(x) &= f(x_0,y_0) + f_x(x_0,y_0(x-x_0)) \\
\lambda &= f_y(x_0,y_0)
\end{aligned}
$$

We let $W(x) = y(x) - y_0)$, then (124) becomes

$$W'(x) = \lambda W(x) + k(x) \tag{125}$$

which is the test equation for the IVP (123). We perturb the initial value problem and then observe the difference in the solution

$$W'_\varepsilon(x) = \lambda W_\varepsilon(x) + k(x) \tag{126}$$

Subtracting Eq. (125) with the initial condition $W(x_0) = x_0$ from (126) we get

$$\begin{aligned} W'_\varepsilon(x) - W'(x) &= \lambda(W_\varepsilon(x) - W(x)) \\ W_\varepsilon(x_0) - W(x_0) &= \varepsilon. \end{aligned} \tag{127}$$

If we let $H = W'_\varepsilon(x) - W'(x)$ we have

$$\begin{aligned} H' &= \lambda H \\ H(x_0) &= \varepsilon. \end{aligned}$$

These shows the convergence and stability of the test problem. We now look for a more general problem for a system using the test equation. It follows the same analysis.
Let consider an IVP of n differential equation's system given by

$$\begin{aligned} \frac{d\mathbf{Y}}{dx}(x) = \mathbf{F}(x, \mathbf{Y}), x_0 \le x \le x_w \\ Y(x_0) = \mathbf{Y}_0. \end{aligned}$$

With a test problem defined as

$$\frac{d\mathbf{Y}}{dx}(x) = \Lambda \mathbf{Y}(x) + K(x)$$

where $\Lambda = \mathbf{F}_y(x_0, y_0)$, if the function f is differentiable, then $\mathbf{F}_y(x, \mathbf{Y})$ represent a Jacobian matrix

$$\mathbf{F}_y(x, \mathbf{Y}) = \frac{\partial f_s(x, y_1, y_2, \ldots, y_w)}{\partial y_t}, 1 \le s, t \le w.$$

Therefore, the system (111) can be written as

$$\frac{d\mathbf{Y}}{dx}(x) = \Lambda \mathbf{Y}(x) + K(x)$$

which can be reduced to a system of first order differential equations

$$v'_s = \lambda_s v_s + \eta_s(x), 1 \le s \le w$$

with $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ being the eigenvalues of $\Lambda = F_y(x_0, y_0)$.
In order to analyze the stability of multi-step method (33), we consider a special model equation[9]

$$\begin{aligned} y'(x) &= \lambda y(x) \\ y_0 &= 1. \end{aligned} \tag{128}$$

Applying Eq.(128) into the multi-step method (33) we obtain

$$\begin{aligned} y_{w+1} &= \sum_{i=0}^{r} c_i y_{w-i} + h\lambda \sum_{i=-1}^{r} d_i y_{w-i}, w \geq r \\ y_{w+1} &= \sum_{i=0}^{r} c_i y_{w-i} + h\lambda d_{-1} y_{w+1} + h\lambda \sum_{i=0}^{r} d_i y_{w-i} \\ y_{w+1} &= h\lambda d_{-1} y_{w+1} + \sum_{i=0}^{r} (d_i + h\lambda) y_{w-i} \end{aligned} \tag{129}$$

which on putting together the coefficients of y_{w+1} we get

$$(1 - h\lambda d_{-1}) y_{w+1} = \sum_{i=0}^{r} (d_i + h\lambda) y_{w-i}.$$

Which is known as linear difference equation of order $(r+1)$
We further extend this to a general solution of the form

$$y_w = a^w, w \geq 0. \tag{130}$$

Now substituting Eq.(130) into (129) and multiplying both sides of (129) by a^{r-w} we get

$$a^{r+1} = \sum_{i=0}^{r} c_i a^{m-i} + h\lambda \sum_{i=-1}^{r} d_i a^{r-i} \tag{131}$$

which is referred to as characteristic equation.
The first root of polynomials is

$$\alpha(a) = a^{r+1} - \sum_{i=0}^{r} a_i a^{r-i} \tag{132}$$

while the second characteristic roots are

$$\gamma(a) = \sum_{i=-1}^{r} d_i a^{r-i} = d_{-1} a^{r+1} + \sum_{i=0}^{r} a^{r-i}. \tag{133}$$

The first and second characteristic polynomials of multi-step method (33) are given by

$$\rho(a, h\lambda) = \alpha(a) - h\lambda \gamma(a) = 0. \tag{134}$$

We now analyze the region of stability for multi-step method (33), its characteristic polynomial is given by Eq. (131) which can be written as

$$a^{r+1} - \sum_{i=0}^{r} c_i a^{r-i} - h\lambda \sum_{i=-1}^{r} d_i a^{r-i} \tag{135}$$

for the model obtained.
For the method to be absolutely stable, we need that all roots of characteristic equation be of magnitude 1.Now obtaining $h\lambda$ from Eq.(135) we get the region of stability for the method(33) which is given as

$$\begin{aligned} a^{r+1} - \sum_{i=0}^{r} c_i a^{r-i} &= h\lambda \sum_{i=-1}^{m} d_i a^{m-i} \\ h\lambda &= \frac{a^{r+1} - \sum_{i=0}^{r} c_i a^{r-i}}{\sum_{i=-1}^{r} d_i a^{r-i}} \end{aligned} \tag{136}$$

where $\lambda < 0$ if λ is a real constant or λ is complex, assuming that $Re(\lambda) < 0$ in stable differential equation problems[9]. The exact solution to the given model problem is

$$y(x) = x_0 e^{\lambda x} = e^{\lambda x}. \tag{137}$$

Thus, the solution of the test problem goes to zero as $x \to +\infty$, that is

$$y(x) \to 0, x \to +\infty. \tag{138}$$

When any numerical scheme is applied to the test problem and the numerical solution satisfy condition

$$y(x_w) \to 0, x_w \to +\infty. \tag{139}$$

If $h\lambda$ is satisfied for the above definition in any numerical scheme, then it is called the region of absolute stability of the numerical method[9].

5 A Mathematical Model for Unemployment

5.1 The Mathematical Model

We consider a proposed model for unemployment by *Raneah*, *Ashi* and *Sarah*. The main variables considered are the number of people employed represented by E, the number of people who are unemployed represented by U and the number of available vacancies represented by V at any time t. In the process of developing the model, we take a number of assumptions.

5.1.1 Assumptions

1 All the unemployed are qualified and competent to be employed.

2 Some unemployed people may join employed class.

3 The rate at which one transits from being unemployed to employed is jointly proportional to the number of unemployed people together with the number of job vacancies available represented by $U(t)$ and $V(t)$ respectively.

4 Some employees represented by $E(t)$ may be affected as some may get fired or dismissed from their current jobs.But some may decide to resign,all in all they join the unemployed class.

5 Only death, migration, or retirement of the employed can lead to creation of job vacancies.

6 The mortality and migration rate of the unemployed class is taken to be proportional to their numbers.

Thus, the rate at which the number of people who are unemployed,the employed and number of job vacancies available change at any time t can be described using the following model:

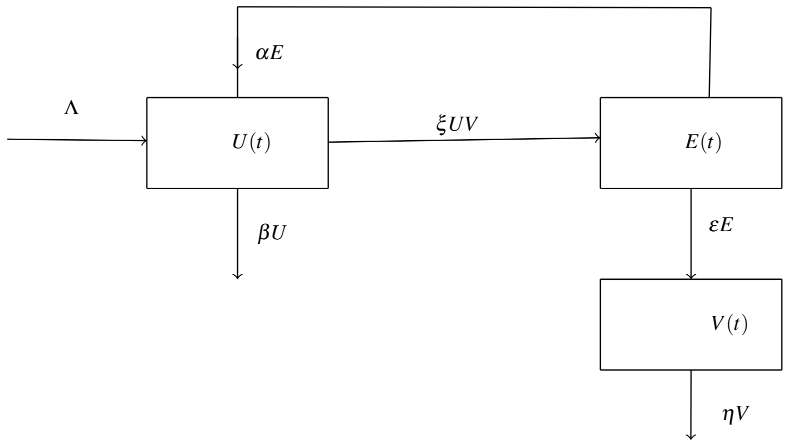


Figure 3. Unemployment model

5.1.2 The governing equations

From the Unemployment Model of Fig.3, we derive the following first order nonlinear system of three ODes:

$$\begin{cases} \frac{\mathrm{d}U}{\mathrm{d}t} = \Lambda - \xi U(t)V(t) + \alpha E(t) - \beta U(t) \\ \frac{\mathrm{d}E}{\mathrm{d}t} = \xi U(t)V(t) - \alpha E(t) - \varepsilon E(t) \\ \frac{\mathrm{d}V}{\mathrm{d}t} = \varepsilon E(t) - \eta V(t) \end{cases} \tag{140}$$

The system of Eq.(140) is nonlinear because of the product involving the dependent variables U and V in the first and second equation. It is a first order since the highest derivative present is of order one.

The meaning of the parameters of our model is given by Tab. 11.

Parameters	Meaning
Λ	Rate at which the number of people who are unemployed increase[2].
ξ	Rate at which the number of people who are unemployed change and become employed[2].
α	Rate at which employed persons are joining the unemployed people due to firing or dismissal from their current jobs[2].
β	Death rate as well as migration of people who are unemployed[2].
ε	The exit rate from labour force[2].
η	Rate of diminishing of available vacancies because of lack of government funds[2].

Table 11. Meaning of parameters used for the Unemployment model.

Where $\Lambda, \xi, \alpha, \beta, \varepsilon$ and η are positive constants.

Theorem 5.1.1. *If all the solutions of the functions* U, E, V $\mathbb{R}_3^+$ *, then we have a set given by*

Θ=*{* $U(t), E(t), V(t) : 0 \leq U(t) + E(t) \leq \frac{\Lambda}{\delta}, 0 \leq V(t) \leq \frac{\varepsilon\Lambda}{\delta\eta}$*}, where* $\delta = min(\beta, \varepsilon)$ *which is bounded as well as positively invariant.*

`Proof.` It can be seen that

$$\frac{\mathrm{d}U}{\mathrm{d}t}|_{U(t)=0} = \Lambda + \alpha E(t) > 0, \frac{\mathrm{d}E}{\mathrm{d}t}|_{E(t)=0} = \xi U(t)V(t) > 0, \frac{\mathrm{d}V}{\mathrm{d}t}|_{V(t)=0} = \varepsilon E(t) > 0.$$

This shows that for $t \geq 0$, all solutions remain non negative. Combining the first two equations of Eq. (140) we arrive at

$$\frac{\mathrm{d}U}{\mathrm{d}t} + \frac{\mathrm{d}E}{\mathrm{d}t} = \Lambda - \beta U(t) - \varepsilon E(t) = \Lambda - \delta(U(t) + E(t))$$

where δ=min(β, ε). Now having the limit supremum we have

$$\lim_{t\to\infty} sup(U(t) + E(t)) \leq \frac{\Lambda}{\delta}.$$

We see from the last equation of the Eq. (140) that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \varepsilon E(t) - \eta V(t) \leq \varepsilon\frac{\Lambda}{\delta} - \eta V(t).$$

Therefore,

$$V(t) \leq \frac{\varepsilon\Lambda}{\delta\eta}.$$

Thus

$$\lim_{t\to\infty} sup(V(t)) \leq \frac{\varepsilon\Lambda}{\delta\eta}.$$

This shows that all the solutions of the Eq. (140) are bounded and lies inside region Θ. Therefore, Θ is positively invariant. □

The values to use in numerical simulation are given by Tab. 12.

Parameters	Value	Reference
Λ	3000	Raneah, Ashi and Sarah (2018)[2].
ξ	0.00000864	Raneah, Ashi and Sarah (2018)[2].
β	0.048	Raneah, Ashi and Sarah (2018)[2].
α	0.01	Raneah, Ashi and Sarah (2018)[[2].
ε	0.05	Raneah, Ashi and Sarah (2018)[2].
η	0.1125	Raneah, Ashi and Sarah (2018)[2].

Table 12. Values of parameters used for the Unemployment model.

5.2 The Numerical Solver

In this section, we give a brief description of the numerical method which will be implemented in MATLAB. We are going to use a fourth order RK (RK4) method to solve the nonlinear system of three equations since the local truncation error is of order $O(h^5)$ which means that we are going to get a better result compared to the methods of lower order.
For a single equation of first order ODE together with initial conditions:

$$\begin{aligned} \frac{dy}{dx} &= f(x,y(x)) \\ y(x_0) &= y_0, \end{aligned} \tag{141}$$

then RK method of fourth order is given by

$$y_{w+1} = y_w + (K_1 + 2K_2 + 2K_3 + K_4)/6.$$

Where

$$K_1 = hf(x_w, y_w)$$

$$K_2 = hf(x_w + h/2, y_w + K_1/2)$$

$$K_3 = hf(x_w + h/2, y_w + K_2/2)$$

$$K_4 = hf(x_w + h, y_w + K_3).$$

If we have a two system of equations together with their initial conditions :

$$\frac{dy}{dx} = f(x, y(x), z(x)) \\ y(x_0) = y_0 \tag{142}$$

$$\frac{dz}{dx} = g(x, y(x), z(x)) \\ z(x_0) = z_0. \tag{143}$$

Then, we solve using

$$y_{w+1} = y_w + (K_1 + 2K_2 + 2K_3 + K_4)/6 \\ z_{w+1} = z_w + (L_1 + 2L_2 + 2L_3 + L_4)/6 \tag{144}$$

where

$$\begin{aligned} K_1 &= hf(x_w, y_w, z_w) \\ L_1 &= hg(x_w, y_w, z_w) \\ K_2 &= hf(x_w + h/2, y_w + K_1/2, z_w + L_1/2) \\ L_2 &= hg(x_w + h/2, y_w + K_1/2, z_w + L_1/2) \\ K_3 &= hf(x_w + h/2, y_w + K_2/2, z_w + L_2/2) \\ L_3 &= hg(x_w + h/2, y_w + K_2/2, z_w + L_2/2) \\ K_4 &= hf(x_w + h, y_w + K_3, z_w + L_3) \\ L_4 &= hg(x_w + h, y_w + K_3, z_w + L_3). \end{aligned} \tag{145}$$

Here, we have a system of three equations and to obtain the numerical solution, we are going to use [8] RK4 method for a three system of equations given by:

$$\begin{aligned} y_{w+1} &= x_w + (K_1 + 2K_2 + 2K_3 + K_4)/6 \\ z_{w+1} &= z_w + (L_1 + 2L_2 + 2L_3 + L_4)/6 \\ q_{w+1} &= q_w + (M_1 + 2M_2 + 2M_3 + M_4)/6 \end{aligned} \tag{146}$$

where

$$
\begin{aligned}
K_1 &= hf(x_w, y_w, z_w, q_w) \\
L_1 &= hg(x_w, y_w, z_w, q_w) \\
M_1 &= hp(x_w, y_w, z_w, q_w) \\
K_2 &= hf(x_w + h/2, y_w + K_1/2, z_w + L_1/2, q_w + M_1/2) \\
L_2 &= hg(x_w + h/2, y_w + K_1/2, z_w + L_1/2, , q_w + M_1/2) \\
M_2 &= hp(x_w + h/2, y_w + K_1/2, z_w + L_1/2, , q_w + M_1/2) \\
K_3 &= hf(x_w + h/2, y_w + K_2/2, z_w + L_2/2, q_w + M_2/2) \\
L_3 &= hg(x_w + h/2, y_w + K_2/2, z_w + L_2/2, q_w + M_2/2) \\
M_3 &= hp(x_w + h/2, y_w + K_2/2, z_w + L_2/2, q_w + M_2/2) \\
K_4 &= hf(x_w + h, y_w + K_3, z_w + L_3, q_w + M_3) \\
L_4 &= hg(x_w + h, y_w + K_3, z_w + L_3, q_w + M_3) \\
M_4 &= hp(x_w + h, y_w + K_3, z_w + L_3, q_w + M_3).
\end{aligned} \tag{147}
$$

In general, for any first order system of ODEs, the formula of RK4 method is as follows

$$
\begin{aligned}
y_{w+1} &= y_w + (K_1 + 2K_2 + 2K_3 + K_4)/6 \\
z_{w+1} &= z_w + (L_1 + 2L_2 + 2L_3 + L_4)/6 \\
q_{w+1} &= q_w + (M_1 + 2M_2 + 2M_3 + M_4)/6 \\
\vdots \quad &= \quad \vdots \\
r_{w+1} &= r_w + (N_1 + 2N_2 + 2N_3 + N_4)/6
\end{aligned} \tag{148}
$$

where

$$
\begin{aligned}
K_1 &= hf(x_w, y_w, z_w, q_w, ..., r_w)\\
L_1 &= hg(x_w, y_w, z_w, q_w, ..., r_w)\\
M_1 &= hp(x_w, y_w, z_w, q_w, ..., r_w)\\
\vdots &= \qquad \vdots\\
N_1 &= hj(x_w, y_w, z_w, q_w, ..., r_w)\\
K_2 &= hf(x_w + \frac{h}{2}, y_w + \frac{K_1}{2}, z_w + \frac{L_1}{2}, q_w + \frac{M_1}{2}, ..., r_w + \frac{N_1}{2})\\
L_2 &= hg(x_w + \frac{h}{2}, y_w + \frac{K_1}{2}, z_w + \frac{L_1}{2},, q_w + \frac{M_1}{2}, ..., r_w + \frac{N_1}{2})\\
M_2 &= hp(x_w + \frac{h}{2}, y_w + \frac{K_1}{2}, z_w + \frac{L_1}{2},, q_w + \frac{M_1}{2}, ..., r_w + \frac{N_1}{2})\\
\vdots &= \qquad \vdots\\
N_2 &= hj(x_w + \frac{h}{2}, y_w + \frac{K_1}{2}, z_w + \frac{L_1}{2},, q_w + \frac{M_1}{2}, ..., r_w + \frac{N_1}{2})\\
K_3 &= hf(x_w + \frac{h}{2}, y_w + \frac{K_2}{2}, z_w + \frac{L_2}{2}, q_w + \frac{M_2}{2}, ..., r_w + \frac{N_2}{2})\\
L_3 &= hg(x_w + \frac{h}{2}, y_w + \frac{K_2}{2}, z_w + \frac{L_2}{2}, q_w + \frac{M_2}{2}, ..., r_w + \frac{N_2}{2})\\
M_3 &= hp(x_w + \frac{h}{2}, y_w + \frac{K_2}{2}, z_w + \frac{L_2}{2}, q_w + \frac{M_2}{2}, ..., r_w + \frac{N_2}{2})\\
\vdots &= \qquad \vdots\\
N_3 &= hj(x_w + \frac{h}{2}, y_w + \frac{K_2}{2}, z_w + \frac{L_2}{2}, q_w + \frac{M_2}{2}, ..., r_w + \frac{N_2}{2})\\
K_4 &= hf(x_w + h, y_w + K_3, z_w + L_3, q_w + M_3, ..., r_w + N_3)\\
L_4 &= hg(x_w + h, y_w + K_3, z_w + L_3, q_w + M_3, ..., r_w + N_3)\\
M_4 &= hp(x_w + h, y_w + K_3, z_w + L_3, q_w + M_3, ..., r_w + N_3)\\
\vdots &= \qquad \vdots\\
N_4 &= hj\left(x_w + h, y_w + K_3, z_w + L_3, q_w + M_3, ..., r_w + N_3\right).
\end{aligned}
\tag{149}
$$

Consequently, for our case we have

$$
\begin{cases}
\frac{dU}{dt} = \Lambda - \xi U(t)V(t) + \alpha E(t) - \beta U(t)\\
\frac{dE}{dt} = \xi U(t)V(t) - \alpha E(t) - \varepsilon E(t)\\
\frac{dV}{dt} = \varepsilon E(t) - \eta V(t)
\end{cases}
\tag{150}
$$

with $U(0) = 10000, E(0) = 1000, V(0) = 100$[15].A time interval of $t \in [0, 150]$[15], then we solve the system using the following formula in order to get the approximate solutions of $U_i \approx U(t_i), E_i \approx E(t_i)$ and $V_i \approx V(t_i)$, $i = w, i = 1, 2, ..., N-1$.

$$\begin{aligned} U_{i+1} &= U_i + (K_1 + 2K_2 + 2K_3 + K_4)/6 \\ E_{i+1} &= E_i + (L_1 + 2L_2 + 2L_3 + L_4)/6 \\ V_{i+1} &= V_i + (M_1 + 2M_2 + 2M_3 + M_4)/6 \end{aligned} \tag{151}$$

where

$$\begin{aligned} K_1 &= hf(t_i, U_i, E_i, V_i) \\ L_1 &= hg(t_i, U_i, E_i, V_i) \\ M_1 &= hp(t_i, U_i, E_i, V_i) \\ K_2 &= hf(t_i + \frac{h}{2}, U_i + \frac{K_1}{2}, E_i + \frac{L_1}{2}, V_i + \frac{M_1}{2}) \\ L_2 &= hg(t_i + \frac{h}{2}, U_i + \frac{K_1}{2}, E_i + \frac{L_1}{2}, V_i + \frac{M_1}{2}) \\ M_2 &= hp(t_i + \frac{h}{2}, U_i + \frac{K_1}{2}, E_i + \frac{L_1}{2}, V_i + \frac{M_1}{2}) \\ K_3 &= hf(t_i + \frac{h}{2}, U_i + \frac{K_2}{2}, E_i + \frac{L_2}{2}, V_i + \frac{M_2}{2}) \\ L_3 &= hg(t_i + \frac{h}{2}, U_i + \frac{K_2}{2}, E_i + \frac{L_2}{2}, V_i + \frac{M_2}{2}) \\ M_3 &= hp(t_i + \frac{h}{2}, U_i + \frac{K_2}{2}, E_i + \frac{L_2}{2}, V_i + \frac{M_2}{2}) \\ K_4 &= hf(t_i + h, E_i + K_3, U_i + L_3, V_i + M_3) \\ L_4 &= hg(t_i + h, E_i + K_3, U_i + L_3, V_i + M_3) \\ M_4 &= hp(t_i + h, E_i + K_3, U_i + L_3, V_i + M_3) \end{aligned} \tag{152}$$

For $i = 0, 1, ..., k^{th}$ iterate.

6 Numerical Results and Discussion

6.1 Discussion on Unemployment

We now implement the method in Matlab to solve Eq.(140) together with its initial conditions and the plot of the results are represented on a graph. From Fig. 4 we observe

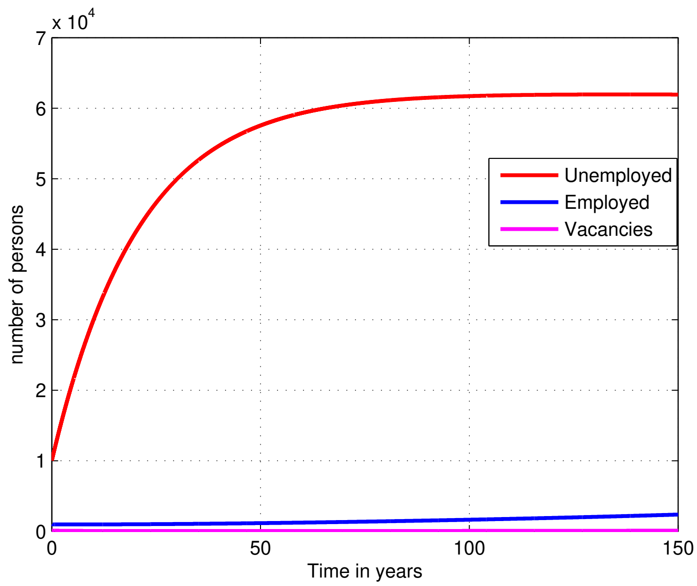


Figure 4. A plot of Eq. (140) **with the initial conditions.**

that the number of unemployed people kept on rise due to limited number of available vacancies compared to the number of unemployed at that time and also due to the rate of increase in number of unemployed individuals.Let now consider different cases of the model Eq. (140) by varying the values of parameters U, E and V with other parameters being constant $h = 0.0001$.For all the cases,we consider when available vacancies exceeds the rate of number of unemployed people,otherwise we will have a plot of Fig.4.

Case 1: $U = E = V$.
In this case, we have equal number of unemployed, employed and vacancies created.

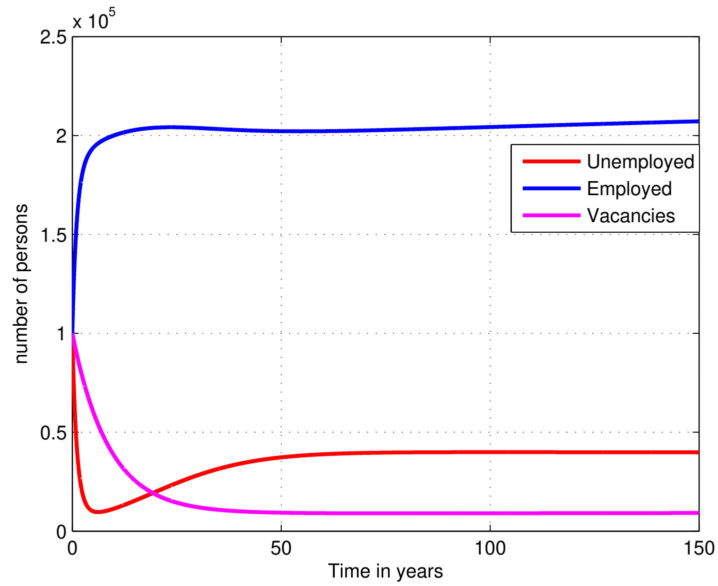


Figure 5. A plot of case 1

From Fig.5,the number of unemployed population reduced over the first few years since there were more vacancies created, thereafter unemployment rose as available vacancies were diminishing because many people got employed.

Case 2: $E < U < V$.
In this case,we have more vacancies created than the number of unemployed people.

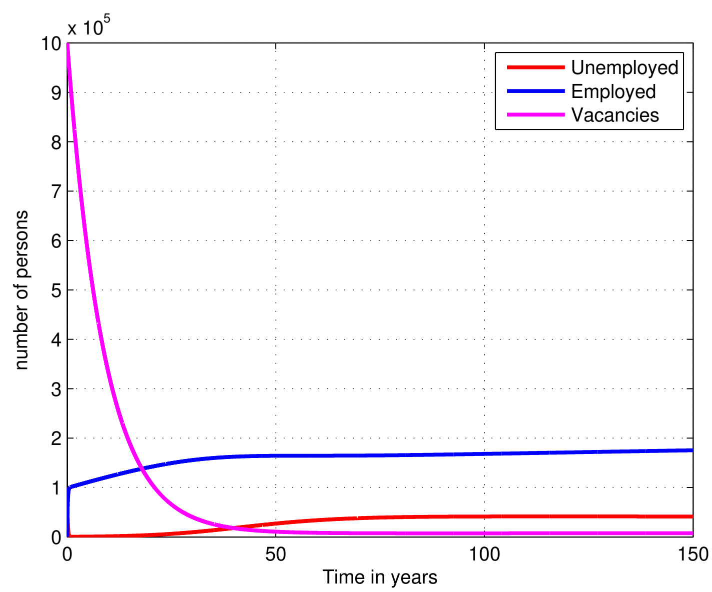


Figure 6. A plot of case 2

From Fig. 6, the number of employed increased significantly and the available vacancies diminished as a result, the unemployment started rising.

Case 3: $V < E \leq V$.
In this case, the number of unemployed people exceeds the available vacancies.

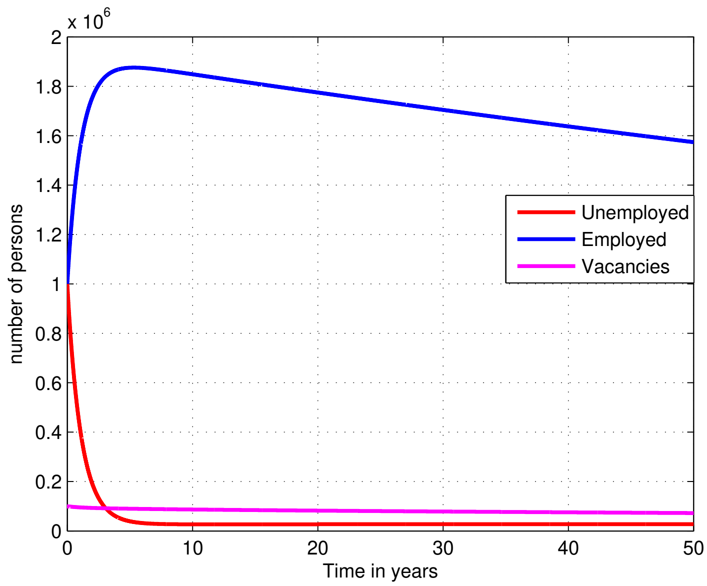
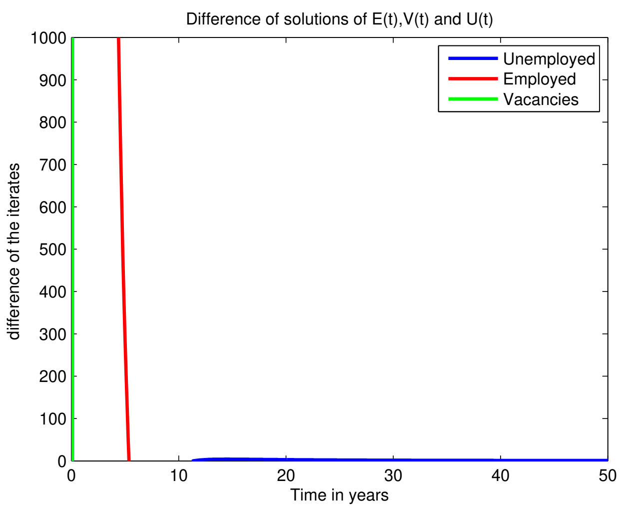


Figure 7. A plot of case 3

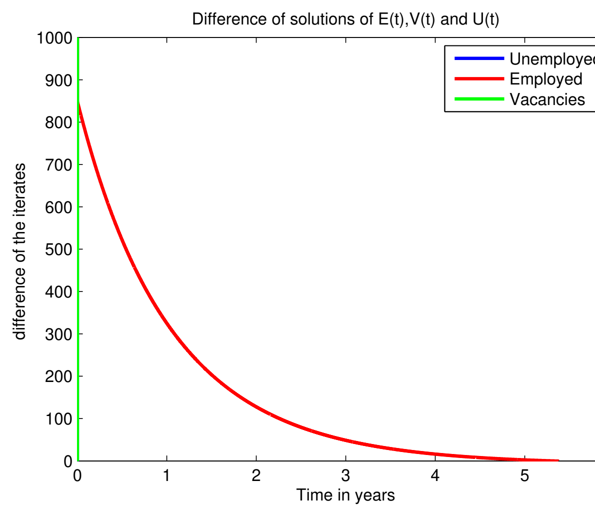
From Fig. 7,there was a significant increase in the number of employed population over the first few years which drastically reduced the number of unemployed population.

6.2 Discussion on Numerical method

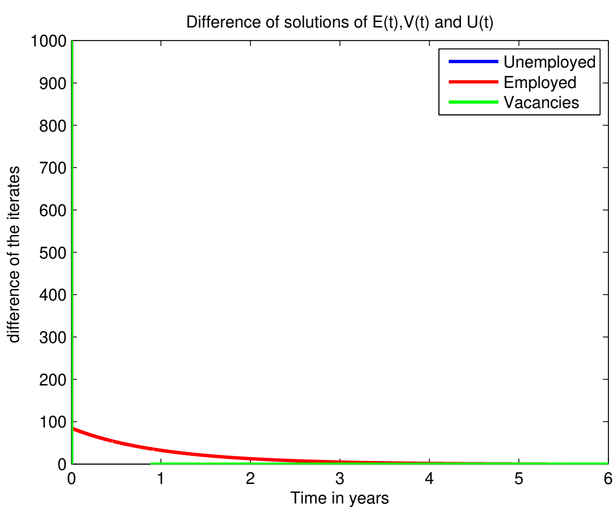
From the previous section we see that the method do converge.we now consider the consistency of the method by using different values of the step length h. We see that



(a) when step length $h = 0.1$



(b) when step length $h = 0.0001$



as the step length h is minimized or as it approaches to zero then also the difference of the solutions tends to zero and we can conclude that RK method is consistent and converges.

7 Conclusion and Future research

7.1 Conclusion

The research was mainly to study numerical methods for solving nonlinear systems of ODES and in particular the unemployment model. Various derivations of RK methods have been obtained. The results have shown that RK4 method is consistent, stable and also convergent. There is a significant decrease in the number of unemployed people and consequently the rate of unemployment when more job vacancies are created. If more funds are set aside for creation of vacancies then, unemployment can be managed.

7.2 Future Research

We were unable to obtain data on unemployment from Kenyan government so as to obtain deterministic results but we hope to get them in future. The study can be extended to the study of numerical methods for solving systems of higher order ODEs of at least second order. Also we would like to investigate on how to choose a suitable step length to use for a numerical method which helps to regulate the "expected error".

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Appendices

.1 Matlab codes

Listing 1. matlab codes for (140) **with** $U(1) = 10000, E(1) = 1000, V(1) = 100$.

```
clc,clear all,close all
max_t=150;
h=0.0001;
t=0:h:max_t;
n=length(t);
U=zeros(n,1); E=zeros(n,1); V=zeros(n,1);
U(1)=10000; E(1)=1000; V(1)=100;
a=3000;b=0.00000864;c=0.01;d=0.048;e=0.005;q=0.1125;

for i=1:n-1
    [f1,g1,p1]=system_rk(U(i),E(i),V(i));
    K1=h*f1;
    L1=h*g1;
    M1=h*p1;

    [f2,g2,p2]=system_rk(U(i)+K1/2,E(i)+L1/2,V(i)+M1/2);
    K2=h*f2;
    L2=h*g2;
    M2=h*p2;
    [f3,g3,p3]=system_rk(U(i)+K2/2,E(i)+L2/2,V(i)+M2/2);
    K3=h*f3;
    L3=h*g3;
    M3=h*p3;
    [f4,g4,p4]=system_rk(U(i)+K3,E(i)+L3,V(i)+M3);
    K4=h*f4;
    L4=h*g4;
    M4=h*p4;
    U(i+1)=U(i)+1/6*(K1+2*K2+2*K3+K4);
    E(i+1)=E(i)+1/6*(L1+2*L2+2*L3+L4);
    V(i+1)=V(i)+1/6*(M1+2*M2+2*M3+M4);
end
figure;
 plot(t,U,'r','LineWidth',2);
 hold on
plot(t,b,'b','LineWidth',2);
 plot(t,V,'m','LineWidth',2);
 set(gcf,'Color','w')
 legend('Unemployed','Employed','Vacancies');
grid on;
```

```
xlabel ('Time in years');
ylabel ('number of persons');
 % plot L_2 norms
 difU=[U(1);diff(U)];
  difE=[E(1);diff(E)];
  difV=[V(1);diff(V)];
  figure;

 plot(t,difU,'b','LineWidth',2);
 hold on
  plot(t,difE,'r','LineWidth',2);
   plot(t,difV,'g','LineWidth',2);
   ylim([0 1000])
 set(gcf,'Color','w')
legend('Unemployed','Employed','Vacancies');
grid on;
xlabel ('Time in years');
ylabel ('number of persons');
```