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Master Thesis

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Applied Mathematics

Submitted to: The Graduate School, University of Nairobi, Kenya

Abstract

The object of the present paper is to study Lorentzian Para Sasakian manifolds admitting a W_3-curvature tensor. We have shown that a W_3-flat LP-Sasakian manifold is an Einstein manifold, a W_3-symmetric and W_3-flat LP-Sasakian manifold is a flat space, and a W_3-semi-symmetric LP-Sasakian manifold is a W_3-symmetric manifold. We have also considered the geometry of an LP-Sasakian manifold endowed with a conservative W_3 curvature tensor and shown that it is an Einstein manifold.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

10/09/2021

Signature Date

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In our capacity as supervisors of the candidate's dissertation, we certify that this dissertation has our approval for submission.

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Dedication

I dedicate this thesis to my family.

Contents

Acknowledgments

To my parents, I wish to thank you for encouraging and supporting me both financially and mentally to further my studies. I thank also my siblings for your kind words of encouragement and being there for me many times I have turned to you for help.

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Felix Mutinda Isaiah

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1 Introduction

Curvature tensors are important structures in differential geometry used to define the geometry of a manifold. In a Riemmanian manifold, the curvature of the space is described by the Riemmanian curvature tensor. In general relativity theory, where the spacetime is represented by a Lorentzian manifold, curvature tensors play a significant role in determining the geometric and physical properties of the spacetime.

1.1 Differentiable manifolds

Definition 1.1.1. *Let V_1 and V_2 be open subsets of $\mathbb{R}^n$ and $f : V_1 \to V_2$ a homeomorphism. We say f is of class C^k if all its partial derivatives of orders $j \leq k$ exist and are continous.*

Definition 1.1.2. *A topological space M which is second countable and Hausdorff is called a topological manifold if $\forall p \in M$, there exists an open neighbourhood $U \subset M$ of p homeomorphic to an open subset $V \subset \mathbb{R}^n$. The dimension of M is n.*

Definition 1.1.3. *Let $x : U \to V$ be a homeomorphism. Then the pair (U, x) is called a chart. Where $U \subset M$ (M a topological manifold) and $V \subset \mathbb{R}^n$. U is called a coordinate neighbourhood of $p \in U$ and $x(p) = (x^1, ..., x^n) \in V$ the local coordinates of p.*

Definition 1.1.4. *We call the set $\mathcal{A}$ containing the charts (U_i, x_i) an atlas of class C^k if the union of all the neighbourhoods $\cup U_i = M$ and whenever $U_i \cap U_j \neq O$ the function $x_j \circ x_i^{-1} : x_i(U_i \cap U_j) \to x_j(U_i \cap U_j)$ is of class C^k.*

An atlas is called complete if it is not contained in any other atlas of the same class.

Definition 1.1.5. *A topological manifold M is called differentiable if it is endowed with a complete atlas $\mathcal{A}$. It is called smooth if $\mathcal{A}$ is of class C^∞.*

Definition 1.1.6. *Let M and N be differentiable manifolds of class C^k. A mapping $f : M \to N$ is said to be differentiable of class $C^a, a \leq k$, if, for every chart (U_i, x_i) of M and every chart (V_j, y_j) of N such that $f(U_i) \subset V_j$ the mapping $y_j \circ f \circ x_i^{-1} : x_i(U_i) \to y_j(V_j)$ is differentiable of class C^a.*

The manifolds M and N are said to be diffeomorphic if f is a homeomorphism such that f and f^{-1} are differentiable. By a differentiable function f of class C^k on M we shall mean a differentiable mapping of class C^k from M into $\mathbb{R}$. Let $\gamma : (U \subset \mathbb{R}) \to M$ be a differentiable mapping of class C^k. We shall refer to the restriction of γ to the closed interval $[a, b] \subset U$ as a differentiable curve of class C^k on M.

1.2 Tangent Vectors and Covectors

Definition 1.2.1. *Let $\gamma(t) : [a,b] \to M$ be a differentiable curve of class C^1 and $p \in U \subset M$ such that $\gamma(t_0) = p$. The tangent vector X to the curve $\gamma(t)$ at p is the mapping $X : C^1(U \subset M) \to \mathbb{R}$ defined as [10]*

$$Xf = \left(\frac{df(\gamma(t))}{dt} \right)_{t_0}$$

.

The tangent vector X satisfies

1. $X(\alpha f + \beta g) = \alpha X(f) + \beta X(g)$;
2. $X(fg) = (Xf)g(p) + f(p)(Xg)$ $\forall \alpha, \beta \in \mathbb{R}$ and $\forall f, g \in C^1(U)$.

The set of all the tangent vectors to M at p denoted by T_pM forms a vector space of dimension n called the tangent space at p. We call the map $X : M \to T_pM$ a vector field if for every $p \in M, X(p)$ is a vector in T_pM. The dual vector space of T_pM denoted by T_p^*M, is called the covector space of M at p. A 1-form (differential form of degree 1) is thus an assignment of a covector at each point p of M.

1.3 Tensors

Definition 1.3.1. *Let V be a vector space over a field F and V^* its dual space. We shall define a tensor of type (r,s) as a multilinear map $T_s^r : (V^* \times \ldots \times V^*)_{r-times} \times (V \times \ldots \times V)_{s-times} \to F$.*

The tensor T_s^r is said to be of contravariant degree r and covariant degree s. We call the scalar $(r+s)$ the rank of the tensor.

Given M as a differentiable manifold of dimension n, a tensor of type (r,s) at $p \in M$ is the multilinear map $T_s^r : (T_p^*M \times \ldots \times T_p^*M)_{r-times} \times (T_pM \times \ldots \times T_pM)_{s-times} \to \mathbb{R}$.

A tensor field of type (r,s) is the assignment of a tensor of type (r,s) to each point $p \in M$.

Contraction is the process by which we set equal a contravariant index and a covariant index of a tensor and sum over the common index thus reducing its rank by 2.

For example, an inner product g on a real vector space V is a $(0,2)$ tensor or simply a covariant tensor of degree 2 satisfying

1. $g(u_1, u_2) \geq 0$ *and* $g(u_1, u_1) = 0 \iff u_1 = 0$

2. $g(u_1, u_2) = g(u_2, u_1)$.

A tensor of type (r,s) is said to be symmetric if swapping any of its indices leaves it unchanged and skew-symmetric if it only alters the sign of its components.

Definition 1.3.2. *A Riemannian manifold is a differentiable manifold with a smooth vector field which assigns to each T_pM an inner product g with properties (1) and (2) as defined above. g is called the metric tensor.*

1.4 Connections and curvature

Definition 1.4.1. *Given M a smooth manifold, we define a connection or covariant differentiation on M as the operator ∇ which maps a pair of smooth vector fields X, Y with domain $\mathcal{D}$ into a smooth vector field $\nabla_X Y$ with the same domain.*

The connection ∇ satisfies the following axioms

1. $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$
2. $\nabla_{fX} Y = f \nabla_X Y$
3. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$
4. $\nabla_X (fY) = (Xf)Y + f \nabla_X Y$.
   $\forall X, Y, Z$ smooth vector fields with domain $\mathcal{D}$ and $\forall f$ smooth functions with the same domain.

Having defined a connection, we say that a smooth vector field X is parallel transported along a smooth curve γ if $\nabla_T X = 0$. Where T is the tangent field of γ. We then say γ is a geodesic if it parallel transports its tangent vector field i.e. $\nabla_T T = 0$.

Definition 1.4.2. *Given two vector fields X and Y, the Lie derivative or commutator of the two fields denoted by $[X, Y]$ is a vector fields defined by*

$$[X, Y] = XY - YX$$

Definition 1.4.3. *The torsion tensor of a connection is given as*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

.

A connection ∇ is said to be torsion free or symmetric if

$$\nabla_X Y - \nabla_Y X = [X,Y]$$

.

Definition 1.4.4. *A connection ∇ is said to be a Riemannian connection if*

1. $\nabla_X Y - \nabla_Y X = [X,Y]$
2. $\nabla_X g = 0.$

Definition 1.4.5. *Given a Riemannian connection ∇, we define the Riemannian curvature tensor as*

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X - \nabla_{[X,Y]} Z$$

It satisfies the following properties

1. $R(X,Y,Z) = -R(Y,X,Z)$ *(skew-symmetric)*
2. $R(X,Y,Z) - R(Y,Z,X) - R(Z,X,Y) = 0$ *(Bianchi's first identity)*
3. $(\nabla_X R)(Y,Z,W) - (\nabla_Y R)(Z,X,W) - (\nabla_Z R)(X,Y,W) = 0$ *(Bianchi's second identity)*

The tensor defined as

$$R(X,Y,Z,W) = g(R(X,Y,Z),W)$$

satistifies the properties

1. $R(X,Y,Z,W) = -R(Y,X,Z,W)$
2. $R(X,Y,Z,W) = -R(X,Y,W,Z)$
3. Bianchi's first and second identities

Definition 1.4.6. *The Ricci curvature tensor $R(X,Y)$ is a $(0,2)$ type tensor obtained by contracting the Riemannian curvature tensor. In index notation*

$$R_{ij} = g^{kl} R_{ikjl}$$

.

Definition 1.4.7. *The scalar curvature tensor R is a $(0,0)$ type tensor obtained by further contracting the Ricci curvature tensor. In index notation*

$$R = g^{ij} R_{ij}$$

Below we give the definitions of some significant curvature tensors.

$$C(X,Y,Z,T) = R(X,Y,Z,T) - \frac{R}{n(n-1)}\left[g(X,T)g(Y,Z) - g(Y,T)g(X,Z)\right] \tag{1}$$

$$\begin{aligned} L(X,Y,Z,T) = R(X,Y,Z,T) - \frac{1}{n-2}&\left[g(Y,Z)Ric(X,T) - g(X,Z)Ric(Y,T)\right. \\ &\left. + g(X,T)Ric(Y,Z) - g(Y,T)Ric(X,Z)\right] \end{aligned} \tag{2}$$

$$\begin{aligned} V(X,Y,Z,T) = R(X,Y,Z,T) - \frac{1}{n-2}&\left[g(X,T)Ric(Y,Z) - g(Y,T)Ric(X,Z)\right. \\ &\left. + g(Y,Z)Ric(X,T) - g(X,Z)Ric(Y,T)\right] \\ &\frac{R}{(n-1)(n-2)}\left[g(X,T)g(Y,Z) - g(Y,T)g(X,Z)\right] \end{aligned} \tag{3}$$

$$W(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(X,Z)Ric(Y,T) - g(X,T)Ric(Y,Z)\right]. \tag{4}$$

These tensor are refered to as *concircular, conharmonic, conformal and weyl projective tensor* respectively [15].

Pokhariyal and Mishra [23, 22], and Pokhariyal [18, 19, 21], have defined and studied new curvature tensors as defined below.

$$W_1(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(X,T)Ric(Y,Z) - g(Y,T)Ric(X,Z)\right] \tag{5}$$

$$W_2(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(X,Z)Ric(Y,T) - g(Y,Z)Ric(X,T)\right] \tag{6}$$

$$W_3(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(X,Z)Ric(X,T) - g(Y,T)Ric(X,Z)\right] \tag{7}$$

$$W_4(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(X,Z)Ric(Y,T) - g(X,Y)Ric(Z,T)\right] \tag{8}$$

$$W_5(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(X,Y)Ric(Z,T) - g(Y,T)Ric(Y,Z)\right] \tag{9}$$

$$W_6(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(X,T)Ric(Z,Y) - g(X,Z)Ric(Y,T)\right] \tag{10}$$

$$W_7(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(Y,Z)Ric(X,T) - g(X,T)Ric(Y,Z)\right] \tag{11}$$

$$W_8(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(Z,T)Ric(X,Y) - g(X,T)Ric(Y,Z)\right] \tag{12}$$

$$W_9(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(Z,T)Ric(X,Y) - g(Y,Z)Ric(X,T)\right] \tag{13}$$

Where $R(X,Y,X,T)$ is the Riemann curvature tensor, g the metric tensor and $Ric(X,Y)$ the Ricci tensor.

2 Riemannian and Contact Manifolds

2.0.1 Riemannian Manifolds

Let T_pM be tangent space at a point p of a differentiable manifold M^n. Let us single out in M^n a real valued bilinear symmetric and positive definite function g on the ordered pair of tangent vectors at each point p on M^n. Then M^n is called Riemannian manifold and g is called the metric tensor of M^n.

g satisfies the following properties

1. $g(X,Y) \in \mathbb{R}$
2. $g(X,Y) = g(Y,X)$
3. $g(aX+bY,Z) = ag(X,Z)+bg(Y,Z)$
4. $g(X,X) > 0$

The angle θ between two vectors is defined by

$$||X||.||Y||cos\theta = g(X,Y)$$

where

$$||X|| = g(X,X)$$

Thus two vector X and Y are perpendicular if $g(X,Y) = 0$

A connection ∇ is said to be Riemannian if it satisfies

1. ∇ is symmetric
$$\nabla_X Y - \nabla_Y X = [X,Y]$$
2. g is covariant constant with respect to ∇ which gives
$$\nabla_X g = 0$$
and
$$g(\nabla_X Y,Z) + g(Y,\nabla_X Z) = X(g(Y,Z))$$

An affine connection ∇ is said to be metric if $\nabla_X g = 0$.

The Riemannian manifold is said to be Einsteinian manifold if

$$Ric(X,Y) = \lambda g(X,Y)$$

A Riemannian manifold is said to be flat if

$$R(X,Y)Z = 0$$

The torsion tensor T is a vector valued linear function and is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

if the torsion tensor vanished the connection is said to be torsin free or symmetric.

Riemannian curvature tensor

The curvature tensor with respect to the Riemannian connection is called the Riemannian curvature tensor.

It is defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X - \nabla_{[X,Y]} Z$$

Riemannian connection

Let X and Y be tangent vectors at $p \in M^n$. Let A and B be C^∞ vector fields about p and let f be a C^∞ real valued function about p, the we have

1. $\nabla_X (A+B) = \nabla_X A + \nabla_Y Z$
2. $\nabla_{X+Y} A = \nabla_X A + \nabla_Y A$
3. $\nabla_{fX} Y = f \nabla_X Y$
4. $\nabla_X fA = (Xf)A + f \nabla_X A$

Using ∇ we can define parallel vector fields along a curve and geodesics. Let r be a C^∞ curve on M^n with tangent vector field X and let Y be a vector field that is parallel along r if $\nabla_X Y = 0$ along r.

The curve γ is geodesic if $\nabla_r T = 0$, that is, if its tangent T is parallel along γ. Thus generalization of a definition of covariant differentiation or connection on C^∞ manifold is clear i.e. We merely need the existence of operator ∇ which satisfies all four condition of above properties listed for ∇ and assigns to C^∞ vector fields X and Y with domain D, a C^∞ field $\nabla_X Y$ on D.

Note that a manifold can have more than one connection.

Let us denote the dot or inner product of tangent vectors X and Y by

$$< X, Y >= \Sigma_{i=1}^{n} X_i Y_i$$

If X and Y are $C^i nfty$ fields the $< X, Y >$ is also $C^i nfty$ field and if D is the domain of X, Y and X, Y are $C^i nfty$ fields then we have

$$\nabla_Y Z - \nabla_Z Y = [Y, Z]$$

and

$$X < Y, Z >=< \nabla_X Y, Z > + < Y, \nabla_X Z >$$

for every X at p in D.

A Riemannian manifold is a C^∞ manifold M on which one has singled out a C^∞ real valued, bilinear, symmetric and positive definite function $<,>$ on ordered pair of tangent vector at each point. Thus if X, Y and Z are in $T_p M$ then

1. $< X, Y >=< Y, X >$
2. $< X + Y, Z >=< X, Z > + < Y, Z >$ and $< aX, Y >= a < X, Y >$
3. $< X, X >> 0$ for all $X \neq 0$
4. If X and Y are C^∞ fields with domain D then $< X, Y >_p =< X_p, Y_p >$ is a C^∞ function on D. If we replace the third property above with $< X, Y >= 0$ for all X implied $Y = 0$ then M^n is a semi-Riemannian (or pseudo Riemannian) manifold. In either case the function is inner product, metric tensor, the Riemannian metric or infinite semi metric on M^n not the topological metric function.

If ∇ is C^∞ connection in semi-Riemannian manifold M^n then ∇ is Riemannian connection if it satisfies above properties.

Properties of Riemannian curvature tensor

The Riemannian curvature tensor is linear over the ring of smooth functions and satisfies below properties

1. $R(X,Y)Z = -R(Y,X)Z$
2. $R(fX,Y)Z = -fR(Y,X)Z$ where f is a smooth function

Let us define
$$R'(X,Y,Z,T) = g(R(X,Y)Z,T)$$
. Then R' is skew symmetric in the first two slots and the last two slots. The Riemannian curvature tensor R satisfies Bianchi's first identity and Bianchi's second identity.

Curvature Tensors

In a Riemannian manifold the Weyl projective tensor reduces to

$$W(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[Ric(X,Z)Y - Ric(Y,Z)X]$$

Conformal curvature tensor

The tensor C defined by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z + \frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)QY + g(Y,Z)QX] \\ + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \end{aligned}$$

is the same for manifolds in conformal correspondence. This tensor is called the conformal curvature tensor.

A manifold whose conformal curvature tensor vanished at every point is said to be conformaly flat. A conformal curvature tensor C satisfies Bianchi's first identity

$$C(X,Y)Z + C(Y,Z)X + C(Z,X)Y = 0$$

Concircular curvature tensor

The concircular curvature tensor is defined by

$$\bar{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]$$

Conharmonic curvature tensor

The conharmonic curvature tensor is defined by

$$L(X,Y)Z = R(X,Y)Z + \frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]$$

Riemannian curvature

Let X and Y be unit tangent vectors at a point p of Riemannian manifold M^n, these vector determine a pensil of direction at p if the unit vectors along that direction are U then

$$U = fX + gY$$

where $f, g \in F$

and

$$f^2 + g^2 = 1$$

the geodesic of M^n whose unit tangent vector are U, generate a two dimensional sub manifold of the tangent manifold T at p.

The gaussian curvature $K(X,Y)$ at p of this two dimensional sub manifold was defined by Riemannian as sectional curvature at p of M^n in direction of X and Y. Thus

$$K = \frac{-K(X,Y)}{||X||^2||Y||^2[1 - cos^2\theta]}$$

where θ is angle between X and Y.

A necessary and sufficient condition on M^n to be locally flat in the neighbourhood U of a point p is that Riemannian curvature of M^n at p vanishes.

If the Riemannian curvature R of M^n at p of the direction X and Y then

$$R(X,Y)Z = K[g(Y,Z)X - g(X,Z)Y] \quad (14)$$

contracting we get

$$Ric = K(n-1)g \tag{15}$$

$$R = [K(n-1)]n \tag{16}$$

contracting (16) we get

$$R = Kn(n-1)$$

hence a Riemannian manifold of constant curvature is an Einstein manifold.

Shur's theorem

If a Riemannian curvature R of M^n at every point of neighbourhood U of M^n is independent of the direction choosen then R is constant throughout the neighbourhood U provided $n > 2$.

Putting (14) and (16) together we get $W = 0$.

Conversely, if $W = 0$

$$R(X,Y)Z = \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY]$$

contracting above equation we get

$$Ric(Y,Z) = \frac{r}{n}g(Y,Z)$$

which sometimes expressed as $RX = \frac{r}{n}X$ and putting the two equations into the first one we get

$$R(X,Y)Z = \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]$$

which shows that a manifold is of constant Riemannian curvature. Hence a necessary and sufficient condition for the manifold M^n to be of constant Riemannian curvature is the Weyl projective curvatue tensor vanishing identically throughout M^n.

Similary the conformal curvature tensor vanished from a manifold with constant Riemannian curvature.

Difference tensor of two connections

Consider a smooth manifold M and let ∇ and $na\bar{b}la$ to be two connections on M. Then for two vector fields X and Y on M, we define the difference tensor by

$$B(X,Y) = \bar{\nabla}_X Y - \nabla_X Y$$

The tensor $B(X,Y)$ is linear from the properties of a connection. Let f be C^∞ function, then

$$B(X,fY) = (Xf)Y + f\nabla_X Y - (Xf)Y - f\bar{\nabla}_X Y = fB(X,Y)$$

If we decompose $B(X,Y)$ into symmetric and skew symmetric pieces we have

$$B(X,Y) = S(X,Y) + A(X,Y)$$

where

$$S(X,Y) = \frac{1}{2}[B(X,Y) - B(Y,X)]$$

is the symmetric part and

$$A(X,Y) = \frac{1}{2}[B(X,Y) - B(Y,X)]$$

is the skew symmetric part.

Then we can express A in terms of torsion tensors T and $\bar{T}$ of connections ∇ and $\bar{\nabla}$ respectively as follows

$$\begin{aligned} 2A(X,Y) &= B(X,Y) - B(Y,X) \\ &= \bar{\nabla}_X Y - \nabla_X Y - \bar{\nabla}_Y X - \nabla_Y X \\ &= \bar{(}T)(X,Y) - T(X,Y) + [X,Y] - [X,Y] \\ &= \bar{(}T)(X,Y) - T(X,Y) \end{aligned}$$

Theorem

The following statements are equivalent

1. The connections ∇ and $\bar{\nabla}$ have the same geodesic
2. $B(X,X) = 0$ for all X
3. $S = 0$
4. $B = A$

Theorem

The connections ∇ and $\bar{\nabla}$ are equal if they have the same geodesic and the same torsion tensors.

Proof

That the first part implies the second is trivial. Conversely, if the geodesic are the same then $S=0$ and if the torsion tensors are equal then $A=0$, hence $B=0$ and $\nabla=\bar{\nabla}$.

Riemannian curvature tensor

The curvature tensor of conneciton ∇ is a linear transformation valued tensor R that assigns to each pair of vector X and Y a linear transformation $R(X,Y)$ of M^n into itself. We define $R(X,Y)Z$ by imbedding X,Y and Z in C^∞ field about M and setting

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z \tag{17}$$

Hence we notice that $R(X,Y)=-R(Y,X)$ and if f is C^∞ then

$$\begin{aligned} R(fX,Y)Z &= f\nabla_X\nabla_Y Z-(Yf)\nabla_X Z-f\nabla_Y\nabla_X Z \\ &\quad +(Yf)\nabla_X Z-f\nabla_{[X,Y]}Z \\ &= fR(X,Y)Z \end{aligned} \tag{18}$$

also

$$\begin{aligned} R(X,Y)(fZ) &= \nabla_X(Yf)X+f\nabla_Y Z-\nabla_Y((Xf)Z-f\nabla_X Z)-([X,Y]f)Z-f\nabla_{[X,Y]}Z \\ &= (XY)(fZ)+(Yf)\nabla_X Z+(Xf)\nabla_Y Z+f\nabla_X\nabla_Y Z-(YX)(fZ) \\ &\quad -(Xf)\nabla_Y Z-(Yf)\nabla_X Z-f\nabla_Y\nabla_X Z-([X,Y]f)Z-f\nabla_{[X,Y]}Z \\ &= fR(X,Y)Z \end{aligned} \tag{19}$$

The linearity of $R(X,Y)Z$ with respect to addition in each slot is trivial to check. The curvature of symmetric linear connection on M^n satisfies Bianchi's identities

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0 \tag{20}$$

for all vector $X,Y,Z\in M^n$ for which the left hand side is defined to prove this, we recall that for symmetric connection

$$\nabla_A B-\nabla_B A=[A,B]$$

$$
\begin{aligned}
R(X,Y)Z+R(Y,Z)X+R(Z,X)Y &= \nabla_X[Y,Z]+\nabla_Y[Z,X]+\nabla_Z[X,Y] \\
&\quad -\nabla_[Y,Z]X-\nabla_[Z,X]Y-\nabla_[X,Y]Z \\
&= [X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] \\
&= 0
\end{aligned}
$$

by Jacobi identity.

If we define

$$Z < X,Y > = < \nabla_Z X,Y > + < X,\nabla_Z Y > \tag{21}$$

from all vectors X,Y,Z with common domain, then using above definition we can define a rank 4 convariant tensor called Riemann-Christoffel curvature tensor as

$$R'(X,Y,Z,T) = < X,R(Z,T)Y > \tag{22}$$

for all X,Y,Z and T in the same domain

Thus from the above definition the following results arise

1. $R'(X,Y,Z,T) = -R'(Y,X,Z,T)$
2. $R'(X,Y,Z,T) = -R'(Y,X,T,Z)$
3. $R'(X,Y,Z,T) = R'(Z,T,X,Y)$

Theorem Let M^n be a Riemann manifold, then there exists a unique torsion free connection ∇ such that

1. ∇ is symmetric
2. $\nabla_X g = 0$ for all X

Parallel translation preserves inner products, this connection is called the Riemannian or Levi-Civita connection.

Proof

Uniqueness from proposition we obtain

$$Xg(Y,Z) - g(\nabla_X Y,Z) - g(Y,\nabla_X Z) = 0$$

using ∇ is torsion free this yields

1. $Xg(Y,Z) = g(\nabla_X Y,Z)$
$$= g([X,Y],Z) + g(Y,\nabla_X Z)$$
cyclically permuting X,Y and Z we get

2. $Yg(Z,X) = g(\nabla_X Y,X) + g([Y,Z],X) + g(Z,\nabla_Y X)$

3. $Zg(X,Y) = g(\nabla_X Z,Y) + g([Z,X],Y) + g(X,\nabla_Z Y)$

substituting (1) from (2) and (3) we get

$$\begin{aligned} 2g(\nabla_Z Y,X) = &-X < Y,Z > +Y < Z,X > +Z < X,Y > \\ &- < [Z,X],Y > - < [Y,Z],X > + < [X,Y],Z > \end{aligned}$$

The right hand of this last expression does not involve ∇, so we have a formula for $g(\nabla_Z Y)$ on X, as $<,>$ is non singular i.e. The map $T_pM \to T_p^*M$ induced by g being an isomorphism and X is arbitrary. $\nabla_Z Y$ is uniquely determined so ∇ is unique. If we define $\nabla_Z Y$ by using the expression $2g$ above then ∇ is a connection and we find condition (1) and (2) of the theorem satisfied.

2.0.2 Complex Manifolds

Complex manifold

An even dimensional differentiable manifold M^n, (n=2m), which can be endowed by a system of complex coordinate neighbourhood (U,α) in such a way that in the intersection $U \cap U'$ of two complex coordinate patches (U,α), (U',α') such that $\alpha' \circ \alpha$ is a complex analytic function is called a complex manifold.

Almost complex manifold

If on an even dimensional differentiable manifold M^n of differentiability class C^{r+1} there extist a vector valued real linear function J of differentiability class C^r satisfying

1. $J^2 + I_n = 0$
2. $\bar{X} + X = 0$ where $\bar{X} = JX$

then M^n is said to be an almost complex manifold and J is said to be an almost complex structure on M^n.

2.0.3 Contact Manifolds

Contact geometry is the study of a geometric structure on smooth manifolds given by a hyperplane distribution in the tangent bundle and specified by a one-form, both of which satisfy a 'maximum non-degeneracy' condition called 'complete non-integrability'.

Contact geometry is in many ways an odd dimensional counter part of Symplectic geometry, which belongs to the even-dimensional world. Both Contact and Symplectic geometry are motivated by the mathematical formulation of classical mechanics, where one can consider either the even-dimensional phase space of a mechanical system or the odd dimensional extended phase space that includes the time variable.

Contact geometry has broad applications in physics, e.g., geometrical optics, classical mechanics, thermodynamics, geometric quantization and applied mathematics such as control theory.

Contact manifolds are old dimensional manifolds which are studied using the complex structure and the differential 1-form on the manifold. By giving additional structures to the above one obtains almost Sasakian, quasi-Sasakian, Sasakian, almost cosymplectic, cosymplectic, conformal K-contact, Kenmotsu and trans-Sasakian manifolds.We give the basic definitions of these manifolds in this chapter.

Definition 2.0.1. *A $2n+1$ dimensional smooth manifold M is called a contact manifold if it has 1-form η such that $\eta \wedge (d\eta)^n \neq 0$.*

Additionally a contact manifold admits a vector field ξ, a tensor field ϕ of type $(1,1)$ and a Riemannian metric g such that

$$\begin{aligned} \eta(\phi) &= 1 \\ \phi^2 X &= X - \eta(X)\psi \\ g(X,\xi) &= \eta(X) \\ g(X,\phi Y) &= d\eta(X,Y) \end{aligned}$$

A Contact metric manifold M is said to be Einstein manifold if its Ricci tensor Ric is of the form $Ric(X,Y) = ag(X,Y)$, where a is a constant and a contact metric manifold M is said to be η-Einstein if its Ricci tensor Ric is of the form

$$Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$$

where a and b are some smooth functions on M.

Definition 2.0.2. *A contact manifold is then a Sasakian manifold given that*

$$(\nabla_X \phi)Y - g(X,Y)\xi - \eta(Y)X$$

Therefore in a Sasakian manifold,

$$\nabla_X \xi = -\phi X$$
$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y$$

Definition 2.0.3. *Given a contact manifold (M,g,η,ξ,ϕ), such that ξ is a Killing vector field, that is ξ satisfies*

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0$$

then we call the manifold M a K-contact Riemannian manifold, where X and Y are arbitrary vector fields.

Note that in a K-contact Riemannian manifold

$$\nabla_X \xi = -\phi X$$

A K-contact Riemannian manifold is a Sasakian manifold if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$$

Definition 2.0.4. *A contact metric manifold with structure tensors (ϕ,ξ,η,g) in which the vector field ξ is a conformal Killing vector field, if it satisfies,*

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 2\alpha g(X,Y)$$

where α is a scalar, then M is called a conformal K-contact manifold.

So, in particular if α = 0, then M becomes K - contact manifold.

Definition 2.0.5. *A Riemannian manifold of dimension $(2n+1)$ is called an almost contact manifold if it admits a 1-form η, a vector field ξ and a type $(1,1)$ tensor ϕ such that*

$$\begin{aligned}
\phi^2 X = -X + \eta(X)\xi, \phi\xi = 0, \eta(\xi) = 1, \eta(\phi X) = 0 \\
g(X,\xi) = \eta(X) \\
g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \\
g(\phi X, Y) = -g(X, \phi Y), g(X,X) = 0 \\
(\nabla_X \eta)Y = g(\nabla_X \xi, Y)
\end{aligned}$$

An almost contact manifold is to be

- a contact manifold if
$$d\eta(X,Y) - \Phi(X,Y) = g(X,\phi Y)$$
where Φ is called the fundamental two form of the manifold.

- K-contact manifold if it is contact and ξ is a Killing vector field

- Sasakian manifold if and only if it is a contact manifold satisfying

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$$

Definition 2.0.6. *An almost contact metric manifold (M,ϕ,ξ,η,g) is said to be an almost co-symplectic manifold, if the fundamental 2-form Φ defined by*

$$\Phi(X,Y) = g(X,\Phi Y)$$

and the 1-form η are closed, that is

$$d\Phi = 0 \;\; and \;\; d\eta = 0$$

where d is the operator of exterior differentiation. If in an almost co-symplectic manifold, the almost contact metric structure is normal then it is called co-symplectic.

Definition 2.0.7. *A generalized Sasakian space form is an almost contact metric manifold whose curvature tensor R is given by*

$$\begin{aligned}
R(X,Y)Z = f_1 &[g(Y,Z)X - g(X,Z)Y] \\
& f_2 [g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z] \\
& f_3 [g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X]
\end{aligned}$$

where f_1, f_2, and f_3, are differentiable functions on the manifold.

Definition 2.0.8. *An almost contact metric manifold is referred to as a Kenmotsu manifold if it satisfies*

$$\begin{aligned}(\nabla_X\phi)Y &= -g(X,\phi Y)\xi - \eta(Y)\phi X\\ (\nabla_X\xi) &= X - \eta(X)\xi\\ (\nabla_X\eta)Y &= g(X,Y) - \eta(X)\eta(Y)\end{aligned}$$

Definition 2.0.9. *Let M be an n dimensional manifold admitting a type $(1,1)$ tensor ϕ, 1-form η, vector field ξ and a Riemannian metric g. Then M is said to be para-contact if*

$$\begin{aligned}\phi^2X = X - \eta(X)\xi, \phi\xi = 0, \eta(\xi) = 1, \eta(\phi X) = 0\\ g(X,\xi) = \eta(X)\\ g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y)\end{aligned}$$

If in addition to the above properties M satisfies

$$\begin{aligned}d\eta = 0, \nabla_X\xi = \phi X\\ (\nabla_X\phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi\end{aligned}$$

then it is called a Para-Sasakian (P-Sasakian) manifold. A P-Sasakian manifold is then called Special Para-Sasakian (SP-Sasakian) manifold if

$$(\nabla_X\eta)Y = -g(X,Y) + \eta(X)\eta(Y)$$

Definition 2.0.10. *Let M be an n dimensional manifold admitting a $(1,1)$ type tensor field ϕ, vector field ξ, 1-form η and a Lorentzian metric g. The we call M a Lorentzian Para-Sasakian (LP-Sasakian) manifold if it satisfies*

$$\begin{aligned}\phi^2X = X + \eta(X)\xi, \phi\xi = 0, \eta(\xi) = -1, \eta(\phi X) = 0\\ g(X,\xi) = \eta(X)\\ g(\phi X,\phi Y) = g(X,Y) + \eta(X)\eta(Y)\\ (\nabla_X\phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi\\ \nabla_X\xi = \phi X\end{aligned}$$

3 Literature Review

The concept of curvature arises naturally in the study of curves and surfaces in Euclidean space. For instance, in $\mathbb{R}^3$, we can define various forms of curvatures of a given surface. Such as the *principal curvatures* k_1, k_2, the *mean curvature* $H = \frac{k_1+k_2}{2}$ and the *Gaussian curvature* $K = k_1 k_2$. Of equal importance is the sign of these curvatures, since it shows how a surface is oriented at a particular point. Given a point p on a surface S, one can think of the principal curvatures as the maximum and minimum rates S bends at p respectively. However, these forms of curvature except for the Gaussian curvature are extrinsic in the sense that they are defined in terms of the ambient space around the surface S. Intrinsic quantities are those defined completely using parameters only on the surface S. Studies have shown that the two forms of describing quantities are equivalent.

However, in describing quantities intrinsically, we require notions of length and angles similar to those of Euclidean spaces. This was addressed by Riemann by introducing the so called Riemannian manifolds. He developed the notion of curvature in an abstract way which is now referred to as Riemann curvature tensor. The same can be defined on semi-Riemannian manifolds. Due to the complexity of the Riemann curvature tensor, other curvature tensors have been defined some been derived from it, like the Ricci curvature and Weyl curvature tensor. The geometric properties of curvature tensors have been studied extensively on various types of manifolds.

The study and classification of Riemann manifolds as symmetric spaces was pioneered by E. Cartan during the late $1920's$. He showed that a Riemann manifold is locally symmetric if $\nabla R = 0$. He also studied semi symmetric Riemannian spaces. i.e. $R(X,Y).R = 0$. A classification of semi symmetric Riemannian spaces was given by Szabo [33].

Since then, many authors have studied this notion of local symmetry and have come up with weaker versions of it. For instance, Walker [35] in 1950 studied recurrent spaces where the Riemann curvature tensor satisfies

$$R_{hijk,l} = R_{hijk}\lambda_l$$

where λ_l is a non-zero vector and comma denotes covariant derivative. This were originally considered by Ruse [26]. Motivated by this result, Patterson [17] has studied a Ricci-Recurrent spaces which are Riemann spaces whose Ricci tensor satisfies

$$R_{ij} \neq 0,\ R_{ij,k} = R_{ij}\lambda_k$$

for some non-zero vector λ_k. It follows that a Recurrent space is a Ricci-Recurrent space but the converse is not true in general.

If the Weyl curvature tensor satisfies the recurrence relation above, the space is known as a conformally recurrent and they were introduced by Adati and Miyazawa [2] in 1967. On the other hand, Chaki and Gupta [12] had studied conformally symmetric manifolds in 1963.

In literature there exist two distinct notions of pseudo symmetric manifolds. One introduced by Chaki [3] in 1987 and the other by Deszcz [5] in 1992. According to Chaki [3], a non-flat semi-Riemannian manifold (M,g) of dimension greater than one is pseudo symmetric if

$$\begin{aligned}(\nabla_X R)(Y,Z)W &= 2A(X)R(Y,Z)W + A(Y)R(X,Z)W \\ &\quad + A(Z)R(Y,X)W + A(W)R(Y,Z)X \\ &\quad + g(R(Y,Z)W,X)\rho,\end{aligned}$$

where A is a non-zero 1-form, ρ is a vector field defined by

$$g(X,\rho) = A(X), \forall X$$

and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. The 1-form A is called the associated 1-form of the manifold. If $A = 0$, then the manifold reduces to a symmetric manifold in the sense of E. Cartan. In 1989, Tamassy and Binh [11] introduced weakly symmetric and weakly projectively symmetric Riemannian manifolds. On the analogy, these notions of symmetry, recurrency, weak symmetry and pseudo symmetry have been studied and loosened in various directions by many authors.

S. Tanno [34] in 1969 gave a classification of connected almost contact Riemannian manifolds whose automorphism groups have the maximum dimension. Inspired by the results of S. Tanno, Kenmotsu [9] in 1972 studied one of the classes given by S. Tanno and showed that the structure of such manifolds is not Sasakian. Such a structure is usually called a Kenmotsu structure. He proved that if a Kenmotsu manifold satisfies the condition $R(X,Y).R = 0$, then the manifold is of constant negative curvature -1. In 1976, Sato [27] introduced and studied the notion of an almost para-contact structure on a Riemannian manifold. The Adati and Matsumoto [1] defined and studied a P-Sasakian manifold and an SP-Sasakian manifold which are considered as special cases of an almost para-contact manifold. The authors showed that a conformally symmetric P-Sasakian manifold of dimension n, $n > 3$, is conformally flat.

The authors Sinha and Sai Prasad [30] in 1995 defined a class of almost para-contact metric manifolds namely Para-Kenmotsu and Special Para-Kenmotsu manifolds. Recently in 2013, Satyanarayana and Sai Prasad [28] have considered the P-Kenmotsu manifoldin which $R(X,Y).C = 0$ where C is the conformal curvature tensor and $R(X,Y).C$ acts as a derivation of the tensor algebra at each point of the manifold for tangent vectors X and Y. They showed that a conformally symmetric P-Kenmotsu manifold (M,g), $n > 3$ is an SP-Kenmotsu manifold. The same authors in 2015 considered the properties of the Weyl projective curvature tensor on a P-Kenmotsu manifold. They showed that a semi-symmetric P-Kenmotsu manifold is of constant curvature and hence is an SP-Kenmotsu manifold [29]. Additionally, they proved that a Weyl projectively semi-symmetric P-Kenmotsu manifold is projectively flat.

In 1970 Pokhariyal and Mishra [23] introduced new tensor fields, called W_2 and E-tensor fields, in a Riemannian manifold, and studied their physical and geometric properties. Pokhariyal [20] then considered these tensor fields in a Sasakian manifold. Later, in 1986, Matsumoto, Ianus and Mihai [8] studied the same tensor fields in a P-Sasakian manifold. In a recent study, U. C. De and Sarkar have studied a P-Sasakian manifold admitting a W_2-curvature tensor. They have shown that among others, a W_2-symmetric P-Sasakian manifold is of constant curvature, hence it is an SP-Sasakian manifold [4].

The curvature tensors introduced by Pokhariyal and Mishra have also been studied in Kenmotsu manifolds and spacetimes. For instance, Yildiz and De [36] have studied Kenmotsu manifolds admitting a W_2-curvature tensor. In their study they have shown that a W_2-semisymmetric Kenmotsu manifold is an Einstein manifold. The authors Singh, S. K. Pandey and G. Pandey have also conducted similar studies [25]. Mallick and De [13] have studied spacetimes admitting W_2-curvature tensor in 2014.

In line with the W_2-curvature tensor, Pokhariyal [18], [19], [21], has introduced new tensor fields W_3 to W_9 and obtained their properties. In [6] the authors have studied the W_3-curvature tensor on relativistic spacetimes. More recently, Njori, Moindi and Pokhariyal [16] have considered W_6-curvature tensor on Kenmotsu manifold admitting semi-symmetric metric connection. The authors have also investigated the geometrical relationship between W_6-curvature tensor and W_8-curvature tensor in Kenmotsu manifold.

4 A study of W_3-curvature tensor in Lorentzian Para Sasakian Manifolds

4.1 Introduction

Pokhariyal [24] defined the W_3-curvature tensor and studied its physical and geometrical properties in a Riemannian manifold. This tensor is defined as

$$\begin{aligned} W_3(X,Y,Z,T) &= R(X,Y,Z,T) \\ &\quad + \frac{1}{n-1}[g(Y,Z)Ric(X,T) - g(Y,T)Ric(X,Z)] \end{aligned} \tag{23}$$

where R is the Riemannian curvature tensor of type $(0,4)$, g is the Riemannian metric and Ric is the Ricci tensor of type $(0,2)$. The tensor $W_3(X,Y,Z,T)$ is skew-symmetric in Z,T and does not satisfy the cyclic property. That is

$$W_3(X,Y,Z,T) = -W_3(X,Y,T,Z) \tag{24}$$

and

$$W_3(X,Y,Z,T) + W_3(Y,Z,X,T) + W_3(Z,X,Y,T) \neq 0 \tag{25}$$

We can express this tensor in index notation as

$$W_{3ijkl} = R_{ijkl} + \frac{1}{n-1}[g_{jk}R_{il} - g_{jl}R_{ik}] \tag{26}$$

In 2018, the authors S.K. Moindi, F. Njui and G.P. Pokhariyal [31] have studied the geometrical properties of W_3-curvature tensor in a K-contact Riemannian manifold. On the other hand, S.O. Pambo, S.K. Moindi and B.M. Nzimbi [32] have studied η-Ricci solition on W_3-semi symmetric LP-Sasakian manifolds. Recently, H. A. Donia, S. Shenawy and A. A. Syied [7] have considered the role of W_3-curvature tensor on relativistic space-times.

Motivated by the above results, in this paper we will investigate certain curvature properties of LP-Sasakian manifolds admitting W_3-curvature tensor.

4.2 Preliminaries

A manifold M^n of dimension n is called an LP-Sasakian manifold if it admits a tensor field ϕ of type $(1,1)$, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy the following properties [14]

$$\eta(\xi) = -1, \tag{27}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{28}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{29}$$

$$g(X,\xi) = \eta(X),\ \nabla_X \xi = \phi X, \tag{30}$$

$$(\nabla_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{31}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Given that M^n is an LP-Sasakian manifold with the structure (ϕ, ξ, η, g), we can deduce the following [14] .

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{32}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X), \tag{33}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{34}$$

$$R(\xi,X)\xi = X + \eta(X)\xi, \tag{35}$$

$$Ric(X,\xi) = (n-1)\eta(X), \tag{36}$$

$$Ric(\phi X, \phi Y) = Ric(X,Y) + (n-1)\eta(X)\eta(Y) \tag{37}$$

for any vector fields X, Y, Z.

We shall use the above results in the following sections.

4.3 W_3-flat LP-Sasakian manifold

Definition 4.3.1. *An LP-Sasakian manifold is said to be flat if* $R(X,Y)Z = 0$.

Definition 4.3.2. *An LP-Sasakian manifold is called W_3-flat if the curvature tensor W_3 vanishes everywhere i.e.* $W_3(X,Y)Z = 0$.

Theorem 4.3.3. *A W_3-flat LP-Sasakian manifold is an Einstein manifold.*

Proof
From 23, if

$$W_3'(X,Y,Z,U) = 0 \tag{38}$$

$$\Longrightarrow R(X,Y,Z,U) = \frac{1}{n-1}[g(Y,U)Ric(X,Z) - g(Y,Z)Ric(X,U)] \tag{39}$$

Taking contraction over X and U we have

$$Ric(Y,Z) = \frac{1}{n-1}[Ric(Y,Z) - g(Y,Z)r] \tag{40}$$

$$Ric(Y,Z) = \frac{-r}{n-2}g(Y,Z) \tag{41}$$

Hence the theorem. Here r denotes the scalar curvature.

4.4 W_3-symmetric LP-Sasakian manifold

Definition 4.4.1. *An LP-Sasakian space is called symmetric if* $\nabla_U R(X,Y)Z = 0$.

Definition 4.4.2. *An LP-Sasakian space is called* W_3*-symmetric if* $\nabla_U W_3(X,Y)Z = 0$

Theorem 4.4.3. *A* W_3*-symmetric and* W_3*-flat LP-Sasakian manifold is a flat space.*

Proof

If M^n is a W_3-symmetric LP-Sasakian manifold, then we have

$$\nabla_U W_3(X,Y)Z = W_3'(X,Y,Z,U) = 0 \tag{42}$$

$$\begin{aligned} \implies R(X,Y,W_3(Z,U,V)) - W_3(R(X,Y,Z),U,V) \\ -W_3(Z,R(X,Y,U),V) - W_3(Z,U,R(X,Y,V)) = 0 \end{aligned} \tag{43}$$

Expanding the terms in the above expression we get

$$\begin{aligned} g(R(X,Y,W_3(Z,U,V)),\xi) &= R'(X,Y,W_3(Z,U,V),\xi) \\ &= g(X,\xi)g(Y,W_3(Z,U,V)) - g(Y,\xi)g(X,W_3(Z,U,V)) \\ &= \eta(X)W_3'(Y,Z,U,V) - \eta(Y)W_3'(X,Z,U,V) \end{aligned} \tag{44}$$

$$
\begin{aligned}
g(W_3(R(X,Y,Z),U,V),\xi) &= W_3'(R(X,Y,Z),U,V,\xi) \\
&= R(R(X,Y,Z),U,V,\xi) + \frac{1}{n-1}[g(U,V)Ric(R(X,Y,Z),\xi) \\
&\quad - g(U,\xi)Ric(R(X,Y,Z),V)] \\
&= R(R(X,Y,Z),U,V,\xi) + \frac{1}{n-1}[g(U,V)(n-1)g(R(X,Y,Z),\xi) \\
&\quad - g(U,\xi)(n-1)g(R(X,Y,Z),V)] \\
&= R(R(X,Y,Z),U,V,\xi) + [g(U,V)R'(X,Y,Z,\xi) \\
&\quad - \eta(U)R'(X,Y,Z,V)] \\
&= g(R(X,Y,Z),\xi)g(U,V) - g(U,\xi)g(R(X,Y,Z),V) \\
&\quad + g(U,V)R'(X,Y,Z,\xi) - \eta(U)R'(X,Y,Z,V) \\
&= g(U,V)R'(X,Y,Z,\xi) - \eta(U)R'(X,Y,Z,V) \\
&\quad + g(U,V)R'(X,Y,Z,\xi) - \eta(U)R'(X,Y,Z,V) \\
&= 2[g(U,V)R'(X,Y,Z,\xi) - \eta(U)R'(X,Y,Z,V)] \qquad (45)
\end{aligned}
$$

$$
\begin{aligned}
g(W_3(Z,R(X,Y,U),V,\xi) &= W_3'(Z,R(X,Y,U),V,\xi) \\
= R(Z,R(X,Y,U),V,T) &= \frac{1}{n-1}[g(R(X,Y,U),V)Ric(Z,\xi) \\
&\quad - g(R(X,Y,U),\xi)Ric(Z,V)] \\
= g(Z,\xi)g(R(X,Y,U),V) &- g(Z,V)g(R(X,Y,U),\xi) \\
+\frac{1}{n-1}[R'(X,Y,U,V)(n-1)g(Z,\xi) &- R'(X,Y,U,\xi)(n-1)g(Z,V)] \\
= g(Z,\xi)R'(X,Y,U,V) &- g(Z,V)R'(X,Y,U,\xi) \\
+g(Z,\xi)R'(X,Y,U,V) &- g(Z,V)R'(X,Y,U,\xi) \\
= \eta(Z)R'(X,Y,U,V) &- g(Z,V)R'(X,Y,U,\xi) \\
+\eta(Z)R'(X,Y,U,V) &- g(Z,V)R'(X,Y,U,\xi)
\end{aligned}
$$

$$
= 2[\eta(Z)R'(X,Y,U,V) - g(Z,V)R'(X,Y,U,\xi)] \qquad (46)
$$

$$\begin{aligned}
g(W_3(Z,U,R(X,Y,V),\xi) &= W_3'(Z,U,R(X,Y,V),\xi) \\
&= R'(Z,U,R(X,Y,V),\xi) + \frac{1}{n-1}[g(U,R(X,Y,V))Ric(Z,\xi) \\
&\qquad - g(U,\xi)Ric(Z,R(X,Y,V))] \\
&= g(Z,\xi)g(U,R(X,Y,V)) - g(Z,R(X,Y,V))g(U,\xi) \\
&+ \frac{1}{n-1}[R'(X,Y,V,U)(n-1)g(Z,\xi) - g(U,\xi)(n-1)g(Z,R(X,Y,V))] \\
&= g(Z,\xi)R'(X,Y,V,U) - g(U,\xi)R'(X,Y,V,Z) \\
&+ g(Z,\xi)R'(X,Y,V,U) - g(U,\xi)R'(X,Y,V,Z) \\
&= \eta(Z)R'(X,Y,V,U) - \eta(U)R'(X,Y,V,Z) \\
&+ \eta(Z)R'(X,Y,V,U) - \eta(U)R'(X,Y,V,Z) \\
&= 2[\eta(Z)R'(X,Y,V,U) - \eta(U)R'(X,Y,V,Z)] \qquad (47)
\end{aligned}$$

Using 44, 45, 46 and 47 in 43 we have

$$\begin{aligned}
&\eta(X)W_3'(Y,Z,U,V) - \eta(Y)W_3'(X,Z,U,V) \\
&-2[g(U,V)R'(X,Y,Z,\xi) - \eta(U)R'(X,Y,Z,V)] \\
&-2[\eta(Z)R'(X,Y,U,V) - g(Z,V)R'(X,Y,U,\xi)] \\
&-2[\eta(Z)R'(X,Y,V,U) - \eta(U)R'(X,Y,V,Z)] = 0 \qquad (48)
\end{aligned}$$

In a W_3-flat manifold, $W_3' = 0$, hence the first two terms vanish. Coefficients of $\eta(Z)$ and $\eta(U)$ vanish due to R' begin skew-symmetric with respect to the last two variables. We thus have

$$2[g(Z,V)R'(X,Y,U,\xi) - g(U,V)R'(X,Y,Z,\xi)] = 0 \qquad (49)$$

Since $g(U,V) \neq g(Z,V) \neq 0$ for arbitrary vectors U,V,Z, this implies that if W_3 is symmetric then

$$R'(X,Y,Z,\xi) = 0 \qquad (50)$$

. This completes the proof.

4.5 W_3-semi-symmetric LP-Sasakian manifold.

Definition 4.5.1. *An LP-Sasakian manifold is called semi-symmetric if* $R(X,Y)R(U,V)Z = 0$.

Definition 4.5.2. *An LP-Sasakian manifold is called W_3-semi-symmetric if $R(X,Y)W_3(U,V)Z = 0$.*

Theorem 4.5.3. *A W_3-semi-symmetric LP-Sasakian manifold is a W_3-symmetric manifold.*

Proof
Taking the inner product

$$g(R(X,Y)W_3(U,V)Z,\xi) = R'(X,Y,W_3(U,V,Z),\xi) = 0 \tag{51}$$
$$g(X,\xi)g(Y,W_3(U,V,Z)) - g(X,W_3(U,V,Z))g(Y,\xi) = 0$$
$$\eta(X)W_3'(U,V,Z,Y) - \eta(Y)W_3'(U,V,Z,X) = 0 \tag{52}$$

Since $\eta(X)$ and $\eta(Y)$ are non-zero, $\implies W_3' = 0$. i.e.

$$\nabla_U W_3(X,Y)Z = W_3'(X,Y,Z,U) = 0$$

Hence a W_3-semi-symmetric LP-Sasakian manifold is a W_3-symmetric manifold.

4.6 W_3-recurrent LP-Sasakian manifold

Definition 4.6.1. *An LP-Sasakian manifold is called recurrent if $(\nabla_U R')(X,Y,Z,\xi) = B(U)R'(X,Y,Z,\xi)$. Where $B(U)$ is the associated recurrence parameter.*

Definition 4.6.2. *An LP-Sasakian manifold is called Ricci recurrent if $(\nabla_U Ric)(X,Y) = B(U)Ric(X,Y)$. Where $B(U)$ is the associated recurrence parameter.*

Definition 4.6.3. *An LP-Sasakian manifold is called W_3-recurrent if $(\nabla_U W_3')(X,Y,Z,\xi) = B(U)W_3'(X,Y,Z,\xi)$. Where $B(U)$ is the associated recurrence parameter.*

Theorem 4.6.4. *If an LP-Sasakian manifold is W_3-recurrent and Ricci recurrent, then for the same recurrence parameter, it is a recurrent manifold.*

Proof

$$
\begin{aligned}
(\nabla_U W_3')(X,Y,Z,\xi) &= B(U)W_3'(X,Y,Z,\xi) \\
B(U)W_3'(X,Y,Z,\xi) &= (\nabla_U R')(X,Y,Z,\xi) + \frac{1}{n-1}[g(Y,Z)(\nabla_U Ric)(X,\xi) \\
&\quad - g(Y,\xi)(\nabla_U Ric)(X,Z)] \\
B(U)W_3'(X,Y,Z,\xi) &= (\nabla_U R')(X,Y,Z,\xi) + \frac{B(U)}{n-1}[g(Y,Z)Ric(X,\xi) \\
&\quad - g(Y,\xi)Ric(X,Z)] \\
(\nabla_U R')(X,Y,Z,\xi) &= B(U)\{W_3'(X,Y,Z,\xi) - \frac{1}{n-1}[g(Y,Z)Ric(X,\xi) \\
&\quad - g(Y,\xi)Ric(X,Z)]\} \\
(\nabla_U R')(X,Y,Z,\xi) &= B(U)R'(X,Y,Z,\xi)
\end{aligned}
$$

Hence the theorem.

5 On the geometry of Lorentzian Para Sasakian Manifolds with Conservative and Irrotational W_3-curvature tensor

In chapter four we have studied the geometrical properties of $W_3(X,Y)Z$ curvature tensor in Lorentzian Para Sasakian manifold and proved some important results. The present chapter deals with the conservativeness of $W_3(X,Y)Z$ and irrotationality of $R(X,Y)Z$ in the same manifold.

It consists of two sections, in the first section we consider the vanishing of the divergence of $W_3(X,Y)Z$ i.e. $(div\ W_3)(X,Y)Z = 0$ and show that the manifold defines Einstein structure and the space is of constant scalar curvature.

In the second section we consider a Lorentzian Para Sasakian manifold which admits irrotational $R(X,Y)Z$ curvature tensor. It is shown that if $(curl\ R)(X,Y)Z = 0$, then the manifold is an Einstein manifold.

5.1 Lorentzian Para Sasakian manifold satisfying $(div\ W_3)(X,Y)Z = 0$

Given that

$$W_3(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(Y,Z)QX - S(X,Z)Y] \tag{53}$$

where Q is the symmetric endomorphism of the tangent space at every point and S is the Ricci tensor of type $(0,2)$.

Differentiating (53) covariantly, we have

$$(\nabla_U W_3)(X,Y)Z = (\nabla_U R)(X,Y)Z + \frac{1}{n-1}[g(Y,Z)(\nabla_U Q)(X) - (\nabla_U S)(X,Z)Y] \tag{54}$$

Contracting equation (54), we obtain

$$(div\ W_3)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) + \frac{1}{n-1}[g(Y,Z)dr(X) - (\nabla_X S)(Y,Z)]$$

$$(div\ W_3)(X,Y)Z = \frac{n-2}{n-1}(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) + \frac{1}{n-1}g(Y,Z)dr(X) \tag{55}$$

Let us suppose that $(div\, W_3)(X,Y)Z = 0$, then equation (55) becomes

$$\frac{n-2}{n-1}(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = -\frac{1}{n-1}g(Y,Z)dr(X) \tag{56}$$

Putting $X = \xi$ in (56), we get

$$\frac{n-2}{n-1}(\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z) = -\frac{1}{n-1}g(Y,Z)dr(\xi) \tag{57}$$

Using (30) and the fact that $L_\xi S = 0$, the first term in the left hand side of equation (57) can be expressed as

$$\begin{aligned}
(\nabla_\xi S)(Y,Z) &= \nabla_\xi S(Y,Z) - S(\nabla_\xi Y,Z) - S(Y,\nabla_\xi Z) \\
&= \nabla_\xi S(Y,Z) - S([\xi,Y] + \nabla_Y \xi, Z) + S(Y,[\xi,Z] + \nabla_Z \xi) \\
&= \nabla_\xi S(Y,Z) - S([\xi,Y],Z) - S(\nabla_Y \xi,Z) - S(Y,[\xi,Z]) - S(Y,\nabla_Z \xi) \\
&= (L_\xi S)(Y,Z) - S(\nabla_Y \xi,Z) - S(Y,\nabla_Z \xi) \\
&= -S(\phi Y,Z) - S(Y,\phi Z) \\
&= -S(\phi Y,Z) + S(\phi Y,Z), \phi \text{ is skew symmetric} \\
&= 0
\end{aligned} \tag{58}$$

We can also expand the second term and using (??) and (36), to get

$$\begin{aligned}
(\nabla_Y S)(\xi,Z) &= \nabla_Y S(\xi,Z) - S(\nabla_Y \xi,Z) - S(\xi,\nabla_Y Z) \\
&= \nabla_Y [(n-1)g(\xi,Z)] - S(\phi Y,Z) - (n-1)g(\xi,\nabla_Y Z) \\
&= (n-1)[g(\nabla_Y \xi,Z) + g(\xi,\nabla_Y Z)] - S(\phi Y,Z) - (n-1)g(\xi,\nabla_Y Z) \\
&= (n-1)g(\phi Y,Z) - S(\phi Y,Z)
\end{aligned} \tag{59}$$

Lastly

$$g(Y,Z)dr(\xi) = 0, \text{since } dr(\xi) = 0 \tag{60}$$

Now using equations (58), (59) and (60) in (57), we obtain

$$S(\phi Y,Z) = (n-1)g(\phi Y,Z) \tag{61}$$

Replacing Z with ϕZ in (61), leads to

$$\begin{aligned}
S(\phi Y,\phi Z) &= (n-1)g(\phi Y,\phi Z) \\
\implies S(Y,Z) &= (n-1)g(Y,Z)
\end{aligned} \tag{62}$$

which on contracting yields $r = n(n-1)$.

This leads to the following theorem

Theorem 5.1.1. *If $(div\ W_3)(X,Y)Z = 0$ in a Lorentzian para sasakian manifold M^n, then the manifold is Einstein and it is of constant scalar curvature $n(n-1)$.*

5.2 Lorentzian Para Sasakian manifold satisfying $(curl\ R)(X,Y)Z = 0$

The rotation (curl) of the curvature tensor $R(X,Y)Z$ on a Riemannian manifold is given by

$$\begin{aligned}(curl\ R)(X,Y)Z = (\nabla_U R)(X,Y)Z - (\nabla_X R)(U,Y)Z + (\nabla_Y R)(U,X)Z \\ - (\nabla_Z R)(X,Y)U\end{aligned} \tag{63}$$

By virtue of Bianchi's second identity

$$(\nabla_U R)(X,Y)Z + (\nabla_X R)(Y,U)Z + (\nabla_Y R)(U,X)Z = 0 \tag{64}$$

equation (63) reduces to

$$(curl\ R)(X,Y)Z = -(\nabla_Z R)(X,Y)U \tag{65}$$

If the curvature tensor is irrotational then $(curl\ R)(X,Y)Z = 0$ and thus

$$(\nabla_Z R)(X,Y)U = 0 \tag{66}$$

which implies

$$\nabla_Z R(X,Y)U = R(\nabla_Z X,Y)U + R(X,\nabla_Z Y)U + R(X,Y)\nabla_Z U \tag{67}$$

Putting $U = \xi$ in (67) we have

$$\nabla_Z R(X,Y)\xi = R(\nabla_Z X,Y)\xi + R(X,\nabla_Z Y)\xi + R(X,Y)\nabla_Z \xi \tag{68}$$

Using (30) and (34) in (68) obtain

$$\begin{aligned}\nabla_Z[\eta(Y)X - \eta(X)Y] = \eta(Y)\nabla_Z X - \eta(\nabla_Z X)Y + \eta(\nabla_Z Y)X - \eta(X)\nabla_Z Y \\ + R(X,Y)\phi Z\end{aligned} \tag{69}$$

Simplifying the above equation we get

$$g(Y,\phi Z)X - g(X,\phi Z)Y = R(X,Y)\phi Z \tag{70}$$

Replacing ϕZ by Z in the above equation it reduces to

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y \tag{71}$$

Contracting this equation leads to the following thorem

Theorem 5.2.1. *If the curvature tensor in Lorentzian Para Sasakian manifold is irrotational, then the manifold is Einstein and it is of constant scalar curvature n(n-1)*

6 Conclusion

6.1 Conclusion

The W_3-curvature tensor is symmetrical in its last pair of variables. The above propeties are of interest especially in the study of electromagnetism and the theory of general relativity.

In particular the Rainich conditions for existence of non- null electro variance can be obtained by the contracted part of this tensor. Thus we can use $W_3(X,Y,Z,T)$ in place of Weyl projective tensor in the study of physical significance and geometry of manifolds.

6.2 Future Research

According to Pokhariyal, [18], $W_3(X,Y,Z,T)$ can be broken down into two parts

$$\alpha(X,Y,Z,T) = \frac{1}{2}[W_3(X,Y,Z,T) - W_3(X,Y,Z,T)] \tag{72}$$

and

$$\beta(X,Y,Z,T) = \frac{1}{2}[W_3(X,Y,Z,T) + W_3(X,Y,Z,T)] \tag{73}$$

which are respectively skew-symmetric and symmetric in X,Y.

From (23), it follows that

$$\begin{aligned} \alpha(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{2(n-1)}[g(Y,Z)Ric(X,T) \\ - g(Y,T)Ric(X,Z) - g(X,Z)Ric(Y,T) + g(X,T)Ric(Y,Z)] \end{aligned} \tag{74}$$

and

$$\begin{aligned} \beta(X,Y,Z,T) = \frac{1}{2(n-1)}[g(Y,Z)Ric(X,T) - g(Y,T)Ric(X,Z) \\ + g(X,Z)Ric(Y,T) - g(X,T)Ric(Y,Z)] \end{aligned} \tag{75}$$

From (74), we see that $\alpha(X,Y,Z,T)$ possesses all the symmetric and skew symmetric properties of $R(X,Y,Z,T)$ as well as the cyclic property

$$\alpha(X,Y,Z,T) + \alpha(Y,Z,X,T) + \alpha(Z,X,Y,T) = 0 \tag{76}$$

.

The author studied the geometric and physical properties of these tensors in a Riemannian manifold and showed that the contracted part of $\alpha(X,Y,Z,T)$ does not vanish in an Einstein space. Thus it is not possible to extend the Pirani formalism of gravitational wave to the Einstein space with the help of $\alpha(X,Y,Z,T)$. He further showed that the vanishing of $\beta(X,Y,Z,T)$ is the necessary and sufficient condition for a space to be Einstein space.

Therefore one can explore the geometric and physical significance of $W_3(X,Y,Z,T)$ and its symmetric and skew symmetric parts in other manifolds to unearth properties which help study the geometry of those manifolds.

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