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Lines on Cubic Surface

Research Report in Mathematics, Number 04, 2021

Fredrick KOECH

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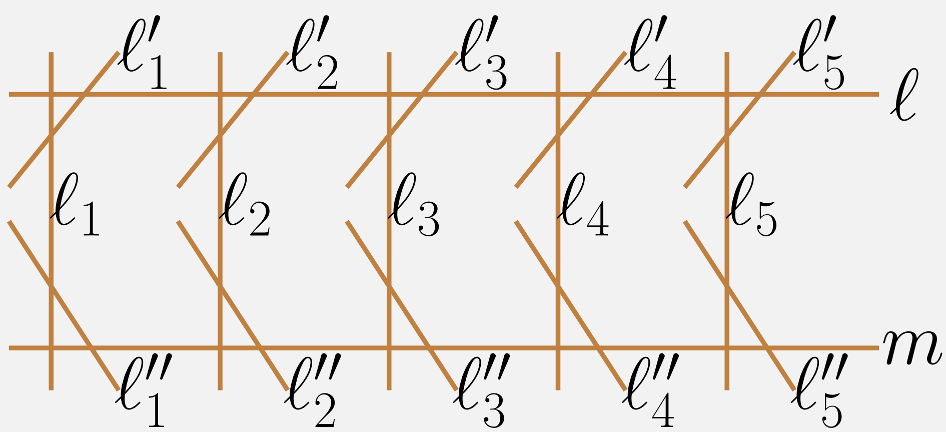

Configuration of lines on a smooth complex cubic surface

Submitted to the Department of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

Master Project in Pure Mathematics **University of Nairobi**
June 2021

Lines on Cubic Surface

Research Report in Mathematics, Number 04, 2021

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Master Thesis

Submitted to the Department of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

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Abstract

In this dissertation, we enumerate the 27 lines on a smooth cubic surface $X \subset \mathbb{P}^3$. We do this by understanding the combinatorics of the subset S of disjoint lines on X of the Grassmanian $Gr(2,4)$ of lines on $\mathbb{P}^3$. Further, using the incidence correspondence defined by the projection $(X,\ell) \mapsto X$ where ℓ is a line on X, we show that the relation is a 27-sheeted covering map by studying the inverse image of lines on a smooth cubic X under the blowup map

$$B\ell_6 : B\ell_6\mathbb{P}^2 \dashrightarrow X.$$

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

	08/09/2021
Signature	Date

FREDRICK KOECH
Reg No. I56/34301/2019

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

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Dedication

I dedicate this project to my lovely parents and colleagues.

Contents

Acknowledgments

I would like to start by expressing profound thanks and praises to God for the blessings in His answers to my silent plea about and throughout this dissertation.

My deep and sincere gratitude is hereby extended to my supervisors Jared Ongaro and Benjamin Kikwai for their guidance and encouragement which have resulted in this project. I am greatly indebted to Geoffrey Mboya for his selfless and invaluable guidance and support towards the completion of this project.

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Last but not least, I acknowledge the moral, financial and spiritual support of my family which turned the challenging road towards completion of this dissertation an adventurous part of my life. My gratitude to my parents Julius Maiyo and Monica Chelimo, sisters Beatrice Koech and Mercy Koech and brothers Boniface Biwott, Edwin Kipkemboi and Fr Kevin Kipkorir.

Fredrick KOECH

Nairobi, 2021.

1 Introduction

The main goal of this thesis is to study the configuration of lines on a nonsingular complex cubic surfaces and be able to demonstrate that there are 27 lines on such a surface.

We do this in two ways

(i) We explore the geometry of Grasmaniann $Gr(2,4)$ of Lines on $\mathbb{P}^3$ through the Plűcker embedding

$$Gr(2,4) \hookrightarrow \mathbb{P}^5$$

and the representation of lines in $Gr(2,4)$. We then show that there is at least a line on a cubic surface X in $\mathbb{P}^3$ and that one can find two such lines ℓ and $\mathfrak{m}$ which are disjoint on X. Further, we demonstrate that the set of lines intersecting a given arbitrary line ℓ in $\mathbb{P}^3$ is a subset of $Gr(2,4)$ and that the set has exactly 5 pairs of disjoint lines. We also appreciate that if one line of the 5 pairs of lines that intersect ℓ also intersects a line $\mathfrak{m}$ disjoint to ℓ, then 5 disjoint lines of the five pairs of lines also do intersect $\mathfrak{m}$. This gives 17 disjoint lines on X: ℓ, $\mathfrak{m}$, the 5 disjoint lines intersecting both ℓ and $\mathfrak{m}$, the 5 disjoint lines that only intersect ℓ and the other 5 disjoint lines that only intersect $\mathfrak{m}$. We then convince the reader and ourselves that any line on $X \setminus \{\text{The 17 lines above}\}$ would intersect exactly 3 of the five lines intersecting the line ℓ; for if more, the line would be ℓ or $\mathfrak{m}$ and if less then the line would intersect atleast 3 of the five lines. There are therefore $\binom{5}{3} = 10$ disjoint lines on $X \setminus \{\text{The 17 lines above}\}$.

(ii) we appreciating that a smooth cubic is birational to $\mathbb{P}^2$ and that the blow up $B\ell_6\mathbb{P}^2$ of $\mathbb{P}^2$ at 6 points in general position is isomorphic

$$B\ell_6 : B\ell_6\mathbb{P}^2 \dashrightarrow X$$

to a smooth cubic X. We further demonstrate that the space of smooth cubics is a dense open subset U of $\mathbb{P}^{19}$. Now by taking a subset M of the product $U \times Gr(2,4)$ consisting of pairs (X,ℓ) of a smooth cubic X and a Line ℓ on X, we define the incidence correspondence

$$\pi : M \to U$$

by the projection $(X,\ell) \mapsto X$. The goal of the thesis is then equivalent to counting the number of inverse images of π ; that is, showing that π is a 27-sheeted covering map. Finally, we count these inverse images of π by counting the images under the blow-up map $B\ell_6$, of the 6 exceptional hypersurfaces, the strict transform of

$\binom{6}{2} := 15$ lines through any two of the six blown up points and the strict transform of $\binom{6}{5} := 6$ conics through any five of the six blown up points.

The outline of the thesis is as follows:

Chapter 2:
In this chapter, we introduce algebra-geometry dictionary and the geometric object of our study: *Irreducible Smooth* cubic.
Chapter 3:
This chapter sets up the stage for the thesis' enumerative goal by focusing on notions such as blowup of $\mathbb{P}^2$ at 6 points, space of smooth cubics and the geometry of Grassmanian $Gr(2,4)$ of lines on $\mathbb{P}^3$.
Chapter 4:
Here, we enumerate lines on smooth cubic in two ways: though combinatorics of lines on smooth cubic or through the analysis of the preimage, through the blowup map

$$B\ell_6 : B\ell_6\mathbb{P}^2 \dashrightarrow X,$$

of lines on the smooth cubic X.

2 Preliminaries

This chapter is a quick basic overview based on [SKKT20], [GethA02] and [Sha13] of the subject: Algebraic Geometry. This is intended to set up notation as well as introduce the object study, smooth cubic in $\mathbb{P}^3$. We will occasionally remind ourselves some Commutative Algebra or refer the reader to [AM69].

2.1 What is Algebraic Geometry?

Let k be an algebraically closed field of characteristic 0.

Definition 2.1.1. *[Projective Space] For a positive integer n, an affine n-space $\mathbb{A}^n$ is the coordinate space $V = k^n$ without a vector space structure. Projective n-space is the quotient*

$$\mathbb{P}^n = \mathbb{P}(k^{n+1}) = k^{n+1}_{x_j} \setminus \{\bar{0}\}/k^*$$

with a point $p = [x_0 : x_1 : \ldots : x_n] = \{(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) \in k^{n+1} \setminus \{\bar{0}\} : \lambda \in k^*\}$ in $\mathbb{P}^n$ is the equivalence class of the nonzero vector $(x_i) \in k^{n+1}$. If we are not in doubt of the identity of the underlying field, we would write $\mathbb{P}^n$ for $\mathbb{P}^n_k$.

The projective n-space can be considered as the parameter space of lines through the origin in affine $n+1$ space or the affine n-spaces with it's $n-1$ compactifying hyperplanes at infinity. That is, on the j-th affine chart $\{x_j \neq 0\}$ for example,

$$\mathbb{P}^n_{<x_0,x_1,\ldots,x_n>} = \{x_j \neq 0\} \cup \{x_j = 0\} = \mathbb{A}^n_{\left\langle \frac{x_0}{x_j}, \frac{x_1}{x_j}, \ldots, \widehat{\frac{x_j}{x_j}}, \ldots, \frac{x_n}{x_j} \right\rangle} \cup \mathbb{P}^{n-1}_{\left[x_0 : x_1 : \ldots : \widehat{x_j} : \ldots : x_n\right]}$$

Therefore, each chart $\{x_j \neq 0\}$ for $j = 0, \ldots, n$ gives an embedding

$$\tau : \{x_j \neq 0\} = \mathbb{A}^n \hookrightarrow \mathbb{P}^n$$

$$(y_0, \ldots : y_{j-1}, y_{j+1}, \ldots, y_n) \mapsto [y_0 : \ldots : y_{j-1} : 1 : y_{j+1} : \ldots : y_n].$$

2.1.1 Affine and Projective Algebraic Varieties

Proposition 2.1.2. *Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in finite variable x_i over algebraically closed field k and S a finitely generated k-algebra. Then*

$$S \cong R/I$$

for an ideal $I \triangleleft R$.

Proof. Let $s_1, \dots, s_n \in S$ be k-algebra generators of S. The ring homomorphism

$$\varphi : R \to S$$

defined by $\varphi(x_j) = s_j$ surjective since s_j generate S. By considering $I = \ker \varphi \triangleleft R$, the result follows from The First Isomorphism Theorem of Rings. $\square$

Definition 2.1.3. *A subset $X \subset \mathbb{A}^n_k$ is an affine variety if $X = \mathbb{V}(I)$ for some ideal I of $R = k[x_1, \dots, x_n]$ where the vanishing set $\mathbb{V}(I)$ of I is defined as*

$$\mathbb{V}(I) = \{(a_1, \dots, a_n) = a \in \mathbb{A}^n_k : f(a) = 0 \text{ for all } f \in I \triangleleft R = k[x_1, \dots, x_n]\} \subseteq \mathbb{A}^n_k.$$

Example 2.1.4. *[Some Basic Vanishing Sets]*

1. *For the Zero ideal, $\mathbb{V}(0) = \mathbb{A}^n_k$.*
2. *For the ideal $(1) = R$, $\mathbb{V}(R) = \emptyset$.*
3. *For a nonconstant polynomial $f \in R \setminus k$ generating a principle ideal $(f) \triangleleft R$, we get the hypersurface*

$$V_f = \mathbb{V}((f)) = \{(a_1, \dots, a_n) = a \in \mathbb{A}^n_k : f(a) = 0\} \subset \mathbb{A}^n_k$$

   defined by zeros of f.

4. *Let $(a_1, \dots, a_n) = a \in \mathbb{A}^n_k$ be a point in k^n and define*

$$\mathfrak{m}_a = (x_1 - a_1, \dots, x_n - a_n) \triangleleft R = k[x_1, \dots, x_n].$$

   We can realize $\mathfrak{m}_a$ as the kernel $\mathfrak{m}_a = \ker ev_a$ of the evaluation-at-$'a'$ map $ev_a : R \to k$ defined by $f \mapsto f(a)$. By the Proposition 2.1.2 above, we have that $R/\mathfrak{m}_a \cong k$ so that $\mathfrak{m}_a$ is a maximal ideal of R corresponding to points of $\mathbb{A}^n_k$.

Theorem 2.1.5. *[Week Nullstellensatz]*
Assume k is algebraically closed. Then every maximal ideal of $R = k[x_1, \dots, x_n]$ is of the form $\mathfrak{m}_a = (x_1 - a_1, \dots, x_n - a_n) \triangleleft R$ for some $(a_1, \dots, a_n) = a \in \mathbb{A}^n_k$.

Example 2.1.6. *Let $k = \mathbb{R}$ which is not algebraically closed. The principle ideal $I = (x^2 + 1) = ker \left(ev_i : \mathbb{R}[x] \xrightarrow{f \mapsto f(i)} \mathbb{C} \right) \triangleleft \mathbb{R}[x]$. Since $\mathbb{R}[x]/I \cong \mathbb{C}$ is a field, we have that I is maximal so that not all maximal ideals of $\mathbb{R}[x]$ are of the form proposed in Theorem 2.1.5 .*

Lemma 2.1.7. *[Hilbert Basis Theorem]*
The polynomial ring $R = k[x_1, \dots, x_n]$ is a Noetherian Ring. That is

(i) Every ideal I of R is finitely generated. That is

$$I = \langle f_1, \dots, f_m \rangle = \left\{ \sum_{i=1}^{m} r_i f_i : r_i \in R \text{ and finitely many generators } f_i \in R \right\}$$

(ii) Ascending Chain Condition on Ideals (ACC): Every Ascending chain of ideals $I_1 \subset I_2 \subset \dots$ terminates. That is, eventually $I_N = I_{N+1} = \dots$

That $R = k[x_1, \dots, x_n]$ is Noetherian is very convenient for us since consequently

- every quotient R/I is Noetherian where $I \lhd R$ is an ideal of R, hence every finitely generated k-algebra is Noetherian and
- every ideal $I \lhd R$ is contained in a maximal ideal thanks to ACC condition on ideals of R.

2.1.2 The Zariski Topology on Affine Varieties

Proposition 2.1.8. *[Basic Properties of Vanishing Sets]*
It is easy to show that

(i) $\mathbb{V}$ is inclusion reversing. That is, for ideals $I \subset J \implies \mathbb{V}(I) \supset \mathbb{V}(J)$.

(ii) Finite union of varieties is a variety. That is,

$$\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I.J) = \mathbb{V}(I \cap J)$$

and by induction on n that

$$\bigcup_{i=1}^{n} \mathbb{V}(I_i) = \mathbb{V}\left(\bigcap_{i=1}^{n} I_i \right).$$

(iii) Arbitrary Intersection of varieties is a variety. That is, let $\{I_\alpha | \alpha \in \Omega\} \subset R$ be an arbitrary collection of ideals then

$$\bigcap_{\alpha \in \Omega} \mathbb{V}(I_\alpha) = \mathbb{V}\left(\left\langle \bigcup_{\alpha \in \Omega} I_\alpha \right\rangle \right) = \mathbb{V}\left(\sum_{\alpha \in \Omega} I_\alpha \right).$$

(iv) $V(I) \cap \mathbb{V}(J) = \mathbb{V}(\langle IUJ \rangle) = \mathbb{V}(1+J) = \emptyset \iff I$ *and* J *are coprime* $I+J=(1)$.

Lemma 2.1.9. *Zariski Topology on* $\mathbb{A}_k^n$ *is defined by setting the affine varieties* $\mathbb{V}(I) \subseteq \mathbb{A}_k^n$ *as its closed sets.*

Proof. This is a consequence of Proposition 2.1.8 . □

We note that the open sets of this topology look like this:

$$\begin{aligned}
U_I &= k^n \setminus \mathbb{V}(I) \\
&= k^n \setminus (\mathbb{V}(f_1) \cap \ldots \cap \mathbb{V}(f_m)); \text{ Theorem 2.1.7 and Proposition 2.1.8 (iii)} \\
&= (k^n \setminus (\mathbb{V}(f_1)) \cup \ldots \cup (k^n \setminus (\mathbb{V}(f_m)); \text{ De Morgan set law} \\
&= D_{f_1} \cup \ldots \cup D_{f_m},
\end{aligned}$$

where

$$D_f = k^n \setminus (\mathbb{V}(f) = \{(a_1, \ldots, a_n) = a \in k^n : f(a) \neq 0\}$$

is a (very large) basic open set.

We formally define affine n-space as the topological space $\mathbb{A}_k^n := (k^n,\ \text{Zariski Topology})$ whereas an affine variety $X = \mathbb{V}(I) \hookrightarrow \mathbb{A}_k^n$ is equipped with Zariski topology induced from $\mathbb{A}_k^n$. The closed subsets $\mathbb{V}(J) = Y$ of $X = V(I) \subset \mathbb{A}_k^n$ corresponds to ideals J containing I.

Example 2.1.10.

1. *Let* k *be algebraically closed field. Since* $k[x]$ *is a PID, we have that the Zariski closed subsets of* $\mathbb{A}_k^1$ *are* $\emptyset, \mathbb{V}(f) = \{m \text{ roots } a_1, \ldots, a_m \text{ of } f_m = f \in [x]\}$ *and* $\mathbb{A}_k^1$.

2. *On* $\mathbb{A}_k^2$ *the Zariski closed subsets are* $\emptyset$, *hypersurfaces* $\mathbb{V}(f)$, *finite set of points*

$$\mathbb{V}\left(\prod_{i=1}^{m}(x_i - a_i, y_i - b_i)\right) = \{(a_1, b_1), \ldots, (a_n, b_n)\}$$

and the union of hypersurfaces and finite points. There are no others and this is why: Take polynomials f, g *with no common factor in a PID* $k[x][y]$ *(respectively in a PID* $k[y][x]$*). By Bezout's Theorem followed by rescaling by* $c \in k[x]$ *of some degree* m*, one can write* $fp + gq = c$ *for some* $p, q \in k[x]$ *(respectively* $p, q \in k[y]$*). We then have that* $V(f, g) \subset \mathbb{V}(c) = \{(a_1, 0), \ldots, (a_m, 0)\}$ *finite choice of points on x-axis (respectively* $V(f, g) \subset \mathbb{V}(c) = \{(0, b_1), \ldots, (0, b_m)\}$ *finite choice of points on y-axis) of the* k^2 *plane.*

2.1.3 The Ideal $\mathbb{I}$ -Variety $\mathbb{V}$ Correspondence

Definition 2.1.11. *[Vanishing Ideal]*
The vanishing ideal $\mathbb{I}(X)$ of a subset $X \subset \mathbb{A}_k^n$ consists of functions

$$\mathbb{I}(X) = \{f \in R : f(a) = 0 \text{ for all } a \in X\}$$

on $\mathbb{A}_k^n$ that vanish on X.

Example 2.1.12.

1. $\mathbb{I}(\mathbb{A}_k^n) = R$.
2. $\mathbb{I}(\{a\}) = \mathfrak{m}_a$.
3. $\mathbb{I}(\mathbb{V}(x^2)) = (x)$, *hence* $\mathbb{I}(\mathbb{V}(I)) \neq I$ *in general.*
4. $f^m \in \mathbb{I}(X) \implies f \in \mathbb{I}(X)$ *so that* $\mathbb{I}(X)$ *is a radical ideal.*

Proposition 2.1.13. *[Properties of Vanishing ideals]*

1. $\mathbb{I}$ *is inclusion reversing. That is,* $X \subset Y \implies \mathbb{I}(X) \supset \mathbb{I}(Y)$.
2. $I \subset \mathbb{I}(\mathbb{V}(I))$, *for example* $(x^2) \subset \mathbb{I}(\mathbb{V}((x^2)))$.
3. $\mathbb{V}(\mathbb{I}(\mathbb{V}(I))) = \mathbb{V}(I)$ *and consequently* $\mathbb{V}(\mathbb{I}(X)) = X$ *for any affine variety* X.

Theorem 2.1.14. *[Hilbert's Nullstellensatz]*
Let k be algenraically closed field. For any ideal $I \lhd R = k[x_1, \ldots, x_n]$, we have

$$\mathbb{I}(\mathbb{V}(I)) = \sqrt{I} = \{f \in R : f^m \in I \text{ for all } m \geq 1\}.$$

In particular, $\mathbb{I}(\mathbb{V}(I)) = I$ if R/I has no nilpotents elements.

Definition 2.1.15. *[Coordinate Ring k[X] of affine variety X]*
Let $X \subset \mathbb{A}_k^n$ and $R = k[x_1, \ldots, x_n]$. Coordinate ring of X is the quotient

$$k[X] = R/\mathbb{I}(X)$$

representing polynomial functions on $\mathbb{A}_k^n$ that vanish on X.

Remark 2.1.16. *Coordinate ring $k[X]$ has the properties*

1. *it is finitely generated (Noetherian) reduced k-algebra.*
2. *it is an integral domain if and only if X is irreducible.*

3. *if k is algebraically closed then $k[\{a\}] \cong k$ and it is the only case when coordinate ring is a field.*

4. $k[\mathbb{A}_k^n] = R/\mathbb{I}(0) \cong R$ *whereas* $k[\mathbb{V}(f)] = R/(f) \cong \left\langle \bar{1}, \bar{f}, \ldots, \bar{f}^{\deg f - 1} \right\rangle$.

Definition 2.1.17. *[Irreducibility]*
An affine variety X is said to be reducible if $X = X_2 \cup X_2$ for proper closed subsets $X_i \subsetneq X$. Otherwise, X is irreducible in which case

- $\mathbb{I}(X) \triangleleft R$ *is a prime ideal on condition that* $X \neq \emptyset$.
- *any nonempty open subset $U \neq \emptyset$ of X is dense.*
- *any two nonempty open subsets $U, V \subset X$ intersect $U \cap V \neq \emptyset$.*
- *it is necessary and sufficient that $k[X]$ is an integral domain.*

Example 2.1.18.

1. $X = \mathbb{V}(xy) = \mathbb{V}(x) \cup \mathbb{V}(y) \subset \mathbb{A}_k^1$ *is reducible as it can be expressed as union of two coordinate axes in k^2.*

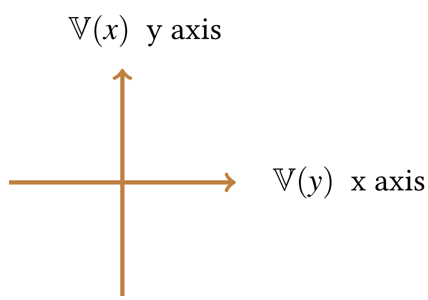

Figure 1. The reducible variety $X = \mathbb{V}(xy)$.

2. $X = \mathbb{V}(< xy, xz >) = \mathbb{V}(x) \cup \mathbb{V}(y,z)$ *is reducible as it can be expressed as union of yz-plane and x-axes. We know that $R/(y,z) \cong k[x]$ is an integral domain hence $\mathbb{V}(y,z)$ is irreducible.*

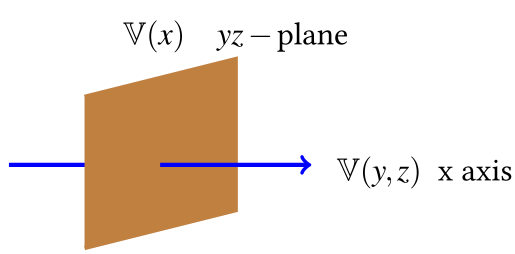

Figure 2. The reducible variety $X = \mathbb{V}(xy, xz)$.

Corollary 2.1.19. *[The $\mathbb{I}$ - $\mathbb{V}$ Correspondence]*
There are order-reversing bijections between ideals $I \lhd R$ and affine varieties X. That is

$$\begin{aligned}\{\textit{varieties}\} &\longleftrightarrow \{\textit{radical ideals}\}\\ \{\textit{irreducible varieties}\} &\longleftrightarrow \{\textit{prime ideals } \mathfrak{p} \lhd k[X]\} = \mathbf{Spec}(R)\\ \{\textit{points}\} &\longleftrightarrow \{\textit{maximal ideals } \mathfrak{m}_a \lhd k[X]\} = \mathbf{Spec}\, m(R)\\ X &\mapsto \mathbb{I}(X)\\ \mathbb{V}(I) &\leftarrow\!\shortmid I.\end{aligned}$$

2.1.4 Nonsingularity of Algebraic Varieties

Definition 2.1.20. *Let q be a point in a projective variety X then the* ***tangent space*** *to X at q is the projective variety*

$$T_qX := \bigcap_{f \in I(V)} \mathbb{V}\left(\sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(q).x_i\right)$$

When f is homogeneous polynomial of degree $d-1$ then each $\frac{\partial f}{\partial x_i}$ will be either 0 of an homogeneous of degree $d-1$.
Taking a different representation of q as $(bq_0 : \ldots : bq_n)$ will scale each term of

$$\left(\sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(q).x_i\right)$$

by a constant term b^{d-1} that is it will scale

$$\left(\sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(q).x_i\right)$$

by a constant and hence not affecting it is zero set.

Example 2.1.21. *A projective line ℓ is it is own tangent space at any point of ℓ, for a change of coordinates ,any $\ell \in \mathbb{P}^n$ is simply a variety cut out by the $(n-1)$ polynomials $(x_2, \ldots, x_n)$ that is $\ell := \mathbb{V}(x_2, \ldots, x_n) := \mathbb{V}((x_2, \ldots, x_n))$,here the last quantity denote the variety associated to the ideal generated by $(x_2, \ldots, x_n)$*
Observe that $\mathbb{I}(\ell) := \mathbb{I}(\mathbb{V}((x_2, \ldots, x_n)) := (x_2, \ldots, x_n)$ by homogeneous Nullstellensatz hence

$$T_q\ell := \bigcap_{f \in (x_2, \ldots, x_n)} \mathbb{V}\left(\sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(q).x_i\right) := \bigcap_{j=2}^{n} \mathbb{V}(x_j) := \mathbb{V}(x_2, \ldots, x_n) := \ell$$

Definition 2.1.22. *The **tangent plane** T_qX, of $X = \mathbb{V}(f_1, \ldots, f_m) \subset \mathbb{P}^n$ at q is given by*

$$T_qX = \mathbb{V}\left(\left.\frac{\partial f_1}{\partial x_0}\right|_q, \ldots, \left.\frac{\partial f_1}{\partial x_n}\right|_q, \ldots, \left.\frac{\partial f_m}{\partial x_0}\right|_q, \ldots, \left.\frac{\partial f_m}{\partial x_n}\right|_q \right).$$

*The points q is called **smooth** when the tangent plane at q is defined while the points is called is **singular** when the tangent plane at q is not defined.*

Neatly and alternatively, we have the following definition.

Definition 2.1.23. *A point $q \in X := \mathbb{V}(f_1, \ldots, f_m) \subset \mathbb{P}^n$ is **nonsingular** if the **Jacobian***

$$J(X)_P := \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_0}\right|_q & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_q \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial f_m}{\partial x_0}\right|_q & \cdots & \left.\frac{\partial f_m}{\partial x_n}\right|_q \end{bmatrix}_{m \times (n+1)}$$

of X at q if the rank of $J(X)_q := m$. Otherwise q is singular. X is nonsingular (smooth) if it has no singular points.

An isolated singularity is a singularity for which there exists a small real number δ such that there are no other singularity within a neighborhood of radius δ centered about the singularity also known as conic double points.
The simple singularities is the ordinary double point where double point occurs at the point $q = (0,0,1)$ on quadratic cone X given by the equation $f(x,y,z) = x^2 + y^2 - z^2$. The union of all generating lines meeting in the singular point q is quadratic cone. The surface becomes cylinder by taking a blow up of X at q with disjoint union of lines of X as it is underlying set hence q is blown up to a circle on the cylinder which is exceptional divisor of the blow up.

Example 2.1.24.

i. *Fermat cubic $X_3 := \mathbb{V}(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subset \mathbb{P}^3$ is nonsingular.*

ii. *Nodal cubic $X_3 = \mathbb{V}(x^3 + x^2 - y^2) \subset \mathbb{A}^2$ is singular at $(0,0)$.*

iii. *The double cone $\mathbb{V}(x^2 + y^2 - z^2) \subset \mathbb{A}^3$ is singular point at $(0,0,0)$, a point where the tangent plane is not defined.*

2.2 Blowup of Algbraic Varieties at Points

Definition 2.2.1. *The blowup of $B = B\ell_0\mathbb{A}^n$ at the origin $0 = (0, \ldots, 0) \in \mathbb{A}^n$ of $\mathbb{A}^n$ is a the variety defined by*

$$B = \{(p, \ell) | p \in \ell\} \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$$

together with the projection

$$\pi : B \longrightarrow \mathbb{A}^n$$

which is one-to-one over $\mathbb{A}^n \setminus \{0\}$. In coordinates, we have that

$$\begin{aligned} B = & \left\{ ((x_1, \cdots, x_n); [y_1, \cdots, y_n]) \middle| rnk \begin{bmatrix} x_1 \cdots x_n \\ y_1 \cdots y_n \end{bmatrix} \leq 1 \right\} \\ = & \mathbb{V}\left(2 \times 2 \textit{ minors of } \begin{bmatrix} x_1 \cdots x_n \\ y_1 \cdots y_n \end{bmatrix} \right) \\ = & \mathbb{V}(\{x_i y_j - x_j y_i | i \leq i < j \leq n\}) \subset \mathbb{A}^n_{(x_i)} \times \mathbb{P}^{n-1}_{[y_i]} \end{aligned}$$

Remark 2.2.2. *The projection map π is a birational map,that is to say π is rational map with a rational inverse*

$$\pi^{-1} : \mathbb{A}^n \dashrightarrow B \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$$

$$(x_1, \ldots, x_n) \longmapsto ((x_1, \ldots, x_n) : [x_1 : \cdots : x_n]).$$

ii. The blowup B is the graph of the rational map

$$\rho : \mathbb{A}^n \dashrightarrow \mathbb{P}^{n-1}$$

$$(x_1, \ldots, x_n) \longmapsto [x_1 : \ldots x_n].$$

We then have, from the composition

$$B \xrightarrow{\pi} \mathbb{A}^n \xrightarrow{\rho} \mathbb{P}^{n-1}$$

of projection to $\mathbb{A}^n$ with the graph, that intuitively; the blowup $B = B\ell_0\mathbb{A}^n$ is like $\mathbb{A}^n$ except at the origin 0 where in B, the origin 0 is replace by the set of all directions approaching the origin.

Proposition 2.2.3. *B is a smooth (irreducible) variety of the dimension n.*

`Proof.` We have

$$B \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1} \supseteq (\mathbb{A}^n \times U_i),$$

where

$$U_i = \mathbb{A}^{n-1}$$

is a standard affine chart. It suffices to check that each

$$B \cap (\mathbb{A}^n \times U_i)$$

is smooth.

For simplicity, we do the case $i = n$ □

Proposition 2.2.4. $B \cap (\mathbb{A}^n \times \mathbb{A}^{n-1}) \cong \mathbb{A}^n$

`Sketch Proof.` Observe that

$$B \cap (\mathbb{A}^n \times \mathbb{A}^{n-1}) = \{(x_1, \cdots, x_n); [y_1 : \cdots : y_n] | y_n \neq 0, x_i y_j = x_j y_i\}$$

$$= \left\{ (x_1, \cdots, x_n); \left[\frac{y_1}{y_n} : \cdots : \frac{y_n - 1}{y_n} : 1 \right] \middle| x_j = x_n(\frac{y_j}{y_n}) \right\}.$$

We have an isomorphism

$$\rho : B \cap U \longrightarrow \mathbb{A}^n$$

defined by

$$\left((x_1, \cdots, x_n); \left[\frac{y_1}{y_n} : \cdots : \frac{y_n - 1}{y_n} : 1 \right] \right) \longmapsto \left(\frac{y_1}{y_n} : \cdots : \frac{y_n - 1}{y_n} : x_n \right)$$

with inverse

$$\rho^{-1} : \mathbb{A}^n \longrightarrow B \cap U$$

defined by

$$((t_n t_1, \cdots, t_n t_{n-1} t_n, t_n); [t_1 : \cdots : t_{n-1} : 1] \longleftarrow (t_1, \cdots, t_{n-1}, t_n).$$

□

Definition 2.2.5. *Blowup of a variety $X \subset \mathbb{A}^n$ at a point $p \in X$ is the the closure of the inverse image of $X \setminus \{p\}$ under the projection $\pi : B\ell_p\mathbb{A}^n \dashrightarrow \mathbb{A}^n$ with projective dimension $n-1$ **exceptional locus** $E_p(X) = \pi^{-1}(p) \subset B\ell_p\mathbb{A}^n$.*

Example 2.2.6. *We consider the blowup a point*

$$p := (0,0,0) \in X := V(x^2+y^2-z^2) \subseteq \mathbb{C}^3_{(x,y,z)}$$

$$\pi^{-1}(X \setminus \{p\}) := \left\{ (\bar{x} = (x,y,z), \ell) \in \mathbb{C}^3_{(x,y,z)} \times \mathbb{P}^2_{[u:v:w]} \middle| \bar{x} \in \ell, x^2+y^2-z^2 = 0 \right\}$$

Using projective coordinates $[u : v : w]$ and blowing up along the chart $w \neq 0$ We obtain $z^2(u^2+v^2-1) = 0 \cong z^2(x^2+y^2-1) = 0$ thus the closure $\overline{\pi^{-1}(X \setminus \{p\})} \subset Bl_0\mathbb{C}^3$ given by $V(x^2+y^2-1)$ is a circle.

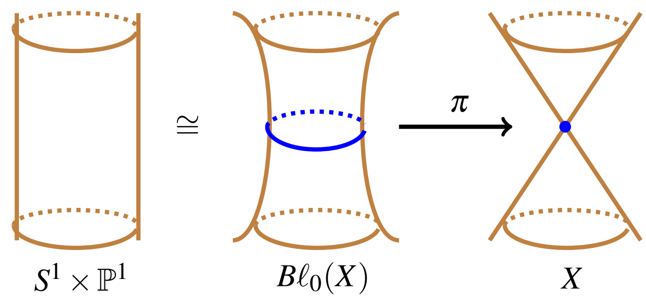

Figure 3. Blowup of a double cone at the origin is a cylinder

3 The Geometry of Lines on Smooth Cubics in $\mathbb{P}^3$

3.1 The Grassmannian $Gr(2,4)$ of lines in $\mathbb{P}^3$ and Some Useful Enumerative Lemmas

Definition 3.1.1. *As a set, the Grassmannian of $d-$ planes in $\mathbb{K}^n$ is*

$$Gr(d,n) = \{d - \textit{dimensional vector spaces } U \subset \mathbb{K}^n\}$$

Example 3.1.2.

1. $Gr(1,n) = \mathbb{P}(k^n) \cong \mathbb{P}^{n-1}$
2. $Gr(n-1,n) = \mathbb{P}((k^n)^*) \cong \mathbb{P}^{n-1}$
3. *Projective duality more generally says*

$$Gr(d,n) \cong Gr(n-d,n)$$

The above examples suggest that the Grassmannian $Gr(d,n)$ may always have the structure of a projective variety. We will shortly demonstrate this and work out some equations, at least in the simple case $Gr(2,4)$.

Proposition 3.1.3. *There is a one-to-one correspondence*

$$Gr(d,n) \longleftrightarrow Max(d,n)/GL(d)$$

where $Max(d,n) \subset M_k(d,n)$ is the subset of matrices of maximal rand d.

Example 3.1.4. *When $d = 1$, this says*

$$Gr(1,n) \longleftrightarrow Max(1,n)/GL(1).$$

Here $Max(1,n)$ is row vectors except the zero vector so

$$Max(1,n) = k^n \backslash 0.$$

Also

$$GL(1) = k^*.$$

We recover

$$\mathbb{P}^{n-1} \longleftrightarrow (k^n \backslash 0)/k^*.$$

3.1.1 The Plűcker embedding

Theorem 3.1.5. *Associating to a subspace U the collection $d \times d$ minors of its representing matrix A_U gives a closed embedding*

$$Gr(d,n) \hookrightarrow \mathbb{P}^{\binom{n}{d}-1},$$

whose image is a projective subvariety of $\mathbb{P}^{\binom{n}{d}-1}$. In particular, $Gr(d,n)$ is projective.

We will not discuss the general proof. Instead, we will look at the simplest nontrivial example.

If $n \leq 3$, all Grassmannian are either points or projective spaces. So the first interesting case when $n = 4$ and $d = 2$.

Example 3.1.6. *Let $d = 2,\ n = 4$. The Plűcker map is $Gr(2,4) \to \mathbb{P}^5$ given in matrix form by*

$$\left[\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}\right] \longmapsto [af - be : ag - ce : ah - de : bg - cf : bh - df : ch - dg]$$

Lemma 3.1.7. *The plűcker map is an embedding*

$$Gr(2,4) \hookrightarrow \mathbb{P}^5$$

with image

$$\mathbb{V}(y_0 y_5 - y_1 y_4 + y_2 y_3) \subset \mathbb{P}^5.$$

`Proof.` It is easy to check that the image of the plucker map satisfy this quadratic relation, the Plűcker relation. To complete the proof, we consider affine charts.

The Plűcker map of $Gr(2,4)$ has image contained in

$$\mathbb{V}(y_0 y_5 - y_1 y_4 + y_2 y_3) \subset \mathbb{P}^5.$$

Let us consider the affine open set.

$$Gr(2,4)_0 \subset \mathbb{V}(y_0y_5 - y_1y_4 + y_2y_3) \cap \{y_0 \neq 0\} \subset \{y_0 \neq 0\} = \mathbb{A}^5 \subset \mathbb{P}^5.$$

In the matrix coordinates used before, this means $af - be \neq 0$. So for the corresponding 2-dimensional subspace $U \subset k^4$, the first two columns of the matrix A_U are linear independent. This means that we can pre-multiply the matrix A_U by a unique change of basis matrix P so that the first two columns become the standard basis vectors of a 2-dimensional vector space.

We get an equivalence of matrices

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & C & D \\ 0 & 1 & G & H \end{pmatrix}$$

For $U \in Gr(2,4)_0$, we have the representing matrix,

$$\begin{pmatrix} 1 & 0 & C & D \\ 0 & 1 & G & H \end{pmatrix}$$

we can then read off the affine Plucker coordinates of this subspace U as

$$U \longmapsto (G, H, -C, -D, CH - DG).$$

We deduce the following

1. The affine plűcker relation

$$y_5 - y_1y_4 + y_2y_3 = 0$$

   indeed holds.

2. There are no further equations involving the plűcker coordinates.

3. In this open set, we recover the subspace U uniquely from its plűcker image.

Considering all such affine charts, we deduce that over the whole Grassmannian $Gr(2,4)$, the plűcker map is an embedding, and its image equals

$$\mathbb{V}(y_0y_5 - y_1y_4 + y_2y_3) \subset \mathbb{P}^5.$$

$\square$

3.1.2 Irreducibility

Theorem 3.1.8. *The Grassmannian $Gr(d,n)$ is an irreducible variety.*

We need a lemma.

Lemma 3.1.9. *Let $GL_n(k) \subset \mathbb{A}^{n^2}$ be the space of invertible linear matrices inside the affine space of all $(n \times n)$ matrices over k. Then $GL_n(k)$ is an irreducible affine variety.*

`Proof.` Let $\triangle \in k[\mathbb{A}^{n^2}]$ be the determinant polynomial on the space of matrices. Then

$$GL_n(k) = D_\triangle,$$

the principal open subset defined by the non-vanishing of $\triangle$. The first statement is then a general instance of the phenomenon that a basic open set is an affine variety is affine.

Also $GL_n(k) = D_\triangle \subset \mathbb{A}^{n^2}$ i.e dense, as affine space $\mathbb{A}^{n^2}$ is irreducible. A dense subset of an irreducible variety must itself be irreducible. This concludes the proof. $\square$

`Proof.` (Proof of theorem 3.1.8) We define a surjective polynomial

$$\rho : GL_n(k) \longrightarrow Gr(d,n).$$

A surjective image of an irreducible variety must be irreducible, so the existence of the map ρ proves the irreducibility of $Gr(d,n)$.

Inside the vector space k^n with fixed basis $\{e_1, \cdots, e_n\}$. Let $W = \langle e_1, \cdots, e_d \rangle$ be a reference $d-$dimensional subspace.

Suppose that $V \subset k^n$ is an arbitrary $d-$dimension linear subspace. Choose a basis $\{v_1, \cdots, v_d\}$ for it, and complete to a basis $\{v_1, \cdots, v_n\}$ of k^n.

Then $A \in GL_n(k)$ with column v_i will map W to V. This defines the surjective polynomial map ρ. This concludes the proof.

$\square$

Lemma 3.1.10. *Let ℓ_1, ℓ_2 be two distinct lines in $\mathbb{P}^3$ then they intersect at a unique point.And if Π is a projective plane such that $\ell \not\subseteq \Pi$ then ℓ intersects Π at a unique point.*

`Proof.` We suppose that the projective plane Π is $\mathbb{P}^2$ by change of coordinates and since $\ell_1 \neq \ell_2$, by another change of coordinates we let $\ell_1 = \mathbb{V}(x_2)$ and $\ell_2 = \mathbb{V}(x_1)$.

Therefore we get the unique point in $\ell_1 \cap \ell_2$ to be $(1:0:0)$. On the other hand consider,$\ell \not\subseteq \Pi$ then by suitable change of coordinates we take $\Pi := \mathbb{V}(x_3)$ and $\ell = V(x_1, x_2)$, thus $(1:0:0:0)$ is a unique point on $\ell \cap \Pi$. $\square$

Lemma 3.1.11. *Consider a quadratic form f in $K[x,y,z,t]$ passing through quadratic surface $Q \in \mathbb{P}^3$ then there exist a matrix $N = N^T$ in $N_4(K)$ such that $f(x) = X^T N X$ for each $x \in K^4$. And Q is singular whenever N is singular.*

`Proof.` Let

$$f(x,y,z,t) = Ax^2 + By^2 + Cz^2 + Dt^2 + Exy + Fxz + Gxt + Hyz + Iyt + Jzt.$$

Then any matrix $(a_{ij}) \in N_4(K)$, $(x,y,z,t) \in K^4$ we have

$$\begin{bmatrix} x, & y, & z, & t \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} x, & y, & z, & t \end{bmatrix} \begin{bmatrix} a_{11}x & a_{12}y & a_{13}z & a_{14}t \\ a_{21}x & a_{22}y & a_{23}z & a_{24}t \\ a_{31}x & a_{32}y & a_{33}z & a_{34}t \\ a_{41}x & a_{42}y & a_{43}z & a_{44}t \end{bmatrix}$$

$$:= x(a_{11}x + a_{12}y + a_{13}z + a_{14}t) + y(a_{21}x + a_{22}y + a_{23}z + a_{24}t)$$
$$+ z(a_{31}x + a_{32}y + a_{33}z + a_{34}t) + t(a_{41}x + a_{42}y + a_{43}z + a_{44}t).$$

Rearranging them we get,

$$= a_{11}x^2 + a_{12}y^2 + a_{13}z^2 + a_{14}t^2 + (a_{12} + a_{21})xy + (a_{13} + a_{31})xz.$$

$$+ (a_{14} + a_{41})xt + (a_{23} + a_{32})yz + (a_{24} + a_{42})yt + (a_{34} + a_{43})zt$$

Observe that

$$a_{11} = A, a_{22} = B, a_{33} = C, a_{44} = D.$$

Also

$$a_{12} = a_{21} = \frac{E}{2},$$
$$a_{13} = a_{31} = \frac{F}{2},$$
$$a_{14} = a_{41} = \frac{G}{2},$$
$$a_{23} = a_{32} = \frac{H}{2},$$
$$a_{24} = a_{42} = \frac{I}{2},$$
$$a_{34} = a_{43} = \frac{J}{2}.$$

Thus giving us a symmetric matrix by taking $N = (a_{ij})$ hence $f(x) = X^T NX$. Finally, Q is singular whenever $x_1, y_1, z_1, t_1 \in K$ are not all zero such that,

$$\begin{bmatrix} \frac{\partial f}{\partial x}(x_1,y_1,z_1,t_1) \\ \frac{\partial f}{\partial y}(x_1,y_1,z_1,t_1) \\ \frac{\partial f}{\partial z}(x_1,y_1,z_1,t_1) \\ \frac{\partial f}{\partial t}(x_1,y_1,z_1,t_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and only happens when there exist $(x_1, y_1, z_1, t_1) \in K$ not all zeros such that

$$\begin{bmatrix} 2(a_{11}x_1 + a_{12}y_1 + a_{13}z_1 + a_{14}t_1) \\ 2(a_{21}x_1 + a_{22}y_1 + a_{23}z_1 + a_{24}t_1) \\ 2(a_{31}x_1 + a_{32}y_1 + a_{33}z_1 + a_{34}t_1) \\ 2(a_{41}x_1 + a_{42}y_1 + a_{43}z_1 + a_{44}t_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

implying that there are some trivial vector$(x_1, y_1, z_1, t_1) \in K^4$ such that $2N(x_1, y_1, z_1, t_1)^T = 0$ indicating that N has nullity atleat 1 and hence it is singular.
Therefore we conclude that Q singular whenever N is singular. □

Corollary 3.1.12. *Let $f \in K[x,y,z,t]$ be homogeneous polynomial of degree 2 and let $Q = \mathbb{V}(f)$ where Q is a quadratic surface in $\mathbb{P}^3$ and let $\ell \in \mathbb{P}^3$ be a line then $\ell \in Q$ and $\ell \subset Q$ if and only if $\ell \cap Q$ contains three points of $\mathbb{P}^3$.*

Lemma 3.1.13. *Let $\ell_i \in \mathbb{P}^3, i = 1,2,3$ be lines which are pairwise disjoint. Then there exist nonsingular quadratic surface Q subset of $\mathbb{P}^3$ which contains $\ell_i, i = 1,2,3$.*

`Proof.` Suppose the three distinct points on ℓ_i are q_i, q_i', q_i'' for $i \in \{1,2,3\}$ then,

$$q_1, q_2, q_3, q_1', q_2', q_3', q_1'', q_2'', q_3''$$

are 9 distinct points. since ℓ_1, ℓ_2, ℓ_3 are disjoint
writing these points as,

$$(x_1 : y_1 : z_1, t_1), \ldots, (x_9 : y_9 : z_9, t_9)$$

then we say that there exist 9 quadratic surface surface consisting of these points and since by the fact that quadratic surface are defined $f \in K[x,y,z,t]$ of these form

$$f(x,y,z,t) = Ax^2 + By^2 + Cz^2 + Dt^2 + Exy + Fxz + Gxt + Hyz + Iyt + Jzt.$$

Therefore there is a quadratic surface containing all these 9 points if and only if there are some coefficients $A, B, C, \ldots, J$ not all zero in such a way that all the 9 equations below hold,

$$\begin{aligned}
Ax_1^2 + By_1^2 + Cz_1^2 + Dt_1^2 + Ex_1y_1 + Fx_1z_1 + Gx_1t_1 + Hy_1z_1 + Iy_1t_1 + Jz_1t_1 &= 0 \\
Ax_2^2 + By_2^2 + Cz_2^2 + Dt_2^2 + Ex_2y_1 + Fx_2z_1 + Gx_2t_1 + Hy_2z_2 + Iy_2t_2 + Jz_2t_2 &= 0 \\
&\vdots \\
Ax_9^2 + By_9^2 + Cz_9^2 + Dt_9^2 + Ex_9y_9 + Fx_9z_9 + Gx_9t_9 + Hy_9z_9 + Iy_9t_9 + Jz_9t_9 &= 0.
\end{aligned}$$

This shows that there is nontrivial solution to this system of 9 homogeneous linear equations where $A, B, C, \ldots, J$ are not all zero and hence there is a quadratic surface Q consisting of 9 points.

Now Q contain the the three lines ℓ_1, ℓ_2, ℓ_3 for $i = \{1,2,3\}$ having that Q contain 3 distinct points of ℓ_i and implies $\ell_i \subseteq Q$ by 3.1.12. Consequently,we note that Q cannot contain any projective plane and thus it is irreducible.

Suppose by contradiction that Q contains a plane Π then we observe that f factors into two possibly equal linear planes in $\mathbb{P}^3$, Π_1 and Π_2 . without loss of generality assume that either $\ell_1, \ell_2 \subseteq \Pi_1$ and $\ell_3 \subseteq \Pi_2$ or $\ell_1, \ell_2, \ell_3 \subseteq \Pi_1$ sice $\ell_1, \ell_2, \ell_3 \subseteq Q$ and we find a projective plane containing at least two of the lines ℓ_1, ℓ_2, ℓ_3 in either case but these lines are pairwise disjoint implying that we found a projective plane containing two disjoint lines, a contradiction.

Finally to show that Q is nonsingular by contradiction, we now suppose that Q is singular then there exist a matrix $N \in N_4(K)$ such that for each $x \in K^4$ we have $f(x) = X^T N x$ by 3.1.11 that is there exist $y \neq 0 \in K^4$ such that $Ny = 0$ then $y^T N = 0$ since N is symmetric and hence for each $x \in K^4$ we have $x^T N y = y^T N x = 0$ note that there must exist $i = \{1,2,3\}$ such that $y \notin \ell_i$ because ℓ_1, ℓ_2, ℓ_3 are disjoint.

Now assume $y \notin \ell_i$ then for each $x \in K^4$ whose corresponding point in $\mathbb{A}^3$ is ℓ_1 and for each $u, v \neq 0 \in K$ we have

$$f(ux+vy) = (ux+vy)^T N(ux+vy) = ux^T Nux + ux^T Nvy + vy^T Nux + vy^T Nvy.$$

$$f(ux+vy) = 0 = f(ux).$$

Because any scalar multiple of x satisfy f since the corresponding point in $\mathbb{P}^3$ of x is in ℓ_1 and the remaining terms are zero hence by suitable change of coordinates we see that the set of all corresponding point in $\mathbb{P}^3$ of the form $ux+vy$ is a unique projective plane containing y and ℓ_1 implying that f is zero hence a contradiction and therefore Q is singular. □

3.2 Smooth Cubic is birational to $\mathbb{P}^1 \times \mathbb{P}^1$ or to $\mathbb{P}^2$

Proposition 3.2.1. *Any smooth cubic surface is birational to* $\mathbb{P}^2$

`Proof.` Consider two disjoint lines $\ell_1, \ell_2 \subset X$. The following mutually inverse rational map $X \dashrightarrow \ell_1 \times \ell_2$ and $\ell_1 \times \ell_2 \dashrightarrow X$ show that X is a birational to $\ell_1 \times \ell_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and hence $\mathbb{P}^2$.

$X \dashrightarrow \ell_1 \times \ell_2$ for every point a not on ℓ_1 and ℓ_2 there is a unique line ℓ in $\mathbb{P}^3$ through ℓ_1, ℓ_2 and a.

Take the rational map from X to $\ell_1 \times \ell_2$ sending a to $(a_1, a_2) := (\ell_1 \cap \ell, \ell_2 \cap \ell)$, which is obviously well defined away from $\ell_1 \cup \ell_2$

On the other hand $\ell_1 \times \ell_2 \dashrightarrow X$ map any pair of points $(a_1, a_2) \in \ell_1 \times \ell_2$ to the third intersection point of X with the line ℓ through a_1 and a_2. This is well defined whenever ℓ is not contained in X. □

Remark 3.2.2. *There are two disjoint lines on X such that there are exactly* 10 *lines on X meeting any given one and exactly* 5 *lines on X meeting any two disjoint given onces,whereas in $\mathbb{P}^2$ any two curves intersect hence X is birational to $\mathbb{P}^2$ and that is infact isomorphic to blowup of $\mathbb{P}^2$ at* 6 *points.*

3.3 Blowup of $\mathbb{P}^2$ at 6 Points in General Position

Proposition 3.3.1. *Any smooth cubic surface is isomorphic to $\mathbb{P}^2$ blown up in* 6*(suitable chosen) points.*

`Proof.` We only sketch the proof. Let X be a smooth cubic surface and consider a rational map $f : X \dashrightarrow \ell_1 \times \ell_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$. First of all we claim that f is actually a morphism.

To see this, note that there is a different description for f. If $a \in X \setminus \ell$, let it be the unique plane in $\mathbb{P}^3$ that contains ℓ_1 and a and set $f_2(a) = H \cap \ell_2$. If one defines $f_1(a)$ analogously, then $f(a) = (f_1(a), f_2(a))$. Now if the point a lies on ℓ_1, let H be the tangent plane to X at a and again set $f_2(a) = H \cap \ell_2$. Extending f_1 similarly, one can show that this extends $f := (f_1, f_2)$ to a well defined morphism $X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ on all of X.

Now let us investigate the inverse map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X$ is not well defined. As already mentioned in the proof of 3.2.1 this is the case if the point $(a_1, a_2) \in \ell_1 \times \ell_2$ is chosen so that $a_1 a_2 \subset X$. In this case the whole line $a_1 a_2$ will be mapped to (a_1, a_2) by f, and it can be checked that f is actually locally the blow up of this point. By remark 3.2.2 there are exactly 5 such lines $a_1 a_2$ on X. Hence X is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in 5 points i.e the blow up of $\mathbb{P}^2$ in 6 suitable chosen points. $\square$

4 The 17+10 or the 6+15+6 Lines on Smooth Cubic

Definition 4.0.1. *[Smooth cubic surface]*

Let $X \subset \mathbb{P}^3$ be a cubic surface.Then X is said to be smooth cubic when it has no singularities.This means that there will be no point $q = (x : y : z : t)$ for which all the first order partial derivatives are zero at the point q that is no point q such that

$$\left(\frac{\partial f}{\partial x}(q) := \frac{\partial f}{\partial y}(q) := \frac{\partial f}{\partial z}(q) := \frac{\partial f}{\partial t}(p) := 0 \right).$$

Proposition 4.0.2. *Let ℓ be a line and q be a point of X then the number of lines contained in X and contain q is 3. Also T_qX is a plane containing the three lines*

`Proof.` Let $\ell \subseteq X$ and we know that $\ell = T_q\ell$ by 2.1.21 then $\mathbb{I}(X) \subseteq \mathbb{I}(\ell)$ thus,

$$T_q\ell := \bigcap_{g \in \mathbb{I}(\ell)} \mathbb{V}\left(\sum_{i=0}^{3} \frac{\partial g}{\partial x_i}(q).x_i \right) \subseteq \bigcap_{f \in \mathbb{I}(X)} \mathbb{V}\left(\sum_{i=0}^{3} \frac{\partial f}{\partial x_i}(q).x_i \right) := T_qX$$

by homogeneous Nullstellensatz we have $\mathbb{I}(X) := (f)$ since $X := \mathbb{V}(f) := V((f))$ that is

$$T_qX := \mathbb{V}\left(\sum_{i=0}^{3} \frac{\partial f}{\partial x_i}(q).x_i \right)$$

to wrap up $\ell \subseteq T_q\ell \subseteq T_qX$,$\ell$ is the line line through $q \subseteq T_qX$. Also $\ell \subseteq X$ so that $\ell \subseteq T_qX \cap X$ meaning any line ℓ through q and contained in X must be contained in $T_qX \cap X$. and hence there are 3 lines through q which are contained in X. □

We will henceforth demonstrate that there are exactly 27 lines on any smooth cubic surface $X \subseteq \mathbb{P}^3$ which contains at least one line.

Theorem 4.0.3. *Let X be a smooth cubic surface in $\mathbb{P}^3$ then X contains exactly 27 lines.*

The following is a proof by **Reids** [AM69].

Proposition 4.0.4. *Through any point $q \in X$ there exist atmost 3 lines of X and coplanar if there are 2 or 3 as shown below.*

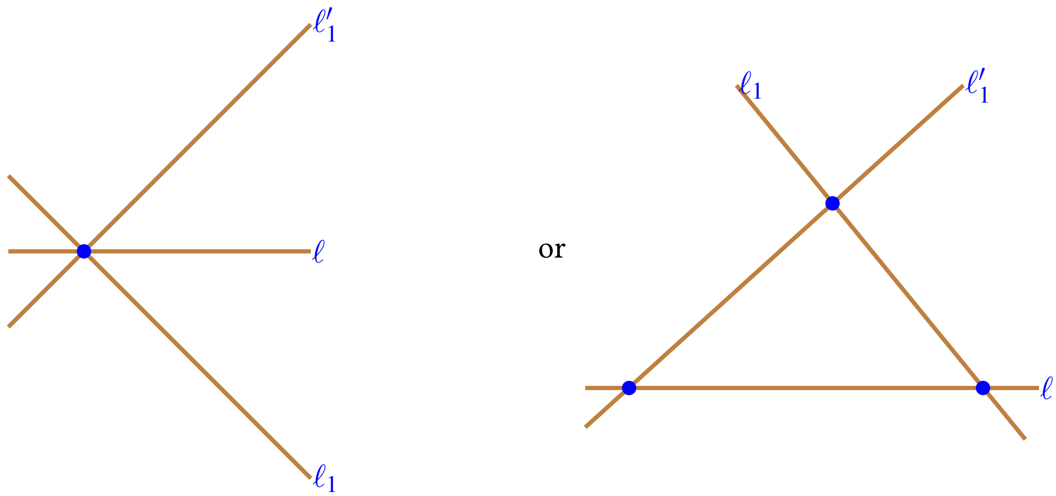

Figure 4. The pictures of lines through the point q.

Proof. Let q be a point, then at most three lines of X go through the point q and every intersection between the plane and the surface X will not give a multiple lines.

Consider a line $\ell \subset X$ on X through the point q then the tangent plane at the point q denoted as T_qX will contain line ℓ. Since X is cubic $T_qX \cap X$ will consist of at most 3 lines giving maximum of 3 lines through the point q. The intersection of the plane Π and X will give a double line and will be singular. $\square$

Proposition 4.0.5. *Let $\Pi \in \mathbb{P}^3$ be a plane. Then $\Pi \cap X$ in one of the following,*

1. *An irreducible cubic curve*
2. *an irreducible conic and a line*
3. *three distinct lines*

Proof. Assume that Π is the plane $Z(t)$ under the change of coordinates, and let g be a homogeneous polynomial of degree 3 then X is cut out from $\mathbb{P}^3$ therefore $X \cap \Pi$ is a cubic plane with defining polynomial $h \in k[x,y,z]$ which is either irreducible or factors as a product of an irreducible quadratic form and a linear form or it factors as a product of three linear forms.

We want to show that $h \in k[x,y,z]$ factors into three distinct linear forms.

For consider $\ell \in X \cap \Pi$, then we show that the linear form in $k[x,y,z]$ defining ℓ factor out of the cube h only once.

Take $\ell := \mathbb{V}(z,t)$ to be another change of coordinates such that linear form defining ℓ in Π is z
by contradiction assume that,

$$h(x,y,z,t) := z^2.a(x,y,z)$$

where a is a linear form and by reversing the process from which we got h from g, then we see that g can be expressed as,

$$g\,(x,y,z,t) := z^2.p\,(x,y,z,t) + t.r\,(x,y,z,t)\,.$$

Also p is a linear form and r is a quadratic form. Now by finding partial derivatives of g we see that,

$$\frac{\partial g}{\partial x}\left(z^2.p(x,y,z,t) + t.r(x,y,z,t)\right) := z^2p_x + tr_x$$
$$\frac{\partial g}{\partial y}\left(z^2.p(x,y,z,t) + t.r(x,y,z,t\right) := z^2p_y + tr_y$$
$$\frac{\partial g}{\partial z}\left(z^2.p(x,y,z,t) + t.r(x,y,z,t\right) := 2zp_z + z^2p_z + tr_z$$
$$\frac{\partial g}{\partial t}\left(z^2.p(x,y,z,t) + t.r(x,y,z,t\right) := z^2p_t + r + tr_t$$

thus $\mathbb{V}(g)$ is singular at any point $(x : y : z : t)$ where $z = t = 0$ and $r(x,y,z,t) = 0$. To restrict to the line ℓ in the plane Π we substituting $z = t = 0$ in r we get a polynomial r_1, where r_1 is identically zero or an homogeneous quadratic in x, y.

That is any point $(x : y)$ is a root of r_1 or r_1 has at least one root $(x : y)$ showing that r has some root $(x : y : 0 : 0)$ along ℓ. Therefore $\mathbb{V}(r_1) \neq \emptyset$ hence X has singular point, a contradiction. □

Proposition 4.0.6. *Any smooth cubic surface X contains at least one line ℓ.*

Proof. Consider any arbitrary point q on the surface X . Taking the intersection of X and the tangent plane at the point q denoted as T_qX,that is

$$T_qX \cap X := C$$

where C is the curve with a nodal or cuspidal singularity at the point q when the curve C is irreducible or the curve C is reducible and contains a line which lie on X. Therefore there exist a linear change of cooordinates where by $q := (0 : 0 : 1 : 0)$ and $T_qX := (t = 0)$. Then the curve $C := xyz = x^3 + y^3$ if the point q is nodal or the curve $C := x^2z = y^3$ when the point q is cuspidal. They are similar cases.

Now lets consider the cuspidal case,

$$f := x^2z - y^3 + g_2(x,y,z,t)t.$$

Where g_2 is homogeneous of degree two and in the coordinates (x,y,z,t). It follows that $g_2(0 : 0 : 1 : 0) \neq 0$ by nonsingularity at the point q.

Now assume $g_2(0 : 0 : 1 : 0) := 1$ then every line ℓ through the point $q_\lambda := (1 : \lambda : \lambda^3 : 0)$

on the curve C goes through the point $r := (0,y,z,t)$ on the plane $X := 0$. A line through q_λ and $r, q_\lambda r$ can be parametrized by writing out,

$$f(\alpha q+\mu r) := A(y,z,t)+B(y,z,t)+C(y,z,t) := 0.$$

Where $g_i \in k(\lambda)$ is homogeneous of degree i in the variables (y,z,t). Then

$$q_\lambda r \subset X \iff A(y,z,t) := B(y,z,t) := C(y,z,t) := 0.$$

Where A is a form of degree one, B is a form of degree two and C is a form of degree three in (y,z,t) with coefficients involving λ.

Claim:
There is some resultant polynomial $X_{27}(\lambda)$ such that $X_{27}(\lambda) := 0 \iff A,B,C$ have a common zero (ξ,η,ζ) in $\mathbb{P}^2$.

We now define the polar form of f as the form in two variables (x,y,z,t) and (x',y',z',t') as

$$f_1\left(x,y,z,t : x',y',z',t'\right) := \left(\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' + \frac{\partial f}{\partial t}t'\right)$$

since for $q(x,y,z,t)$ elements of X and $q \neq r := (x',y',z',t')$ an element of $\mathbb{P}^3$ by the definition of the tangent space we have $f_1(q,r) := 0$ if and only if the line qr is tangent to X at a point q.

Precisely,

$$f(\alpha q+\mu r) := \alpha^3 f(q)+\alpha^2\mu f_1(q,r)+\alpha\mu^2 f_1(r,q)+\mu^3 f(r).$$

So that $q \neq r \in \mathbb{P}^3$.

The following four conditions are the equations for the line qr to be contained in $X : (f = 0)$ so that $f(r) := f_1(r,q) := f_1(q,r) := f(q)$.

Geometry line qr is tangent to X at both q and r, hence $f|_{qr}$ has double roots at the point q and r, therefore $qr \in X$.

The polar of $f := x^2z - y^3 + g_2(x,y,z,t)t$ is,

$$f_1 = 2xzx' - 3yy' + x^2z + g_2(x,y,z,t)t' + tg_1(x,y,z,t;x',y',z',t')$$

is the polar form of g shown in the same way as above, here g is quadratic, g_1 is symmetric billinear form such that $g_1(q_1q) := 2g(q)$.

Given,

$$A := z - 3\alpha^2 y + g(1,\alpha,\alpha^3,0)t$$

$$B := -3\alpha y^2 + g_1(1,\alpha,\alpha^3,0:0,y,z,t)t$$

$$C := -y^3 + g(0,y,z,t)t.$$

And substituting $q_\lambda = (1,\lambda,\lambda^3,0)$,$r = (0,y,z,t)$ then it give rise to the equation $q_\lambda r \in S$ as $A = B = C = 0$.

Finally, we now eliminate y,z,t from the equations above considering the highest power of α.

Therefore it shows that $g(1,\alpha,\alpha^3,0) := \alpha^6 + ... = a^6$ because $g(0,0,1,0) = 1$ hence a^6 is a polynomial whose leading coefficient is one with degree 6.

Then $A := 0$

$$\Rightarrow A := z - 3\alpha^2 y + g(1,\alpha,\alpha^3,0)t.$$

$$0 := z - 3\alpha^2 y + a^6 t.$$

Therefore

$$z := 3\alpha^2 y - a^6 t.$$

Also by substituting z in B and then using the bilinearity of g_1, we get

$$B := -3\alpha y^2 + g_1(1,\alpha,\alpha^3,0;0,1,3\alpha^2 y - a^6 t,t)t$$

$$= b_0 y^2 + b_1 yt + b_2 t^2,$$

where

$$b_0 := -3\alpha$$

$$b_1 := g_1(1,\alpha,\alpha^3,0;0,1,3\alpha^2,o) := 6a^5 + ...$$

$$b_2 := g_1(1,\alpha,\alpha^3,0;0,0,-a^6,1) := -2a^9 + ...$$

And by similar process of substituting z in C and expanding the quadratic form g results to

$$C := -y^3 + g(0,y,3\alpha^2 y - a^6 t)t$$

$$= c_0 y^3 + c_1 y^2 t + c_2 y t^2 + c_3 t^3$$

where ,

$$C_0 := -1$$

$$c_1 = g(0,1,3\alpha^2,0 := 9\alpha^4 + ...$$

$$c_2 := g(0,1,3\alpha^2,0;0,0,-a^6,1) := -6a^8 + \dots$$
$$c_3 := g(0,0,-a^6,1) := \alpha^{12} + \dots$$

Hence we see that B' and C' have common zero (η,τ) as stated by Sylvester's determinant formula.

Theorem 4.0.7. *[Sylvester's determinant formular]*

Let k be algebraic closed field and suppose given a quadratic and cubic form U,V as, $b(U,V) := b_0U^2 + b_1UV + b_2V^2$, $c(U,V) := c_0U^3 + c_1U^2V + c_2UV^2 + c_3V^3$. Then b and c have common zero $(\eta,\tau) \in \mathbb{P}$ if and only if,

$$det \begin{vmatrix} a_0 & a_1 & a_2 & & \\ & a_0 & a_1 & a_2 & \\ & & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & b_3 & \\ & b_0 & b_1 & b_2 & b_3 \end{vmatrix} := 0.$$

Therefore from our result we have that B' and C' have common zero (η,τ) if and only if,

$$det \begin{vmatrix} b_0 & b_1 & b_2 & & \\ & b_0 & b_1 & b_2 & \\ & & b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 & c_3 & \\ & c_0 & c_1 & c_2 & c_3 \end{vmatrix} := 0$$

which shows that the determinant is a polynomial in α and it is easy to see the leading terms is coming from taking leading terms in the each entry of the determinant and hence

$$b_0 := -3a, b_1 := 6a^5, b_2 := -2a^9$$

$$c_0 := -1, c_1 := 9a^4, c_2 := -6a^8, c_3 := a^{12}.$$

Therefore,

$$\begin{vmatrix} -3\alpha & 6\alpha^5 & -2\alpha^9 & & \\ & -3\alpha & 6\alpha^5 & -2\alpha^9 & \\ & & -3\alpha & 6\alpha^5 & -2\alpha^9 \\ -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} & \\ & -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} \end{vmatrix} := \alpha^{27}.det \begin{vmatrix} -3 & 6 & -2 & & \\ & -3 & 6 & -2 & \\ & & -3 & 6 & -2 \\ -1 & 9 & -6 & 1 & \\ & -1 & 9 & -6 & 1 \end{vmatrix} := \alpha^{27}.$$

Then there are atmost 27 roots if the roots are unique therefore giving a maximum of 27 lines on a smooth cubic surface. $\square$

4.1 17+10 Lines on a Smooth Cubic in $\mathbb{P}^3$

Proposition 4.1.1. *Let ℓ be a line on X. Then there exist exactly five pairs of lines (ℓ_i, ℓ'_i) such that every pair of lines (ℓ_i, ℓ'_i) is coplanar with the line ℓ and $(\ell_i \cup \ell'_i) \cap (\ell_j \cup \ell'_j) = \emptyset \forall i \neq j$*

`Proof.` Consider the given line ℓ on X and the plane Π such that $\ell \subset \Pi$. Then the intersection $\Pi \cap X$ is a conic and a line. If $\Pi \cap X$ the intersection is singular, then it consist of 3 lines as shown,

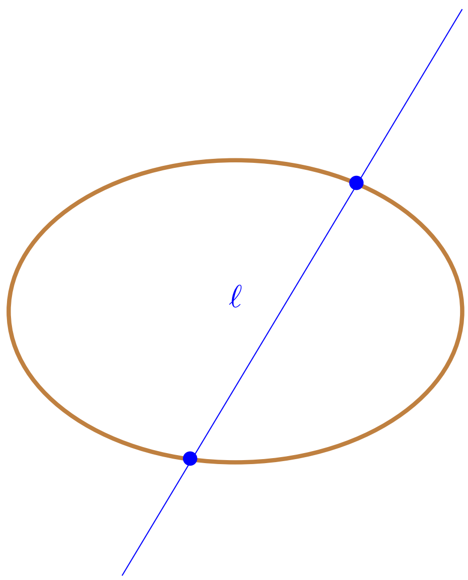

Figure 5. Intersection of a conic and a straight line.

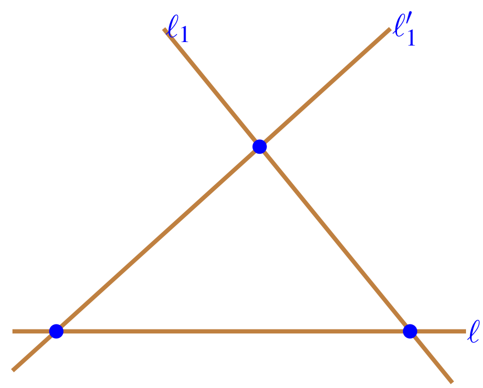

Figure 6. Intersection of three straight line.

In order to show that there are exactly 5 distinct planes $\ell \subset \Pi_\lambda$ for a singular case we suppose that $\ell := (z = t = 0)$ this is two linear equations $z = 0$ and $t = 0$. This defines two planes so when i set both to 0.

That is the intersection of planes equal to a ℓ and so i get this line ℓ by change of coordinates.

Therefore, the equation for the plane containing ℓ only contains z and t,(this is an homogeneous equation in two variables) and they have to be of the form $\Pi : \mu z := \lambda t$, when $\mu := 0$.

Now suppose $\mu \neq 0$ and assume $\mu := 1$. Then it implies that $z := \lambda t$. By expressing it in terms of homogeneous coordinates (x, y, t) on the plane Π, and rewriting the equation

$$f(x,y,z,t) = A(z,t)x^2 + B(z,t)xy + C(z,t)y^2 + D(z,t)x + E(z,t)y + F(z,t). \tag{1}$$

Where A,B,C are linear forms in (z,t), D,E are quadratic in (z,t) and F is cubic form in (z,t).

Replacing z with λt we get;

$$A(\lambda t,t) = tA(\lambda,1), B(\lambda t,t) = tB(\lambda,1), C(\lambda t,t) = tC(\lambda,1);$$

$$D(\lambda t,t) = t^2 D(\lambda,1), E(\lambda t,t) = t^2 E(\lambda,1)$$

And

$$F(\lambda t,t) = t^3 F(\lambda,1).$$

Hence we have restricted this equation to a plane Π. Substituting everything in 1 above and dividing out by t we get an equation say g, where

$$g := Ax^2 + Bxy + Cy^2 + Dtx + Ety + Ft^2. \tag{2}$$

By taking this equation as the conic in x and y then it becomes singular when the discriminant is zero and we will get the rest of intersection which is now exact the conic which degenerate into two lines when the conic is not smooth.

Therefore, we check degeneracy by checking the smoothness g_x,g_y and g_z;

$$\begin{bmatrix} g_x \\ g_y \\ z \end{bmatrix} := \begin{bmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{bmatrix} \begin{bmatrix} x \\ y \\ t \end{bmatrix}$$

The discriminant denoted by Δ is a polynomial g of degree 5 in (z,t),

$$\Delta(z,t) := 4ACF + BDE - AE^2 - B^2F - CD^2 := 0 \tag{3}$$

It has zero exactly 5 times with multiplicity.

To prove the claim, it suffices to show that this has simple roots. Thus every such root λ will give the plane Π_λ through the line ℓ such that $\Pi_\lambda \cap X$ consist of three lines.

By the simpleness of the roots of g, 5 such planes exist and hence every line ℓ on X intersect with exactly 10 lines. $\square$

Proposition 4.1.2. *There are atleat 5 disjoint pairs of lines (ℓ_i, ℓ_i') which intersects with the line ℓ and any other line $n \subset X$ will meet exactly one of ℓ_i and ℓ_i' for $i := 1,2,3,4,5$. A line n will intersect the plane Π in $\mathbb{P}^3$, $\Pi_i \cap X := \ell \cup \ell_i \cup \ell_i'$ showing that n intersect one of the lines and it cannot intersect all since the lines which intersect ℓ are found, therefore n intersect ℓ_i or ℓ_i' and it cannot intersect both because it will lie on the plane Π_i and the intersection of Π_i and X will give four lines.*

Proposition 4.1.3. *If there are 4 disjoint lines $\ell_1, \ell_2, \ell_3, \ell_4$ in $\mathbb{P}^3$ then they lie on the quadratic and have an infinite number of lines intersecting all lines or they do not lie on a quadric and have one or two.*

Proof. Let ℓ_1, ℓ_2, ℓ_3 be disjoint lines then through them there always passes a smooth quadric Q. The quadric has two set of lines J_1, J_2, since ℓ_1, ℓ_2, ℓ_3 are disjoint they belong to one set of lines J_1 and every line which intersects all the three lines on Q and belong to J_2 hence ℓ_4 is disjoint and lie on Q and it belongs to J_1 and the infinite family of lines J_2 will all intersect the four lines. When ℓ_4 does not lie on Q then the line ℓ_4 intersects the Q in one or two points and the lines from J_2 passing through these points intersect all the four lines.

Finally, let ℓ and m be two disjoint lines on X then every pair (ℓ_i, ℓ_i'), $i = 1,2,3,4,5$ which intersects ℓ and one of them intersects m. Now let ℓ_i intersect both ℓ and m then m intersects with the pairs (ℓ_i, ℓ_i''),$l = 1,2,3,4,5$. Thus gives 17 lines on X that is ℓ, m, the 5 lines intersecting both, 5 lines which intersect only l and 5 lines which intersect only m of the pair as shown below,

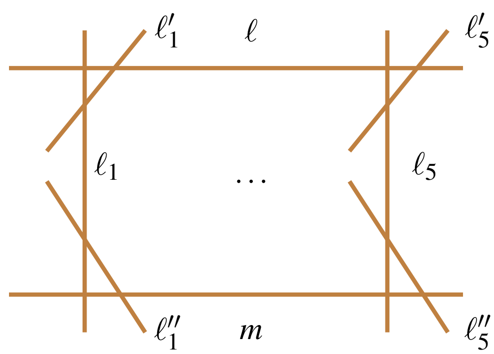

Figure 7. Configuration of lines on $X_3 \subset \mathbb{P}^3$.

□

Any line $\ell \in X$ not included in the 17 lines as above will intersect 3 of the lines $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$.

Here, there are no 4 of the lines that will lie on a quadric because then X would be reducible.

Line n cannot meet more than 3 of the lines ℓ_i since it will be ℓ or m by proposition 4.1.3.

If it intersect with less than 3 of the lines ℓ_i then it will intersect with 3 or more of ℓ_i. Therefore,it means either $\ell_2', \ell_3'\ell_4', \ell_5'$ or $\ell_1, \ell_3', \ell_4', \ell_5'$ but then ℓ and ℓ_1'' intersect these four and through the same argument in proposition 4.1.3, n can not intersect all four lines. Therefore n intersect 3 of lines.

The combination of all the tree lines of ℓ_i gives a line not contained in the 17 found lines.

There are 10 lines intersecting ℓ_1 by proposition 4.1.3, Only four lines ℓ,ℓ_1',ℓ_1'',m are found showing that there are 6 more by above, each one of them will intersect 2 lines $\ell_2,\ell_3,\ell_4,\ell_5$.

Hence there are only 6 possibilities therefore they all occur.

Now by adding all the lines found it gives line ℓ and line m the 15 lines ℓ_i,ℓ_i',ℓ_i'' for $i=1,2,3,4,5$ and $\binom{5}{2}:=10$ of the intersecting ℓ_i,ℓ_j,ℓ_k lines where $i\neq j\neq k$ where $i,j,k=1,2,3,4,5$. Thus gives 27 lines of a maximum 27 and hence all the lines are found.

4.2 6+15+6 Lines on a Smooth Cubic in $\mathbb{P}^3$

In this section, we give an alternative enumeration of the 27 lines on a smooth cubic surface by counting strict transforms in the blow-up map

$$B\ell_{\{p_1,\ldots,p_6\}}:B\ell_{\{p_1,\ldots,p_6\}}\mathbb{P}^2 \dashrightarrow \mathbb{P}^2.$$

Proposition 4.2.1. *The space of smooth cubics surfaces in $\mathbb{P}^3$ is a dense open $U\subset\mathbb{P}^{19}$.*

`Sketch Proof.` The vector subspace

$$\mathbb{C}\left[\mathbb{P}^3_{[x_i]}\right]_3 = \langle x_0^3,\ x_1^3,\ x_2^3,\ x_3^3,\ x_0^2x_1,\ x_0^2x_2,\ x_0^2x_3,\ x_0x_1^2,\ x_0x_2^2,\ x_0x_3^2,\ x_1^2x_2,\ x_1^2x_3,\ x_1x_2^2,\ x_1x_3^2,\ x_2^2x_3,\ x_2x_3^2,\ x_0x_1x_2,\ x_0x_1x_3,\ x_0x_2x_3,\ x_1x_2x_3\rangle \cong \mathbb{C}^{20}_{(u_i)}$$

is spanned by 20 degree 3 monomials in $\mathbb{C}\left[\mathbb{P}^3_{[x_i]}\right]=\mathbb{C}[x_0,x_1,x_2,x_3]$. Now the projective space of the vector subspace

$$\mathbb{P}\left(\mathbb{C}\left[\mathbb{P}^3_{[x_i]}\right]_3\right)=\mathbb{C}\left[\mathbb{P}^3_{[x_i]}\right]_3\setminus\{0\}/\mathbb{C}^*\cong\mathbb{P}^{19}_{[u_i]}=\mathbb{C}^{19}_{\left(\widehat{\frac{u_0}{u_0}},\frac{u_1}{u_0},\ldots,\frac{u_{20}}{u_0}\right)}\cup\mathbb{P}^{18}_{[\widehat{u_0},u_1,\ldots,u_n]}.$$

By setting $U=\mathbb{C}^{19}_{\left(\widehat{\frac{u_0}{u_0}},\frac{u_1}{u_0},\ldots,\frac{u_{20}}{u_0}\right)}$, the result follows from the embedding $U\hookrightarrow\mathbb{P}^{19}_{[u_i]}$. $\square$

Remark 4.2.2. *Let U be as in Proposition 4.2.1 and a subset $M\subset U\times G(2,4)$ consisting of pairs (X,ℓ) of a smooth cubic X and a Line ℓ on it. We would like to understand the incidence correspondence from the projection*

$$\pi: M\xrightarrow{(X,\ell)\to\ell}U$$

The goal of the thesis is equivalent to counting the number of inverse images of π, that is, showing that π is a $27:1$ map, 27-sheeted covering map.

Theorem 4.2.3. *There are 27 lines on a smooth complex cubic surface.*

Proof. It is interesting to see the lines on a cubic surface X in a picture of 3.3.1. In which we think of X as a blow up of $\mathbb{P}^2$ in 6 points. It turns out that the 27 lines correspond to the following curves that we already (and that are all isomorphic to $\mathbb{P}^1$).

- the 6 exceptional hypersurfaces.
- the strict transforms of the $\binom{6}{2} := 15$ lines through two of the blown up points.
- the strict transforms of the $\binom{6}{2} := 6$ conics through five of the blown up points.

In fact; it is easy to check by the above explicit description of the isomorphism of X with the blow up of $\mathbb{P}^2$ that these curves on the blow-up actually correspond to lines on the cubic surface.

It is also interesting to see again in this picture that every such line meets 10 of the other lines as mentioned in Remark 3.2.2.

Every exceptional hypersurface intersects the 5 lines and the 5 conics that pass through this blown-up point.

Every line through two of the blown-up points meet.

- the 2 exceptional hypersurface of the blown up points.
- the $\binom{4}{2} := 6$ lines through two of the four remaining points.
- the 2 conics through the four remaining points and one of the blown up points.

Every conic through five of the blown up points meet the 5 exceptional hypersurface at these points, as well as the 5 lines through one of these five points and the remaining point. $\square$

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