University of Nairobi

School of Mathematics

ISSN: 2410-1397

Master Project in Mathematical Statistics

ON FINITE GAMMA MIXTURES AND THEIR PROPERTIES

Research Report in Mathematics, Number 16, 2021

Mutua Robert Munywoki July 2021

Submitted to the School of Mathematics in partial fulfillment for a degree in Master of Science in Mathematical Statistics

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Master Project

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Submitted to: The Graduate School, University of Nairobi, Kenya

Abstract

The concept of finite mixture has contributed immensely in search of more flexible distributions that are in a position to capture data heterogeneity. One parameter Lindley distribution as the first case of a finite mixed gamma distribution has been generalized up to five parameters and goodness of fit measures done. Based on the available literature, Lindley and its generalizations has been extensively applied in modeling of lifetime data. Generalized cases of Lindley distribution prove to be more flexible than one parameter Lindley in modeling lifetime data. However, Lindley and its generalizations have not been extensively compared to other finite gamma mixtures. In this project, the goal is to study finite gamma mixtures and their applications to lifetime data. Similarly, finite gamma mixtures have been constructed up to three component and their statistical properties studied.

Selected constructed finite gamma mixtures were fitted to a lifetime data regarding carbon fiber breaking stress recorded by Nichols and Padget (2006). The model parameters were estimated using method of moments (MOM) and maximum likelihood (MLE) techniques. The results of one parameter selected distributions proved that Suja, Rama, Aradhana, Sujatha, Akash and Shanker were better fit than Lindley distribution. Based on the selected distributions two parameter, it was established that QSD, AG2PAD, QAD, AG2PSD were more flexible than AG2PLD while G2PSD performed worse than AG2PLD. Based on the selected three parameter distributions fitted, AG3PLD was a better fit than AG3PAD and AG3PSD.

Master Thesis in Mathematics at the University of Nairobi, Kenya.
ISSN 2410-1397: Research Report in Mathematics

DISTRIBUTOR: School of Mathematics, University of Nairobi, Kenya

Declaration and Approval

I the undersigned declare that this project is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

23/08/2021

Signature Date

MUTUA ROBERT MUNYWOKI

Reg No. I56/33944/2019

In my capacity as a supervisor of the candidate's project, I certify that this project has my approval for submission.

23/08/2021

Signature Date

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Dedication

To those who believe in God and power of prayer

Contents

Acknowledgments

First, I thank the almighty God for His strength and grace through out my masters period. I would like to acknowledge my supervisor Dr. Isaac C. Kipchirchir for his guidance and tireless efforts in supervising this project work. I also extent my deepest gratitude to Prof JAM Ottieno for his invaluable contribution to my project. It is through their patience, faith in me and continual encouragement that the project has been a success.

Secondly, I would like to thank the lecturers in the school of mathematics that have taught me during my masters period. Further I would like to thank my classmates in the Msc. Mathematical Statistics for their invaluable contribution in class discussions. My hope is that our friendship shall continue past academic life. Finally, I would like to express sincere gratitude to my parents for their prayers and support and encouragement through out my masters period.
God Bless you all.

MUTUA ROBERT MUNYWOKI

Nairobi, 2021.

1 GENERAL INTRODUCTION

1.1 Background Information

A finite mixed distribution is a convex combination of two or more density functions of different random variables. Combining two or more probability density functions enhance flexibility of the mixed distribution in analyzing complex data. Mixed distributions have wide application in statistical analysis such as survival analysis, classification, modeling among others. The concept of finite mixed distributions was first encountered in modeling of outliers through combining observations primarily to have the best outcome (Newcomp, 1886). The second incidence of finite mixed model is the combination of two univariate Gaussian distributions (Pearson, 1894). MOME was applied in the estimation of the newly constructed distribution.

Vardi et al. (1985) applied finite Poisson mixtures on Positron emission tomography (PET) to study occurrence of emissions in a single line in photon detectors. The estimation methods in photon detectors were maximum likelihood estimation, least square estimation and method of moment estimation. Another case of finite mixed distribution is a finite mixture of multivariate Poisson distribution introduced by Karlis and Meligkotsidou (2007) as an alternative model to fit count data. The proposed model was flexible to allow estimation of parameters in over-dispersed data.

The first incidence of finite gamma mixture in search of flexible probability distributions to fit a lifetime data was encountered by Lindley (1958) one parameter Lindley distribution. One parameter Lindley distribution is derived from a combination of Gamma $(1, \theta)$ and Gamma $(2, \theta)$ through the concept of finite mixtures. Ghitany et al. (2008) further discussed statistical properties of LD. Further it was proved that one parameter Lindley distribution was a better fit than the exponential distribution. Exponential distribution has a constant hazard function thus unable to model different shapes of hazard functions (increasing, decreasing and bathtub).

The concept of finite mixture leads to development of flexible distributions to model different shapes of hazard function. In this research project, we shall provide a comprehensive work on finite gamma mixtures application on lifetime data.

1.2 Definitions, Notations and Terminologies

Definitions

Generalization : This is a concept of constructing a new distribution with more parameters than the original one mainly to increase the model flexibility in order to accommodate more data.

Finite Mixture: A k-component finite mixture is defined as;

$$f(x) = \sum_{j=1}^{k} \omega_j f_j(x|\vartheta_j) \tag{1.1}$$

where $\omega_j > 0$ and $\sum_{j=1}^{k} \omega_j = 1$ and $f_j(x|\vartheta_j)$ is the j^{th} component pdf. Further, ω_j is a vector of mixing weights or mixing probabilities and ϑ is the parameter vector of the j^{th} component. There are three sets of parameters to be estimated. To begin with, if k, ω and ϑ are unknown then they have to be estimated. If only k is known then ω and ϑ are estimated. Lastly, if k and ω are known only ϑ is estimated.
A finite mixture is also known as **mixed distribution**

Survival function : Survival function is denoted by $S(x)$ which is known as the probability of an individual to survive beyond say time x.

hazard function : hazard function is denoted by $h(x)$ which is the instantaneous rate that an event occurs.

Equilibrium distribution: This is mathematical function that is useful in evaluating a lifetime of a random variable say x at given period say t.

Mean excess loss : It is denoted as $m(x)$ that is mean of random variable say x which represents remaining lifetime of a given lifetime process.

Notations

1. $\Gamma\alpha$: This is a representation of an ordinary gamma function.
2. $\Gamma(\alpha, y)$: This is a representation of upper incomplete gamma function.
3. $\gamma(\alpha, y)$: This is a representation of lower incomplete gamma function.
4. $\Phi(\alpha, y)$: This is a representation of lower incomplete gamma function ratio.
5. $\vartheta(\alpha, y)$: This is a representation of upper incomplete gamma function ratio.
6. $S(x)$: Is the survival probability
7. $h(x)$: Is the hazard function
8. $f_l(x)$: Pdf of the excess loss function.
9. $m(x)$: Mean residual lifetime (mean excess loss).
10. $f_e(x)$: Equilibrium distribution.
11. $S_e(x)$: Survival function based on the equilibrium distribution.
12. $h_e(x)$: hazard function of the equilibrium distribution.
13. $P(.)$: Probability of an event to occur.

Terminologies

Pdf : Probability Density Function

Cdf : Cumulative Density Function

MLE : Maximum Likelihood Estimation

MOME : Method of Moment Estimation

QSD : Quasi Sujatha distribution

QLD : Quasi Lindley distribution

NQLD : New Quasi Lindley distribution

LD : Lindley distribution

ED : Exponential distribution

AG2PAD : A generalized two parameter Aradhana distribution

QAD : Quasi Aradhana distribution

AG2PSD : A Generalized two parameter Sujatha distribution

AG2PLD :A Generalized two parameter Lindley distribution

G2PSD : Generalized two parameter Shanker distribution

AG3PLD : A generalized three parameter Lindley distribution

AG3PAD : A generalized three parameter Aradhana distribution

AG4PLD : A generalized four parameter Lindley distribution

AG5PLD : A generalized five parameter Lindley distribution

G2PAD : Generalized two parameter Akash distribution

G3PAD : Generalized three parameter Akash distribution

G3PSD : Generalized three parameter Shanker distribution

AG2PRD : A generalized two parameter Rama distribution

AG3PRD : A generalized three parameter Rama distribution

1.3 Research Problem

Lindley (1958) introduced one parameter Lindley distribution as a two component finite gamma mixture ($k = 2$). Studies have been done on Lindley and its generalizations in terms of goodness of fit measures . However, Lindley and its generalizations have not been compared to other finite gamma mixtures. The problem in the study is to construct other finite gamma mixtures up to three components ($k = 3$) and compare accuracy measures with Lindley and its generalizations.

1.4 Objectives of the Study

1. The broad objective of the study is to study finite gamma mixtures and their applications.

2. To construct finite gamma mixtures and study their statistical properties.

3. To estimate parameters of selected distributions from 2 above using MOME and MLE methods.

4. To compare goodness of fit measures of the fitted distributions in 3 above.

1.5 Methodology

In this section, we discuss mathematical tools to be employed in the research.

Special functions: The following highlighted special functions will be used in the research.

- A gamma function is denoted by $\Gamma(\alpha)$ is defined as;

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx; \quad \alpha > 0 \tag{1.2}$$

- An incomplete upper gamma function is denoted by $\Gamma(\alpha, y)$ is defined as;

$$\Gamma(\alpha, y) = \int_y^\infty x^{\alpha-1} e^{-x} dx; \quad \alpha > 0 \tag{1.3}$$

- incomplete lower gamma function denoted by $\gamma_{(\alpha,y)}$ is defined as;

$$\gamma(\alpha, y) = \int_0^y x^{\alpha-1} e^{-x} dx; \quad \alpha > 0 \tag{1.4}$$

- An incomplete upper gamma function ratio denoted by $\Gamma\alpha(y)$ is defined as;

$$\vartheta(\alpha, y) = \frac{\Gamma(\alpha, y)}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)} \int_y^\infty x^{\alpha-1} e^{-x} dx; \quad \alpha > 0 \tag{1.5}$$

- An incomplete lower gamma function ratio denoted by $\Phi(\alpha, y)$ is defined as;

$$\Phi(\alpha, y) = \frac{\gamma(\alpha, y)}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)} \int_0^y x^{\alpha-1} e^{-x} dx; \quad \alpha > 0 \tag{1.6}$$

We can note the following in special functions applied in the research.

$$\Gamma(\alpha,y)+\gamma(\alpha,y)=\Gamma(\alpha)$$
$$\vartheta(\alpha,y)+\Phi(\alpha,y)=1$$
$$\lim_{y\to\infty}\gamma(\alpha,y)=\Gamma(\alpha,0)=\Gamma(\alpha)$$

- let $g(y)$ be a probability density function of a generalized two parameter gamma distribution, then $g(y)$ is defined as;

$$g(y)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}e^{-\beta y}y^{\alpha-1}; \quad y>0,\beta>0,\alpha>0 \tag{1.7}$$

Reliability analysis functions: For the reliability analysis functions to be considered are survival function and hazard function.

- **Survival function:** Let $S(x)$ be the probability of an individual to survive beyond time x or the event of has not occurred by time x. Survival function $S(x)$ given as;

$$\begin{aligned} S(x) &= P(X>x) \\ &= 1-P(X\leq x) \end{aligned}$$

$$S(x)=1-F(x) \tag{1.8}$$

- **hazard function:** Let $h(x)$ be a conditional density, provided that the event that the researcher is interested in has not yet occurred prior to time say x.hazard function $h(x)$ is given by;

$$\begin{aligned} h(x) &= \lim_{\Delta x\to 0}\frac{P(x\leq X\leq x+\Delta x|X>x)}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{P(x\leq X\leq x+\Delta x, X>x)}{\Delta x P(X>x)} \\ &= \lim_{\Delta x\to 0}\frac{P(x\leq X\leq x+\Delta x)}{\Delta x S(x)} \\ &= \frac{1}{S(x)}\lim_{\Delta x\to 0}\frac{P(x\leq X\leq x+\Delta x)}{\Delta x} \end{aligned}$$

moreover,

$$h(x)=\frac{f(x)}{S(x)} \tag{1.9}$$

Excess loss functions: The following excess loss functions to be used in the study are:-

- **Pdf of excess loss distribution**
  The pdf of excess loss function denoted by $f_l(x)$ is given as;

$$\begin{aligned}
P(X > z) &= \int_z^{\infty} f(x)dx \\
1 - P(X \leq z) &= \int_z^{\infty} f(x)dx \\
1 - F(z) &= \int_z^{\infty} f(x)dx \\
1 &= \frac{\int_z^{\infty} f(x)dx}{1 - F(z)}
\end{aligned}$$

$$f_l(x) = \frac{f(x)}{1 - F(z)}; \quad x > z \tag{1.10}$$

- **Mean excess loss (Mean residual lifetime)**
  Mean excess loss of a random variable X denoted by $m(x)$ is given as;

$$\begin{aligned}
m(x) &= E[T - x | T > x] \\
&= \int_x^{\infty} (t - x) \frac{f(t)}{1 - F(x)} dt \\
&= \frac{1}{1 - F(x)} \int_x^{\infty} (t - x) f(t) dt
\end{aligned}$$

where $f(t) = \frac{-dS(t)}{dt}$ and then;

$$\begin{aligned}
&= \frac{1}{1 - F(x)} \int_x^{\infty} (t - x) \left(\frac{-dS(t)}{dt} \right) dt \\
&= \frac{-1}{1 - F(x)} \int_x^{\infty} (t - x) dS(t)
\end{aligned}$$

Using integration by parts technique we have the following;

$$\begin{aligned}
u &= (t - x) \Longrightarrow du = dt \\
dv &= dS(t) \Longrightarrow v = S(t) \\
&= -\frac{-1}{1 - F(x)} \left[\left((t - x) S(t) \right)_x^{\infty} - \int_x^{\infty} S(t) dt \right] \\
&= \frac{-1}{1 - F(x)} \left[0 - \int_x^{\infty} S(t) dt \right] \\
&= \frac{\int_x^{\infty} S(t) dt}{1 - F(x)} \\
&= \frac{1}{1 - F(x)} \int_x^{\infty} S(t) dt
\end{aligned}$$

$$m(x) = \frac{1}{1 - F(x)} \int_x^{\infty} (1 - F(t)) dt \tag{1.11}$$

- **Equilibrium distribution**
  By definition, the expected value of X is defined as;

$$\begin{aligned} E(x) &= \int_0^\infty x f(x) dx \\ &= \int_0^\infty x \left(\frac{-dS(x)}{dx} \right) dx \\ &= -\int_0^\infty x dS(x) \end{aligned}$$

Using integration by parts technique then;

$$\begin{aligned} u = x &\Longrightarrow du = dx \\ dv = dS(x) &\Longrightarrow v = S(x) \\ &= -\left[xS(x)|_0^\infty - \int_0^\infty S(x)dx \right] \\ &= -\left[0 - \int_0^\infty S(x)dx \right] \\ E(x) &= \int_0^\infty S(x)dx \\ 1 &= \frac{\int_0^\infty S(x)dx}{E(x)} \end{aligned}$$

therefore;

$$f_e(x) = \frac{S(x)}{E(x)}; \quad x > 0 \tag{1.12}$$

- **survival function of the equilibrium distribution**
  $S_e(x)$ of the equilibrium distribution is given by the following relation;

$$S_e(x) = \frac{1}{E(x)} \int_x^\infty S(t)dt \tag{1.13}$$

- **hazard function based on the equilibrium distribution**
  hazard function of the equilibrium distribution is denoted by $h_e(x)$ and is given as;

$$h_e(x) = \frac{f_e(x)}{S_e(x)} \tag{1.14}$$

Alternatively, hazard function of the equilibrium distribution is obtained as inverse of the mean residual lifetime. From Eq.1.12 and Eq.1.13 the inverse is obtained as;

$$
\begin{aligned}
&= \frac{S(x)}{E(x)} \frac{1}{\int_x^{\infty} \frac{S(t)}{E(x)} dt} \\
&= \frac{S(x)}{\int_x^{\infty} S(t) dt} \\
&= \frac{1}{\int_x^{\infty} \frac{S(t)}{S(x)} dt} \\
&= \frac{1}{\frac{\int_x^{\infty} (1-F(t)) dt}{1-F(x)}} \\
&= \frac{1}{\frac{1}{1-F(x)} \int_x^{\infty} (1-F(t)) dt}
\end{aligned}
$$

Therefore;

$$h_e(x) = \frac{1}{m(x)} \tag{1.15}$$

r^{th} Moment technique: The r^{th} moment is defined as;

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx \tag{1.16}$$

Mgf Technique: The mgf of a random variable X is defined as;

$$M_x(t) = E(e^{tx}) \tag{1.17}$$

Where $E(x) = M_x^I(0)$ and $E(x^2) = M^{II}(0)$ Using mgf technique, the r^{th} moment is defined as;

$$E(X^r) = \frac{d^r}{dt^r} M_x(t)|_{t=0}$$

Method of Moment Estimation (MOME)

Maximum Likelihood Estimation (MLE)

1.6 Literature Review

1.6.1 Lindley Distribution and its Generalizations

Lindley (1958) introduced one parameter Lindley distribution as a two component mixed distribution of Gamma $(1, \theta)$ and Gamma $(2, \theta)$ through finite mixture concept with mixing probability as $\omega = \frac{\theta}{\theta+1}$. Further Ghitany et al. (2008), discussed statistical properties of one parameter Lindley distribution. The parameter of the introduced LD was estimated using maximum likelihood estimation method. The proposed one parameter LD was applied on both simulated data ($N = 10,000$ times) and real time data on waiting times of 100 bank customers. LD proved to be a better fit than exponential distribution (Gamma $(1, \theta)$). Therefore, One parameter Lindley distribution was more flexible than exponential distribution.

A generalized three parameter Lindley distribution (Case 1) was introduced by Zakerzadeh and Dolati (2009) as a two-component finite mixed distribution of Gamma (α, θ) and Gamma $(\alpha + 1, \theta)$ with mixing weight as $\omega = \frac{\theta}{\theta+\gamma}$. Statistical properties such as order statistics and random variable generation were discussed. The parameters of the distribution were estimated using maximum likelihood estimation. The distribution was fitted to two data sets on failure rates of 15 electronic components and 25 100-centimeters yarn specimens. The generalized three parameter Lindley distribution was established to be best fit among log-normal, weibull and Gamma (α, θ) distributions.

Deniz and Ojeda (2011) proposed a discrete LD through reparameterization. Properties such as reversed hazard rate, fisher's information about λ, moments and related measures (skewness and kurtosis) among others of a discrete LD were discussed. The discrete Lindley distribution was estimated using MLE technique and it proved to be a better fit than poison distribution.

AG2PLD was proposed and introduced by Shanker et al.(2013c) as a mixed distribution of Gamma $(1,\theta)$ and Gamma $(2,\theta)$ through finite mixture concept with mixing proportion as $\omega = \frac{\theta}{\theta+\alpha}$. Statistical properties such as stochastic ordering, moments, kurtosis, skewness, hazard function and mean residual lifetime (MRL) were discussed. Parameter estimation was done using MLE technique. A generalized two-parameter Lindley distribution (AG2PLD) proved to be a better fit than LD when applied on both lifetime data.

Shanker and Mishra (2013 a) proposed and introduced a quasi Lindley distribution (QLD) as a two component finite gamma mixed distribution of Gamma $(1,\theta)$ and Gamma $(2,\theta)$ with mixing proportion as $\omega = \frac{\alpha}{\alpha+1}$. Statistical properties such as stochastic orderings, failure rate, mean residual lifetime and moments were discussed. The model was fitted to two sets of data. The data sets were on survival times of 72 guinea pigs that were exposed to virulent tubercle bacterial and grouped mortality data of blackbird species. QLD was a better fit than LD.

Similarly, another form of QLD was proposed by Shanker and Amanuel (2013) as a two component mixed distribution of Gamma $(1,\theta)$ and Gamma $(2,\theta)$ with mixing weight as $\omega = \frac{\alpha}{\theta^2+\alpha}$. The properties stochastic orderings, moments, kurtosis, skewness failure rate, mean deviations, Lorenz curve and mean residual lifetime were discussed. New QLD was fitted to survival data in days of 72 guinea pigs, waiting times of 100 bank customers and grouped mortality data. The study proved that new QLD is a better fit than QLD for modeling both survival and waiting times data.

Another case of a generalized two parameter Lindley distribution was introduced by Shanker and Mishra (2013 b). The two parameter Lindley distribution is two component finite gamma mixed distribution of Gamma $(1,\theta)$ and Gamma $(2,\theta)$ with mixing weight as $\omega = \frac{\alpha\theta}{\alpha\theta+1}$. Statistical properties such as stochastic orderings, moments, failure rate and mean residual lifetime were discussed. The distribution was fitted to both survival and waiting times data sets with estimation technique as maximum likelihood and method of moments. Evidently, the two parameter Lindley distribution was more flexible than LD in modeling both survival and waiting times data.

Al-Babtain et el. (2015) introduced two cases of a generalized four parameter Lindley distribution as special cases of a generalized five parameter Lindley distribution. Case 1 version of a generalized Lindley distribution was introduced as a two component finite mixed distribution of Gamma (α,θ) and Gamma (β,θ) with mixing proportion as $\omega = \frac{\theta}{\theta+k}$. Similarly, case 2 version of a generalized four parameter Lindley distribution was introduced as a finite mixed distribution of Gamma (α, θ) and Gamma (β, θ) with mixing weight as $\omega = \frac{\theta}{\theta+\eta}$. Both statistical properties and estimation were discussed.

A generalized five parameter Lindley distribution was introduced as a two component finite mixed distribution of Gamma (α, θ) and Gamma (β, θ) with mixing proportion as $\omega = \frac{k\theta}{\eta+k\theta}$ (Al-Babtain et el. 2015). Statistical properties such as survival function, kurtosis, skewness, hazard rate, reversed hazard rate, lorenz and Bonferroni curves, moments among others were discussed. The parameters were estimated using both MLE and MOME. The distribution was fitted to lifetime data on stress breaking of carbon fibers. A generalized five parameter Lindley distribution was best fitted among exponential, LD and AG3PLD.

Another generalized three parameter Lindley distribution (Case 2) was introduced by Shanker et al. (2017) as a two component finite mixed distribution of Gamma $(1, \theta)$ and Gamma $(2, \theta)$ with mixing proportion as $\omega = \frac{\alpha\theta}{\alpha\theta+1}$. Some of the statistical properties discussed are stochastic orderings, mean deviations, failure rate, mean residual lifetime among others. The distribution was fitted on survival data in days of 72 guinea pigs using maximum likelihood estimation technique. Case 2 version of a generalized three parameter Lindley distribution proved to be a better fit than case 1 version of a generalized three parameter Lindley distribution by Zakerzadeh and

Dolati (2009).

1.6.2 Other two-Component Finite Gamma Mixtures

Akash Distribution and its Generalizations

Shanker (2015a) introduced one parameter Akash distribution as a two component finite mixed distribution of Gamma $(1,\theta)$ and Gamma $(3,\theta)$ with mixing weight as $\omega = \frac{\theta^2}{\theta^2+2}$. Both statistical and mathematical properties such as moments, Lorenz curve, mean deviations, stochastic orderings, mean residual lifetime among others were discussed. Estimation technique was MLE and MOME. The distribution was fitted to two lifetime data sets. One parameter Akash distribution was well fitted than LD and exponential distribution.
Another case of one parameter Akash distribution introduced in discrete Akash (Abebe and Shanker (2018)). properties of the distribution were estimated using MLE technique and fitted to a lifetime data. Discrete Akash distribution was found to be of better fit than other discrete distributions such as discrete Lindley , discrete Shanker among others.
Shanker et el. (2018) introduced Aksh distribution with two parameters as a finite mixed distribution of Gamma $(1,\theta)$ and Gamma $(2,\theta)$ with mixing probability as $\omega = \frac{\alpha\theta^2}{\alpha\theta^2+2}$. Statistical and mathematical properties such as moments, stress-strength analysis, mean residual lifetime, hazard rate among others. The distribution was fitted on a failure times of 15 electric components. A generalized two parameter Akash distribution was found to be best fitted distribution than exponential, one parameter Akash and log-normal distributions.

Shanker Distribution and its Generalizations

Shanker (2015b) introduced one parameter Shanker distribution as a finite mixed distribution of Gamma $(1,\theta)$ and Gamma $(2,\theta)$ with mixing probability as $\omega = \frac{\theta^2}{\theta^2+1}$. Statistical and mathematical properties such as Lorenz curve,. order statistics, Renyi Entropy, stochastic orderings, failure rate, mean residual lifetime among others were discussed. The distribution was fitted to three types of lifetime data. The lifetime data are waiting times of 100 bank customers, relief times of 20 patients in minutes and glass strength data of aircraft window. One parameter Shanker distribution was a better fit than LD and exponential distribution.

Rama Distribution and its Generalizations

A two component finite mixed distribution of Gamma $(1,\theta)$ and Gamma $(4,\theta)$ with mixing proportion as $\omega = \frac{\theta^3}{\theta^3+6}$ known as Rama distribution (Shanker, 2017a). Statistical properties such as Lorenz curve, failure rate, moments and related measures, mean deviations, stochastic orderings, stress- strength analysis and Bonferroni curve. One parameter Rama distribution was fitted to a glass strength data of aircraft window. The estimation methods were both MOME and MLE. It was found that one parameter Rama distribution gave a better fit than other one parameter distributions such as exponential, LD, Sujatha, Amarendra, Akash, Shanker and Aradhana.

A generalized two parameter Rama distribution is introduced as a finite mixed distribution of Gamma $(1,\theta)$ and Gamma $(4,\theta)$ with mixing probability as $\omega=\frac{\alpha\theta^3}{\alpha\theta^3+6}$ (Edith et al., 2019). Mathematical and statistical properties such as Lorenz curve, mean deviations, moments, skewness, kurtosis, failure rate, survival function, mean residual lifetime and stochastic orderings were constructed. The generalized two parameter Rama distribution was fitted to a lifetime data of 200 patients that were diagnosed with hepatitis B during their second hospital visit to a general hospital located in Eastern Nigeria. The parameters were estimated using both MLE and MOME. It was found that a generalized two parameter Rama distribution provided a better fit than exponential, LD, Shanker, Akash, two-parameter Akash and Ishita.

Suja Distribution and its Generalizations

A two-component finite mixed distribution of Gamma $(1,\theta)$ and Gamma $(5,\theta)$ with mixing weight as $\omega=\frac{\theta^4}{\theta^4+24}$. The distribution would be called Suja distribution (Shanker, 2017b). Both statistical and mathematical properties such as moments and related measures, Bonferroni and Lorenz curves, failure rate, mean residual lifetime among others have been discussed. The introduced distribution was fitted to a lifetime data (strength data of aircraft window). Estimation method was MOME and MLE techniques. The proposed distribution was well fitted than Akash, Sujatha, Shanker, Devya, Amarendra, LD, exponential and Aradhana distributions.

A generalized two parameter Suja distribution known as "Power Length-Biased Suja distribution" was introduced (Al-Omari et al., 2019). Statistical properties were studied. The distribution was fitted to a lifetime data (failure times of 84 windshield of an aircraft). The model parameters were estimated using MLE technique. The power Length-Biased Suja Distribution (PLBSD) was a better fit than Garima, Rama, Ishita, Sushila, Suja and length-biased Suja distributions.

1.6.3 Three-Component Finite Gamma Mixtures

Sujatha Distribution and its Generalizations

A three component finite mixture of Gamma $(1,\theta)$, Gamma $(2,\theta)$ and Gamma $(3,\theta)$ with mixing proportions $\omega_1 = \frac{\theta^2}{\theta^2+\theta+2}$ and $\omega_2 = \frac{\theta}{\theta^2+\theta+2}$ known as Sujatha distribution was introduced (Shanker,2016a). Statistical and mathematical properties such as shapes of the pdf and cdf, moments and related measures, stochastic orderings, among others were discussed. The proposed distribution was fitted to three lifetime data sets (relief data of 20 patients,glass strength data of aircraft window and tensile strength of 69 carbon fibers recorded in GPa). The model parameter was estimated using both MOME and MLE estimation techniques. Sujatha distribution provided a better fit than exponential, LD, Shanker and Akash distributions.

Shanker (2016b) introduced a generalized two parameter Sujatha distribution as a three component finite mixture of Gamma

$(1,\theta)$, Gamma $(2,\theta)$ and Gamma $(3,\theta)$ with mixing probabilities as $\omega = \frac{\alpha\theta}{\alpha\theta+\theta+2}$ and $\omega_2 = \frac{\theta}{\alpha\theta+\theta+2}$. The generalized two parameter Sujatha distribution is referred to as Quasi Sujatha distribution (QSD). Some of the statistical properties derived are mean deviations, Lorenz and Bonferroni curves, moments and failure rate. The proposed distribution was fitted to a lifetime data on failure times on 15 electric components using both MOME and MLE techniques. QSD was found to be a better fit than WLD, weibull, log-normal, Sujatha, LD, exponential and Gamma distribution.

Another form of a generalized two parameter Sujatha distribution (AG2PSD) was introduced a three component finite mixture of Gamma $(1,\theta)$, Gamma $(2,\theta)$ and Gamma $(3,\theta)$ with mixing probability as $\omega_1 = \frac{\theta^2}{\theta^2+\theta+2\alpha}$ and $\omega_2 = \frac{\theta}{\theta^2+\theta+2\alpha}$ (Shanker et al., 2017c). The properties of the distribution were discussed. The two parameter distribution was fitted to a lifetime data sets using both MOME and MLE estimation techniques. AG2PSD provided a better fit than LD, Sujatha, Aradhana and Exponential distributions.

Similarly, another form of a generalized two parameter Sujatha distribution was introduced as three component finite mixture of Gamma $(1,\theta)$, Gamma $(2,\theta)$ and Gamma $(3,\theta)$ with mixing proportions as $\omega_1 = \frac{\theta^3}{\theta^3+\alpha\theta+2\alpha}$ and $\omega_2 = \frac{\alpha\theta}{\theta^3+\alpha\theta+2\alpha}$ (Shanker, 2020). The generalized two parameter Sujatha distribution is known as New Quasi Sujatha distribution (NQSD). Statistical properties derived were mean deviations, kurtosis, stochastic orderings, skewness, Lorenz and Bonferroni curves, moments among others were discussed. The distribution was fitted to a lifetime data (on tensile strength of 69 carbon fibers) using MLE and MOME techniques. NQSD provided a better fit than QSD, exponential, LD and Sujatha distribution in modeling lifetime data.

Aradhana Distribution and its Generalizations

A three component finite mixed distribution of Gamma $(1, \theta)$, Gamma $(2, \theta)$ and Gamma $(3, \theta)$ with mixing probabilities as $\omega_1 = \frac{\theta^2}{\theta^2+2\theta+2}$ and $\omega_2 = \frac{2\theta}{\theta^2+2\theta+2}$ (Shanker, 2016c). Properties such as mean deviations, skewness, index of dispersion, Bonferroni and Lorenz curves, stochastic orderings, order statistics, stress-strength analysis were discussed. The introduced distribution was fitted to lifetime data sets using both MOME and MLE estimation techniques. One parameter Aradhana distribution was found to be a better fit than exponential, LD, Akash and Shanker distributions.

Weldon and Shanker (2018) introduced a generalized two parameter Aradhana distribution (AG2PAD) as a finite mixed distribution of Gamma $(1, \theta)$, Gamma $(2, \theta)$ and Gamma $(3, \theta)$ with mixing proportions as $\omega_1 = \frac{\theta^2}{\theta^2+2\alpha\theta+2\alpha}$ and $\omega_2 = \frac{2\alpha\theta}{\theta^2+2\alpha\theta+2\alpha}$. Mathematical and statistical and mathematical properties were discussed. AG2PAD was fitted to a lifetime data using MLE as the main estimation method. AG2PAD was a better fit than QSD, Aradhana, LD and exponential distributions.

Similarly, another form of generalized two parameter distribution was introduced as a three component finite finite mixture of Gamma $(1,\theta)$, Gamma $(2,\theta)$ and Gamma $(3,\theta)$ with mixing proportions as $\omega_1 = \frac{\theta^2}{\theta^2+2\alpha\theta+2\alpha}$ and $\omega_2 = \frac{2\alpha\theta}{\theta^2+2\alpha\theta+2\alpha}$. The distribution is known as Quasi Aradhana distribution (Shanker et al.,2018). Vital statistical and mathematical properties such as CDF and PDF shapes of varying parameters, Lorenz and Bonferroni curves, moments and related measures, stochastic orderings, reliability measures and mean deviations were derived and discussed. QAD was fitted to a lifetime data on failure times of 15 electronic components using both MOME and MLE estimation techniques. QAD provided a better fit than exponential, LD, Weibull, Aradhana, Gamma, log-normal and exponentiated exponential distribution.

1.7 Significance of the Study

1. There are data that do not fit to the already known distributions. As such, finite gamma mixed distributions are applied to fit such data.
2. Finite gamma mixtures have been used as mixing distributions to come up with more flexible distributions.
3. Generalizing a finite gamma mixed distribution increases flexibility of the distribution. A flexible model capable in capturing heterogeneity in the data.
4. Finite gamma mixtures play a significant role in reliability analysis in engineering to understand failure rate of a machine.
5. Finite gamma mixtures can be used to study frequency of insurance claims within a given period.

2 TWO COMPONENT FINITE GAMMA MIXTURE (Case of Lindley Distribution)

2.1 Introduction

A two component finite gamma mixture a case of Lindley distribution and its generalizations is considered in this chapter. We shall construct and derive statistical properties of Lindley distribution and its generalizations. The mixed distribution is expressed in terms of pdf and Cdf. Reliability measures such as hazard function, survival function, mean residual function (MRL), equilibrium distribution and pdf of excess loss function have been derived for Lindley distribution and its generalizations. The moments both about mean and centralized ones are derived.

2.2 One parameter Lindley distribution

2.2.1 Construction of one parameter Lindley distribution

Proposition 2.2.1. *One parameter Lindley distribution is a finite mixed distribution of Gamma(1,θ) and Gamma(2,θ) with weighing probability as* $\omega = \frac{\theta}{\theta+1}$. *The pdf* $f(x;\theta)$ *and Cdf* $F(x;\theta)$ *of one parameter LD are;*

$$f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}; \quad (\theta+1)>0, x>0, \theta>0 \tag{2.1}$$

$$F(x;\theta) = 1 - \frac{(\theta+\theta x+1)e^{-\theta x}}{1+\theta}; \quad (\theta+1)>0, x>0, \theta>0 \tag{2.2}$$

Proof. Using Eq.1.1, the pdf of LD is obtained as;

$$
\begin{aligned}
f(x;\theta) &= \frac{\theta}{\theta+1}(\theta e^{-\theta x}) + \frac{1}{\theta+1}(\theta^2 e^{-\theta x} x) \\
&= \frac{\theta^2}{\theta+1} e^{-\theta x} + \frac{\theta^2 x}{\theta+1} e^{-\theta x} \\
&= \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}; \quad (\theta+1) > 0, x > 0, \theta > 0
\end{aligned}
$$

The pdf is a shown in Eq.2.1 above.
The Cdf of LD in Eq.2.1 is obtained as shown below;

$$
\begin{aligned}
F(x;\theta) &= \int_0^\infty f(x)dx \\
&= \int_0^\infty \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}dx \\
&= \frac{\theta^2}{\theta+1} I_1 \\
I_1 &= \int_0^\infty (1+x)e^{-\theta x} \\
u &= (1+x) \implies du = dx \\
dv &= e^{-\theta x} \implies v = \frac{-e^{-\theta x}}{\theta} \\
I_1 &= -(1+x)\frac{e^{-\theta x}}{\theta} - \frac{1}{\theta}\int_0^\infty -e^{-\theta x}dx \\
I_1 &= -(1+x)\frac{e^{-\theta x}}{\theta} - \frac{1}{\theta} I_2 \\
I_2 &= \int_0^\infty -e^{-\theta x}dx \\
Let\ u &= -e^{-\theta x} \implies du = -\theta dx \implies \frac{du}{-\theta} = dx
\end{aligned}
$$

$$I_2 = \int_0^{\infty} -e^u \frac{du}{-\theta}$$
$$I_2 = \frac{1}{\theta}\int_0^{\infty} e^u du$$
$$I_2 = \frac{1}{\theta}e^{-\theta x}$$

Combining I_1 and I_2 we have the following

$$= \frac{\theta^2}{\theta+1}\int_0^{\infty}(1+x)e^{-\theta x}dx$$
$$= \frac{\theta^2}{\theta+1}[\frac{-(1+x)e^{-\theta x}}{\theta} - \frac{e^{-\theta x}}{\theta^2}]$$
$$= \frac{-\theta(1+x)e^{-\theta x}}{1+\theta} - \frac{e^{-\theta x}}{1+\theta}$$
$$= \frac{-(\theta+\theta x+1)e^{-\theta x}}{1+\theta}$$
$$F(x;\theta) = 1 - \frac{(\theta+\theta x+1)e^{-\theta x}}{1+\theta}; \quad (\theta+1)>0, x>0, \theta>0$$

The Cdf of LD is as shown in Eq. 2.2. □

2.2.2 Reliability Analysis

Proposition 2.2.2. *The survival function denoted by $S(x,\theta)$ and hazard function denoted by $h(x;\theta)$ of one parameter Lindley distribution Eq. 2.1 are:*

$$S(x;\theta) = \frac{(\theta+\theta x+1)e^{-\theta x}}{1+\theta}; \quad (\theta+1)>0, x>0, \theta>0 \tag{2.3}$$
$$h(x;\theta) = \frac{\theta^2(1+x)}{\theta+\theta x+1}; \quad x>0, \theta>0 \tag{2.4}$$

Proof. Using Eq.1.8, the survival function of LD is obtained as;

$$S(x;\theta) = 1-(1-\frac{(\theta+\theta x+1)e^{-\theta x}}{1+\theta})$$
$$= \frac{(\theta+\theta x+1)e^{-\theta x}}{1+\theta}; \quad (\theta+1)>0, x>0, \theta>0$$

By definition of hazard function Eq.1.9, the hazard function denoted by $h(x;\theta)$ of LD Eq.2.1 is obtained as;

$$h(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\frac{(1+\theta)}{(\theta+\theta x+1)e^{-\theta x}}$$
$$= \frac{\theta^2(1+x)}{\theta+\theta x+1}; \quad x>0, \theta>0$$

□

2.2.3 Moments and related measures

Proposition 2.2.3. *The r^{th} moments of LD about the origin are obtained using both moment generating technique and methods of moments are:*

$$\mu_r^1 * = \frac{r!(\theta+r+1)}{\theta^r(\theta+1)} \quad ; \quad r=1,2,3,\cdots \tag{2.5}$$

Proof. By definition of method of moments Eq.1.16, the moment of LD are obtained as;

$$= \int_0^\infty x^r \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}dx$$
$$= \frac{\theta^2}{\theta+1}\left[\int_0^\infty x^r e^{-\theta x}dx + \int_0^\infty x^{r+1}e^{-\theta x}dx\right]$$

Using the relation $\frac{\Gamma\alpha}{\theta^\alpha} = \int_0^\infty t^{\alpha-1}e^{-\theta t}dt$

$$\begin{aligned}
\int_0^\infty x^r e^{-\theta x}dx &= \frac{\Gamma(r+1)}{\theta^{r+1}} \\
\int_0^\infty x^{r+1} e^{-\theta x}dx &= \frac{\Gamma(r+2)}{\theta^{r+2}} \\
&= \frac{\theta^2}{1+\theta}\left[\frac{\Gamma(r+1)}{\theta^{r+1}} + \frac{\Gamma(r+2)}{\theta^{r+2}}\right] \\
&= \frac{\theta^2}{1+\theta}\left[\frac{r\Gamma r}{\theta^{r+1}} + \frac{(r+1)r\Gamma r}{\theta^{r+2}}\right] \\
&= \frac{r\Gamma r(\theta+r+1)}{\theta^r(\theta+1)} \\
E(X^r) &= \frac{r!(\theta+r+1)}{\theta^r(\theta+1)}; \quad r = 1,2,3,\cdots
\end{aligned}$$

Similarly, by definition of mgf Eq.1.17, moments about the origin of LD are obtained as;

$$\begin{aligned}
M_x(t) &= \int_0^\infty e^{tx}\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}dx \\
&= \frac{\theta^2}{\theta+1}\int_0^\infty e^{-(\theta-t)x}(1+x)dx \\
&= \frac{\theta^2}{\theta+1}\left[\frac{1}{\theta-t} + \frac{1}{(\theta-t)^2}\right] \\
&= \frac{\theta^2}{\theta+1}\left[\frac{1}{\theta}\sum_{k=0}^\infty (\frac{t}{\theta})^k + \frac{1}{\theta^2}\sum_{k=0}^\infty \binom{k+1}{k}(\frac{t}{\theta})^k\right] \\
M_x(t) &= \sum_{k=0}^\infty \frac{(\theta+(k+1))}{\theta+1}(\frac{t}{\theta})^k
\end{aligned}$$

The r^{th} moments of a one parameter Lindley distribution are obtained as a coefficient of $\frac{t^r}{r!}$ of the moment generating function $m_x(t)$ as;

$$\mu_r^1 = \frac{r!(\theta+r+1)}{\theta^r(\theta+1)}; \quad r = 1,2,3,\cdots$$

□

setting $r = 1,2,3$ and 4 in Eq.2.5 we obtain the moments about the origin for one parameter Lindley distribution are as:

$$\mu_1^1 = \frac{\theta+2}{\theta(\theta+1)}, \quad \mu_2^1 = \frac{2(\theta+3)}{\theta^2(\theta+1)}$$
$$\mu_3^1 = \frac{6(\theta+4)}{\theta^3(\theta+1)}, \quad \mu_4^1 = \frac{24(\theta+5)}{\theta^4(\theta+1)}$$

The centralized moments of LD are derived as;

$$
\begin{aligned}
\mu_1 &= \mu_1^1 \\
\mu_2 &= \mu_2^1 - (\mu_1^1)^2 \\
\mu_2 &= \frac{2(\theta+3)}{\theta^2(\theta+1)} + \frac{(\theta+2)^2}{\theta^2(\theta+1)^2} \\
\mu_2 &= \frac{2(\theta+3)(\theta+1) - (\theta+2)^2}{\theta^2(\theta+1)^2} \\
\mu_2 &= \frac{\theta^2+4\theta+2}{\theta^2(\theta+1)^2} \Longrightarrow \sigma^2 \\
\mu_3 &= u_3^1 - 3u_2^1 u_1^1 + 2(u_1^1)^3 \\
\mu_3 &= \frac{6(\theta+4)}{\theta^3(\theta+1)} - 3(\frac{2(\theta+3)}{\theta^2(\theta+1)}\frac{\theta+2}{\theta(\theta+1)}) + 2(\frac{\theta+2}{\theta(\theta+1)})^3 \\
&= \frac{6(\theta+4)(\theta+1)^2 - (\theta+1)(6\theta^2+30\theta+36) + (2\theta^3+12\theta^2+24\theta}{\theta^3(1+\theta)^3} \\
\mu_3 &= \frac{2\theta^3+12\theta^2+12\theta+4}{\theta^3(1+\theta)^3} \\
\mu_4 &= \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2 \\
&= \frac{24(\theta+5)}{\theta^4(\theta+1)} - 24(\frac{(\theta+4)}{\theta^3(\theta+1)}\frac{(\theta+2)}{\theta(\theta+1)}) + 12(\frac{(\theta+2)}{\theta(\theta+1)}\frac{(\theta+3)}{\theta^2(\theta+1)}) - 3 \\
\mu_4 &= \frac{9\theta^4+72\theta^3+132\theta^2+96\theta+24}{\theta^4(\theta+1)^4} \\
\mu_4 &= \frac{3\left(3\theta^4+24\theta^3+44\theta^2+32\theta+8\right)}{\theta^4(\theta+1)^4}
\end{aligned}
$$

Proposition 2.2.4. *Other related measures such as variation coefficient (c.v), skewness (υ_1),kurtosis (υ_2) and dispersion index (υ_3) of a one parameter Lindley distribution Eq.2.1 are derived* *as:*

$$C.v = \frac{\sqrt{\theta^2+4\theta+2}}{(\theta+2)} \tag{2.6}$$

$$\upsilon_1 = \frac{2\theta^3+12\theta^2+12\theta+4}{(\theta^2+4\theta+2)^{\frac{3}{2}}} \tag{2.7}$$

$$\upsilon_2 = \frac{9\theta^4+72\theta^3+132\theta^2+96\theta+24}{(\theta^2+4\theta+2)^2} \tag{2.8}$$

$$\upsilon_3 = \frac{\theta^2+4\theta+2}{\theta^3+3\theta^2+2\theta} \tag{2.9}$$

The variation coefficient of LD Eq.2.1 is obtained as;

Proof.

$$C.v = \frac{\sigma}{\mu_1}$$
$$= \frac{\sqrt{\theta^2+4\theta+2}}{\theta(\theta+1)}\frac{\theta(\theta+1)}{\theta+2} \Longrightarrow \frac{\sqrt{\theta^2+4\theta+2}}{\theta+2}$$

Skewness value Eq.2.7is given as;

$$\upsilon_1 = \frac{\mu_3}{(\sigma^2)^{\frac{3}{2}}}$$
$$= \frac{2\theta^3+12\theta^2+12\theta+4}{\theta^3(1+\theta)^3}\left(\frac{\theta^2(\theta+1)^2}{\theta^2+4\theta+2}\right)^{\frac{3}{2}} \Longrightarrow \frac{2\theta^3+12\theta^2+12\theta+4}{(\theta^2+4\theta+2)^{\frac{3}{2}}}$$

Kurtosis value denoted by υ_2 in Eq.2.8is obtained as;

$$\upsilon_2 = \frac{\mu_4}{(\mu_2)^2}$$
$$= \frac{3\left(3\theta^4+24\theta^3+44\theta^2+32\theta+8\right)}{\theta^4(\theta+1)^4}\left(\frac{\theta^2(\theta+1)^2}{\theta^2+4\theta+2}\right)^2 \Longrightarrow \frac{3(3\theta^4+24\theta^3}{(\theta^2 -}$$

Dispersion index denoted by υ_3 of LD Eq.2.1 is obtained as;

$$\upsilon_3 = \frac{\sigma^2}{\mu_1^1}$$
$$= \frac{(\theta^2+4\theta+2)}{\theta^2(\theta+1)^2}\frac{(\theta(\theta+1))}{\theta+2} \Longrightarrow \frac{\theta^2+4\theta+2}{\theta(\theta+1)(\theta+2)} \Longrightarrow \frac{\theta^2+4\theta+2}{\theta^3+3\theta^2+2\theta}$$

□

2.2.4 Excess loss distribution

Proposition 2.2.5. *The probability density function of excess loss distribution $f_e(x;\theta)$, mean residual lifetime (m(x)), survival function based on equilibrium distribution $S_e(x;\theta)$ and hazard function based on the equilibrium distribution $h_e(x;\theta)$ for one parameter Lindley distribution Eq.2.1 are:*

$$f_l(x;\theta) = \frac{\theta^2(1+x)}{\theta+\theta z+1}e^{-(x-z)\theta}; \quad x > z \tag{2.10}$$

$$m(x) = \frac{\theta+\theta x+2}{\theta(\theta+\theta x+1)} \tag{2.11}$$

$$f_e(x,\theta) = \frac{\theta(\theta+\theta x+1)e^{-\theta x}}{(\theta+2)} \tag{2.12}$$

$$S_e(x;\theta) = \frac{1}{\theta+2}(\theta+\theta x+2)e^{-\theta x} \tag{2.13}$$

$$h_e(x;\theta) = \frac{\theta(\theta+\theta x+1)}{\theta x+\theta+2} \tag{2.14}$$

Using the relation in Eq.1.10, pdf of excess of LD Eq.2.1 is obtained as;

Proof.

$$\begin{aligned} f_l(x) &= \frac{f(x)}{S(z)}; \quad x > z \\ &= \frac{\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}}{\frac{\theta+\theta z+1}{\theta+1}e^{-\theta z}} \\ &= \frac{\theta^2(1+x)e^{-\theta x}}{(\theta+\theta z+1)e^{-\theta z}} \\ &= \frac{\theta^2(1+x)}{\theta+\theta z+1}e^{-(x-z)\theta}; \quad x > z \end{aligned}$$

As shown in Eq.2.10.
Applying the relation in Eq.1.11, mean excess loss of LD Eq.2.1 is obtained as;

$$\begin{aligned} m(x) &= \frac{1+\theta}{(\theta+\theta x+1)e^{-\theta x}} \int_x^\infty [\frac{\theta+\theta t+1}{1+\theta}]e^{-\theta t}dt \\ &= \frac{1}{(\theta+\theta x+1)e^{-\theta x}} \int_x^\infty [\theta+\theta t+1]e^{-\theta t}dt \end{aligned}$$

Taking the part $\int_x^\infty [\theta+\theta t+1]e^{-\theta t}dt$ and integrating using integration by parts technique we have;

$$\begin{aligned}
u &= \theta+\theta t+1 \implies du = \theta dt \\
dv &= e^{-\theta t} \implies v = \frac{-e^{-\theta t}}{\theta} \\
&= -(\theta+\theta t+1)\frac{e^{-\theta t}}{\theta} + \theta \int_x^\infty \frac{e^{-\theta t}}{\theta} dt \\
&= \frac{e^{-\theta x}(\theta+\theta x+2)}{\theta} \\
&= \frac{\theta+\theta x+2}{\theta(\theta+\theta x+1)}
\end{aligned}$$

As shown in Eq.2.11.

Using the relation Eq.1.12, equilibrium distribution of LD Eq.2.1 is obtained as;

$$\begin{aligned}
f_e(x;\theta) &= \frac{(\theta+\theta x+1)e^{-\theta x}}{(1+\theta)} \frac{\theta(\theta+1)}{(\theta+2)} \\
&= \frac{\theta(\theta+\theta x+1)e^{-\theta x}}{(\theta+2)}
\end{aligned}$$

By definition Eq.1.13, survival function based on the equilibrium distribution denoted by $S_e(x;\theta)$ of LD Eq.2.1 is obtained as;

$$
\begin{aligned}
S_e(x;\theta) &= \frac{\int_x^\infty S(t)dt}{E(x)}, \quad x > 0 \\
&= \frac{\int_x^\infty \frac{(\theta+\theta t+1)}{(1+\theta)} e^{-\theta t}}{\frac{\theta+2}{\theta(\theta+1)}} dt \\
&= \frac{\theta}{\theta+2} \int_x^\infty (\theta+\theta t+1) e^{-\theta t} dt
\end{aligned}
$$

Using integration by parts technique we have:

$$
\begin{aligned}
u = (\theta+\theta t+1) &\Longrightarrow du = \theta dt \\
dv = e^{-\theta t} &\Longrightarrow v = \frac{-e^{-\theta t}}{\theta} \\
&= \frac{-e^{-\theta t}}{\theta}(\theta+\theta t+1) + \int_x^\infty \frac{e^{-\theta t}}{\theta} \theta dt \\
&= \left[\frac{-e^{-\theta t}}{\theta}(\theta+\theta t+1) - \frac{e^{-\theta t}}{\theta} \right]_x^\infty \\
&= \frac{e^{-\theta x}}{\theta}(\theta+\theta x+2) \\
&= \frac{\theta}{\theta+2} \frac{e^{-\theta x}}{\theta}(\theta+\theta x+2) \\
S_e(x;\theta) &= \frac{1}{\theta+2}(\theta+\theta x+2) e^{-\theta x}
\end{aligned}
$$

As shown in Eq.2.13 above.
Using the relation Eq.1.14, equilibrium distribution hazard func-

tion of LD Eq.2.1 is obtained as;

$$h_e(x;\theta) = \frac{\theta(\theta+\theta x+1)e^{-\theta x}}{(\theta+2)} \frac{(\theta+2)}{(\theta+\theta x+2)e^{-\theta x}} \Longrightarrow \frac{\theta(\theta+\theta x+1)}{\theta x+\theta+2}$$

□

2.3 Two Parameter Lindley Distribution

2.3.1 Construction of Two Parameter Lindley Distribution

Proposition 2.3.1. *AG2PLD is a two component finite mixed distribution of Gamma (1, θ) and Gamma (2, θ) with weighing proportion $\omega = \frac{\theta}{\theta+\beta}$, pdf and Cdf of AG2PLD are;*

$$f(x;\beta,\theta) = \frac{\theta^2}{\theta+\beta}(1+\beta x)e^{-\theta x}, \theta > 0, \beta > 0; (\theta+\beta) > 0, x > 0 \tag{2.15}$$

$$F(x;\beta,\theta) = 1 - \left[1 + \frac{\theta\beta x}{(\theta+\beta)}\right]e^{-\theta x}, \theta > 0, \beta > 0; (\theta+\beta) > 0, x > 0 \tag{2.16}$$

The pdf of AG2PLD Eq.2.15 is obtained through the concept of finite mixture defined in Eq.1.1 as shown below;

Proof.

$$\begin{aligned} f(x;\beta,\theta) &= \frac{\theta}{\theta+\beta}[\theta e^{-\theta x}] + \frac{\beta}{\theta+\beta}(\theta^2 e^{-\theta x}x) \\ &= \frac{\theta^2}{\theta+\beta}e^{-\theta x} + \frac{\beta\theta^2}{\theta+\beta}xe^{-\theta x} \\ &= \frac{\theta^2}{\theta+\beta}(1+\beta x)e^{-\theta x}; \theta > 0, \beta > 0(\theta+\beta) > 0, x > 0 \end{aligned}$$

As shown in Eq. 2.16.

Remark 2.3.2. *AG2PLD is nested with two distributions that are one parameter LD and exponential distribution at different values of* β*.*
putting $\beta = 1$*, a generalized two parameter Lindley distribution Eq.2.15 turns to one parameter Lindley distribution Eq.2.1 that is a two-component finite mixed distribution of* $Gamma(1, \theta)$ *and* $Gamma(2, \theta)$
Similarly, putting $\beta = 0$ *a generalized two parameter Lindley distribution Eq.2.15 reduces to an exponential distribution.*

$$f(x;\theta) = \theta e^{-\theta x}; \quad x > 0, \theta > 0 \tag{2.17}$$

This shows that a generalized two parameter Lindley distribution Eq.2.15 is more flexible than one parameter Lindley distribution Eq.2.1.

Similarly, the Cdf of AG2PLD Eq.2.15 is obtained as;

$$\begin{aligned} F(x, \beta, \theta) &= \int_0^\infty \frac{\theta^2}{\theta + \beta}(1 + \beta x)e^{-\theta x}dx \\ &= \frac{\theta^2}{\theta + \beta}\int_0^\infty (1 + \beta x)e^{-\theta x}dx \\ &= \frac{\theta^2}{\theta + \beta}I_1 \end{aligned}$$

Where $I_1 = \int_0^\infty (1+\beta x)e^{-\theta x}dx$

$$du = 1+\beta x \implies du = \beta dx$$
$$dv = e^{-\theta x} \implies v = \frac{-e^{-\theta x}}{\theta}$$
$$I_1 = \frac{-e^{-\theta x}}{\theta}(1+\beta x) - \int_0^\infty \frac{-e^{-\theta x}}{\theta}\beta dx$$
$$I_1 = \frac{-e^{-\theta x}}{\theta}(1+\beta x) - I_2$$
$$I_2 = \frac{\beta}{\theta}\int_0^\infty -e^{-\theta x}dx$$

Let $u = -\theta x$ then,

$$du = -\theta x dx \implies \frac{du}{-\theta} = dx$$
$$= \frac{\beta}{\theta}\int_0^\infty -e^u \frac{du}{-\theta} \implies \frac{\beta}{\theta^2}\int_o^\infty e^u du$$
$$I_2 = \frac{\beta}{\theta^2}e^{-\theta x}$$

Combining I_1 and I_2 we have,

$$\begin{aligned}
I_1 &= -(1+\beta x)\frac{e^{-\theta x}}{\theta} - \frac{\beta(e^{-\theta x})}{\theta^2} \\
&= \frac{\theta^2}{\theta+\beta}\int_0^{\infty}(1+\beta x)e^{-\theta x}dx \\
&= 1 + -\frac{\theta(1+\beta x)e^{-\theta x}}{\theta+\beta} - \frac{\beta e^{-\theta x}}{\theta+\beta} \\
&= 1 - \frac{\theta e^{-\theta x} - \theta\beta e^{-\theta x}}{\theta+\beta} - \frac{\beta e^{-\theta x}}{\theta+\beta} \\
&= 1 - [\frac{\theta+\beta+\theta\beta x}{\theta+\beta}]e^{-\theta x} \\
F(x;\beta,\theta) &= 1 - [1 + \frac{\theta\beta x}{\theta+\beta}]e^{-\theta x}; \theta > 0, \beta > 0, (\theta+\beta) > 0, x > 0
\end{aligned}$$

□

2.3.2 Reliability Analysis

Proposition 2.3.3. *The survival function denoted by $S(x;\beta,\theta)$ and hazard function denoted by $h(x;\beta,\theta)$ for the two parameter Lindley distribution Eq.2.15 are:*

$$S(x;\beta,\theta) = (\frac{\theta+\beta+\theta\beta x}{\theta+\beta})e^{-\theta x}; \beta > 0, \theta > 0, (\theta+\beta) > 0, x > 0 \tag{2.18}$$

$$h(x;\beta,\theta) = \frac{\theta^2(1+\beta x)}{\theta+\beta+\theta\beta x}; \beta > 0, \theta > 0, x > 0 \tag{2.19}$$

By definition of survival function in Eq.1.8, survival function of AG2PLD Eq.2.15 is obtained as;

Proof.

$$S(x;\beta,\theta) = 1-[1-[1+\frac{\theta\beta x}{\theta+\beta}]e^{-\theta x}]$$
$$= 1-1+[1+\frac{\theta\beta x}{\theta+\beta}]e^{-\theta x}$$
$$= [1+\frac{\theta\beta x}{\theta+\beta}]e^{-\theta x}, \theta>0, \beta>0, x>0$$

As shown in Eq. 2.18. Using the relation Eq.1.9, hazard function of AG2PLD is obtained as;

$$h(x;\beta,\theta) = \frac{\frac{\theta^2}{\theta+\beta}(1+\beta x)e^{-\theta x}}{(\frac{\theta+\beta+\theta\beta x}{\theta+\theta})}$$
$$= \frac{\theta^2(1+\beta x)}{\theta+\beta+\theta\beta x}; \beta>0, \theta>0, x>0$$

As shown in Eq. 2.19. □

2.3.3 Moments and related measures

Proposition 2.3.4. *In this section, we use both moment generating function (mgf) and r^{th} moments techniques to get moments of a generalized two parameter Lindley distribution Eq.2.15. The expression for the r^{th} moments using both moment generating function (mgf) and method of moments is given as:*

$$\mu_r^1 * = \frac{r!(\theta+\beta(r+1))}{\theta^r(\theta+\beta)}, \quad for \quad r=1,2,3,\cdots \qquad (2.20)$$

Proof. By definition of moments in Eq.1.16, r^{th} moments of AG2PLD Eq.2.15 are obtained as;

$$
\begin{aligned}
E(X^r) &= \int_0^\infty x^r \frac{\theta^2}{\theta+\beta}(1+\beta x)e^{-\theta x}dx \\
&= \frac{\theta^2}{\theta+\beta}\int_0^\infty x^r(1+\beta x)e^{-\theta x}dx \\
&= \frac{\theta^2}{\theta+\beta}\left[\int_0^\infty x^r e^{-\theta x}dx+\beta\int_0^\infty x^{r+1}e^{-\theta x}dx\right]
\end{aligned}
$$

Using the relation $\int_0^\infty t^{r-1}e^{-\theta t}dt = \frac{\Gamma r}{\theta^r}$ we have;

$$
\begin{aligned}
\int_0^\infty x^r e^{-\theta x}dx &= \frac{\Gamma(r+1)}{\theta^{r+1}} \\
\beta\int_0^\infty x^{r+1}e^{-\theta x}dx &= \beta\frac{\Gamma(r+2)}{\theta^{r+2}} \\
&= \frac{\theta^2}{\theta+\beta}\frac{\Gamma(r+1)}{\theta^{r+1}+\frac{\beta(\Gamma r+2)}{\theta^{r+2}}} \\
&= \frac{\theta^2}{\theta+\beta}[\frac{r\Gamma r}{\theta^{r+1}}+\frac{\beta(r+1)r\Gamma r}{\theta^{r+2}}] \\
&= \frac{r\Gamma r(\theta+\beta(r+1))}{\theta^r(\theta+\beta)}; \quad for \quad r=1,2,3,\cdots \\
E(X^r) &= \frac{r!(\theta+\beta(r+1))}{\theta^r(\theta+\beta)}; \quad r=1,2,3,\cdots
\end{aligned}
$$

Using the relation in Eq.1.17, mgf of AG2PLD is obtained as;

$$
\begin{aligned}
m_x(t) &= \int_0^{\infty} e^{tx}\frac{\theta^2}{\theta+\beta}(1+\beta x)e^{-\theta x}dx \\
&= \frac{\theta^2}{\theta+\beta}\int_0^{\infty} e^{tx}(1+\beta x)e^{-\theta x}dx \\
&= \frac{\theta^2}{\theta+\beta}[\frac{1}{(\theta-t)}+\frac{\beta}{(\theta-t)^2}] \\
&= \frac{\theta^2}{\theta+\beta}[(\frac{1}{\theta})\sum_{k=0}^{\infty}(\frac{t}{\theta})^k+(\frac{\beta}{\theta^2})\sum_{k=0}^{\infty}\binom{k+1}{k}(\frac{t}{\theta})^k] \\
&= \sum_{k=0}^{\infty}\frac{\theta+\beta(k+1)}{\theta+\beta}(\frac{t}{\theta})^k
\end{aligned}
$$

The none centralized moments of AG2PLD.2.15 are obtained as a coefficient of $\frac{t^r}{r!}$ of the moment generating function $m_x(t)$ as shown below;

$$
\mu_r^1 = \frac{r!(\theta+\beta(r+1))}{\theta^r(\theta+\beta)}; \quad r = 1,2,3,\cdots
$$

□

we now obtain the first four moments about the origin of a generalized two parameter Lindley distribution Eq.2.15 by putting $r = 1,2,3$ and 4 in Eq.2.20 as shown below:

$$
\begin{aligned}
\mu_1^1 &= \frac{(\theta+2\beta)}{\theta(\theta+\beta)}, \quad \mu_2^1 = \frac{2(\theta+3\beta)}{\theta^2(\theta+\beta)} \\
\mu_3^1 &= \frac{6(\theta+4\beta)}{\theta^3(\theta+\beta)}, \quad \mu_4^1 = \frac{24(\theta(\theta+5\beta))}{\theta^4(\theta+\beta)}
\end{aligned}
$$

The first four moments about the mean are calculated as;

$$\mu_1 = \mu_1^i$$
$$\mu_2 = \mu_2^1 - (\mu_1^1)^2$$
$$= \frac{2(\theta+3\beta)}{\theta^2(\theta+\beta)} - (\frac{\theta+2\beta}{\theta(\theta+\beta)})^2$$
$$\mu_2 = \frac{\theta^2+4\beta\theta+2\beta^2}{\theta^2(\theta+\beta)^2}$$
$$\mu_3 = \mu_3^1 - 3\mu_2^1\mu_1^1 + 2(\mu_1^1)^3$$
$$= \frac{6\theta+24\beta}{\theta^3(\theta+\beta)} - 3\left(\frac{(2\theta+6\beta)}{\theta^2(\theta+\beta)}\frac{(\theta+2\beta)}{\theta(\theta+\beta)}\right) + 2\left(\frac{\theta+2\beta}{\theta(\theta+\beta)^2}\right)$$
$$\mu_3 = \frac{2\theta^3+12\beta\theta^2+12\beta^2\theta+4\beta^3}{\theta^3(\theta+\beta)^3}$$
$$\mu_4 = \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)$$
$$= \frac{24(\theta+5\beta)}{\theta^4(\theta+\beta)} - 4[\frac{6(\theta+4\beta)}{\theta^3(\theta+\beta)}\frac{\theta+2\beta}{\theta(\theta+\beta)}] + 6[\frac{\theta+2\beta}{\theta(\theta+\beta)}\frac{2(\theta+3\theta)}{\theta^2(\theta+\beta)}] - 3[\frac{\theta+}{\theta(\theta}$$
$$\mu_4 = \frac{9\theta^4+72\theta^3\beta+72\theta^2\beta+60\theta\beta^2-264\theta\beta^3+480\theta\beta^2+24\beta^4}{\theta^4(\theta+\beta)^4}$$

Proposition 2.3.5. *Other related measures such as variation coefficient (c.v), skewness (υ_1), kurtosis (υ_2) and index of dispersion*

(υ_3) are;

$$C.v = \frac{\sqrt{\theta^2+4\beta\theta+2\beta^2}}{\theta+2\beta} \tag{2.21}$$

$$\upsilon_1 = \frac{2\theta^3+12\theta^2\beta+12\theta\beta^2+4\beta^3}{(\theta^2+4\beta\theta+2\beta^2)^{\frac{3}{2}}} \tag{2.22}$$

$$\upsilon_2 = \frac{9\theta^4+72\theta^3\beta+72\theta^2\beta+60\theta\beta^2-264\theta\beta^3+480\theta\beta^2+24\beta^4}{(\theta^2+4\beta\theta+2\beta^2)^2} \tag{2.23}$$

$$\upsilon_3 = \frac{\theta^2+4\beta\theta+2\beta^2}{\theta(\theta+\beta)(\theta+2\beta)} \tag{2.24}$$

The $C.V$ coefficient In 2.21 is obtained as;

Proof.

$$\begin{aligned} C.v &= \frac{\sigma}{\mu_1^1} \\ &= \frac{\sqrt{\theta^2+4\beta\theta+2\beta^2}}{\theta(\theta+\beta)}\frac{\theta(\theta+\beta)}{\theta+2\beta} \Longrightarrow \frac{\sqrt{\theta^2+4\beta\theta+2\beta^2}}{\theta+2\beta} \end{aligned}$$

Skewness coefficient υ_1 in 2.22 is obtained as;

$$\begin{aligned} \upsilon_1 &= \frac{\mu_4}{(\mu_2)^{\frac{3}{2}}} \\ &= \frac{2\theta^3+12\theta^2\beta+12\theta\beta^2+4\beta^3}{\theta^3(\theta+\beta)^3}\left(\frac{\theta^2(\theta+\beta)^2}{\theta^2+4\beta\theta+2\theta^2}\right)^{\frac{3}{2}} \\ &= \frac{2\theta^3+12\theta^2\beta+12\theta\beta^2+4\beta^3}{(\theta^2+4\beta\theta+2\beta^2)^{\frac{3}{2}}} \end{aligned}$$

Kurtosis coefficient υ_2 in 2.23 is obtained as;

$$\upsilon_2 = \frac{\mu_4}{(\mu_2)^2}$$
$$= \frac{9\theta^4 + 72\theta^3\beta + 72\theta^2\beta + 60\theta\beta^2 - 264\theta\beta^3 + 480\theta\beta^2 + 24\beta^4}{\theta^4(\theta+\beta)^4}\left(\frac{\theta^2(\theta +}{\theta^2 + 4\theta\beta}\right.$$
$$= \frac{9\theta^4 + 72\beta\theta^3 + 72\beta\theta^2 + 60\beta^2\theta^2 - 264\beta^3 + 360\beta^2 + 24\beta^4}{(\theta^2 + 4\theta\beta + 2\beta^2)^2}$$

Dispersion index υ_3 in 2.24 is derived as;

$$\upsilon_3 = \frac{\sigma^2}{\mu_1^1}$$
$$= \frac{\theta^2 + 4\beta\theta + 2\beta^2}{\theta^2(\theta+\beta)^2}\frac{\theta(\theta+\beta)}{\theta+2\beta} \Longrightarrow \frac{\theta^2 + 4\beta\theta + 2\beta^2}{\theta(\theta+\beta)(\theta+2\beta)} \Longrightarrow \frac{\theta^2 + 4\beta\theta + 2\beta^2}{\theta^3 + 3\beta\theta^2 + 2\beta^2}$$

□

2.3.4 Excess Loss and Equilibrium Distribution

Proposition 2.3.6. *The pdf of excess loss distribution $f_l(x;\beta,\theta)$, Mean residual lifetime (m(x)), equilibrium distribution $f_e(x;\beta,\theta)$,* *survival function $S_e(x;\beta,\theta)$ and hazard function $h_e(x;\beta,\theta)$ are:*

$$f_l(x;\beta,\theta)=\frac{\theta^2(1+\beta x)}{\theta+\beta+\theta\beta}e^{-(x-z)\theta};\quad x>z \tag{2.25}$$

$$m(x)=\frac{(\theta+2\beta+\theta\beta x)}{\theta(\theta+\beta+\theta\beta x)} \tag{2.26}$$

$$f_e(x;\beta,\theta)=\frac{\theta(\theta+\beta+\theta\beta x)e^{-\theta x}}{2\beta+\theta} \tag{2.27}$$

$$S_e(x;\beta,\theta)=\frac{(\theta+2\beta+\theta\beta x)e^{-\theta x}}{\theta+2\theta} \tag{2.28}$$

$$h_e(x;\beta,\theta)=\frac{\theta(\theta+\beta+\theta\beta x)}{\theta+2\beta+\theta\beta x} \tag{2.29}$$

Proof. Using the relation Eq.1.10, the excess loss pdf of AG2PLD Eq.2.15 is obtained as;

$$\begin{aligned}
f_l(x;\beta,\theta)&=\frac{f(x)}{S(z)};\quad x>z\\
&=\frac{\frac{\theta^2}{\theta+\beta}(1+\beta x)e^{-\theta x}}{\frac{\theta+\beta+\theta\beta z}{\theta+\beta}e^{-\theta z}};\quad x>z\\
&=\frac{\theta^2(1+\beta x)}{\theta+\beta+\theta\beta}e^{-(x-z)\theta};\quad x>z
\end{aligned}$$

By definition of excess loss distribution Eq.1.11, mean residual time of AG2PLD Eq.2.15 is obtained as;

$$\begin{aligned}
m(x)&=\frac{\theta+\beta}{(\theta+\beta+\theta\beta x)e^{-\theta x}}\int_x^{\infty}\frac{(\theta+\beta+\theta\beta t)e^{-\theta t}}{\theta+\beta}dt\\
&=\frac{1}{(\theta+\beta+\theta\beta x)e^{-\theta x}}\int_x^{\infty}(\theta+\beta+\theta\beta t)e^{-\theta t}dt
\end{aligned}$$

Taking the part $\int_x^\infty (\theta+\beta+\theta\beta t)e^{-\theta t}dt$ and using integration by parts technique we have the following:

$$
\begin{aligned}
u &= (\theta+\beta+\theta\beta t) \Longrightarrow du = \theta\beta dt \\
dv &= e^{-\theta t} \Longrightarrow \frac{-e^{-\theta t}}{\theta} \\
&= -(\theta+\beta+\theta\beta t)\frac{e^{-\theta t}}{\theta} + \int_x^\infty \frac{e^{-\theta t}}{\theta}\theta\beta dt \\
&= -(\theta+\beta+\theta\beta t)\frac{e^{-\theta t}}{\theta} + \beta\int_x^\infty e^{-\theta t}dt \\
&= -(\theta+\beta+\theta\beta t)\frac{e^{-\theta t}}{\theta} - \frac{\beta e^{-\theta t}}{\theta} \\
&= -\frac{e^{-\theta t}}{\theta}[\theta+\beta+\theta\beta t+\beta]_x^\infty \\
&= \frac{e^{-\theta x}}{\theta}(\theta+2\beta+\theta\beta x) \\
&= \frac{1}{(\theta+\beta+\theta\beta x)e^{-\theta x}}\frac{e^{-\theta x}}{\theta}(\theta+2\beta+\theta\beta x) \\
&= \frac{(\theta+2\beta+\theta\beta x)}{\theta(\theta+\beta+\theta\beta x)}
\end{aligned}
$$

As shown in 2.26. We obtain equilibrium distribution in Eq.2.27 using the relation defined in Eq.1.12 as;

$$
\begin{aligned}
f_e(x;\beta,\theta) &= \frac{S(x)}{E(x)} \\
&= \frac{(\theta+\beta+\theta\beta x)e^(-\theta x)}{(\theta+\beta)}\frac{\theta(\theta+\beta)}{\theta+2\beta} \\
&= \frac{\theta(\theta+\beta+\theta\beta x)e^{-\theta x}}{2\beta+\theta}
\end{aligned}
$$

Using the relation in Eq. 1.13, $S_e(x;\beta,\theta)$ in Eq.2.28 is obtained as;

$$
\begin{aligned}
S_e(x;\beta,\theta) &= \frac{\int_x^\infty \frac{(\theta+\beta+\theta\beta t)e^{-\theta t}}{\theta+\beta}dt}{\frac{\theta+2\theta}{\theta(\theta+\beta)}} \\
S(t;\beta,\theta) &= \frac{1}{\theta+\beta}\int_x^\infty (\theta+\beta+\beta\theta t)e^{-\theta t} = \frac{e^{-\theta x(\theta+2\beta+\theta\beta x)}}{\theta(\theta+\beta)} \\
S_e(x;\beta,\theta) &= \frac{e^{-\theta x(\theta+2\beta+\theta\beta x)}}{\theta(\theta+\beta)}\frac{\theta(\theta+\beta)}{\theta+2\beta} \\
&= \frac{(\theta+2\beta+\theta\beta x)e^{-\theta x}}{\theta+2\theta}
\end{aligned}
$$

Applying the formula defined in Eq.1.14, hazard function based on the equilibrium distribution of AG2PLD 2.29 is obtained as;

$$
\begin{aligned}
h_e(x;\beta,\theta) &= \frac{\theta(\theta+\beta+\theta\beta x)e^{-\theta x}}{2\beta+\theta}\frac{\theta+2\beta}{(\theta+2\beta+\theta\beta x)e^{-\theta x}} \\
&= \frac{\theta(\theta+\beta+\theta\beta x)}{\theta+2\beta+\theta\beta x}
\end{aligned}
$$

As shown in Eq.2.29. □

Remark 2.3.7. *It is noted that mean residual lifetime (MRL) and hazard function of the equilibrium distribution of AG2PLD Eq.2.15 satisfy the following;*

$$m(0) = \frac{\theta + 2\beta}{\theta(\theta + \beta)} = \mu_1^1$$

$$h(x;\beta,\theta) = \frac{1}{m(x)} = \frac{\theta(\theta + \beta + \beta\theta x)}{\theta + 2\beta + \beta\theta x}$$

2.4 Quasi Lindley Distribution

2.4.1 Construction of Quasi Lindley distribution

Proposition 2.4.1. *Quasi Lindley is a finite mixed distribution of Gamma (1,θ) and Gamma (2,θ) with mixing weight as $\omega = \frac{\alpha}{\alpha+1}$, pdf and Cdf of QLD are:*

$$f(x;\alpha,\theta) = \frac{\theta}{\alpha + 1}(\alpha + \theta x)e^{-\theta x}; \alpha > 0, \theta > 0, (\alpha + 1) > 0, x > 0 \tag{2.30}$$

$$F(x;\alpha,\theta) = 1 - \left[1 + \frac{\theta x}{\alpha + 1}\right]e^{-\theta x}; \alpha > 0, \theta > 0, (\alpha + 1) > 0, x > 0 \tag{2.31}$$

Proof. Applying the concept of finite mixtures defined in Eq.1.1, the pdf of QLD is derived as;

$$\begin{aligned} f(x;\alpha,\theta) &= \frac{\alpha}{\alpha+1}(\theta e^{-\theta x}) + \frac{1}{\alpha+1}(\theta^2 e^{-\theta x}x) \\ &= \frac{\alpha\theta}{\alpha+1}e^{-\theta x} + \frac{\theta}{\alpha+1}e^{-\theta x}x \\ &= \frac{\theta}{\alpha+1}\left[\alpha e^{-\theta x} + e^{-\theta x}\theta x\right] \\ &= \frac{\theta}{\alpha+1}(\alpha+\theta x)e^{-\theta x}\alpha > 0, \theta > 0, (\alpha+1) > 0, x > 0 \end{aligned}$$

As shown in Eq.2.30 above.

Remark 2.4.2. *Quasi Lindley distribution Eq.2.30 is nested with two Lindley family distributions at* $\alpha = 0$ *and* $\alpha = \theta$*. Putting* $\alpha = \theta$*, the Quasi Lindley Eq.2.30 reduces to a one parameter Lindley distribution Eq.2.1.*
Similarly, putting $\alpha = 0$ *the Quasi Lindley distribution Eq.2.30 reduces to a gamma distribution of the form;*

$$f(x;\theta) = \theta^2 x e^{-\theta x}; \quad x > 0, \theta > 0 \tag{2.32}$$

Further the Cdf of QLD Eq.2.30 is obtained as;

$$\begin{aligned} F(x;\alpha,\theta) &= \frac{\theta}{\alpha+1}\int_0^\infty (\alpha+\theta x)e^{-\theta x}dx \\ F(x;\alpha,\theta) &= \frac{\theta}{\alpha+1}I_1 \end{aligned}$$

Where $I_1 = \int_0^{\infty}(\alpha+\theta x)e^{-\theta x}dx$ using integration by parts technique we have:

$$u = (\alpha+\theta x) \Longrightarrow du = \theta dx$$
$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$
$$I_1 = -(\alpha+\theta x)\frac{e^{-\theta x}}{\theta} - \int_0^{\infty} -e^{-\theta x}dx$$
$$I_1 = -(\alpha+\theta x)\frac{e^{-\theta x}}{\theta} - I_2$$

Where $I_2 = \int_0^{\infty} -e^{-\theta x}dx$ let $u = -\theta x$ we have;

$$u = -\theta x \Longrightarrow \frac{du}{-\theta} = dx$$
$$= \int_0^{\infty} -e^u \frac{du}{-\theta} = \int_0^{\infty} \frac{e^u}{\theta}du = \frac{1}{\theta}\int_0^{\infty} e^u du$$
$$I_2 = \frac{1}{\theta}e^{-\theta x}$$

Combining I_1 and I_2 we have ;

$$I_1 = -(\alpha+\theta x)\frac{e^{-\theta x}}{\theta} - \frac{e^{-\theta x}}{\theta}$$
$$F(x;\alpha,\theta) = \frac{\theta}{\alpha+1}\int_0^{\infty}(\alpha+\theta x)e^{-\theta x}dx$$
$$= 1 + -\frac{(\alpha+\theta x)e^{-\theta x}}{\alpha+1} - \frac{e^{-\theta x}}{\alpha+1}$$
$$= 1 + -\frac{\alpha e^{-\theta x} - \theta x e^{-\theta x} - e^{-\theta x}}{\alpha+1}$$
$$= 1 + -(\frac{\alpha+1+\theta x}{\alpha+1})e^{-\theta x}$$
$$F(x;\alpha,\theta) = 1 - \left[1 + \frac{\theta x}{\alpha+1}\right]e^{-\theta x}; \alpha > 0, \theta > 0, (\alpha+1) > 0, x > 0$$

As shown in Eq.2.31. □

2.4.2 Reliability Analysis

Proposition 2.4.3. *Survival function denoted by $S(x;\alpha,\theta)$ and hazard function denoted by $h(x;\alpha,\theta)$ of Quasi Lindley distribution Eq. 2.30 are;*

$$S(x;\alpha,\theta)=\left[1+\frac{\theta x}{\alpha+1}\right]e^{-\theta x};\theta>0,\alpha>0,(\alpha+1)>0,x>0 \tag{2.33}$$

$$h(x;\alpha,\theta)=\frac{\theta(\alpha+\theta x)}{\alpha+\theta x+1};\theta>0,\alpha>0,x>0 \tag{2.34}$$

Proof. Survival function denoted by $S(x;\alpha,\theta)$ in Eq.2.33 is obtained using the relation Eq. 1.8 as;

$$\begin{aligned}S(x;\alpha,\theta)&=1-F(x)\\&=1-\left[1-\left[1+\frac{\theta x}{\alpha+1}\right]e^{-\theta x}\right]\\&=\left[1+\frac{\theta x}{\alpha+1}\right]e^{-\theta x};\theta>0,\alpha>0,(\alpha+1)>0,x>0\end{aligned}$$

hazard function of QLD presented in Eq.2.34 is derived using the relation Eq.1.9 as;

$$\begin{aligned}h(x;\alpha,\theta)&=\frac{\frac{\theta}{\alpha+1}(\alpha+\theta x)e^{-\theta x}}{\left[\frac{\alpha+1+\theta x}{\alpha+1}\right]e^{-\theta x}}\\&=\frac{\theta(\alpha+\theta x)}{\alpha+\theta x+1};\theta>0,\alpha>0,x>0\end{aligned}$$

□

2.4.3 Moments and other measures

Proposition 2.4.4. *The moments about origin of the Quasi Lindley distribution Eq.2.30 are obtained using both mgf and moment technique. The moments are:*

$$\mu_r^1 * = \frac{r!(\alpha+r+1)}{\theta^r(\alpha+1)}, \quad r = 1,2,3,\cdots \tag{2.35}$$

`Proof.` Using the relation in 1.16, the moments of QLD Eq.2.30 are obtained as;

$$= \frac{\theta}{\alpha+1}\int_0^\infty x^r(\alpha+\theta x)e^{-\theta x}dx$$
$$= \frac{\theta}{\alpha+1}\left[\alpha\int_0^\infty x^r e^{-\theta x}dx + \theta\int_0^\infty x^{r+1}e^{-\theta x}dx\right]$$

Using the relation $\int_0^\infty t^{\alpha-1}e^{-\theta t}dt = \frac{\Gamma(\alpha)}{\theta^\alpha}$ we have the following;

$$\alpha\int_0^\infty x^r e^{-\theta x}dx = \frac{\alpha\Gamma(r+1)}{\theta^{r+1}}$$
$$\theta\int_0^\infty x^{r+1}e^{-\theta x}dx = \frac{\theta\Gamma(r+2)}{\theta^{r+2}}$$
$$= \frac{\theta}{\alpha+1}\left[\frac{\alpha\Gamma(r+1)}{\theta^{r+1}} + \frac{\theta\Gamma(r+2)}{\theta^{r+2}}\right]$$
$$= \frac{\theta}{\alpha+1}\left[\frac{\alpha r\Gamma r}{\theta^{r+1}} + \frac{\theta(r+1)r\Gamma r}{\theta^{r+2}}\right]$$
$$= \frac{r\Gamma r}{\theta^r(\alpha+1)}\big(\alpha+(r+1)\big)$$
$$E(X^r) = \frac{r!(\alpha+r+1)}{\theta^r(\alpha+1)}, r = 1,2,3,\cdots$$

By definition of mgf in Eq.1.17, mgf of a random variable from QLD is derived as;

$$
\begin{aligned}
&= \frac{\theta}{\alpha+1}\int_0^\infty e^{-(\theta-t)x}(\alpha+\theta x)dx \\
&= \frac{\theta}{\alpha+1}\left[\frac{\alpha}{(\theta-t)}+\frac{\theta}{(\theta-t)^2}\right] \\
&= \frac{\theta}{\alpha+1}\left[\frac{\alpha}{\theta}\sum_{k=0}^{\infty}(\frac{t}{\theta})^k+\frac{1}{\theta}\sum_{k=0}^{\infty}\binom{k+1}{k}(\frac{t}{\theta})^k\right] \\
m_x(t) &= \sum_{k=0}^{\infty}\frac{(\alpha+(k+1))}{\theta^r(\alpha+1)}(\frac{t}{\theta})^k
\end{aligned}
$$

The r^{th} moments of Quasi Lindley distribution Eq.2.30 are obtained as a coefficient of $\frac{t^r}{r!}$ of the moment generating function $m_x(t)$ as shown below;

$$
\mu_r^1 = \frac{r!(\alpha+(r+1))}{\theta^r(\alpha+1)}, r = 1,2,3,\cdots
$$

□

The four moments of a generalized Quasi Lindley distribution Eq.2.30 about the origin are obtained by putting $r = 1,2,3$ and 4 in Eq.2.35 as:

$$
\mu_1^1 = \frac{\alpha+2}{\theta(\alpha+1)}, \quad \mu_2^1 = \frac{2(\alpha+3)}{\theta^2(\alpha+1)}
$$

$$
\mu_3^1 = \frac{6(\alpha+4)}{\theta^3(\alpha+1)}, \quad \mu_4^1 = \frac{24(\alpha+5)}{\theta^4(\alpha+1)}
$$

The centralized moments are:

$$\mu_1 = \mu_1^1 = \frac{\alpha+2}{\theta(\alpha+1)}$$
$$\mu_2 = \mu_2^1 - (\mu_1^1)^2$$
$$= \frac{2(\alpha+3)}{\theta^2(\alpha+1)} - \left[\frac{\alpha+2}{\theta^2(\alpha+1)}\right]^2$$
$$\mu_2 = \frac{\alpha^2+4\alpha+2}{(\theta(\alpha+1))^2}$$
$$\mu_3 = 2(\mu_1^1)^2\mu_1^1 - 3\mu_1^1\mu_2^1 + \mu_3^1$$
$$\mu_3 = \frac{2\alpha^3+12\alpha^2+12\alpha+4}{\theta^3(\alpha+1)^3}$$
$$\mu_4 = \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2$$
$$\mu_4 = \frac{9\alpha^4+72\alpha^3+132\alpha^2+96\alpha+24}{\theta^4(\alpha+1)^4}$$

Proposition 2.4.5. *Other related measures such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) of a Quasi Lindley distribution Eq.2.30 are as;*

$$C.v = \frac{\sqrt{(\alpha^2+4\alpha+2)}}{\alpha+2} \tag{2.36}$$
$$\upsilon_1 = \frac{2\alpha^3+12\alpha^2+12\alpha+4}{(\alpha^2+4\alpha+2)^{\frac{3}{2}}} \tag{2.37}$$
$$\upsilon_2 = \frac{9\alpha^4+72\alpha^3+132\alpha^2+96\alpha+24}{(\alpha^2+4\alpha+2)^2} \tag{2.38}$$
$$\upsilon_3 = \frac{\alpha^2+4\alpha+2}{\theta(\alpha+1)(\alpha+2)} \tag{2.39}$$

Proof. The variation coefficient 2.36 of QLD Eq.2.30 is obtained as;

$$C.v = \frac{\sigma}{\mu_1}$$
$$= (\frac{\sqrt{\alpha^2+4\alpha+2}}{\theta(\alpha+1)})\frac{\theta(\alpha+1)}{\alpha+2} \Longrightarrow \frac{\sqrt{(\alpha^2+4\alpha+2)}}{\alpha+2}$$

The skewness coefficient in Eq.2.37 is obtained as;

$$\upsilon_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}}$$
$$= \frac{2\alpha^3+12\alpha^2+12\alpha+4}{\theta^3(\alpha+1)^3}\left[\frac{\theta^3(\alpha+1)^3}{\alpha^2+4\alpha+2}\right]^{\frac{3}{2}} \Longrightarrow \frac{2\alpha^3+12\alpha^2+12\alpha+4}{(\alpha^2+4\alpha+2)^{\frac{3}{2}}}$$

Kurtosis coefficient 2.38 is obtained as;

$$\upsilon_2 = \frac{\mu_4}{(\mu_2)^2}$$
$$\upsilon_2 = \frac{9\alpha^4+72\alpha^3+132\alpha^2+96\alpha+24}{\theta^4(\alpha+1)^4}\left[\frac{\theta^2(\alpha+1)^2}{\alpha^2+4\alpha+2}\right]^2 \Longrightarrow \frac{9\alpha^4+72\alpha^3 -}{(\alpha^2}$$

Index of dispersion in Eq.2.39 is obtained as;

$$\upsilon_3 = \frac{\sigma^2}{\mu_1^1}$$
$$\upsilon_3 = \frac{\alpha^2+4\alpha+2}{\theta^2(\alpha+1)^2} * \frac{\theta(\alpha+1)}{\alpha+2}$$
$$\upsilon_3 = \frac{\alpha^2+4\alpha+2}{\theta(\alpha+1)(\alpha+2)}$$

□

2.4.4 Excess loss distribution

Proposition 2.4.6. *The probability density function of excess loss distribution $f_e(x;\alpha,\theta)$, mean residual lifetime (m(x)), survival function based on equilibrium distribution $S_e(x;\alpha,\theta)$ and hazard function based on the equilibrium distribution $h_e(x;\alpha,\theta)$ for of Quasi Lindley distribution 2.30 are:*

$$f_e(x;\alpha,\theta) = \frac{\theta(\alpha+\theta x)e^{-(x-z)\theta}}{(\alpha+1+\theta z)}; \quad x > z \tag{2.40}$$

$$m(x) = \frac{\alpha+\theta x+2}{\theta(\alpha+\theta x+1)} \tag{2.41}$$

$$f_e(x;\alpha,\theta) = \frac{\theta(\alpha+\theta x+1)e^{-\theta x}}{\alpha+2} \tag{2.42}$$

$$S_e(x;\alpha,\theta) = \frac{(\alpha+\theta x+2)e^{-\theta x}}{\alpha+2} \tag{2.43}$$

$$h_e(x;\alpha,\theta) = \frac{\theta(\alpha+1)}{\alpha+\theta x+2} \tag{2.44}$$

Proof. Using the relation in Eq.1.10 we obtain pdf of excess loss in Eq.2.40 as;

$$\begin{aligned} f_e(x) &= \frac{f(x)}{S(z)}, \quad x > z \\ &= \frac{\frac{\theta}{\alpha+1}(\alpha+\theta x)e^{-\theta x}}{(\frac{\alpha+1+\theta z}{\alpha+1})e^{-\theta z}} \\ f_e(x;\alpha,\theta) &= \frac{\theta(\alpha+\theta x)e^{-(x-z)\theta}}{(\alpha+1+\theta z)}; \quad x > z \end{aligned}$$

By definition of mean residual lifetime in Eq.1.11, mean residual lifetime of QLD Eq.2.30 in Eq.2.41 is obtained as;

$$\begin{aligned}
m(x) &= \frac{\alpha+1}{(\alpha+\theta x+1)e^{-\theta x}}\int_x^\infty \frac{\alpha+\theta t+1}{\alpha+1}e^{-\theta t}dt \\
&= \frac{\alpha+1}{(\alpha+\theta x+1)e^{-\theta x}}\frac{1}{\alpha+1}\int_x^\infty \alpha+\theta t+1)e^{-\theta t}dt \\
&= \frac{1}{(\alpha x+1)e^{-\theta x}}\int_x^\infty (\alpha+\theta t+1)e^{-\theta t}dt
\end{aligned}$$

Taking the part $\int_x^\infty (\alpha+\theta t+1)e^{-\theta t}dt$ and using integration by parts then;

$$\begin{aligned}
u &= (\alpha+\theta t+1) \Longrightarrow du = \theta dt \\
dv &= e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta} \\
&= -\left[(\alpha+\theta t+1)\frac{e^{-\theta t}}{\theta}\right]_x^\infty + \int_x^\infty e^{-\theta t}dt \\
&= -\frac{-e^{-\theta t}}{\theta}[\alpha+\theta t+2]|_x^\infty \\
&= \frac{e^{-\theta x}}{\theta}(\alpha+\theta x+2) \\
&= \frac{1}{(\alpha+\theta x+1)e^{-\theta x}}\frac{e^{-\theta x}}{\theta}(\alpha+\theta x+2) \\
m(x) &= \frac{\alpha+\theta x+2}{\theta(\alpha+\theta x+1)}
\end{aligned}$$

By the definition of equilibrium distribution in Eq.1.12, equilibrium distribution in Eq.2.42 of QLD Eq.2.30 is obtained as;

$$\begin{aligned} f_e(x;\alpha,\theta) &= \frac{(\alpha+\theta x+1)e^{-\theta x}}{\alpha+1}\frac{\theta(\alpha+1)}{\alpha+2} \\ &= \frac{\theta(\alpha+\theta x+1)e^{-\theta x}}{\alpha+2} \end{aligned}$$

Equilibrium distribution survival function in Eq.2.43 is obtained using the relation in Eq.1.13 as;

$$\begin{aligned} \int_x^{\infty} S(t;\alpha,\theta)dt &= \frac{(\alpha+\theta x+2)e^{-\theta x}}{\theta(\alpha+1)} \\ &= \frac{(\alpha+\theta x+2)e^{-\theta x}}{\theta(\alpha+1)}\frac{\theta(\alpha+1)}{\alpha+2} \\ S_e(x;\alpha,\theta) &= \frac{(\alpha+\theta x+2)e^{-\theta x}}{\alpha+2} \end{aligned}$$

hazard function based on the equilibrium distribution in Eq.2.44 is obtained using the relation Eq.1.14 as;

$$\begin{aligned} h_e(x) &= \frac{f_e(x)}{S_e(x)} \\ &= \frac{\theta(\alpha+\theta x+1)e^{-\theta x}}{\alpha+2}\frac{\alpha+2}{(\alpha+\theta x+2)e^{-\theta x}} \\ h_e(x;\alpha,\theta) &= \frac{\theta(\alpha+1)}{\alpha+\theta x+2} \end{aligned}$$

□

Remark 2.4.7. *It is noted that mean residual lifetime and hazard function of the equilibrium distribution satisfy the following;*

$$m(0) = \frac{\alpha+2}{\theta(\alpha+1)} = \mu_1^1$$
$$h_e(x;\alpha,\theta) = \frac{1}{m(x)} = \frac{\theta(\alpha+1)}{\alpha+\theta x+2}$$

2.5 Three Parameter Lindley distribution

2.5.1 Construction of three parameter Lindley distribution

Proposition 2.5.1. *AG3PLD is constructed as a finite mixed distribution of Gamma(1,θ) and Gamma(2,θ) with weighing proportion as $\omega = \frac{\alpha\theta}{\alpha\theta+\beta}$, Pdf and Cdf of AG3PLD Eq.2.45 are;*

$$f(x;\alpha,\beta,\theta) = \frac{\theta^2}{\alpha\theta+\beta}(\alpha+\beta x)e^{-\theta x}; x>0, \beta>0, \alpha>0, (\alpha\theta+\beta)>0 \tag{2.45}$$

$$F(x;\alpha,\beta,\theta) = 1-[1+\frac{\beta\theta x}{\alpha\theta+\beta}]e^{-\theta x}; x>0, \beta>0, \alpha>0, \quad (\alpha\theta+\beta)> \tag{2.46}$$

Proof. The pdf of AG3PLD Eq.2.45 is constructed as a two component finite mixture using the relation Eq.1.1 as;

$$= \frac{\alpha\theta^2}{\alpha\theta+\beta}(\theta e^{-\theta x}) + \frac{\beta}{\alpha\theta+\beta}(\theta^2 e^{-\theta x}x)$$
$$= \frac{\theta^2\alpha e^{-\theta x}}{\alpha\theta+\beta} + \frac{\theta^2\beta e^{-\theta x}x}{\alpha\theta+\beta}$$
$$f(x;\alpha,\beta,\theta) = \frac{\theta^2}{\alpha\theta+\beta}(\alpha+\beta x)e^{-\theta x}; x>0, \quad \beta>0, \alpha>0, (\alpha\theta+\beta)>$$

Remark 2.5.2. *A generalized three parameter Lindley distribution Eq.2.45 is nested with six Lindley family distributions at different values of* β *and* α.
Putting $\beta = \theta$, *AG3PLD reduces to a QLD in Eq.2.30.*
Putting $\beta = 1$, *AG3PLD Eq.2.45 reduces to AG2PLD of the form;*

$$f(x;\alpha,\theta) = \frac{\theta^2}{\alpha\theta+1}\left[\alpha+x\right]e^{-\theta x}; \quad x>0, \alpha>0, \theta>0, (\alpha\theta+1)>0 \tag{2.47}$$

Putting $\alpha = 1$, *AG3PLD reduces to AG2PLD Eq.2.15.*
Similarly, putting ($\alpha = \theta, \quad \beta = \alpha$) *AG3PLD Eq.2.45 reduces to a NQLD of the form;*

$$f(x;\theta,\alpha) = \frac{\theta^2}{\theta^2+\alpha}\left[\theta^2+\alpha x\right]e^{-\theta x}; \quad \alpha>0, \theta>0, (\theta^2+\alpha)>0, x>0 \tag{2.48}$$

A generalized new two parameter Quasi Lindley Eq.2.48was introduced by Shanker and Amanuel (2013) as two component finite mixture of $x_1 \sim$ *gamma (1,*θ*) and* $x_2 \sim$ *gamma (2,*θ*) with mixing proportion as* $\omega = \frac{\alpha}{\theta^2+\alpha}$.
Putting ($\alpha = \beta = 1$) *and* ($\alpha = \theta, \beta = 0$) *a generalized three parameter Lindley distribution Eq.2.45 reduces to one parameter Lindley distribution Eq.2.1 and gamma (1,*θ*) respectively.*

Further the Cdf Eq.2.46 is obtained as;

$$
\begin{aligned}
F(x;\alpha,\beta,\theta) &= \frac{\theta^2}{\alpha\theta+\beta}\int_0^\infty (\alpha\theta+\beta x)e^{-\theta x}dx \\
&= \frac{\theta^2}{\alpha\theta+\beta} I_1 \\
I_1 &= \int_0^\infty (\alpha\theta+\beta x)e^{-\theta x}dx \\
u &= (\alpha + x\beta) \Longrightarrow du = \beta dx \\
dv &= e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta} \\
I_1 &= -(\alpha + x\beta)\frac{e^{-\theta x}}{\theta} - \frac{\beta}{\theta}\int_0^\infty -e^{-\theta x}dx \\
I_1 &= -(\alpha + x\beta)\frac{e^{-\theta x}}{\theta} - \frac{\beta}{\theta} I_2 \\
I_2 &= \int_0^\infty -e^{-\theta x}dx
\end{aligned}
$$

let $u = -\theta x \Longrightarrow \frac{du}{-\theta} = dx$ then we have;

$$
\begin{aligned}
I_2 &= \int_0^\infty \frac{-e^u}{\theta} du \\
I_2 &= \frac{e^{-\theta x}}{\theta}
\end{aligned}
$$

Combining I_1 and I_2 we have the following;

$$
\begin{aligned}
I_1 &= -(\alpha + x\beta)\frac{e^{-\theta x}}{\theta} - \frac{\beta e^{-\theta x}}{\theta^2} \\
&= \frac{\theta^2}{\alpha\theta + \beta}\int_0^{\infty}(\alpha + x\beta)e^{-\theta x} \\
&= 1 - \frac{\theta(\alpha + x\beta)e^{-\theta x}}{\alpha\theta + \beta} - \frac{\beta e^{-\theta x}}{\alpha\theta + \beta} \\
&= 1 - [\frac{\alpha\theta + \beta + \beta\theta x}{\alpha\theta + \beta}]e^{-\theta x}; x > 0, \beta > 0, \alpha > 0, (\alpha\theta + \beta) > 0
\end{aligned}
$$

□

2.5.2 Reliability Analysis

Proposition 2.5.3. *The survival function denoted by* $S(x;\alpha,\beta,\theta)$ *and hazard function denoted by* $h(x;\alpha,\beta,\theta)$ *of a generalized three parameter Lindley distribution (AG3PLD) Eq.2.45 are;*

$$
S(x;\alpha,\beta,\theta) = \frac{(\alpha\theta + \beta + \theta\beta x)e^{-\theta x}}{\alpha\theta + \beta}; x > 0, \beta > 0, \alpha > 0, (\alpha\theta + \beta) > 0 \tag{2.49}
$$

$$
h(x;\alpha,\beta,\theta) = \frac{\theta^2(\alpha + x\beta)}{\alpha\theta + \beta + \theta\beta x}; \quad x > 0, \beta > 0, \alpha > 0, \theta > 0 \tag{2.50}
$$

Proof. Using the relation defined in Eq.1.8, survival function in Eq.2.49 of AG3PLD Eq.2.45 as;

$$
\begin{aligned}
S(x;\alpha,\beta,\theta) &= 1-[1-[\frac{\alpha\theta+\beta+\beta\theta x}{\alpha\theta+\beta}]e^{-\theta x}] \\
&= \left(1+\frac{\beta\theta x}{\alpha\theta+\beta}\right)e^{-\theta x} \\
&= \frac{\alpha\theta+\beta+\beta\theta x}{\alpha\theta+\beta}e^{-\theta x}, x>0, \beta>0, \alpha>0, (\alpha\theta+\beta)>0
\end{aligned}
$$

Further hazard function in Eq.2.50 is obtained using the relation in Eq.1.9 as;

$$
\begin{aligned}
h(x;\alpha,\beta,\theta) &= \frac{\frac{\theta^2}{\alpha\theta+\beta}(\alpha+x\beta)e^{-\theta x}}{\frac{(\alpha\theta+\beta+\theta\beta x)e^{-\theta x}}{\alpha\theta+\beta}} \\
&= \frac{\theta^2(\alpha+x\beta)}{\alpha\theta+\beta+\theta\beta x}, \quad x>0, \quad \beta>0, \quad \alpha>0
\end{aligned}
$$

□

2.5.3 Moments and related measures

Proposition 2.5.4. *The moments about the origin for a generalized three parameter Lindley distribution in Eq.2.45 are obtained using method of moments and moment generating function technique as;*

$$
\mu_r^1* = \frac{r!(\alpha\theta+\beta(r+1))}{\theta^r(\alpha\theta+\beta)}; \quad r=1,2,3,\cdots \tag{2.51}
$$

Proof. Using Eq.1.16, the r^{th} moments of AG3PLD are obtained as;

$$E(X^r) = \frac{\theta^2}{\alpha\theta+\beta}\int_0^\infty x^r(\alpha+x\beta)e^{-\theta x}dx$$
$$= \frac{\theta^2}{\alpha\theta+\beta}[\alpha\int_0^\infty x^r e^{-\theta x}dx + \beta\int_0^\infty x^{r+1}e^{-\theta x}dx]$$

Using the relation $\int_0^\infty t^{\alpha-1}e^{-\theta t}dt = \frac{\Gamma(\alpha)}{\theta^\alpha}$ we have the following;

$$\alpha\int_0^\infty x^r e^{-\theta x}dx = \frac{\alpha\Gamma(r+1)}{\theta^{r+1}}$$
$$\beta\int_0^\infty x^{r+1}e^{-\theta x}dx = \frac{\beta\Gamma(r+2)}{\theta^{r+2}}$$
$$= \frac{\theta^2}{\alpha\theta+\beta}[\frac{\alpha r\Gamma r}{\theta^{r+1}} + \frac{\beta(r+1)r\Gamma r}{\theta^{r+2}}]$$
$$= \frac{r\Gamma r}{\theta^r(\alpha\theta+\beta)}(\alpha\theta+\beta(r+1))$$
$$= \frac{r!(\alpha\theta+\beta(r+1))}{\theta^r(\alpha\theta+\beta)}; \quad r = 1,2,3,\cdots$$

The mgf of AG3PLD is obtained using Eq.1.17 as shown below;

$$m_x(t) = \frac{\theta^2}{\alpha\theta+\beta}[\int_0^\infty e^{-(\theta-t)x}(\alpha+x\beta)dx]$$
$$= \frac{\theta^2}{\alpha\beta+\beta}[\frac{\alpha}{(\theta-t)} + \frac{\beta}{(\theta-t)^2}]$$
$$= \frac{\theta^2}{\alpha\theta+\beta}\left[\frac{\alpha}{\theta}\sum_{k=0}^\infty(\frac{t}{\theta})^k + \frac{\beta}{\theta^2}\sum_{k=0}^\infty\binom{k+1}{k}(\frac{t}{\theta})^k\right]$$

The moment generating function of a generalized three parameter Lindley distribution Eq.2.45 is as;

$$m_x(t) = \sum_{k=0}^{\infty} \frac{\alpha\theta + \beta(k+1)}{\theta^r(\alpha\theta+\beta)} (\frac{t}{\theta})^k \tag{2.52}$$

□

The r^{th} moments about the origin for a generalized three parameter Lindley distribution in Eq.2.45 are obtained as a coefficient of $\frac{t^r}{r!}$ in Eq.2.52 as:

$$\mu_r^1 = \frac{r!(\alpha\theta + \beta(r+1))}{\theta^r(\alpha\theta+\beta)}; r = 1, 2, 3, \cdots$$

The four none centralized moments are;

$$\mu_1^1 = \frac{\alpha\theta + 2\beta}{\theta(\alpha\theta+\beta)}, \quad \mu_2^1 = \frac{2(\alpha\theta+3\beta)}{\theta^2(\alpha\theta+\beta)}$$
$$\mu_3^1 = \frac{6(\alpha\theta+4\beta)}{\theta^3(\alpha\theta+\beta)}, \quad \mu_4^1 = \frac{24(\alpha\theta+5\beta)}{\theta^4(\alpha\theta+\beta)}$$

The centralized moments of a generalized three parameter Lindley distribution are obtained as;

$$
\begin{aligned}
\mu_1 &= \mu_1^1 = \frac{\alpha\theta + 2\beta}{\theta(\alpha\theta+\beta)} \\
\mu_2 &= \mu_2^1 - (\mu_1^1)^2 \\
\mu_2 &= \frac{\alpha^2\theta^2 + 4\alpha\beta\theta + 2\beta^2}{\theta^2(\alpha\theta+\beta)^2} \\
\mu_3 &= 2(\mu_1^1)^2\mu_1^1 - 3\mu_1^1\mu_2^1 + \mu_3^1 \\
\mu_3 &= \frac{2\alpha^3\theta^3 + 12\alpha^2\beta\theta^2 + 12\alpha\beta^2\theta + 4\beta^3}{\theta^3(\alpha\theta+\beta)^3} \\
\mu_4 &= \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2 \\
\mu_4 &= \frac{9\theta^4\alpha^4 + 72\alpha^3\theta^3\beta + 132\alpha^2\beta^2\theta^2 + 96\alpha\beta^3\theta + 24\beta^4}{\theta^4(\alpha\theta+\beta)^4}
\end{aligned}
$$

Proposition 2.5.5. *Other related measures such as variation coefficient (C.v),skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) of a generalized three parameter Lindley distribution Eq.2.45 are obtained as:*

$$C.v = \frac{\sqrt{\alpha^2\theta^2 + 4\alpha\beta\theta + 2\beta^2}}{\alpha\theta + 2\beta} \tag{2.53}$$

$$\upsilon_1 = \frac{2\alpha^3\theta^3 + 12\alpha^2\beta\theta^2 + 12\alpha\beta^2\theta + 4\beta^3}{(\alpha^2\theta^2 + 4\alpha\beta\theta + 2\beta^2)^{\frac{3}{2}}} \tag{2.54}$$

$$\upsilon_2 = \frac{(72\beta^2\theta + 24\alpha\beta^2\theta + 36\alpha\beta\theta^2 + 12\alpha^2\beta\theta^2 - 84\beta^2 - 12\alpha\beta\theta - 3\alpha^2\theta^2)(}{(\alpha^2\theta^2 + 4\alpha\beta\theta + 2\beta^2)^2} \tag{2.55}$$

$$\upsilon_3 = \frac{\alpha^2\theta^2 + 4\alpha\beta\theta + 2\beta^2}{\theta(\alpha\theta+\beta)(\alpha\beta + 2\beta)} \tag{2.56}$$

Proof. : Coefficient of variation 2.53 is obtained as;

$$
\begin{aligned}
C.v &= \frac{\sigma}{\mu_1} \\
&= \frac{\sqrt{\alpha^2\theta^2+4\alpha\beta\theta+2\beta^2}}{\theta(\alpha\theta+\beta)} \frac{\theta(\alpha\theta+\beta)}{\alpha\theta+2\beta} \\
&= \frac{\sqrt{\alpha^2\theta^2+4\alpha\beta\theta+2\beta^2}}{\alpha\theta+2\beta}
\end{aligned}
$$

Skewness coefficient 2.54of AG3PLD 2.45 is given as;

$$
\begin{aligned}
\upsilon_1 &= \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} \\
&= \frac{2\alpha^3\theta^3+12\alpha^2\beta\theta^2+12\alpha\beta^2\theta+4\beta^3}{\theta^3(\alpha\theta+\beta)^3} \frac{\theta^3(\alpha\theta+\beta)^3}{(\alpha^2\theta^2+4\alpha\beta\theta+2\beta^2)^{\frac{3}{2}}} \\
&= \frac{2\alpha^3\theta^3+12\theta^2\beta\theta^2+12\alpha\beta^2\theta+4\beta^3}{(\alpha^2\theta^2+4\alpha\beta\theta+2\beta^2)^{\frac{3}{2}}}
\end{aligned}
$$

Kurtosis coefficient 2.55 is obtained as;

$$
\begin{aligned}
\upsilon_2 &= \frac{\mu_4}{(\mu_2)^2} \\
&= \frac{9\theta^4\alpha^4+72\alpha^3\theta^3\beta+132\alpha^2\beta^2\theta^2+96\alpha\beta^3\theta+24\beta^4}{\theta^4(\alpha\theta+\beta)^4} \left[\frac{\theta^2(\alpha\theta+}{\alpha^2\theta^2+4\alpha\beta\theta} \right. \\
&= \frac{9\theta^4\alpha^4+72\alpha^3\theta^3\beta+132\alpha^2\beta^2\theta^2+96\alpha\beta^3\theta+24\beta^4}{\left(\alpha^2\theta^2+4\alpha\beta\theta+2\beta^2\right)^2}
\end{aligned}
$$

Dispersion index 2.56 of AG3PLD is obtained as;

$$
\begin{aligned}
\upsilon_3 &= \frac{\sigma^2}{\mu_1^1} \\
&= \frac{\alpha^2\theta^2+4\alpha\beta\theta+2\beta^2}{\theta^2(\alpha\theta+\beta)^2} \frac{\theta(\alpha\theta+\beta)}{\alpha\theta+2\beta} \Longrightarrow \frac{\alpha^2\theta^2+4\alpha\beta\theta+2\beta^2}{\theta(\alpha\theta+\beta)(\alpha\beta+2\beta)}
\end{aligned}
$$

□

2.5.4 Excess loss distribution

Proposition 2.5.6. *The probability density function of excess loss distribution $f_e(x;\alpha,\beta,\theta)$, mean residual lifetime $(m(x))$, survival function based on equilibrium distribution $S_e(x,\alpha,\beta,\theta)$ and hazard function based on the equilibrium distribution $h_e(x;\alpha,\beta,\theta)$ for a generalized three parameter Lindley distribution Eq.2.45 are:*

$$f_l(x,\alpha,\beta,\theta) = \frac{\theta^2(\alpha+x\beta)e^{-(x-z)\theta}}{\alpha\theta+\beta+\theta\beta z}; \quad x > z \tag{2.57}$$

$$m(x) = \frac{\alpha\theta+\theta\beta x+2\beta}{\theta(\alpha(\alpha\theta+\theta\beta x+\beta))} \tag{2.58}$$

$$f_e(x) = \frac{\theta(\alpha\theta+\beta+\theta\beta x)e^{-\theta x}}{\alpha\theta+2\beta} \tag{2.59}$$

$$S_e(x;\alpha,\beta,\theta) = \frac{(\alpha\theta+2\beta+\theta\beta x)e^{-\theta x}}{\alpha\theta+2\beta} \tag{2.60}$$

$$h_e(x;\alpha,\beta,\theta) = \frac{\theta(\alpha\theta+\beta+\theta\beta x)}{\alpha\theta+2\beta+\theta\beta x} \tag{2.61}$$

Proof. Using Eq.1.10, the pdf of excess loss Eq.2.57 is obtained as;

$$\begin{aligned} f_l(x;\alpha,\beta,\theta) &= \frac{\frac{\theta^2}{\alpha\theta+\beta}(\alpha+x\beta)e^{-\theta x}}{(\frac{\alpha\theta+\beta+\theta\beta z}{\alpha\theta+\beta})e^{-\theta z}} \\ &= \frac{\theta^2(\alpha+x\beta)e^{-(x-z)\theta}}{\alpha\theta+\beta+\theta\beta z}; \quad x > z \end{aligned}$$

Mean residual lifetime Eq.2.58 is obtained using Eq.1.11 as;

$$\begin{aligned} m(x) &= \frac{\alpha\theta+\beta}{(\alpha\theta+\beta+\theta\beta)e^{-\theta x}} \frac{1}{\alpha\theta+\beta} \int_x^\infty (\alpha\theta+\beta+\theta\beta t)e^{-\theta t} dt \\ &= \frac{1}{(\alpha\theta+\beta+\theta x)e^{-\theta x}} \int_x^\infty (\alpha\theta+\beta+\theta\beta t)e^{-\theta t} dt \end{aligned}$$

Taking the part $\int_x^\infty (\alpha\theta+\theta\beta t)e^{-\theta t}$ and integrating using integration by parts technique we have;

$$\begin{aligned} u &= (\alpha\theta+\beta+\theta\beta t) \Longrightarrow du = \theta\beta dt \\ dv &= e^{-\theta t} \Longrightarrow v = \frac{-e-\theta t}{\theta} \\ &= -(\alpha\theta+\beta+\theta\beta t)\frac{e^{-\theta t}}{\theta} + \int_x^\infty \beta e^{-\theta t} dt \\ &= -(\alpha\theta+\beta+\theta\beta t)\frac{e^{-\theta t}}{\theta} - \frac{\beta}{\theta}e^{-\theta t} \\ &= -\frac{e^{-\theta t}}{\theta}(\alpha\theta+\theta\beta t+2\beta)\Big|_x^\infty \\ &= \frac{e^{-\theta x}}{\theta}(\alpha\theta+\theta\beta x+2\beta) \\ &= \frac{1}{(\alpha\theta+\beta+\theta\beta x)e^{-\theta x}} \frac{e^{-\theta x}}{\theta}(\alpha\theta+\theta\beta x+2\beta) \\ &= \frac{\alpha\theta+\theta\beta x+2\beta}{\theta(\alpha\theta+\theta\beta x+\beta)} \end{aligned}$$

Using the relation Eq.1.12, equilibrium distribution Eq.2.59 is obtained as;

$$f_e(x;\alpha,\beta,\theta) = \frac{(\alpha\theta+\beta+\theta\beta x)e^{-\theta x}}{\alpha\theta+\beta} \frac{\theta(\alpha\theta+\beta)}{\alpha\theta+2\beta} \Longrightarrow \frac{\theta(\alpha\theta+\beta+\theta\beta x)e}{\alpha\theta+2\beta}$$

Survival function based on equilibrium distribution Eq. 2.60 is obtained using Eq.1.13 as;

$$\int_x^\infty S(t;\alpha,\beta,\theta)dt = \frac{e^{-\theta x}(\alpha\theta+2\beta+\theta\beta x)}{\theta(\alpha\theta+\beta)}$$

$$S_e(x;\alpha,\beta,\theta) = \frac{e^{-\theta x}(\alpha\theta+2\beta+\theta\beta x)}{\theta(\alpha\theta+\beta)}\frac{\theta(\alpha\theta+\beta)}{\alpha\theta+2\beta} \Longrightarrow \frac{(\alpha\theta+2\beta+\theta\beta x}{\alpha\theta+2\beta}$$

Using relation Eq.1.14, the hazard function based on equilibrium distribution Eq.2.61 is obtained as;

$$h_e(x;\alpha,\beta,\theta) = \frac{\theta(\alpha\theta+\beta+\theta\beta x)e^{-\theta x}}{\alpha\theta+2\beta}\frac{\alpha\theta+2\beta}{(\alpha\theta+2\beta+\theta\beta x)e^{-\theta x}} \Longrightarrow \frac{\theta(\alpha\theta+\beta}{\alpha\theta+2\beta}$$

□

2.6 Four parameter Lindley distribution

2.6.1 Construction of four parameter Lindley distribution

Proposition 2.6.1. *AG4PLD is a finite mixed distribution of Gamma (α,θ) and Gamma (β,θ) with weighing proportion $\omega = \frac{\theta}{\theta+k}$ the pdf and Cdf of a generalized four parameter Lindley distribution are;*

$$f(x;\alpha,\beta,\theta,k) = \frac{\theta^2}{(\theta+k)}\left[\frac{(\theta x)^{\alpha-1}}{\Gamma\alpha}+\frac{k(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta x}, \alpha>0, \beta>0, \theta>0, k> \tag{2.62}$$

$$F(x;\alpha,\beta,\theta,k) = \frac{1}{\theta+k}\left[\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right], \alpha>0, \beta>0, \theta>0, k>0, x>0 \tag{2.63}$$

Proof. The pdf Eq.2.62 is obtained using finite mixture Eq.1.1 concept as;

$$
\begin{aligned}
f(x;\alpha,\beta,\theta,k) &= \frac{\theta}{\theta+k}\left[\frac{\theta^{\alpha}}{\Gamma\alpha}e^{-\theta x}x^{\alpha-1}\right]+\frac{k}{\theta+k}[\frac{\theta^{\beta}}{\Gamma\beta}e^{-\theta x}x^{\beta-1}] \\
&= \frac{\theta^{\alpha+1}e^{-\theta x}x^{\alpha-1}}{(\theta+k)\Gamma\alpha}+\frac{k\theta^{\beta}e^{-\theta x}x^{\beta-1}}{(\theta+k)\Gamma\beta} \\
&= \frac{\theta^{2}\theta^{\alpha-1}e^{-\theta x}x^{\alpha-1}}{(\theta+k)\Gamma\alpha}+\frac{k\theta^{2}\theta^{\beta-1}e^{-\theta x}x^{\beta-1}}{(\theta+k)\theta\Gamma\beta} \\
&= \frac{\theta^{2}}{(\theta+k)}\left[\frac{(\theta x)^{\alpha-1}}{\Gamma\alpha}+\frac{k(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta x};\alpha>0,\beta>0,\theta>
\end{aligned}
$$

Remark 2.6.2. *Putting $k=1$, $\alpha=1$ and $\beta=2$ a generalized four parameter Lindley distribution Eq.2.62 reduces to one parameter Lindley distribution Eq.2.1. Similarly, putting $\alpha=1$, $\beta=2$ and $k=\beta$ a generalized four parameter Lindley distribution Eq.2.62 reduces to a generalized two parameter Lindley distribution Eq.2.15.*

moreover, cumulative density function Eq.2.63 as;

$$
\begin{aligned}
F(x;\alpha,\beta,\theta,k) &= \frac{\theta^{2}}{(\theta+k)}\int_{0}^{\infty}\left[\frac{(\theta x)^{\alpha-1}}{\Gamma\alpha}+\frac{k(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta x}dx \\
&= \frac{\theta^{2}}{(\theta+k)}\left[\int_{0}^{\infty}\frac{(\theta x)^{\alpha-1}}{\Gamma\alpha}e^{-\theta x}dx+\int_{0}^{\infty}\frac{k(\theta x)^{\beta-1}}{\theta\Gamma\beta}e^{-\theta x}dx\right] \\
&= \frac{1}{(\theta+k)}\left[\frac{\theta}{\Gamma\alpha}\int_{0}^{\infty}(\theta x)^{\alpha-1}e^{-\theta x}dx+\frac{k}{\Gamma\beta}\int_{0}^{\infty}(\theta x)^{\beta-1}e^{-\theta x}\right. \\
&= \frac{1}{\theta+k}\left[\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right];\alpha>0,\beta>0,\theta>0,k>0,x
\end{aligned}
$$

where $\gamma_{\alpha(\theta x)}=\frac{1}{\Gamma\alpha}\int_{0}^{\infty}(\theta x)^{\alpha-1}e^{-\theta x}dx$ and $\gamma_{\beta(\theta x)}=\frac{1}{\Gamma\beta}\int_{0}^{\infty}(\theta x)^{\beta-1}e^{-\theta x}dx$ this completes the proof. □

2.6.2 Reliability Analysis

Proposition 2.6.3. *The survival function denoted by $S(x;\alpha,\beta,\theta,k)$ and hazard function denoted by $h(x;\alpha,\beta,\theta,k)$ of a four parameter generalized Lindley distribution Eq.2.62 are;*

$$S(x;\alpha,\beta,\theta,k) = \frac{\theta+k-(\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)})}{\theta+k}; \alpha>0,\beta>0,\theta>0,k>0,x>0 \tag{2.64}$$

$$h(x;\alpha,\beta,\theta,k) = \frac{\theta^2\big(\frac{(\theta x)^{\alpha-1}}{\Gamma\alpha}+\frac{k(\theta x)^{\beta-1}}{\Gamma\beta}\big)e^{-\theta x}}{\theta+k-\big[\theta\gamma_{\alpha(\theta x)+k\gamma_{\beta(\theta x)}}\big]}; \alpha>0,\beta>0,\theta>0,k>0,x>0 \tag{2.65}$$

Proof. survival function Eq.2.64 is obtained using the relation Eq.1.8 as;

$$\begin{aligned} S(x;\alpha,\beta,\theta,k) &= 1-\frac{1}{\theta+k}\big[\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\big] \\ &= \frac{\theta+k-\big[\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\big]}{\theta+k}; \alpha>0,\beta>0,\theta>0,k>0,x>0 \end{aligned}$$

hazard function Eq.2.65 is obtained using the relation Eq.1.9 as;

$$\begin{aligned} h(x;\alpha,\beta,\theta,k) &= \frac{\frac{\theta^2}{(\theta+k)}\left[\frac{(\theta x)^{\alpha-1}}{\Gamma\alpha}+\frac{k(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta x}}{\frac{\theta+k-\left[\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right]}{\theta+k}} \\ &= \frac{\theta^2\left(\frac{(\theta x)^{\alpha-1}}{\Gamma\alpha}+\frac{k(\theta x)^{\beta-1}}{\Gamma\beta}\right)e^{-\theta x}}{\theta+k-\left[\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right]}; \alpha>0,\beta>0,\theta>0,k>0,x>0 \end{aligned}$$

□

2.6.3 Moments and related measures

Proposition 2.6.4. *The r^{th} moments of a generalized four parameter Lindley distribution Eq.2.62 about the origin are obtained using both moment generating technique and methods of moments as;*

$$E(X^r) = \frac{1}{\theta^r(\theta+k)}\left[\frac{\theta}{\Gamma\alpha}\Gamma\alpha+r+\frac{k}{\Gamma\beta}\Gamma\beta+r\right], \quad for \quad r=1,2,3,\cdots \tag{2.66}$$

$$m_x(t) = \frac{1}{(\theta+k)}\left[\sum_{r=0}^{\infty}\frac{(-1)^r}{\theta^r}\left(\theta\binom{-\alpha}{r}+k\binom{-\beta}{r}\right)t^r\right], \quad where \quad \theta > t \tag{2.67}$$

Proof. By definition of moments 1.16, moments Eq.2.66 obtained as;

$$\begin{aligned}
E(X^r) &= \frac{\theta^2}{(\theta+k)}\int_0^{\infty} x^r\left[\frac{(\theta x)^{\alpha-1}}{\Gamma\alpha}+\frac{k(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta x}dx \\
&= \frac{\theta^2}{(\theta+k)}\left[\int_0^{\infty} x^r\frac{(\theta x)^{\alpha-1}}{\Gamma\alpha}e^{-\theta x}dx+\int_0^{\infty}\frac{k(\theta x)^{\beta-1}x^r e^{-\theta x}dx}{\theta\Gamma\beta}\right] \\
&= \frac{\theta^2}{(\theta+k)}\left[\frac{\theta^{\alpha-1}}{\Gamma\alpha}\int_0^{\infty} x^{\alpha+r-1}e^{-\theta x}dx+\frac{k\theta^{\beta-2}}{\Gamma\beta}\int_0^{\infty} x^{\beta+r-1}e^{-\theta x}dx\right] \\
&= \frac{\theta^2}{(\theta+k)}\left[\frac{\theta^{\alpha-1}}{\Gamma\alpha}\frac{\Gamma(\alpha+r)}{\theta^{\alpha+r}}+\frac{k\theta^{\beta-1}}{\theta\Gamma\beta}\frac{\Gamma(\beta+r)}{\theta^{\beta+r}}\right] \\
&= \frac{1}{\theta^r(\theta+k)}\left[\frac{\theta}{\Gamma\alpha}\Gamma(\alpha+r)+\frac{k}{\Gamma\beta}\Gamma(\beta+r)\right], \quad for \quad r=1,2,3,\cdots
\end{aligned}$$

By definition of mgf 1.17, mgf Eq.2.67 is obtained as;

$$\begin{aligned}
m_x(t) &= \frac{\theta^2}{(\theta+k)}\left[\int_0^\infty e^{tx}\left[\frac{(\theta x)^{\alpha-1}}{\Gamma\alpha}+\frac{k(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta x}dx\right] \\
&= \frac{\theta^2}{(\theta+k)}\left[\int_0^\infty \frac{(\theta x)^{\alpha-1}e^{-(\theta-t)x}}{\Gamma\alpha}dx+\int_0^\infty \frac{k(\theta x)^{\beta-1}e^{-(\theta-t)x}}{\theta\Gamma\beta}dx\right] \\
&= \frac{\theta^2}{(\theta+k)}\left[\frac{\theta^{\alpha-1}}{\Gamma\alpha}\int_0^\infty e^{-(\theta-t)x}x^{\alpha-1}dx+\frac{k\theta^{\beta-2}}{\Gamma\beta}\int_0^\infty e^{-(\theta-t)x}x^{\beta-1}dx\right] \\
&= \frac{1}{(\theta+k)}\left[\frac{\theta^{\alpha+1}}{\Gamma\alpha}\int_0^\infty e^{-(\theta-t)x}x^{\alpha-1}dx+\frac{k\theta^{\beta}}{\beta}\int_0^\infty e^{-(\theta-t)x}x^{\beta-1}dx\right]
\end{aligned}$$

Using the relation Eq.1.7 we have;

$$\int_0^\infty x^{\alpha-1}e^{-(\theta-t)x}dx = \frac{\Gamma\alpha}{(\theta-t)^\alpha}$$

$$\int_0^\infty x^{\beta-1}e^{-(\theta-t)x}dx = \frac{\Gamma\beta}{(\theta-t)^\beta}$$

$$\begin{aligned}
&= \frac{1}{(\theta+k)}\left[\frac{\theta^\alpha\theta}{\Gamma\alpha}\frac{\Gamma\alpha}{(\theta-t)^\alpha}+\frac{k\theta^\beta}{\Gamma\beta}\frac{\Gamma\beta}{(\theta-t)^\beta}\right] \\
&= \frac{1}{(\theta+k)}\left[\theta(\frac{\theta}{\theta-t})^\alpha+k(\frac{\theta}{\theta-t})^\beta\right] \\
&= \frac{1}{(\theta+k)}\left[\theta(\frac{\theta-t}{\theta})^{-\alpha}+k(\frac{\theta-t}{\theta})^{-\beta}\right] \\
&= \frac{1}{(\theta+k)}\left[\theta\left(1-\frac{t}{\theta}\right)^{-\alpha}+k\left(1-\frac{t}{\theta}\right)^{-\beta}\right]
\end{aligned}$$

Using the identity function $(1-x)^{-d} = \sum_{r=0}^{\infty}(-1)^r\binom{-d}{r}x^r$ we have the following;

$$
\begin{aligned}
(1-\frac{t}{\theta})^{-\alpha} &= \sum_{r=0}^{\infty}(-1)^r\binom{-\alpha}{r}(\frac{t}{\theta})^r \\
(1-\frac{t}{\theta})^{-\beta} &= \sum_{r=0}^{\infty}(-1)^r\binom{-\beta}{r}(\frac{t}{\theta})^r \\
&= \frac{1}{(\theta+k)}\left[\theta\sum_{r=0}^{\infty}(-1)^r\binom{-\alpha}{r}(\frac{t}{\theta})^r + k\sum_{r=0}^{\infty}(-1)^r\binom{-\beta}{r}(\frac{t}{\theta})^r\right] \\
m_x(t) &= \frac{1}{(\theta+k)}\left[\sum_{r=0}^{\infty}\frac{(-1)^r}{\theta^r}\left(\theta\binom{-\alpha}{r}+k\binom{-\beta}{r}\right)t^r\right], \quad where \quad \theta
\end{aligned}
$$

□

We now use the moment generating function (mgf) Eq.2.67 to derive the first four moments about the origin of a generalized four parameter Lindley distribution Eq.2.62 as:

$$m_x(t) = \frac{1}{(\theta+k)}\left[\theta(1-\frac{t}{\theta})^{-\alpha} + k(1-\frac{t}{\theta})^{-\beta}\right]$$

$$m_x^i(t)\frac{1}{(\theta+k)}\left[-\theta\alpha(1-\frac{t}{\theta})^{-(\alpha+1)}\frac{-1}{\theta} + -\beta k(1-\frac{t}{\theta})^{-(\beta+1)}\frac{-1}{\theta}\right]$$

$$= \frac{1}{\theta(\theta+k)}\left[\alpha\theta(1-\frac{t}{\theta})^{-(\alpha+1)} + k\beta(1-\frac{t}{\theta})^{-(\beta+1)}\right]$$

$$m_x^1(0) = \frac{1}{\theta(\theta+k)}\left[\alpha\theta + \beta k\right] = \frac{\alpha\theta + k\beta}{\theta(\theta+k)} = \mu_1^1$$

$$m_x^{ii}(t)\frac{1}{\theta(\theta+k)}\left[-(\alpha+1)\alpha\theta(1-\frac{t}{\theta})^{-(\alpha+2)}(\frac{-1}{\theta}) + -(\beta+1)\beta k(1-\frac{t}{\theta})^{-(\beta+2}\right.$$

$$= \frac{1}{\theta^2(\theta+k)}\left[(\alpha+1)\alpha\theta(1-\frac{t}{\theta})^{-(\alpha+2)} + (\beta+1)\beta k(1-\frac{t}{\theta})^{-(\beta+2)}\right]$$

$$m_x^{ii}(0) = \frac{1}{\theta^2(\theta+k)}\left[(\alpha+1)\alpha\theta + (\beta+1)\beta k\right] = \mu_2^1$$

$$m_x^{iii}(t) = \frac{1}{\theta^2(\theta+k)}\left[-(\alpha+1)(\alpha+2)\alpha\theta(1-\frac{t}{\theta})^{-(\alpha+3)}(\frac{-1}{\theta}) + -(\beta+1)(\beta -\right.$$

$$= \frac{1}{\theta^3(\theta+k)}\left[(\alpha+1)(\alpha+2)\alpha\theta(1-\frac{t}{\theta})^{-(\alpha+3)} + (\beta+1)(\beta+2)\beta k(1 -\right.$$

$$m_x^{iii}(0) = \frac{1}{\theta^3(\theta+k)}\left[(\alpha+1)(\alpha+2)\alpha\theta + (\beta+1)(\beta+2)\beta k\right] = \mu_3^1$$

$$m_x^{iv}(t) = \frac{1}{\theta^3(\theta+k)}\left[-(\alpha+1)(\alpha+2)(\alpha+3)(1-\frac{t}{\theta})^{-(\alpha+4)}(\frac{-1}{\theta}) + -(\beta+1)\right.$$

$$= \frac{1}{\theta^4(\theta+k)}\left[(\alpha+1)(\alpha+2)(\alpha+3)\alpha\theta(1-\frac{t}{\theta})^{-(\alpha+4)} + (\beta+1)(\beta+2\right.$$

$$m_x^{iv}(0) = \frac{1}{\theta^4(\theta+k)}\left[(\alpha+1)(\alpha+2)(\alpha+3)\alpha\theta + (\beta+1)(\beta+2)(\beta+3)\beta k\right] =$$

We now obtain the centralized moments of AGF4LD 2.62 as:

$$\mu_1 = \mu_1^1 \Longrightarrow \frac{\alpha\theta + k\beta}{\theta(\theta+k)}$$

$$\mu_2 = \frac{\alpha\theta^2 + \beta^2\theta k + \alpha^2\theta k + \alpha\theta k - (\beta k)^2 + \beta k^2 - 2\alpha\beta\theta k - (\beta k)^2}{\theta^2(\theta+k)^2} = \sigma^2$$

$$\mu_3 = \mu_3^1 - 3\mu_2^1\mu_1^1 + 2(\mu_1^1)^3$$

$$\mu_4 = \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2$$

Proposition 2.6.5. *Other related measures such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) of a generalized four parameter Lindley distribution Eq.2.62 are obtained as:*

$$C.v = \frac{\sqrt{\alpha\theta^2 + \beta^2\theta k + \alpha^2\theta k + \alpha\theta k - (\beta k)^2 + \beta k^2 - 2\alpha\beta\theta k - (\beta k)^2}}{\alpha\theta + k\beta} \tag{2.68}$$

$$\upsilon_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} \tag{2.69}$$

$$\upsilon_2 = \frac{\mu_4}{(\mu_2)^2} \tag{2.70}$$

$$\upsilon_3 = \frac{\alpha\theta^2 + \beta^2\theta k + \alpha^2\theta k + \alpha\theta k - (\beta k)^2 + \beta k^2 - 2\alpha\beta\theta k - (\beta k)^2}{\theta(\theta+k)(\alpha\theta + k\beta)} \tag{2.71}$$

`Proof.` Coefficient of variation 2.68

$$C.v = \frac{\sigma}{\mu_1^1}$$

$$= \frac{\sqrt{\alpha\theta^2 + \beta^2\theta k + \alpha^2\theta k + \alpha\theta k - (\beta k)^2 + \beta k^2 - 2\alpha\beta\theta k - (\beta k)^2}}{\theta(\theta+k)} \frac{\theta(}{\alpha\theta}$$

$$= \frac{\sqrt{\alpha\theta^2 + \beta^2\theta k + \alpha^2\theta k + \alpha\theta k - (\beta k)^2 + \beta k^2 - 2\alpha\beta\theta k - (\beta k)^2}}{\alpha\theta + k\beta}$$

Skewness coefficient 2.69 is obtained as;

$$\upsilon_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} \Longrightarrow \frac{\mu_3^1 - 3\mu_2^1\mu_1^1 + 2(\mu_1^1)^3}{(\mu_2^1 - (\mu_1^1))^{\frac{3}{2}}}$$

Kurtosis coefficient 2.70 is obtained as;

$$\upsilon_2 = \frac{\mu_4}{(\mu_2)^2} \Longrightarrow \frac{\mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2}{(\mu_2^1 - (\mu_1^1)^2}$$

Index of dispersion 2.71 is obtained as;

$$\begin{aligned}\upsilon_3 &= \frac{\sigma^2}{\mu_1^1} \\ &= \frac{\alpha\theta^2 + \beta^2\theta k + \alpha^2\theta k + \alpha\theta k - (\beta k)^2 + \beta k^2 - 2\alpha\beta\theta k - (\beta k)^2}{\theta^2(\theta+k)^2} * \frac{\theta(\theta+k)}{\alpha\theta + k\beta} \\ &= \frac{\alpha\theta^2 + \beta^2\theta k + \alpha^2\theta k + \alpha\theta k - (\beta k)^2 + \beta k^2 - 2\alpha\beta\theta k - (\beta k)^2}{\theta(\theta+k)(\alpha\theta + k\beta)}\end{aligned}$$

□

2.6.4 Excess loss distribution

Proposition 2.6.6. *The pdf of excess loss distribution $f_l(x)$, mean residual lifetime (MRL), equlibrium distribution $f_e(x;\alpha,\beta,\theta,k)$, survival distribution based on equilibrium distribution $S_e(x;\alpha,\beta,\theta,k)$ and hazard function based on equilibrium distribution $h_e(x;\alpha,\beta,\theta,k)$*

of a generalized four parameter Lindley distribution Eq.2.62 are;

$$f_l(x;\alpha,\beta,\theta,k) = \frac{\left[\frac{\theta^{\alpha-1}}{\Gamma\alpha}\left(\theta x^{\alpha-1}+(\alpha-1)x^{\alpha-2}\right)+\frac{k\theta^{\beta-2}}{\Gamma\beta}\left(\theta x^{\beta-1}+(\beta-\right.\right.}{\theta+k-\left[\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right]}$$

(2.72)

$$m(x) = \frac{\alpha\theta\Gamma(\alpha+1)(\theta x)+\beta k\Gamma(\beta+1)(\theta x)}{\theta\left[\theta+k-\left(\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right)\right]}-x$$

(2.73)

$$f_e(x;\alpha,\beta,\theta,k) = \frac{\theta\left[\theta+k-(\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)})\right]}{\alpha\theta+\beta k} \quad \textbf{(2.74)}$$

$$S_e(x;\alpha,\beta,\theta,k) = \frac{\theta\left[\alpha\theta\Gamma(\alpha+1)(\theta x)+k\beta\Gamma(\beta+1)(\theta x)\right]-x}{\alpha\theta+\beta k}$$

(2.75)

$$h_e(x;\alpha,\beta,\theta,k) = \frac{\theta+k-\left[\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right]}{\alpha\theta^2\Gamma(\alpha+1)(\theta x)+k\beta\theta\Gamma(\beta+1)(\theta x)-x}$$

(2.76)

Proof. Pdf of excess distribution Eq.2.72 is obtained as;

$$
\begin{aligned}
f_l(x,\alpha,\beta,\theta,k) &= \frac{\int_x^\infty f(t)dt}{S(x)} \\
&= \frac{\frac{\theta^2}{\theta+k}\left[\int_x^\infty \frac{(\theta t)^{\alpha-1}}{\Gamma\alpha} + \frac{k(\theta t)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta t dt}}{\frac{\theta+k-\left(\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right)}{\theta+k}} \\
&= \frac{\theta^2\int_x^\infty\left[\frac{(\theta t)^{\alpha-1}}{\Gamma\alpha} + \frac{k(\theta t)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta t}dt}{\theta+k-\left[\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right]}
\end{aligned}
$$

Taking the numerator part $\theta^2\int_x^\infty\left[\frac{(\theta t)^{\alpha-1}}{\Gamma\alpha} + \frac{k(\theta t)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta t}dt$ we have

$$
\begin{aligned}
\theta^2\int_x^\infty \frac{(\theta t)^{\alpha-1}}{\Gamma\alpha}e^{-\theta t}dt &= \frac{\theta^{\alpha+1}}{\Gamma\alpha}\int_x^\infty t^{\alpha-1}e^{-\theta t}dt \\
&= \frac{\theta^{\alpha-1}}{\Gamma\alpha}(\theta x^{\alpha-1}+(\alpha-1)x^{\alpha-2})e^{-\theta x} \\
\theta^2\int_x^\infty \frac{k(\theta t)^{\beta-1}}{\theta\Gamma\beta}e^{-\theta t}dt &= \frac{k\theta^{\beta-2}}{\Gamma\beta}\left(\theta x^{\beta-1}+(\beta-1)x^{\beta-2}\right)e^{-\theta x} \\
&= \frac{\left[\frac{\theta^{\alpha-1}}{\Gamma\alpha}(\theta x^{\alpha-1}+(\alpha-1)x^{\alpha-2}) + \frac{k\theta^{\beta-2}}{\Gamma\beta}(\theta x^{\beta-1}+(\beta-1\right.}{\theta+k-\left[\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right]}
\end{aligned}
$$

MRL Eq.2.73 is obtained using relation Eq.1.11 as;

$$m(x = \frac{\frac{\theta^2}{\theta+k}\int_x^\infty t\left[\frac{(\theta t)^{\alpha-1}}{\Gamma\alpha}+\frac{k(\theta t)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta t}}{\frac{\theta+k-\left[\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right]}{\theta+k}} - x$$

$$= \frac{\theta^2\int_x^\infty t\left[\frac{(\theta t)^{\alpha-1}}{\Gamma\alpha}+\frac{k(\theta t)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta t}dt}{\theta+k-\left[\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right]} - x$$

$$= \frac{\alpha\theta\Gamma(\alpha+1)(\theta x)+\beta k\Gamma(\beta+1)(\theta x)}{\theta\left[\theta+k-\left(\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right)\right]} - x$$

Using the relation Eq.1.12, equilibrium distribution Eq.2.74 as;

$$f_e(x;\alpha,\beta,\theta,k) = \frac{\theta+k-\left[\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right]}{\theta+k}\frac{\theta(\theta+k)}{\alpha\theta+\beta k}$$

$$= \frac{\theta\left[\theta+k-(\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)})\right]}{\alpha\theta+\beta k}$$

Survival function based on equilibrium distribution Eq.2.75 is obtained using relation Eq.1.13 as;

$$\begin{aligned}
\int_x^\infty f_e(x;\alpha,\beta,\theta,k)dt &= \int_x^\infty 1dt - \int_x^\infty \left[\frac{1}{\theta+k}\left(\theta\gamma_{\alpha(\theta t)} + k\gamma_{\beta(\theta t)}\right)\right]dt \\
&= -x + \frac{\theta}{\theta+k}\int_x^\infty \gamma_{\alpha(\theta t)}dt + \frac{k}{\theta+k}\int_x^\infty \gamma_{\beta(\theta t)}dt \\
&= -x + \frac{\theta}{\theta+k}\alpha\Gamma(\alpha+1)(\theta x) + \frac{k}{\theta+k}\beta\Gamma(\beta+1)(\theta x) \\
&= \frac{1}{\theta+k}\left[\alpha\theta\Gamma(\alpha+1)(\theta x) + k\beta\Gamma(\beta+1)(\theta x)\right] - x \\
S_e(x;\alpha,\beta,\theta,k) &= \frac{\frac{1}{\theta+k}\left[\alpha\theta\Gamma(\alpha+1)(\theta x) + k\beta\Gamma(\beta+1)(\theta x)\right] - x}{\frac{\alpha\theta+\beta k}{\theta(\theta+k)}} \\
&= \frac{\theta\left[\alpha\theta\Gamma(\alpha+1)(\theta x) + k\beta\Gamma(\beta+1)(\theta x)\right] - x}{\alpha\theta+\beta k}
\end{aligned}$$

Using the relation Eq.1.14, the hazard function based on the equilibrium distribution is obtained as;

$$\begin{aligned}
h_e(x;\alpha,\beta,\theta,k) &= \frac{\frac{\theta+k-\left[\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right]}{\alpha\theta+\beta k}}{\frac{\alpha\theta^2\Gamma(\alpha+1)(\theta x)+k\beta\theta\Gamma(\beta+1)(\theta x)-x}{\alpha\theta+k\beta}} \\
&= \frac{\theta+k-\left[\theta\gamma_{\alpha(\theta x)}+k\gamma_{\beta(\theta x)}\right]}{\alpha\theta^2\Gamma(\alpha+1)(\theta x)+k\beta\theta\Gamma(\beta+1)(\theta x)-x} = \frac{1}{m(x)}
\end{aligned}$$

□

2.7 Five parameter Lindley distribution

2.7.1 Construction of five parameter Lindley distribution

Proposition 2.7.1. *Let $\omega = \frac{k\theta}{\eta + k\theta}$ be a mixing proportion , AG5PLD is constructed as finite mixed distribution of Gamma (θ, α) and Gamma (θ, β). The pdf and Cdf of a generalized five parameter Lindley (AG5PLD) are;*

$$f(x;\alpha,\beta,\theta,k,\eta) = \frac{\theta^2}{k\theta+\eta}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma\alpha} + \frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta x}, \alpha > 0, \beta > 0, \theta \tag{2.77}$$

$$F(x;\alpha,\beta,\theta,k,\eta) = \frac{1}{k\theta+\eta}\left[k\theta\gamma_{\alpha(\theta x)} + \eta\gamma_{\beta(\theta x)}\right], \alpha > 0, \beta > 0, \theta > 0, k > \tag{2.78}$$

Proof. Using the concept of finite mixtures Eq.1.1 is obtained as;

$$\begin{aligned}
f(x;\alpha,\beta,\theta,k,\eta) &= \frac{k\theta}{\eta+k\theta}\left[\frac{\theta^\alpha}{\Gamma\alpha}e^{-\theta x}x^{\alpha-1}\right] + \frac{\eta}{\eta+k\theta}\left[\frac{\theta^\beta}{\Gamma\beta}e^{-\theta x}x^{\beta-1}\right] \\
&= \frac{k\theta^{\alpha+1}e^{-\theta x}x^{\alpha-1}}{(\eta+k\theta)\Gamma\alpha} + \frac{\eta\theta^2\theta^{\beta-1}e^{-\theta x}x^{\beta-1}}{\theta(\eta+k\theta)\Gamma\beta} \\
&= \frac{k\theta^2\theta^{\alpha-1}e^{-\theta x}x^{\alpha-1}}{(\eta+k\theta)\Gamma\alpha} + \frac{\eta\theta^2\theta^{\beta-1}e^{-\theta x}x^{\beta-1}}{\theta(\eta+k\theta)\Gamma\beta} \\
&= \frac{\theta^2}{(k\theta+\eta)}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma\alpha} + \frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta x}, \alpha > 0, \beta > 0
\end{aligned}$$

Remark 2.7.2. *Flexibility of a generalized five parameter Lindley distribution Eq.2.77 is attested through assigning the parameters with particular numerical values to obtain various Lindley distribution generalizations.*
Putting $\beta = k = 1$ and $\eta = 0$, AG5PLD reduces to a two parameter gamma distribution of the form;

$$f(x;\alpha,\theta) = \frac{\theta^\alpha}{\Gamma\alpha}e^{-\theta x}x^{\alpha-1}; \quad x > 0, \alpha > 0, \theta > 0 \tag{2.79}$$

Putting $\alpha = \beta = k = 1$ and $\eta = 0$, AG5PLD reduces to exponential distribution Eq.2.17 with parameter θ .
Similarly, putting $\alpha = k = \eta = 1$ and $\beta = 2$, AG5PLD reduces to one parameter Lindley Eq.2.1. We can easily confirm that AG5PLD reduces to a Quasi Lindley distribution Eq.2.30 by putting $\alpha = 1, \beta = 2, \eta = \theta$ and $k = \alpha$.
Putting $\beta = \alpha + 1, k = 1$ and $\eta = \gamma$ AG5PLD reduces to a generalized three parameter Lindley distribution introduced by Dolati and Zakerzadeh (2009) of the form;

$$f(x;\alpha,\theta,\gamma) = \frac{\theta^2(\theta x)^{\alpha-1}\big(\alpha+\gamma x\big)e^{-\theta x}}{(\theta+\gamma)\Gamma(\alpha+1)}; \quad x>0, \alpha>0, \theta>0, \gamma>0 \tag{2.80}$$

A generalized three parameter Lindley distribution Eq.2.80 is a two finite component mixed distribution of Gamma (α,θ) and Gamma ($\alpha+1$, θ) with weight proportion as $\omega = \frac{\theta}{\theta+\gamma}$.
Putting $\theta = \frac{\theta}{\alpha}$, $\quad \alpha = 1$, $\quad \beta = 2$ and $k = 1$ AG5PLD reduces to Janardan distribution introduced by Shanker et al (2013) of the form;

$$f(x;\alpha,\theta) = \frac{\theta^2}{\alpha(\theta+\alpha^2)}\big[1+\alpha x\big]e^{-\left(\frac{\theta}{\alpha}\right)x}; \quad x>0, \alpha>0, \theta>0 \tag{2.81}$$

Janardan distribution Eq.2.81 is a two component finite mixed distribution of Gamma ($1, \frac{\theta}{\alpha}$) and Gamma ($2, \frac{\theta}{\alpha}$) with weight proportion as $\omega = \frac{\theta}{\theta+\alpha^2}$.
Putting $\eta = k = 1$, AGF5LD reduces to a Lindley distribution with three parameters of the form;

$$f(x;\alpha,\beta,\theta) = \frac{1}{\theta+1}\left[\frac{\theta^{\alpha+1}x^{\alpha-1}}{\Gamma\alpha} + \frac{\theta^{\beta}x^{\beta-1}}{\Gamma\beta}\right]e^{-\theta x}; \quad x>0, \alpha>0, \beta>0, \theta>0 \tag{2.82}$$

AG3PLD Eq.2.82 was introduced by Elbatal et al (2013). It is a two component finite mixed distribution of $x_1 \sim$ *Gamma*(α, θ) *and* $x_2 \sim$ *Gamma* (β, θ) *with mixing proportion as* $\omega = \frac{\theta}{\theta+1}$. *Putting* $\theta = \frac{1}{2}, \quad \alpha = \frac{v}{2}, \quad \beta = k = 1, \quad \eta = 0$ *AG5PLD reduces to a chi square distribution of the form;*

$$\chi^2_{(v)} = 2\left[\frac{\left(\frac{1}{2}\right)^{\frac{v}{2}} x^{\frac{v}{2}-1}}{\Gamma(\frac{v}{2})}\right] e^{-\frac{1}{2}x}; \quad x > 0, v > 0 \qquad (2.83)$$

Putting $v, \quad v \in \mathbb{N}, \quad \beta = k = 1, \quad \eta = 0$ *AG5PLD reduces to Erlang distribution that was introduced by Erlang (1917) of the form;*

$$f(x; v, \theta) = \frac{\theta^v}{\Gamma v} e^{-\theta x} x^{v-1}; \quad x > 0, \theta > 0, v = 1, 2, 3, \cdots \qquad (2.84)$$

Putting $k = 1$ *AG5PLD reduces to four parameter Lindley distribution that was introduced by Al-Babtain (2015) of the form;*

$$f(x; \alpha, \beta, \theta, \eta) = \frac{\theta^2}{\theta + \eta}\left[\frac{(\theta x)^{\alpha-1}}{\Gamma\alpha} + \frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right] e^{-\theta x}; \quad x > 0, \alpha > 0, \beta > \qquad (2.85)$$

Similarly, putting $\eta = 1$ *AG5PLD reduces to a four parameter Lindley introduced by Al-Babtain (2015) of the form;*

$$f(x; \alpha, \beta, \theta, \theta, k) = \frac{\theta^2}{k\theta + 1}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma\alpha} + \frac{(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right] e^{-\theta x}; \quad x > 0, \alpha > 0, \beta \qquad (2.86)$$

Setting $\alpha = 1, \quad \beta = 2, \quad \eta = 1$ *AG5PLD reduces to AG2PLD introduced by Al-Babtain (2015) of the form;*

$$f(x; \theta, k) = \frac{\theta^2}{k\theta + 1}\left[k + x\right] e^{-\theta x}; \quad x > 0, \theta > 0, k > 0 \qquad (2.87)$$

Similarly, Cdf 2.63 is obtained as;

$$F(x;\alpha,\beta,\theta,k,\eta) = \frac{\theta^2}{(k\theta+\eta)}\int_0^\infty \left[\frac{k(\theta x)^{\alpha-1}}{\Gamma\alpha}+\frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta x}dx$$
$$= \frac{1}{(k\theta+\eta)}\left[\frac{\theta^2 k}{\Gamma\alpha}\int_0^\infty (\theta x)^{\alpha-1}e^{-\theta x}dx + \frac{\theta^2\eta}{\theta\Gamma\beta}\int_0^\infty (\theta x)^{\beta-1}e^{-\theta x}\right.$$
$$= \frac{1}{(k\theta+\eta)}\left[\frac{\theta k}{\Gamma\alpha}\int_0^\infty \theta(\theta x)^{\alpha-1}e^{-\theta x}dx + \frac{\eta}{\Gamma\beta}\int_0^\infty \theta(\theta x)^{\beta-1}e^{-}\right.$$
$$= \frac{1}{(k\theta+\eta)}\left[k\theta\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right], \alpha>0, \beta>0, \theta>0, k>0,$$

□

2.7.2 Reliability Analysis

Proposition 2.7.3. *The survival function $S(x;\alpha,\beta,\theta,k,\eta)$ and hazard function $h(x;\alpha,\beta,\theta,k,\eta)$ of a generalized five parameter Lindley distribution Eq.2.77 are;*

$$S(x;\alpha,\beta,\theta,k,\eta) = \frac{k\theta+\eta-\left[k\theta\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right]}{k\theta+\eta}; \alpha>0, \beta>0, \theta>0, k>0,$$

(2.88)

$$h(x;\alpha,\beta,\theta,k,\eta) = \frac{\theta^2\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma\alpha}+\frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta x}}{k\theta+\eta-\left[k\theta\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right]}; \alpha>0, \beta>0, \theta>0, k>0,$$

(2.89)

Proof. survival function Eq.2.88 is obtained using Eq.1.8 as;

$$S(x;\alpha,\beta,\theta,k,\eta) = 1-\frac{1}{k\theta+\eta}\left[k\theta\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right]$$
$$= \frac{k\theta+\eta-\left[k\theta\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right]}{k\theta+\eta}; \alpha>0, \beta>0, \theta>0, k>0,$$

hazard function Eq. 2.89 is obtained using the relation Eq.1.9 as;

$$h(x;\alpha,\beta,\theta,k,\eta) = \frac{\theta^2}{k\theta+\eta}\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma\alpha} + \frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta x} * \frac{k\theta \cdots}{k\theta+\eta-\left[k\gamma_{\alpha} \cdots\right.}$$

$$= \frac{\theta^2\left[\frac{k(\theta x)^{\alpha-1}}{\Gamma\alpha} + \frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta x}}{k\theta+\eta-\left[k\theta\gamma_{\alpha(\theta x)} + \eta\gamma_{\beta(\theta x)}\right]}; \alpha>0, \beta>0, \theta>0, k \cdots$$

□

2.7.3 Moments and related measures

Proposition 2.7.4. *The r^{th} moments of a generalized five parameter Lindley distribution Eq. 2.77 about the origin are obtained using both moment generating technique and methods of moments as;*

$$E(X^r) = \frac{1}{k\theta+\eta}\left[\frac{k\theta^{\alpha+1}\Gamma(\alpha+r)}{\theta^{\alpha}\Gamma\alpha} + \frac{\eta\theta^{\beta}\Gamma(\beta+r)}{\theta^{\beta}\Gamma\beta}\right], \quad r = 1,2,3,\cdots \tag{2.90}$$

$$m_x(t) = \frac{1}{k\theta+\eta}\left[k\theta(1-\frac{t}{\theta})^{-\alpha} + \eta(1-\frac{t}{\theta})^{-\beta}\right], \quad \theta > t \tag{2.91}$$

Proof. By definition of moments Eq.1.16, moments Eq.2.90 are obtained as;

$$\begin{aligned}
E(X^r) &= \frac{\theta^2}{k\theta+\eta}\left[\int_0^{\infty} x^r\left(\frac{(\theta x)^{\alpha-1}}{\Gamma\alpha}+\frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right)\right]e^{-\theta x}dx \\
&= \frac{\theta^2}{k\theta+\eta}\left[\frac{k\theta^{\alpha-1}}{\Gamma\alpha}\int_0^{\infty} x^{\alpha+r-1}e^{-\theta x}dx+\frac{\eta\theta^{\beta-1}}{\theta\Gamma\beta}\int_0^{\infty} x^{\beta+r-1}e^{-\theta x}dx\right] \\
&= \frac{\theta^2}{k\theta+\eta}\left[\frac{k\theta^{\alpha-1}\Gamma(\alpha+r)}{\Gamma\alpha\theta^{\alpha+r}}+\frac{\eta\theta^{\beta-2}\Gamma(\beta+r)}{\Gamma\beta\theta^{\beta+r}}\right] \\
&= \frac{1}{k\theta+\eta}\left[\frac{k\theta^{\alpha+1}\Gamma(\alpha+r)}{\Gamma\alpha\theta^{\alpha+r}}+\frac{\eta\theta^{\beta}\Gamma(\beta+r)}{\Gamma\beta\theta^{\beta+r}}\right] \\
&= \frac{1}{\theta^r(k\theta+\eta)}\left[\frac{k\theta^{\alpha+1}\Gamma(\alpha+r)}{\theta^r\Gamma\alpha}+\frac{\eta\theta^{\beta}\Gamma(\beta+r)}{\theta^{\beta}\Gamma\beta}\right], \quad r=1,2,3,\cdots
\end{aligned}$$

Mgf Eq.2.91 is obtained by use of relation Eq.1.17 as;

$$
\begin{aligned}
m_x(t) &= \frac{\theta^2}{k\theta+\eta}\left[\int_0^\infty e^{tx}\left(\frac{k(\theta x)^{\alpha-1}}{\Gamma\alpha}+\frac{\eta(\theta x)^{\beta-1}}{\theta\Gamma\beta}\right)e^{-\theta x}dx\right] \\
&= \frac{\theta^2}{k\theta+\eta}\left[\frac{k\theta^{\alpha-1}}{\Gamma\alpha}\int_0^\infty e^{-(\theta-t)x}x^{\alpha-1}dx+\frac{\eta\theta^{\beta-2}}{\Gamma\beta}\int_0^\infty e^{-(\theta-t)x}x^{\beta-1}dx\right] \\
&= \frac{1}{k\theta+\eta}\left[\frac{k\theta^{\alpha-1}}{\Gamma\alpha}\int_0^\infty e^{-(\theta-t)x}x^{\alpha-1}dx+\frac{\eta\theta^{\beta}}{\Gamma\beta}\int_0^\infty e^{-(\theta-t)x}x^{\beta-1}dx\right] \\
&= \frac{1}{k\theta+\eta}\left[\frac{k\theta^{\alpha+1}\Gamma\alpha}{\Gamma\alpha(\theta-t)^{\alpha}}+\frac{\eta\theta^{\beta}\Gamma\beta}{\Gamma\beta(\theta-t)^{\beta}}\right] \\
&= \frac{1}{k\theta+\eta}\left[\frac{k\theta^{\alpha}\theta}{(\theta-t)^{\alpha}}+\frac{\eta\theta^{\beta}}{(\theta-t)^{\beta}}\right] \\
&= \frac{1}{k\theta+\eta}\left[(\frac{\theta}{\theta-t})^{\alpha}k\theta+\eta(\frac{\theta}{\theta-t})^{\beta}\right] \\
&= \frac{1}{k\theta+\eta}\left[k\theta(\frac{\theta-t}{\theta})^{-\alpha}+\eta(\frac{\theta-t}{\theta})^{-\beta}\right] \\
m_x(t) &= \frac{1}{k\theta+\eta}\left[k\theta(1-\frac{t}{\theta})^{-\alpha}+\eta(1-\frac{t}{\theta})^{-\beta}\right], \quad \theta > t
\end{aligned}
$$

using the identity relation $(1-x)^{-d} = \sum_{r=0}^{\infty}(-1)^r\binom{-d}{r}x^r$, the moment generating function can be expressed as; $m_x(t) = \frac{1}{k\theta+\eta}\Bigg[\sum_{r=0}^{\infty}\frac{(-1)}{\theta^r}$ $\eta\binom{-\beta}{r}\Bigg)t^r\Bigg]$ where

$$
\begin{aligned}
(1-\frac{t}{\theta})^{-\alpha} &= \sum_{r=0}^{\infty}(-1)^r\binom{-\alpha}{r}(\frac{t}{\theta})^r \\
(1-\frac{t}{\theta})^{-\beta} &= \sum_{r=0}^{\infty}(-1)^r\binom{-\beta}{r}(\frac{t}{\theta})^r
\end{aligned}
$$

□

The moment generating function (mgf) Eq.2.91 is applied to derive the first four moments about the origin of a generalized five parameter Lindley distribution Eq.2.77 as:

$$m_x(t) = \frac{1}{k\theta+\eta}\left[k\theta\big(1-\frac{t}{\theta}\big)^{-\alpha} + \eta\big(1-\frac{t}{\theta}\big)^{-\beta}\right]$$

$$m_x^i(t) = \frac{1}{k\theta+\eta}\left[-\alpha\theta k(1-\frac{t}{\theta})^{-(\alpha+1)}(-\frac{1}{\theta}) - \beta\eta(1-\frac{t}{\theta})^{-(\beta+1)}(-\frac{1}{\theta})\right]$$

$$m_x^i(t) = \frac{1}{\theta(k\theta+\eta)}\left[\alpha\theta k\big(1-\frac{t}{\theta}\big)^{-(\alpha+1)} + \beta\eta\big(1-\frac{t}{\theta}\big)^{-(\beta+1)}\right]$$

$$m_x^i(0) = \frac{1}{\theta(k\theta+\eta)}\left[\alpha\theta k + \beta\eta\right] = \mu_1^1$$

$$m_x^{ii}(t) = \frac{1}{\theta(k\theta+\eta)}\left[-(\alpha+1)\alpha\theta k\big(1-\frac{t}{\theta}\big)^{-(\alpha+2)}(-\frac{1}{\theta}) - (\beta+1)\beta\eta\big(1-\frac{t}{\theta}\big)\right.$$

$$= \frac{1}{\theta^2(k\theta+\eta)}\left[(\alpha+1)\alpha\theta k\big(1-\frac{t}{\theta}\big)^{-(\alpha+2)} + (\beta+1)\beta\eta\big(1-\frac{t}{\theta}\big)^{-(\beta+2)}\right]$$

$$m_x^{ii}(0) = \frac{1}{\theta^2(k\theta+\eta)}\left[(\alpha+1)\alpha\theta k + (\beta+1)\beta\eta\right] = \mu_2^1$$

$$m_x^{iii}(t) = \frac{1}{\theta^2(k\theta+\eta)}\left[-(\alpha+1)(\alpha+2)\alpha\theta k\big(1-\frac{t}{\theta}\big)^{-(\alpha+3)} - (\beta+1)(\beta+2)\beta\right.$$

$$= \frac{1}{\theta^3(k\theta+\eta)}\left[(\alpha+1)(\alpha+2)\alpha\theta k\big(1-\frac{t}{\theta}\big)^{-(\alpha+3)} + (\beta+1)(\beta+2)\beta\eta\right.$$

$$m_x^{iii}(0) = \frac{1}{\theta^3(k\theta+\eta)}\left[(\alpha+1)(\alpha+2)\alpha\theta k + (\beta+1)(\beta+2)\beta\eta\right] = \mu_3^1$$

$$m_x^{iv}(t) = \frac{1}{\theta^3(k\theta+\eta)}\left[-(\alpha+1)(\alpha+2)(\alpha+3)\alpha\theta k\big(1-\frac{t}{\theta}\big)^{-(\alpha+4)}(\frac{-1}{\theta}) - (\beta+1\right.$$

$$m_x^{iv}(0) = \frac{1}{\theta^4(k\theta+\eta)}\left[(\alpha+1)(\alpha+2)(\alpha+3)\alpha\theta k+(\beta+1)(\beta+2)(\beta+3)\right.$$

2.7.4 Excess loss distribution

Proposition 2.7.5. *The expressions of probability density function (pdf) of excess distribution $f_l(x;\alpha,\beta,\theta,k,\eta)$, mean residual lifetime (MRL), equilibrium distribution $f_e(x;\alpha,\beta,\theta,k,\eta)$, survival function based on equilibrium distribution $S_e(x;\alpha,\beta,\theta,k,\eta)$, hazard function based on equilibrium distribution $h_e(x;\alpha,\beta,\theta,k,\eta)$ of a generalized five parameter Lindley distribution Eq.2.77 are as:*

$$f_l(x;\alpha,\beta,\theta,k,\eta) = \frac{\left[\frac{k\theta^{\alpha-1}}{\Gamma\alpha}\left(\theta x^{\alpha-1}+(\alpha-1)x^{\alpha-2}\right)+\frac{\eta\theta^{\beta-2}}{\Gamma\beta}\left(\theta x^{\beta-1}+(\beta\right.\right.}{\eta+\theta k-\left[\theta k\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right]} \tag{2.92}$$

$$m(x) = \frac{\alpha\theta k\Gamma(\alpha+1)(\theta x)+\eta\beta\Gamma(\beta+1)(\theta x)-x}{\theta\left[\eta+\theta k-\left(\theta k\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right)\right]} \tag{2.93}$$

$$f_e(x;\alpha,\beta,\theta,k,\eta) = \frac{\eta+\theta k-\left[k\theta\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right]}{\alpha\theta k+\beta\eta} \tag{2.94}$$

$$S_e(x;\alpha,\beta,\theta,k,\eta) = \frac{\theta\left[\theta k\alpha\Gamma(\alpha+1)(\theta x)+\eta\beta\Gamma(\beta+1)(\theta x)\right]-x}{\alpha\theta k+\beta\eta} \tag{2.95}$$

$$h_e(x;\alpha,\beta,\theta,k,\eta) = \frac{\eta+\theta k-\left[k\theta\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right]}{\theta\left[\theta k\alpha\Gamma(\alpha+1)(\theta x)+\eta\beta\Gamma(\beta+1)(\theta x)\right]-x} \tag{2.96}$$

Proof. The pdf of excess loss distribution Eq.2.92 is obtained as;

$$
\begin{aligned}
f_l(x) &= \frac{\int_x^\infty f(t)dt}{S(x)} \\
&= \frac{\frac{\theta^2}{k\theta+\eta}\int_x^\infty \left[\frac{k(\theta t)^{\alpha-1}}{\Gamma\alpha}+\frac{\eta(\theta t)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta t}dt}{\frac{k\theta+\eta-\left[k\theta\gamma_{\alpha(\theta x)}+\gamma_{\beta(\theta x)}\right]}{k\theta+\eta}} \\
&= \frac{\theta^2\int_x^\infty \left[\frac{k(\theta t)^{\alpha-1}}{\Gamma\alpha}+\frac{\eta(\theta t)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta t}dt}{k\theta+\eta-\left[k\theta\gamma_{\alpha(\theta x)}+\gamma_{\beta(\theta x)}\right]}
\end{aligned}
$$

Working on the numerator we have the following;

$$
\begin{aligned}
\theta^2\int_x^\infty \left[\frac{k(\theta t)^{\alpha-1}}{\Gamma\alpha}e^{-\theta x}\right]dt &= \frac{k\theta^{\alpha+1}}{\Gamma\alpha}\int_x^\infty t^{\alpha-1}e^{-\theta t}dt \\
&= \frac{k\theta^{\alpha+1}}{\Gamma\alpha}\left(\frac{(\alpha-1)x^{\alpha-2}+\theta x^{\alpha-1}}{\theta^2}\right)e^{-\theta x} \\
&= \frac{k\theta^{\alpha-1}}{\Gamma\alpha}\left[\theta x^{\alpha-1}+(\alpha-1)x^{\alpha-2}\right]e^{-\theta x}
\end{aligned}
$$

Similarly;

$$
\begin{aligned}
\theta^2\int_x^\infty \frac{\eta(\theta t)^{\beta-1}}{\theta\Gamma\beta}e^{-\theta t}dt &= \frac{\theta^\beta\eta}{\Gamma\beta}\int_x^\infty t^{\beta-1}e^{-\theta t}dt \\
&= \frac{\eta\theta^\beta}{\Gamma\beta}\left[\frac{\theta x^{\beta-1}+(\beta-1)x^{\beta-2}}{\theta^2}\right]e^{-\theta x} \\
&= \frac{\eta\theta^{\beta-2}}{\Gamma\beta}\left[\theta x^{\beta-1}+(\beta-1)x^{\beta-2}\right]e^{-\theta x}
\end{aligned}
$$

Then $f_l(x;\alpha,\beta,\theta,k,\eta)$ becomes;

$$f_l(x;\alpha,\beta,\theta,k,\eta)=\frac{\left[\frac{k\theta^{\alpha-1}}{\Gamma\alpha}\left(\theta x^{\alpha-1}+(\alpha-1)x^{\alpha-2}\right)+\frac{\eta\theta^{\beta-2}}{\Gamma\beta}\left(\theta x^{\beta-1}+(\beta\right.\right.}{\eta+\theta k-\left[\theta k\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right]}$$

Mean residual lifetime Eq.2.93 is obtained as;

$$\begin{aligned}
m(x)&=\frac{\int_x^\infty tf(t)dt-xa}{S(x)},x>0\\
&=\frac{\frac{\theta^2}{\eta+\theta k}\int_x^\infty t\left[\frac{k(\theta t)^{\alpha-1}}{\Gamma\alpha}+\frac{\eta(\theta t)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta t}dt}{\frac{\eta+\theta k-\left[\theta k\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right]}{\eta+\theta k}}-x\\
&=\frac{\theta^2\int_x^\infty t\left[\frac{k(\theta t)^{\alpha-1}}{\Gamma\alpha}+\frac{\eta(\theta t)^{\beta-1}}{\theta\Gamma\beta}\right]e^{-\theta t}dt}{\eta+\theta k-\left[\theta\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right]}-x\\
&=\frac{\frac{\theta^{\alpha+1}k}{\Gamma\alpha}\int_x^\infty t^\alpha e^{-\theta t}dt+\frac{\eta\theta^\beta}{\Gamma\beta}\int_x^\infty t^\beta e^{-\theta t}}{\eta+\theta k-\left[\theta\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right]}-x\\
&=\frac{\theta^2k\alpha\Gamma(\alpha+1)(\theta x)+\eta\theta\beta\Gamma(\beta+1)(\theta x)}{\eta+\theta k-\left[k\theta\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right]}-x\\
&=\frac{\alpha\theta k\Gamma(\alpha+1)(\theta x)+\eta\beta\Gamma(\beta+1)(\theta x)}{\theta\left(\eta+\theta-\left[\theta k\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right]\right)}-x
\end{aligned}$$

Using the relation 1.12, equilibrium distribution Eq.2.94 is obtained as;

$$\begin{aligned}
f_e(x;\alpha,\beta,\theta,k,\eta)&=\frac{\eta+\theta k-\left[k\theta\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right]}{k\theta+\eta}\frac{\theta(k\theta+\eta)}{\alpha\theta+\beta\eta}\\
&=\frac{\theta\left[\eta+\theta k-\left(k\theta\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right)\right]}{\alpha\theta k+\beta\eta}
\end{aligned}$$

Survival function based on the equilibrium distribution Eq.2.95 is obtained using relation Eq. 1.13 as;

$$\int_x^\infty S(t;\alpha,\beta,\theta,k,\eta)dt = \int_x^\infty 1dt - \int_x^\infty \frac{1}{\theta k+\eta}\left[\theta k\gamma_{\alpha(\theta t)} + \eta\gamma_{\beta(\theta t)}\right]dt$$

$$= -x + \frac{1}{\theta k+\eta}\int_x^\infty \left(\theta k\gamma_{\alpha(\theta t)}\right)dt + \frac{1}{\theta k+\eta}\int_{x0}^\infty \eta\gamma_{\beta(\theta t)}d$$

$$= -x + \frac{1}{\theta k+\eta}\left[\theta k\alpha\Gamma(\alpha+1)(\theta x)\right] + \frac{1}{\theta k+\eta}\left[\eta\beta\Gamma(\right.$$

$$= \frac{1}{\theta k+\eta}\left[\theta k\alpha\Gamma(\alpha+1)(\theta x) + \eta\beta\Gamma(\beta+1)(\theta x)\right] - x$$

$$= \frac{\frac{1}{\theta k+\eta}\left[\theta k\alpha\Gamma(\alpha+1) + \eta\beta\Gamma(\beta+1)\right] - x}{\frac{\alpha\theta k+\beta\eta}{\theta(\theta k+\eta)}}$$

$$= \frac{\theta\left(\theta k\alpha\Gamma(\alpha+1)(\theta x) + \eta\beta\Gamma(\beta+1)(\theta x)\right) - x}{\alpha\theta k+\beta\eta}$$

Using the relation Eq.1.14, hazard function on equilibrium distribution Eq.2.96 as;

$$h_e(x;\alpha,\beta,\theta,k,\eta) = \frac{\frac{\eta+\theta k-\left[k\theta\gamma_{\alpha(\theta x)}+\eta\gamma_{\beta(\theta x)}\right]}{\alpha\theta k+\beta\eta}}{\frac{\theta\left[\theta k\alpha\Gamma(\alpha+1)(\theta x)+\eta\gamma_{\beta(\theta x)}\right]-x}{\alpha\theta k+\beta\eta}}$$

$$= \frac{\eta+\theta k-\left[k\theta\gamma_{\alpha(\theta x)}+\eta\gamma_{\theta x}\right]}{\theta\left[\theta k\alpha\Gamma(\alpha+1)(\theta x)+\eta\beta\Gamma(\beta+1)(\theta x)\right]-x}$$

□

3 TWO COMPONENT FINITE GAMMA MIXTURE (Case of Akash distribution)

3.1 Introduction

The objective is to construct and derive statistical properties of a two component finite gamma mixed distribution a case of Akash distribution. Akash distribution is generalized up to three parameters. The mixed distribution is expressed in terms of pdf and Cdf. we later derive statistical properties such as reliability measures, equilibrium distribution properties, moments both centralized and none-centralized moments.

3.2 One parameter Akash distribution

3.2.1 Construction of one parameter Akash distribution

Proposition 3.2.1. *One parameter Akash distribution is a two component finite mixed distribution of Gamma(1,θ) and Gamma(3,θ) with weight proportion as $\omega = \frac{\theta^2}{\theta^2+2}$, the pdf and Cdf of one parameter Akash distribution are;*

$$f(x;\theta) = \frac{\theta^3}{\theta^2+2}\left[1+x^2\right]e^{-\theta x}; x>0, \theta>0 \quad (3.1)$$

$$F(x;\theta) = 1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}; x>0, \theta>0 \quad (3.2)$$

Proof. Using the concept of finite mixtures Eq.1.1, the pdf of one parameter Akash distribution is obtained as;

$$
\begin{aligned}
f(x;\theta) &= \frac{\theta^2}{\theta^2+2}\left[\theta e^{-\theta x}\right] + \frac{2}{\theta^2+2}\left[\frac{\theta^3}{\Gamma 3} e^{-\theta x} x^2\right] \\
&= \frac{\theta^3}{\theta^2+2} e^{-\theta x} + \frac{2\theta^3}{(\theta^2+2)\Gamma 3} \\
&= \frac{\theta^3 e^{-\theta x}}{\theta^2+2} + \frac{2\theta^3 e^{-\theta x} x^2}{2(\theta^2+2)} \\
&= \frac{\theta^3}{\theta^2+2}\left[1+x^2\right] e^{-\theta x}, x>0, \theta>0
\end{aligned}
$$

further Cdf Eq.3.2 is obtained as;

$$
\begin{aligned}
F(x;\theta) &= \frac{\theta^3}{\theta^2+2}\int_0^\infty (1+x^2)e^{-\theta x}dx \\
&= \frac{\theta^3}{\theta^2+2} I_1 \\
I_1 &= \int_0^\infty (1+x^2)e^{-\theta x}dx \\
u &= (1+x^2) \implies du = 2xdx \\
dv &= e^{-\theta x} \implies v = \frac{-e^{-\theta x}}{\theta} \\
I_1 &= -(1+x^2)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}\int_0^\infty -xe^{-\theta x}dx \\
I_1 &= -(1+x^2)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta} * I_2 \\
I_2 &= \int_0^\infty -xe^{-\theta x}dx \\
u &= -x \implies du = -dx \\
dv &= e^{\theta x} \implies v = \frac{-e^{-\theta x}}{\theta}
\end{aligned}
$$

$$I_2 = \frac{xe^{-\theta x}}{\theta} - \frac{1}{\theta}\int_0^\infty e^{-\theta x}dx$$
$$I_2 = \frac{xe^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2}$$

From I_1 and I_2 we have;

$$I_1 = -(1+x^2)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}[\frac{xe^{-\theta x}}{\theta} + \frac{e^{-\theta x}}{\theta^2}]$$
$$I_1 = \frac{-(1+x^2)e^{-\theta x}}{\theta} - \frac{2xe^{-\theta x}}{\theta^2} - \frac{2e^{-\theta x}}{\theta^3}$$
$$I_1 = \frac{-(\theta^2+\theta^3x^2+2x\theta+2)e^{-\theta x}}{\theta^3}$$
$$= \frac{\theta^3}{\theta^2+2}\int_0^\infty (1+x^2)e^{-\theta x}dx$$
$$= 1+-\frac{\theta^3}{\theta^2+2}\left[\frac{\theta^2+2+\theta^2x^2+2x\theta}{\theta^3}\right]e^{-\theta x}$$
$$= 1-\left[\frac{\theta^2+2+\theta^2x^2+2x\theta}{\theta^2+2}\right]e^{-\theta x}$$
$$= 1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}; x>, \theta>0$$

□

3.2.2 Reliability analysis

Proposition 3.2.2. *Survival function denoted by* $S(x;\theta)$ *and hazard function denoted by* $h(x;\theta)$ *of one parameter Akash dis-*

tribution Eq.3.1 are;

$$S(x;\theta)=\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}; x>0, \theta>0 \tag{3.3}$$

$$h(x;\theta)=\frac{\theta^3(1+x^2)}{\theta^2+2+\theta x(\theta x+2)}; x>0, \theta>0 \tag{3.4}$$

Proof. Applying the relation Eq.1.8, survival function $S(x;\theta)$ Eq.3.3 is obtained as;

$$\begin{aligned} S(x;\theta) &= 1-\left[1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\right] \\ &= \left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x} \end{aligned}$$

hazard function $h(x;\theta)$ Eq.3.4 is obtained using Eq.1.9 as;

$$\begin{aligned} h(x;\theta) &= \frac{\frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}}{\left[\frac{\theta^2+2+\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}} \\ &= \frac{\theta^3(1+x^2)}{\theta^2+2+\theta x(\theta x+2)} \end{aligned}$$

□

3.2.3 Moments and related measures

Proposition 3.2.3. *The r^{th} moments of one parameter Akash distribution are obtained using both method of moments and moment generating function as shown below:*

$$\mu_r^1 * = \frac{r!(\theta^2+(r+1)(r+2))}{\theta^r(\theta^2+2)} \tag{3.5}$$

Proof. By definition of moments Eq.1.16, moments of one parameter Akash are obtained as;

$$
\begin{aligned}
E(X^r) &= \frac{\theta^3}{\theta^2+2}\int_0^\infty x^r(1+x^2)e^{-\theta x}dx \\
&= \frac{\theta^3}{\theta^2+2}\left[\int_0^\infty x^r e^{-\theta x}dx + \int_0^\infty x^{r+2}e^{-\theta x}dx\right] \\
&= \frac{\theta^3}{\theta^2+2}\left[\frac{\Gamma(r+1)}{\theta^{r+1}} + \frac{\Gamma(r+3)}{\theta^3}\right] \\
&= \frac{r!(\theta^2+(r+1)(r+2))}{\theta^r(\theta^2+2)}; r = 1,2,3,\cdots
\end{aligned}
$$

Applying the definition of mgf Eq.1.17, mgf of one parameter Akash distribution is obtained as;

$$
\begin{aligned}
m_x(t) &= \frac{\theta^3}{\theta^2+2}\int_0^\infty e^{tx}(1+x^2)e^{-\theta x}dx \\
&= \frac{\theta^3}{\theta^2+2}\int_0^\infty e^{-(\theta-t)x}(1+x^2)dx \\
&= \frac{\theta^3}{\theta^2+2}\left[\frac{1}{(\theta-t)} + \frac{2}{(\theta-t)^3}\right] \\
&= \frac{\theta^3}{\theta+2}\left[\frac{1}{\theta}\sum_{k=0}^{\infty}(\frac{t}{\theta})^k + \frac{2}{\theta^3}\sum_{k=0}^{\infty}\binom{k+2}{k}(\frac{t}{\theta})^k\right] \\
&= \sum_{k=0}^{\infty}\frac{\theta^2+(k+1)(k+2)}{\theta^2+2}(\frac{t}{\theta})^k
\end{aligned}
$$

□

The r^{th} moments about the origin of one parameter Akash distribution Eq.3.1 are obtained as a coefficient of $\frac{t^r}{r!}$ in $m_x(t)$ as:

$$\mu_r^1 = \frac{r!(\theta^2 + (r+1)(r+2))}{\theta^r(\theta^2+2)}; r = 1,2,3,\cdots$$

Setting $r = 1,2,3$ and 4 in Eq.3.5 the none centralized moments are:

$$\mu_1^1 = \frac{\theta^2+6}{\theta(\theta^2+2)}, \quad \mu_2^1 = \frac{2(\theta^2+12)}{\theta^2(\theta^2+2)}$$
$$\mu_3^1 = \frac{6(\theta^2+20)}{\theta^3(\theta^2+2)}, \quad \mu_4^1 = \frac{24(\theta^2+30)}{\theta^4(\theta^2+2)}$$

We now derive centralized moments as;

$$\mu_1 = \mu_1^1$$
$$\mu_2 = \mu_2^1 - (\mu_1^1)^2$$
$$\mu_2 = \frac{2(\theta^2+12)}{\theta^2(\theta^2+2)} - \left(\frac{\theta^2+6}{\theta(\theta^2+2)}\right)^2$$
$$\mu_2 = \frac{\theta^4+16\theta^2+12}{\theta^2(\theta^2+2)^2}$$
$$\mu_3 = \mu_3^1 - 3\mu_2^1\mu_1^1 + 2(\mu_1^1)$$
$$= \frac{6(\theta^2+20)}{\theta^3(\theta^2+2)} - 3\left(\frac{2(\theta^2+12)}{\theta^2(\theta^2+2)}\frac{\theta^2+6}{\theta(\theta^2+2)}\right) + 2\left(\frac{\theta^2+6}{\theta(\theta^2+2)}\right)^2$$
$$\mu_3 \frac{2\theta^6+60\theta^4+72\theta^2+48}{\theta^3(\theta^2+2)^3}$$
$$\mu_4 = \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^!)^2$$
$$\mu_4 = \frac{9\theta^8+384\theta^6+1224\theta^4+1728\theta^2+720}{\theta^4(\theta^2+2)^4}$$

Proposition 3.2.4. *Other related measures such as variation coefficient (C.v),skewness (υ_1), kurtosis (υ_2) and dispersion index*

(υ_3) of a one parameter Akash distribution Eq.3.1 are:

$$C.v = \frac{\sqrt{\theta^4 + 16\theta^2 + 12}}{\theta^2 + 6} \tag{3.6}$$

$$\upsilon_1 = \frac{2\theta^6 + 60\theta^4 + 72\theta^2 + 48}{(\theta^4 + 16\theta^2 + 12)^{\frac{3}{2}}} \tag{3.7}$$

$$\upsilon_2 = \frac{9\theta^8 + 384\theta^6 + 1224\theta^4 + 1728\theta^2 + 720}{(\theta^4 + 16\theta^2 + 12)^2} \tag{3.8}$$

$$\upsilon_3 = \frac{\theta^4 + 16\theta^2 + 12}{\theta^5 + 8\theta^3 + 12\theta} \tag{3.9}$$

Proof. Coefficient of variation Eq.3.6 is obtained as;

$$C.v = \frac{\sigma}{\mu_1^1}$$
$$= \frac{\sqrt{\theta^4 + 16\theta^2 + 12}}{\theta(\theta^2 + 2)} \frac{\theta(\theta^2 + 2)}{\theta^2 + 6} \Longrightarrow \frac{\sqrt{\theta^4 + 16\theta^2 + 12}}{\theta^2 + 6}$$

Skewness coefficient in Eq.3.7 is obtained as;

$$\upsilon_1 = \frac{\mu_3}{(\mu_2)^2}$$
$$= \frac{\theta^4 + 16\theta^2 + 12}{\theta^5 + 8\theta^3 + 12\theta} \left[\frac{\theta^2(\theta^2 + 2)^2}{\theta^4 + 16\theta^2 + 12} \right]^{\frac{3}{2}} \Longrightarrow \frac{2\theta^6 + 60\theta^4 + 72\theta^2 + 48}{(\theta^4 + 16\theta^2 + 12)^{\frac{3}{2}}}$$

Coefficient of kurtosis Eq.3.8 is obtained as ;

$$\upsilon_2 = \frac{\mu_4}{(\mu_2)^2}$$
$$= \frac{9\theta^8 + 384\theta^6 + 1224\theta^4 + 1728\theta^2 + 720}{\theta^4(\theta^2 + 2)^4} \left(\frac{\theta^2(\theta^2 + 2)^2}{\theta^4 + 16\theta^2 + 12} \right)$$
$$= \frac{9\theta^8 + 384\theta^6 + 1224\theta^4 + 1728\theta^2 + 720}{(\theta^4 + 16\theta^2 + 12)^2}$$

Index of dispersion Eq.3.9 is obtained as;

$$\upsilon_3 = \frac{\sigma^2}{\mu_1^1}$$
$$= \frac{\theta^4+16\theta^2+12}{\theta^2(\theta^2+2)^2}\frac{\theta(\theta^2+6)}{\theta^2+6} \Longrightarrow \frac{\theta^4+16\theta^2+12}{\theta^5+8\theta^3+12\theta}$$

□

3.2.4 Excess Loss Distribution

Proposition 3.2.5. *In this section, the pdf of excess loss distribution $f_l(x;\theta)$, mean residual lifetime (MRL), equilibrium distribution $f_e(x;\theta)$, survival function based on the equilibrium distribution $S_e(x;\theta)$ and hazard function based on the equilibrium distribution $h_e(x;\theta)$ of one parameter Akash distribution Eq.3.1 are stated as:*

$$f_l(x;\theta) = \frac{\theta^3(1+x^2)e^{-(x-z)\theta}}{\theta^2+2+\theta z(\theta z+2)}; x > z \tag{3.10}$$

$$m(x) = \frac{\theta(\theta^2+2+\theta^2X^2+2\theta x)+2(\theta^2+6+\theta^2 x+\theta)+2\theta}{\theta^2(\theta^2+2+\theta^2x^2+2\theta x)} \tag{3.11}$$

$$f_e(x;\theta) = \frac{(\theta^2+2+\theta x(\theta x+2))e^{-\theta x}}{\theta^2+6} \tag{3.12}$$

$$S_e(x;\theta) = \frac{(\theta^3+6\theta+4\theta^2x+\theta^3x^2)e^{-\theta x}}{\theta(\theta^2+6)} \tag{3.13}$$

$$h_e(x;\theta) = \frac{\theta(\theta^2+2+\theta^2x^2+2\theta x)}{\theta^3+6\theta+4\theta^2x+\theta^3x^2} \tag{3.14}$$

`Proof.` Pdf of excess los distribution $f_l(x;\theta)$ in Eq.3.10 is obtained using the relation Eq.1.10 as;

$$f_l(x;\theta) = \frac{\frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}}{\frac{(\theta^2+2+\theta z(\theta z+2))e^{-\theta z}}{\theta^2+2}}$$

$$= \frac{\theta^3(1+x^2)e^{-(x-z)\theta}}{\theta^2+2+\theta z(\theta z+2)}; x > z$$

Using the relation Eq.1.11, the mean residual lifetime Eq.3.11 is obtained as;

$$m(x)\frac{\theta^2+2}{(\theta^2+2+\theta^2x^2+2\theta x)e^{-\theta x}}\int_x^\infty \left(\frac{\theta^2+2+\theta^2t^2+2\theta t}{\theta^2+2}\right)e^{-\theta t}dt$$

$$\frac{1}{(\theta^2+2+\theta^2x^2+2\theta x)e^{-\theta x}}\int_x^\infty (\theta^2+2+\theta^2t^2+2\theta t)e^{-\theta t}dt$$

we now use integration by parts technique on the part $\int_x^\infty (\theta^2 + 2 + \theta^2 t^2 + 2\theta t) e^{-\theta t} dt$;

$$I_1 = \int_x^\infty (\theta^2 + 2 + \theta^2 t^2 + 2\theta t) e^{-\theta t} dt$$

$$u = (\theta^2 + 2 + \theta^2 t^2 + 2\theta t) \Longrightarrow du = 2\theta^2 t + 2\theta$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{\theta t}}{\theta}$$

$$= -(\theta^2 + 2 + \theta^2 t^2 + 2\theta t)\frac{e^{-\theta t}}{\theta} + \frac{2}{\theta}\int_x^\infty (\theta^2 t + \theta) e^{-\theta t} dt$$

$$I_1 = -(\theta^2 + 2 + \theta^2 t^2 + 2\theta t)\frac{e^{-\theta t}}{\theta} + \frac{2}{\theta} I_2$$

$$I_2 = \int_x^\infty (\theta^2 t + \theta) e^{-\theta t} dt$$

$$u = \theta^2 t + \theta \Longrightarrow du = \theta^2 dt$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_2 = -(\theta^2 t + \theta)\frac{e^{-\theta t}}{\theta} + \theta \int_x^\infty e^{-\theta t} dt$$

$$I_2 = -(\theta^2 t + \theta)\frac{e^{-\theta t}}{\theta} + \theta I_3$$

$$I_3 = \int_x^\infty e^{-\theta t} dt$$

$$I_3 = \frac{-e^{-\theta t}}{\theta}$$

From I_1, I_2 and I_3 we have;

$$I_1 = \left[-(\theta(\theta^2+2+\theta^2 t^2+2\theta t)+2(\theta^2 t+\theta)+2\theta)\frac{e^{-\theta t}}{\theta} \right]_x^{\infty}$$

$$\begin{aligned} I_1 &= (\theta(\theta^2+2+\theta^2x^2+2\theta x)+2(\theta^2 x+\theta)+2\theta)\frac{e^{-\theta x}}{\theta} \\ &= \frac{1}{(\theta^2+2+\theta^2x^2+2\theta x)e^{-\theta x}} I_1 \\ &= \frac{\theta(\theta^2+2+\theta^2x^2+2\theta x)+2(\theta^2x+\theta)+2\theta}{\theta^2(\theta^2+2+\theta^2x^2+2\theta x)} \\ &= \frac{\theta^2+6+\theta^2x^2+4\theta x}{\theta(\theta^2+2+\theta^2x^2+2\theta x)} \end{aligned}$$

By definition Eq.1.12, the equilibrium distribution $f_e(x;\theta)$ in Eq.3.12 is obtained as;

$$f_e(x;\theta) = \frac{\theta^2+2+\theta x(\theta x+2)}{\theta^2+2}\frac{\theta(\theta^2+2)}{\theta^2+6} \Longrightarrow \frac{\theta(\theta^2+2+\theta x(\theta x+2)e^{-\theta}}{\theta^2+6}$$

Using the relation Eq.1.13, survival function based on the equilibrium distribution Eq.3.13is obtained as;

$$\begin{aligned} \int_x^{\infty} s(t;\theta)dt &= \frac{(\theta^3+6\theta+4\theta^2x+\theta^3x)e^{-\theta x}}{\theta(\theta^2+2)} \\ &= \frac{(\theta^2+6+4\theta x+\theta^2x^2)e^{-\theta x}}{\theta(\theta^2+6)}\frac{\theta(\theta^2+2)}{\theta^2+6} \\ S_e(x;\theta) &= \frac{(\theta^2+6+4\theta^2+\theta^2x^2)e^{-\theta x}}{\theta^2+6} \end{aligned}$$

By definition in Eq.1.14, hazard function based on the equilibrium distribution Eq.3.14 is obtained as;

$$
\begin{aligned}
h_e(x;\theta) &= \frac{\theta(\theta^2+2+\theta x(\theta x+2))e^{-\theta x}}{\theta^2+6}\frac{\theta^2+6}{(\theta^2+6+4\theta^2x^2+4\theta x)e^{-\theta x}} \\
&= \frac{\theta(\theta^2+2+\theta^2x^2+2\theta x)}{\theta^2+6+\theta^2x^2+4\theta x}
\end{aligned}
$$

□

3.3 Two parameter Akash distribution

3.3.1 construction of two parameter Akash distribution

Proposition 3.3.1. *Let $\omega = \frac{\alpha\theta^2}{\alpha\theta^2+2}$ be the mixing probability, a generalized two parameter Akash distribution (G2PAD) is a two component finite mixed distribution of Gamma (1,θ) and Gamma (3,θ). The pdf and Cdf of G2PAD are stated as;*

$$
f(x;\alpha,\theta) = \frac{\theta^3}{\alpha\theta^2+2}(\alpha+x^2)e^{-\theta x}; x>0, \alpha>0, \theta>0 \quad (3.15)
$$

$$
F(x;\alpha,\theta) = 1-\left[1+\frac{\theta^2x^2+2\theta x}{\alpha\theta^2+2}\right]; x>0, \alpha>0, \theta>0 \quad (3.16)
$$

Proof. Using the concept of finite mixture defined in Eq.1.1, the pdf of G2PAD is obtained as;

$$
\begin{aligned}
f(x;\alpha,\theta) &= \frac{\alpha\theta^2}{\alpha\theta^2+2}\left[\theta e^{-\theta x}\right] + \frac{2}{\alpha\theta^2+2}\left[\frac{\theta^3e^{-\theta x}x^2}{\Gamma 3}\right] \\
&= \frac{\alpha\theta^3e^{-\theta x}}{\alpha\theta^2+2} + \frac{\theta^3e^{-\theta x}x^2}{\alpha\theta^2+2} \\
&= \frac{\theta^3}{\alpha\theta^2+2}\left[\alpha+x^2\right]e^{-\theta x}; x>0, \alpha>0, \theta>0
\end{aligned}
$$

Remark 3.3.2. *Setting $\alpha = 1$ a generalized two parameter Akash distribution Eq.3.15 reduces to a one parameter Akash distribution Eq.3.1.*

Further Cdf Eq.3.16 is obtained as;

$$
\begin{aligned}
F(x;\alpha,\theta) &= \int_0^\infty f(x;\alpha,\theta)dx \\
&= \frac{\theta^3}{\alpha\theta^2+2}\int_0^\infty (\alpha+x^2)e^{-\theta x}dx \\
&= \frac{\theta^3}{\alpha\theta^2+2}I_1 \\
I_1 &= \int_0^\infty (\alpha+x^2)e^{-\theta x}dx \\
u &= (\alpha+x^2) \Longrightarrow du = 2xdx \\
dv &= e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta} \\
I_1 &= -(\alpha+x^2)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}\int_0^\infty -xe^{-\theta x}dx \\
I_1 &= -(\alpha+x^2)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}I_2 \\
I_2 &= \int_0^\infty -xe^{-\theta x}dx \\
u &= -x \Longrightarrow du = -dx \\
dv &= e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta} \\
I_2 &= \frac{xe^{-\theta x}}{\theta} - \frac{1}{\theta}\int_0^\infty e^{-\theta x}dx
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \frac{xe^{-\theta x}}{\theta} - \frac{1}{\theta} I_3 \\
I_3 &= \int_0^{\infty} e^{-\theta x} dx \\
u &= -\theta x \Longrightarrow du = -\theta dx \Longrightarrow \frac{du}{-\theta} = dx \\
I_3 &= \int_0^{\infty} e^u \frac{du}{-\theta} = \frac{1}{-\theta} e^u \Longrightarrow \frac{-e^{-\theta x}}{\theta}
\end{aligned}
$$

From I_1, I_2 and I_3 we have the following;

$$
\begin{aligned}
I_1 &= -(\alpha + x^2)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}\left[\frac{xe^{-\theta x}}{\theta} - \frac{1}{\theta}\left(\frac{e^{-\theta x}}{-\theta}\right)\right] \\
I_1 &= -(\alpha\theta^2 + x^2\theta^2 + 2\theta x + 2)\frac{e^{-\theta x}}{\theta^3} \\
&= 1 + -(\alpha\theta^2 + x^2\theta^2 + 2\theta x + 2)\frac{e^{-\theta x}}{\theta^3}\frac{\theta^3}{\alpha\theta^2 + 2} \\
&= 1 - \left[1 + \frac{x^2\theta^2 + 2x\theta}{\alpha\theta^2 + 2}\right] e^{-\theta x}; x > 0, \alpha > 0, \theta > 0
\end{aligned}
$$

□

3.3.2 Reliability Analysis

Proposition 3.3.3. *In this section, we state survival function $S(x;\alpha,\theta)$ and hazard function $h(x;\alpha,\theta)$ of a generalized two parameter Akash distribution Eq.3.15 as:*

$$
S(x;\alpha,\theta) = \left[\frac{\alpha\theta^2 + 2 + x^2\theta^2 + 2x\theta}{\alpha\theta^2 + 2}\right] e^{-\theta x}; x > 0, \alpha > 0, \theta > 0 \tag{3.17}
$$

$$
h(x;\alpha,\theta) = \frac{\theta^3(\alpha + x^2)}{\alpha\theta^2 + 2 + x^2\theta^2 + 2x\theta}; x > 0, \alpha > 0, \theta > 0 \tag{3.18}
$$

Proof . By definition Eq.1.8, survival function $S(x;\alpha,\theta)$ Eq.3.17 is obtained as;

$$S(x;\alpha,\theta) = 1 - \left[1 - \left[1 + \frac{x^2\theta^2 + 2x\theta}{\alpha\theta^2 + 2}\right]e^{-\theta x}\right]$$
$$= \left[1 + \frac{x^2\theta^2 + 2x\theta}{\alpha\theta^2 + 2}\right]e^{-\theta x}, x > 0, \alpha > 0, \theta > 0$$

Using the relation Eq.1.9, hazard function $h(x;\alpha\theta)$ in Eq.3.18 is obtained as;

$$h(x;\alpha,\theta) = \frac{\theta^3}{\alpha\theta^2 + 2}(\alpha + x^2)e^{-\theta x} * \left[\frac{\alpha\theta^2 + 2}{(\alpha\theta^2 + 2 + x^2\theta^2 + 2x\theta)e^{-\theta x}}\right]$$
$$= \frac{\theta^3(\alpha + x^2)}{\alpha\theta^2 + 2 + x^2\theta^2 + 2x\theta}$$

□

3.3.3 Moments and related measures

Proposition 3.3.4. *The r^{th} moments of a generalized two parameter Akash distribution are obtained using both method of moments and moment generating function as shown below:*

$$\mu_r^1 * = \frac{r!\left(\alpha\theta^2 + (r+1)(r+2)\right)}{\theta^r(\alpha\theta^2 + 2)}; r = 1, 2, 3, \cdots \qquad (3.19)$$

Proof. By definition of moments Eq.1.16, the moments of G2PAD are obtained as;

$$
\begin{aligned}
E(X^r) &= \frac{\theta^3}{\alpha\theta^2+2}\int_0^\infty x^r(\alpha+x^2)e^{-\theta x}dx \\
&= \frac{\theta^3}{\alpha\theta^2+2}\left[\alpha\int_0^\infty x^r e^{-\theta x}dx+\int_0^\infty x^{r+2}e^{-\theta x}dx\right] \\
&= \frac{\theta^3}{\alpha\theta^2+2}\left[\frac{\alpha\Gamma(r+1)}{\theta^{r+1}}+\frac{\Gamma(r+3)}{\theta^{r+3}}\right] \\
&= \frac{r!\big(\alpha\theta^2+(r+1)(r+2)\big)}{\theta^r(\alpha\theta^2+2)};r=1,2,3,\cdots
\end{aligned}
$$

Using the relation Eq.1.17, mgf of G2PAD is obtained as;

$$
\begin{aligned}
m_x(t) &= \frac{\theta^3}{\alpha\theta^2+2}\int_0^\infty (\alpha+x^2)e^{-\theta x}etxdx \\
&= \frac{\theta^3}{\alpha\theta^2+2}\int_0^\infty e^{-(t-\theta)x}(\alpha+x^2)dx \\
&= \frac{\theta^3}{\alpha\theta^2+2}\left[\frac{\alpha}{(t-\theta)}+\frac{2}{(t-\theta)^3}\right] \\
&= \frac{\theta^3}{\alpha\theta^2+2}\left[\frac{\alpha}{\theta}\sum_{k=0}^{\infty}\Big(\frac{t}{\theta}\Big)^k+\frac{2}{\theta^3}\sum_{k=0}^{\infty}\binom{k+2}{k}\Big(\frac{t}{\theta}\Big)^k\right] \\
&= \sum_{k=0}^{\infty}\frac{\alpha\theta^2+(k+1)(k+2)}{\alpha\theta^2+2}\Big(\frac{t}{\theta}\Big)^k
\end{aligned}
$$

□

The r^{th} moments about the origin of a generalized two parameter Akash distribution are obtained by putting $\frac{t^r}{r!}$ in $m_x(t)$ as shown below;

$$\mu_r^1 = \frac{r!\big(\alpha\theta^2 + (r+1)(r+2)\big)}{\theta^r(\alpha\theta^2+2)}; r = 1,2,3,\cdots$$

The first four moments about the origin of a generalized two parameter Akash distribution Eq.3.15 are obtained by putting $r = 1,2,3 \quad and \quad$ 4 in Eq.3.19 as;

$$\mu_1^1 = \frac{\alpha\theta^2+6}{\theta(\alpha\theta^2+2)}, \quad \mu_2^1 = \frac{2(\alpha\theta^2+12)}{\theta^2(\alpha\theta^2+2)}$$
$$\mu_3^1 = \frac{6(\alpha\theta^2+20)}{\theta^3(\alpha\theta^2+2)}, \quad \mu_4^1 = \frac{24(\alpha\theta^2+30)}{\theta^4(\alpha\theta^2+2)}$$

We now derive expression for the first four moments about the mean of a generalized two parameter Akash distribution as:

$$\mu_1 = \mu_1^1$$
$$\mu_2 = \mu_2^1 - (\mu_1^1)^2$$
$$= \frac{2(\alpha\theta^2+12)}{\theta^2(\alpha\theta^2+2)} - \left(\frac{\alpha\theta^2+6}{\theta(\alpha\theta^2+2)}\right)^2$$
$$\mu_2 = \frac{\alpha^2\theta^4+16\alpha\theta^2+12}{\theta^2(\alpha\theta^2+2)^2}$$
$$\mu_3 = \mu_3^1 - 3\mu_2^1\mu_1^1 + 2(\mu_1^1)^3$$
$$= \frac{6(\alpha\theta^2+20)}{\theta^3(\alpha\theta^2+2)} - 3\left(\frac{2(\alpha\theta^2+12)}{\theta^2(\alpha\theta^2+2)}\frac{\alpha\theta^2+6}{\theta(\alpha\theta^2+2)}\right) + 2\left(\frac{\alpha\theta^2+6}{\theta(\alpha\theta^2+2)}\right)^3$$
$$\mu_3 = \frac{2\alpha^3\theta^6+6\alpha^2\theta^4+108\alpha\theta^2+48}{\theta^3(\alpha\theta^2+2)^3}$$
$$\mu_4 = \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2$$
$$= \frac{24(\alpha\theta^2+30)}{\theta^4(\alpha\theta^2+2)} - 4\left[\frac{6(\alpha\theta^2+20)}{\theta^3(\alpha\theta^2+2)}\frac{\alpha\theta^2+6}{\theta(\alpha\theta^2+2)}\right] + 6\left[\frac{\alpha\theta^2+6}{\theta(\alpha\theta^2+2)}\frac{2(\alpha\theta^2+1}{\theta^2(\alpha\theta^2+}\right.$$
$$= \frac{9\alpha^4\theta^8+384\alpha^3\theta^6+1224\alpha^2\theta^4+1728\alpha\theta^2+720}{\theta^4(\alpha\theta^2+2)^4}$$

Proposition 3.3.5. *Other related measures of a generalized two parameter AKash distribution such as coefficient of variation (C.v), coefficient of skewness (υ_1), coefficient of kurtosis (υ_2) and* *dispersion index (υ_3) are stated as;*

$$C.v = \frac{\sqrt{\alpha^2\theta^4 + 16\alpha\theta^2 + 12}}{\alpha\theta^2 + 6} \tag{3.20}$$

$$\upsilon_1 = \frac{2\alpha^3\theta^6 + 60\alpha^2\theta^4 + 72\alpha\theta^2 + 48}{(\alpha^2\theta^4 + 16\alpha\theta^2 + 12)^{\frac{3}{2}}} \tag{3.21}$$

$$\upsilon_2 = \frac{9\alpha^4\theta^8 + 384\alpha^2\theta^6 + 1224\alpha^2\theta^4 + 1728\alpha\theta^2 + 720}{(\alpha^2\theta^4 + 16\alpha\theta^2 + 12)^2} \tag{3.22}$$

$$\upsilon_3 = \frac{\alpha^2\theta^4 + 16\alpha\theta^2 + 12}{\alpha^2\theta^5 + 8\alpha\theta^3 + 12\theta} \tag{3.23}$$

Coefficient of variation in Eq.3.20 is obtained as;

Proof.

$$\begin{aligned} C.v &= \frac{\sigma}{\mu_1^1} \\ &= \frac{\sqrt{\alpha^2\theta^4 + 16\alpha\theta^2 + 12}}{\theta(\alpha\theta^2 + 2)} \frac{\theta(\alpha\theta^2 + 2)}{\alpha\theta^2 + 6} \Longrightarrow \frac{\sqrt{\alpha^2\theta^4 + 16\alpha\theta^2 + 12}}{\alpha\theta^2 + 6} \end{aligned}$$

Skewness coefficient in 3.21 is obtained as;

$$\begin{aligned} \upsilon_1 &= \frac{\mu_3}{(\mu_2)^2} \\ &= \frac{2\alpha^3\theta^6 + 6\alpha^2\theta^4 + 108\alpha\theta^2 + 48}{\theta^3(\alpha\theta^2 + 2)^3} \left[\frac{\theta^2(\alpha\theta^2 + 2)^2}{\alpha^2\theta^4 + 16\alpha\theta^2 + 12}\right]^{\frac{3}{2}} \\ &= \frac{2\alpha^3\theta^6 + 60\alpha^2\theta^4 + 72\alpha\theta^2 + 48}{(\alpha^2\theta^4 + 16\alpha\theta^2 + 12)^{\frac{3}{2}}} \end{aligned}$$

Further kurtosis coefficient Eq.3.22 is obtained as;

$$\begin{aligned}
\upsilon_2 &= \frac{\mu_4}{(\mu_2)^2} \\
&= \frac{9\alpha^4\theta^8 + 384\alpha^3\theta^6 + 1224\alpha^2\theta^4 + 1728\alpha\theta^2 + 720}{\theta^4(\alpha\theta^2+2)^4}\left[\frac{\theta^2(\alpha\theta^2+2)^2}{\alpha^2\theta^4+16\alpha\theta^2+12}\right] \\
&= \frac{9\alpha^4\theta^8 + 384\alpha^2\theta^6 + 1224\alpha^2\theta^4 + 1728\alpha\theta^2 + 720}{(\alpha^2\theta^4+16\alpha\theta^2+12)^2}
\end{aligned}$$

Index of dispersion 3.23 is obtained as;

$$\begin{aligned}
\upsilon_3 &= \frac{\sigma^2}{\mu_1^1} \\
&= \frac{\alpha^2\theta^4+16\alpha\theta^2+12}{\theta^2(\alpha\theta^2+2)^2}\frac{\theta(\alpha\theta^2+2)}{\alpha\theta^2+6} \Longrightarrow \frac{\alpha^2\theta^4+16\alpha\theta^2+12}{\alpha^2\theta^5+8\alpha\theta^3+12\theta}
\end{aligned}$$

□

3.3.4 Excess Loss distribution

Proposition 3.3.6. *The probability density function of excess loss function $f_l(x;\alpha,\theta)$, Mean residual lifetime (MRL), equilibrium distribution $f_e(x;\alpha,\theta)$, survival function of equilibrium distribution $S_e(x;\alpha,\theta)$ and hazard function of equilibrium distribution $h_e(x;\alpha,\theta)$ of a generalized two parameter Akash dis-*

tribution Eq.3.15 are stated as;

$$f_l(x;\alpha,\theta) = \frac{\theta^3(\alpha+x^2)e^{-(x-z)\theta}}{\alpha\theta^2+2+z^2\theta^2+2z\theta}; x > z \quad (3.24)$$

$$m(x) = \frac{6+4\theta x+\alpha\theta^2+x^2\theta^2}{\theta(\alpha\theta^2+2+x^2\theta^2+2\theta x)} \quad (3.25)$$

$$f_e(x;\alpha,\theta) = \frac{\theta(\alpha\theta^2+2+x^2\theta^2+2\theta x)e^{-\theta x}}{\alpha\theta^2+6} \quad (3.26)$$

$$S_e(x;\alpha,\theta) = \frac{(6+4\theta x+\alpha\theta^2+\theta^2x^2)e^{-\theta x}}{\alpha\theta^2+6} \quad (3.27)$$

$$h_e(x;\alpha,\theta) = \frac{\theta(\alpha\theta^2+2+\theta^2x^2+2\theta x)}{6+4\theta x+\alpha\theta^2+\theta^2x^2} \quad (3.28)$$

Proof. First, pdf of excess loss distribution Eq.3.24 is obtained using the relation Eq.1.10 as;

$$\begin{aligned} f_l(x;\alpha,\theta) &= \frac{\theta^3}{\alpha\theta^2+2}(\alpha+x^2)e^{-\theta x}\frac{\alpha\theta^2+2}{(\alpha\theta^2+2+\theta^2z^2+2\theta z)e^{-\theta z}} \\ &= \frac{\theta^3(\alpha+x^2)e^{-(x-z)\theta}}{\alpha\theta^2+2+z^2\theta^2+2z\theta}; \quad x > z \end{aligned}$$

secondly, mean excess loss Eq.3.25 is obtained using relation in Eq.1.11 as;

$$\begin{aligned} m(x) &= \frac{\alpha\theta^2+2}{(\alpha\theta^2+2+\theta^2x^2+2\theta x)e^{-\theta x}}\int_x^\infty \frac{\alpha\theta^2+2+\theta^2t^2+2\theta t}{\alpha\theta^2+2}e^{-\theta t}dt \\ &= \frac{1}{(\alpha\theta^2+2+\theta^2x^2+2\theta x)e^{-\theta x}}\int_x^\infty (\alpha\theta^2+2+\theta^2t^2+2\theta t)e^{-\theta t}dt \end{aligned}$$

Taking the part $\int_x^\infty(\alpha\theta^2+2+\theta^2t^2+2\theta t)e^{-\theta t}dt$ and using integration by parts technique we have the following;

$$u=(\alpha\theta^2+2+\theta^2t^2+2\theta t)\Longrightarrow du=2\theta^2t+2\theta$$

$$dv=e^{-\theta t}\Longrightarrow v=\frac{-e^{-\theta t}}{\theta}$$

$$I_1=-(\alpha\theta^2+2+\theta^2t^2+2\theta t)\frac{e^{-\theta t}}{\theta}+\frac{2}{\theta}*I_2$$

$$I_2=\int_x^\infty(\theta^2t+\theta)e^{-\theta t}dt$$

$$u=\theta^2t+\theta\Longrightarrow du=\theta^2dt$$

$$dv=e^{-\theta t}\Longrightarrow v=\frac{-e^{-\theta t}}{\theta}$$

$$I_2=-(\theta^2t+\theta)\frac{e^{-\theta t}}{\theta}+\theta\int_x^\infty e^{-\theta t}dt$$

$$I_2=-(\theta^2t+\theta)\frac{e^{-\theta t}}{\theta}+\theta*I_3$$

$$I_3=\int_x^\infty e^{-\theta t}dt=\theta\left(\frac{e^{-\theta t}}{\theta}\right)\Longrightarrow -e^{-\theta t}$$

From I_1,I_2 I_3 we have the following;

$$
\begin{aligned}
I_1 &= -(\alpha\theta^2+2+\theta^2t^2+2\theta t)\frac{e^{-\theta t}}{\theta}+\frac{2}{\theta}\left[-(\theta^2t+\theta)\frac{e^{-\theta t}}{\theta}-e^{-\theta t}\right]\\
&= \left[\frac{\big(-\theta(\alpha\theta^2+2+\theta^2t^2+2\theta t)-2(\theta^2t+\theta)-2\theta\big)e^{-\theta t}}{\theta^2}\right]_x^\infty\\
&= \frac{\big(6+4\theta x+\alpha\theta^2+\theta^2x^2\big)e^{-\theta x}}{\theta}\\
&= \frac{1}{(\alpha\theta^2+2+\theta^2x^2+2\theta x)e^{-\theta x}} * \frac{(6+4\theta x+\alpha\theta^2+\theta^2x^2)e^{-\theta x}}{\theta}\\
&= \frac{6+4\theta x+\alpha\theta^2+\theta^2x^2}{\theta(\alpha\theta^2+2+\theta^2x^2+2\theta x)}
\end{aligned}
$$

Using relation Eq.1.12, equilibrium distribution $f_e(x;\alpha,\theta)$ in Eq.3.26 is obtained as;

$$
\begin{aligned}
f_e(x;\alpha,\theta) &= \left[\frac{\alpha\theta^2+2+\theta^2x^2+2\theta x}{\alpha\theta^2+2}\right]e^{-\theta x}\frac{\theta(\alpha\theta^2+2)}{\alpha\theta^2+6}\\
&= \frac{\theta(\alpha\theta^2+2+x^2\theta^2+2\theta x)e^{-\theta x}}{\alpha\theta^2+6}
\end{aligned}
$$

By definition Eq.1.13, survival function based on equilibrium distribution Eq.3.27 is obtained as;

$$
\begin{aligned}
\int_x^\infty S(t;\alpha,\theta)dt &= \frac{(6+4\theta x+\alpha\theta^2+\theta^2x^2)e^{-\theta x}}{\theta(\alpha\theta^2+2)}\\
S_e(x;\alpha,\theta) &= \frac{(6+4\theta x+\alpha\theta^2+\theta^2x^2)e^{-\theta x}}{\theta(\alpha\theta^2+2)}\frac{\theta(\alpha\theta^2+2)}{\alpha\theta^2+6}\\
&= \frac{(6+4\theta x+\alpha\theta^2+\theta^2x^2)e^{-\theta x}}{\alpha\theta^2+6}
\end{aligned}
$$

Using relation Eq.1.14, $h_e(x;\alpha,\theta)$ in Eq.3.28 is obtained as;

$$h_e(x;\alpha,\theta) = \frac{(6+4\theta x+\alpha\theta^2+\theta^2x^2)e^{-\theta x}}{\alpha\theta^2+6}\frac{\alpha\theta^2+6}{(6+4\theta x+\alpha\theta^2+\theta^2x^2)e^{-\theta x}}$$
$$= \frac{\theta(\alpha\theta^2+2+\theta^2x^2+2\theta x)}{6+4\theta x+\alpha\theta^2+\theta^2x^2}$$

□

3.4 Three parameter Akash distribution

3.4.1 Construction of three parameter Akash distribution

Proposition 3.4.1. *Let* $\omega = \frac{\alpha\theta^2}{\alpha\theta^2+2\beta}$ *be mixing weight a generalized three parameter Akash distribution is a two component finite mixed distribution of Gamma*$(1,\theta)$ *and Gamma*$(3,\theta)$*. The pdf and Cdf of G3PAD are stated as;*

$$f(x;\alpha,\beta,\theta) = \frac{\theta^3}{\alpha\theta^2+2\beta}\left[\alpha+\beta x\right]e^{-\theta x}; x>0, \alpha>0, \beta>0, \theta>0 \tag{3.29}$$

$$F(x;\alpha,\beta,\theta) = 1-\left[1+\frac{\beta\theta^2x^2+2\beta\theta x}{\alpha\theta^2+2\beta}\right]e^{-\theta x}; x>0, \alpha>0, \beta>0, \theta>0 \tag{3.30}$$

Proof. Using the concept of finite mixture defined in Eq.1.1, the pdf Eq.3.29 of G2PAD is obtained as;

$$f(x;\alpha,\beta,\theta) = \frac{\alpha\theta^2}{\alpha\theta^2+2\beta}\left[\theta e^{-\theta x}\right]+\frac{2\beta}{\alpha\theta^2+2\beta}\left[\frac{\theta^3}{\Gamma 3}e^{-\theta x}x^2\right]$$
$$= \frac{\alpha\theta^2e^{-\theta x}}{\alpha\theta^2+2\beta}+\frac{2\beta\theta^3e^{-\theta x}x^2}{\Gamma 3(\alpha\theta^2+2\beta)}$$
$$= \frac{\theta^3}{\alpha\theta^2+2\beta}\left[\alpha+\beta x\right]e^{-\theta x}; x>0, \alpha>0, \beta>0, \theta>0$$

Remark 3.4.2. *Putting* $\beta = 0, \quad \alpha = 1$ *a generalized three parameter Akash distribution Eq.3.29 reduces to exponential distribution 2.17.*
Similarly, putting $\beta = 1$ *a generalized three parameter Akash distribution Eq.3.29 reduces to a generalized two parameter Akash distribution Eq.3.15.*
Putting $\alpha = \beta = 1$ *a generalized three parameter Akash distribution Eq.3.29 reduces to one parameter Akash distribution Eq.3.1.*

A generalized Akash distribution Eq.3.29 is modification of a generalized two parameter Akash Eq.3.15 introduced by Shanker et al.(2018).

further Cdf Eq.3.30 is obtained as;

$$
\begin{aligned}
F(x;\alpha,\beta,\theta) &= \frac{\theta^3}{\alpha\theta^2+2\beta}\int_0^\infty(\alpha+\beta x)e^{-\theta x}dx \\
&= \frac{\theta^3}{\alpha\theta^2+2\beta}I_1 \\
I_1 &= \int_0^\infty(\alpha+\beta x)e^{-\theta x}dx \\
u &= (\alpha+\beta x) \Longrightarrow du = 2\beta x \\
dv &= e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta} \\
&= -(\alpha+\beta x)\frac{e^{-\theta x}}{\theta} - \frac{2\beta}{\theta}\int_0^\infty -xe^{-\theta x}dx \\
&= -(\alpha+\beta x)\frac{e^{-\theta x}}{\theta} - \frac{2\beta}{\theta} * I_2 \\
I_2 &= \int_0^\infty -xe^{-\theta x}dx \\
u &= -x \Longrightarrow du = -dx \\
dv &= e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta} \\
I_2 &= \frac{xe^{-\theta x}}{\theta} - \frac{1}{\theta}\int_0^\infty e^{-\theta x}dx \\
I_2 &= \frac{xe^{-\theta x}}{\theta} - \frac{1}{\theta}I_3 \\
I_3 &= \int_0^\infty e^{-\theta x}dx \\
u &= -\theta x \Longrightarrow du = -\theta dx \Longrightarrow \frac{du}{-\theta} = dx \\
&= \frac{-1}{\theta}\int_0^\infty e^u du \Longrightarrow \frac{-1}{\theta}e^u \Longrightarrow \frac{-1}{\theta}e^{-\theta x}
\end{aligned}
$$

From I_1,I_2, I_3 we have the following;

$$= -(\alpha+\beta x^2)\frac{e^{-\theta x}}{\theta} - \frac{2\beta}{\theta}\left[\frac{xe^{-\theta}}{\theta} - \frac{1}{\theta}(\frac{-e^{-\theta x}}{\theta})\right]$$

$$= -\left[\alpha\theta^2+2\beta+\beta\theta^2x^2+2\beta\theta x\right]\frac{e^{-\theta x}}{\theta^3}$$

$$= 1+-\left[\alpha\theta^2+2\beta+\beta\theta^2x^2+2\beta\theta x\right]\frac{e^{-\theta x}}{\theta^3}\frac{\theta^3}{\alpha\theta^2+2\beta}$$

$$F(x;\alpha,\beta,\theta) = 1-\left[1+\frac{\beta\theta^2x^2+2\beta\theta x}{\alpha\theta^2+2\beta}\right]e^{-\theta x}; x>0, \alpha>0, \beta>0, \theta>0$$

□

3.4.2 Reliability Analysis

Proposition 3.4.3. *In this section, survival function denoted by $S(x;\alpha,\beta,\theta)$ and hazard function denoted by $h(x;\alpha,\beta,\theta)$ of a generalized three parameter Akash distribution Eq.3.29 are stated as;*

$$S(x;\alpha,\beta,\theta) = \left[\frac{\alpha\theta^2+2\beta+\beta\theta^2x^2+2\beta\theta x}{\alpha\theta^2+2\beta}\right]e^{-\theta x}; x>0, \alpha>0, \beta>0, \theta \tag{3.31}$$

$$h(x;\alpha,\beta,\theta) = \frac{\theta^3(\alpha+\beta x^2)}{\alpha\theta^2+2\beta+\beta\theta^2x^2+2\beta\theta x}; x>0, \alpha>0, \beta>0, \theta>0 \tag{3.32}$$

Proof. First by definition in Eq.1.8, survival function $S(x;\alpha,\beta,\theta)$ in Eq.3.31 is obtained as;

$$S(x;\alpha,\beta,\theta) = 1-\left[1-\left[1+\frac{\beta\theta^2x^2+2\beta\theta x}{\alpha\theta^2+2\beta}\right]e^{-\theta x}\right]$$

$$= \left[\frac{\alpha\theta^2+2\beta+\beta\theta^2x^2+2\beta\theta x}{\alpha\theta^2+2\beta}\right]e^{-\theta x}; x>0, \alpha>0, \beta>0, \theta$$

further hazard function $h(x;\alpha,\beta,\theta)$ in Eq.3.32 is obtained using the relation defined in Eq.3.32 as;

$$h(x;\alpha,\beta,\theta)=\frac{\frac{\theta^3}{\alpha\theta^2+2\beta}\left[\alpha+\beta x\right]e^{-\theta x}}{\left[\frac{\alpha\theta^2+2\beta+\beta\theta^2x^2+2\beta\theta x}{\alpha\theta^2+2\beta}\right]e^{-\theta x}}$$
$$=\frac{\theta^3(\alpha+\beta x^2)}{\alpha\theta^2+2\beta+\beta\theta^2x^2+2\beta\theta x};x>0,\alpha>0,\beta>0,\theta>0$$

□

3.4.3 Moments and related measures

Proposition 3.4.4. *The r^{th} moments of a generalized three parameter Akash distribution Eq.3.29 are derived using both method of moments and moment generating function (mgf) as shown below;*

$$\mu_r^1*=\frac{r!\big(\alpha\theta^2+\beta(r+1)(r+2)\big)}{\theta^r(\alpha\theta^2+2\beta)};\quad r=1,2,3,\cdots \tag{3.33}$$

Proof. By definition in Eq.1.16, the moments of G3PAD are obtained as;

$$
\begin{aligned}
E(X^r) &= \frac{\theta^3}{\alpha\theta^2+2\beta}\int_0^\infty x^r(\alpha+\beta x^2)e^{-\theta x}dx \\
&= \frac{\theta^3}{\alpha\theta^2+2\beta}\left[\alpha\int_0^\infty x^r e^{-\theta x}dx+\beta\int_0^\infty x^{r+2}e^{-\theta x}dx\right] \\
&= \frac{\theta^3}{\alpha\theta^2+2\beta}\left[\frac{\alpha\Gamma(r+1)}{\theta^{r+1}}+\frac{\beta\Gamma(r+3)}{\theta^{r+3}}\right] \\
&= \frac{\theta^3}{\alpha\theta^2+2\beta}\left[\frac{\alpha r\Gamma r}{\theta^{r+1}}+\frac{\beta(r+2)(r+1)r\Gamma r}{\theta^{r+3}}\right] \\
&= \frac{\theta^3 r\Gamma r}{\alpha\theta^2+2\beta}\left[\frac{\alpha}{\theta}+\frac{\beta(r+2)(r+1)}{\theta^3}\right] \\
&= \frac{r!\big(\alpha\theta^2+\beta(r+1)(r+2)\big)}{\theta^r(\alpha\theta^2+2\beta)};\quad r=1,2,3,\cdots
\end{aligned}
$$

Similarly, by definition in Eq.1.17 mgf of G3PAD is obtained as;

$$
\begin{aligned}
m_x(t)&\frac{\theta^3}{\alpha\theta^2+2\beta}\int_0^\infty e^{tx}(\alpha+\beta x^2)e^{-\theta x}dx \\
&= \frac{\theta^3}{\alpha\theta^2+2\beta}\int_0^\infty e^{-(\theta-t)x}(\alpha+\beta x)dx \\
&= \frac{\theta^3}{\alpha\theta^2+2\beta}\left[\frac{\alpha}{(\theta-t)}+\frac{\beta}{(\theta-t)^3}\right] \\
&= \frac{\theta^3}{\alpha\theta^2+2\beta}\left[\frac{\alpha}{\theta}\sum_{k=0}^{\infty}\left(\frac{t}{\theta}\right)^k+\frac{\beta}{\theta^3}\sum_{k=0}^{\infty}\binom{k+2}{k}\right] \\
&= \sum_{k=0}^{\infty}\frac{\alpha\theta^2+\beta(k+1)(k+2)}{\alpha\theta^2+2\beta}\left(\frac{t}{\theta}\right)^k
\end{aligned}
$$

The moments about the origin of a generalized three parameter Akash distribution Eq.3.29 are obtained as a coefficient $\frac{t^r}{r!}$

in $m_x(t)$ as;

$$\mu_r^1 = \frac{r!\big(\alpha\theta^2 + \beta(r+1)(r+2)\big)}{\theta^r(\alpha\theta^2 + 2\beta)}; \quad r = 1, 2, 3, \cdots$$

□

The four moments of a generalized three parameter Akash distribution Eq.3.29 about the origin are obtained by putting $r = 1, 2, 3$ and 4 in Eq.3.33 as shown below;

$$\mu_1^1 = \frac{\alpha\theta^2 + 6\beta}{\theta(\alpha\theta^2 + 2\beta)}, \quad \mu_2^1 = \frac{2(\alpha\theta^2 + 12\beta)}{\theta^2(\alpha\theta^2 + 2\beta)}$$
$$\mu_3^1 = \frac{6(\alpha\theta^2 + 20\beta)}{\theta^3(\alpha\theta^2 + 2\beta)}, \quad \mu_4^1 = \frac{24(\alpha\theta^2 + 30\beta)}{\theta^4(\alpha\theta^2 + 2\beta)}$$

We now derive centralized moments of a generalized Akash distribution as;

$$\mu_1 = \mu_1^1$$
$$\mu_2 = \mu_2^1 - (\mu_1^1)^2$$
$$= \frac{2(\alpha\theta^2 + 12\beta)}{\theta^2(\alpha\theta^2 + 2\beta)} - \left[\frac{\alpha\theta^2 + 6\beta}{\theta^2(\alpha\theta^2 + 2\beta)}\right]^2$$
$$\mu_2 = \frac{\alpha^2\theta^4 + 16\alpha\beta\theta^2 + 12\beta^2}{\theta^2(\alpha\theta^2 + 2\beta)^2}$$
$$\mu_3 = \mu_3^1 - 3\mu_2^1\mu_1^1 + 2(\mu_1^1)^3$$
$$= \frac{6(\alpha\theta^2 + 20\beta)}{\theta^3(\alpha\theta^3 + 2\beta)} - 3\left[\frac{2(\alpha\theta^2 + 12\beta)}{\theta^2(\alpha\theta^2 + 2\beta)}\frac{(\alpha\theta^2 + 6\beta)}{\theta(\alpha\theta^2 + 2\beta)}\right] - 2\left[\frac{\alpha\theta^2 + 6\beta}{\theta(\alpha\theta^2 + 2\beta}\right.$$
$$\mu_3 = \frac{2\alpha^3\theta^6 + 132\alpha^2\beta\theta^4 + 28\alpha^2\beta\theta^5 + 72\alpha\beta^2\theta^3 + 48\beta^3\theta - 144\alpha\beta^2\theta^2}{\theta^3(\alpha\theta^2 + 2\beta)^3}$$
$$\mu_4 = \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2$$

Proposition 3.4.5. *Other related measures of a generalized three parameter AKash distribution Eq.3.29 such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) are stated as;*

$$C.v = \frac{\sqrt{\alpha^2\theta^4 + 16\alpha\beta\theta^2 + 12\beta^2}}{\alpha\theta^2 + 6\beta} \tag{3.34}$$

$$\upsilon_1 = \frac{2\alpha^3\theta^6 + 132\alpha^2\beta\theta^4 + 28\alpha^2\beta\theta^5 + 72\alpha\beta^2\theta^3 + 48\beta^3\theta - 144\alpha\beta^2\theta^2}{(\alpha^2\theta^4 + 16\alpha\beta\theta^2 + 12\beta^2)^{\frac{3}{2}}} \tag{3.35}$$

$$\upsilon_2 = \frac{\mu_4}{(\mu_2)^2} \tag{3.36}$$

$$\upsilon_3 = \frac{\alpha^2\theta^4 + 16\alpha\beta\theta^2 + 12\beta^2}{\theta(\alpha\theta^2 + 2\beta)(\alpha\theta^2 + 6\beta)} \tag{3.37}$$

Proof. To begin with, coefficient of variation Eq.3.34 is obtained as;

$$
\begin{aligned}
C.v &= \frac{\sigma}{\mu_1^1} \\
&= \frac{\sqrt{\alpha^2\theta^4 + 16\alpha\beta\theta^2 + 12\beta^2}}{\theta(\alpha\theta^2+2\beta)} \frac{\theta(\alpha\theta^2+2\beta)}{\alpha\theta^2+6\beta} \Longrightarrow \frac{\sqrt{\alpha^2\theta^4 + 16\alpha\beta\theta^2 + 12\beta^2}}{\alpha\theta^2+6\beta}
\end{aligned}
$$

Secondly, skewness Eq.3.35 is derived as;

$$
\begin{aligned}
\upsilon_1 &= \frac{\mu_3^1 - 3\mu_2^1\mu_1^1 + 2(\mu_1^1)^3}{(\mu_2)^{\frac{3}{2}}} \\
&= \frac{\frac{2\alpha^3\theta^6+132\alpha^2\beta\theta^4+28\alpha^2\beta\theta^5+72\alpha\beta^2\theta^3+48\beta^3\theta-144\alpha\beta^2\theta^2-384\beta^3-108\alpha^2\beta\theta^2}{\theta^3(\alpha\theta^2+2\beta)^3}}{\left[\frac{\alpha^2\theta^4+16\alpha\beta\theta^2+12\beta^2}{\theta^2(\alpha\theta^2+2\beta)^2}\right]^{\frac{3}{2}}} \\
&= \frac{2\alpha^3\theta^6 + 132\alpha^2\beta\theta^4 + 28\alpha^2\beta\theta^5 + 72\alpha\beta^2\theta^3 + 48\beta^3\theta - 144\alpha\beta^2\theta^2 - 384}{(\alpha^2\theta^4 + 16\alpha\beta\theta^2 + 12\beta^2)^{\frac{3}{2}}}
\end{aligned}
$$

kurtosis coefficient Eq.3.36 is obtained as;

$$
\upsilon_2 = \frac{\mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2}{\left[\mu_2^1 - (\mu_1^1)^2\right]^2}
$$

Lastly, index of dispersion Eq.3.37 is obtained as;

$$
\begin{aligned}
\upsilon_3 &= \frac{\sigma^2}{\mu_1^1} \\
&= \frac{\alpha^2\theta^4 + 16\alpha\beta\theta^2 + 12\beta^2}{\theta^2(\alpha\theta^2+2\beta)^2} \frac{\theta(\alpha\theta^2+2\beta)}{\alpha\theta^2+6\beta} \Longrightarrow \frac{\alpha^2\theta^4 + 16\alpha\beta\theta^2 + 12\beta^2}{\theta(\alpha\theta^2+2\beta)(\alpha\theta^2+6\beta)}
\end{aligned}
$$

□

3.4.4 Excess Loss Distribution

Proposition 3.4.6. *The probability density function of excess loss function $f_l(x;\alpha,\beta,\theta)$, Mean residual lifetime (MRL), equilibrium distribution $f_e(x;\alpha,\beta,\theta)$, survival function of equilibrium distribution $S_e(x;\alpha,\beta,\theta)$ and hazard function of equilibrium distribution $h_e(x;\alpha,\beta,\theta)$ of a generalized three parameter Akash distribution Eq.3.29 are stated as;*

$$f_l(x;\alpha,\beta,\theta) = \frac{\theta^3(\alpha+\beta x)e^{-(x-z)\theta}}{\alpha\theta^2+2\beta+\beta\theta^2z^2+2\beta\theta z}; \quad x > z \tag{3.38}$$

$$m(x)\frac{\alpha\theta^2+\theta^2\beta x^2+6\beta+4\beta\theta x}{\theta(\alpha\theta^2+2\beta+\theta^2\beta x^2+2\beta\theta x)} \tag{3.39}$$

$$f_e(x;\alpha,\beta,\theta) = \frac{\theta(\alpha\theta^2+2\beta+\theta^2\beta x+2\beta\theta x)e^{-\theta x}}{\alpha\theta^2+6\beta} \tag{3.40}$$

$$S_e(x;\alpha,\beta,\theta) = \frac{(\alpha\theta^2+\beta\theta^2x^2+6\beta+4\beta\theta x)e^{-\theta x}}{\alpha\theta^2+6\beta} \tag{3.41}$$

$$h_e(x;\alpha,\beta,\theta) = \frac{\alpha\theta^2+2\beta+\theta^2\beta x^2+2\beta\theta x}{\theta(\alpha\theta^2+\theta^2\beta x^2+6\beta+4\beta\theta x)} \tag{3.42}$$

Proof. To begin with, pdf of excess loss distribution Eq.3.38 is obtained using the relation in Eq.1.10 as;

$$\begin{aligned} f_l(x;\alpha,\beta,\theta) &= \frac{\frac{\theta^3}{\alpha\theta^2+2\beta}\left[\alpha+\beta x\right]e^{-\theta x}}{\left[\frac{\alpha\theta^2+2\beta+\beta\theta^2z^2+2\beta\theta z}{\alpha\theta^2+2\beta}\right]e^{-\theta z}} \\ &= \frac{\theta^3(\alpha+\beta x)e^{-(x-z)\theta}}{\alpha\theta^2+2\beta+\beta\theta^2z^2+2\beta\theta z}; \quad x > z \end{aligned}$$

By definition of mean excess loss in Eq.1.11, $m(x)$ in Eq.3.39 is derived as;

$$
\begin{aligned}
m(x) &= \frac{\alpha\theta^2+2\beta}{(\alpha\theta^2+2\beta+\theta^2\beta x+2\beta\theta x)e^{-\theta x}}\int_x^\infty \Big[\frac{\alpha\theta^2+2\beta+\theta^2\beta t^2+2\beta\theta t}{\alpha\theta^2+2\beta}\Big]e^{-\theta t}d \\
&= \frac{1}{(\alpha\theta^2+2\beta+\theta^2\beta x+2\beta\theta x)e^{-\theta x}}\int_x^\infty (\alpha\theta^2+2\beta+\theta^2\beta t^2+2\beta\theta t)e^{-\theta t}d
\end{aligned}
$$

Taking the part $\int_x^\infty(\alpha\theta^2+2\beta+\theta^2\beta t^2+2\beta\theta t)e^{-\theta t}dt$ we have;

$$u = (\alpha\theta^2+2\beta+\theta^2\beta t^2+2\beta\theta t) \Longrightarrow du = (2\theta^2\beta t+2\beta\theta)dt$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$= -(\alpha\theta^2+2\beta+\theta^2\beta t^2+2\beta\theta t)\frac{e^{-\theta t}}{\theta}+2\beta\int_x^\infty(\theta t+1)e^{-\theta t}dt$$

Taking the part $\int_x^\infty (\theta t+1)e^{-\theta t}dt$ we have the following;

$$u=(\theta t+1) \Longrightarrow du=\theta dt$$

$$dv=e^{-\theta t} \Longrightarrow v=\frac{-e^{-\theta t}}{\theta}$$

$$=-(\theta t+1)\frac{e^{-\theta t}}{\theta}+\int_x^\infty e^{-\theta t}dt$$

$$=2\beta\left[-(\theta t+1)\frac{e^{-\theta t}}{\theta}-\frac{e^{-\theta t}}{\theta}\right]$$

$$=-(\alpha\theta^2+2\beta+\theta^2\beta t^2+2\beta\theta t)\frac{e^{-\theta t}}{\theta}+2\beta\left[-(\theta t+1)\frac{e^{-\theta t}}{\theta}-\frac{e^{-\theta t}}{\theta}\right]$$

$$=-\left[\frac{(\alpha\theta^2+2\beta+\theta^2\beta t+2\beta\theta t+2\beta+2\beta)e^{-\theta t}}{\theta}\right]_x^\infty$$

$$=\left[\frac{(\alpha\theta^2+\theta^2\beta x^2+6\beta+4\beta\theta x)e^{-\theta x}}{\theta}\right]$$

$$=\frac{1}{(\alpha\theta^2+2\beta+\theta^2\beta x+2\beta\theta x)e^{-\theta x}}\left[\frac{(\alpha\theta^2+\theta^2\beta x^2+6\beta+4\beta\theta x)e^{-\theta x}}{\theta}\right.$$

$$\frac{\alpha\theta^2+\theta^2\beta x^2+6\beta+4\beta\theta x}{\theta(\alpha\theta^2+2\beta+\theta^2\beta x^2+2\beta\theta x)}$$

Using the relation in Eq.1.12, $f_e(x;\alpha,\beta,\theta)$ in Eq.3.40 is obtained as;

$$f_e(x;\alpha,\beta,\theta)=\frac{\left[\frac{\alpha\theta^2+2\beta+\beta\theta^2x^2+2\beta\theta x}{\alpha\theta^2+2\beta}\right]e^{-\theta x}}{\frac{\alpha\theta^2+6\beta}{\theta(\alpha\theta^2+2\beta)}}$$

$$=\frac{\theta(\alpha\theta^2+2\beta+\theta^2\beta x+2\beta\theta x)e^{-\theta x}}{\alpha\theta^2+6\beta}$$

By definition in Eq.1.13, survival function based on the equilibrium distribution $S_e(x;\alpha,\beta,\theta)$ in Eq.3.41 is obtained as;

$$\begin{aligned}
\int_x^\infty S(t;\alpha,\beta,\theta)dt &= \frac{(\alpha\theta^2+\theta^2\beta x^2+6\beta+4\beta\theta x)e^{-\theta x}}{\theta(\alpha\theta^2+2\beta)} \\
&= \frac{(\alpha\theta^2+\theta^2\beta x^2+6\beta+4\beta\theta x)e^{-\theta x}}{\theta(\alpha\theta^2+2\beta)}\frac{\theta(\alpha\theta^2+2\beta)}{\alpha\theta^2+6\beta} \\
S_e(x;\alpha,\beta,\theta) &= \frac{(\alpha\theta^2+\beta\theta^2 x^2+6\beta+4\beta\theta x)e^{-\theta x}}{\alpha\theta^2+6\beta}
\end{aligned}$$

Using the relation Eq.1.14, hazard function based on equilibrium distribution Eq.3.42 is obtained as;

$$\begin{aligned}
h_e(x;\alpha,\beta,\theta) &= \frac{\theta(\alpha\theta^2+2\beta+\theta^2\beta x+2\beta\theta x)e^{-\theta x}}{\alpha\theta^2+6\beta}\frac{\alpha\theta^2+6\beta}{(\alpha\theta^2+\theta^2\beta x^2+6\beta+4\beta\theta x} \\
&= \frac{\alpha\theta^2+2\beta+\theta^2\beta x^2+2\beta\theta x}{\theta(\alpha\theta^2+\theta^2\beta x^2+6\beta+4\beta\theta x)}
\end{aligned}$$

□

4 TWO COMPONENT FINITE GAMMA MIXTURE (Case of Shanker distribution)

4.1 Introduction

In this chapter, a two component finite gamma mixed a case of Shanker distribution is considered. we shall construct and derive properties of Shanker distribution and its generalizations. The mixed distribution is expressed in terms ofpdf and Cdf. We later derive statistical properties such as reliability analysis measures, equilibrium properties, moments about both the origin and mean.

4.2 One parameter Shanker distribution

4.2.1 Construction of a one parameter Shanker distribution

Proposition 4.2.1. *Let* $\omega = \frac{\theta^2}{\theta^2+1}$ *be mixing probability, one parameter Shanker distribution is a finite mixed distribution of Gamma*$(1,\theta)$ *and Gamma*$(2,\theta)$ *. The pdf and Cdf of one parameter Shanker distribution are stated as;*

$$f(x;\theta) = \frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}; x>0, \theta>0 \tag{4.1}$$

$$F(x;\theta) = 1-\left[1+\frac{\theta x}{\theta^2+1}\right]e^{-\theta x}; x>0, \theta>0 \tag{4.2}$$

Proof. By the use of finite mixed concept in Eq.1.1, the pdf Eq.4.1of one parameter Shanker distribution is obtained as;

$$
\begin{aligned}
f(x;\theta) &= \frac{\theta^2}{\theta^2+1}\left(\theta e^{-\theta x}\right) + \frac{1}{\theta^2+1}\left(\frac{\theta^2}{\Gamma 2}e^{-\theta x}x\right) \\
&= \frac{\theta^3}{\theta^2+1}e^{-\theta x} + \frac{\theta^2}{\theta^2+1}e^{-\theta x}x \\
&= \frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}; x>0, \theta>0
\end{aligned}
$$

further Cdf in Eq.4.2 is obtained as;

$$
\begin{aligned}
F(x;\theta) &= \frac{\theta^2}{\theta^2+1}\int_0^\infty (\theta+x)e^{-\theta x}dx \\
&= \frac{\theta^2}{\theta^2+1}I_1 \\
I_1 &= \int_0^\infty (\theta+x)e^{-\theta x}dx \\
u &= (\theta+x) \Longrightarrow du = dx \\
dv &= e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta} \\
I_1 &= -(\theta+x)\frac{e^{-\theta x}}{\theta} - \frac{1}{\theta}\int_0^\infty -e^{-\theta x}dx \\
I_1 &= -(\theta+x)\frac{e^{-\theta x}}{\theta} - \frac{1}{\theta}I_2 \\
I_2 &= \int_0^\infty -e^{-\theta x}dx \\
u &= -\theta x \Longrightarrow du = -\theta dx \\
&= \int_0^\infty -e^u \frac{du}{-\theta} = \frac{1}{\theta}\int_0^\infty e^u du \\
I_2 &= \frac{1}{\theta}e^{-\theta x}
\end{aligned}
$$

From I_1 and I_2 we have;

$$
\begin{aligned}
I_1 &= -(\theta + x)\frac{e^{-\theta x}}{\theta} - \frac{1}{\theta^2}e^{-\theta x} \\
&= 1 + -(\theta + x)\frac{e^{-\theta x}}{\theta}\frac{\theta^2}{\theta^2+1} - \frac{1}{\theta^2}e^{-\theta x}\frac{\theta^2}{\theta^2+1} \\
&= 1 + -\frac{\theta(\theta + x)e^{-\theta x}}{\theta^2+1} - \frac{e^{-\theta x}}{\theta^2+1} \\
&= 1 - \left[\frac{\theta^2 + \theta x + 1}{\theta^2+1}\right]e^{-\theta x} \\
F(x;\theta) &= 1 - \left[1 + \frac{\theta x}{\theta^2+1}\right]e^{-\theta x}; x > 0, \theta > 0
\end{aligned}
$$

□

4.2.2 Reliability Analysis

Proposition 4.2.2. *The survival function denoted by $S(x;\theta)$ and hazard function denoted by $h(x;\theta)$ of one parameter Shanker distribution Eq.4.1 are stated as;*

$$
S(x;\theta) = \left[\frac{\theta^2 + 1 + \theta x}{\theta^2+1}\right]e^{-\theta x}; x > 0, \theta > 0 \tag{4.3}
$$

$$
h(x;\theta) = \frac{\theta^2(\theta + x)}{\theta^2 + 1 + \theta x}; x > 0, \theta > 0 \tag{4.4}
$$

By definition in Eq.1.8, survival function $S(x;\theta)$ in Eq.4.3 is obtained as;

Proof.

$$S(x;\theta) = 1 - \left[1 - \left[1 + \frac{\theta x}{\theta^2+1}\right]e^{-\theta x}\right]$$
$$= \left[\frac{\theta^2+1+\theta x}{\theta^2+1}\right]e^{-\theta x}; x > 0, \theta > 0$$

Using the relation in Eq.1.9 , hazard function Eq.4.4 is obtained as;

$$h(x;\theta) = \frac{\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}}{\left[\frac{\theta^2+1+\theta x}{\theta^2+1}\right]e^{-\theta x}}$$
$$= \frac{\theta^2(\theta+x)}{\theta^2+1+\theta x}; x > 0, \theta > 0$$

□

4.2.3 Moments and related measures

Proposition 4.2.3. *The r^{th} moments of one parameter Shanker distribution are derived using both moment generating function (mgf) and methods of moments are stated as;*

$$\mu_r^1 * = \frac{r!\left(\theta^2+(r+1)\right)}{\theta^r(\theta^2+1)}; r = 1,2,3,\cdots \qquad (4.5)$$

Proof. By definition Eq.1.16, moments of one parameter Shanker distribution are obtained as;

$$\begin{aligned}
E(X^r) &= \frac{\theta^2}{\theta^2+1}\left[x^r(\theta+x)e^{-\theta x}dx\right] \\
&= \frac{\theta^2}{\theta^2+1}\left[\theta\int_0^\infty x^r e^{-\theta x}dx + \int_0^\infty x^{r+1}e^{-\theta x}dx\right] \\
&= \frac{\theta^2}{\theta^2+1}\left[\frac{\theta\Gamma(r+1)}{\theta^{r+1}} + \frac{\Gamma(r+2)}{\theta^{r+2}}\right] \\
&= \frac{\theta^2}{\theta^r(\theta^2+1)}\left[r\Gamma r + \frac{(r+1)r\Gamma r}{\theta^2}\right] \\
&= \frac{r!\big(\theta^2+(r+1)\big)}{\theta^r(\theta^2+1)}; r = 1,2,3,\cdots
\end{aligned}$$

further the mgf of Shanker distribution is obtained using the relation Eq.1.17 as;

$$\begin{aligned}
m_x(t) &= \frac{\theta^2}{\theta^2+1}\int_0^\infty e^{-(\theta-t)x}(\theta+x)dx \\
&= \frac{\theta^2}{\theta^2+1}\left[\frac{\theta}{\theta-t} + \frac{1}{(\theta-t)^2}\right] \\
&= \frac{\theta^2}{\theta^2+1}\left[\sum_{k=0}^\infty \left(\frac{t}{\theta}\right)^k + \frac{1}{\theta^2}\sum_{k=0}^\infty \binom{k+1}{k}\left(\frac{t}{\theta}\right)^k\right] \\
&= \sum_{k=0}^\infty \frac{\big(\theta^2+(k+1)\big)}{\theta^2+1}\left(\frac{t}{\theta}\right)^k
\end{aligned}$$

□

The moments about origin about the origin of one parameter Shanker distribution Eq.4.1 are obtained as

$$\mu_r^1 = \frac{r!\big(\theta^2+(r+1)\big)}{\theta^r(\theta^2+1)}$$

We obtain the four none centralized moments of one parameter Shanker distribution Eq.4.1 by putting $r = 1, 2, 3$ and 4 in Eq.4.5 as;

$$\mu_1^1 = \frac{\theta^2+2}{\theta(\theta^2+1)}, \quad \mu_2^1 = \frac{2(\theta^2+3)}{\theta^2(\theta^2+1)}$$
$$\mu_3^1 = \frac{6(\theta^2+2)}{\theta^3(\theta^2+1)}, \quad \mu_4^1 = \frac{24(\theta^2+5)}{\theta^4(\theta^2+1)}$$

The centralized moments are derived as;

$$\mu_1 = \mu_1^1$$
$$\mu_2 = \mu_2^1 - (\mu_1^1)^2$$
$$\mu_2 = \frac{2(\theta^2+3)}{\theta^2(\theta^2+1)} - \left[\frac{\theta^2+2}{\theta(\theta^2+1)}\right]^2 = \frac{\theta^4+4\theta^2+2}{\theta^2(\theta^2+1)^2}$$
$$\mu_3 = \mu_3^1 - 3\mu_2^1\mu_1^1 + 2(\mu_1^1)^3$$
$$\mu_3 = \frac{6(\theta^2+2)}{\theta^3(\theta^2+1)} - 3\left[\frac{2(\theta^2+3)}{\theta^2(\theta^2+1)}\frac{\theta^2+2}{\theta(\theta^2+1)}\right] + 2\left[\frac{\theta^2+2}{\theta(\theta^2+1)}\right]^3 = \frac{2\theta^6+12\theta^4 -}{\theta^3(\theta^2+1)}$$
$$\mu_4 = \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2$$
$$= \frac{24(\theta^2+5)}{\theta^4(\theta^2+1)} - 4\left[\frac{6(\theta^2+2)}{\theta^3(\theta^2+1)}\frac{\theta^2+2}{\theta(\theta^2+1)}\right] + 6\left[\frac{\theta^2+2}{\theta(\theta^2+1)}\frac{2(\theta^2+3)}{\theta^2(\theta^2+1)}\right] - 3\Big[$$
$$\mu_4 = \frac{9\theta^8+72\theta^6+132\theta^4+96\theta^2+24}{\theta^4(\theta^2+1)^4}$$

Proposition 4.2.4. *Other related measures of one parameter Shanker distribution such as variation coefficient (C.v), skewness*

(υ_1), kurtosis (υ_2) and dispersion index (υ_3) are stated as;

$$C.v = \frac{\sqrt{\theta^4+4\theta^2+2}}{\theta^2+2} \tag{4.6}$$

$$\upsilon_1 = \frac{2\theta^6+12\theta^4+4}{(\theta^4+4\theta^2+2)^{\frac{3}{2}}} \tag{4.7}$$

$$\upsilon_2 = \frac{9\theta^8+72\theta^6+132\theta^4+96\theta^2+24}{(\theta^4+4\theta^2+4)^2} \tag{4.8}$$

$$\upsilon_3 = \frac{\theta^4+4\theta^2+2}{\theta(\theta^2+1)(\theta^2+2)} \tag{4.9}$$

`Proof.` To begin with, coefficient of variation Eq.4.6 is obtained as;

$$\begin{aligned} C.v &= \frac{\sigma}{\mu_1^1} \\ &= \frac{\sqrt{\theta^4+4\theta^2+2}}{\theta(\theta^2+1)} \frac{\theta(\theta^2+1)}{\theta^2+2} \Longrightarrow \frac{\sqrt{\theta^4+4\theta^2+2}}{\theta^2+2} \end{aligned}$$

further, skewness coefficient Eq.4.7 is obtained as;

$$\begin{aligned} \upsilon_1 &= \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} \\ &= \frac{2\theta^6+12\theta^4+4}{\theta^3(\theta^2+1)^3}\left[\frac{\theta^2(\theta^2+1)^2}{\theta^4+4\theta^2+2}\right]^{\frac{3}{2}} \Longrightarrow \frac{2\theta^6+12\theta^4+4}{(\theta^4+4\theta^2+2)^{\frac{3}{2}}} \end{aligned}$$

Kurtosis coefficient Eq.4.8 is obtained as;

$$\begin{aligned} \upsilon_2 &= \frac{\mu_4}{(\mu_2)^2} \\ &= \frac{9\theta^8+72\theta^6+132\theta^4+96\theta^2+24}{\theta^4(\theta^2+1)^4}\left[\frac{\theta^2(\theta^2+1)}{\theta^4+4\theta^2+2}\right]^2 \Longrightarrow \frac{9\theta^8+72\theta^6 \ldots}{(\theta^4 \ldots} \end{aligned}$$

Lastly, index of dispersion Eq.4.9 is obtained as;

$$\upsilon_3 = \frac{\sigma^2}{\mu_1^1}$$
$$= \frac{\theta^4+4\theta^2+2}{\theta^2(\theta^2+1)^2}\frac{\theta(\theta^2+1)}{\theta^2+2} \Longrightarrow \frac{\theta^4+4\theta^2+2}{\theta(\theta^2+1)(\theta^2+2)}$$

□

4.2.4 Excess Loss Distribution

Proposition 4.2.5. *In this section, we state probability density function of excess function $f_l(x;\theta)$, Mean residual lifetime (MRL), equilibrium distribution $f_e(x;\theta)$, survival function of equilibrium distribution $S_e(x;\theta)$, hazard function of equilibrium distribution $h_e(x;\theta)$ of one parameter Shanker distribution Eq.4.1 as;*

$$f_l(x;\theta) = \frac{\theta^2(\theta+x)e^{-(x-z)\theta}}{\theta^2+1+\theta z}; \quad x > z \tag{4.10}$$

$$m(x) = \frac{\theta^2+\theta x+2}{\theta(\theta^2+1+\theta x)} \tag{4.11}$$

$$f_e(x;\theta) = \frac{\theta(\theta^2+1+\theta x)e^{-\theta x}}{\theta^2+2} \tag{4.12}$$

$$S_e(x;\theta) = \frac{(\theta^2+\theta x+2)e^{-\theta x}}{\theta^2+2} \tag{4.13}$$

$$h_e(x;\theta) = \frac{\theta(\theta^2+1+\theta x)}{\theta^2+\theta x+2} \tag{4.14}$$

Proof. Firstly, pdf of excess loss distribution $f_l(x;\theta)$ Eq.4.10 is obtained using relation Eq.1.10 as;

$$\begin{aligned} f_l(x;\theta) &= \frac{\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}}{\left[\frac{\theta^2+1+\theta z}{\theta^2+1}\right]e^{-\theta z}} \\ &= \frac{\theta^2(\theta+x)e^{-(x-z)\theta}}{\theta^2+1+\theta z}; \quad x > z \end{aligned}$$

Using the relation Eq.1.11, mean residual lifetime $m(x)$ in Eq.4.11 is obtained as;

$$\begin{aligned} m(x) &= \frac{\theta^2+1}{(\theta^2+1+\theta x)e^{-\theta x}}\int_x^{\infty}\frac{(\theta^2+1+\theta t)e^{-\theta t}}{\theta^2+1}dt \\ &= \frac{1}{(\theta^2+1+\theta x)e^{-\theta x}}\int_x^{\infty}(\theta^2+1+\theta t)e^{-\theta t}dt \end{aligned}$$

Taking the part $\int_x^{\infty}(\theta^2+1+\theta t)e^{-\theta t}dt$ we have;

$$\begin{aligned} u &= (\theta^2+1+\theta t) \Longrightarrow du = \theta dt \\ dv &= e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta} \\ &= -(\theta^2+1+\theta t)\frac{-e^{-\theta t}}{\theta} + \int_x^{\infty} e^{-\theta t}dt \\ &= -(\theta^2+1+\theta t)\frac{-e^{-\theta t}}{\theta} - \frac{e^{-\theta t}}{\theta} \\ &= -\left[\frac{e^{-\theta t}}{\theta}\left(\theta^2+2+\theta t\right)\right]_x^{\infty} \Longrightarrow \frac{e^{-\theta x}}{\theta}(\theta^2+2+\theta x) \\ &= \frac{1}{(\theta^2+1+\theta x)e^{-\theta x}}\frac{e^{-\theta x}}{\theta}(\theta^2+2+\theta x) \Longrightarrow \frac{\theta^2+\theta x+2}{\theta(\theta^2+1+\theta x)} \end{aligned}$$

By definition in Eq.1.12, equilibrium distribution $f_e(x;\theta)$ Eq.4.12 is obtained as;

$$f_e(x;\theta) = \frac{\left[\frac{\theta^2+1+\theta x}{\theta^2+1}\right]e^{-\theta x}}{\frac{\theta^2+2}{\theta(\theta^2+1)}}$$
$$= \frac{\theta(\theta^2+1+\theta x)e^{-\theta x}}{\theta^2+2}$$

further, using the relation Eq.1.13 survival function based on equilibrium distribution Eq.4.13 is obtained as;

$$\int_x^{\infty} S(t;\theta)dt = (\theta^2+\theta x+2)\frac{e^{-\theta x}}{\theta(\theta^2+1)}$$
$$S(x;\theta) = (\theta^2+\theta x+2)\frac{e^{-\theta x}}{\theta(\theta^2+1)}\frac{\theta^2+2}{\theta(\theta^2+1)} \Longrightarrow \frac{(\theta^2+\theta x+2)e^{-\theta x}}{\theta^2+2}$$

hazard function based on equilibrium distribution $h_e(x;\theta)$ Eq.4.14 is obtained using Eq.1.14 as;

$$h_e(x;\theta) = \frac{\theta(\theta^2+1+\theta x)e^{-\theta x}}{\theta^2+2}\frac{\theta^2+2}{(\theta^2+\theta x+2)e^{-\theta x}} \Longrightarrow \frac{\theta(\theta^2+1+\theta x)}{\theta^2+\theta x+2}$$

□

4.3 Two parameter Shanker distribution

4.3.1 Construction of a two parameter Shanker distribution

Proposition 4.3.1. *Let $\omega = \frac{\theta^2}{\theta^2+\beta}$ be the mixing weight, generalized two parameter Shanker distribution (G2PSD) is a two component finite mixed distribution of Gamma($1,\theta$) and Gamma($2,\theta$).*

The pdf and Cdf of G2PSD are stated as;

$$f(x;\beta,\theta) = \frac{\theta^2}{\theta^2+\beta}\left[\theta+\beta x\right]e^{-\theta x};\beta>0,\theta>0,x>0 \quad (4.15)$$

$$F(x;\beta,\theta) = 1-\left[1+\frac{\beta\theta x}{\theta^2+\beta}\right]e^{-\theta x};\beta>0,\theta>0,x>0 \quad (4.16)$$

Proof. By the concept of finite mixtures Eq.1.1, the pdf Eq.4.15 is obtained as;

$$\begin{aligned} f(x;\beta,\theta) &= \frac{\theta^2}{\theta^2+\beta}\left[\theta e^{-\theta x}\right]+\frac{\beta}{\theta^2+\beta}\left[\frac{\theta^2 e^{-\theta x}x}{\Gamma 2}\right] \\ &= \frac{\theta^3}{\theta^2+\beta}e^{-\theta x}+\frac{\beta\theta^2}{\theta^2+\beta}e^{-\theta x}x \\ &= \frac{\theta^2}{\theta^2+\beta}\left[\theta+\beta x\right]e^{-\theta x};\beta>0,\theta>0,x>0 \end{aligned}$$

Remark 4.3.2. *One parameter Shanker distribution Eq.4.1 is a particular case of a generalized two parameter Shanker distribution Eq.4.15 by putting $\beta = 1$.*

G2PSD Eq.4.15 is a modification of one parameter Shanker distribution Eq.4.1 introduced by Shanker (2015b).

further Cdf Eq.4.16 is obtained as;

$$
\begin{aligned}
F(x;\beta,\theta) &= \frac{\theta^2}{\theta^2+\beta}\int_0^\infty(\theta+\beta x)e^{-\theta x}dx \\
&= \frac{\theta^2}{\theta^2+\beta}I_1 \\
I_1 &= \int_0^\infty(\theta+\beta x)e^{-\theta x}dx \\
u &= (\theta+\beta x) \Longrightarrow du = \beta dx \\
dv &= e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta} \\
I_1 &= -(\theta+\beta x)\frac{-e^{-\theta x}}{\theta} - \frac{\beta}{\theta}\int_0^\infty -e^{-\theta x}dx \\
I_1 &= -(\theta+\beta x)\frac{-e^{-\theta x}}{\theta} - \frac{\beta}{\theta}I_2 \\
I_2 &= \int_0^\infty -e^{-\theta x}dx \\
u &= -\theta x \Longrightarrow du = -\theta dx \Longrightarrow dx = \frac{-du}{\theta} \\
&= \int_0^\infty -e^u\frac{-du}{\theta} \Longrightarrow \frac{1}{\theta}\int_0^\infty e^u du \Longrightarrow \frac{1}{\theta}e^{-\theta x}
\end{aligned}
$$

From I_1 and I_2 then;

$$
\begin{aligned}
I_1 &= -(\theta+\beta x)\frac{e^{-\theta x}}{\theta} - \frac{\beta}{\theta}(\frac{1}{\theta}e^{-\theta x}) \\
&= -(\theta+\beta x)\frac{e^{-\theta x}}{\theta} - \frac{\beta}{\theta^2}e^{-\theta x} \\
&= 1+-(\theta+\beta x)\frac{e^{-\theta x}}{\theta}\frac{\theta^2}{\theta^2+\beta} - \frac{\beta}{\theta^2}\frac{\theta^2}{\theta^2+\beta} \\
&= 1\left[\frac{\theta^2+\beta\theta x+\beta}{\theta^2+\beta}\right]e^{-\theta x} \\
&= 1-\left[1+\frac{\beta\theta x}{\theta^2+\beta}\right]e^{-\theta x}; \beta>0, \theta>0, x>0
\end{aligned}
$$

□

4.3.2 Reliability Analysis

Proposition 4.3.3. *Survival function denoted by* $S(x;\beta,\theta)$ *and hazard function denoted by* $h(x;\beta,\theta)$ *of generalized two parameter Shanker distribution 4.15 are stated as;*

$$
S(x;\beta,\theta) = \left[1+\frac{\beta\theta x}{\theta^2+\beta}\right]e^{-\theta x}; x>0, \beta>0, \theta>0 \tag{4.17}
$$

$$
h(x;\beta,\theta) = \frac{\theta^2(\theta+\beta x)}{\theta^2+\beta+\beta\theta x}; x>0, \beta>0, \theta>0 \tag{4.18}
$$

Proof. To begin with, survival function $S(x;\beta,\theta)$ Eq.4.17 is derived using the relation Eq.1.8 as;

$$
\begin{aligned}
S(x;\beta,\theta) &= 1-\left[1-\left[1+\frac{\beta\theta x}{\theta^2+\beta}\right]e^{-\theta x}\right] \\
&= \left[1+\frac{\beta\theta x}{\theta^2+\beta}\right]e^{-\theta x}; x>0, \beta>0, \theta>0
\end{aligned}
$$

further hazard function $h(x;\beta,\theta)$ Eq.4.18 is obtained using relation Eq.1.9 as;

$$h(x;\beta,\theta)=\frac{\frac{\theta^2}{\theta^2+\beta}\left[\theta+\beta x\right]e^{-\theta x}}{\left[1+\frac{\beta\theta x}{\theta^2+\beta}\right]e^{-\theta x}}$$
$$=\frac{\theta^2(\theta+\beta x)}{\theta^2+\beta+\beta\theta x};x>0,\beta>0,\theta>0$$

□

4.3.3 Moments and related measures

Proposition 4.3.4. *The r^{th} moments of a generalized two parameter Shanker distribution are derived using both moment generating function (mgf) and methods of moments are stated as;*

$$\mu_r^1*=\frac{r!\big(\theta^2+\beta(r+1)\big)}{\theta^r(\theta^2+\beta)};r=1,2,3,\cdots \tag{4.19}$$

Proof. By definition of moments Eq.1.16, moments of G2PSD are obtained as;

$$E(X^r)=\frac{\theta^2}{\theta^2+\beta}\int_0^\infty x^r(\theta+\beta x)e^{-\theta x}dx$$
$$=\frac{\theta^2}{\theta^2+\beta}\left[\theta\int_0^\infty x^r e^{-\theta x}dx+\beta\int_0^\infty x^{r+1}e^{-\theta x}dx\right]$$
$$=\frac{\theta^2}{\theta^2+\beta}\left[\frac{\theta\Gamma(r+1)}{\theta^{r+1}}+\frac{\beta\Gamma(r+2)}{\theta^{r+2}}\right]\implies\frac{1}{\theta^r(\theta^2+\beta)}\big[\theta^2 r\Gamma r+\beta(r+1)r$$
$$=\frac{r\Gamma r}{\theta^r(\theta^2+\beta)}\big(\theta^2+\beta(r+1)\big)$$
$$=\frac{r!\big(\theta^2+\beta(r+1)\big)}{\theta^r(\theta^2+\beta)};r=1,2,3,\cdots$$

By mgf formula Eq.1.17, mgf of G2PSD is obtained as;

$$\begin{aligned}
m_x(t) &= \frac{\theta^2}{\theta^2+\beta}\int_0^\infty e^{tx}(\theta+\beta x)e^{-\theta x}dx \Longrightarrow \frac{\theta^2}{\theta^2+\beta}\left[e^{-(\theta-t)x(\theta+\beta x)}dx\right] \\
&= \frac{\theta^2}{\theta^2+\beta}\left[\frac{\theta}{\theta-t}+\frac{\beta}{(\theta-t)^2}\right] \\
&= \frac{\theta^2}{\theta^2+\beta}\left[\sum_{k=0}^{\infty}\left(\frac{t}{\theta}\right)^k+\frac{\beta}{\theta^2}\sum_{k=0}^{\infty}\binom{k+1}{k}\left(\frac{t}{\theta}\right)^k\right] \\
&= \sum_{k=0}^{\infty}\frac{\theta^2+\beta(k+1)}{\theta^2+\beta}\left(\frac{t}{\theta}\right)^k
\end{aligned}$$

The respective moments are derived as a coefficient of $\frac{t^r}{r!}$ in $m_x(t)$ above as;

$$\mu_r^1 = \frac{r!(\theta^2+\beta(r+1))}{\theta^r(\theta^2+\beta)}; r = 1,2,3,\cdots$$

□

The four none centralized moments are obtained by putting r=1,2,3 and 4 in Eq.4.19 as shown below;

$$\mu_1^1 = \frac{\theta^2+2\beta}{\theta(\theta^2+\beta)}, \quad \mu_2^1 = \frac{2(\theta^2+3\beta)}{\theta^2(\theta^2+\beta)}$$
$$\mu_3^1 = \frac{6(\theta^2+4\beta)}{\theta^3(\theta^2+\beta)}, \quad \mu_4^1 = \frac{24(\theta^2+5\beta)}{\theta^4(\theta^2+\beta)}$$

The centralized moments of a generalized two parameter Shanker distribution are obtained as;

$$\mu_1 = \mu_1^1$$
$$\mu_2 = \mu_2^1 - (\mu_1^1)^2$$
$$\mu_2 = \frac{2\theta^2+6\beta}{\theta^2(\theta^2+\beta)} - \left[\frac{\theta^2+2\beta}{\theta(\theta^2+\beta)}\right]^2 \Longrightarrow \frac{\theta^4+4\beta\theta^2+2\beta^2}{\theta^2(\theta^2+\beta)^2}$$
$$\mu_3 = \mu_3^1 - 3\mu_2^1\mu_1^1 + 2\mu_1^1$$
$$\mu_3 = \frac{6(\theta^2+4\beta)}{\theta^3(\theta^2+\beta)} - 3\left[\frac{2(\theta^2+3\beta)}{\theta^2(\theta^2+\beta)}\frac{\theta^2+2\beta}{\theta(\theta^2+\beta)}\right] + 2\left[\frac{\theta^2+2\beta}{\theta(\theta^2+\beta)}\right]^3 \Longrightarrow \frac{2\theta^6+12}{}$$
$$\mu_4 = \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2$$
$$= \frac{24(\theta^2+5\beta)}{\theta^4(\theta^2+\beta)} - 4\left[\frac{6(\theta^2+4\beta)}{\theta^3(\theta^2+\beta)}\frac{\theta^2+2\beta}{\theta(\theta^2+\beta)}\right] + 6\left[\frac{\theta^2+2\beta}{\theta(\theta^2+\beta)}\frac{2(\theta^2+3\beta)}{\theta^2(\theta^2+\beta)}\right] -$$

$$\mu_4 = \frac{96\beta\theta^6+420\beta^2\theta^4+348\beta^2\theta^2-72\beta^4+12\theta^9+72\beta^4\theta-3\theta^10-18\beta\theta^8 -}{\theta^4(\theta^2+\beta)^4}$$

Proposition 4.3.5. *Other related measures of a generalized two parameter Shanker distribution such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) are stated as;*

$$C.v = \frac{\sqrt{\theta^4+4\beta\theta^2+2\beta^2}}{\theta^2+2\beta} \tag{4.20}$$
$$\upsilon_1 = \frac{2\theta^6+12\beta\theta^4+84\beta^2\theta^2+4\beta^3}{\left(\theta^4+4\beta\theta^2+2\beta^2\right)^{\frac{3}{2}}} \tag{4.21}$$
$$\upsilon_2 = \frac{\mu_4}{(\mu_2)^2} \tag{4.22}$$
$$\upsilon_3 = \frac{\theta^4+4\beta\theta^2+2\beta^2}{\theta(\theta^2+\beta)(\theta^2+2\beta)} \tag{4.23}$$

Proof. Variation coefficient Eq.4.20 is obtained as;

$$
\begin{aligned}
C.v &= \frac{\sigma}{\mu_1^1} \\
&= \frac{\sqrt{\theta^4+4\beta\theta^2+2\beta^2}}{\theta(\theta^2+\beta)}\frac{\theta(\theta^2+\beta)}{\theta^2+2\beta} \Longrightarrow \frac{\sqrt{\theta^4+4\beta\theta^2+2\beta^2}}{\theta^2+2\beta}
\end{aligned}
$$

Skewness coefficient Eq.4.21 is obtained as;

$$
\begin{aligned}
\upsilon_1 &= \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} \\
&= \frac{2\theta^6+12\beta\theta^4+84\beta^2\theta^2+4\beta^3}{\theta^3(\theta^3+\beta)^3}\left[\frac{\theta^2(\theta^2+\beta)^2}{\theta^4+4\beta\theta^2+2\beta^2}\right]^{\frac{3}{2}} \Longrightarrow \frac{2\theta^6+12\beta}{(\theta^4+}
\end{aligned}
$$

Similarly, kurtosis coefficient Eq.4.22 is obtained as;

$$
\begin{aligned}
\upsilon_2 &= \frac{\mu_4}{(\mu_2)^2} \\
&= \frac{\mu_4}{(\frac{\theta^4+4\beta\theta^2+2\beta^2}{\theta^2(\theta^2+\beta)^2})^2}
\end{aligned}
$$

Where μ_4 is defined in the section centralized moments of a generalized two parameter Shanker distribution. Index of dispersion

$$
\begin{aligned}
\upsilon_3 &= \frac{\sigma^2}{\mu_1^1} \\
&= \frac{\theta^4+4\beta\theta^2+2\beta^2}{\theta^2(\theta^2+\beta)^2}\frac{\theta(\theta^2+\beta)}{\theta^2+2\beta} \Longrightarrow \frac{\theta^4+4\beta\theta^2+2\beta^2}{\theta(\theta^2+\beta)(\theta^2+2\beta)}
\end{aligned}
$$

□

4.3.4 Excess Loss Distribution

Proposition 4.3.6. *In this section, we state probability density function of excess function $f_l(x;\beta,\theta)$, Mean residual lifetime (MRL), equilibrium distribution $f_e(x;\beta,\theta)$, survival function of equilibrium distribution $S_e(x;\beta,\theta)$, hazard function of equilibrium distribution $h_e(x;\beta,\theta)$ of a generalized two parameter Shanker distribution Eq.4.15 as;*

$$f_l(x;\beta,\theta) = \frac{\theta^2(\theta+\beta x)e^{-(x-z)\theta}}{\theta^2+\beta+\beta\theta z}; x > z \tag{4.24}$$

$$m(x) = \frac{\theta^2+2\beta+\beta\theta x}{\theta(\theta^2+\beta+\beta\theta x)} \tag{4.25}$$

$$f_e(x;\beta,\theta) = \frac{\theta(\theta^2+\beta+\beta\theta x)e^{-\theta x}}{\theta^2+2\beta} \tag{4.26}$$

$$S_e(x;\beta,\theta = \frac{(\theta^2+2\beta+\beta\theta x)e^{-\theta x}}{\theta^2+2\beta} \tag{4.27}$$

$$h_e(x;\beta,\theta) = \frac{\theta(\theta^2+\beta+\beta\theta x)}{\theta^2+2\beta+\beta\theta x} \tag{4.28}$$

Proof. Using the relation Eq.1.10, $f_l(x;\beta,\theta)$ Eq.4.24 is obtained as;

$$\begin{aligned} f_l(x;\beta,\theta) &= \frac{\frac{\theta^2}{\theta^2+\beta}\left[\theta+\beta x\right]e^{-\theta x}}{\left[\frac{\theta^2+\beta+\beta\theta z}{\theta^2+\beta}\right]e^{-\theta z}} \\ &= \frac{\theta^2(\theta+\beta x)e^{-(x-z)\theta}}{\theta^2+\beta+\beta\theta z}; x > z \end{aligned}$$

By definition Eq.1.11, $m(x)$ Eq.4.25 is obtained as;

$$
\begin{aligned}
m(x) &= \frac{\theta^2+\beta}{(\theta^2+\beta+\beta\theta x)e^{-\theta x}} \int_x^{\infty} \frac{(\theta^2+\beta+\beta\theta t)e^{-\theta t}}{\theta^2+\beta} dt \\
&= \frac{1}{\theta^2+\beta+\beta\theta x)e^{-\theta x}} \int_x^{\infty} \left[\theta^2+\beta+\beta\theta t\right] e^{-\theta t} dt
\end{aligned}
$$

Taking the part $\int_x^{\infty} \left[\theta^2+\beta+\beta\theta t\right] e^{-\theta t} dt$ we have;

$$
\begin{aligned}
u &= (\theta^2+\beta+\beta\theta t) \Longrightarrow du = \beta\theta dt \\
dv &= e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta} \\
&= -(\theta^2+\beta+\beta\theta t)\frac{-e^{-\theta t}}{\theta} + \beta \int_x^{\infty} e^{-\theta t} dt \\
&= -\left[\frac{e^{-\theta t}}{\theta}\left[\theta^2+2\beta+\beta\theta t\right]\right]_x^{\infty} \\
&= \left[\frac{e^{-\theta x}}{\theta}\left[\theta^2+2\beta+\beta\theta x\right]\right] \\
&= \frac{1}{\theta^2+\beta+\beta\theta x)e^{-\theta x}}\left[\frac{e^{-\theta x}}{\theta}\left[\theta^2+2\beta+\beta\theta x\right]\right] \Longrightarrow \frac{\theta^2+2\beta+\beta\theta x}{\theta(\theta^2+\beta+\beta\theta x)}
\end{aligned}
$$

Using the relation Eq.1.12, equilibrium definition Eq.4.26 is obtained as;

$$
\begin{aligned}
f_e(x;\beta,\theta) &= \frac{(\theta^2+\beta+\beta\theta x)e^{-\theta x}}{\theta^2+\beta} \frac{\theta(\theta^2+\beta)}{\theta^2+2\beta} \\
&= \frac{\theta(\theta^2+\beta+\beta\theta x)e^{-\theta x}}{\theta^2+2\beta}
\end{aligned}
$$

By definition Eq.1.13, survival function based on equilibrium distribution Eq.4.27 is obtained as;

$$\int_x^{\infty} S(t;\beta,\theta)dt = \frac{(\theta^2+2\beta+\beta\theta x)e^{-\theta x}}{\theta(\theta^2+\beta)}$$

$$S_e(x;\beta,\theta) = \frac{(\theta^2+2\beta+\beta\theta x)e^{-\theta x}}{\theta(\theta^2+\beta)}\frac{\theta(\theta^2+\beta)}{\theta^2+2\beta}$$

$$= \frac{(\theta^2+2\beta+\beta\theta x)e^{-\theta x}}{\theta^2+2\beta}$$

Further using the relation Eq.1.14, $h_e(x;\beta,\theta)$ in Eq.4.28 is obtained as;

$$h_e(x;\beta,\theta) = \frac{\theta(\theta^2+\beta+\beta\theta x)e^{-\theta x}}{\theta^2+2\beta}\frac{\theta^2+2\beta}{e^{-\theta x}(\theta^2+2\beta+\beta\theta x)}$$

$$= \frac{\theta(\theta^2+\beta+\beta\theta x)}{\theta^2+2\beta+\beta\theta x}$$

□

4.4 Three parameter Shanker distribution

4.4.1 Construction of a three parameter Shanker distribution

Proposition 4.4.1. *Let $\omega = \frac{\alpha\theta^2}{\alpha\theta^2+\beta}$ be mixing proportion, generalized three parameter Shanker distribution is a two component finite mixed distribution of Gamma$(1,\theta)$ and Gamma$(2,\theta)$. The pdf and Cdf of G3PSD are stated as;*

$$f(x;\alpha,\beta,\theta) = \frac{\theta^2}{\alpha\theta^2+\beta}\left[\alpha\theta+\beta x\right]e^{-\theta x}; x>0, \alpha>0, \beta>0, \theta>0 \tag{4.29}$$

$$F(x;\alpha,\beta,\theta) = 1-\left[1+\frac{\beta\theta x}{\alpha\theta^2+\beta}\right]e^{-\theta x}; x>0, \alpha>0, \beta>0, \theta>0 \tag{4.30}$$

Proof. By definition of finite mixed distribution Eq.1.1, pdf Eq.4.29 is obtained as;

$$
\begin{aligned}
f(x;\alpha,\beta,\theta) &= \frac{\alpha\theta^2}{\alpha\theta^2+\beta}\left[\theta e^{-\theta x}\right] + \frac{\beta}{\alpha\theta^2+\beta}\left[\frac{\theta^2}{\Gamma 2}e^{-\theta x}\right] \\
&= \frac{\alpha\theta^3 e^{-\theta x}}{\alpha\theta^2+\beta} + \frac{\beta\theta^2 e^{-\theta x}x}{\alpha\theta^2+\beta} \\
&= \frac{\theta^2}{\alpha\theta^2+\beta}\left[\alpha\theta+\beta x\right]e^{-\theta x}; x>0, \alpha>0, \beta>0, \theta>0
\end{aligned}
$$

Remark 4.4.2. *Putting* $\alpha = \beta = 1$ *a generalized three parameter Shanker distribution reduces to a one parameter Shanker distribution Eq.4.1.Similarly, putting* $\alpha = 1$*, a generalized three parameter Shanker distribution Eq.4.29 reduces to a generalized two parameter Shanker distribution Eq.4.15.*

A generalized three parameter Shanker distribution Eq.4.29 is a modification of proposed two parameter Shanker distribution Eq.4.15.

further Cdf Eq.4.30 is obtained as;

$$F(x;\alpha,\beta,\theta)=\frac{\theta^2}{\alpha\theta^2+\beta}\int_0^\infty(\alpha\theta+\beta x)e^{-\theta x}dx$$

$$F(x;\alpha,\beta,\theta)=\frac{\theta^2}{\alpha\theta^2+\beta}I_1$$

$$I_1=\int_0^\infty(\alpha\theta+\beta x)e^{-\theta x}dx$$

$$u=(\alpha\theta+\beta x)\Longrightarrow du=\beta dx$$

$$dv=e^{-\theta x}\Longrightarrow v=\frac{-e^{-\theta x}}{\theta}$$

$$I_1=-(\alpha\theta+\beta x)\frac{-e^{-\theta x}}{\theta}-\frac{\beta}{\theta}\int_0^\infty e^{-\theta x}dx$$

$$I_1=-(\alpha\theta+\beta x)\frac{-e^{-\theta x}}{\theta}-\frac{\beta}{\theta}I_2$$

$$I_2=\int_0^\infty e^{-\theta x}dx$$

$$u=-\theta x\Longrightarrow\frac{du}{-\theta}=dx$$

$$I_2=\int_0^\infty -e^u\frac{du}{-\theta}\Longrightarrow\frac{1}{\theta}\int_0^\infty e^u du\Longrightarrow\frac{1}{\theta}e^{-\theta x}$$

From I_1 and I_2 we have;

$$\begin{aligned}I_1&=-(\alpha\theta+\beta x)\frac{e^{-\theta x}}{\theta}-\frac{\beta}{\theta}e^{-\theta x}\\&=1+-\frac{\theta^2}{\alpha\theta^2+\beta}\left[\alpha\theta+\beta x\right]\frac{e^{-\theta x}}{\theta}-\frac{\beta}{\alpha\theta^2+\beta}e^{-\theta x}\frac{\theta^2}{\alpha\theta^2+\beta}\\&=1-\frac{\theta(\alpha\theta+\beta x)e^{-\theta x}}{\alpha\theta^2+\beta}-\frac{\beta e^{-\theta x}}{\alpha\theta^2+\beta}\\&=1-\left[1+\frac{\beta\theta x}{\alpha\theta^2+\beta}\right]e^{-\theta x};x>0,\alpha>0,\beta>0,\theta>0\end{aligned}$$

□

4.4.2 Reliability Analysis

Proposition 4.4.3. *Survival function denoted by $S(x;\alpha,\beta,\theta)$ and hazard function denoted by $h(x;\alpha,\beta,\theta)$ of a generalized three parameter Shanker distribution Eq.4.29 are stated as;*

$$S(x;\alpha,\beta,\theta)=\left[1+\frac{\beta\theta x}{\alpha\theta^2+\beta}\right]e^{-\theta x};x>0,\alpha>0,\beta>0,\theta>0 \tag{4.31}$$

$$h(x;\alpha,\beta,\theta)=\frac{\theta^2(\alpha\theta+\beta x)e^{-\theta x}}{\alpha\theta^2+\beta+\beta\theta x};x>0,\alpha>0,\beta>0,\theta>0 \tag{4.32}$$

`Proof.` Using the relation Eq.1.8, survival function Eq.4.31 is obtained as;

$$\begin{aligned}S(x;\alpha,\beta,\theta)&=1-\left[1-\left[1+\frac{\beta\theta x}{\alpha\theta^2+\beta}\right]e^{-\theta x}\right]\\&=\left[1+\frac{\beta\theta x}{\alpha\theta^2+\beta}\right]e^{-\theta x};x>0,\alpha>0,\beta>0,\theta>0\end{aligned}$$

further using the relation Eq.1.9, hazard function Eq.4.32 is obtained as;

$$\begin{aligned}h(x;\alpha,\beta,\theta)&=\frac{\frac{\theta^2}{\alpha\theta^2+\beta}\left[\alpha\theta+\beta x\right]e^{-\theta x}}{\left[\frac{\alpha\theta^2+\beta+\beta\theta x}{\alpha\theta^2+\beta}\right]e^{-\theta x}}\\&=\frac{\theta^2(\alpha\theta+\beta x)e^{-\theta x}}{\alpha\theta^2+\beta+\beta\theta x};x>0,\alpha>0,\beta>0,\theta>0\end{aligned}$$

□

4.4.3 Moments and related measures

Proposition 4.4.4. *The r^{th} moments of a generalized three parameter Shanker distribution Eq.4.29 are derived using both moment generating function (mgf) and methods of moments are stated as;*

$$\mu_r^1 * = \frac{r!\left[\alpha\theta^2 + \beta(r+1)\right]}{\theta^r(\alpha\theta^2+\beta)}; r = 1,2,3,\cdots \tag{4.33}$$

Proof. By definition Eq.1.16, r^{th} moments of G3PSD are obtained as;

$$\begin{aligned}
E(X^r) &= \frac{\theta^2}{\alpha\theta^2+\beta}\left[\int_0^\infty x^r(\alpha\theta+\beta x)e^{-\theta x}dx\right] \\
&= \frac{\theta^2}{\alpha\theta^2+\beta}\left[\alpha\theta\int_0^\infty x^r e^{-\theta x}dx + \beta\int_0^\infty x^{r+1}e^{-\theta x}dx\right] \\
&= \frac{\theta^2}{\alpha\theta^2+\beta}\left[\frac{\alpha\theta\Gamma(r+1)}{\theta^{r+1}} + \frac{\beta\Gamma(r+2)}{\theta^{r+2}}\right] \Longrightarrow \frac{\theta^2}{\alpha\theta^2+\beta}\left[\frac{\alpha\theta r\Gamma r}{\theta^{r+1}} + \frac{\beta(r+1}{\theta^{r+}}\right. \\
&= \frac{r!\left[\alpha\theta^2+\beta(r+1)\right]}{\theta^r(\alpha\theta^2+\beta)}; r = 1,2,3,\cdots
\end{aligned}$$

Using Eq.1.17, mgf of G3PSD is given as;

$$\begin{aligned}
m_x(t) &= \frac{\theta^2}{\alpha\theta^2+\beta}\left[\int_0^\infty e^{tx}(\alpha\theta+\beta x)e^{-\theta x}dx\right] \Longrightarrow \frac{\theta^2}{\alpha\theta^2+\beta}\int_0^\infty \left[e^{-(\theta-t)x}(\alpha\theta+\right. \\
&= \frac{\theta^2}{\alpha\theta^2+\beta}\left[\frac{\alpha\theta}{(\theta-t)} + \frac{\beta}{(\theta-t)^2}\right] \\
&= \frac{\theta^2}{\alpha\theta^2+\beta}\left[\frac{\alpha\theta}{\theta}\sum_{k=0}^\infty\left(\frac{t}{\theta}\right)^k + \frac{\beta}{\theta^2}\sum_{k=0}^\infty\binom{k+1}{k}\left(\frac{t}{\theta}\right)^k\right] \\
&= \sum_{k=0}^\infty \frac{\alpha\theta^2+\beta(k+1)}{\alpha\theta^2+\beta}\left(\frac{t}{\theta}\right)^k
\end{aligned}$$

The respective moments are derived as a coefficient of $\frac{t^r}{r!}$ in $m_x(t)$ as;

$$\mu_r^1 = \frac{r!\left(\alpha\theta^2+\beta(r+1)\right)}{\theta^r(\alpha\theta^2+\beta)}; r = 1,2,3,\cdots$$

□

4.4.4 Excess Loss Distribution

Proposition 4.4.5. *In this section, we state probability density function of excess function $f_l(x;\alpha,\beta,\theta)$, Mean residual lifetime (MRL), equilibrium distribution $f_e(x;\alpha,\beta,\theta)$, survival function of equilibrium distribution $S_e(x;\alpha,\beta,\theta)$, hazard function of equilibrium distribution $h_e(x;\alpha,\beta,\theta)$ of one parameter Shanker distribution Eq.4.29 as;*

$$f_l(x;\alpha,\beta,\theta) = \frac{\theta^2(\theta+\beta x)e^{-(x-z)\theta}}{\theta^2+\beta+\beta\theta z}; x > z \tag{4.34}$$

$$m(x) = \frac{\alpha\theta^2+2\beta+\beta\theta x}{\theta\left[\alpha\theta^2+\beta+\beta\theta x\right]} \tag{4.35}$$

$$f_e(x;\alpha,\beta,\theta) = \frac{\theta(\alpha\theta^2+\beta+\beta\theta x)e^{-\theta x}}{\alpha\theta^2+2\beta} \tag{4.36}$$

$$S_e(x;\alpha,\beta,\theta) = \frac{(\alpha\theta^2+2\beta+\beta\theta x)e^{-\theta x}}{\alpha\theta^2+2\beta} \tag{4.37}$$

$$h_e(x;\alpha,\beta,\theta) = \frac{\theta(\alpha\theta^2+\beta+\beta\theta x)}{\alpha\theta^2+2\beta+\beta\theta x} \tag{4.38}$$

Proof. Applying the formula Eq.1.10, the pdf of excess loss distribution $f_l(x;\alpha,\beta,\theta)$ Eq.4.34 is obtained as;

$$
\begin{aligned}
f_l(x;\alpha,\beta,\theta) &= \frac{\frac{\theta^2}{\alpha\theta^2+\beta}\left[\alpha\theta+\beta x\right]e^{-\theta x}}{\left[\frac{\theta^2+\beta+\beta\theta z}{\theta^2+\beta}\right]e^{-\theta z}} \\
&= \frac{\theta^2(\theta+\beta x)e^{-(x-z)\theta}}{\theta^2+\beta+\beta\theta z}; x > z
\end{aligned}
$$

further mean excess loss Eq.4.35 is is obtained using Eq.1.11 as;

$$
m(x) = \frac{1}{(\alpha\theta^2+\beta+\beta\theta x)e^{-\theta x}}\int_x^{\infty}(\alpha\theta^2+\beta+\beta\theta t)e^{-\theta t}dt
$$

Taking the part $\int_x^\infty(\alpha\theta^2+\beta+\beta\theta t)e^{-\theta t}dt$ we have the following;

$$
\begin{aligned}
u &= (\alpha\theta^2+\beta+\beta\theta t) \Longrightarrow du = \beta\theta dt \\
dv &= e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta} \\
&= -(\alpha\theta^2+\beta+\beta\theta t)\frac{e^{-\theta t}}{\theta} + \beta\int_x^\infty e^{-\theta t}dt \\
&= -(\alpha\theta^2+\beta+\beta\theta t)\frac{e^{-\theta t}}{\theta} - \frac{\beta}{\theta}e^{-\theta t} \\
&= -\frac{e^{-\theta t}}{\theta}\Big[\alpha\theta^2+2\beta+\beta\theta t\Big]_x^\infty \\
&= \frac{e^{-\theta x}}{\theta}\Big[\alpha\theta^2+2\beta+\beta\theta x\Big] \\
&= \frac{1}{(\alpha\theta^2+\beta+\beta\theta x)e^{-\theta x}} * \frac{e^{-\theta x}}{\theta}\Big[\alpha\theta^2+2\beta+\beta\theta x\Big] \\
&= \frac{\alpha\theta^2+2\beta+\beta\theta x}{\theta\big[\alpha\theta^2+\beta+\beta\theta x\big]}
\end{aligned}
$$

Using the relation Eq.1.12, equilibrium distribution $f_e(x;\alpha,\beta,\theta)$ Eq.4.36 is given as;

$$
\begin{aligned}
f_e(x;\alpha,\beta,\theta) &= \frac{(\alpha\theta^2+\beta+\beta\theta x)e^{-\theta x}}{\alpha\theta^2+\beta}\frac{\theta(\alpha\theta^2+\beta)}{\alpha\theta^2+2\beta} \\
&= \frac{\theta(\alpha\theta^2+\beta+\beta\theta x)e^{-\theta x}}{\alpha\theta^2+2\beta}
\end{aligned}
$$

By definition Eq.1.13, survival function based on equilibrium distribution $S_e(x;\alpha,\beta,\theta)$ Eq.1.13 is obtained as;

$$
\begin{aligned}
S(t;\alpha,\beta,\theta)dt &= \frac{(\alpha\theta^2+2\beta+\beta\theta x)e^{-\theta x}}{\theta(\alpha\theta^2+\beta)} \\
S_e(x;\alpha,\beta,\theta) &= \frac{(\alpha\theta^2+2\beta+\beta\theta x)e^{-\theta x}}{\theta(\alpha\theta^2+\beta)}\frac{\theta(\alpha\theta^2+\beta)}{\alpha\theta^2+2\beta} \\
&= \frac{(\alpha\theta^2+2\beta+\beta\theta x)e^{-\theta x}}{\alpha\theta^2+2\beta}
\end{aligned}
$$

further hazard function based on equilibrium distribution $h_e(x;\alpha,\beta,\theta)$ Eq.4.38 is obtained using Eq.1.14 as;

$$
\begin{aligned}
h_e(x;\alpha,\beta,\theta) &= \frac{\theta(\alpha\theta^2+\beta+\beta\theta x)e^{-\theta x}}{\alpha\theta^2+2\beta}\frac{\alpha\theta^2+2\beta}{(\alpha\theta^2+2\beta+\beta\theta x)e^{-\theta x}} \\
&= ss\frac{\theta(\alpha\theta^2+\beta+\beta\theta x)}{\alpha\theta^2+2\beta+\beta\theta x}
\end{aligned}
$$

□

5 TWO COMPONENT FINITE GAMMA MIXTURE (Case of Rama distribution)

5.1 Introduction

In this chapter we consider a two component finite gamma mixture a case of Rama distribution. We shall construct and derive statistical properties of Rama distribution and its generalizations. The mixed distribution is expressed in terms of-pdf and Cdf. We later derive statistical properties such as reliability analysis measures, equilibrium properties, moments about both the origin and mean.

5.2 One parameter Rama distribution

5.2.1 Construction of one parameter Rama distribution

Proposition 5.2.1. *One parameter Rama distribution is constructed as a finite mixed distribution of Gamma$(1,\theta)$ and Gamma$(4,\theta)$ with mixing proportion as $\omega = \frac{\theta^3}{\theta^3+6}$. The pdf and Cdf of one parameter Rama distribution are defined as;*

$$f(x;\theta) = \frac{\theta^4}{\theta^3+6}\left[1+x^3\right]e^{-\theta x}; x>0, \theta>0 \tag{5.1}$$

$$F(x;\theta) = 1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}; x>0, \theta>0 \tag{5.2}$$

Proof. Applying the concept of finite mixed distribution Eq.1.1 pdf of one parameter Rama distribution is constructed as;

$$
\begin{aligned}
f(x;\theta) &= \frac{\theta^3}{\theta^3+6}\left[\theta e^{-\theta x}\right] + \frac{6}{\theta^3+6}\left[\frac{\theta^4 e^{-\theta x}x^3}{\Gamma 4}\right] \\
&= \frac{\theta^4 e^{-\theta x}}{\theta^3+6} + \frac{6\theta^4 e^{-\theta x}x^3}{6(\theta^3+6)} \\
&= \frac{\theta^4 e^{-\theta x}}{\theta^3+6} + \frac{\theta^4 e^{-\theta x}x^3}{\theta^3+6} \\
&= \frac{\theta^4}{\theta^3+6}\left[1+x^3\right]e^{-\theta x}; x>0, \theta>0
\end{aligned}
$$

further Cdf Eq.5.2 is obtained as;

$$
\begin{aligned}
F(x;\theta) &= \frac{\theta^4}{\theta^3+6}\int_0^\infty (1+x^3)e^{-\theta x}dx \\
F(x;\theta) &= \frac{\theta^4}{\theta^3+6} I_1 \\
I_1 &= \int_0^\infty (1+x^3)e^{-\theta x}dx \\
u &= (1+x^3) \implies du = 3x^2 \\
dv &= e^{-\theta x} \implies v = \frac{-e^{-\theta x}}{\theta x} \\
I_1 &= -(1+x^3)\frac{-e^{-\theta x}}{\theta x} - \frac{3}{\theta}\int_0^\infty -x^2 e^{-\theta x}dx \\
&= -(1+x^3)\frac{-e^{-\theta x}}{\theta x} - \frac{3}{\theta} I_2 \\
I_2 &= \int_0^\infty -x^2 e^{-\theta x}dx \\
u &= -x^2 \implies du = -2x
\end{aligned}
$$

$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$

$$I_2 = \frac{x^2 e^{-\theta x}}{\theta} - \frac{2}{\theta}\int_0^{\infty} xe^{-\theta x}dx$$

$$I_2 = \frac{x^2 e^{-\theta x}}{\theta} - \frac{2}{\theta}I_3$$

$$I_3 = \int_0^{\infty} xe^{-\theta x}dx$$

$$u = x \Longrightarrow du = dx$$

$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$

$$I_3 = \frac{-xe^{-\theta x}}{\theta} - \frac{1}{\theta}\int_0^{\infty} -e^{-\theta x}dx$$

$$I_3 = \frac{-xe^{-\theta x}}{\theta} - \frac{1}{\theta}I_4$$

$$I_4 = \int_0^{\infty} -e^{-\theta x}dx$$

Let $u = -\theta x$, we have;

$$u = -\theta x \Longrightarrow du = -\theta dx \Longrightarrow \frac{du}{-\theta} = dx$$

$$I_4 = \int_0^{\infty} -e^u \frac{du}{-\theta} \Longrightarrow \frac{1}{\theta}\int_0^{\infty} e^u du = \frac{1}{\theta}e^{-\theta x}$$

From I_1, I_2,I_3 and I_4 we have the following;

$$= -(1+x^3)\frac{e^{-\theta x}}{\theta} - \frac{3}{\theta}\left[\frac{x^2 e^{-\theta x}}{\theta} - \frac{2}{\theta}\left(\frac{-xe^{-\theta x}}{\theta} - \frac{1}{\theta}(\frac{e^{-\theta x}}{\theta})\right)\right]$$

$$I_1 = -(\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x)\frac{e^{-\theta x}}{\theta}$$

$$F(x;\theta) = 1 - \left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}; x>0, \theta>0$$

□

5.2.2 Reliability Analysis Measures

Proposition 5.2.2. *Survival function denoted by $S(x;\theta)$ and hazard function denoted by $h(x;\theta)$ of one parameter Rama distribution Eq.5.1 are stated as;*

$$S(x;\theta) = \frac{\left[\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x\right]e^{-\theta x}}{\theta^3+6}; x>0, \theta>0 \tag{5.3}$$

$$h(x;\theta) = \frac{\theta^4(1+x^3)}{\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x}; x>0, \theta>0 \tag{5.4}$$

Proof. Using the relation Eq.1.8 survival function $S(x;\theta)$ Eq.5.3 is obtained as;

$$S(x;\theta) = 1 - \left[1 - \left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}\right]$$

$$= \frac{\left[\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x\right]e^{-\theta x}}{\theta^3+6}; x>0, \theta>0$$

The hazard function $h(x;\theta)$ Eq.5.4 is obtained using Eq.1.9 as;

$$h(x;\theta) = \frac{\theta^4}{\theta^3+6}\left[1+x^3\right]e^{-\theta x}\frac{\theta^3+6}{\left[\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x\right]e^{-\theta x}}$$
$$= \frac{\theta^4(1+x^3)}{\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x};x>0,\theta>0$$

□

5.2.3 Moments and related measures

Proposition 5.2.3. *The r^{th} moments of a one parameter Rama distribution are derived using both moment generating function (mgf) and methods of moments are stated as;*

$$\mu_r^1 * = \frac{r!\left[\theta^3+(r+1)(r+2)(r+3)\right]}{\theta^r(\theta^3+6)};r=1,2,3,\cdots \quad (5.5)$$

Proof. By definition of r^{th} moments Eq.1.16, moments of one parameter Rama distribution Eq.5.1 are obtained as;

$$E(X^r) = \frac{\theta^4}{\theta^3+6}\left[\int_0^\infty x^r e^{-\theta x}dx + \int_0^\infty x^{r+3}e^{r+3}e^{-\theta x}dx\right]$$
$$= \frac{\theta^4}{\theta^3+6}\left[\frac{\Gamma(r+1)}{\theta^{r+1}}+\frac{\Gamma(r+4)}{\theta^{r+4}}\right] \Longrightarrow \frac{\theta^4 r\Gamma r}{\theta^r(\theta^3+6)}\left[\frac{1}{\theta}+\frac{(r+3)(r+2}{\theta^4}\right.$$
$$= \frac{r!\left[\theta^3+(r+1)(r+2)(r+3)\right]}{\theta^r(\theta^3+6)};r=1,2,3,\cdots$$

$$
\begin{aligned}
m_x(t) &= \frac{\theta^4}{\theta^3+6}\left[\int_0^{\infty} e^{-(\theta-t)x}(1+x^3)dx\right] \\
&= \frac{\theta^4}{\theta^3+6}\left[\frac{1}{(\theta-t)}+\frac{6}{(\theta-t)^4}\right] \\
&= \frac{\theta^4}{\theta^3+6}\left[\frac{1}{\theta}\sum_{k=0}^{\infty}\Big(\frac{t}{\theta}\Big)^k+\frac{6}{\theta^4}\sum_{k=0}^{\infty}\binom{k+3}{k}\Big(\frac{t}{\theta}\Big)^k\right] \\
&= \sum_{k=0}^{\infty}\frac{\theta^3+(k+1)(k+2)(k+3)}{\theta^3+6}\Big(\frac{t}{\theta}\Big)^k
\end{aligned}
$$

The none centralized moments are derived as a coefficient of $\frac{t^r}{r!}$ from $m_x(t)$ as;

$$
\mu_r^1 = \frac{r!\left[\theta^3+(r+1)(r+2)(r+3)\right]}{\theta^r(\theta^3+6)}; r = 1,2,3,\cdots
$$

□

For $r = 1,2,3$ and 4 in Eq.5.5 we obtain the four moments about the origin of one parameter Rama distribution Eq.5.1 as;

$$
\mu_1^1 = \frac{\theta^3+24}{\theta(\theta^3+6)}, \quad \mu_2^1 = \frac{2(\theta^3+60)}{\theta^2(\theta^3+6)}
$$

$$
\mu_3^1 = \frac{6(\theta^3+120)}{\theta^3(\theta^3+6)}, \quad \mu_4^1 = \frac{24(\theta^3+210)}{\theta^4(\theta^3+6)}
$$

The centralized moments of Rama distribution Eq.5.1 are;

$$\mu_1 = \mu_1$$
$$\mu_2 = \mu_2^1 - (\mu_1^1)^2$$
$$\mu_2 = \frac{2(\theta^3+60)}{\theta^2(\theta^3+6)} - \left[\frac{\theta^3+24}{\theta(\theta^3+6)}\right]^2 \Longrightarrow \frac{\theta^6+84\theta^3+144}{\theta^2(\theta^3+6)^2}$$
$$\mu_3 = \mu_3^1 - 3\mu_2^1\mu_1^1 + 2(\mu_1^1)^3$$
$$\mu_3 = \frac{6(\theta^3+120)}{\theta^3(\theta^3+6)} - 3\left[\frac{2(\theta^3+60)}{\theta^2(\theta^3+6)}\frac{\theta^3+24}{\theta(\theta^3+6)}\right] + 2\left[\frac{\theta^3+24}{\theta(\theta^3+6)}\right]^3 \Longrightarrow 2\theta^9$$
$$\mu_4 = \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2$$

$$= \frac{24(\theta^3+210)}{\theta^4(\theta^3+6)} - 4\left[\frac{6(\theta^3+120)}{\theta^3(\theta^3+6)}\frac{\theta^3+24}{\theta(\theta^3+6)}\right] + 6\left[\frac{\theta^3+24}{\theta(\theta^3+6)}\frac{2(\theta^3+60}{\theta^2(\theta^3+}\right.$$
$$\mu_4 = \frac{9\theta^{12}+2808\theta^9+20736\theta^6+93312\theta^3+93312}{\theta^4(\theta^3+6)^4}$$

Proposition 5.2.4. *Other related measures of a one parameter Rama distribution Eq.5.1 such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) are stated as;*

$$C.v = \frac{\sqrt{\theta^6+84\theta^3+144}}{\theta^3+24} \tag{5.6}$$
$$\upsilon_1 = \frac{2\theta^9+396\theta^6+648\theta^3+1728}{(\theta^6+84\theta^3+144)^{\frac{3}{2}}} \tag{5.7}$$
$$\upsilon_2 = \frac{9\theta^{12}+2808\theta^9+20736\theta^6+93312\theta^3+93312}{(\theta^6+84\theta^3+144)^2} \tag{5.8}$$
$$\upsilon_3 = \frac{\theta^6+84\theta^3+144}{\theta(\theta^3+6)(\theta^3+24)} \tag{5.9}$$

Proof. variation coefficient Eq.5.6 is obtained as;

$$
\begin{aligned}
C.v &= \frac{\sigma}{\mu_1^1} \\
&= \frac{\sqrt{\theta^6 + 84\theta^3 + 144}}{\theta(\theta^3 + 6)} * \frac{\theta(\theta^3 + 6)}{\theta^3 + 24} \Longrightarrow \frac{\sqrt{\theta^6 + 84\theta^3 + 144}}{\theta^3 + 24}
\end{aligned}
$$

Skewness coefficient Eq.5.7is obtained as;

$$
\begin{aligned}
\upsilon_1 &= \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} \\
&= \frac{2\theta^9 + 396\theta^6 + 648\theta^3 + 1728}{\theta^3(\theta^3 + 6)^3} \left[\frac{\theta^2(\theta^3 + 6)^2}{\theta^6 + 84\theta^3 + 144} \right]^{\frac{3}{2}} \Longrightarrow \frac{2\theta^9 + 396\theta^6 + 64}{(\theta^6 + 84\theta^3 +}
\end{aligned}
$$

further kurtosis coefficient Eq.5.8 is obtained as;

$$
\begin{aligned}
\upsilon_2 &= \frac{\mu_4}{(\mu_2)^2} \\
&= \frac{9\theta^{12} + 2808\theta^9 + 20736\theta^6 + 93312\theta^3 + 93312}{\theta^4(\theta^3 + 6)^4} \left[\frac{\theta^2(\theta^3 + 6)^2}{\theta^6 + 84\theta^3 + 144} \right]^2 \\
&= \frac{9\theta^{12} + 2808\theta^9 + 20736\theta^6 + 93312\theta^3 + 93312}{(\theta^6 + 84\theta^3 + 144)^2}
\end{aligned}
$$

Lastly, index of dispersion Eq.5.9 is obtained as;

$$
\begin{aligned}
\upsilon_3 &= \frac{\sigma^2}{\mu_1^1} \\
&= \frac{\theta^6 + 84\theta^3 + 144}{\theta^2(\theta^3 + 6)^2} \frac{\theta(\theta^3 + 6)}{\theta^3 + 24} \Longrightarrow \frac{\theta^6 + 84\theta^3 + 144}{\theta(\theta^3 + 6)(\theta^3 + 24)}
\end{aligned}
$$

□

5.2.4 Excess Loss Distribution

Proposition 5.2.5. *In this section we state, probability density function of excess loss function $f_l(x;\theta)$, mean residual lifetime (MRL), equilibrium distribution $f_e(x;\theta)$, survival function of the equilibrium distribution $S_e(x;\theta)$ and hazard function of the equilibrium distribution $h_e(x;\theta)$ of one parameter Rama distribution Eq.5.1 as;*

$$f_l(x;\theta) = \frac{\theta^4(1+x^3)e^{-(x-z)\theta}}{\theta^3+6+\theta^3z^3+3\theta^2z^2+6\theta z}y \tag{5.10}$$

$$m(x) = \frac{\theta^3+24+18\theta x+6\theta^2x^2+\theta^3x^3}{\theta(\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x)} \tag{5.11}$$

$$f_e(x;\theta) = \frac{\theta(\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x)e^{-\theta x}}{\theta^3+24} \tag{5.12}$$

$$S_e(x;\theta) = \frac{(\theta^3+24+18\theta x+6\theta^2x^2+\theta^3x^3)e^{-\theta x}}{\theta^3+24} \tag{5.13}$$

$$h_e(x;\theta) = \frac{\theta(\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x)}{\theta^3+24+18\theta x+6\theta^2x^2+\theta^3x^3} \tag{5.14}$$

Proof. By definition Eq.1.10 , the pdf of excess loss distribution Eq.5.10 is obtained as;

$$\begin{aligned} f_l(x;\theta) &= \frac{\theta^4}{\theta^3+6}\left[1+x^3\right]e^{-\theta x}\frac{(\theta^3+6+\theta^3z^3+3\theta^2z^2+6\theta z)e^{-\theta z}}{\theta^3+6} \\ &= \frac{\theta^4(1+x^3)e^{-(x-z)\theta}}{\theta^3+6+\theta^3z^3+3\theta^2z^2+6\theta z}; x > z \end{aligned}$$

further applying the relation Eq.1.11 $m(x)$ Eq.5.11 is obtained as;

$$m(x) = \frac{\theta^3+6}{(\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x)e^{-\theta x}}\int_x^\infty \frac{(\theta^3+6+\theta^3t^3+3\theta^2t^2+6\theta t)e}{\theta^3+6}$$

$$= \frac{1}{(\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x)e^{-\theta x}}\int_x^\infty (\theta^3+6+\theta^3t^3+3\theta^2t^2+6\theta t)e$$

$$I_1 = \int_x^\infty (\theta^3+6+\theta^3t^3+3\theta^2t^2+6\theta t)e^{-\theta t}dt$$

$$u = (\theta^3+6+\theta^3t^3+3\theta^2t^2+6\theta t) \implies du = (6\theta^2 t+3\theta^3t^2+6\theta)dt$$

$$dv = e^{-\theta t} \implies v = \frac{-e^{-\theta t}}{\theta}$$

$$I_1 = -(\theta^3+6+\theta^3t^3+3\theta^2t^2+6\theta t)\frac{e^{-\theta t}}{\theta} + \int_x^\infty (6\theta t+3\theta^2t^2+6)e^{-\theta t}dt$$

$$I_2 = \int_x^\infty (6\theta t+3\theta^2t^2+6)e^{-\theta t}dt$$

$$u = (6\theta t+3\theta^2t^2+6) \implies du = 6\theta+6\theta^2 t$$

$$dv = e^{-\theta t} \implies v = \frac{-e^{-\theta t}}{\theta}$$

$$I_2 = -(6\theta t+3\theta^2t^2+6)\frac{e^{-\theta t}}{\theta} + \int_x^\infty (6+6\theta t)e^{-\theta t}dt$$

$$I_3 = \int_x^\infty (6+6\theta t)e^{-\theta t}dt$$

$$u = (6+6\theta t) \implies du = 6\theta dt$$

$$dv = e^{-\theta t} \implies v = \frac{-e^{-\theta t}}{\theta}$$

$$I_3 = -(6+6\theta t)\frac{e^{-\theta t}}{\theta} + 6\int_x^\infty e^{-\theta t}dt$$
$$I_4 = \int_x^\infty e^{-\theta t}dt \Longrightarrow \frac{-e^{-\theta t}}{\theta}$$

From I_1,I_2,I_3 and I_4 then;

$$I_1 = \frac{-e^{-\theta t}}{\theta}\left[\theta^3+24+18\theta t+6\theta^2t^2+6\theta^2t^2+\theta^3t^3\right]_x^\infty \Longrightarrow \left[\theta^3+24+18\right.$$
$$= \frac{1}{(\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x)e^{-\theta x}}\left[\theta^3+24+18\theta x+6\theta^2x^2+\theta^3x^3\right.$$
$$= \frac{\theta^3+24+18\theta x+6\theta^2x^2+\theta^3x^3}{\theta(\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x)}$$

$f_e(x;\theta)$ in Eq.5.12 is obtained as;

$$f_e(x;\theta) = \frac{\left[\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x\right]e^{-\theta x}}{\theta^3+6}\frac{\theta(\theta^3+6)}{\theta^3+24} \Longrightarrow \frac{\theta(\theta^3+6+}{}$$

By definition Eq.1.13 $S_e(x;\theta)$ Eq.5.13 is obtained as;

$$\int_x^\infty S(t;\theta)dt = \frac{(\theta^3+24+18\theta x+6\theta^2x^2+\theta^3x^3)e^{-\theta x}}{\theta(\theta^3+6)}$$
$$= \frac{(\theta^3+24+18\theta x+6\theta^2x^2+\theta^3x^3)e^{-\theta x}}{\theta(\theta^3+6)}\frac{\theta(\theta^3+6)}{\theta^3+24}$$
$$S_e(x;\theta) = \frac{(\theta^3+24+18\theta x+6\theta^2x^2+\theta^3x^3)e^{-\theta x}}{\theta^3+24}$$

further hazard function based on equilibrium distribution Eq.5.14 is given as;

$$h_e(x;\theta) = \frac{\theta(\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x)e^{-\theta x}}{\theta^3+24}\frac{\theta^3+24}{(\theta^3+24+18\theta x+6\theta^2x^2}$$
$$= \frac{\theta(\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x)}{\theta^3+24+18\theta x+6\theta^2x^2+\theta^3x^3}$$

□

Remark 5.2.6. *It can be verified that hazard function of the equilibrium distribution Eq.5.14 of one parameter Rama distribution Eq.5.1 is the inverse of mean residual lifetime Eq.5.11.*

5.3 Two parameter Rama distribution

5.3.1 Construction of a two parameter Rama distribution

Proposition 5.3.1. *Let $\omega = \frac{\alpha\theta^3}{\alpha\theta^3+6}$ be mixing proportion, a generalized two parameter Rama distribution (AG2PRD) is a finite mixed distribution of Gamma$(1,\theta)$ and Gamma$(4,\theta)$. The pdf and Cdf of a generalized two parameter Rama distribution are defined as;*

$$f(x;\alpha,\theta) = \frac{\theta^4}{\alpha\theta^3+6}\left[\alpha + x^3\right]e^{-\theta x}; x>0, \alpha>0, \theta>0 \qquad (5.15)$$

$$F(x;\alpha,\theta) = 1 - \left[1 + \frac{\theta^3x^3 + 3\theta^2x^2 + 6\theta x}{\alpha\theta^3+6}\right]e^{-\theta x}; x>0, \alpha>0, \theta>0 \qquad (5.16)$$

Proof. Using finite mixture concept in Eq.1.1, the pdf Eq.5.15 is obtained as;

$$\begin{aligned} f(x;\alpha,\theta) &= \frac{\alpha\theta^3}{\alpha\theta^3+6}\left[\theta e^{-\theta x}\right] + \frac{6}{\alpha\theta^3+6}\left[\frac{\theta^4 e^{-\theta x}x^3}{\Gamma 4}\right] \\ &= \frac{\alpha\theta^4 e^{-\theta x}}{\alpha\theta^3+6} + \frac{6\theta^4 e^{-\theta x}x^3}{6(\alpha\theta^3+6)} \\ &= \frac{\theta^4}{\alpha\theta^3+6}\left[\alpha + x^3\right]e^{-\theta x}; x>0, \alpha>0, \theta>0 \end{aligned}$$

Remark 5.3.2. *Putting $\alpha = 1$ a generalized two parameter Rama distribution Eq.5.15 reduces to a one parameter Rama distribu-*

tion Eq.5.1. This shows that a generalized two parameter Rama distribution is flexible over one parameter Rama distribution.

further Cdf Eq.5.16 is obtained as;

$$F(x;\alpha,\theta) = \frac{\theta^4}{\alpha\theta^3+6}\int_0^\infty (\alpha+x^3)e^{-\theta x}dx$$
$$F(x;\alpha,\theta) = \frac{\theta^4}{\alpha\theta^3+6}I_1$$
$$I_1 = \int_0^\infty (\alpha+x^3)e^{-\theta x}dx$$
$$u = (\alpha+x^3) \Longrightarrow du = 3x^2dx$$
$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$
$$I_1 = -(\alpha+x^3)\frac{e^{-\theta x}}{\theta} - \frac{3}{\theta}\int_0^\infty -x^2e^{-\theta x}dx$$
$$I_1 = -(\alpha+x^3)\frac{e^{-\theta x}}{\theta} - \frac{3}{\theta}I_2$$
$$I_2 = \int_0^\infty -x^2e^{-\theta x}dx$$
$$u = -x^2 \Longrightarrow du = -2xdx$$
$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$
$$I_2 = \frac{x^2e^{-\theta x}}{\theta} - \frac{2}{\theta}\int_0^\infty xe^{-\theta x}dx$$
$$I_2 = \frac{x^2e^{-\theta x}}{\theta} - \frac{2}{\theta}I_3$$
$$I_3 = \int_0^\infty xe^{-\theta x}dx$$
$$u = x \Longrightarrow du = dx$$

$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$
$$I_3 = \frac{-xe^{-\theta x}}{\theta} - \frac{1}{\theta}\int_0^\infty -e^{-\theta x}dx$$
$$I_3 = \frac{-xe^{-\theta x}}{\theta} - \frac{1}{\theta} * I_4$$
$$I_4 = \int_0^\infty -e^{-\theta x}dx$$
$$u = -\theta x \Longrightarrow du = -\theta dx \Longrightarrow \frac{du}{-\theta} = dx$$
$$I_4 = \frac{1}{\theta}\int_0^\infty -e^u \frac{du}{-\theta} \Longrightarrow \frac{1}{\theta^2}\int_0^\infty e^u du \Longrightarrow \frac{1}{\theta^2}e^u \Longrightarrow \frac{e^{-\theta x}}{\theta^2}$$

From I_1,I_2,I_3 and I_4 we have;

$$I_1 = -(\alpha + x^3)\frac{e^{-\theta x}}{\theta} - \frac{3}{\theta}\left[\frac{x^2 e^{-\theta x}}{\theta} - \frac{2}{\theta}\left(\frac{-xe^{-\theta x}}{\theta} - \frac{e^{-\theta x}}{\theta^2}\right)\right]$$
$$I_1 = -\left[(\alpha + x^3)\theta^3 + 3\theta^2 x^2 + 6\theta x + 6\right]\frac{e^{-\theta x}}{\theta^4}$$
$$= 1 - \left[1 + \frac{\theta^3 x^3 + 3\theta^2 x^2 + 6\theta x}{\alpha\theta^3 + 6}\right]e^{-\theta x}; x > 0, \alpha > 0, \theta > 0$$

□

5.3.2 Reliability Analysis

Proposition 5.3.3. *Survival function denoted by* $S(x;\alpha,\theta)$ *and hazard function denoted by* $h(x;\alpha,\theta)$ *of a generalized two pa-*

rameter Rama distribution Eq.5.15 are stated as;

$$S(x;\alpha,\theta) = \left[\frac{\alpha\theta^3 + 6 + \theta^3x^3 + 6\theta x}{\alpha\theta^3 + 6}\right] e^{-\theta x}; x > 0, \alpha > 0, \theta \tag{5.17}$$

$$h(x;\alpha,\theta) = \frac{\theta^4(\alpha + x^3)}{\alpha\theta^3 + 6 + \theta^3x^3 + 3\theta^2x^2 + 6\theta x}; x > 0, \alpha > 0, \theta \tag{5.18}$$

`Proof.` survival function $S(x;\alpha,\theta)$ in Eq5.17 is obtained using relation Eq.1.8 as;

$$\begin{aligned} S(x;\alpha,\theta) &= 1 - \left[1 - \left[1 + \frac{\theta^3x^3 + 3\theta^2x^2 + 6\theta x}{\alpha\theta^3 + 6}\right] e^{-\theta x}\right] \\ &= \left[\frac{\alpha\theta^3 + 6 + \theta^3x^3 + 6\theta x}{\alpha\theta^3 + 6}\right] e^{-\theta x}; x > 0, \alpha > 0, \theta \end{aligned}$$

moreover, hazard function $h(x;\alpha,\theta)$ Eq.5.18 is obtained using relation Eq.1.9 as;

$$\begin{aligned} h(x;\alpha,\theta) &= \frac{\theta^4}{\alpha\theta^3 + 6}\left[\alpha + x^3\right] e^{-\theta x} \left[\frac{\alpha\theta^3 + 6}{\alpha\theta^3 + 6 + \theta^3x^3 + 3\theta^2x^2 + 6\theta x}\right] \\ &= \frac{\theta^4(\alpha + x^3)}{\alpha\theta^3 + 6 + \theta^3x^3 + 3\theta^2x^2 + 6\theta x}; x > 0, \alpha > 0, \theta \end{aligned}$$

□

5.3.3 Moments and related measures

Proposition 5.3.4. *The r^{th} moments of a generalized two parameter Rama distribution Eq.5.15 are derived using both method of moments and moment generating function as;*

$$\mu_r^1 * = \frac{r!\big(\alpha\theta^3 + (r+1)(r+2)(r+3)\big)}{\theta^r(\alpha\theta^3 + 6)}; \quad r = 1, 2, 3, \cdots \tag{5.19}$$

Proof. By definition of r^{th} moments Eq.1.16, moments of AG2PRD are given as;

$$
\begin{aligned}
E(X^r) &= \frac{\theta^4}{\alpha\theta^3+6}\left[\int_0^\infty x^r(\alpha+x^3)e^{-\theta x}dx\right] \\
&= \frac{\theta^4}{\alpha\theta^3+6}\left[\alpha\int_0^\infty x^r e^{-\theta x}dx+\int_0^\infty x^{r+3}e^{-\theta x}dx\right] \\
&= \frac{\theta^4}{\alpha\theta^3+6}\left[\frac{\alpha\Gamma(r+1)}{\theta^{r+1}}+\frac{\Gamma(r+4)}{\theta^{r+4}}\right] \\
&= \frac{r!\big(\alpha\theta^3+(r+1)(r+2)(r+3)\big)}{\theta^r(\alpha\theta^3+6)}; \quad r=1,2,3,\cdots
\end{aligned}
$$

Similarly, by definition of mgf Eq.1.17 the mgf of AG2PRD is obtained as;

$$
\begin{aligned}
m_x(t) &= \frac{\theta^4}{\alpha\theta^3+6}\int_0^\infty e^{tx}(\alpha+x^3)e^{-\theta x}dx \\
&= \frac{\theta^4}{\alpha\theta^3+6}\left[\int_0^\infty e^{-(\theta-t)x}(\alpha+x^3)dx\right] \\
&= \frac{\theta^4}{\alpha\theta^3+6}\left[\frac{\alpha}{\theta-t}+\frac{6}{(\theta-t)^4}\right] \\
&= \frac{\theta^4}{\alpha\theta^3+6}\left[\frac{\alpha}{\theta}\sum_{k=0}^{\infty}\left(\frac{t}{\theta}\right)^k+\frac{6}{\theta^4}\sum_{k=0}^{\infty}\binom{k+3}{k}\left(\frac{t}{\theta}\right)^k\right] \\
&= \sum_{k=0}^{\infty}\frac{\alpha\theta^3+(k+1)(k+2)(k+3)}{\alpha\theta^3+6}\left(\frac{t}{\theta}\right)^k
\end{aligned}
$$

The r^{th} moments of a generalized two parameter Rama distribution are obtained as a coefficient of $\frac{t^r}{r!}$ of the moment generating function $m_x(t)$ as;

$$
\mu_r^1 = \frac{r!\big(\alpha\theta^3+(r+1)(r+2)(r+3)\big)}{\theta^r(\alpha\theta^3+6)}; \quad r=1,2,3,\cdots
$$

□

Putting $r = 1,2,3$ and 4 in Eq.5.19 we obtain the four none centralized moments of a generalized two parameter Rama distribution as;

$$\mu_1^1 = \frac{\alpha\theta^3 + 24}{\theta(\alpha\theta^3 + 6)}, \quad \mu_2^1 = \frac{2(\alpha\theta^3 + 60)}{\theta^2(\alpha\theta^3 + 6)}$$
$$\mu_3^1 = \frac{6(\alpha\theta^3 + 120)}{\theta^3(\alpha\theta^3 + 6)}, \quad \mu_4^1 = \frac{24(\alpha\theta^3 + 210)}{\theta^4(\alpha\theta^3 + 6)}$$

Moments about the mean are derived using the relationship between moments about the origin and mean as;

$$\mu_1^1 = \mu_1$$
$$\mu_2 = \mu_2^1 - (\mu_1^1)^2$$
$$\mu_2 = \frac{2(\alpha\theta^3 + 60)}{\theta^2(\alpha\theta^3 + 6)} - \left[\frac{\alpha\theta^3 + 24}{\theta(\alpha\theta^3 + 6)}\right]^2 \Longrightarrow \frac{\alpha^2\theta^6 + 84\alpha\theta^3 + 144}{\theta^2(\alpha\theta^3 + 6)^2}$$
$$\mu_3 = \mu_3^1 - 3\mu_2^1\mu_1^1 + 2(\mu_1^1)^3$$
$$= \frac{6(\alpha\theta^3 + 120)}{\theta^3(\alpha\theta^3 + 6)} - 3\left[\frac{2(\alpha\theta^3 + 60)}{\theta^2(\alpha\theta^3 + 6)}\frac{\alpha\theta^3 + 24}{\theta(\alpha\theta^3 + 6)}\right] + 2\left[\frac{\alpha\theta^3 + 24}{\theta(\alpha\theta^3 + 6)}\right]^3$$
$$\mu_3 = \frac{2\alpha^3\theta^9 + 396\alpha^2\theta^6 + 648\alpha\theta^3 + 1728}{\theta^3(\alpha\theta^3 + 6)^3}$$
$$\mu_4 = \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2$$
$$\mu_4 = \frac{9\alpha^4\theta^{12} + 2808\alpha^3\theta^9 + 2070\alpha^2\theta^6 + 93312\alpha\theta^3 + 93312}{\theta^4(\alpha\theta^3 + 6)^4}$$

Proposition 5.3.5. *Other related measures of a generalized two parameter Rama distribution Eq.5.15 such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3)*

are stated as;

$$C.v = \frac{\sqrt{\alpha^2\theta^6 + 84\alpha\theta^3 + 144}}{\alpha\theta^3 + 24} \tag{5.20}$$

$$\upsilon_1 = \frac{2\alpha^3\theta^9 + 396\alpha^2\theta^6 + 648\alpha\theta^3 + 1728}{(\alpha^2\theta^6 + 84\alpha\theta^3 + 144)^{\frac{3}{2}}} \tag{5.21}$$

$$\upsilon_2 = \frac{9\alpha^4\theta^{12} + 2808\alpha^3\theta^9 + 2070\alpha^2\theta^6 + 93312\alpha\theta^3 + 93312}{(\alpha^2\theta^6 + 84\alpha\theta^3 + 144)^2} \tag{5.22}$$

$$\upsilon_3 = \frac{\alpha^2\theta^6 + 84\alpha\theta^3 + 144}{\theta(\alpha\theta^3 + 6)(\alpha\theta^3 + 24)} \tag{5.23}$$

Proof. To begin with, variation coefficient Eq.5.20 is obtained as;

$$\begin{aligned} C.v &= \frac{\sigma}{\mu_1^1} \\ &= \frac{\sqrt{\alpha^2\theta^6 + 84\alpha\theta^3 + 144}}{\theta(\alpha\theta^3 + 6)} \frac{\theta(\alpha\theta^3 + 6)}{\alpha\theta^3 + 24} \Longrightarrow \frac{\sqrt{\alpha^2\theta^6 + 84\alpha\theta^3 + 144}}{\alpha\theta^3 + 24} \end{aligned}$$

Secondly, skewness coefficient Eq.5.21 is given as;

$$\begin{aligned} \upsilon_1 &= \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} \\ &= \frac{2\alpha^3\theta^9 + 396\alpha^2\theta^6 + 648\alpha\theta^3 + 1728}{\theta^3(\alpha\theta^3 + 6)^3} \left[\frac{\theta^2(\alpha\theta^3 + 6)^2}{\alpha^2\theta^6 + 84\alpha\theta^3 + 144}\right]^{\frac{3}{2}} \\ &= \frac{2\alpha^3\theta^9 + 396\alpha^2\theta^6 + 648\alpha\theta^3 + 1728}{(\alpha^2\theta^6 + 84\alpha\theta^3 + 144)^{\frac{3}{2}}} \end{aligned}$$

further, kurtosis coefficient Eq.5.22 is obtained as;

$$\upsilon_2 = \frac{9\alpha^4\theta^12+2808\alpha^3\theta^9+2070\alpha^2\theta^6+93312\alpha\theta^3+93312}{\theta^4(\alpha\theta^3+6)^4}\left[\frac{\theta^2(\alpha\theta}{\alpha^2\theta^6+84}\right.$$

$$= \frac{9\alpha^4\theta^12+2808\alpha^3\theta^9+2070\alpha^2\theta^6+93312\alpha\theta^3+93312}{(\alpha^2\theta^6+84\alpha\theta^3+144)^2}$$

lastly, index of dispersion Eq.5.23 is obtained as;

$$\upsilon_3 = \frac{\sigma^2}{\mu_1^1}$$

$$= \frac{\alpha^2\theta^6+84\alpha\theta^3+144}{\theta^2(\alpha\theta^3+6)^2}\frac{\theta(\alpha\theta^3+6)}{\alpha\theta^3+24} \Longrightarrow \frac{\alpha^2\theta^6+84\alpha\theta^3+144}{\theta(\alpha\theta^3+6)(\alpha\theta^3+24)}$$

□

5.3.4 Excess Loss Distribution

Proposition 5.3.6. *In this section we state, probability density function of excess loss function $f_l(x;\alpha,\theta)$, mean residual lifetime (MRL), equilibrium distribution $f_e(x;\alpha,\theta)$, survival function based on the equilibrium distribution $S_e(x;\alpha,\theta)$ and hazard function based on the equilibrium distribution $h_e(x;\alpha,\theta)$ of* *a generalized two parameter Rama distribution Eq.5.15 as;*

$$f_l(x;\alpha,\theta)=\frac{\theta^4(\alpha+x^3)e^{-(x-z)\theta}}{\alpha\theta^3+6+\theta^3z^3+3\theta^2z^2+6\theta z};x>z \tag{5.24}$$

$$m(x)=\frac{\alpha\theta^3+24+\theta^3x^3+6\theta^2x^2+18\theta x}{\theta(\alpha\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x)} \tag{5.25}$$

$$f_e(x;\alpha,\theta)=\frac{\theta(\alpha\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x)e^{-\theta x}}{\alpha\theta^3+24} \tag{5.26}$$

$$S_e(x;\alpha,\theta)=\frac{(\alpha\theta^3+24+\theta^3x^3+6\theta^2x^2+18\theta x)e^{-\theta x}}{\alpha\theta^3+24} \tag{5.27}$$

$$h_e(x;\alpha,\theta)=\frac{\theta(\alpha\theta^3+\theta^3x^3+3\theta^2x^2+6\theta x)}{\alpha\theta^3+24+\theta^3x^3+6\theta^2x^2+18\theta x} \tag{5.28}$$

Proof. By definition Eq.1.10, $f_l(x;\alpha,\theta)$ Eq.5.24 is obtained as;

$$\begin{aligned} f_l(x;\alpha,\theta) &= \frac{\theta^4}{\alpha\theta^3+6}\left[\alpha+x^3\right]e^{-\theta x}\frac{\alpha\theta^3+6}{(\alpha\theta^3+6+\theta^3z^3+3\theta^2z^2+6\theta z)e^{-\theta z}} \\ &= \frac{\theta^4(\alpha+x^3)e^{-(x-z)\theta}}{\alpha\theta^3+6+\theta^3z^3+3\theta^2z^2+6\theta z}; \quad x>z \end{aligned}$$

Similarly, by definition Eq.1.11 $m(x)$ in Eq.5.25 is obtained as;

$$m(x) = \frac{\alpha\theta^3+6}{(\alpha\theta^3+6+\theta^3x^3+3\theta^2x^2+6\theta x)e^{-\theta x}}\int_x^\infty \frac{(\alpha\theta^3+6+\theta^3t^3+3\theta^2t^2}{\alpha\theta^3+6}$$

$$= \frac{1}{(\alpha\theta^3+6+\theta^3+3\theta^2x^2+6\theta x)e^{-\theta x}}\int_x^\infty (\alpha\theta^3+6+\theta^3t^3+3\theta^2t^2+$$

$$I_1 = \int_x^\infty (\alpha\theta^3+6+\theta^3t^3+3\theta^2t^2+6\theta t)e^{-\theta t}dt$$

$$u = (\alpha\theta^3+6+\theta^3t^3+3\theta^2t^2+6\theta t) \Longrightarrow du = (3\theta^3t^2+6\theta^2t+6\theta)dt$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_1 = -(\alpha\theta^3+6+\theta^3t^3+3\theta^2t^2+6\theta t)\frac{e^{-\theta t}}{\theta} + \int_x^\infty (3\theta^2t^2+6\theta t+6)e^{-\theta}$$

$$I_1 = -(\alpha\theta^3+6+\theta^3t^3+3\theta^2t^2+6\theta t)\frac{e^{-\theta t}}{\theta} + I_2dt$$

$$I_2 = \int_x^\infty (3\theta^2t^2+6\theta t+6)e^{-\theta t}dt$$

$$u = (3\theta^2t^2+6\theta t+6) \Longrightarrow du = 6\theta^2t+6\theta$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_2 = -(3\theta^2t^2+6\theta t+6)\frac{e^{-\theta t}}{\theta} + \int_x^\infty (6\theta t+6)e^{-\theta t}dt$$

$$I_2 = -(3\theta^2t^2 + 6\theta t + 6)\frac{e^{-\theta t}}{\theta} + I_3 dt$$

$$I_3 = \int_x^\infty (6\theta t + 6)e^{-\theta t}dt$$

$$u = (6\theta t + 6) \Longrightarrow du = 6\theta dt$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_3 = -((6\theta t + 6)\frac{e^{-\theta t}}{\theta} + 6\int_x^\infty e^{-\theta t}dt$$

$$I_3 = -(6\theta t + 6)\frac{e^{-\theta t}}{\theta} + 6 * I_4$$

$$I_4 = \int_x^\infty e^{-\theta t}dt$$

$$I_4 = \int_x^\infty e^{-\theta t}dt \Longrightarrow \frac{-e^{-\theta t}}{\theta}$$

From I_1,I_2,I_3 and I_4 we have the following;

$$I_1 = -\frac{e^{-\theta t}}{\theta}\left[\alpha\theta^3 + 24 + \theta^3t^3 + 6\theta^2t^2 + 18\theta t\right]_x^\infty \Longrightarrow \left[\alpha\theta^3 + 24 + \theta^3x^3 + 6\theta^2x^2\right.$$

$$= \frac{1}{(\alpha\theta^3 + 6 + \theta^3 + 3\theta^2x^2 + 6\theta x)e^{-\theta x}}\left[\alpha\theta^3 + 24 + \theta^3x^3 + 6\theta^2x^2 + 18\theta x\right]\frac{e^{-}}{\theta}$$

$$= \frac{\alpha\theta^3 + 24 + \theta^3x^3 + 6\theta^2x^2 + 18\theta x}{\theta(\alpha\theta^3 + 6 + \theta^3x^3 + 3\theta^2x^2 + 6\theta x)}$$

By definition of equilibrium distribution Eq.1.12, $f_e(x;\alpha,\theta)$ Eq.5.26 is obtained as;

$$f_e(x;\alpha,\theta) = \left[\frac{\alpha\theta^3 + 6 + \theta^3x^3 + 6\theta x}{\alpha\theta^3 + 6}\right]e^{-\theta x}\frac{\theta(\alpha\theta^3 + 6)}{\alpha\theta^3 + 24} \Longrightarrow \frac{(\alpha\theta^3 + 24 + \theta^3x^3}{\alpha\theta}$$

further, $S_e(x;\alpha,\theta)$ in Eq.5.27 is obtained as;

$$\int_x^{\infty} S(t;\alpha,\theta)dt = \left[\frac{(\alpha\theta^3+24+\theta^3x^3+6\theta^2x^2+18\theta x)e^{-\theta x}}{\theta(\alpha\theta^3+6)}\right]$$

$$S_e(x;\alpha,\theta) = \left[\frac{(\alpha\theta^3+24+\theta^3x^3+6\theta^2x^2+18\theta x)e^{-\theta x}}{\theta(\alpha\theta^3+6)}\right]\frac{\theta(\alpha\theta^3+6)}{\alpha\theta^3+24}$$

$$= \frac{(\alpha\theta^3+24+\theta^3x^3+6\theta^2x^2+18\theta x)e^{-\theta x}}{\alpha\theta^3+24}$$

by definition Eq.1.14, $h_e(x;\alpha,\theta)$ in Eq.5.28 is obtained as;

$$h_e(x;\alpha,\theta) = \frac{(\alpha\theta^3+24+\theta^3x^3+6\theta^2x^2+18\theta x)e^{-\theta x}}{\alpha\theta^3+24}\frac{\alpha\theta^3 +}{(\alpha\theta^3+24+\theta^3x^3+}$$

$$= \frac{\theta(\alpha\theta^3+\theta^3x^3+3\theta^2x^2+6\theta x)}{\alpha\theta^3+24+\theta^3x^3+6\theta^2x^2+18\theta x}$$

□

5.4 Three parameter Rama distribution

5.4.1 Construction of a three parameter Rama distribution

Proposition 5.4.1. *Let $\omega = \frac{\alpha\theta^3}{\alpha\theta^3+6\beta}$ be mixing probability, a generalized three parameter Rama distribution (AG3PRD) is a finite mixture of Gamma $(1,\theta)$ and Gamma $(4,\theta)$. The pdf and Cdf of AG3PRD are;*

$$f(x;\alpha,\beta,\theta) = \frac{\theta^4}{\alpha\theta^3+6\beta}\left[\alpha+\beta x\right]e^{-\theta x^3}; \quad x>0,\alpha>0,\beta>0,\theta>0 \tag{5.29}$$

$$F(x;\alpha,\beta,\theta) = 1-\left[1+\frac{\beta\theta^3x^3+3\beta\theta^2x^2+6\beta\theta x}{\alpha\theta^3+6\beta}\right]e^{-\theta x}; \quad x>0,\alpha>0, \tag{5.30}$$

Proof. By definition of a finite mixed distribution Eq.1.1 pdf Eq.5.29 is obtained as;

$$
\begin{aligned}
f(x;\alpha,\beta,\theta) &= \frac{\alpha\theta^3}{\alpha\theta^3+6\beta}\left[\theta e^{-\theta x}\right] + \frac{6\beta}{\alpha\theta^3+6\beta}\left[\frac{\theta^4 e^{-\theta x}x^3}{\Gamma 4}\right] \\
&= \frac{\alpha\theta^4 e^{-\theta x}}{\alpha\theta^3+6\beta} + \frac{6\beta\theta^4 e^{-\theta x}x^3}{(\alpha\theta^3+6\beta)\Gamma 4} \\
&= \frac{\alpha\theta^4 e^{-\theta x}}{\alpha\theta^3+6\beta} + \frac{\beta\theta^4 e^{-\theta x}x^3}{\alpha\theta^3+6\beta} \\
&= \frac{\theta^4}{\alpha\theta^3+6\beta}\left[\alpha+\beta x\right]e^{-\theta x}; \quad x>0,\alpha>0,\beta>0,\theta>0
\end{aligned}
$$

A generalized three parameter Rama distribution Eq.5.29 is a modification of a generalized two parameter Rama Eq.5.15 which was introduced by Umeth et al (2019).

Remark 5.4.2. *Putting $\alpha = 1$ and $\beta = 0$, a generalized three parameter Rama distribution Eq.5.29 reduces to exponential distribution Eq.2.17.*
Similarly, putting $\alpha = \beta = 1$ a generalized three parameter Rama distributionEq.5.29 reduces to one parameter Rama distribution Eq.5.1.
Putting $\beta = 1$ a generalized three parameter Rama reduces to a generalized two parameter Rama distribution Eq.5.15.

A generalized three parameter Rama distribution Eq.5.29 is a modification of a Rama distribution with two parameters Eq.5.15 introduced by Edith et al., (2019)

$$F(x;\alpha,\beta,\theta) = \frac{\theta^4}{\alpha\theta^3+6\beta}\int_0^\infty (\alpha+\beta x^3)e^{-\theta x}dx$$
$$F(x;\alpha,\beta,\theta) = \frac{\theta^4}{\alpha\theta^3+6\beta}I_1$$
$$I_1 = \int_0^\infty (\alpha+\beta x^3)e^{-\theta x}dx$$
$$u = (\alpha+\beta x) \Longrightarrow du = 3\beta x^2$$
$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$
$$I_1 = -(\alpha+\beta x^3)\frac{e^{-\theta x}}{\theta} - \frac{3\beta}{\theta}\int_0^\infty -x^2e^{-\theta x}dx$$
$$I_1 = -(\alpha+\beta x^3)\frac{e^{-\theta x}}{\theta} - \frac{3\beta}{\theta}I_2$$
$$I_2 = \int_0^\infty -x^2e^{-\theta x}dx$$
$$u = -x^2 \Longrightarrow du = -2xdx$$
$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$
$$I_2 = \frac{x^2e^{-\theta x}}{\theta} - \frac{2}{\theta}\int_0^\infty xe^{-\theta x}dx$$
$$I_2 = \frac{x^2e^{-\theta x}}{\theta} - \frac{2}{\theta}I_3$$
$$I_3 = \int_0^\infty xe^{-\theta x}dx$$
$$u = x \Longrightarrow du = dx$$
$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$
$$I_3 = \frac{-xe^{-\theta x}}{\theta} - \frac{1}{\theta}\int_0^\infty -e^{-\theta x}dx$$
$$I_4 = \int_0^\infty -e^{-\theta x}dx$$
$$u = -\theta x \Longrightarrow du = -\theta dx \Longrightarrow \frac{du}{-\theta} = dx$$

From I_1, I_2, I_3 and I_4 we have the following;

$$I_1 = -(\alpha+\beta x)\frac{e^{-\theta x}}{\theta} - \frac{3\beta x^2 e^{-\theta x}}{\theta^2} - \frac{6\beta x e^{-\theta x}}{\theta^3} - \frac{6\beta e^{-\theta x}}{\theta^4}$$
$$I_1 = \frac{-e^{-\theta x}}{\theta^4}\left[\alpha\theta^3 + \beta\theta^3 x^3 + 3\beta\theta^2 x^2 + 6\beta\theta x + 6\beta\right]$$
$$= 1 + -\frac{-e^{-\theta x}}{\theta^4}\left[\alpha\theta^3 + \beta\theta^3 x^3 + 3\beta\theta^2 x^2 + 6\beta\theta x + 6\beta\right]\frac{\theta^4}{\alpha\theta^3 + 6\beta}$$
$$= 1 - \left[1 + \frac{\beta\theta^3 x^3 + 3\beta\theta^2 x^2 + 6\beta\theta x}{\alpha\theta^3 + 6\beta}\right]e^{-\theta x}; \quad x > 0, \alpha > 0, \beta > 0, \theta > 0$$

□

5.4.2 Reliability Analysis

Proposition 5.4.3. *Survival function denoted by $S(x;\alpha,\beta,\theta)$ and hazard function denoted by $h(x;\alpha,\beta,\theta)$ of a generalized three parameter Rama Eq.5.29 are stated as;*

$$S(x;\alpha,\beta,\theta) = \left[\frac{\alpha\theta^3 + 6\beta + \beta\theta^3 x^3 + 3\beta\theta^2 x^2 + 6\beta\theta x}{\alpha\theta^3 + 6\beta}\right]e^{-\theta x}; x > 0, \alpha > 0, \beta > \tag{5.31}$$

$$h(x;\alpha,\beta,\theta) = \frac{\theta^4(\alpha + \beta x^3)}{\alpha\theta^3 + 6\beta + \beta\theta^3 x^3 + 3\beta\theta^2 x^2 + 6\beta\theta x}; x > 0, \alpha > 0, \beta > 0, \theta > \tag{5.32}$$

Proof. survival function $S(x;\alpha,\beta,\theta)$ in Eq.5.31 is obtained using Eq.1.8 as;

$$S(x;\alpha,\beta,\theta) = 1 - \left[1 - \left[1 + \frac{\beta\theta^3 x^3 + 3\beta\theta^2 x^2 + 6\beta\theta x}{\alpha\theta^3 + 6\beta}\right]e^{-\theta x}\right]$$
$$= \left[\frac{\alpha\theta^3 + 6\beta + \beta\theta^3 x^3 + 3\beta\theta^2 x^2 + 6\beta\theta x}{\alpha\theta^3 + 6\beta}\right]e^{-\theta x}; x > 0, \alpha > 0, \beta >$$

further, hazard function $h(x;\alpha,\beta,\theta)$ in Eq.5.32 is obtained using Eq.1.9 as;

$$
\begin{aligned}
h(x;\alpha,\beta,\theta) &= \frac{\theta^4}{\alpha\theta^3+6\beta}\Big[\alpha+\beta x\Big]e^{-\theta x}\frac{\alpha\theta^3+6\beta}{(\alpha\theta^3+6\beta+\beta\theta^3x^3+3\beta\theta^2x^2+6\ldots} \\
&= \frac{\theta^4(\alpha+\beta x^3)}{\alpha\theta^3+6\beta+\beta\theta^3x^3+3\beta\theta^2x^2+6\beta\theta x}; x>0, \alpha>0, \beta>0\ldots
\end{aligned}
$$

□

5.4.3 Moments and related measures

Proposition 5.4.4. *The r^{th} moments of a generalized three parameter Rama distribution Eq.5.29 are derived using both method of moments and moment generating function as;*

$$
\mu_r^1 * = \frac{r!\big[\alpha\theta^3+\beta(r+3)(r+2)(r+1)\big]}{\theta^r(\alpha\theta^3+6\beta)}; \quad r=1,2,3,\cdots \tag{5.33}
$$

Proof. By definition Eq.1.16, moments of AG3PRD are obtained as;

$$
\begin{aligned}
E(X^r) &= \frac{\theta^4}{\alpha\theta^3+6\beta}\left[\int_0^\infty x^r(\alpha+\beta x^3)e^{-\theta x}dx\right] \\
&= \frac{\theta^4}{\alpha\theta^3+6\beta}\left[\alpha\int_0^\infty x^r e^{-\theta x}dx+\beta\int_0^\infty x^{r+3}e^{-\theta x}dx\right] \\
&= \frac{\theta^4}{\alpha\theta^3+6\beta}\left[\frac{\alpha\Gamma(r+1)}{\theta^{r+1}}+\frac{\beta\Gamma(r+4)}{\theta^{r+4}}\right] \\
&= \frac{\theta^4}{\alpha\theta^3+6\beta}\left[\frac{\alpha r\Gamma r}{\theta^{r+1}}+\frac{\beta(r+3)(r+2)(r+1)r\Gamma r}{\theta^{r+4}}\right] \\
&= \frac{r!\big[\alpha\theta^3+\beta(r+3)(r+2)(r+1)\big]}{\theta^r(\alpha\theta^3+6\beta)}; \quad r=1,2,3,\cdots
\end{aligned}
$$

Similarly, by definition Eq.1.17, mgf of AG3PRD is obtained as;

$$\begin{aligned}
m_x(t) &= \frac{\theta^4}{\alpha\theta^3+6\beta}\left[\int_0^\infty e^{tx}(\alpha+\beta x^3)e^{-\theta x}dx\right] \\
&= \frac{\theta^4}{\alpha\theta^3+6\beta}\left[\int_0^\infty e^{-(\theta-t)x}(\alpha+\beta x^3)dx\right] \\
&= \frac{\theta^4}{\alpha\theta^3+6\beta}\left[\frac{\alpha}{\theta-t}+\frac{6\beta}{(\theta-t)^4}\right] \\
&= \frac{\theta^4}{\alpha\theta^3+6\beta}\left[\frac{\alpha}{\theta}\sum_{k=0}^{\infty}\Big(\frac{t}{\theta}\Big)^k+\frac{6\beta}{\theta^4}\sum_{k=0}^{\infty}\binom{k+3}{k}\Big(\frac{t}{\theta}\Big)^k\right] \\
&= \sum_{k=0}^{\infty}\frac{\alpha\theta^3+\beta(k+1)(k+2)(k+3)}{\alpha\theta^3+6\beta}\Big(\frac{t}{\theta}\Big)^k
\end{aligned}$$

The moments about the origin are obtained as a coefficient of $\frac{t^r}{r!}$ in the derived moment generating function $m_x(t)$ as;

$$\mu_r^1 = \frac{r!\left[\alpha\theta^3+\beta(r+1)(r+2)(r+3)\right]}{\theta^r(\alpha\theta^3+6\beta)}; \quad r = 1,2,3,\cdots$$

□

For $r = 1,2,3$ and 4 in Eq.5.33 we obtain the four moments about the origin of a generalized three parameter Rama distribution Eq.5.29 as;

$$\mu_1^1 = \frac{\alpha\theta^3+24\beta}{\theta(\alpha\theta^3+6\beta)}, \quad \mu_2^1 = \frac{2(\alpha\theta^3+60\beta)}{\theta^2(\alpha\theta^3+6\beta)}$$
$$\mu_3^1 = \frac{6(\alpha\theta^3+120\beta)}{\theta^3(\alpha\theta^3+6\beta)}, \quad \mu_4^1 = \frac{24(\alpha\theta^3+210\beta)}{\theta^4(\alpha\theta^3+6\beta)}$$

We now derive the centralized four moments of a generalized three parameter Rama distribution as;

$$\mu_1 = \mu_1^1 = \frac{\alpha\theta^3 + 24\beta}{\theta(\alpha\theta^3 + 6\beta)}$$
$$\mu_2 = \mu_2^1 - (\mu_1^1)^2$$
$$\mu_2 = \frac{2(\alpha\theta^3 + 60\beta)}{\theta^2(\alpha\theta^3 + 6\beta)} - \left[\frac{\alpha\theta^3 + 24\beta}{\theta(\alpha\theta^3 + 6\beta)}\right]^2 \Longrightarrow \frac{\alpha^2\theta^6 + 84\alpha\beta\theta^3 + 144\beta^2}{\theta^2(\alpha\theta^3 + 6\beta)^2}$$
$$\mu_3 = \mu_3^1 - 3\mu_1^1\mu_2^1 + 2(\mu_1^1)^3$$
$$\mu_3 = \frac{2\left[\alpha^3\theta^9 + 342\alpha^2\beta\theta^6 + 5508\alpha\beta^2\theta^3 + 203040\beta^3\right]}{\theta^3(\alpha\theta^3 + 6\beta)^3}$$

The following relation is applied to derive the fourth centralized moment

$$\mu_4 = \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2$$

Proposition 5.4.5. *Other related measures of a generalized three parameter Rama distribution Eq.5.29 such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3)*

are stated as;

$$C.v=\frac{\sqrt{\alpha^2\theta^6+84\alpha\beta\theta^3+144\beta^2}}{\alpha\theta^3+24\beta} \tag{5.34}$$

$$\upsilon_1=\frac{2\left[\alpha^3\theta^9+342\alpha^2\beta\theta^6+5508\alpha\beta^2\theta^3+203040\beta^3\right]}{\left[\alpha^2\theta^6+84\alpha\beta\theta^3+144\beta^2\right]^{\frac{3}{2}}} \tag{5.35}$$

$$\upsilon_2=\frac{\mu_4^1-4\mu_3^1\mu_1^1+6\mu_1^1\mu_2^1-3(\mu_1^1)^2}{(\mu_2^1-(\mu_1^1))^2} \tag{5.36}$$

$$\upsilon_3=\frac{\alpha^2\theta^6+84\alpha\beta\theta^3+144\beta^2}{\theta(\alpha\theta^3+6\beta)(\alpha\theta^3+24\beta)} \tag{5.37}$$

`Proof.` Coefficient of variation in Eq.5.34 is obtained as;

$$C.v=\frac{\sigma}{\mu_1^1}$$

$$C.v=\frac{\sqrt{\alpha^2\theta^6+84\alpha\beta\theta^3+144\beta^2}}{\theta(\alpha\theta^3+6\beta)}\frac{\theta(\alpha\theta^3+6\beta)}{\alpha\theta^3+24\beta}\Longrightarrow\frac{\sqrt{\alpha^2\theta^6+84\alpha\beta\theta^3+144}}{\alpha\theta^3+24\beta}$$

Similarly, skewsness in Eq.5.35 is obtained as;

$$\upsilon_1=\frac{\mu_3}{(\mu_2)^{\frac{3}{2}}}$$

$$=\frac{2\left[\alpha^3\theta^9+342\alpha^2\beta\theta^6+5508\alpha\beta^2\theta^3+203040\beta^3\right]}{\theta^3(\alpha\theta^3+6\beta)^3}\left[\frac{\theta^2(\alpha\theta^3+6\beta)^2}{\alpha^2\theta^6+84\alpha\beta\theta^3+144}\right.$$

$$=\frac{2\left[\alpha^3\theta^9+342\alpha^2\beta\theta^6+5508\alpha\beta^2\theta^3+203040\beta^3\right]}{\left[\alpha^2\theta^6+84\alpha\beta\theta^3+144\beta^2\right]^{\frac{3}{2}}}$$

kurtosis presented in Eq.5.36 can be obtained using the following relation; The following relation is applied to derive coefficient of kurtosis of a generalized three parameter Rama distribution Eq.5.29.

$$\upsilon_2 = \frac{\mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2}{\left[\mu_2^1 - (\mu_1^1)^2\right]^2}$$

further index of dispersion Eq.5.37 is obtained as;

$$\begin{aligned}\upsilon_3 &= \frac{\sigma^2}{\mu_1^1} \\ &= \frac{\alpha^2\theta^6 + 84\alpha\beta\theta^3 + 144\beta^2}{\theta^2(\alpha\theta^3+6\beta)^2}\frac{\theta(\alpha\theta^3+6\beta)}{\alpha\theta^3+24\beta} \Longrightarrow \frac{\alpha^2\theta^6 + 84\alpha\beta\theta^3 + 144}{\theta(\alpha\theta^3+6\beta)(\alpha\theta^3+2}\end{aligned}$$

□

5.4.4 Excess Loss Distribution

Proposition 5.4.6. *In this section, we state probability density function of excess loss function $f_l(x;\alpha,\beta,\theta)$, mean residual lifetime (MRL), equilibrium distribution $f_e(x;\alpha,\beta,\theta)$, survival function of the equilibrium distribution $S_e(x;\alpha,\beta,\theta)$ and hazard function of the equilibrium distribution $h_e(x;\alpha,\beta,\theta)$ of a gen-*

eralized three parameter Rama distribution Eq.5.29 as;

$$f_l(x;\alpha,\beta,\theta)=\frac{\theta^4(\alpha+\beta x^3)e^{-(x-z)\theta}}{\alpha\theta^3+6\beta+\beta\theta^3z^3+3\beta\theta^2z^2+6\beta z};\quad x>z \tag{5.38}$$

$$m(x)=\frac{\alpha\theta^3+24\beta+\beta\theta^3x^3+6\beta\theta^2x^2+18\beta\theta x}{\theta(\alpha\theta^3+6\beta+\beta\theta^3x^3+3\beta\theta^2x^2+6\beta\theta x)} \tag{5.39}$$

$$f_e(x;\alpha,\beta,\theta)=\frac{\theta(\alpha\theta^3+6\beta+\beta\theta^3x^3+3\beta\theta^2x^2+6\beta\theta x)e^{-\theta x}}{\alpha\theta^3+24\beta} \tag{5.40}$$

$$S_e(x;\alpha,\beta,\theta)=\frac{(\alpha\theta^3+24\beta+\beta\theta^3x^3+6\beta\theta^2x^2+18\beta\theta x)e^{-\theta x}}{\alpha\theta^3+24\beta} \tag{5.41}$$

$$h_e(x;\alpha,\beta,\theta)=\frac{\theta(\alpha\theta^3+6\beta+\beta\theta^3x^3+3\beta\theta^2x^2+6\beta\theta x)}{\alpha\theta^3+24\beta+\beta\theta^3x^3+6\beta\theta^2x^2+18\beta\theta x} \tag{5.42}$$

Proof. To begin with, $f_l(x;\alpha,\beta,\theta)$ in Eq.5.38 is obtained using Eq.1.10 as;

$$\begin{aligned}f_l(x;\alpha,\beta,\theta)&=\frac{\theta^4}{\alpha\theta^3+6\beta}\left[\alpha+\beta x^3\right]e^{-\theta x}\frac{\alpha\theta^3+6\beta}{(\alpha\theta^3+6\beta+\beta\theta^3z^3+3\beta\theta^2z^2+6\beta z)}\\&=\frac{\theta^4(\alpha+\beta x^3)e^{-(x-z)\theta}}{\alpha\theta^3+6\beta+\beta\theta^3z^3+3\beta\theta^2z^2+6\beta z};\quad x>z\end{aligned}$$

Similarly, $m(x)$ in Eq.5.39 is obtained using Eq.1.11 as;

$$m(x) = \frac{\alpha\theta^3+6\beta}{(\alpha\theta^3+6\beta+\beta\theta^3x^3+3\beta\theta^2x^2+6\beta\theta x)e^{-\theta x}}\int_x^\infty \frac{(\alpha\theta^3+6\beta+\beta\theta^3$$

$$= \frac{1}{(\alpha\theta^3+6\beta+\beta\theta^3x^3+3\beta\theta^2x^2+6\beta\theta x)e^{-\theta x}}\int_x^\infty (\alpha\theta^3+6\beta+\beta\theta^3$$

$$I_1 = \int_x^\infty (\alpha\theta^3+6\beta+\beta\theta^3t^3+3\beta\theta^2t^2+6\beta\theta t)e^{-\theta t}dt$$

$$u = (\alpha\theta^3+6\beta+\beta\theta^3t^3+3\beta\theta^2t^2+6\beta\theta t) \Longrightarrow 3\beta\theta^3t^2+6\beta\theta^2t+6\beta$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_1 = -(\alpha\theta^3+6\beta+\beta\theta^3t^3+3\beta\theta^2t^2+6\beta\theta t)\frac{e^{-\theta t}}{\theta} + \int_x^\infty (3\beta\theta^2t^2+6\beta$$

$$I_2 = \int_x^\infty (3\beta\theta^2t^2+6\beta\theta t+6\beta)e^{-\theta t}dt$$

$$u = (3\beta\theta^2t^2+6\beta\theta t+6\beta) \Longrightarrow du = 6\beta\theta^2t+6\beta\theta$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_2 = -(3\beta\theta^2t^2+6\beta\theta t+6\beta)\frac{e^{-\theta t}}{\theta} + 6\beta\int_x^\infty (\theta t+1)e^{-\theta t}dt$$

$$
\begin{aligned}
I_3 &= \int_x^\infty (\theta t+1)e^{-\theta t}dt \\
u &= (\theta t+1) \implies du = \theta dt \\
dv &= e^{-\theta t} \implies v = \frac{-e^{-\theta t}}{\theta} \\
I_3 &= -(\theta t+1)\frac{e^{-\theta t}}{\theta} + \int_x^\infty e^{-\theta t}dt \\
I_4 &= \int_x^\infty e^{-\theta t}dt \implies \frac{-e^{-\theta t}}{\theta} \\
I_1 &= -\left[\frac{e^{-\theta t}}{\theta}\left(\alpha\theta^3 + 24\beta + \beta\theta^3 t^3 + 6\beta\theta^2 t^2 + 18\beta\theta t\right)\right]_x^\infty \\
I_1 &= \frac{e^{-\theta x}}{\theta}\left(\alpha\theta^3 + 24\beta + \beta\theta^3 x^3 + 6\beta\theta^2 x^2 + 18\beta\theta x\right) \\
&= \frac{1}{(\alpha\theta^3 + 6\beta + \beta\theta^3 x^3 + 3\beta\theta^2 x^2 + 6\beta\theta x)e^{-\theta x}} I_1 \\
&= \frac{\alpha\theta^3 + 24\beta + \beta\theta^3 x^3 + 6\beta\theta^2 x^2 + 18\beta\theta x}{\theta(\alpha\theta^3 + 6\beta + \beta\theta^3 x^3 + 3\beta\theta^2 x^2 + 6\beta\theta x)}
\end{aligned}
$$

Using the relation Eq.1.12 $f_e(x;\alpha,\beta,\theta)$ in Eq.5.40 is obtained as;

$$
\begin{aligned}
f_e(x;\alpha,\beta,\theta) &= \left[\frac{\alpha\theta^3 + 6\beta + \beta\theta^3 x^3 + 3\beta\theta^2 x^2 + 6\beta\theta x}{\alpha\theta^3 + 6\beta}\right] e^{-\theta x} \frac{\theta(\alpha\theta^3 + 6\beta)}{\alpha\theta^3 + 24\beta} \\
&= \frac{\theta(\alpha\theta^3 + 6\beta + \beta\theta^3 x^3 + 3\beta\theta^2 x^2 + 6\beta\theta x)e^{-\theta x}}{\alpha\theta^3 + 24\beta}
\end{aligned}
$$

survival function based on the equilibrium distribution $S_e(x;\alpha,\beta,\theta)$ in Eq.5.41 is obtained using Eq.1.13 as;

$$\int_x^\infty S(t;\alpha,\beta,\theta)dt = \frac{(\alpha\theta^3+24\beta+\beta\theta^3x^3+6\beta\theta^2x^2+18\beta\theta x)e^{-\theta x}}{\theta(\alpha\theta^3+6\beta)}$$

$$= \frac{(\alpha\theta^3+24\beta+\beta\theta^3x^3+6\beta\theta^2x^2+18\beta\theta x)e^{-\theta x}}{\theta(\alpha\theta^3+6\beta)} \frac{\theta(\alpha\theta}{\alpha\theta^3}$$

$$= \frac{(\alpha\theta^3+24\beta+\beta\theta^3x^3+6\beta\theta^2x^2+18\beta\theta x)e^{-\theta x}}{\alpha\theta^3+24\beta}$$

Lastly, hazard function based on equilibrium distribution denoted by $h_e(x;\alpha,\beta,\theta)$ in Eq.5.42 is obtained using Eq.1.14 as;

$$h_e(x;\alpha,\beta,\theta) = \frac{\theta(\alpha\theta^3+6\beta+\beta\theta^3x^3+3\beta\theta^2x^2+6\beta\theta x)e^{-\theta x}}{\alpha\theta^3+24\beta} \frac{}{(\alpha\theta^3+24\beta}$$

$$= \frac{\theta(\alpha\theta^3+6\beta+\beta\theta^3x^3+3\beta\theta^2x^2+6\beta\theta x)}{\alpha\theta^3+24\beta+\beta\theta^3x^3+6\beta\theta^2x^2+18\beta\theta x}$$

$\square$

6 TWO COMPONENT FINITE GAMMA MIXTURE (Case of Suja distribution)

6.1 Introduction

A two component finite gamma mixture a case of Suja distribution is considered. We shall construct and derive statistical properties of Suja distribution and its generalizations. The mixed distribution is expressed in terms of pdf and Cdf. Statistical properties such as reliability analysis measures, equilibrium properties and moments (both centralized and non centralized).

6.2 One parameter Suja distribution

6.2.1 Construction of a one parameter Suja distribution

Proposition 6.2.1. *Let $\omega = \frac{\theta^4}{\theta^4+24}$ be a mixing proportion, one parameter SUja distribution is a finite mixed distribution of Gamma $(1,\theta)$ and Gamma $(5,\theta)$. The pdf and Cdf of one parameter Suja distribution are stated as;*

$$f(x;\theta) = \frac{\theta^5}{\theta^4+24}\left[1+x^4\right]e^{-\theta x}; x>0, \theta>0 \tag{6.1}$$

$$F(x,\theta) = 1-\left[1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right]e^{-\theta x}; x>0, \theta>02 \tag{6.2}$$

Proof. By definition of finite mixture Eq.1.1, pdf Eq.6.1of one parameter Suja distribution is constructed as;

$$
\begin{aligned}
f(x;\theta &= \frac{\theta^4}{\theta^4+24}\left[\theta e^{-\theta x}\right] + \frac{24}{\theta^4+24}\left[\frac{\theta^5 x^4 e^{-\theta x}}{\Gamma 5}\right] \\
&= \frac{\theta^5 e^{-\theta x}}{\theta^4+24} + \frac{24\theta^5 e^{-\theta x} x^4}{24(\theta^4+24)} \\
&= \frac{\theta^5}{\theta^4+24}\left[1+x^4\right] e^{-\theta x}; x>0, \theta>0
\end{aligned}
$$

further Cdf Eq.6.2 is obtained as;

$$
\begin{aligned}
F(x;\theta) &= \frac{\theta^5}{\theta^4+24}\int_0^\infty \left(1+x^4\right) e^{-\theta x} dx \\
F(x;\theta) &= \frac{\theta^5}{\theta^4+24} I_1 \\
I_1 &= \int_0^\infty \left(1+x^4\right) e^{-\theta x} dx \\
u &= \left(1+x^4\right) \implies du = 4x^3 dx \\
dv &= e^{-\theta x} \implies v = \frac{-e^{-\theta x}}{\theta} \\
I_1 &= -\left(1+x^4\right)\frac{e^{-\theta x}}{\theta} - \frac{4}{\theta}\int_0^\infty -x^3 e^{-\theta x} dx \\
I_1 &= -\left(1+x^4\right)\frac{e^{-\theta x}}{\theta} - \frac{4}{\theta} I_2 \\
I_2 &= \int_0^\infty -x^3 e^{-\theta x} dx \\
u &= -x^3 \implies -3x^2
\end{aligned}
$$

$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$

$$I_2 = \frac{x^3 e^{-\theta x}}{\theta} - \frac{3}{\theta}\int_0^\infty x^2 e^{-\theta x} dx$$

$$I_2 = \frac{x^3 e^{-\theta x}}{\theta} - \frac{3}{\theta} I_3$$

$$I_3 = \int_0^\infty x^2 e^{-\theta x} dx$$

$$u = x^2 \Longrightarrow du = 2x dx$$

$$I_3 = \frac{-x^2 e^{-\theta x}}{\theta} - \frac{2}{\theta}\int_0^\infty -x e^{-\theta x} dx$$

$$I_3 = \frac{-x^2 e^{-\theta x}}{\theta} - \frac{2}{\theta} I_4$$

$$I_4 = \int_0^\infty -x e^{-\theta x} dx$$

$$u = -x \Longrightarrow du = -dx$$

$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$

$$I_4 = \frac{x e^{-\theta x}}{\theta} - \frac{1}{\theta}\int_0^\infty e^{-\theta x}$$

$$I_4 = \frac{x e^{-\theta x}}{\theta} - \frac{1}{\theta} I_5$$

$$I_5 = \int_0^\infty e^{-\theta x} dx$$

$$u = -\theta x \Longrightarrow du = -\theta dx \Longrightarrow \frac{du}{-\theta} = dx$$

From I_1,I_2,I_3,I_4 and I_5 then;

$$\begin{aligned} I_1 &= \frac{-e^{-\theta x}}{\theta^5}\left[\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x\right] \\ &= 1+\frac{-e^{-\theta x}}{\theta^5}\left[\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x\right]\frac{\theta^5}{\theta^4+24} \end{aligned}$$

$$F(x;\theta) = 1-\left[1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right]e^{-\theta x};x>0,\theta>0$$

□

6.2.2 Reliability Analysis

Proposition 6.2.2. *In this section, survival function denoted by $S(x;\theta)$ and hazard function denoted by $h(x;\theta)$ of a one parameter Suja distribution Eq.6.1 are stated as;*

$$S(x;\theta) = \left[1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right]e^{-\theta x};x>0,\theta>0 \tag{6.3}$$

$$h(x;\theta) = \frac{\theta^5(1+x^4)}{\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x};x>0,\theta>0 \tag{6.4}$$

Proof. To begin with, survival function Eq.6.3 is obtained using Eq.1.8 as;

$$\begin{aligned} S(x;\theta) &= 1-\left[1-\left[1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right]e^{-\theta x}\right] \\ &= \left[1+\frac{\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right]e^{-\theta x};x>0,\theta>0 \end{aligned}$$

using the relation Eq.1.9, hazard function Eq.6.4 as;

$$h(x;\theta) = \frac{\frac{\theta^5}{\theta^4+24}\left[1+x^4\right]e^{-\theta x}}{\left[\frac{\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\theta^4+24}\right]e^{-\theta x}}$$

$$= \frac{\theta^5(1+x^4)}{\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}; x>0, \theta>0$$

□

6.2.3 Moments and related measures

Proposition 6.2.3. *The r^{th} moments of a one parameter Suja distribution Eq.6.1 are derived using both method of moments and moment generating function as;*

$$\mu_r^1* = \frac{r!\big(\theta^4+(r+1)(r+2)(r+3)\big)}{\theta^r(\theta^4+24)}; r=1,2,3,\cdots \quad (6.5)$$

Proof. By definition of moments Eq.1.16 r^{th} moments of EQ.6.1 are obtained as;

$$E(X^r) = \frac{\theta^5}{\theta^4+24}\int_0^\infty x^r(1+x^4)e^{-\theta x}dx$$

$$= \frac{\theta^5}{\theta^4+24}\left[\int_0^\infty x^r e^{-\theta x}dx + \int_0^\infty x^{r+4}e^{-\theta x}dx\right]$$

$$= \frac{\theta^5}{\theta^4+24}\left[\frac{\Gamma(r+1)}{\theta^{r+1}} + \frac{\Gamma(r+5)}{\theta^{r+5}}\right]$$

$$= \frac{r\Gamma r\left[\theta^4+(r+1)(r+2)(r+3)(r+4)\right]}{\theta^r(\theta^4+24)}$$

$$= \frac{r!\big(\theta^4+(r+1)(r+2)(r+3)\big)}{\theta^r(\theta^4+24)}; r=1,2,3,\cdots$$

moreover, by definition Eq.1.17, the mgf of 6.1 is given as;

$$\begin{aligned}
m_x(t) &= \frac{\theta^5}{\theta^4+24}\int_0^\infty e^{tx}(1+x^4)e^{-\theta x}dx \\
&= \frac{\theta^5}{\theta^4+24}\left[\int_0^\infty e^{-(\theta-t)x}(1+x^4)dx\right] \\
&= \frac{\theta^5}{\theta^4+24}\left[\frac{1}{\theta-t}+\frac{24}{(\theta-t)^5}\right] \\
&= \frac{\theta^5}{\theta^4+24}\left[\frac{1}{\theta}\sum_{k=0}^{\infty}\left(\frac{t}{\theta}\right)^k+\frac{24}{\theta^5}\sum_{k=0}^{\infty}\binom{k+4}{k}\left(\frac{t}{\theta}\right)^k\right] \\
&= \sum_{k=0}^{\infty}\frac{\theta^4+(k+1)(k+2)(k+3)(k+4)}{\theta^4+24}\left(\frac{t}{\theta}\right)^k
\end{aligned}$$

The moments about the origin are obtained as a coefficient of $\frac{t^r}{r!}$ in the derived moment generating function $m_x(t)$ as;

$$\mu_r^1 = \frac{r!\big(\theta^4+(r+1)(r+2)(r+3)\big)}{\theta^r(\theta^4+24)}; r = 1,2,3,\cdots$$

□

The four none centralized moments of a one parameter Suja distribution Eq.6.1 are obtained by putting $r = 1,2,3$ and 4 in moment generating function Eq.6.5 as;

$$\mu_1^1 = \frac{\theta^4+120}{\theta(\theta^4+24)}, \quad \mu_2^1 = \frac{2(\theta^4+360)}{\theta^2(\theta^4+24)}$$
$$\mu_3^1 = \frac{6(\theta^4+840)}{\theta^3(\theta^4+24)}, \quad \mu_4^1 = \frac{24(\theta^4+1680)}{\theta^4(\theta^4+24)}$$

We now derive the centralized moments of a one parameter Suja distribution Eq.6.1 as;

$$\mu_1 = \mu_1^1 \Longrightarrow \frac{\theta^4+120}{\theta(\theta^4+24)}$$

$$\mu_2 = \mu_2^1 - (\mu_1^1)^2 \Longrightarrow \frac{\theta^8+528\theta^4+2880}{\theta^2(\theta^4+24)^2}$$

$$\mu_3 = \mu_3^1 - 3[\mu_1^1\mu_2^1] + 2[\mu_1^1]^2 \Longrightarrow \frac{2\theta^{12}+3024\theta^8+3456\theta^4+138240}{\theta^3(\theta^4+24)^3}$$

$$\mu_4 = \mu_4^1 - 4\mu_3^1 + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2$$

$$\mu_4 = \frac{9\theta^{16}+23904\theta^{12}+528768\theta^8+111144496\theta^4+34836480}{\theta^4(\theta^4+24)^4}$$

Proposition 6.2.4. *Other related measures of a one parameter Suja distribution such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) are stated as;*

$$C.v = \frac{\sqrt{\theta^8+528\theta^4+2880}}{\theta^4+120} \tag{6.6}$$

$$\upsilon_1 = \frac{2\theta^{12}+3024\theta^8+3456\theta^4+138240}{(\theta^8+528\theta^4+2880)^{\frac{3}{2}}} \tag{6.7}$$

$$\upsilon_2 = \frac{9\theta^{16}+23904\theta^{12}+528768\theta^8+111144496\theta^4+34836480}{(\theta^8+528\theta^4+2880)^2} \tag{6.8}$$

$$\upsilon_3 = \frac{\theta^8+528\theta^4+2880}{\theta(\theta^4+24)(\theta^4+120)} \tag{6.9}$$

Proof. Variation coefficient Eq.6.6 is obtained as;

$$C.v = \frac{\sigma}{\mu_1^1}$$

$$= \frac{\sqrt{\theta^8+528\theta^4+2880}}{\theta(\theta^4+24)}\frac{\theta(\theta^4+24)}{\theta^4+120} \Longrightarrow \frac{\sqrt{\theta^8+528\theta^4+2880}}{\theta^4+120}$$

further, skewness coefficient Eq.6.7 is obtained as;

$$
\begin{aligned}
\upsilon_1 &= \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} \\
&= \frac{2\theta^{12}+3024\theta^8+3456\theta^4+138240}{\theta^3(\theta^4+24)^3}\left[\frac{\theta^2(\theta^4+24)^2}{\theta^8+528\theta^4+2880}\right]^{\frac{3}{2}} \\
&= \frac{2\theta^{12}+3024\theta^8+3456\theta^4+138240}{(\theta^8+528\theta^4+2880)^{\frac{3}{2}}}
\end{aligned}
$$

kurtosis coefficient Eq.6.8 is obtained as;

$$
\begin{aligned}
\upsilon_2 &= \frac{\mu_4}{(\mu_2)^2} \\
&= \frac{9\theta^{16}+23904\theta^{12}+528768\theta^8+111144496\theta^4+34836480}{\theta^4(\theta^4+24)^4}\left[\frac{\theta^2(\theta^4}{\theta^8+528}\right. \\
&= \frac{9\theta^{16}+23904\theta^{12}+528768\theta^8+111144496\theta^4+34836480}{(\theta^8+528\theta^4+2880)^2}
\end{aligned}
$$

lastly, index of dispersion Eq.6.9 is obtained as;

$$
\begin{aligned}
\upsilon_3 &= \frac{\sigma^2}{\mu_1^1} \\
&= \frac{\theta^8+528\theta^4+2880}{\theta^2(\theta^4+24)^2}\frac{\theta(\theta^4+24)}{\theta^4+120} \Longrightarrow \frac{\theta^8+528\theta^4+2880}{\theta(\theta^4+24)(\theta^4+120)}
\end{aligned}
$$

□

6.2.4 Excess loss distribution

Proposition 6.2.5. *In this section, we state probability density function of excess loss function $f_l(x;\theta)$, mean residual lifetime (MRL), equilibrium distribution $f_e(x;\theta)$, survival function of the equilibrium distribution $S_e(x;\theta)$ and hazard function based on* *the equilibrium distribution $h_e(x;\theta)$ of a one parameter Suja distribution Eq.6.1 as;*

$$f_l(x;\theta) = \frac{\theta^5(1+x^4)e^{-(x-z)\theta}}{\theta^4+24+\theta^4z^4+4\theta^3z^3+12\theta^2z^2+24\theta z}; \quad x>z \tag{6.10}$$

$$m(x) = \frac{\theta^4+120+\theta^4x^4+8\theta^3x^3+36\theta^2x^2+96\theta x}{\theta(\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x)} \tag{6.11}$$

$$f_e(x;\theta) = \frac{\theta(\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x)e^{-\theta x}}{\theta+120} \tag{6.12}$$

$$S_e(x;\theta) = \frac{(\theta^4+120+\theta^4x^4+8\theta^3x^3+36\theta^2x^2+96\theta x)e^{-\theta x}}{\theta^4+120} \tag{6.13}$$

$$h_e(x;\theta) = \frac{\theta(\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x)}{\theta^4+120+\theta^4x^4+8\theta^3x^3+36\theta^2x^2+96\theta x} \tag{6.14}$$

Proof. By the definition Eq.1.10, $f_l(x;\theta)$ in Eq.6.10 is obtained as;

$$\begin{aligned} f_l(x;\theta) &= \frac{\frac{\theta^5}{\theta^4+24}\left[1+x^4\right]e^{-\theta x}}{\frac{(\theta^4+24+\theta^4z^4+4\theta^3z^3+12\theta^2z^2+24\theta z)e^{-\theta z}}{\theta^4+24}} \\ &= \frac{\theta^5(1+x^4)e^{-(x-z)\theta}}{\theta^4+24+\theta^4z^4+4\theta^3z^3+12\theta^2z^2+24\theta z}; \quad x>z \end{aligned}$$

using the relation Eq.1.11, the mean residual lifetime is obtained as;

$$m(x) = \frac{\theta^4+24}{(\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x)e^{-\theta x}} \int_x^\infty \frac{(\theta^4+24+\theta^4}{}$$

$$= \frac{1}{(\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x)e^{-\theta x}} \int_x^\infty (\theta^4+24+\theta^4$$

$$I_1 = \int_x^\infty (\theta^4+24+\theta^4t^4+4\theta^3t^3+12\theta^2t^2+24\theta t)e^{-\theta t}dt$$

$$u = (\theta^4+24+\theta^4t^4+4\theta^3t^3+12\theta^2t^2+24\theta t) \Longrightarrow du = (4\theta^4t^3+12\theta$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_1 = -(\theta^4+24+\theta^4t^4+4\theta^3t^3+12\theta^2t^2+24\theta t)\frac{e^{-\theta t}}{\theta}+I_2$$

$$I_2 = \int_x^\infty (4\theta^3t^3+12\theta^2t^2+24\theta t+24)e^{-\theta t}dt$$

$$u = (4\theta^3t^3+12\theta^2t^2+24\theta t+24) \Longrightarrow du = (12\theta^3t^2+24\theta^1tt+24\theta)$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_2 = -(4\theta^3t^3+12\theta^2t^2+24\theta t+24)\frac{e^{-\theta t}}{\theta}+I_3$$

$$I_3 = \int_x^\infty (12\theta^2t^2+24\theta t+24)e^{-\theta t}dt$$

$$u = (12\theta^2t^2+24\theta t+24) \Longrightarrow du = 24\theta^2t+24\theta$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_3 = -(12\theta^2t^2+24\theta t+24)\frac{e^{-\theta t}}{\theta}+\int_x^\infty (24\theta t+24)e^{-\theta t}dt$$

$$I_4 = \int_x^\infty (24\theta t+24)e^{-\theta t}dt$$

$$u = (24\theta t+24) \Longrightarrow du = 24\theta$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_4 = -(24\theta t+24)\frac{e^{-\theta t}}{\theta}+24\int^\infty e^{-\theta t}dt$$

From I_1,I_2,I_3,I_4 and I_5 we have the following;

$$I_1 = \frac{-e^{-\theta t}}{\theta}\left[\theta^4 + 120 + \theta^4 t^4 + 8\theta^3 t^3 + 36\theta^2 t^2 + 96\theta t\right]_x^\infty$$
$$I_1 = \frac{e^{-\theta x}}{\theta}\left[\theta^4 + 120 + \theta^4 x^4 + 8\theta^3 x^3 + 36\theta^2 x^2 + 96\theta x\right]$$
$$= \frac{1}{(\theta^4 + 24 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x)e^{-\theta x}} I_1$$
$$m(x) = \frac{\theta^4 + 120 + \theta^4 x^4 + 8\theta^3 x^3 + 36\theta^2 x^2 + 96\theta x}{\theta(\theta^4 + 24 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x)}$$

further, $f_e(x;\theta)$ in Eq.6.12 is obtained using Eq.1.12 as;

$$f_e(x;\theta) = \frac{(\theta^4 + 24 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x)e^{-\theta x}}{\theta^4 + 24} * \frac{\theta(\theta^4 + 24)}{\theta^4 + 120}$$
$$= \frac{\theta(\theta^4 + 24 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x)e^{-\theta x}}{\theta + 120}$$

applying Eq.1.13, $S_e(x;\theta)$ in Eq.6.13 is obtained as;

$$\int_x^\infty S(t;\theta)dt = \frac{(\theta^4 + 120 + \theta^4 x^4 + 8\theta^3 x^3 + 36\theta^2 x^2 + 96\theta x)e^{-\theta x}}{\theta(\theta^4 + 24)}$$
$$= \frac{(\theta^4 + 120 + \theta^4 x^4 + 8\theta^3 x^3 + 36\theta^2 x^2 + 96\theta x)e^{-\theta x}}{\theta(\theta^4 + 24)} * \frac{\theta(\theta^4 + 24)}{\theta^4 + 120}$$
$$S_e(x;\theta) = \frac{(\theta^4 + 120 + \theta^4 x^4 + 8\theta^3 x^3 + 36\theta^2 x^2 + 96\theta x)e^{-\theta x}}{\theta^4 + 120}$$

lastly, $h_e(x;\theta)$ in Eq.6.14 is obtained using Eq.1.14 as;

$$h_e(x;\theta) = \frac{\theta(\theta^4 + 24 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x)}{\theta^4 + 120 + \theta^4 x^4 + 8\theta^3 x^3 + 36\theta^2 x^2 + 96\theta x} = \frac{1}{m(x)}$$

$\square$

6.3 Two parameter Suja distribution

6.3.1 Construction of a two parameter Suja distribution

Proposition 6.3.1. *A generalized two parameter Suja distribution is a finite mixed distribution of Gamma $(1,\theta)$ and Gamma $(5,\theta)$ with weighing proportion as $\omega = \frac{\alpha\theta^4}{\alpha\theta^4+24}$. The pdf and Cdf of a generalized two parameter Suja distribution are defined as;*

$$f(x;\alpha,\theta) = \frac{\theta^5}{\alpha\theta^4+24}\left[\alpha + x^4\right]e^{-\theta x}; x>0, \alpha>0, \theta>0 \tag{6.15}$$

$$F(x;\alpha,\theta) = 1 - \left[1 + \frac{\theta^4x^4 + 4\theta^3x^3 + 12\theta^2x^2 + 24\theta x}{\alpha\theta^4+24}\right]e^{-\theta x}; x>0, \alpha>0, \tag{6.16}$$

Proof. By definition of a finite mixture Eq.1.1, a two parameter Suja distribution is constructed as;

$$\begin{aligned} f(x;\alpha,\theta) &= \frac{\alpha\theta^4}{\alpha\theta^4+24}\left[\theta e^{-\theta x}\right] + \frac{24}{\alpha\theta^4+24}\left[\frac{\theta^5 e^{-\theta x}}{\Gamma 5}\right] \\ &= \frac{\alpha\theta^5 e^{-\theta x}}{\alpha\theta^4+24} + \frac{24\theta^5 x^4 e^{-\theta x}}{24(\alpha\theta^4+24)} \\ &= \frac{\theta^5}{\alpha\theta^4+24}\left[\alpha + x^4\right]e^{-\theta x}; x>0, \alpha>0, \theta>0 \end{aligned}$$

Remark 6.3.2. *Putting $\alpha = 1$ a generalized two parameter Suja distribution Eq.6.15 reduces to a one parameter Suja distribution Eq.6.1. This Implies that a generalized two parameter Suja distribution is flexible over one parameter Suja distribution.*

The generalized Suja distribution to two parameter distribution Eq.6.15 is a modification of a one parameter Suja distribution Eq.6.1 introduced by Shanker (2017b).

further Cdf Eq.6.16 is obtained as;

$$F(x;\alpha,\theta) = \frac{\theta^5}{\alpha\theta^4+24}\int_0^\infty (\alpha+x^4)e^{-\theta x}dx$$
$$I_1 = \int_0^\infty (\alpha+x^4)e^{-\theta x}dx$$
$$u = (\alpha+x^4) \Longrightarrow du = 4x^3$$
$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$
$$I_1 = -(\alpha+x^4)\frac{e^{-\theta x}}{\theta} - \frac{4}{\theta}\int_0^\infty -x^3e^{-\theta x}dx$$
$$I_1 = -(\alpha+x^4)\frac{e^{-\theta x}}{\theta} - \frac{4}{\theta}I_2$$
$$I_2 = \int_0^\infty -x^3e^{-\theta x}dx$$
$$u = -x^3 \Longrightarrow du = -3x^2$$
$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$
$$I_2 = \frac{x^3e^{-\theta x}}{\theta} - \frac{3}{\theta}\int_0^\infty x^2e^{-\theta x}dx$$
$$I_2 = \frac{x^3e^{-\theta x}}{\theta} - \frac{3}{\theta}I_3$$

$$I_3 = \int_0^\infty x^2 e^{-\theta x} dx$$
$$u = x^2 \implies du = 2x$$
$$dv = e^{-\theta x} \implies v = \frac{-e^{-\theta x}}{\theta}$$
$$I_3 = \frac{-x^2 e^{-\theta x}}{\theta} - \frac{2}{\theta} \int_0^\infty -x e^{-\theta x} dx$$
$$I_3 = \frac{-x^2 e^{-\theta x}}{\theta} - \frac{2}{\theta} I_4$$
$$I_4 = \int_0^\infty -x e^{-\theta x} dx$$
$$u = -x \implies du = -dx$$
$$dv = e^{-\theta x} \implies v = \frac{-e^{-\theta x}}{\theta}$$
$$I_4 = \frac{x e^{-\theta x}}{\theta} - \frac{1}{\theta} \int_0^\infty e^{-\theta x} dx$$
$$I_4 = \frac{x e^{-\theta x}}{\theta} - \frac{1}{\theta} I_5$$
$$I_5 = \int_0^\infty e^{-\theta x} dx$$
$$u = -\theta x \implies du = -\theta dx \implies \frac{du}{-\theta} = dx$$
$$I_5 = \int_0^\infty e^u \frac{du}{-\theta} \implies \frac{-1}{\theta} \int_0^\infty e^u du \implies \frac{1}{-\theta} e^u \implies \frac{e^{-\theta x}}{-\theta}$$

From I_1,I_2,I_3, I_4 and I_5 then;

$$I_1 = \frac{-e^{-\theta x}}{\theta^5}\left[\alpha\theta^4 + 24 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x\right]$$
$$= 1 + \frac{-e^{-\theta x}}{\theta^5}\left[\alpha\theta^4 + 24 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x\right]\frac{\theta^5}{\alpha\theta^5 + 24}$$
$$F(x;\alpha,\theta) = 1 - \left[1 + \frac{\theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x}{\alpha\theta^4 + 24}\right]e^{-\theta x}; x > 0, \alpha > 0, \theta > 0$$

□

6.3.2 Reliability Analysis

Proposition 6.3.3. *We now state survival function denoted by $S(x;\alpha,\theta)$ and hazard function denoted by $h(x;\alpha,\theta)$ of a generalized two parameter Suja distribution Eq.6.15 as;*

$$S(x;\alpha,\theta) = \left[1 + \frac{\theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x}{\alpha\theta^4 + 24}\right]e^{-\theta x}; x > 0, \alpha > 0, \theta > 0 \tag{6.17}$$

$$h(x;\alpha,\theta) = \frac{\theta^5(\alpha + x^4)}{\alpha\theta^4 + 24 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x}; x > 0, \alpha > 0, \theta > 0 \tag{6.18}$$

Proof. To begin with, survival function Eq.6.17 is obtained using Eq.1.8 as;

$$S(x;\alpha,\theta) = 1 - \left[1 - \left[1 + \frac{\theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x}{\alpha\theta^4 + 24}\right]e^{-\theta x}\right]$$
$$= \left[1 + \frac{\theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x}{\alpha\theta^4 + 24}\right]e^{-\theta x}; x > 0, \alpha > 0, \theta > 0$$

further hazard function Eq.6.18 is obtained using Eq.1.9 as;

$$h(x;\alpha,\theta) = \frac{\frac{\theta^5}{\alpha\theta^4+24}\left[\alpha + x^4\right]e^{-\theta x}}{\left[\frac{\alpha\theta^4+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}{\alpha\theta^4+24}\right]e^{-\theta x}}$$
$$= \frac{\theta^5(\alpha+x^4)}{\alpha\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x}; x > 0, \alpha > 0, \theta >$$

□

6.3.3 Moments and related measures

Proposition 6.3.4. *The r^{th} moments of a generalized two parameter Suja distribution Eq.6.15 are derived using both method of moments and moment generating function as;*

$$\mu_r^1 * = \frac{r!\left[\alpha\theta^4 + (r+1)(r+2)(r+3)(r+4)\right]}{\theta^r(\alpha\theta^4+24)}; \quad r = 1,2,3,\cdots \tag{6.19}$$

Proof. By definition Eq.1.16, moments of a generalized two parameter Suja are obtained as;

$$E(X^r) = \frac{\theta^5}{\alpha\theta^4+24}\int_0^\infty x^r(\alpha+x^4)e^{-\theta x}dx$$
$$= \frac{\theta^5}{\alpha\theta^4+24}\left[\alpha\int_0^\infty x^r e^{-\theta x}dx + \int_0^\infty x^{r+4}e^{-\theta x}dx\right]$$
$$= \frac{\theta^5}{\alpha\theta^4+24}\left[\frac{\alpha\Gamma(r+1)}{\theta^{r+1}} + \frac{\Gamma(r+5)}{\theta^{r+5}}\right]$$
$$= \frac{r!\left[\alpha\theta^4 + (r+1)(r+2)(r+3)(r+4)\right]}{\theta^r(\alpha\theta^4+24)}; \quad r = 1,2,3,\cdots$$

Similarly, by definition of mgf Eq.1.17 moment generating function of Eq.6.15 is obtained as;

$$\begin{aligned} m_x(t) &= \frac{\theta^5}{\alpha\theta^4+24}\int_0^\infty e^{tx}(\alpha+x^4)e^{-\theta x}dx \\ &= \frac{\theta^5}{\alpha\theta^4+24}\left[e^{-(\theta-t)x(\alpha+x^4)}dx\right] \\ &= \frac{\theta^5}{\alpha\theta^4+24}\left[\frac{\alpha}{(\theta-t)}+\frac{24}{(\theta-t)^5}\right] \\ &= \frac{\theta^5}{\alpha\theta^4+24}\left[\frac{\alpha}{\theta}\sum_{k=0}^{\infty}\Big(\frac{t}{\theta}\Big)^k+\frac{24}{\theta^5}\sum_{k=0}^{\infty}\binom{k+4}{k}\Big(\frac{t}{\theta}\Big)^k\right] \\ &= \sum_{k=0}^{\infty}\frac{\alpha\theta^4+(k+1)(k+2)(k+3)(k+4)}{\alpha\theta^4+24}\Big(\frac{t}{\theta}\Big)^k \end{aligned}$$

The r^{th} moments of a generalized two parameter Suja distribution Eq.6.15 are obtained as a coefficient of $\frac{t^r}{r!}$ in the moment generating function $m_x(t)$ as;

$$\mu_r^1 = \frac{r!\left[\alpha\theta^4+(r+1)(r+2)(r+3)(r+4)\right]}{\theta^r(\alpha\theta^4+24)}; \quad r = 1,2,3,\cdots$$

□

Setting $r = 1,2,3$ and 4 in Eq.6.19 we derive four none centralized moments as;

$$\begin{aligned} \mu_1^1 &= \frac{\alpha\theta^4+120}{\theta(\alpha\theta^4+24)}, \quad \mu_2^1 = \frac{2(\alpha\theta^4+360)}{\theta^2(\alpha\theta^4+24)} \\ \mu_3^1 &= \frac{6(\alpha\theta^4+840)}{\theta^3(\alpha\theta^4+24)}, \quad \mu_4^1 = \frac{24(\alpha\theta^4+1680)}{\theta^4(\alpha\theta^4+24)} \end{aligned}$$

The centralized moments of a two parameter Suja distribution are obtained as;

$$\mu_1 = \mu_1^1$$
$$\mu_2 = \mu_2^1 - (\mu_1^1)^2$$
$$\mu_2 = \frac{\alpha^2\theta^8 + 528\alpha\theta^4 + 2880}{\theta^2(\alpha\theta^4 + 24)^2}$$
$$\mu_3 = \mu_3^1 - 3[\mu_1^1\mu_2^1] + 2[\mu_1^1]^3$$
$$\mu_3 = \frac{2\alpha^3\theta^{12} + 3024\alpha^2\theta^8 + 517536\alpha\theta^4 - 2972160}{\theta^3(\alpha\theta^4 + 24)^3}$$

The following relation is applied to derive the fourth centralized moment

$$\mu_4 = \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2$$

Proposition 6.3.5. *Other related measures of a generalized two parameter Suja distribution Eq.6.15 such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) are stated as;*

$$C.v = \frac{\sqrt{\alpha^2\theta^3 + 528\alpha\theta^4 + 2880}}{\alpha\theta^4 + 120} \tag{6.20}$$
$$\upsilon_1 = \frac{2\alpha^3\theta^{12} + 3024\alpha^2\theta^8 + 517536\alpha\theta^4 - 2972160}{(\alpha^2\theta^8 + 528\alpha\theta^4 + 2880)^{\frac{3}{2}}} \tag{6.21}$$
$$\upsilon_2 = \frac{\mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2}{(\mu_2^1 - (\mu_1^1))^2} \tag{6.22}$$
$$\upsilon_3 = \frac{\alpha^2\theta^8 + 528\alpha\theta^4 + 2880}{\theta(\alpha\theta^4 + 24)(\alpha\theta^4 + 120)} \tag{6.23}$$

Proof. To begin with, coefficient of variation Eq.6.20 is obtained as;

$$C.v = \frac{\sigma}{\mu_1^1}$$
$$= \frac{\sqrt{\alpha^2\theta^3 + 528\alpha\theta^4 + 2880}}{\theta(\alpha\theta^4 + 24)} \frac{\theta(\alpha\theta^4 + 24)}{\alpha\theta^4 + 120} \Longrightarrow \frac{\sqrt{\alpha^2\theta^3 + 528\alpha\theta^4 + 2880}}{\alpha\theta^4 + 120}$$

similarly, skewness coefficient Eq.6.21is obtained as;

$$\upsilon_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}}$$
$$= \frac{2\alpha^3\theta^{12} + 3024\alpha^2\theta^8 + 517536\alpha\theta^4 - 2972160}{\theta^3(\alpha\theta^4 + 24)^3} \left[\frac{\theta^2(\alpha\theta^4 + 24)^2}{\alpha^2\theta^8 + 528\alpha\theta^4 + 2880} \right]$$
$$= \frac{2\alpha^3\theta^{12} + 3024\alpha^2\theta^8 + 517536\alpha\theta^4 - 2972160}{(\alpha^2\theta^8 + 528\alpha\theta^4 + 2880)^{\frac{3}{2}}}$$

The following relation is applied to derive kurtosis coefficient of a generalized two parameter Suja distribution Eq.6.15;

$$\upsilon_2 = \frac{\mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_1^1\mu_2^1 - 3(\mu_1^1)^2}{(\mu_2^1 - (\mu_1^1))^2}$$

lastly, index of dispersion Eq.6.23 is obtained as;

$$\upsilon_3 = \frac{\sigma^2}{\mu_1^1}$$
$$= \frac{\theta^8 + 528\theta^4 + 2880}{\theta^2(\theta^4 + 24)^2} \frac{\theta(\alpha\theta^4 + 24)}{\alpha\theta^4 + 120} \Longrightarrow \frac{\alpha^2\theta^8 + 528\alpha\theta^4 + 2880}{\theta(\alpha\theta^4 + 24)(\alpha\theta^4 + 120)}$$

□

6.3.4 Excess loss Distribution

Proposition 6.3.6. *In this section, we state probability density function of excess function $f_l(x;\alpha,\theta)$, mean residual lifetime (MRL), equilibrium distribution $f_e(x;\alpha,\theta)$, survival function of the equilibrium distribution $S_e(x;\alpha,\theta)$ and hazard function of equilibrium distribution $h_e(x;\alpha,\theta)$ of a generalized two parameter Suja distribution Eq.6.15 as;*

$$f_l(x;\alpha,\theta) = \frac{\theta^5(\alpha+x^4)e^{-(x-z)\theta}}{\alpha\theta^4+24+\theta^4z^4+4\theta^3z^3+12\theta^2z^2+24\theta z}; \quad x>z \tag{6.24}$$

$$m(x) = \frac{\alpha\theta^4+120+\theta^4x^4+8\theta^3x^3+36\theta^2x^2+96\theta x}{\theta(\alpha\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x)} \tag{6.25}$$

$$f_e(x;\alpha,\theta) = \frac{\theta(\alpha\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x)e^{-\theta x}}{\alpha\theta^4+120} \tag{6.26}$$

$$S_e(x;\alpha,\theta) = \frac{(\alpha\theta^4+120+\theta^4x^4+8\theta^3x^3+36\theta^2x^2+96\theta x)e^{-\theta x}}{\alpha\theta^4+120} \tag{6.27}$$

$$h_e(x;\alpha,\theta) = \frac{\theta(\alpha\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x)}{\alpha\theta^4+120+\theta^4x^4+8\theta^3x^3+36\theta^2x^2+96\theta x} \tag{6.28}$$

Proof. By definition in Eq.1.10, the pdf of excess loss function Eq.6.24 is obtained as;

$$\begin{aligned} f_l(x;\alpha,\theta) &= \frac{\theta^5}{\alpha\theta^4+24}(\alpha+x^4)e^{-\theta x}\frac{\alpha\theta^4+24}{(\alpha\theta^4+24+\theta^4z^4+4\theta^3z^3+12\theta^2z^2+} \\ &= \frac{\theta^5(\alpha+x^4)e^{-(x-z)\theta}}{\alpha\theta^4+24+\theta^4z^4+4\theta^3z^3+12\theta^2z^2+24\theta z}; \quad x>z \end{aligned}$$

Mean residual lifetime Eq.6.25 is obtained using relation Eq.1.11 as;

$$m(x) = \frac{\alpha\theta^4+24}{(\alpha\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x)e^{-\theta x}}\int_x^\infty \frac{(\alpha\theta^4+24+\theta^4t^4}{}$$

$$= \frac{1}{(\alpha\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x)e^{-\theta x}}\int_x^\infty (\alpha\theta^4+24+\theta^4t^4$$

$$I_1 = \int_x^\infty (\alpha\theta^4+24+\theta^4t^4+4\theta^3t^3+12\theta^2t^2+24\theta x)e^{-\theta t}dt$$

$$u = (\alpha\theta^4+24+\theta^4t^4+4\theta^3t^3+12\theta^2t^2+24\theta x) \Longrightarrow du = 4\theta^4t^3+12\theta^3t^2$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_1 = -(\alpha\theta^4+24+\theta^4t^4+4\theta^3t^3+12\theta^2t^2+24\theta x)\frac{e^{-\theta t}}{\theta}+I_2$$

$$I_2 = \int_x^\infty (4\theta^3t^3+12\theta^2t^2+24\theta t+24)e^{-\theta t}dt$$

$$u = (4\theta^3t^3+12\theta^2t^2+24\theta t+24) \Longrightarrow du = (12\theta^3t^2+24\theta^2t+24\theta)$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_2 = -(4\theta^3t^3+12\theta^2t^2+24\theta t+24)\frac{e^{-\theta t}}{\theta}+I_3$$

$$I_3 = \int_x^\infty (12\theta^2t^2+24\theta t+24)e^{-\theta t}dt$$

$$u = (12\theta^2t^2+24\theta t+24) \Longrightarrow du = (24\theta^2t+24\theta)$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_3 = -(12\theta^2t^2+24\theta t+24)\frac{e^{-\theta t}}{\theta}+I_4$$

$$I_4 = \int_x^\infty (24\theta t+24)e^{-\theta t}dt$$

$$u = (24\theta t+24) \Longrightarrow du = 24\theta$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_4 = -(24\theta t + 24)\frac{e^{-\theta t}}{\theta} + 24\int_x^{\infty} e^{-\theta t} dt$$
$$I_5 = \int_x^{\infty} e^{-\theta t} dt = \frac{-e^{-\theta t}}{\theta}$$

From I_1,I_2,I_3,I_4 and I_5 we have the following;

$$I_1 = \frac{-e^{-\theta t}}{\theta}\Big[\alpha\theta^4 + 120 + \theta^4 t^4 + 8\theta^3 t^3 + 36\theta^2 t^2 + 96\theta t\Big]_x^{\infty}$$
$$I_1 = \frac{e^{-\theta x}}{\theta}\Big[\alpha\theta^4 + 120 + \theta^4 x^4 + 8\theta^3 x^3 + 36\theta^2 x^2 + 96\theta x\Big]$$
$$= \frac{1}{(\alpha\theta^4 + 24 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x)e^{-\theta x}} I_1$$
$$m(x) = \frac{\alpha\theta^4 + 120 + \theta^4 x^4 + 8\theta^3 x^3 + 36\theta^2 x^2 + 96\theta x}{\theta(\alpha\theta^4 + 24 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x)}$$

further equilibrium distribution Eq.6.26 is obtained using Eq.1.12 as;

$$f_e(x;\alpha,\theta) = \frac{(\alpha\theta^4 + 24 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x)e^{-\theta x}}{\alpha\theta^4 + 24} \frac{\theta(\alpha\theta^4 + 2}{\alpha\theta^4 + 12}$$
$$= \frac{\theta(\alpha\theta^4 + 24 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x)e^{-\theta x}}{\alpha\theta^4 + 120}$$

using Eq.1.13, $S_e(x;\alpha,\theta)$ in Eq.6.27 is obtained as;

$$\int_x^{\infty} S(t;\alpha,\theta)dt = \frac{(\alpha\theta^4 + 120 + \theta^4 x^4 + 8\theta^3 x^3 + 36\theta^2 x^2 + 96\theta x)e^{-\theta x}}{\theta(\alpha\theta^4 + 24)}$$
$$= \frac{(\alpha\theta^4 + 120 + \theta^4 x^4 + 8\theta^3 x^3 + 36\theta^2 x^2 + 96\theta x)e^{-\theta x}}{\theta(\alpha\theta^4 + 24)} * \frac{\theta(}{\alpha}$$
$$S_e(x;\alpha,\theta) = \frac{(\alpha\theta^4 + 120 + \theta^4 x^4 + 8\theta^3 x^3 + 36\theta^2 x^2 + 96\theta x)e^{-\theta x}}{\alpha\theta^4 + 120}$$

lastly, hazard function based on the equilibrium distribution Eq.6.28 is obtained using relation Eq.1.14 as;

$$h_e(x;\alpha,\theta)=\frac{\theta(\alpha\theta^4+24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x)}{\alpha\theta^4+120+\theta^4x^4+8\theta^3x^3+36\theta^2x^2+96\theta x}$$

□

6.4 Three parameter Suja distribution

6.4.1 Construction of a three parameter Suja distribution

Proposition 6.4.1. *Let $\omega=\frac{\alpha\theta^4}{\alpha\theta^4+24\beta}$ be weight probability a generalized three parameter Suja distribution is constructed as a finite mixture of Gamma $(1,\theta)$ and Gamma $(5,\theta)$. The pdf and Cdf of a generalized three parameter Suja distribution are stated as;*

$$f(x;\alpha,\beta,\theta)=\frac{\theta^5}{\alpha\theta^4+24\beta}\left[\alpha+\beta x^4\right]e^{-\theta x};x>0,\alpha>0,\beta>0,\theta>0 \tag{6.29}$$

$$F(x;\alpha,\beta,\theta)=1-\left[1+\frac{\beta\theta^4x^4+4\beta\theta^3x^3+12\beta\theta^2x^2+24\beta\theta x}{\alpha\theta^4+24\beta}\right]e^{-\theta x};x>0,\alpha \tag{6.30}$$

Proof. By the concept of finite mixture Eq.1.1, the pdf of a generalized three parameter Suja distribution is constructed as;

$$\begin{aligned}f(x;\alpha\beta,\theta)&=\frac{\alpha\theta^4}{\alpha\theta^4+24\beta}\left[\theta e^{-\theta x}\right]+\frac{24\beta}{\alpha\theta^4+24\beta}\left[\frac{\theta^5e^{-\theta x}x^4}{\Gamma 5}\right]\\&=\frac{\alpha\theta^5e^{-\theta x}}{\alpha\theta^4+24\beta}+\frac{24\beta\theta^5e^{-\theta x}e^x}{24(\alpha\theta^4+24\beta)}\\&=\frac{\theta^5}{\alpha\theta^4+24\beta}\left[\alpha+\beta x^4\right]e^{-\theta x};x>0,\alpha>0,\beta>0,\theta>0\end{aligned}$$

A generalized three parameter Suja distribution Eq.6.29 is a modification of proposed two parameter Suja distribution Eq.6.15.

further Cdf Eq.6.30 is obtained as;

$$F(x;\alpha,\beta,\theta) = \frac{\theta^5}{\alpha\theta^4+24\beta}\int_0^\infty(\alpha+\beta x^4)e^{-\theta x}dx$$
$$F(x;\alpha,\beta,\theta) = \frac{\theta^5}{\alpha\theta^4+24\beta}I_1$$
$$I_1 = \int_0^\infty(\alpha+\beta x^4)e^{-\theta x}dx$$
$$u = (\alpha+\beta x^4) \implies du = 4\beta x^3$$
$$dv = e^{-\theta x} \implies v = \frac{-e^{-\theta x}}{\theta}$$
$$I_1 = -(\alpha+\beta x^4)\frac{e^{-\theta x}}{\theta} - \frac{4\beta}{\theta}\int_0^\infty -x^3e^{-\theta x}dx$$
$$I_1 = -(\alpha+\beta x^4)\frac{e^{-\theta x}}{\theta} - \frac{4\beta}{\theta}I_2$$
$$I_2 = \int_0^\infty -x^3e^{-\theta x}dx$$
$$u = -x^3 \implies du = -3x^2$$
$$dv = e^{-\theta x} \implies v = \frac{-e^{-\theta x}}{\theta}$$
$$I_2 = \frac{x^3e^{-\theta x}}{\theta} - \frac{3}{\theta}\int_0^\infty x^2e^{-\theta x}dx$$
$$I_2 = \frac{x^3e^{-\theta x}}{\theta} - \frac{3}{\theta}I_3$$

$$I_3 = \int_0^\infty x^2 e^{-\theta x} dx$$
$$u = x^2 \Longrightarrow du = 2x dx$$
$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$
$$I_3 = \frac{-x^2 e^{-\theta x}}{\theta} - \frac{2}{\theta} \int_0^\infty -x e^{-\theta x} dx$$
$$I_3 = \frac{-x^2 e^{-\theta x}}{\theta} - \frac{2}{\theta} I_4$$
$$I_4 = \int_0^\infty -x e^{-\theta x} dx$$
$$u = -x \Longrightarrow du = -dx$$
$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$
$$I_4 = \frac{x e^{-\theta x}}{\theta} - \frac{1}{\theta} \int_0^\infty e^{-\theta x} dx$$
$$I_4 = \frac{x e^{-\theta x}}{\theta} - \frac{1}{\theta} I_5$$
$$I_5 = \int_0^\infty e^{-\theta x} dx$$
$$u = -\theta x \Longrightarrow du = -\theta dx \Longrightarrow \frac{du}{-\theta} = dx$$
$$I_5 = \int_0^\infty e^u \frac{du}{-\theta} = \frac{-1}{\theta} e^{-\theta x}$$

From I_1,I_2,I_3,I_4 and I_5 we have the following;

$$I_1 = -(\alpha+\beta x)\frac{e^{-\theta x}}{\theta} - \frac{4\beta}{\theta}\left[\frac{x^3 e^{-\theta x}}{\theta} - \frac{3}{\theta}\left(\frac{-x^2 e^{-\theta x}}{\theta} - \frac{2}{\theta}\left(\frac{xe^{-\theta x}}{\theta}\right.\right.\right.$$

$$I_1 = \frac{-e^{-\theta x}}{\theta^5}\left[\alpha\theta^4 + 24\beta + \beta\theta^4 x^4 + 4\beta\theta^3 x^3 + 12\beta\theta^2 x^2 + 24\beta\theta\right.$$

$$= 1 + \frac{-e^{-\theta x}}{\theta^5}\left[\alpha\theta^4 + 24\beta + \beta\theta^4 x^4 + 4\beta\theta^3 x^3 + 12\beta\theta^2 x^2 + 2\right.$$

$$F(x;\alpha,\beta,\theta) = 1 - \left[1 + \frac{\beta\theta^4 x^4 + 4\beta\theta^3 x^3 + 12\beta\theta^2 x^2 + 24\beta\theta x}{\alpha\theta^4 + 24\beta}\right] e^{-\theta x}; x >$$

□

6.4.2 Reliability Analysis

Proposition 6.4.2. *In this section, we state survival function $S(x;\alpha,\beta,\theta)$ and hazard function $h(x;\alpha,\beta,\theta)$ of a generalized three parameter Suja distribution Eq.6.29 as;*

$$S(x;\alpha,\beta,\theta) = \left[1 + \frac{\beta(\theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x)}{\alpha\theta^4 + 24\beta}\right] e^{-\theta x}; x > 0, \alpha >$$

(6.31)

$$h(x;\alpha,\beta,\theta) = \frac{\theta^5(\alpha+\beta x^4)e^{-\theta x}}{\alpha\theta^4 + \beta(24 + \theta^4 x^4 + \theta^3 x^3 + 12\theta^2 x^2 + 24\theta x)}; x > 0, \alpha >$$

(6.32)

Proof. To begin with, survival function Eq.6.31 is obtained using Eq.1.8 as;

$$S(x;\alpha,\beta,\theta) = 1 - \left[1 - \left[1 + \frac{\beta\theta^4 x^4 + 4\beta\theta^3 x^3 + 12\beta\theta^2 x^2 + 24\beta\theta x}{\alpha\theta^4 + 24\beta}\right] e^{-\theta x}\right.$$

$$= \left[1 + \frac{\beta(\theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x)}{\alpha\theta^4 + 24\beta}\right] e^{-\theta x}; x > 0, \alpha >$$

further hazard function Eq.6.32 is obtained bu use of relation Eq.1.9 as;

$$h(x;\alpha,\beta,\theta) = \frac{\frac{\theta^5}{\alpha\theta^4+24\beta}\left[\alpha+\beta x^4\right]e^{-\theta x}}{\left[1+\frac{\beta(\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x)}{\alpha\theta^4+24\beta}\right]e^{-\theta x}}$$
$$= \frac{\theta^5(\alpha+\beta x^4)e^{-\theta x}}{\alpha\theta^4+\beta(24+\theta^4x^4+\theta^3x^3+12\theta^2x^2+24\theta x)};x>0,\alpha>0,\beta >$$

□

6.4.3 Moments and related measures

Proposition 6.4.3. *The r^{th} moments of a generalized three parameter Suja distribution Eq.6.29 are derived using both method of moments and moment generating function as;*

$$\mu_r^1 * = \frac{r!\left[\alpha\theta^4+\beta(r+1)(r+2)(r+3)(r+4)\right]}{\theta^r(\alpha\theta^4+24\beta)} \tag{6.33}$$

Proof. By definition Eq.1.16, moments of a generalized three parameter is obtained as;

$$
\begin{aligned}
E(X^r) &= \frac{\theta^5}{\alpha\theta^4+24\beta}\int_0^\infty x^r(\alpha+\beta x^4)e^{-\theta x}dx \\
&= \frac{\theta^5}{\alpha\theta^4+24\beta}\left[\alpha\int_0^\infty x^r e^{-\theta x}dx+\beta\int_0^\infty x^{r+4}e^{-\theta x}dx\right] \\
&= \frac{\theta^5}{\alpha\theta^4+24\beta}\left[\frac{\alpha\Gamma(r+1)}{\theta^{r+1}}+\frac{\beta\Gamma(r+5)}{\theta^{r+5}}\right] \\
&= \frac{r\Gamma r\left[\alpha\theta^4+\beta(r+1)(r+2)(r+3)(r+4)\right]}{\theta^r(\alpha\theta^4+24\beta)} \\
&= \frac{r!\left[\alpha\theta^4+\beta(r+1)(r+2)(r+3)(r+4)\right]}{\theta^r(\alpha\theta^4+24\beta)}
\end{aligned}
$$

further by definition Eq.1.17, mgf of a generalized three parameter Suja distribution is obtained as;

$$
\begin{aligned}
m_x(t) &= \frac{\theta^5}{\alpha\theta^4+24\beta}\left[\int_0^\infty e^{tx}(\alpha+\beta x^4)e^{-\theta x}dx\right] \\
&= \frac{\theta^5}{\alpha\theta^4+24\beta}\left[\int_0^\infty e^{-(\theta-t)x}(\alpha+\beta x^4)e^{-\theta x}dx\right] \\
&= \frac{\theta^5}{\alpha\theta^4+24\beta}\left[\frac{\alpha}{(\theta-t)}+\frac{24\beta}{(\theta-t)^5}\right] \\
&= \frac{\theta^5}{\alpha\theta^4+24\beta}\left[\frac{\alpha}{\theta}\sum_{k=0}^\infty\left(\frac{t}{\theta}\right)^k+\frac{24\beta}{\theta^5}\sum_{k=0}^\infty\binom{k+4}{k}\left(\frac{t}{\theta}\right)^k\right] \\
&= \sum_{k=0}^\infty\frac{\alpha\theta^4+\beta(k+1)(k+2)(k+3)(k+4)}{\alpha\theta^4+24\beta}\left(\frac{t}{\theta}\right)^k
\end{aligned}
$$

The r^{th} moments of a generalized three parameter Suja distribution Eq.6.29 are obtained as a coefficient of $\frac{t^r}{r!}$ in $m_x(t)$ as shown below;

$$\mu_r^1 = \frac{r!\left[\alpha\theta^4 + \beta(r+1)(r+2)(r+3)(r+4)\right]}{\theta^r(\alpha\theta^4 + 24\beta)}$$

□

For values of r as 1,2,3 and 4 in Eq.6.33 we obtain the four none centralized moments of a generalized three parameter Suja distribution as;

$$\mu_1^1 = \frac{\alpha\theta^4 + 120\beta}{\theta(\alpha\theta^4 + 24\beta)}, \quad \mu_2^1 = \frac{2(\alpha\theta^4 + 360\beta)}{\theta^2(\alpha\theta^4 + 24\beta)}$$
$$\mu_3^1 = \frac{6(\alpha\theta^4 + 840\beta)}{\theta^3(\alpha\theta^4 + 24\beta)}, \quad \mu_4^1 = \frac{24(\alpha\theta^4 + 1680\beta)}{\theta^4(\alpha\theta^4 + 24\beta)}$$

The centralized moments of a generalized three parameter Suja distribution are as;

$$\mu_1 = \mu_1^1$$
$$\mu_2 = \mu_2^1 - [\mu_1^1]^2 \Longrightarrow \frac{\alpha^2\theta^8 + 528\alpha\beta\theta^4 + 5840\beta^2}{\theta^2(\alpha\theta^4 + 24\beta)^2}$$
$$\mu_3 = \mu_3^1 - 3[\mu_1^1\mu_2^1] + 2[\mu_1^1]^3 \Longrightarrow \frac{2(\alpha^3\theta^{12} + 1512\alpha^2\beta\theta^8 + 1728\alpha\beta^2\theta^4 + 69120}{\theta^3(\alpha\theta^4 + 24\beta)^3}$$

The following expression is applied to get the forth moments about the mean of a generalized three parameter Suja distribution;

$$\mu_4 = \mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2$$

Proposition 6.4.4. *Other related measures of a generalized three parameter Suja distribution Eq.6.29 such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) are stated as;*

$$C.v = \frac{\sqrt{\alpha^2\theta^8 + 528\alpha\beta\theta^4 + 5840\beta^2}}{\alpha\theta^4 + 120\beta} \tag{6.34}$$

$$\upsilon_1 = \frac{2(\alpha^3\theta^{12} + 1512\alpha^2\beta\theta^8 + 1728\alpha\beta^2\theta^4 + 69120\beta^3)}{(\alpha^2\theta^8 + 528\alpha\beta\theta^4 + 5840\beta^2)^{\frac{3}{2}}} \tag{6.35}$$

$$\upsilon_2 = \frac{\mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2}{(\mu_2^1 - (\mu_1^1))^2} \tag{6.36}$$

$$\upsilon_3 = \frac{\alpha^2\theta^8 + 528\alpha\beta\theta^4 + 5840\beta^2}{\theta(\alpha\theta^4 + 24\beta)(\alpha\theta^4 + 120\beta)} \tag{6.37}$$

Proof. To begin with, variation coefficient Eq6.34 is obtained as;

$$C.v = \frac{\sigma}{\mu_1^1}$$
$$= \frac{\sqrt{\alpha^2\theta^8 + 528\alpha\beta\theta^4 + 5840\beta^2}}{\theta(\alpha\theta^4 + 24\beta)} \frac{\theta(\alpha\theta^4 + 24\beta)}{\alpha\theta^4 + 120\beta} \Longrightarrow \frac{\sqrt{\alpha^2\theta^8 + 528\alpha}}{\alpha\theta^4 +}$$

further skewness coefficient Eq.6.35 is obtained as;

$$\upsilon_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}}$$
$$= \frac{2(\alpha^3\theta^{12} + 1512\alpha^2\beta\theta^8 + 1728\alpha\beta^2\theta^4 + 69120\beta^3)}{\theta^3(\alpha\theta^4 + 24\beta)^3}\left[\frac{\theta^2(\alpha\theta^4 + 24\beta)\cdots}{\alpha^2\theta^8 + 528\alpha\beta\theta^4 + 5\cdots}\right.$$
$$= \frac{2(\alpha^3\theta^{12} + 1512\alpha^2\beta\theta^8 + 1728\alpha\beta^2\theta^4 + 69120\beta^3)}{(\alpha^2\theta^8 + 528\alpha\beta\theta^4 + 5840\beta^2)^{\frac{3}{2}}}$$

To get coefficient of kurtosis, the following relation is applied;

$$\upsilon_2 = \frac{\mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2}{(\mu_2^1 - (\mu_1^1))^2}$$

lastly, dispersion index Eq.6.37 is obtained as;

$$\upsilon_3 = \frac{\sigma^2}{\mu_1^1}$$
$$= \frac{\alpha^2\theta^8 + 528\alpha\beta\theta^4 + 5840\beta^2}{\theta^2(\alpha\theta^4 + 24\beta)^2}\frac{\theta(\alpha\theta^4 + 24\beta)}{\alpha\theta^4 + 120\beta} \Longrightarrow \frac{\alpha^2\theta^8 + 528\alpha\beta\theta^4 + 584\cdots}{\theta(\alpha\theta^4 + 24\beta)(\alpha\theta^4 + 12\cdots}$$

□

6.4.4 Excess Loss Distribution

Proposition 6.4.5. *In this section, we state probability density function of excess function $f_l(x;\alpha,\beta,\theta)$, mean residual lifetime (MRL), equilibrium distribution $f_e(x;\alpha,\beta,\theta)$, survival function based on the equilibrium distribution $S_e(x;\alpha,\beta,\theta)$ and hazard function of equilibrium distribution $h_e(x;\alpha,\beta,\theta)$ of a general-*

ized three parameter Suja distribution Eq.6.29 as;

$$f_l(x;\alpha,\beta,\theta)=\frac{\theta^5(\alpha+\beta x)e^{-(x-z)\theta}}{\alpha\theta^4+\beta(24+\theta^4z^4+4\theta^3z^3+12\theta^2z^2+24\theta z)};\quad x>z \tag{6.38}$$

$$m(x)=\frac{\alpha\theta^4+\beta(120+\theta^4x^4+8\theta^3x^3+36\theta^2x^2+96\theta x)}{\theta(\alpha\theta^4+\beta(24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x))} \tag{6.39}$$

$$f_e(x;\alpha,\beta,\theta)=\frac{\theta(\alpha\theta^4+\beta(24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x)e^{-\theta x}}{\alpha\theta^4+120\beta} \tag{6.40}$$

$$S_e(x;\alpha,\beta,\theta)=\frac{(\alpha\theta^4+\beta(120+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x))e^{-\theta x}}{\alpha\theta^4+120\beta} \tag{6.41}$$

$$h_e(x;\alpha,\beta,\theta)=\frac{\theta(\alpha\theta^4+\beta(24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x))}{\alpha\theta^4+\beta(120+\theta^4x^4+8\theta^3x^3+36\theta^2x^2+96\theta x)} \tag{6.42}$$

Proof. To begin with, the pdf of excess loss distribution Eq.6.38 is obtained by use of Eq.1.10 as;

$$\begin{aligned}f_l(x;\alpha,\beta,\theta)&=\frac{\frac{\theta^5}{\alpha\theta^4+24\beta}\left[\alpha+\beta x^4\right]e^{-\theta x}}{\frac{(\alpha\theta^4+\beta(24+\theta^4z^4+4\theta^3z^3+12\theta^2z^2+24\theta z))e^{-\theta z}}{\alpha\theta^4+24\beta}}\\&=\frac{\theta^5(\alpha+\beta x)e^{-(x-z)\theta}}{\alpha\theta^4+\beta(24+\theta^4z^4+4\theta^3z^3+12\theta^2z^2+24\theta z)};\quad x>z\end{aligned}$$

By definition Eq.1.11, mean residual lifetime Eq.6.39 is obtained as;

$$m(x) = \frac{\alpha\theta^4 + 24\beta}{(\alpha\theta^4 + \beta(24 + \theta^4x^4 + 4\theta^3x^3 + 12\theta^2x^2 + 24\theta x))e^{-\theta x}} \frac{I_1}{\alpha\theta^4 + 24\beta}$$

$$I_1 = \int_x^\infty (\alpha\theta^4 + \beta(24 + \theta^4t^4 + 4\theta^3t^3 + 12\theta^2t^2 + 24\theta t))e^{-\theta t}dt$$

$$u = (\alpha\theta^4 + \beta(24 + \theta^4t^4 + 4\theta^3t^3 + 12\theta^2t^2 + 24\theta t)) \Longrightarrow du = 4\beta\theta^4t^3 + 12$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_1 = -(\alpha\theta^4 + \beta(24 + \theta^4t^4 + 4\theta^3t^3 + 12\theta^2t^2 + 24\theta t))\frac{e^{-\theta t}}{\theta} + \int_x^\infty (4\beta\theta^3t^3 +$$

$$I_1 = -(\alpha\theta^4 + \beta(24 + \theta^4t^4 + 4\theta^3t^3 + 12\theta^2t^2 + 24\theta t))\frac{e^{-\theta t}}{\theta} + I_2$$

$$I_2 = \int_x^\infty (4\beta\theta^3t^3 + 12\beta\theta^2t^2 + 24\beta\theta t + 24\beta)e^{-\theta t}dt$$

$$u = (4\beta\theta^3t^3 + 12\beta\theta^2t^2 + 24\beta\theta t + 24\beta) \Longrightarrow du = (12\beta\theta^3t^2 + 24\beta\theta^2t + 2$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_2 = -(4\beta\theta^3t^3 + 12\beta\theta^2t^2 + 24\beta\theta t + 24\beta)\frac{e^{-\theta t}}{\theta} + \int_x^\infty (12\beta\theta^2t^2 + 24\beta\theta t +$$

$$I_2 = -(4\beta\theta^3t^3 + 12\beta\theta^2t^2 + 24\beta\theta t + 24\beta)\frac{e^{-\theta t}}{\theta} + I_3$$

$$I_3 = \int_x^\infty (12\beta\theta^2t^2 + 24\beta\theta t + 24\beta)e^{-\theta t}dt$$

$$u = (12\beta\theta^2t^2 + 24\beta\theta t + 24\beta) \Longrightarrow du = 24\beta\theta^2t + 24\beta\theta$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_3 = -(12\beta\theta^2t^2 + 24\beta\theta t + 24\beta)\frac{e^{-\theta t}}{\theta} + \int_x^\infty (24\beta\theta t + 24\beta)e^{-\theta t}dt$$

$$I_3 = -(12\beta\theta^2t^2 + 24\beta\theta t + 24\beta)\frac{e^{-\theta t}}{\theta} + I_4$$

$$I_4 = \int_x^\infty (24\beta\theta t + 24\beta)e^{-\theta t}dt$$

$$u = (24\beta\theta t + 24\beta) \Longrightarrow du = 24\beta\theta$$

$$u = -\theta t \Longrightarrow du = -\theta dt \Longrightarrow \frac{du}{-\theta} = dt$$
$$I_5 = \int_x^\infty e^u \frac{du}{-\theta} \Longrightarrow \frac{-e^{\theta t}}{\theta}$$

From I_1,I_2,I_3, I_4 and I_5 we have the following;

$$I_1 = \left[\frac{-e^{-\theta t}}{\theta}\left[\alpha\theta^4 + 120\beta + \beta\theta^4 t^4 + 8\beta\theta^3 t^3 + 36\beta\theta^2 t^2 + 96\beta\theta t\right]\right]_x^\infty$$
$$I_1 = \frac{e^{-\theta x}}{\theta}\left[\alpha\theta^4 + 120\beta + \beta\theta^4 x^4 + 8\beta\theta^3 x^3 + 36\beta\theta^2 x^2 + 96\beta\theta x\right]$$
$$= I_1 \frac{1}{(\alpha\theta^4 + \beta(24 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x))e^{-\theta x}}$$
$$m(x) = \frac{\alpha\theta^4 + \beta(120 + \theta^4 x^4 + 8\theta^3 x^3 + 36\theta^2 x^2 + 96\theta x)}{\theta(\alpha\theta^4 + \beta(24 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x))}$$

further equilibrium distribution Eq.6.40 is obtained using Eq.1.12 as;

$$f_e(x;\alpha,\beta,\theta) = \frac{(\alpha\theta^4 + 24\beta + \beta\theta^4 x^4 + 4\beta\theta^3 x^3 + 12\beta\theta^2 x^2 + 24\beta\theta x)e^{-\theta x}}{\alpha\theta^4 + 24\beta}$$
$$= \frac{\theta(\alpha\theta^4 + \beta(24 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x)e^{-\theta x}}{\alpha\theta^4 + 120\beta}$$

the survival function based on equilibrium distribution Eq.6.41 is obtained by use of relation Eq.1.13 as;

$$\int_x^\infty S(t;\alpha,\beta,\theta)dt = \frac{(\alpha\theta^4 + 120\beta + \beta\theta^4 x^4 + 8\beta\theta^3 x^3 + 36\beta\theta^2 x^2 + 96\beta\theta}{\theta(\alpha\theta^4 + 24\beta)}$$
$$= \frac{(\alpha\theta^4 + 120\beta + \beta\theta^4 x^4 + 8\beta\theta^3 x^3 + 36\beta\theta^2 x^2 + 96\beta\theta}{\theta(\alpha\theta^4 + 24\beta)}$$
$$S_e(x;\alpha,\beta,\theta) = \frac{(\alpha\theta^4 + \beta(120 + \theta^4 x^4 + 4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x))e^{-}}{\alpha\theta^4 + 120\beta}$$

hazard function based on the equilibrium distribution Eq.6.42 is obtained by use of Eq.1.14 as;

$$h_e(x;\alpha,\beta,\theta) = \frac{\frac{\theta(\alpha\theta^4+\beta(24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x)e^{-\theta x}}{\alpha\theta^4+120\beta}}{\frac{(\alpha\theta^4+\beta(120+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x))e^{-\theta x}}{\alpha\theta^4+120\beta}}$$

$$= \frac{\theta(\alpha\theta^4+\beta(24+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x))}{\alpha\theta^4+\beta(120+\theta^4x^4+8\theta^3x^3+36\theta^2x^2+96\theta x)}$$

□

7 THREE COMPONENT FINITE GAMMA MIXTURE (Case of Sujatha distribution)

7.1 Introduction

A three component finite gamma mixture a case of Sujatha distribution is considered in this chapter. We shall construct and derive statistical properties of Sujatha distribution and its generalizations. The mixed distribution is expressed in terms of pdf and Cdf. Statistical properties such as reliability analysis measures, equilibrium distribution properties and moments (both centralized and none centralized) are studied for this distribution.

7.2 One parameter Sujatha distribution

7.2.1 Construction of a one parameter Sujatha distribution

Proposition 7.2.1. *Let* $\omega_1 = \frac{\theta^2}{\theta^2+\theta+2}$ *and* $\omega_2 = \frac{\theta}{\theta^2+\theta+2}$ *be weight proportions, one parameter Sujatha distribution is constructed as a three component finite mixed distribution of Gamma* $(1,\theta)$*, Gamma* $(2,\theta)$ *and Gamma* $(3,\theta)$*. The pdf and Cdf of a one parameter Sujatha distribution are stated as;*

$$f(x;\theta) = \frac{\theta^3}{\theta^2+\theta+2}\left[1+x+x^2\right]e^{-\theta x}; x>0, \theta>0 \tag{7.1}$$

$$F(x;\theta) = 1-\left[1+\frac{\theta x(\theta+\theta x+2)}{\theta^2+\theta+2}\right]e^{-\theta x}; x>0, \theta>0 \tag{7.2}$$

Proof. To begin with, by definition of a finite mixture Eq.1.1 the pdf Eq.7.1 is constructed as;

$$
\begin{aligned}
f(x;\theta) &= \frac{\theta^2}{\theta^2+\theta+2}\left[\theta e^{-\theta x}\right] + \frac{\theta}{\theta^2+\theta+2}\left[\frac{\theta^2 e^{-\theta x} x}{\Gamma 2}\right] + \frac{2}{\theta^2+\theta+2}\left[\frac{\theta^3 e^{-\theta x} x^2}{\Gamma 3}\right] \\
&= \frac{\theta^3 e^{-\theta x}}{\theta^2+\theta+2} + \frac{\theta^3 e^{-\theta x} x}{\theta^2+\theta+2} + \frac{2\theta^3 e^{-\theta x} x^2}{(\theta^2+\theta+2)\Gamma 3} \\
&= \frac{\theta^3}{\theta^2+\theta+2}\left[1+x+x^2\right] e^{-\theta x}; x>0, \theta>0
\end{aligned}
$$

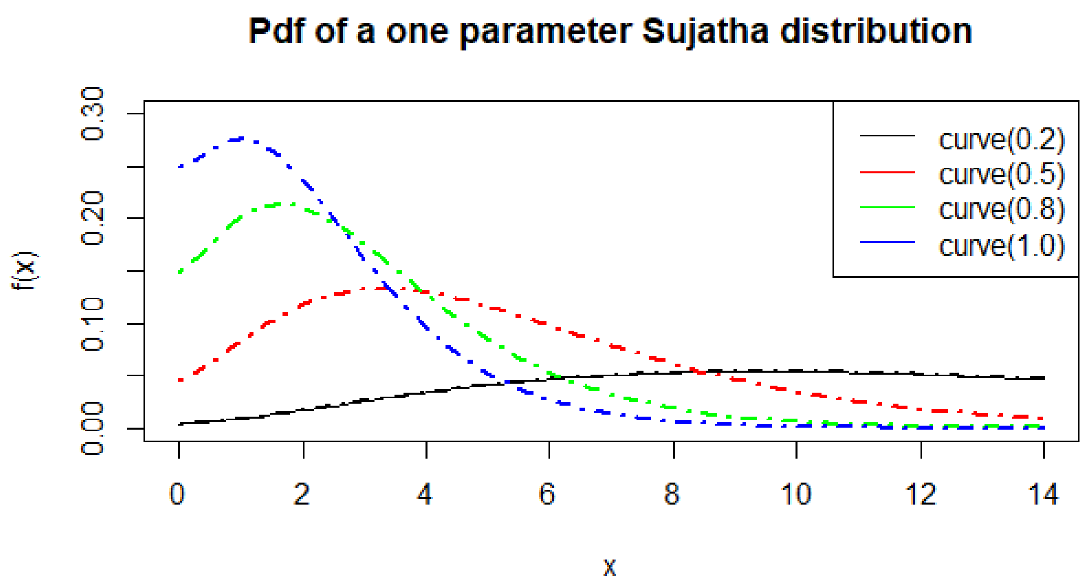

Figure 1. Shapes $f(x)$ of one parameter Sujatha distribution with varying values of θ

further Cdf Eq.7.2 is obtained as;

$$F(x;\theta) = \frac{\theta^3}{\theta^2+\theta+2}\int_0^\infty (1+x+x^2)e^{-\theta x}dx$$
$$F(x;\theta) = \frac{\theta^3}{\theta^2+\theta+2}I_1$$
$$I_1 = \int_0^\infty (1+x+x^2)e^{-\theta x}dx$$
$$u = (1+x+x^2) \Longrightarrow du = (1+2x)$$
$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$
$$I_1 = -(1+x+x^2)\frac{e^{-\theta x}}{\theta} - \frac{1}{\theta}\int_0^\infty -(1+2x)e^{-\theta x}dx$$
$$I_1 = -(1+x+x^2)\frac{e^{-\theta x}}{\theta} - \frac{1}{\theta}I_2$$
$$I_2 = \int_0^\infty -(1+2x)e^{-\theta x}dx$$
$$u = -(1+2x) \Longrightarrow du = -2dx$$
$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$
$$I_2 = (1+2x)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}\int_0^\infty e^{-\theta x}dx$$
$$I_2 = (1+2x)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}I_3$$
$$I_3 = \int_0^\infty e^{-\theta x}dx$$
$$u = -\theta x \Longrightarrow du = -\theta dx \Longrightarrow \frac{du}{-\theta} = dx$$
$$I_3 = \int_0^\infty e^u \frac{du}{-\theta} \Longrightarrow \frac{1}{-\theta}\int_0^\infty e^u du$$
$$I_3 = \frac{-e^{-\theta x}}{\theta}$$

From I_1, I_2, I_3 we have the following;

$$I_1 = -(1+x+x^2)\frac{e^{-\theta x}}{\theta} - \frac{1}{\theta}\left[(1+2x)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}(\frac{-1}{\theta}e^{-\theta x})\right]$$

$$I_1 = \frac{-e^{-\theta x}}{\theta^3}\left[\theta^2+\theta+2+\theta^2 x+\theta^2 x^2+2\theta x\right]$$

$$F(x;\theta) = 1-\left[1+\frac{\theta x(\theta+\theta x+2)}{\theta^2+\theta+2}\right]e^{-\theta x}; x>0, \theta>0$$

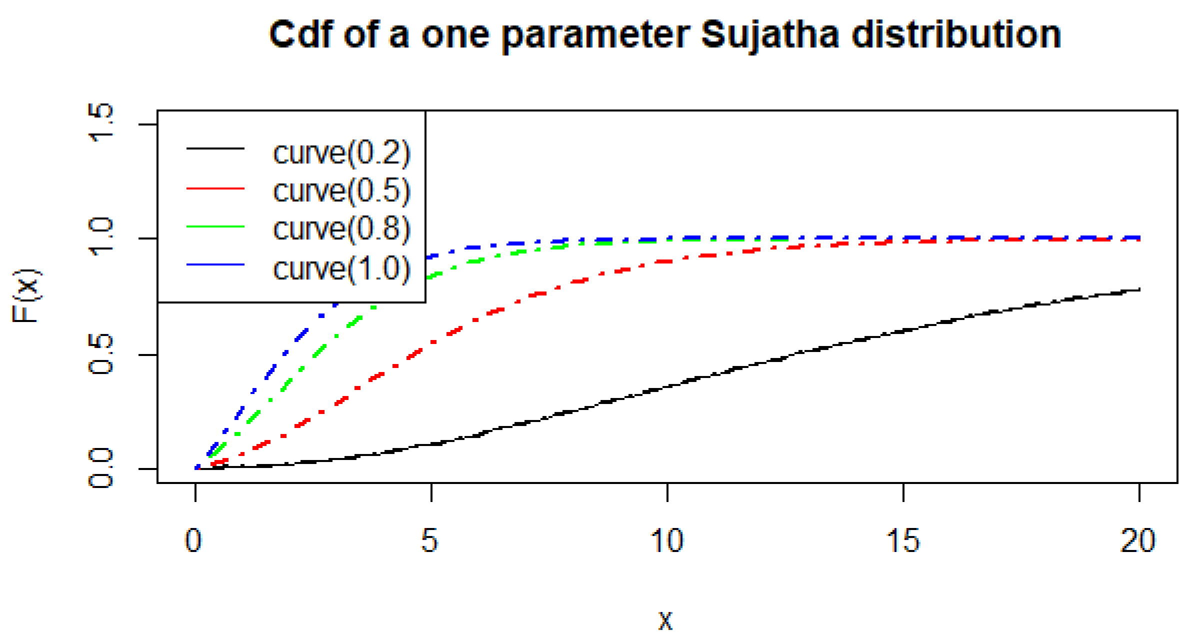

Figure 2. Shapes of CDF of one parameter Sujatha distribution with varying values of θ

□

7.2.2 Reliability Analysis

Proposition 7.2.2. *In this section, we state the survival function denoted by $S(x;\theta)$ and hazard function denoted by $h(x;\theta)$ of a* *one parameter Sujatha distribution Eq.7.1 as;*

$$S(x;\theta) = \left[1 + \frac{\theta x(\theta+\theta x+2)}{\theta^2+\theta+2}\right] e^{-\theta x}; x>0, \theta>0 \tag{7.3}$$

$$h(x;\theta) = \frac{\theta^3(1+x+x^2)}{\theta^2+\theta+2+\theta x(\theta+\theta x+2)}; x>0, \theta>0 \tag{7.4}$$

Proof. To begin with, survival function Eq.7.3 is obtained by use of relation Eq.1.8 as;

$$S(x;\theta) = 1 - \left[1 - \left[1 + \frac{\theta x(\theta+\theta x+2)}{\theta^2+\theta+2}\right] e^{-\theta x}\right]$$
$$= \left[1 + \frac{\theta x(\theta+\theta x+2)}{\theta^2+\theta+2}\right] e^{-\theta x}; x>0, \theta>0$$

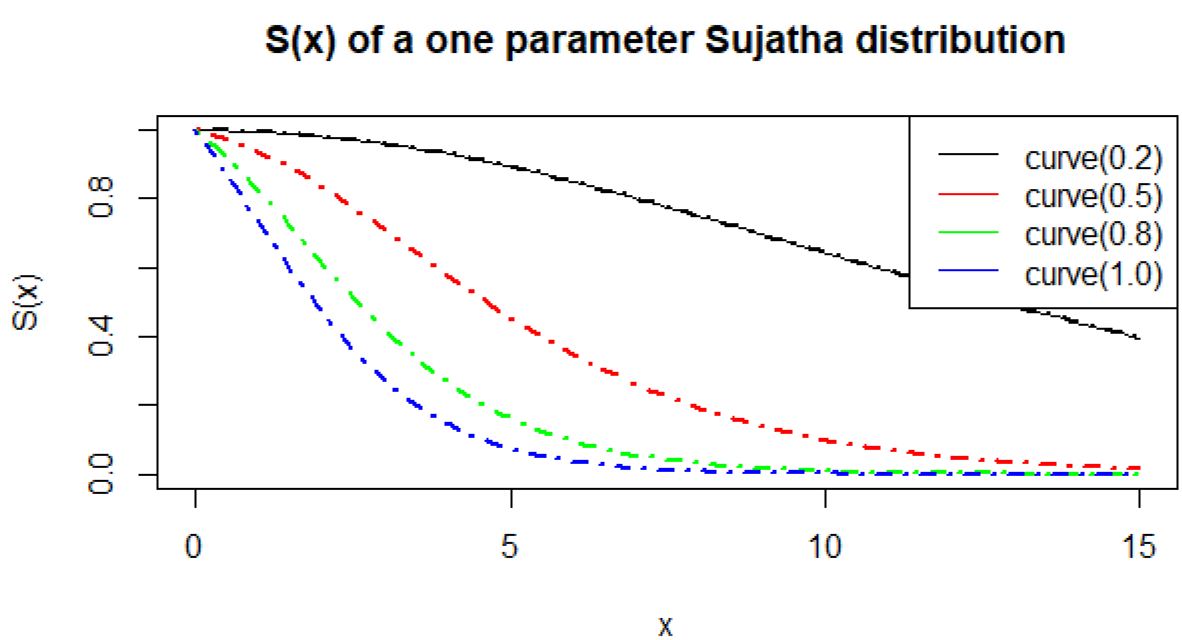

Figure 3. Shapes of $S(x)$ of one parameter Sujatha distribution with varying values of θ

further hazard function Eq.7.4 is obtained using the relation Eq.1.9

$$h(x;\theta) = \frac{\frac{\theta^3}{\theta^2+\theta+2}\left[1+x+x^2\right]e^{-\theta x}}{\left[\frac{\theta^2+\theta+2+\theta x(\theta+\theta x+2)}{\theta^2+\theta+2}\right]e^{-\theta x}}$$

$$= \frac{\theta^3(1+x+x^2)}{\theta^2+\theta+2+\theta x(\theta+\theta x+2)}; x>0, \theta>0$$

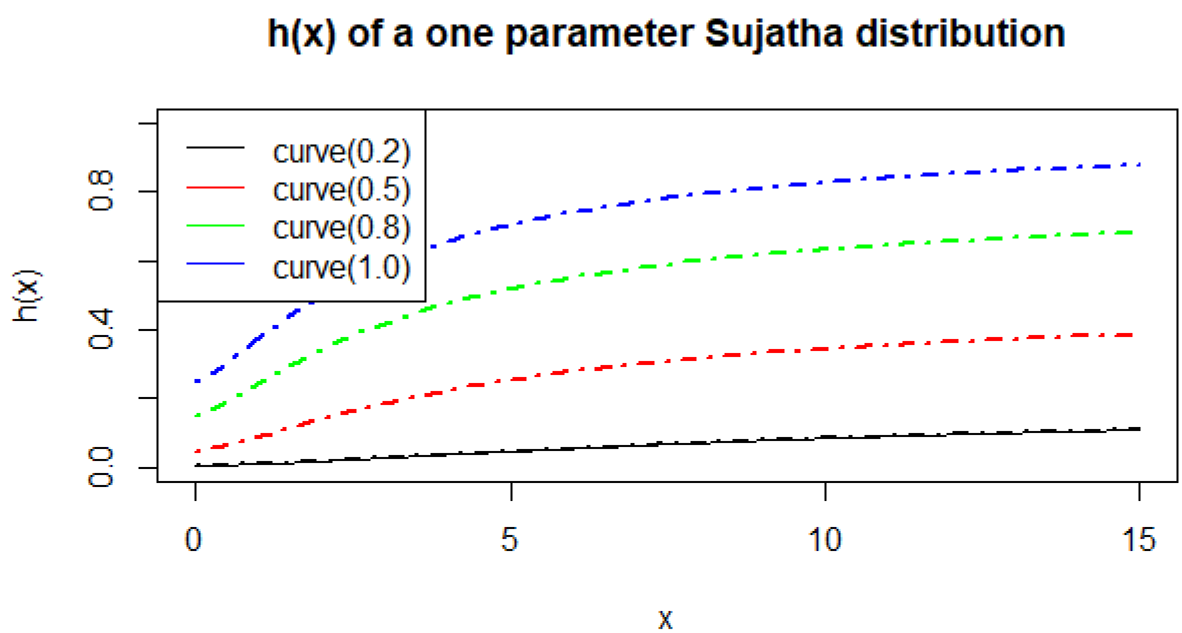

Figure 4. Shapes of $h(x)$ of Sujatha distribution with different values of θ

□

7.2.3 Moments and related measures

Proposition 7.2.3. *The r^{th} moments of a one parameter Sujatha distribution Eq.7.1 are derived using both method of moments* *and moment generating function as;*

$$\mu_r^1 * = \frac{r!\left[\theta^2+\theta(r+1)+(r+1)(r+2)\right]}{\theta^r(\theta^2+\theta+2)}; \quad r=1,2,3,\cdots$$

(7.5)

Proof. By definition Eq.1.16, moments of one parameter Sujatha distribution are obtained as;

$$\begin{aligned}
E(X^r) &= \frac{\theta^3}{\theta^2+\theta+2}\int_0^\infty x^r(1+x+x^2)e^{-\theta x}dx \\
&= \frac{\theta^3}{\theta^2+\theta+2}\left[\int_0^\infty x^r e^{-\theta x}dx+\int_0^\infty x^{r+1}e^{-\theta x}dx+\int_0^\infty x^{r+2}e^{-\theta x}dx\right] \\
&= \frac{\theta^3}{\theta^2+\theta+2}\left[\frac{\Gamma(r+1)}{\theta^{r+1}}+\frac{\Gamma(r+2)}{\theta^{r+2}}+\frac{\Gamma(r+3)}{\theta^{r+3}}\right] \\
&= \frac{r\Gamma r\left[\theta^2+\theta(r+1)+(r+1)(r+2)\right]}{\theta^r(\theta^2+\theta+2)} \\
&= \frac{r!\left[\theta^2+\theta(r+1)+(r+1)(r+2)\right]}{\theta^r(\theta^2+\theta+2)}; \quad r=1,2,3,\cdots
\end{aligned}$$

further by definition Eq.1.17, mgf of one parameter Sujatha is obtained as;

$$\begin{aligned}
m_x(t) &= \frac{\theta^3}{\theta^2+\theta+2}\int_0^\infty e^{tx}(1+x+x^2)e^{-\theta x}dx \\
&= \frac{\theta^3}{\theta^2+\theta+2}\int_0^\infty e^{-(\theta-t)x}(1+x+x^2)dx \\
&= \frac{\theta^3}{\theta^2+\theta+2}\left[\frac{1}{\theta-t}+\frac{1}{(\theta-t)^2+\frac{2}{(\theta-t)^3}}\right] \\
&= \frac{\theta^3}{\theta^2+\theta+2}\left[(\frac{1}{\theta})\sum_{k=0}^{\infty}(\frac{t}{\theta})^k+(\frac{1}{\theta^2})\sum_{k=0}^{\infty}\binom{k+1}{k}(\frac{t}{\theta})^k+(\frac{2}{\theta^3})\sum_{k=0}^{\infty}\binom{k+2}{k}\right. \\
&= \sum_{k=0}^{\infty}\frac{\theta^2+\theta(k+1)+(k+)(k+2)}{\theta^2+\theta+2}(\frac{t}{\theta})^k
\end{aligned}$$

The none centralized moments of a one parameter Sujatha distribution Eq.7.1 are obtained as a coefficient $\frac{t^r}{r!}$ in the moment generating function $m_x(t)$ as;

$$\mu_r^1 = \frac{r!\left[\theta^2+\theta(r+1)+(r+1)(r+2)\right]}{\theta^r(\theta^2+\theta+2)}; \quad r = 1,2,3,\cdots$$

□

For r values as 1,2,3 and 4 in Eq.7.5 we obtain four moments about the origin of a one parameter Sujatha distribution as;

$$\mu_1^1 = \frac{\theta^2+2\theta+6}{\theta(\theta^2+\theta+2)}, \quad \mu_2^1 = \frac{2(\theta^2+3\theta+12)}{\theta^2(\theta^2+\theta+2)}$$
$$\mu_3^1 = \frac{6(\theta^2+4\theta+20)}{\theta^3(\theta^2+\theta+2)}, \quad \mu_4^1 = \frac{24(\theta^2+5\theta+30)}{\theta^4(\theta^2+\theta+2)}$$

The centralized moments of a one parameter Sujatha distribution are;

$$\mu_1 = \mu_1^1$$

$$\mu_2 = \mu_2^1 - [\mu_1^1]^2 \Longrightarrow \frac{\theta^4 + 4\theta^3 + 18\theta^2 + 12\theta + 12}{\theta^2(\theta^2 + \theta + 2)^2}$$

$$\mu_3 = \mu_3^1 - 3[\mu_1^1 \mu_2^1] + 2[\mu_1^1]^3$$

$$\mu_3 = \frac{2\theta^6 + 12\theta^5 + 72\theta^4 + 88\theta^3 + 108\theta^2 + 72\theta + 48}{\theta^3(\theta^2 + \theta + 2)^3}$$

$$\mu_4 = \mu_4^1 - 4[\mu_3^1 \mu_1^1] + 6[\mu_1^1 \mu_2^1] - 3[\mu_1^1]^2$$

$$\mu_4 = \frac{3(3\theta^8 + 24\theta^7 + 172\theta^6 + 376\theta^5 + 736\theta^4 + 864\theta^3 + 912\theta^2 + 480\theta +}{\theta^4(\theta^2 + \theta + 2)^4}$$

Proposition 7.2.4. *Other related measures of a one parameter Sujatha distribution Eq.7.1 such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) are stated as;*

$$C.v = \frac{\sqrt{\theta^4 + 4\theta^3 + 18\theta^2 + 12\theta + 12}}{\theta^2 + 2\theta + 6} \tag{7.6}$$

$$\upsilon_1 = \frac{2\theta^6 + 12\theta^5 + 72\theta^4 + 88\theta^3 + 108\theta^2 + 72\theta + 48}{(\theta^4 + 4\theta^3 + 18\theta^2 + 12\theta + 12)^{\frac{3}{2}}} \tag{7.7}$$

$$\upsilon_2 = \frac{3(3\theta^8 + 24\theta^7 + 172\theta^6 + 376\theta^5 + 736\theta^4 + 864\theta^3 + 912\theta^2 + 480\theta}{(\theta^4 + 4\theta^3 + 18\theta^2 + 12\theta + 12)^2} \tag{7.8}$$

$$\upsilon_3 = \frac{\theta^4 + 4\theta^3 + 18\theta^2 + 12\theta + 12}{\theta(\theta^2 + \theta + 2)(\theta^2 + 2\theta + 6)} \tag{7.9}$$

Proof. To begin with, coefficient of variation Eq.7.6 is obtained as;

$$C.v = \frac{\sigma}{\mu_1^1}$$
$$= \frac{\sqrt{\theta^4+4\theta^3+18\theta^2+12\theta+12}}{\theta(\theta^2+\theta+2)} \frac{\theta(\theta^2+\theta+2)}{\theta^2+2\theta+6} \Longrightarrow \frac{\sqrt{\theta^4+4\theta^3+18\theta^2+}}{\theta^2+2\theta+6}$$

Secondly, skewness coefficient Eq.7.7 is obtained as;

$$\upsilon_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}}$$
$$= \frac{2\theta^6+12\theta^5+72\theta^4+88\theta^3+108\theta^2+72\theta+48}{\theta^3(\theta^2+\theta+2)^3} \left[\frac{\theta^2(\theta^2+\theta+2)^2}{\theta^4+4\theta^3+18\theta^2+12\theta} \right.$$
$$= \frac{2\theta^6+12\theta^5+72\theta^4+88\theta^3+108\theta^2+72\theta+48}{(\theta^4+4\theta^3+18\theta^2+12\theta+12)^{\frac{3}{2}}}$$

Thirdly, kurtosis coefficient Eq.7.8 is obtained as;

$$\upsilon_2 = \frac{\mu_4}{(\mu_2)^2}$$
$$= \frac{3(3\theta^8+24\theta^7+172\theta^6+376\theta^5+736\theta^4+864\theta^3+912\theta^2+480\theta+240)}{\theta^4(\theta^2+\theta+2)^4}$$
$$= \frac{3(3\theta^8+24\theta^7+172\theta^6+376\theta^5+736\theta^4+864\theta^3+912\theta^2+480\theta+240)}{(\theta^4+4\theta^3+18\theta^2+12\theta+12)^2}$$

lastly, index of dispersion Eq.7.9is obtained as;

$$\upsilon_3 = \frac{\sigma^2}{\mu_1^1}$$
$$= \frac{\theta^4+4\theta^3+18\theta^2+12\theta+12}{\theta^2(\theta^2+\theta+2)^2} \frac{\theta(\theta^2+\theta+2)}{\theta^2+2\theta+6} \Longrightarrow \frac{\theta^4+4\theta^3+18\theta^2+12\theta}{\theta(\theta^2+\theta+2)(\theta^2+2\theta}$$

□

7.2.4 Excess Loss Distribution

Proposition 7.2.5. *In this section, we state probability density function of excess function $f_l(x;\theta)$, mean residual lifetime (MRL), equilibrium distribution $f_e(x;\theta)$, survival function based on the equilibrium distribution $S_e(x;\theta)$ and hazard function based on the equilibrium distribution $h_e(x;\theta)$ of one parameter Sujatha distribution Eq.7.1 as;*

$$f_e(x;\theta) = \frac{\theta^3(1+x+x^2)e^{-(x-z)\theta}}{\theta^2+\theta+2+\theta z(\theta+\theta z+2)}; \quad x > z \tag{7.10}$$

$$m(x) = \frac{\theta^2+2\theta+\theta^2x^2+\theta^2x+4\theta x+6}{\theta(\theta^2+\theta+2+\theta x(\theta x+\theta+2))} \tag{7.11}$$

$$f_e(x;\theta) = \frac{\theta(\theta^2+\theta+2+\theta^2x+\theta^2x^2+2\theta x)e^{-\theta x}}{\theta^2+2\theta+6} \tag{7.12}$$

$$S_e(x;\theta) = \frac{(\theta^2+2\theta+\theta^2x^2+\theta^2x+4\theta x+6)e^{-\theta x}}{\theta^2+2\theta+6} \tag{7.13}$$

$$h_e(x;\theta) = \frac{\theta(\theta^2+\theta+2+\theta^2x+\theta^2x^2+2\theta x)}{\theta^2+2\theta+\theta^2x^2+\theta^2x+4\theta x+6} \tag{7.14}$$

Proof . To begin with, pdf of excess loss distribution Eq.7.10 is obtained by use of Eq.1.10 as;

$$\begin{aligned} f_e(x;\theta) &= \frac{\frac{\theta^3}{\theta^2+\theta+2}\left[1+x+x^2\right]e^{-\theta x}}{\left[\frac{\theta^2+\theta+2+\theta z(\theta+\theta z+2)}{\theta^2+\theta+2}\right]e^{-\theta z}} \\ &= \frac{\theta^3(1+x+x^2)e^{-(x-z)\theta}}{\theta^2+\theta+2+\theta z(\theta+\theta z+2)}; \quad x > z \end{aligned}$$

Secondly, the mean residual lifetime Eq.7.11 is obtained by the use of Eq.1.11 as;

$$m(x) = \frac{\theta^2+\theta+2}{[\theta^2+\theta+2+\theta x(\theta x+\theta+2)]e^{-\theta x}} \int_x^\infty \frac{[\theta^2+\theta+2+\theta t(\theta t+\theta+2)]e^{-}}{\theta^2+\theta+2}$$
$$= \frac{1}{[\theta^2+\theta+2+\theta x(\theta x+\theta+2)]e^{-\theta x}} \int_x^\infty [\theta^2+\theta+2+\theta t(\theta t+\theta+2)]e^{-}$$

Taking the part $I_1 = \int_x^\infty [\theta^2+\theta+2+\theta t(\theta t+\theta+2)]e^{-\theta t}dt$ and using integration by parts technique we have the following;

$$u = \theta^2+\theta+2+\theta t(\theta t+\theta+2) \Longrightarrow du = (2\theta^2 t+\theta^2 t+2\theta)$$
$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$
$$I_1 = -(\theta^2+\theta+2+\theta t(\theta t+\theta+2))\frac{e^{-\theta t}}{\theta} + \int_x^\infty (2\theta t+\theta+2)e^{-\theta t}dt$$
$$I_1 = -(\theta^2+\theta+2+\theta t(\theta t+\theta+2))\frac{e^{-\theta t}}{\theta} + I_2$$
$$I_2 = \int_x^\infty (2\theta t+\theta+2)e^{-\theta t}dt$$
$$u = (2\theta t+\theta+2) \Longrightarrow du = 2\theta dt$$
$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$
$$I_2 = -(2\theta t+\theta+2)\frac{-e^{\theta t}}{\theta} + 2\int_x^\infty e^{-\theta t}dt$$
$$I_2 = -(2\theta t+\theta+2)\frac{-e^{\theta t}}{\theta} + 2I_3$$
$$I_3 = \int_x^\infty e^{-\theta t}dt \Longrightarrow \frac{-e^{-\theta t}}{\theta}$$

From I_1, I_2 and I_3 we have the following;

$$I_1 = \left[\frac{-e^{\theta t}}{\theta}\left[\theta^2+2\theta+\theta^2t^2+\theta^2t+4\theta t+6\right]\right]_x^\infty$$

$$I_1 = \frac{e^{-\theta x}}{\theta}\left[\theta^2+2\theta+\theta^2x^2+\theta^2x+4\theta x+6\right]$$

$$= \frac{1}{[\theta^2+\theta+2+\theta x(\theta x+\theta+2)]e^{-\theta x}} * \frac{e^{-\theta x}}{\theta}\left[\theta^2+2\theta+\theta^2x^2+\theta^2x\right.$$

$$m(x) = \frac{\theta^2+2\theta+\theta^2x^2+\theta^2x+4\theta x+6}{\theta(\theta^2+\theta+2+\theta x(\theta x+\theta+2))}$$

by definition of equilibrium distribution Eq.1.12, $f_e(x;\theta)$ is obtained as;

$$f_e(x;\theta) = \frac{(\theta^2+\theta+2+\theta^2x+\theta^2x^2+2\theta x)e^{-\theta x}}{\theta^2+2\theta+2}\frac{\theta(\theta^2+\theta+2)}{\theta^2+2\theta+6}$$

$$= \frac{\theta(\theta^2+\theta+2+\theta^2x+\theta^2x^2+2\theta x)e^{-\theta x}}{\theta^2+2\theta+6}$$

further by definition Eq.1.13, $S_e(x;\theta)$ in Eq.7.13 is obtained as;

$$\int_x^\infty S(t;\theta)dt = \frac{[\theta^2+2\theta+\theta^2x^2+\theta^2x+4\theta x+6]e^{-\theta x}}{\theta(\theta^2+\theta+2)}$$

$$= \frac{[\theta^2+2\theta+\theta^2x^2+\theta^2x+4\theta x+6]e^{-\theta x}}{\theta(\theta^2+\theta+2)}\frac{\theta(\theta^2+\theta+2)}{\theta^2+2\theta+6}$$

$$S_e(x;\theta) = \frac{(\theta^2+2\theta+\theta^2x^2+\theta^2x+4\theta x+6)e^{-\theta x}}{\theta^2+2\theta+6}$$

lastly, hazard function based on equilibrium distribution Eq.7.14 is obtained by use of the relation Eq.1.14 as;

$$h_e(x;\theta) = \frac{\theta(\theta^2+\theta+2+\theta^2x+\theta^2x^2+2\theta x)e^{-\theta x}}{\theta^2+2\theta+6}\frac{\theta^2+2\theta+}{(\theta^2+2\theta+\theta^2x^2+\theta^2x}$$

$$= \frac{\theta(\theta^2+\theta+2+\theta^2x+\theta^2x^2+2\theta x)}{\theta^2+2\theta+\theta^2x^2+\theta^2x+4\theta x+6}$$

$\square$

7.2.5 Estimation of one parameter Sujatha distribution

In this section, we shall discuss two modes of estimation that are MOME and MLE.

MLE technique

Let $x_1, x_2, x_3, \cdots n$ denote a random sample drawn of size n from a One Parameter Sujatha Distribution Eq.7.1, the L function is obtained as:

$$L(x;\theta) = \prod_{i=1}^{n} f(x;\theta)$$

$$= \prod_{i=1}^{n} \left[\frac{\theta^3}{\theta^2+\theta+2}(1+x_i+x_i^2)e^{-\theta x_i}\right]$$

$$L(x;\theta) = \left[\frac{\theta^3}{\theta^2+\theta+2}\right]^n \prod_{i=1}^{n}(1+x_i+x_i^2)e^{-\theta \Sigma_{i=1}^{n} x_i}$$

$$lnL = nln\left(\frac{\theta^3}{\theta^2+\theta+2}\right) + \sum_{i=1}^{n} ln(1+x_i+x_i^2) - \theta\sum_{i=1}^{n} x_i$$

$$= n[3ln\theta - ln(\theta^2+\theta+2)] + \sum_{i=1}^{n} ln(1+x_i+x_i^2) - n\theta\bar{x}$$

$$\frac{\partial lnL}{\partial\theta} = \frac{3n}{\theta} - \frac{n(2\theta+1)}{\theta^2+\theta+2} - n\bar{x} = 0$$

The value $\bar{x}$ is the mean of the sample. We obtain the MLE of $\hat{\theta}$ of θ of a One parameter Sujatha distribution Eq.7.1 by equating $\frac{\partial lnL}{\partial\theta} = 0$, which is obtained as a solution of the polynomial of degree three in θ.

$$-\bar{x}\theta^3-(\bar{x}-1)\theta^2-2(\bar{x}-1)\theta+6=0 \tag{7.15}$$

Method of Moments (MOME)

The population mean of a one parameter Sujatha distribution Eq.7.1 is equated with the mean of the sample to give MOME of $\hat{\theta}$ of θ which is equal to the solution of Eq.7.15.

7.2.6 Application

We shall apply lifetime data regarding carbon fiber breaking stress presented in table 1 that was recorded in GPa (Nichols and Padgett, 2006). For illustrative purposes, one parameter distributions Aradhana, Sujatha, Akash, Shanker, Lindley and Exponential are fitted and the best fitted distribution is selected based on accuracy measures.

Table 1. Stress breaking data of 66 carbon fibers measured in GPa

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47	3.11	3.56
4.42	2.41	3.19	3.22	1.69	3.28	3.09	1.87	3.15	4.90	1.57
2.67	2.93	3.22	3.39	2.81	4.20	3.33	2.55	3.31	3.31	2.85
1.25	4.38	1.84	0.39	3.68	2.48	0.85	1.61	2.79	4.70	2.03
1.89	2.88	2.82	2.05	3.65	3.75	2.43	2.95	2.97	3.39	2.96
2.35	2.55	2.59	2.03	1.61	2.12	3.15	1.08	2.56	1.80	2.53

Table 2. Descriptive Summary of the data

Min	1 st Qu.	Median	Mean	3 rd Qu.	Max	Var
0.390	2.178	2.835	2.760	3.277	4.900	0.7938

Illustration in table 2 displays descriptive summary statistics of the lifetime data. The mean value is greater than the variance hence, the lifetime data is over-dispersed.
The goodness of fit measures like -2lnL, AIC, BIC and AICC have been used to select the best fitted distribution. The goodness of fit measures are calculated as;

$$AIC = -2lnL + 2p$$

$$AICC = \frac{2p(p+1)}{n-p-1} + AIC$$

$$BIC = -2lnL + pln(n)$$

Where p and n are the parameter number included in the distribution and size of the sample respectively.

Table 3. MLE estimates, -2lnL, AIC, AICC and BIC of the fitted one parameter distributions

Model	Estimate	-2lnL	AIC	AICC	BIC
Suja	$\hat{\theta}$=1.5378	207.9748	209.9748	210.0373	212.1645
Rama	$\hat{\theta}$=1.2043	218.23	220.230	220.2925	222.4197
Aradhana	$\hat{\theta}$= 0.8332	228.8408	228.8408	230.9033	233.0305
Sujatha	$\hat{\theta}$=0.8533	229.5849	231.5849	231.6474	233.7446
Akash	$\hat{\theta}$=0.8835	230.6726	232.6726	232.7351	234.8623
Shanker	$\hat{\theta}$=0.6028	239.4447	241.4447	241.5072	243.6344
Lindley	$\hat{\theta}$=0.5902	244.7740	246.7740	246.8365	248.9637
Exponential	$\hat{\theta}$=0.3623	266.0032	268.0032	268.0657	270.1929

Distribution with lower values of -2lnL, BIC, AIC and AICC is the best fitted one. Suja distribution is best fitted among the estimated distributions. In figure 5 below, the density of One Parameter Suja Distribution has the highest curve than other fits. Therefore, One Parameter Suja is the best fitted model.

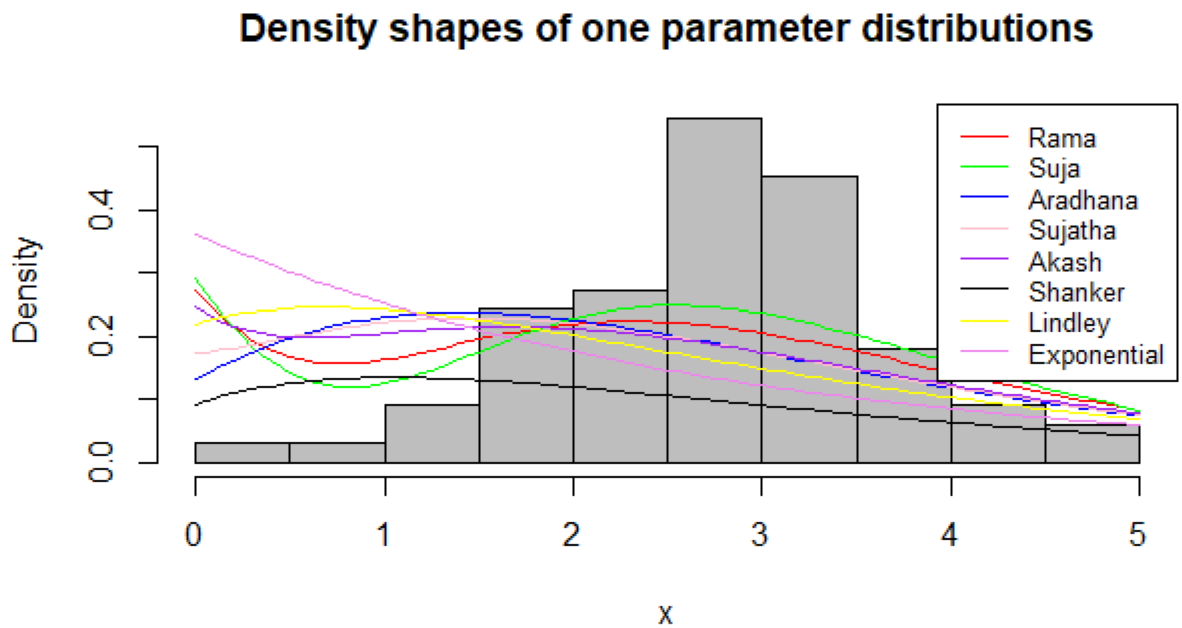

Figure 5. Estimated Densities of fitted One Parameter Distributions

7.3 Two parameter Sujatha distribution

7.3.1 Construction of a two parameter Sujatha distribution

Proposition 7.3.1. *Let* $\omega_1 = \frac{\theta^2}{\theta^2+\alpha\theta+2\alpha}$ *and* $\omega_2 = \frac{\alpha\theta}{\theta^2+\alpha\theta+2\alpha}$ *be the mixing weights , AG2PSD is a three component finite mixed distribution of Gamma* $(1,\theta)$ *Gamma* $(2,\theta)$ *and Gamma* $(3,\theta)$*. The pdf and Cdf of AG2PSD are stated as;*

$$f(x;\alpha,\theta) = \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\left[1+\alpha x+\alpha x^2\right]e^{-\theta x}; x>0, \alpha>0, \theta>0 \tag{7.16}$$

$$F(x;\alpha,\theta) = 1-\left[1+\frac{\alpha\theta^2 x+\alpha\theta^2 x^2+2\alpha\theta x}{\theta^2+\alpha\theta+2\alpha}\right]e^{-\theta x}; x>0, \alpha>0, \theta>0 \tag{7.17}$$

Proof. By definition of finite mixture Eq.1.1, pdf Eq.7.16 is constructed as;

$$f(x;\alpha,\theta) = \frac{\theta^2}{\theta^2+\alpha\theta+2\alpha}\left[\theta e^{-\theta x}\right] + \frac{\alpha\theta}{\theta^2+\alpha\theta+2\alpha}\left[\frac{\theta^2 e^{-\theta x}x}{\Gamma 2}\right] + \frac{2\alpha}{\theta^2+\alpha\theta+}$$
$$= \frac{\theta^3 e^{-\theta x}}{\theta^2+\alpha\theta+2\alpha} + \frac{\alpha\theta^3 x e^{-\theta x}}{\theta^2+\alpha\theta+2\alpha} + \frac{2\alpha\theta^3 e^{-\theta x}x^2}{2(\theta^2+\alpha\theta+2\alpha)}$$
$$= \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\left[1+\alpha x+\alpha x^2\right]e^{-\theta x}; x>0, \alpha>0, \theta>0$$

Remark 7.3.2. *A generalized two parameter Sujatha distribution Eq.7.16 is nested with two distributions. Putting* $\alpha = 1$, *turns to one parameter Sujatha distribution Eq.7.1. Similarly, putting* $\alpha = 0$, *turns to exponential distribution Eq.2.17.*

AG2PSD is a modification of Sujatha by Shanker et al., 2017c.

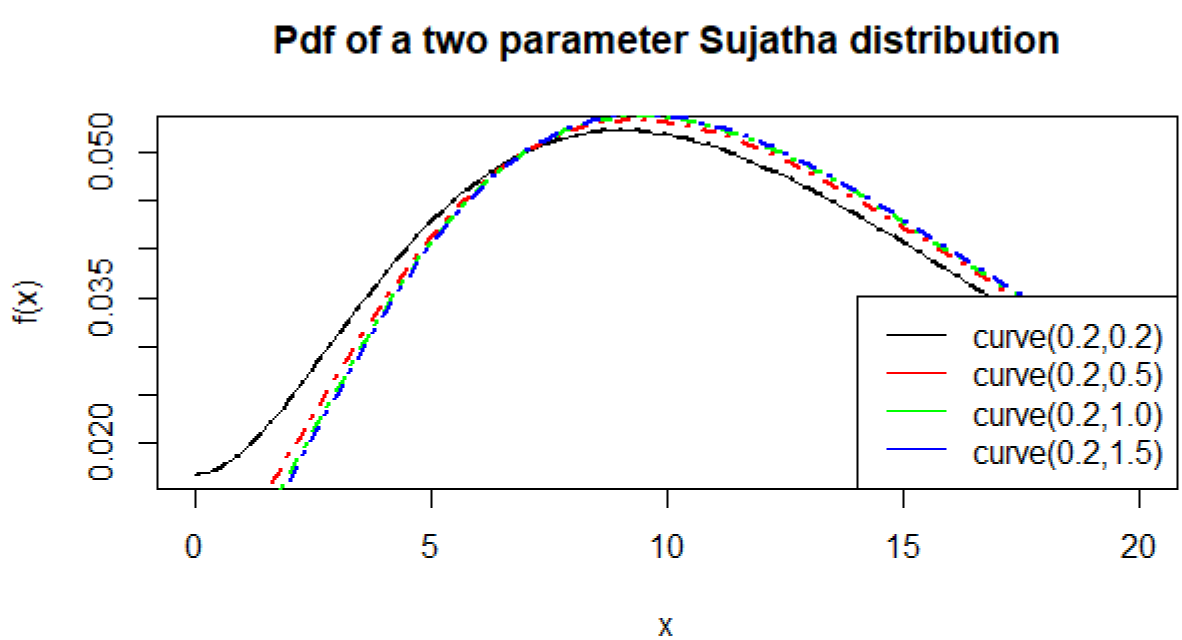

Figure 6. Shapes of $f(x)$ of two parameter Sujatha distribution with varying values of α and constant θ

further Cdf Eq.7.17 is obtained as;

$$F(x;\alpha,\theta) = \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\int_0^\infty (1+\alpha x+\alpha x^2)e^{-\theta x}dx$$

$$F(x;\alpha,\theta) = \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}I_1$$

$$I_1 = \int_0^\infty (1+\alpha x+\alpha x^2)e^{-\theta x}dx$$

$$u = (1+\alpha x+\alpha x^2) \implies du = (\alpha+2\alpha x)$$

$$dv = e^{-\theta x} \implies v = \frac{-e^{-\theta x}}{\theta}$$

$$I_1 = -(1+\alpha x+\alpha x^2)\frac{e^{-\theta x}}{\theta} - \frac{\alpha}{\theta}\int_0^\infty -(1+2x)e^{-\theta x}dx$$

$$I_1 = -(1+\alpha x+\alpha x^2)\frac{e^{-\theta x}}{\theta} - \frac{\alpha}{\theta}I_2$$

$$I_2 = \int_0^\infty -(1+2x)e^{-\theta x}dx$$

$$u = -(1+2x) \implies du = -2dx$$

$$dv = e^{-\theta x} \implies v = \frac{-e^{-\theta x}}{\theta}$$

$$I_2 = (1+2x)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}\int_0^\infty e^{-\theta x}dx$$

$$I_2 = (1+2x)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}I_3$$

$$I_3 = \int_0^\infty e^{-\theta x}dx$$

$$u = -\theta x \implies \frac{du}{-\theta} = dx$$

$$I_3 = \int_0^\infty e^u \frac{du}{-\theta} = \frac{1}{-\theta}\int_0^\infty e^u du = \frac{e^{-\theta x}}{-\theta}$$

From I_1, I_2 and I_3 we have the following;

$$I_1 = -(1+\alpha x+\alpha x^2)\frac{e^{-\theta x}}{\theta} - \frac{\alpha}{\theta}\left[(1+2x)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}(\frac{1}{-\theta})e^{-\theta x}\right]$$

$$I_1 = \frac{-e^{-\theta x}}{\theta^3}\left[2\alpha+\alpha\theta+\theta^2+\alpha\theta^2 x+\alpha\theta^2 x^2+2\alpha\theta x\right]$$

$$F(x;\alpha,\theta) = 1 - \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\frac{-e^{-\theta x}}{\theta^3}\left[2\alpha+\alpha\theta+\theta^2+\alpha\theta^2 x+\alpha\theta^2 x^2+2\alpha\theta x\right]$$

$$= 1 - \left[1+\frac{\alpha\theta^2 x+\alpha\theta^2 x^2+2\alpha\theta x}{\theta^2+\alpha\theta+2\alpha}\right]e^{-\theta x}; x>0, \alpha>0, \theta>0$$

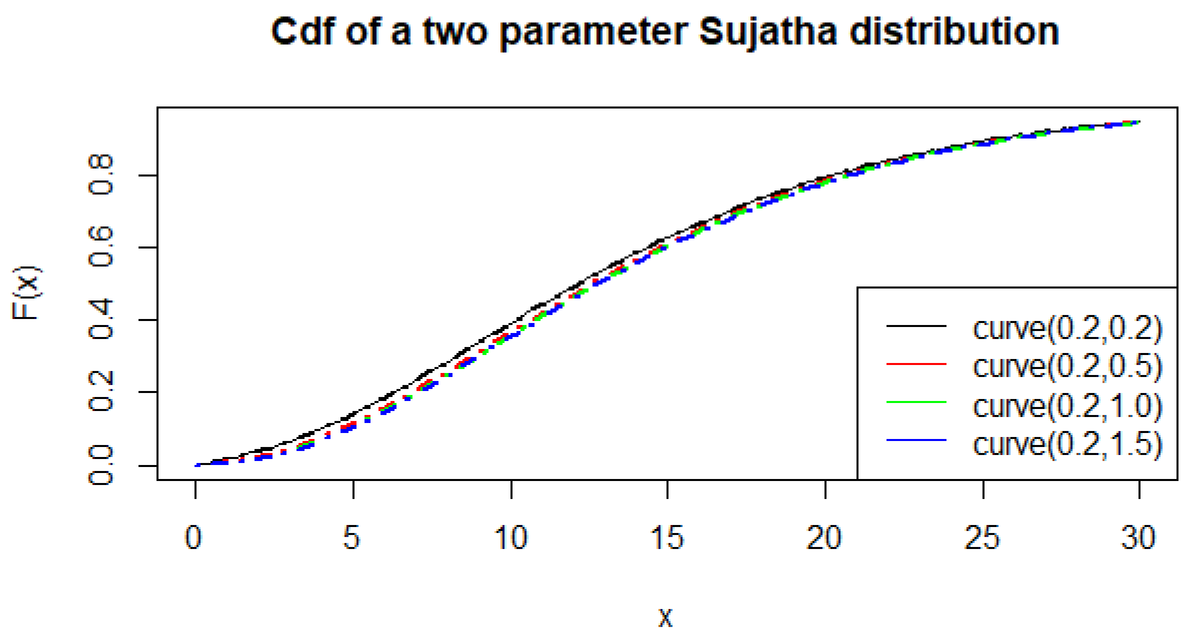

Figure 7. Shapes of $F(x)$ of two parameter Sujatha distribution with varying values of α and constant θ

□

7.3.2 Reliability Analysis

Proposition 7.3.3. *In this section, we state survival function denoted by $S(x;\alpha,\theta)$ and hazard function denoted by $h(x;\alpha,\theta)$*

of a generalized two parameter Sujatha distribution Eq.7.16 as;

$$S(x;\alpha,\theta)=\left[1+\frac{\alpha(\theta^2x+\theta^2x^2+2\theta x)}{\theta^2+\alpha\theta+2\alpha}\right]e^{-\theta x};x>0,\alpha>0,\theta>0 \tag{7.18}$$

$$h(x;\alpha,\theta)=\frac{\theta^3(1+\alpha(x+x^2))}{\theta^2+\alpha(\theta+2+\theta^2x+\theta^2x^2+2\theta x)};x>0,\alpha>0,\theta>0 \tag{7.19}$$

Proof. To begin with, survival function Eq.7.18 is obtained by the use of relation Eq.1.8 as;

$$\begin{aligned}S(x;\alpha,\theta)&=1-\left[1-\left[1+\frac{\alpha\theta^2x+\alpha\theta^2x^2+2\alpha\theta x}{\theta^2+\alpha\theta+2\alpha}\right]e^{-\theta x}\right]\\&=\left[1+\frac{\alpha(\theta^2x+\theta^2x^2+2\theta x)}{\theta^2+\alpha\theta+2\alpha}\right]e^{-\theta x};x>0,\alpha>0,\theta>0\end{aligned}$$

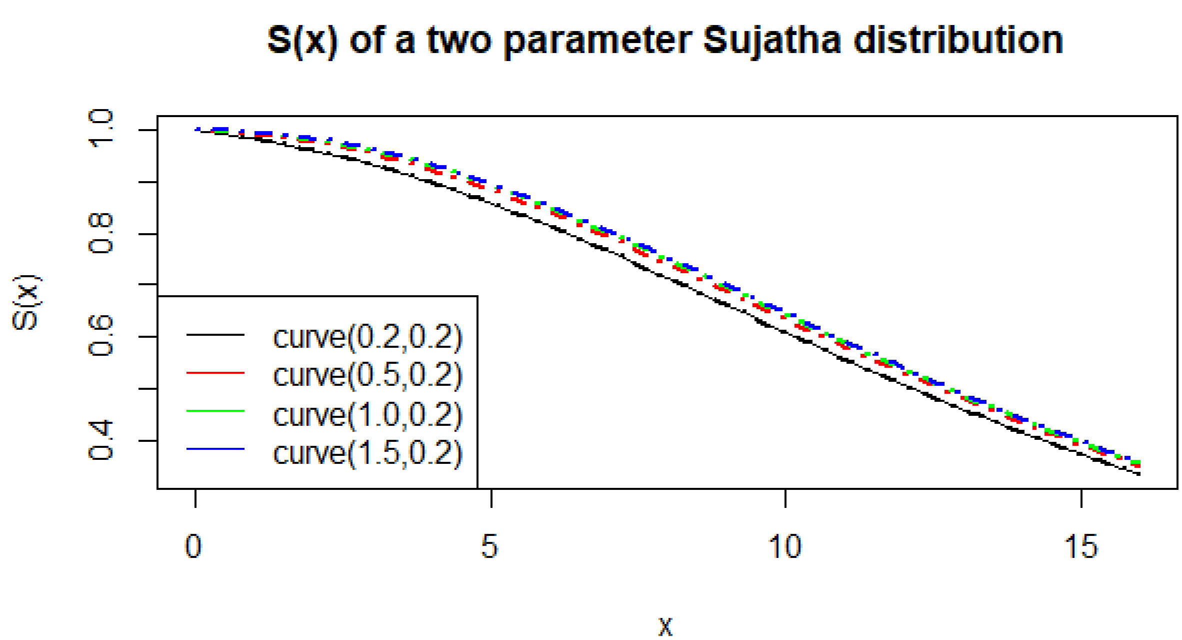

Figure 8. Shapes of $S(x)$ of AG2PSD with varying values of α and constant θ

Similarly, by definition Eq.1.9 the hazard function Eq.7.19 is obtained as;

$$h(x;\alpha,\theta) = \frac{\frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\left[1+\alpha x+\alpha x^2\right]e^{-\theta x}}{\left[\frac{\theta^2+\alpha(\theta+2+\theta^2 x+\theta^2 x^2+2\theta x)}{\theta^2+\alpha\theta+2\alpha}\right]e^{-\theta x}}$$

$$= \frac{\theta^3(1+\alpha(x+x^2))}{\theta^2+\alpha(\theta+2+\theta^2 x+\theta^2 x^2+2\theta x)}; x>0, \alpha>0, \theta>0$$

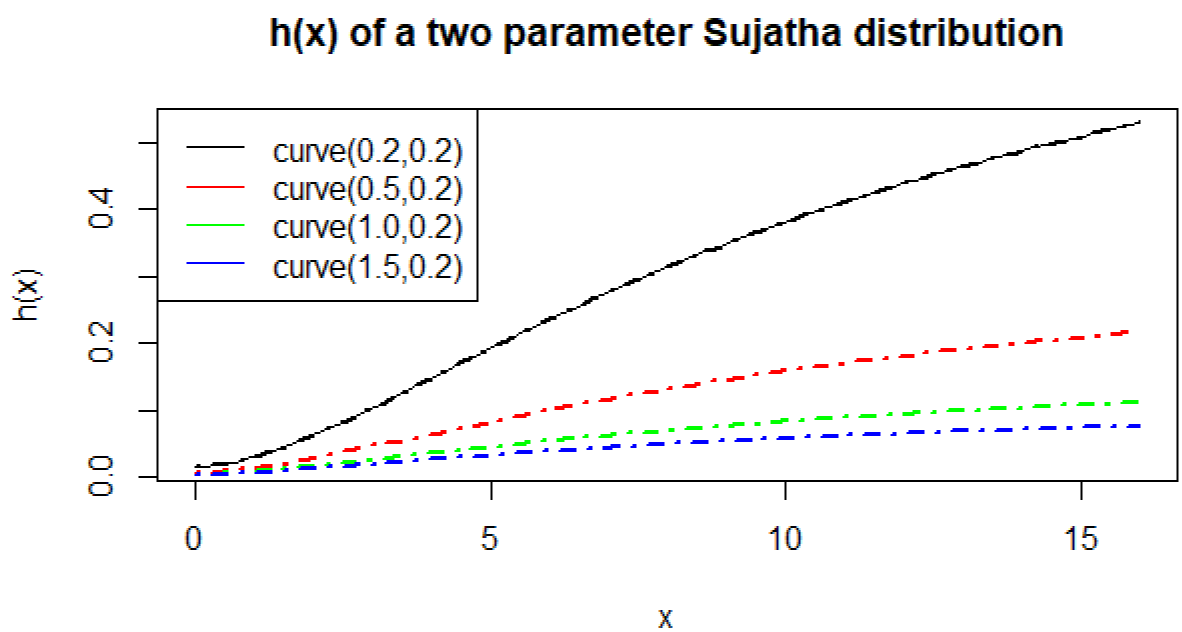

Figure 9. Shapes of $h(x)$ of two parameter Sujatha distribution with varying values of α and constant θ

□

7.3.3 Moments and related measures

Proposition 7.3.4. *The r^{th} moments of a generalized two parameter Sujatha distribution Eq.7.16 are derived using method* *of moments and moment generating function (mgf) as;*

$$\mu_r^1 * = \frac{r!\left[\theta^2+\alpha\theta(r+1)+\alpha(r+1)(r+2)\right]}{\theta^r(\theta^2+\alpha\theta+2\alpha)}; \quad r=1,2,3,\cdots$$

(7.20)

Proof. By definition Eq.1.16, moments of AG2PSD are obtained as;

$$\begin{aligned}
E(X^r) &= \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\int_0^\infty(1+\alpha(x+x^2))e^{-\theta x}dx \\
&= \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\left[\int_0^\infty x^r(1+\alpha(x+x^2))e^{-\theta x}dx\right] \\
&= \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\left[\int_0^\infty x^r e^{-\theta x}dx+\alpha\int_0^\infty x^{r+1}e^{-\theta x}dx+\alpha\int_0^\infty x^{r+2}e^{-\theta
\end{aligned}$$

$$\begin{aligned}
&= \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\left[\frac{\Gamma(r+1)}{\theta^{r+1}}+\frac{\alpha\Gamma(r+2)}{\theta^{r+2}}+\frac{\alpha\Gamma(r+3)}{\theta^{r+3}}\right] \\
&= \frac{r\Gamma r\left[\theta^2+\alpha\theta(r+1)+\alpha(r+1)(r+2)\right]}{\theta^r(\theta^2+\alpha\theta+2\alpha)} \\
&= \frac{r!\left[\theta^2+\alpha\theta(r+1)+\alpha(r+1)(r+2)\right]}{\theta^r(\theta^2+\alpha\theta+2\alpha)}; \quad r=1,2,3,\cdots
\end{aligned}$$

further by definition Eq.1.17, mgf of AG2PSD is given as;

$$
\begin{aligned}
m_x(t) &= \int_0^{\infty} e^{tx}\frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}(1+\alpha x+\alpha x^2)e^{-\theta x}dx \\
&= \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\int_0^{\infty} e^{-(\theta-t)x}(1+\alpha x+\alpha x^2)dx \\
&= \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\left[\frac{1}{(\theta-t)}+\frac{\alpha}{(\theta-t)^2}+\frac{2\alpha}{(\theta-t)^3}\right] \\
&= \frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\Bigg[(\frac{1}{\theta})\sum_{k=0}^{\infty}(\frac{t}{\theta})^k+(\frac{\alpha}{\theta^2})^k\sum_{k=0}^{\infty}\binom{k+1}{k}(\frac{\alpha}{\theta^2})^k+\frac{2\alpha}{\theta^3}\sum_{k=0}^{\infty}\begin{pmatrix}k+ \\ k\end{pmatrix} \\
&= \sum_{k=0}^{\infty}\frac{\theta^2+\alpha\theta(k+1)+\alpha(k+1)(k+2)}{\theta^2+\alpha\theta+2\alpha}(\frac{t}{\theta})^k
\end{aligned}
$$

The r^{th} moments of AG2PSD Eq.7.16 are obtained as a coefficient of $\frac{t^r}{r!}$ as;

$$
\mu_r^1 = \frac{r!\left[\theta^2+\alpha\theta(r+1)+\alpha(r+1)(r+2)\right]}{\theta^r(\theta^2+\alpha\theta+2\alpha)}; \quad r=1,2,3,\cdots
$$

□

For the values of r as 1,2,3 and 4 in Eq.7.20 the none centralized moments are obtained as;

$$
\mu_1^1 = \frac{\theta^2+2\alpha\theta+6\alpha}{\theta(\theta^2+\alpha\theta+2\alpha)}, \quad \mu_2^1 = \frac{2(\theta^2+3\alpha\theta+12\alpha)}{\theta^2(\theta^2+\alpha\theta+2\alpha)}
$$

$$
\mu_3^1 = \frac{3(\theta^2+4\alpha\theta+20\alpha)}{\theta^3(\theta^2+\alpha\theta+2\alpha)}, \quad \mu_4^1 = \frac{4(\theta^2+5\alpha\theta+30\alpha)}{\theta^4(\theta^2+\alpha\theta+2\alpha)}
$$

The centralized moments of AG2PSD are obtained as;

$$\mu_1 = \mu_1^1$$
$$\mu_2 = \mu_2^1 - [\mu_1^1]^2 \implies \frac{2\alpha^2\theta^2 + 4\alpha\theta^3 + 12\alpha^2\theta + 16\alpha\theta^2 + 12\alpha^2}{\theta^2(\theta^2 + \alpha\theta + 2\alpha)^2}$$
$$\mu_3 = \mu_3^1 - 3[\mu_1^1\mu_2^1] + 2[\mu_1^1]^3$$
$$\mu_3 = \frac{2(2\alpha^3\theta^3 + 6\alpha^2\theta^4 + 6\alpha\theta^5 + \theta^6 + 18\alpha^3\theta^2 + 42\alpha^2\theta^3 + 30\alpha\theta^4 + 36\alpha}{\theta^3(\theta^2 + \alpha\theta + 2\alpha)^2}$$
$$\mu_4 = \mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2$$

Proposition 7.3.5. *Other related measures of a generalized two parameter Sujatha distribution Eq.7.16 such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) are stated as;*

$$C.v = \frac{\sqrt{2\alpha^2\theta^2 + 4\alpha\theta^3 + 12\alpha^2\theta + 16\alpha\theta^2 + 12\alpha^2}}{\theta^2 + 2\alpha\theta + 6\alpha} \tag{7.21}$$
$$\upsilon_1 = \frac{2(2\alpha^3\theta^3 + 6\alpha^2\theta^4 + 6\alpha\theta^5 + \theta^6 + 18\alpha^3\theta^2 + 42\alpha^2\theta^3 + 30\alpha\theta^4 + 36}{(2\alpha^2\theta^2 + 4\alpha\theta^3 + 12\alpha^2\theta + 16\alpha\theta^2 + 12\alpha^2)^{\frac{3}{2}}} \tag{7.22}$$
$$\upsilon_2 = \frac{\mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2}{[\mu_2^1 - [\mu_1^1]^2]^2} \tag{7.23}$$
$$\upsilon_3 = \frac{2\alpha^2\theta^2 + 4\alpha\theta^3 + 12\alpha^2\theta + 16\alpha\theta^2 + 12\alpha^2}{\theta(\theta^2 + \alpha\theta + 2\alpha)(\theta^2 + 2\alpha\theta + 6\alpha)} \tag{7.24}$$

Proof. To begin with, variation coefficient Eq.7.21 is obtained as;

$$
\begin{aligned}
C.v &= \frac{\sigma}{\mu_1^1} \\
&= \frac{\sqrt{2\alpha^2\theta^2+4\alpha\theta^3+12\alpha^2\theta+16\alpha\theta^2+12\alpha^2}}{\theta(\theta^2+\alpha\theta+2\alpha)} * \frac{\theta(\theta^2+\alpha\theta+2\alpha)}{\theta^2+2\alpha\theta+6\alpha} \\
&= \frac{\sqrt{2\alpha^2\theta^2+4\alpha\theta^3+12\alpha^2\theta+16\alpha\theta^2+12\alpha^2}}{\theta^2+2\alpha\theta+6\alpha}
\end{aligned}
$$

Secondly, skewness value Eq.7.22 is obtained as;

$$
\begin{aligned}
\upsilon_1 &= \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} \\
&= \frac{\frac{2(2\alpha^3\theta^3+6\alpha^2\theta^4+6\alpha\theta^5+\theta^6+18\alpha^3\theta^2+42\alpha^2\theta^3+30\alpha\theta^4+36\alpha^2\theta^2+24\alpha^3)}{\theta^3(\theta^2+\alpha\theta+2\alpha)^2}}{\left[\frac{2\alpha^2\theta^2+4\alpha\theta^3+12\alpha^2\theta+16\alpha\theta^2+12\alpha^2}{\theta^2(\theta^2+\alpha\theta+2\alpha)^2}\right]^{\frac{3}{2}}} \\
&= \frac{2(2\alpha^3\theta^3+6\alpha^2\theta^4+6\alpha\theta^5+\theta^6+18\alpha^3\theta^2+42\alpha^2\theta^3+30\alpha\theta^4+36\alpha^2\theta^2 +}{(2\alpha^2\theta^2+4\alpha\theta^3+12\alpha^2\theta+16\alpha\theta^2+12\alpha^2)^{\frac{3}{2}}}
\end{aligned}
$$

The following expression is applied to derive coefficient of kurtosis of a generalized two parameter Sujatha distribution Eq.7.16.

$$
\upsilon_2 = \frac{\mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2}{[\mu_2^1 - [\mu_1^1]^2]^2}
$$

lastly, index of dispersion Eq.7.24 is obtained as;

$$\begin{aligned}
\upsilon_3 &= \frac{\sigma^2}{\mu_1^1} \\
&= \frac{2\alpha^2\theta^2 + 4\alpha\theta^3 + 12\alpha^2\theta + 16\alpha\theta^2 + 12\alpha^2}{\theta^2(\theta^2 + \alpha\theta + 2\alpha)^2} \frac{\theta(\theta^2 + \alpha\theta + 2\alpha)}{\theta^2 + 2\alpha\theta + 6\alpha} \\
&= \frac{2\alpha^2\theta^2 + 4\alpha\theta^3 + 12\alpha^2\theta + 16\alpha\theta^2 + 12\alpha^2}{\theta(\theta^2 + \alpha\theta + 2\alpha)(\theta^2 + 2\alpha\theta + 6\alpha)}
\end{aligned}$$

□

7.3.4 Excess Loss Distribution

Proposition 7.3.6. *The probability density function of excess loss function $f_l(x;\alpha,\theta)$, mean residual lifetime (MRL), equilibrium distribution $f_e(x;\alpha,\theta)$, survival function of the equilibrium distribution $S_e(x;\alpha,\theta)$ and hazard function of equilibrium distribution $h_e(x;\alpha,\theta)$ of a generalized two parameter Sujatha* *distribution Eq.7.16 are stated as;*

$$f_l(x;\alpha,\theta) = \frac{\theta^3(1+\alpha x+\alpha x^2)e^{-(x-z)\theta}}{\theta^2+\alpha\theta+2\alpha+\alpha\theta^2 z+\alpha\theta^2 z^2+2\alpha\theta z}; \quad x > z \tag{7.25}$$

$$m(x) = \frac{\theta^2+2\alpha\theta+6\alpha\alpha\theta^2 x+\alpha\theta^2 x^2+4\alpha\theta x}{\theta(\theta^2+\alpha\theta+2\alpha+\alpha\theta^2 x+\alpha\theta^2 x^2+2\alpha\theta x)} \tag{7.26}$$

$$f_e(x;\alpha,\theta) = \frac{\theta(\theta^2+\alpha\theta+2\alpha+\alpha\theta^2 x+\alpha\theta^2 x^2+2\alpha\theta x)e^{-\theta x}}{\theta^2+2\alpha\theta+6\alpha} \tag{7.27}$$

$$S_e(x;\alpha,\theta) = \frac{(\theta^2+2\alpha\theta+6\alpha+\alpha\theta^2 x+\alpha\theta^2 x^2+4\alpha\theta x)e^{-\theta x}}{\theta^2+2\alpha\theta+6\alpha} \tag{7.28}$$

$$h_e(x;\alpha,\theta) = \frac{\theta(\theta^2+\alpha\theta+2\alpha+\alpha\theta^2 x+\alpha\theta^2 x^2+2\alpha\theta x)}{\theta^2+2\alpha\theta+6\alpha+\alpha\theta^2 x+\alpha\theta^2 x^2+4\alpha\theta x} \tag{7.29}$$

Proof. To begin with, pdf of excess loss distribution Eq.7.25 is obtained by use of relation Eq.1.10 as;

$$\begin{aligned} f_l(x;\alpha,\theta) &= \frac{\frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\left[1+\alpha x+\alpha x^2\right]e^{-\theta x}}{\left[\frac{\theta^2+\alpha\theta+2\alpha+\alpha\theta^2 z+\alpha\theta^2 z^2+2\alpha\theta z}{\theta^2+\alpha\theta+2\alpha}\right]e^{-\theta z}} \\ &= \frac{\theta^3(1+\alpha x+\alpha x^2)e^{-(x-z)\theta}}{\theta^2+\alpha\theta+2\alpha+\alpha\theta^2 z+\alpha\theta^2 z^2+2\alpha\theta z}; \quad x > z \end{aligned}$$

Secondly, mean excess loss Eq.7.26 is obtained by use of the relation Eq.1.11 as;

$$m(x) = \frac{\theta^2+\alpha\theta+2\alpha}{(\theta^2+\alpha\theta+2\alpha+\alpha\theta^2x+\alpha\theta^2x^2+2\alpha\theta x)e^{-\theta x}} \int_x^\infty \frac{(\theta^2+\alpha\theta+2\alpha}{}$$

$$= \frac{1}{(\theta^2+\alpha\theta+2\alpha+\alpha\theta^2x+\alpha\theta^2x^2+2\alpha\theta x)e^{-\theta x}} \int_x^\infty (\theta^2+\alpha\theta+2\alpha$$

$$I_1 = \int_x^\infty (\theta^2+\alpha\theta+2\alpha+\alpha\theta^2t+\alpha\theta^2t^2+2\alpha\theta t)e^{-\theta t}dt$$

$$u = (\theta^2+\alpha\theta+2\alpha+\alpha\theta^2t+\alpha\theta^2t^2+2\alpha\theta t) \Longrightarrow du = 2\alpha\theta^2t+2\alpha\theta$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_1 = -(\theta^2+\alpha\theta+2\alpha+\alpha\theta^2t+\alpha\theta^2t^2+2\alpha\theta t)\frac{e^{-\theta t}}{\theta} + \frac{\alpha\theta}{\theta}\int_x^\infty (2\theta t+2$$

$$I_1 = -(\theta^2+\alpha\theta+2\alpha+\alpha\theta^2t+\alpha\theta^2t^2+2\alpha\theta t)\frac{e^{-\theta t}}{\theta} + \alpha I_2$$

$$I_2 = \int_x^\infty (2\theta t+2+\theta)e^{-\theta t}dt$$

$$u = (2\theta t+2+\theta) \Longrightarrow du = 2\theta dt$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_2 = -(2\theta t+2+\theta)\frac{e^{-\theta t}}{\theta} + 2\int_x^\infty e^{-\theta t}dt$$

$$I_2 = -(2\theta t+2+\theta)\frac{e^{-\theta t}}{\theta} + 2I_3$$

$$I_3 = \int_x^\infty e^{-\theta t}dt = \frac{-e^{-\theta t}}{\theta}$$

From I_1, I_2 and I_3 we have the following;

$$I_1 = -(\theta^2+\alpha\theta+2\alpha+\alpha\theta^2 t+\alpha\theta^2 t^2+2\alpha\theta t)\frac{e^{-\theta t}}{\theta}+\alpha\Big[-(2\theta t+2+\theta)\frac{e}{}$$

$$I_1 = \left[\frac{-e^{-\theta t}}{\theta}\left(\theta^2+2\alpha\theta+6\alpha+\alpha\theta^2 t+\alpha\theta^2 t^2+4\alpha\theta t\right)\right]_x^\infty$$

$$I_1 = \frac{e^{-\theta x}}{\theta}\left(\theta^2+2\alpha\theta+6\alpha+\alpha\theta^2 x+\alpha\theta^2 x^2+4\alpha\theta x\right)$$

$$= I_1\frac{1}{(\theta^2+\alpha\theta+2\alpha+\alpha\theta^2 x+\alpha\theta^2 x^2+2\alpha\theta x)e^{-\theta x}}$$

$$m(x) = \frac{\theta^2+2\alpha\theta+6\alpha\alpha\theta^2 x+\alpha\theta^2 x^2+4\alpha\theta x}{\theta(\theta^2+\alpha\theta+2\alpha+\alpha\theta^2 x+\alpha\theta^2 x^2+2\alpha\theta x)}$$

by definition Eq.1.12, equilibrium distribution Eq.7.27 is obtained as;

$$f_e(x;\alpha,\theta) = \frac{(\theta^2+\alpha\theta+2\alpha+\alpha\theta^2 x+\alpha\theta^2 x^2+2\alpha\theta x)e^{-\theta x}}{\theta^2+\alpha\theta+2\alpha}\frac{\theta(\theta^2+\alpha\theta+2\alpha)}{\theta^2+2\alpha\theta+6\alpha}$$

$$= \frac{\theta(\theta^2+\alpha\theta+2\alpha+\alpha\theta^2 x+\alpha\theta^2 x^2+2\alpha\theta x)e^{-\theta x}}{\theta^2+2\alpha\theta+6\alpha}$$

Similarly, by definition Eq.1.13, $S_e(x;\alpha,\theta)$ in Eq.7.28 is given as;

$$\int_x^\infty S(t;\alpha,\theta)dt = \frac{(\theta^2+2\alpha\theta+6\alpha+\alpha\theta^2 x+\alpha\theta^2 x^2+4\alpha\theta x)e^{-\theta x}}{\theta(\theta^2+\alpha\theta+2\alpha)}$$

$$= \frac{(\theta^2+2\alpha\theta+6\alpha+\alpha\theta^2 x+\alpha\theta^2 x^2+4\alpha\theta x)e^{-\theta x}}{\theta(\theta^2+\alpha\theta+2\alpha)}\frac{\theta(\theta^2+\alpha\theta-}{\theta^2+2\alpha\theta+}$$

$$S_e(x;\alpha,\theta) = \frac{(\theta^2+2\alpha\theta+6\alpha+\alpha\theta^2 x+\alpha\theta^2 x^2+4\alpha\theta x)e^{-\theta x}}{\theta^2+2\alpha\theta+6\alpha}$$

lastly, by definition Eq.1.14 $h_e(x;\alpha,\theta)$ in Eq.7.29 is obtained as;

$$h_e(x;\alpha,\theta) = \frac{\frac{\theta(\theta^2+\alpha\theta+2\alpha+\alpha\theta^2x+\alpha\theta^2x^2+2\alpha\theta x)e^{-\theta x}}{\theta^2+2\alpha\theta+6\alpha}}{\left[\frac{(\theta^2+2\alpha\theta+6\alpha+\alpha\theta^2x+\alpha\theta^2x^2+4\alpha\theta x)}{\theta^2+2\alpha\theta+6\alpha}\right]e^{-\theta x}}$$
$$= \frac{\theta(\theta^2+\alpha\theta+2\alpha+\alpha\theta^2x+\alpha\theta^2x^2+2\alpha\theta x)}{\theta^2+2\alpha\theta+6\alpha+\alpha\theta^2x+\alpha\theta^2x^2+4\alpha\theta x}$$

□

7.3.5 Estimation of Two Parameter Sujatha Distribution

In this section, we discuss two methods of estimation that are MOME and MLE.

Method of Moment (MOME)

In this section, we shall apply the first two moments about the origin of a generalized two parameter Sujatha distribution Eq.7.16 to derive MOME of α and θ. The value μ_1^1 is equated to the mean of the sample as;

$$\bar{x} = \frac{\theta^2 + 2\alpha\theta + 6\alpha}{\theta(\theta^2 + \alpha\theta + 2\alpha)}$$
$$\bar{x} = \frac{\theta^2 + \alpha\theta + 2\alpha}{\theta(\theta^2 + \alpha\theta + 2\alpha)} + \frac{\alpha\theta + 4\alpha}{\theta(\theta^2 + \alpha\theta + 2\alpha)}$$
$$\bar{x} = \frac{1}{\theta} + \frac{\alpha\theta + 4\alpha}{\theta(\theta^2 + \alpha\theta + 2\alpha)}$$
$$\bar{x} - \frac{1}{\theta} = \frac{\alpha\theta + 4\alpha}{\theta(\theta^2 + \alpha\theta + 2\alpha)}$$

$$(\theta^2 + \alpha\theta + 2\alpha) = \frac{\alpha(\theta + 4)}{\theta\bar{x} - 1} \quad (7.30)$$

The second population mean is equated with the second moment of the sample as;

$$m_2^1 = \frac{2(\theta^2 + 3\alpha\theta + 12\alpha)}{\theta^2(\theta^2 + \alpha\theta + 2\alpha)}$$
$$m_2^1 = \frac{2(\theta^2 + \alpha\theta + 2\alpha)}{\theta^2(\theta^2 + \alpha\theta + 2\alpha)} + \frac{4\alpha\theta + 20\alpha}{\theta^2(\theta^2 + \alpha\theta + 2\alpha)}$$
$$m_2^1 = \frac{2}{\theta^2} + \frac{4\alpha\theta + 20\alpha}{\theta^2(\theta^2 + \alpha\theta + 2\alpha)}$$
$$m_2^1 - \frac{2}{\theta^2} = \frac{4\alpha\theta + 20\alpha}{\theta^2(\theta^2 + \alpha\theta + 2\alpha)}$$
$$\frac{\theta^2 m_2^1 - 2}{\theta^2} = \frac{4\alpha\theta + 20\alpha}{\theta^2(\theta^2 + \alpha\theta + 2\alpha)}$$

$$(\theta^2 + \alpha\theta + 2\alpha) = \frac{4\alpha\theta + 20\alpha}{\theta^2 m_2^1 - 2} \quad (7.31)$$

Equating Eq.7.30 and Eq. 7.31, we obtain a polynomial of degree three in θ as;

$$-m_2^1\alpha\theta^3+4\alpha(\bar{x}-m_2^1)\theta^2+2\alpha(10\bar{x}-1)\theta-12\alpha=0 \tag{7.32}$$

Equation 7.32 can be solved using an iterative technique such as Bisection method to obtain MOME of θ as $\hat{\theta}$. The method of moment of $\hat{\alpha}$ of the parameter α is obtained by putting the value of $\hat{\theta}$ in equation Eq.7.30 as;

$$\hat{\alpha}=\frac{(1-\hat{\theta}\bar{x})(\hat{\theta})^2}{-(\hat{\theta})^2\bar{x}-\hat{\theta}-2(\bar{x}-1)(\hat{\theta})+6} \tag{7.33}$$

Maximum Likelihood Estimation

Let $(x_1,x_2,x_3\cdots$ n) be a random sample of size n drawn from AG2PSD Eq.7.16, the L function is derived as;

$$\prod_{i=1}^{n}f(x;\alpha,\theta)=\prod_{i=1}^{n}\left[\frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}(1+\alpha x_i+\alpha x_i^2)e^{-\theta x_i}\right]$$

$$L=\left(\frac{\theta^3}{\theta^2+\alpha\theta+2\alpha}\right)^n\prod_{i=1}^{n}(1+\alpha x_i+\alpha x_i^2)e^{-\theta\sum_{i=1}^{n}x_i}$$

$$lnL=n[3ln\theta-ln(\theta^2+\alpha\theta+2\alpha)]+\sum_{i=1}^{n}ln(1+\alpha x_i+\alpha x_i^2)-n\theta$$

The MLEs of the estimates $(\hat{\alpha},\hat{\theta})$ of the parameters (α,θ) are obtained by solving the following two equations;

$$\frac{\partial lnL}{\partial\theta}=\frac{3n}{\theta}-\frac{n(2\theta+\alpha)}{\theta^2+\alpha\theta+2\alpha}-n\bar{x}=0 \tag{7.34}$$

$$\frac{lnL}{\partial\alpha}=\frac{-n(\theta+2)}{\theta^2+\alpha\theta+2\alpha}+\sum_{i=1}^{n}\frac{(1+x_i)x_i}{(1+\alpha x_i+\alpha x_i^2)}=0 \tag{7.35}$$

The value $\bar{x}$ is the population mean of a generalized two parameter Sujatha Eq.7.16

Equations 7.34 and 7.35 can not be solved directly. However, fisher scoring technique is applied on the following equations;

$$\frac{\partial^2 lnL}{\partial\theta^2} = \frac{-3n}{\theta^2} + \frac{2n}{\theta^2+\alpha\theta+2\alpha} + \frac{n\alpha(4\theta-\alpha)}{(\theta^2+\alpha\theta+2\alpha)^2}$$
$$\frac{\partial^2 lnL}{\partial\theta\partial\alpha} = \frac{n(\theta^2+4\theta)}{(\theta^2+\alpha\theta+2\alpha)^2}$$
$$\frac{\partial^2 lnL}{\partial\alpha^2} = \frac{n(\theta+2)^2}{(\theta^2+\alpha\theta+2\alpha)^2} - \sum_{i=1}^{n}\frac{(x_i+x_i^2)^2}{(1+\alpha x_i+\alpha x_i^2)^2}$$

To get MLEs of the parameters α and θ of a two parameter Sujatha distribution Eq.7.16 are obtained as $\hat{\theta}$ and $\hat{\alpha}$ by solving the following set of equations.

$$\begin{bmatrix} \frac{\partial^2 lnL}{\partial\theta^2} & \frac{\partial^2 lnL}{\partial\theta\partial\alpha} \\ \frac{\partial^2 lnL}{\partial\theta\partial\alpha} & \frac{\partial^2 lnL}{\partial\alpha^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial lnL}{\partial\theta} \\ \frac{\partial lnL}{\partial\alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}$$

The values θ_0 and α_0 are initials of the parameters θ and α respectively. The equations are solved using iterative technique till maximized values are obtained for $\hat{\theta}$ and $\hat{\alpha}$.

7.3.6 Application

We shall apply data on carbon fiber breaking stress presented in table 1 to fit two parameter distributions. Two parameter distributions fitted are Quasi Sujatha, Aradhana, Quasi Aradhana, Sujatha, Lindley and Shanker distributions.

Table 4. MLE estimates, -2lnL, AIC, AICC and BIC values of fitted Two Parameter Distributions

Model	Estimates	-2lnL	AIC	AICC	BIC
QSD	$\hat{\alpha}$=-0.5504 $\hat{\theta}$=1.1031	201.5952	205.5952	205.7857	209.9745
AG2PAD	$\hat{\alpha}$=44.6250 $\hat{\theta}$=1.0801	205.0829	209.0829	210.1650	213.4622
QAD	$\hat{\alpha}$=41.7732 $\hat{\theta}$=1.0798	205.1277	209.1277	209.3182	213.5070
AG2PSD	$\hat{\alpha}$=54.2032 $\hat{\theta}$=0.9637	213.8446	217.8446	218.0351	222.2239
AG2PLD	$\hat{\beta}$=64.7808 $\hat{\theta}$=0.7224	224.5887	228.5887	228.7792	232.9680
G2PSD	$\hat{\beta}$=54.6149 $\hat{\theta}$=0.7173	224.5061	228.5061	228.6966	232.8854

Best fitted distribution has lower values of -2lnL, AIC, AICC and BIC measures. Based on the estimated distributions in table 4, Quasi Sujatha distribution is a better among the six fitted distributions. In figure 10 below, fitted density of QSD has the highest curve than the other fits.

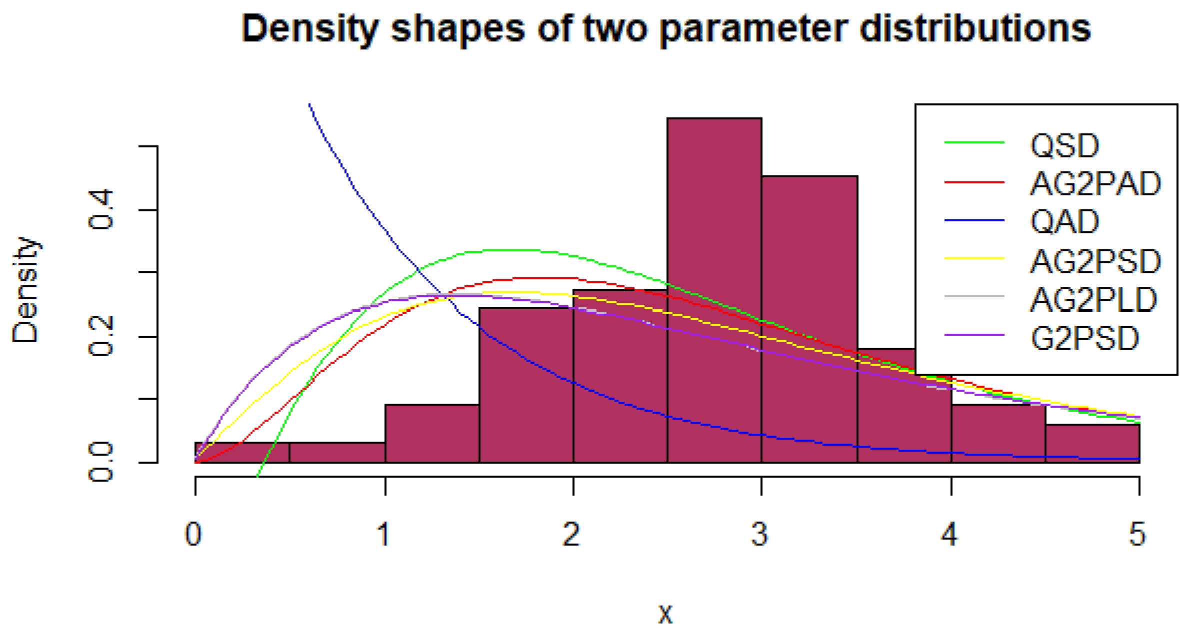

Figure 10. Estimated densities of fitted two parameter distributions

7.3.7 Quasi Sujatha distribution

7.3.8 Construction of a Quasi Sujatha distribution

Proposition 7.3.7. *Let $\omega_1 = \frac{\alpha\theta}{\alpha\theta+\theta+2}$ and $\omega_2 = \frac{\theta}{\alpha\theta+\theta+2}$ be mixing proportions, Quasi Sujatha distribution (QSD) is a finite mixed distribution of Gamma $(1,\theta)$, Gamma $(2,\theta)$ and Gamma $(3,\theta)$. The pdf and Cdf of QSD are defined as;*

$$f(x;\alpha,\theta) = \frac{\theta^2}{\alpha\theta+\theta+2}\left[\alpha+\theta x+\theta x^2\right]e^{-\theta x}; x>0, \alpha>0, \theta>0 \tag{7.36}$$

$$F(x;\alpha,\theta) = 1-\left[1+\frac{\theta x(\theta+\theta x+2)}{\alpha\theta+\theta+2}\right]e^{-\theta x}; x>0, \alpha>0, \theta>0 \tag{7.37}$$

Proof . By definition of a finite mixture in Eq.1.1, pdf Eq.7.36 is obtained as;

$$f(x;\alpha,\theta)=\frac{\alpha\theta}{\alpha\theta+\theta+2}\left[\theta e^{-\theta x}\right]+\frac{\theta}{\alpha\theta+\theta+2}\left[\frac{\theta^2 e^{-\theta x}x}{\Gamma 2}\right]+\frac{2}{\alpha\theta+\theta+2}$$
$$=\frac{\alpha\theta^2 e^{-\theta x}}{\alpha\theta+\theta+2}+\frac{\theta^3 e^{-\theta x}x}{\alpha\theta+\theta+2}+\frac{2\theta^3 e^{-\theta x}x^2}{2(\alpha\theta+\theta+2)}$$
$$=\frac{\theta^2}{\alpha\theta+\theta+2}\left[\alpha+\theta x+\theta x^2\right]e^{-\theta x};x>0,\alpha>0,\theta>0$$

Remark 7.3.8. *It is noted that a generalized two parameter Quasi Sujatha distribution Eq.7.36 is nested with two distributions.*
Putting $\alpha=\theta$; the Quasi Sujatha distribution turns to be one parameter Sujatha Eq.7.1
Putting $\alpha=0$; the Quasi Sujatha distribution turns to be a one parameter size-biased Lindley distribution (SBLD) given as;

$$f(x;\theta)=\frac{\theta^3}{\theta+2}x\left[1+x\right]e^{-\theta x};x>0,\theta>0 \qquad (7.38)$$

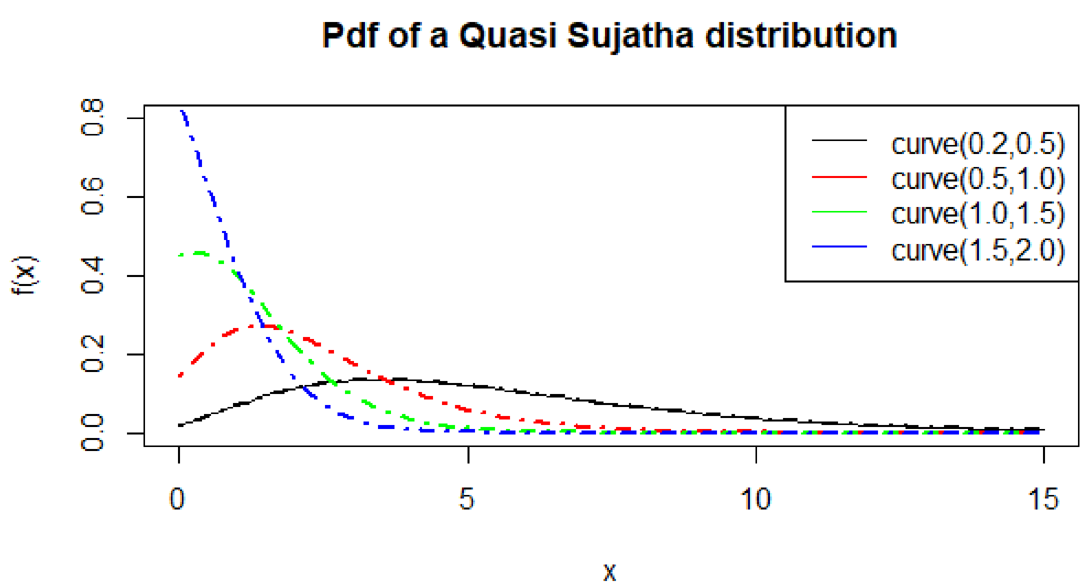

Figure 11. Shapes of $f(x)$ of a Quasi Sujatha distribution with varying values of both α and θ

further Cdf Eq.7.37 is obtained as;

$$F(x;\alpha,\theta) = \frac{\theta^2}{\alpha\theta+\theta+2}\int_0^\infty(\alpha+\theta x+\theta x^2)e^{-\theta x}dx$$

$$F(x;\alpha,\theta) = \frac{\theta^2}{\alpha\theta+\theta+2}I_1$$

$$I_1 = \int_0^\infty(\alpha+\theta x+\theta x^2)e^{-\theta x}dx$$

$$u = (\alpha+\theta x+\theta x^2) \Longrightarrow du = (\theta+2\theta x)$$

$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$

$$I_1 = -(\alpha+\theta x+\theta x^2)\frac{e^{-\theta x}}{\theta} - \int_0^\infty -(1+2x)e^{-\theta x}dx$$

$$I_1 = -(\alpha+\theta x+\theta x^2)\frac{e^{-\theta x}}{\theta} - I_2$$

$$I_2 = \int_0^\infty -(1+2x)e^{-\theta x}dx$$

$$u = -(1+2x) \Longrightarrow du = -2dx$$

$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$

$$I_2 = (1+2x)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}\int_0^\infty e^{-\theta x}dx$$

$$I_2 = (1+2x)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}I_3$$

$$I_3 = \int_0^\infty e^{-\theta x}dx$$

$$u = -\theta x \Longrightarrow du = -\theta dx \Longrightarrow \frac{du}{-\theta} = dx$$

$$I_3 = \int_0^\infty e^u\frac{du}{-\theta} \Longrightarrow \frac{1}{-\theta}\int_0^\infty e^u du$$

$$I_3 = \frac{e^{-\theta x}}{-\theta}$$

From I_1, I_2 and I_3 we have the following;

$$I_1 = -(\alpha+\theta x+\theta x^2)\frac{e^{-\theta x}}{\theta} - \left[(1+2x)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}\left(\frac{e^{-\theta x}}{-\theta}\right)\right]$$

$$I_1 = \frac{-e^{-\theta x}}{\theta^2}\left[\alpha\theta+\theta^2 x+\theta^2 x^2+\theta+2\theta x+2\right]$$

$$= 1+\frac{-e^{-\theta x}}{\theta^2}\left[\alpha\theta+\theta^2 x+\theta^2 x^2+\theta+2\theta x+2\right]\frac{\theta^2}{\alpha\theta+\theta+2}$$

$$F(x;\alpha,\theta) = 1-\left[1+\frac{\theta x(\theta+\theta x+2)}{\alpha\theta+\theta+2}\right]e^{-\theta x}; x>0, \alpha>0, \theta>0$$

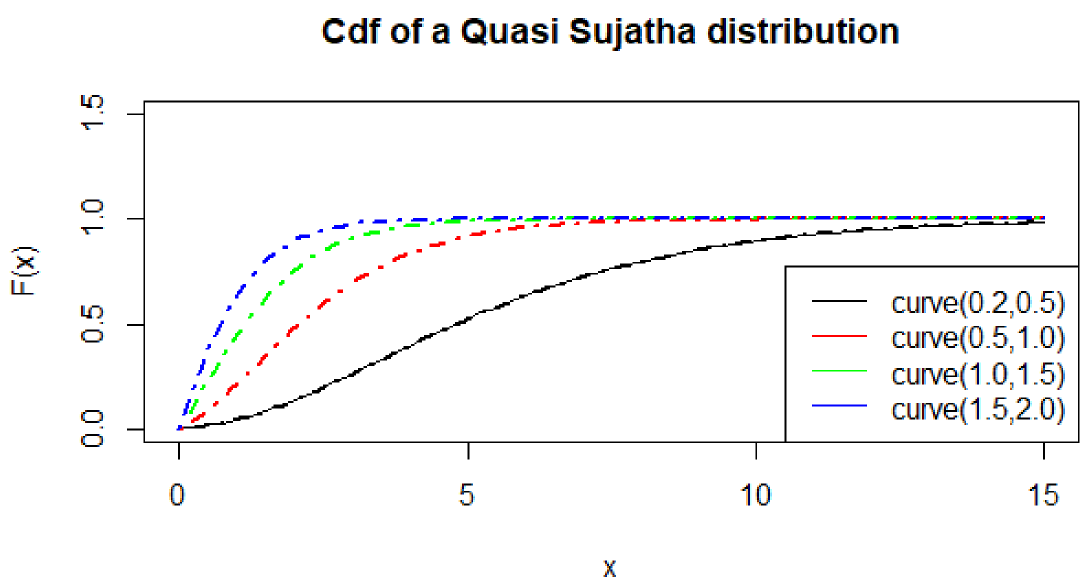

Figure 12. Shapes of $F(x)$ of a Quasi Sujatha distribution with varying values of both α and θ

□

7.3.9 Reliability Analysis

Proposition 7.3.9. *In this section we state survival function denoted by* $S(x;\alpha,\theta)$ *and hazard function denoted by* $h(x;\alpha,\theta)$ *of* *a generalized two parameter Quasi Sujatha distribution Eq.7.36 as;*

$$S(x;\alpha,\theta)=\left[1+\frac{\theta x(\theta+\theta x+2)}{\alpha\theta+\theta+2}\right]e^{-\theta x};x>0,\alpha>0,\theta>0 \tag{7.39}$$

$$h(x;\alpha,\theta)=\frac{\theta^2(\alpha+\theta x+\theta x^2)}{\alpha\theta+\theta+2+\theta x(\theta+\theta x+2)};x>0,\alpha>0,\theta>0 \tag{7.40}$$

Proof. Using the relation Eq.1.8, survival function Eq.7.39 is obtained as;

$$\begin{aligned}S(x;\alpha,\theta)&=1-\left[1-\left[1+\frac{\theta x(\theta+\theta x+2)}{\alpha\theta+\theta+2}\right]e^{-\theta x}\right]\\&=\left[1+\frac{\theta x(\theta+\theta x+2)}{\alpha\theta+\theta+2}\right]e^{-\theta x};x>0,\alpha>0,\theta>0\end{aligned}$$

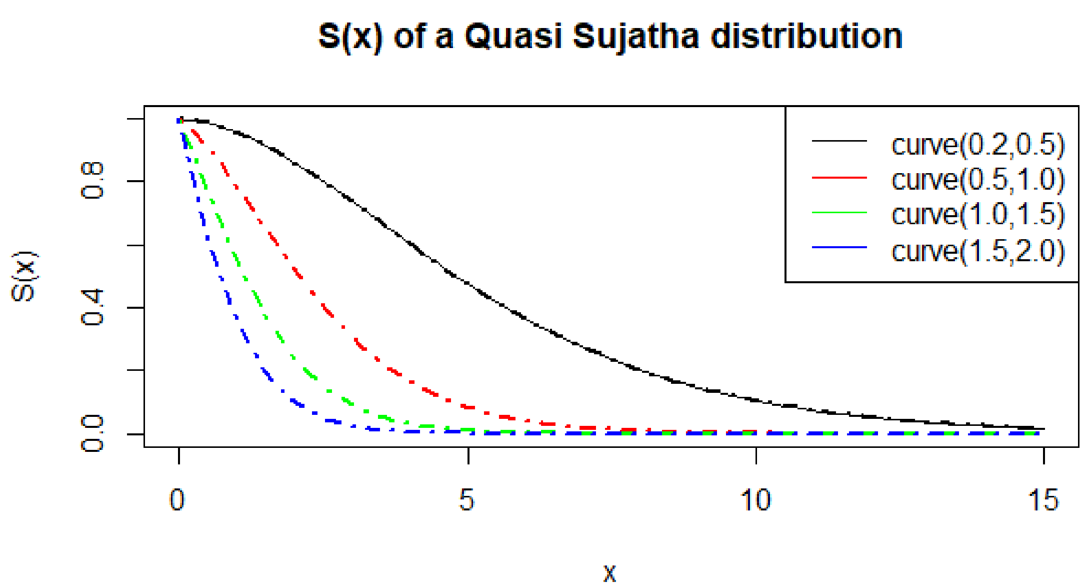

Figure 13. Shapes of $S(x)$ of a Quasi Sujatha distribution with varying values of both α and θ

further hazard function Eq.7.40 is obtained by use of Eq.1.9 as;

$$h(x;\alpha,\theta) = \frac{\frac{\theta^2}{\alpha\theta+\theta+2}\left[\alpha+\theta x+\theta x^2\right]e^{-\theta x}}{\left[\frac{\alpha\theta+\theta+2+\theta x(\theta+\theta x+2)}{\alpha\theta+\theta+2}\right]e^{-\theta x}}$$

$$= \frac{\theta^2(\alpha+\theta x+\theta x^2)}{\alpha\theta+\theta+2+\theta x(\theta+\theta x+2)}; x>0, \alpha>0, \theta>0$$

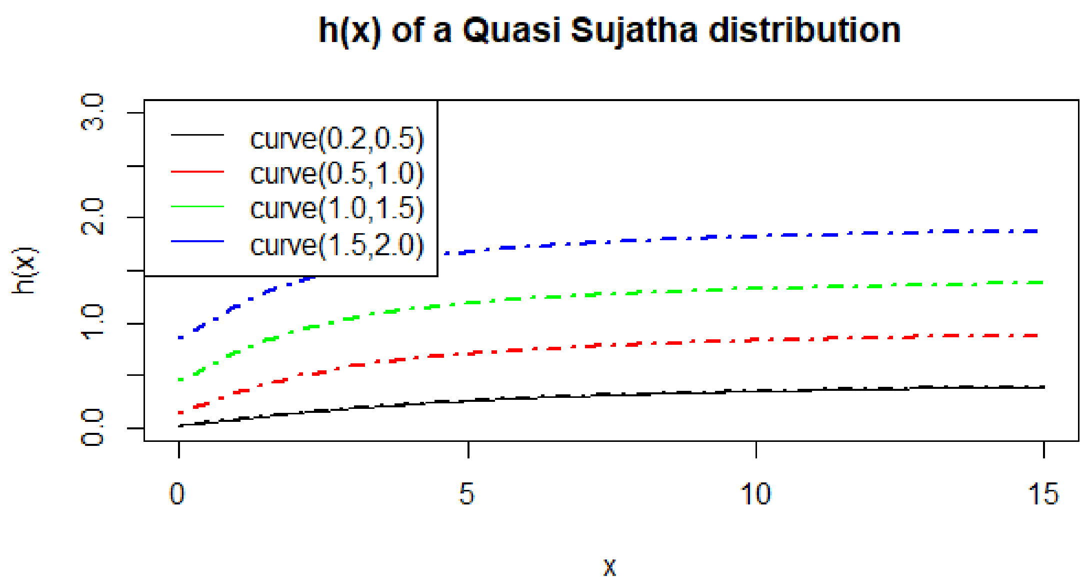

Figure 14. Shapes of $h(x)$ of a QSD with varying values of both α and θ

□

7.3.10 Moments and related measures

Proposition 7.3.10. *The r^{th} moments of a generalized two parameter Quasi Sujatha distribution Eq.7.36 are derived using method* *of moments and moment generating function as;*

$$\mu_r^1 * = \frac{r!\left(\alpha\theta + \theta(r+1) + (r+1)(r+2)\right)}{\theta^r(\alpha\theta + \theta + 2)}; \quad r = 1, 2, 3, \cdots$$

(7.41)

Proof. To begin with, by the definition Eq.1.16 r^{th} moments of Quasi Sujatha distribution are given as;

$$\begin{aligned}
E(X^r) &= \frac{\theta^2}{\alpha\theta + \theta + 2}\int_0^\infty x^r(\alpha + \theta x + \theta x^2)e^{-\theta x}dx \\
&= \frac{\theta^2}{\alpha\theta + \theta + 2}\left[\alpha\int_0^\infty x^r e^{-\theta x}dx + \theta\int_0^\infty x^{r+1}e^{-\theta x}dx + \theta\int_0^\infty x^{r+2}e^{-\theta x}dx\right] \\
&= \frac{\theta^2}{\alpha\theta + \theta + 2}\left[\frac{\alpha\Gamma(r+1)}{\theta^{r+1}} + \frac{\theta\Gamma(r+2)}{\theta^{(r+2)}} + \frac{\theta\Gamma(r+3)}{\theta^{r+3}}\right] \\
&= \frac{\theta^2}{\theta^r(\alpha\theta + \theta + 2)}\left[\frac{\alpha r\Gamma r}{\theta} + \frac{\theta(r+1)r\Gamma r}{\theta^2} + \frac{\theta(r+2)(r+1)r\Gamma r}{\theta^3}\right] \\
&= \frac{r!\left(\alpha\theta + \theta(r+1) + (r+1)(r+2)\right)}{\theta^r(\alpha\theta + \theta + 2)}; \quad r = 1, 2, 3, \cdots
\end{aligned}$$

further by definition Eq.1.17, mgf of a Quasi Sujatha is given as;

$$
\begin{aligned}
m_x(t) &= \frac{\theta^2}{\alpha\theta+\theta+2}\int_0^\infty e^{tx}(\alpha+\theta x+\theta x^2)e^{-\theta x}dx \\
&= \frac{\theta^2}{\alpha\theta+\theta+2}\left[\int_0^\infty e^{-(\theta-t)x}(\alpha+\theta x+\theta x^2)dx\right] \\
&= \frac{\theta^2}{\alpha\theta+\theta+2}\left[\frac{\alpha}{(\theta-t)}+\frac{\theta}{(\theta-t)^2}+\frac{2\theta}{(\theta-t)^3}\right] \\
&= \frac{\theta^2}{\alpha\theta+\theta+2}\left[\frac{\alpha}{\theta}\sum_{k=0}^{\infty}\left(\frac{t}{\theta}\right)^k+\frac{1}{\theta}\sum_{k=0}^{\infty}\binom{k+1}{k}\left(\frac{t}{\theta}\right)^k+\frac{2}{\theta^2}\sum_{k=0}^{\infty}\binom{k+2}{k}\right. \\
&= \sum_{k=0}^{\infty}\frac{\alpha\theta+\theta(k+1)+(k+1)(k+2)}{\alpha\theta+\theta+2}\left(\frac{t}{\theta}\right)^k
\end{aligned}
$$

The r^{th} moments about the origin of a generalized two parameter Quasi Sujatha Eq.7.36 are obtained as a coefficient of $\frac{t^r}{r!}$ of the moment generating function $m_x(t)$ as shown below;

$$
\mu_r^1 = \frac{r!\Big(\alpha\theta+\theta(r+1)+(r+1)(r+2)\Big)}{\theta^r(\alpha\theta+\theta+2)}; \quad r=1,2,3,\cdots
$$

□

For the values of r as 1,2,3 and 4 in Eq.7.41 we get the first four none centralized moments of QSD Eq.7.36 as;

$$
\mu_1^1 = \frac{\alpha\theta+2\theta+6}{\theta(\alpha\theta+\theta+2)}, \quad \mu_2^1 = \frac{2(\alpha\theta+3\theta+12)}{\theta^2(\alpha\theta+\theta+2)}
$$

$$
\mu_3^1 = \frac{6(\alpha\theta+4\theta+20)}{\theta^3(\alpha\theta+\theta+2)}, \quad \mu_4^1 = \frac{24(\alpha\theta+5\theta+30)}{\theta^4(\alpha\theta+\theta+2)}
$$

The first four centralized moments (moments about the mean) are obtained as;

$$\mu_1 = \mu_1^1$$
$$\mu_2 = \mu_2^1 - [\mu_1^1]^2 \Longrightarrow \frac{\theta^2(\alpha^2+4\alpha+2)+16\alpha\theta+12(\theta+1)}{\theta^2(\alpha\theta+\theta+2)^2}$$
$$\mu_3 = \mu_3^1 - 3[\mu_1^1\mu_2^1] + 2[\mu_1^1]^3$$
$$\mu_3 = \frac{2\left[\theta^3(\alpha^3+6\alpha^2+6\alpha+2)+6\theta^2(5\alpha^2+7\alpha+3)+36\alpha\theta+12(3\theta+2)\right]}{\theta^3(\alpha\theta+\theta+2)^3}$$
$$\mu_4 = \mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2$$

Proposition 7.3.11. *We now define Other related measures of a generalized two parameter Quasi Sujatha distribution Eq.7.36 such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) are stated as;*

$$C.v = \frac{\sqrt{\theta^2(\alpha^2+4\alpha+2)+16\alpha\theta+12(\theta+1)}}{\alpha\theta+2\theta+6} \tag{7.42}$$

$$\upsilon_1 = \frac{2\left[\theta^3(\alpha^3+6\alpha^2+6\alpha+2)+6\theta^2(5\alpha^2+7\alpha+3)+36\alpha\theta+12(3\theta+2)\right]}{\left[\theta^2(\alpha^2+4\alpha+2)+16\alpha\theta+12(\theta+1)\right]^{\frac{3}{2}}} \tag{7.43}$$

$$\upsilon_2 = \frac{\mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2}{[\mu_2^1 - [\mu_1^1]^2]^2} \tag{7.44}$$

$$\upsilon_3 = \frac{\theta^2(\alpha^2+4\alpha+2)+16\alpha\theta+12(\theta+1)}{\theta(\alpha\theta+\theta+2)(\alpha\theta+2\theta+6)} \tag{7.45}$$

`Proof.` To begin with, variation coefficient Eq.7.42 is obtained as;

$$
\begin{aligned}
C.v &= \frac{\sigma}{\mu_1^1} \\
&= \frac{\sqrt{\theta^2(\alpha^2+4\alpha+2)+16\alpha\theta+12(\theta+1)}}{\theta(\alpha\theta+\theta+2)} \frac{\theta(\alpha\theta+\theta+2)}{\alpha\theta+2\theta+6} \\
&= \frac{\sqrt{\theta^2(\alpha^2+4\alpha+2)+16\alpha\theta+12(\theta+1)}}{\alpha\theta+2\theta+6}
\end{aligned}
$$

Secondly, skewness coefficient Eq.7.43 is given as;

$$
\begin{aligned}
\upsilon_1 &= \frac{\mu_3}{[\mu_2]^{\frac{3}{2}}} \\
&= \frac{\frac{2\left[\theta^3(\alpha^3+6\alpha^2+6\alpha+2)+6\theta^2(5\alpha^2+7\alpha+3)+36\alpha\theta+12(3\theta+2)\right]}{\theta^3(\alpha\theta+\theta+2)^3}}{\left[\frac{\theta^2(\alpha^2+4\alpha+2)+16\alpha\theta+12(\theta+1)}{\theta^2(\alpha\theta+\theta+2)^2}\right]^{\frac{3}{2}}} \\
&= \frac{2\left[\theta^3(\alpha^3+6\alpha^2+6\alpha+2)+6\theta^2(5\alpha^2+7\alpha+3)+36\alpha\theta+12(3\theta+\right.}{\left[\theta^2(\alpha^2+4\alpha+2)+16\alpha\theta+12(\theta+1)\right]^{\frac{3}{2}}}
\end{aligned}
$$

the following relation is applied to derive coefficient of kurtosis expression of a generalized two parameter Quasi Sujatha distribution Eq.7.36.

$$
\upsilon_2 = \frac{\mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2}{[\mu_2^1 - [\mu_1^1]^2]^2}
$$

lastly, index of dispersion coefficient Eq.7.45 is obtained as;

$$\begin{aligned}
\upsilon_3 &= \frac{\sigma^2}{\mu_1^1} \\
&= \frac{\theta^2(\alpha^2+4\alpha+2)+16\alpha\theta+12(\theta+1)}{\theta^2(\alpha\theta+\theta+2)^2}\frac{\theta(\alpha\theta+\theta+2)}{\alpha\theta+2\theta+6} \\
&= \frac{\theta^2(\alpha^2+4\alpha+2)+16\alpha\theta+12(\theta+1)}{\theta(\alpha\theta+\theta+2)(\alpha\theta+2\theta+6)}
\end{aligned}$$

□

7.3.11 Excess loss Distribution

Proposition 7.3.12. *In this section we state probability density function of excess loss distribution $f_l(x;\alpha,\theta)$, mean residual lifetime (MRL), equilibrium distribution $f_e(x;\alpha,\theta)$, survival function based on the equilibrium distribution $S_e(x,\alpha,\theta)$ and hazard function of the equilibrium distribution $h_e(x;\alpha,\theta)$ of a Quasi Sujatha distribution Eq.7.36 as;*

$$f_l(x;\alpha,\theta) = \frac{\theta^2(\alpha+\theta x+\theta x^2)e^{-(x-z)\theta}}{\alpha\theta+\theta+2+\theta z(\theta+\theta z+2)}; \quad x>z \tag{7.46}$$

$$m(x) = \frac{\alpha\theta+2\theta+2\theta^2x+\theta^2x^2+4\theta x+6}{\theta(\alpha\theta+\theta+2+\theta^2x+\theta^2x^2+2\theta x)} \tag{7.47}$$

$$f_e(x;\alpha,\theta) = \frac{\theta(\alpha\theta+\theta+2+\theta^2x+\theta^2x^2+2\theta x)e^{-\theta x}}{\alpha\theta+2\theta+6} \tag{7.48}$$

$$S_e(x,\alpha,\theta) = \frac{(\alpha\theta+2\theta+2\theta^2x+\theta^2x^2+4\theta x+6)e^{-\theta x}}{\alpha\theta+2\theta+6} \tag{7.49}$$

$$h_e(x;\alpha,\theta) = \frac{\theta(\alpha\theta+\theta+2+\theta^2x+\theta^2x^2+2\theta x)}{\alpha\theta+2\theta+2\theta^2x+\theta^2x^2+4\theta x+6} \tag{7.50}$$

Proof. By definition Eq.1.10, the pdf of excess loss distribution Eq.7.46 is obtained as;

$$f_l(x;\alpha,\theta) = \frac{\frac{\theta^2}{\alpha\theta+\theta+2}\left[\alpha+\theta x+\theta x^2\right]e^{-\theta x}}{\left[\frac{\alpha\theta+\theta+2+\theta z(\theta+\theta z+2)}{\alpha\theta+\theta+2}\right]e^{-\theta z}}$$

$$= \frac{\theta^2(\alpha+\theta x+\theta x^2)e^{-(x-z)\theta}}{\alpha\theta+\theta+2+\theta z(\theta+\theta z+2)}; \quad x > z$$

further, by definition Eq.1.12 mean residual lifetime Eq.7.47 is obtained as;

$$m(x) = \frac{\alpha\theta+\theta+2}{(\alpha\theta+\theta+2+\theta^2 x+\theta^2 x^2+2\theta x)e^{-\theta x}}\int_x^\infty \frac{(\alpha\theta+\theta+2+\theta^2 t+\theta}{\alpha\theta+\theta+}$$

$$= \frac{1}{(\alpha\theta+\theta+2+\theta^2 x+\theta^2 x^2+2\theta x)e^{-\theta x}}\int_x^\infty (\alpha\theta+\theta+2+\theta^2 t+\theta^2$$

$$= \frac{1}{(\alpha\theta+\theta+2+\theta^2 x+\theta^2 x^2+2\theta x)e^{-\theta x}} I_1$$

$$I_1 = \int_x^\infty (\alpha\theta+\theta+2+\theta^2 t+\theta^2 t^2+2\theta t)e^{-\theta t}dt$$

$$u = (\alpha\theta+\theta+2+\theta^2 t+\theta^2 t^2+2\theta t) \Longrightarrow du = (\theta^2+2\theta^2 t+2\theta)$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_1 = -(\alpha\theta+\theta+2+\theta^2 t+\theta^2 t^2+2\theta t)\frac{e^{-\theta t}}{\theta} + \int_x^\infty (\theta+2\theta t+2)e^{-\theta t}dt$$

$$I_1 = -(\alpha\theta+\theta+2+\theta^2 t+\theta^2 t^2+2\theta t)\frac{e^{-\theta t}}{\theta} + I_2$$

$$
\begin{aligned}
I_2 &= \int_x^{\infty} (\theta + 2\theta t + 2) e^{-\theta t} dt \\
u &= (\theta + 2\theta t + 2) \Longrightarrow du = 2\theta dt \\
dv &= e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta} \\
I_2 &= -(\theta + 2\theta t + 2)\frac{e^{-\theta t}}{\theta} + 2\int_x^{\infty} e^{-\theta t} dt \\
I_2 &= -(\theta + 2\theta t + 2)\frac{e^{-\theta t}}{\theta} + 2 * I_3 \\
I_3 &= \int_x^{\infty} e^{-\theta t} dt = \frac{-e^{[} - \theta t]}{\theta}
\end{aligned}
$$

Combining I_1, I_2 and I_3 then;

$$
\begin{aligned}
I_1 &= -(\alpha\theta + \theta + 2 + \theta^2 t + \theta^2 t^2 + 2\theta t)\frac{e^{-\theta t}}{\theta} - (\theta + 2\theta t + 2)\frac{e^{-\theta t}}{\theta} - \frac{2e^{-\theta t}}{\theta} \\
I_1 &= \left[\frac{-e^{-\theta t}}{\theta}\left[\alpha\theta + 2\theta + 2\theta^2 t + \theta^2 t^2 + 4\theta t + 6\right]\right]_x^{\infty} \\
I_1 &= \frac{e^{-\theta x}}{\theta}\left[\alpha\theta + 2\theta + 2\theta^2 x + \theta^2 x^2 + 4\theta x + 6\right] \\
&= \frac{1}{(\alpha\theta + \theta + 2 + \theta^2 x + \theta^2 x^2 + 2\theta x)e^{-\theta x}} \frac{e^{-\theta x}}{\theta}\left[\alpha\theta + 2\theta + 2\theta^2 x + \theta^2 x^2 + \right. \\
m(x) &= \frac{\alpha\theta + 2\theta + 2\theta^2 x + \theta^2 x^2 + 4\theta x + 6}{\theta(\alpha\theta + \theta + 2 + \theta^2 x + \theta^2 x^2 + 2\theta x)}
\end{aligned}
$$

equilibrium distribution defined in Eq.7.48 is obtained by use of relation Eq.1.12 as;

$$
\begin{aligned}
f_e(x;\alpha,\theta) &= \frac{(\alpha\theta + \theta + 2 + \theta^2 x + \theta^2 x^2 + 2\theta x)e^{-\theta x}}{\alpha\theta + \theta + 2} \frac{\theta(\alpha\theta + \theta + 2)}{\alpha\theta + 2\theta + 6} \\
&= \frac{\theta(\alpha\theta + \theta + 2 + \theta^2 x + \theta^2 x^2 + 2\theta x)e^{-\theta x}}{\alpha\theta + 2\theta + 6}
\end{aligned}
$$

further survival function based on the equilibrium distribution Eq.7.49 is obtained by use relation Eq.1.13 as;

$$\int_x^\infty S(t;\alpha,\theta)dt = \frac{(\alpha\theta+2\theta+2\theta^2x+\theta^2x^2+4\theta+6)e^{-\theta x}}{\theta(\alpha\theta+\theta+2)}$$
$$= \frac{(\alpha\theta+2\theta+2\theta^2x+\theta^2x^2+4\theta+6)e^{-\theta x}}{\theta(\alpha\theta+\theta+2)}\frac{\theta(\alpha\theta+\theta+2)}{\alpha\theta+2\theta+6}$$
$$S_e(x,\alpha,\theta) = \frac{(\alpha\theta+2\theta+2\theta^2x+\theta^2x^2+4\theta x+6)e^{-\theta x}}{\alpha\theta+2\theta+6}$$

lastly, by definition Eq.1.14 the hazard function based on equilibrium distribution Eq.7.50 is given as;

$$h_e(x;\alpha,\theta) = \frac{\frac{\theta(\alpha\theta+\theta+2+\theta^2x+\theta^2x^2+2\theta x)e^{-\theta x}}{\alpha\theta+2\theta+6}}{\left[\frac{(\alpha\theta+2\theta+2\theta^2x+\theta^2x^2+4\theta x+6)e^{-\theta x}}{\alpha\theta+2\theta+6}\right]}$$
$$= \frac{\theta(\alpha\theta+\theta+2+\theta^2x+\theta^2x^2+2\theta x)}{\alpha\theta+2\theta+2\theta^2x+\theta^2x^2+4\theta x+6}$$

□

7.4 Three parameter Sujatha distribution

7.4.1 Construction of a three parameter Sujatha distribution

Proposition 7.4.1. *A generalized three parameter Sujatha distribution (AG2PSD) is a three component finite mixed distribution of Gamma* $(1,\theta)$*, Gamma* $(2,\theta)$ *and Gamma* $(3,\theta)$ *with mixing proportions as* $\omega_1 = \frac{\theta^2}{\theta^2+\alpha\beta\theta+2\alpha\beta}$ *and* $\omega_2 = \frac{\alpha\beta\theta}{\theta^2+\alpha\beta\theta+2\alpha\beta}$.

The pdf and Cdf of AG3PSD are stated as;

$$f(x;\alpha,\beta,\theta)=\frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}\left[1+\alpha\beta x+\alpha\beta x^2\right]e^{-\theta x};x>0,\alpha>0,\beta>0, \tag{7.51}$$

$$F(x;\alpha,\beta,\theta)=1-\left[1+\frac{\alpha\beta(\theta^2x+\theta^2x^2+2\theta x)}{\theta^2+\alpha\beta\theta+2\alpha\beta}\right]e^{-\theta x};x>0,\alpha>0,\beta>0,\theta> \tag{7.52}$$

By the definition of a finite mixture Eq.1.1, the pdf Eq.7.51 is constructed as;

Proof.

$$\begin{aligned}
f(x;\alpha,\beta,\theta) &= \frac{\theta^2}{\theta^2+\alpha\beta\theta+2\alpha\beta}\left[\theta e^{-\theta x}\right]+\frac{\alpha\beta\theta}{\theta^2+\alpha\beta\theta+2\alpha\beta}\left[\frac{\theta^2e^{-\theta x}x}{\Gamma 2}\right]+\frac{}{\theta^2 \cdots}\\
&= \frac{\theta^3e^{-\theta x}}{\theta^2+\alpha\beta\theta+2\alpha\beta}+\frac{\alpha\beta e^{-\theta x}x}{\theta^2+\alpha\beta\theta+2\alpha\beta}+\frac{2\alpha\beta\theta^3e^{-\theta x}x^2}{2(\theta^2+\alpha\beta\theta+2\alpha\beta)}\\
&= \frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}\left[1+\alpha\beta x+\alpha\beta x^2\right]e^{-\theta x};x>0,\alpha>0,\beta>0,
\end{aligned}$$

Remark 7.4.2. *The proposed generalized three parameter Sujatha distribution Eq.7.51 is nested with three distributions. To begin with, putting $\alpha=\beta=0$, a generalized three parameter Sujatha distribution reduces to exponential distribution in Eq.2.17. Putting $\alpha=\beta=1$, a generalized three parameter Sujatha distribution reduces to a one parameter Sujatha distribution in Eq.7.1. Similarly, putting $\beta=1$, AG3PSD reduces to AG2PSD in Eq.7.16.*

AG3PSD Eq.4.29 is a modification of proposed AG2PSD Eq.4.15.

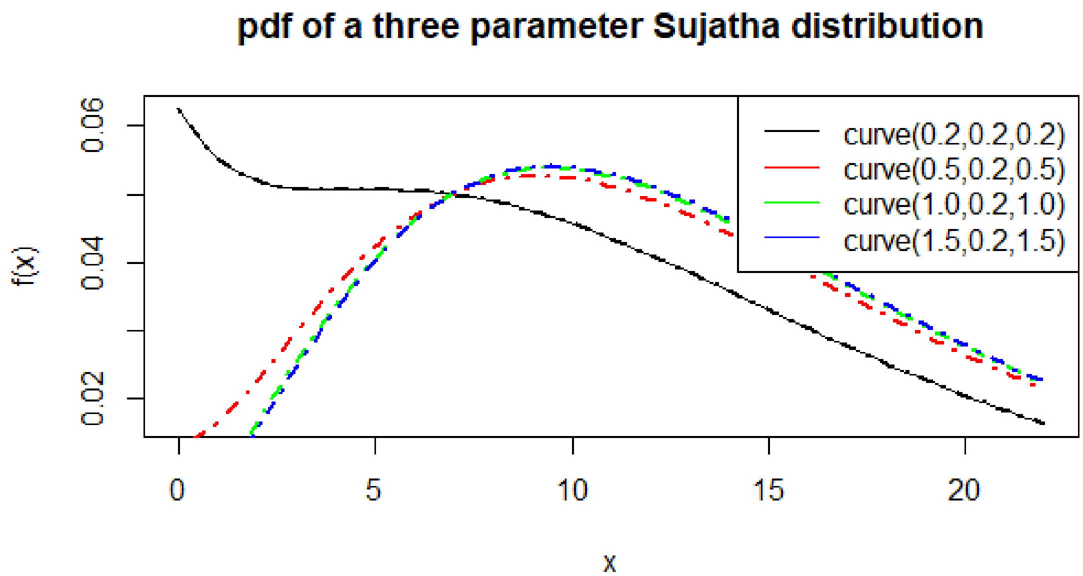

Figure 15. Shapes of $f(x)$ of a generalized three parameter Sujatha distribution with varying values of α, β and a constant θ

further Cdf Eq.7.52 is given as;

$$F(x;\alpha,\beta,\theta) = \frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}\int_0^\infty (1+\alpha\beta x+\alpha\beta x^2)e^{-\theta x}dx$$

$$F(x;\alpha,\beta,\theta) = \frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}I_1$$

$$I_1 = \int_0^\infty (1+\alpha\beta x+\alpha\beta x^2)e^{-\theta x}dx$$

$$u = (1+\alpha\beta x+\alpha\beta x^2) \Longrightarrow du = (\alpha\beta+2\alpha\beta x)$$

$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$

$$I_1 = -(1+\alpha\beta x+\alpha\beta x^2)\frac{e^{-\theta x}}{\theta} - \frac{1}{\theta}\int_0^\infty -(\alpha\beta+2\alpha\beta x)e^{-\theta x}dx$$

$$I_1 = -(1+\alpha\beta x+\alpha\beta x^2)\frac{e^{-\theta x}}{\theta} - \frac{1}{\theta}I_2$$

$$I_2 = \int_0^\infty -(\alpha\beta+2\alpha\beta x)e^{-\theta x}dx$$

$$u = -(\alpha\beta+2\alpha\beta x) \Longrightarrow du = -2\alpha\beta$$

$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$

$$I_2 = (\alpha\beta+2\alpha\beta x)\frac{e^{-\theta x}}{\theta} - \frac{2\alpha\beta}{\theta}\int_0^\infty e^{-\theta x}dx$$

$$I_2 = (\alpha\beta+2\alpha\beta x)\frac{e^{-\theta x}}{\theta} - \frac{2\alpha\beta}{\theta}I_3$$

$$I_3 = \int_0^\infty e^{-\theta x}dx$$

$$u = -\theta x \Longrightarrow du = -\theta dx \Longrightarrow \frac{du}{-\theta} = dx$$

$$I_3 = \int_0^\infty e^u\frac{du}{-\theta} \Longrightarrow \frac{1}{-\theta}\int_0^\infty e^u du \Longrightarrow \frac{e^{-\theta x}}{\theta}$$

From I_1,I_2 and I_3 we have the following;

$$I_1 = -(1+\alpha\beta x+\alpha\beta x^2)\frac{e^{-\theta x}}{\theta} - \frac{1}{\theta}\left[(\alpha\beta+2\alpha\beta x)\frac{e^{-\theta x}}{\theta} - \frac{2\alpha\beta}{\theta}\right.$$

$$I_1 = -\left[\theta^2(1+\alpha\beta x+\alpha\beta x^2)+\theta(\alpha\beta+2\alpha\beta x)+2\alpha\beta\right]\frac{e^{-\theta x}}{\theta^3}$$

$$= 1 + -\left[\frac{\theta^2(1+\alpha\beta x+\alpha\beta x^2)+\theta(\alpha\beta+2\alpha\beta x)+2\alpha\beta}{\theta^2+\alpha\beta\theta+2\alpha\beta}\right]e^{-}$$

$$F(x;\alpha,\beta,\theta) = 1 - \left[1+\frac{\alpha\beta(\theta^2 x+\theta^2 x^2+2\theta x)}{\theta^2+\alpha\beta\theta+2\alpha\beta}\right]e^{-\theta x}; x>0, \alpha>0, \beta>$$

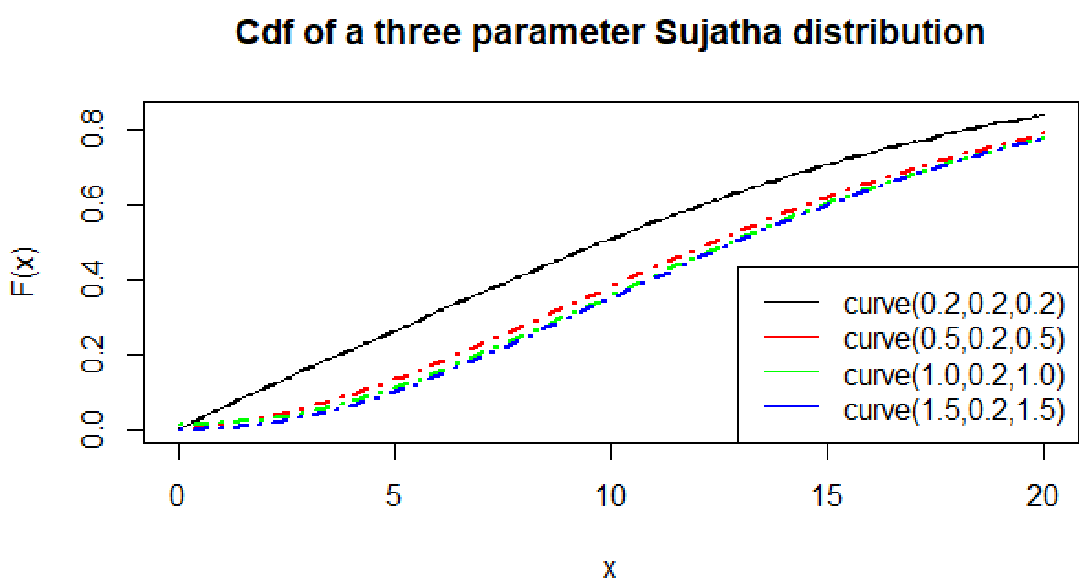

Figure 16. Shapes of $F(x)$ of a generalized three parameter Sujatha distribution with varying values of α, β and constant θ

□

7.4.2 Reliability Analysis

Proposition 7.4.3. *We now state the survival function denoted by $S(x;\alpha,\beta,\theta)$ and hazard function denoted by $h(x;\alpha,\beta,\theta)$ of* *a generalized three parameter Sujatha distribution Eq.7.51 as;*

$$S(x;\alpha,\beta,\theta) = \left[1 + \frac{\alpha\beta(\theta^2 x + \theta^2 x^2 + 2\theta x)}{\theta^2 + \alpha\beta\theta + 2\alpha\beta}\right] e^{-\theta x}; x > 0, \alpha > 0, \beta > 0, \theta > 0 \quad (7.53)$$

$$h(x;\alpha,\beta,\theta) = \frac{\theta^3(1 + \alpha\beta(x + x^2))}{\theta^2 + \alpha\beta(\theta + 2 + \theta^2 x + \theta^2 x^2 + 2\theta x)}; x > 0, \alpha > 0, \beta > 0, \theta > 0 \quad (7.54)$$

Proof. To begin with, survival function Eq.7.53 is obtained by use of relation Eq.1.8 as;

$$\begin{aligned} S(x;\alpha,\beta,\theta) &= 1 - \left[1 - \left[1 + \frac{\alpha\beta(\theta^2 x + \theta^2 x^2 + 2\theta x)}{\theta^2 + \alpha\beta\theta + 2\alpha\beta}\right] e^{-\theta x}\right] \\ &= \left[1 + \frac{\alpha\beta(\theta^2 x + \theta^2 x^2 + 2\theta x)}{\theta^2 + \alpha\beta\theta + 2\alpha\beta}\right] e^{-\theta x}; x > 0, \alpha > 0, \beta > 0, \theta > 0 \end{aligned}$$

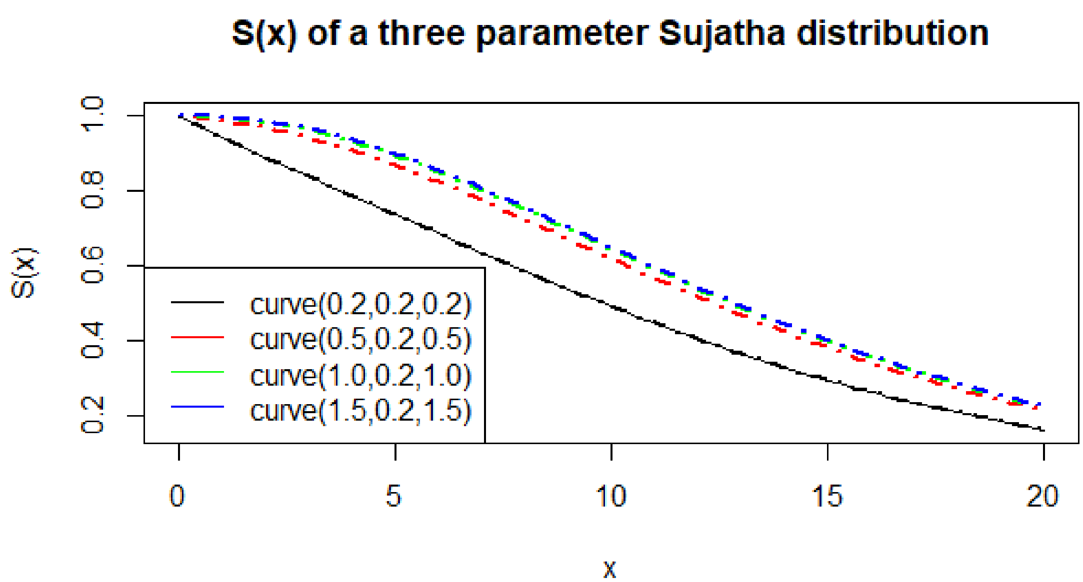

Figure 17. Shapes of $S(x)$ of a generalized three parameter Sujatha distribution with varying values of α, β and constant θ

further be definition Eq.1.9 , the hazard function Eq.7.53 is obtained as;

$$h(x;\alpha,\beta,\theta)=\frac{\frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}\left[1+\alpha\beta x+\alpha\beta x^2\right]e^{-\theta x}}{\left[\frac{\theta^2+\alpha\beta(\theta+2+\theta^2x+\theta^2x^2+2\theta x)}{\theta^2+\alpha\beta\theta+2\alpha\beta}\right]e^{-\theta x}}$$

$$=\frac{\theta^3(1+\alpha\beta(x+x^2))}{\theta^2+\alpha\beta(\theta+2+\theta^2x+\theta^2x^2+2\theta x)};x>0,\alpha>0,\beta>0,$$

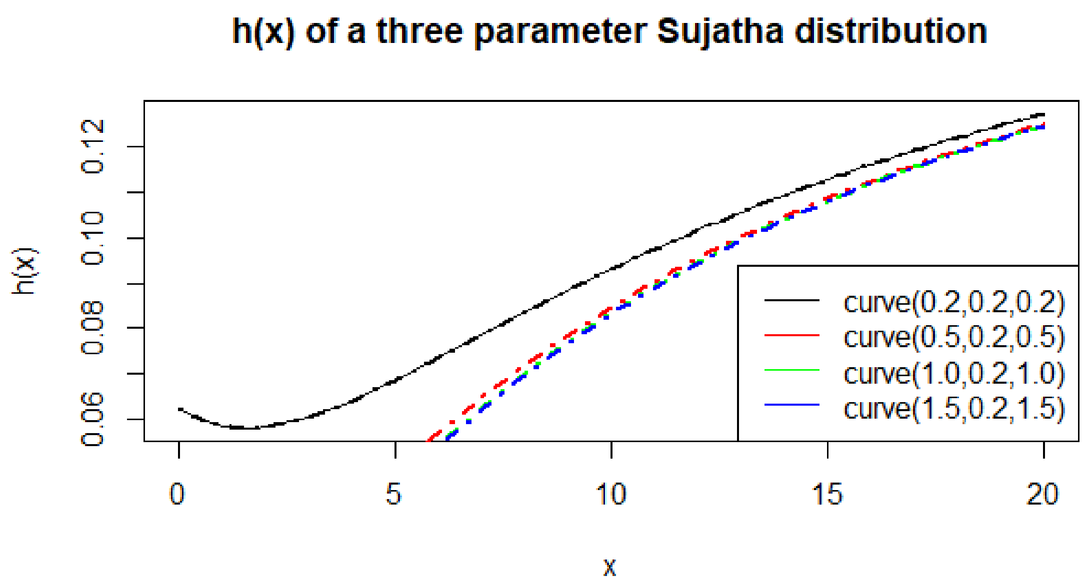

Figure 18. Shapes of $h(x)$ of a generalized three parameter Sujatha distribution with varying values of α, β and constant θ

□

7.4.3 Moments and related measures

Proposition 7.4.4. *The r^{th} moments of a generalized three parameter Sujatha distribution Eq.7.51 are derived using method of* *moments and moment generating function as;*

$$\mu_r^1 * = \frac{r!\left[\theta^2+\alpha\beta\theta(r+1)+\alpha\beta(r+1)(r+2)\right]}{\theta^r(\theta^2+\alpha\beta\theta+2\alpha\beta)}; \quad r=1,2,3,\cdots$$

(7.55)

Proof. By definition Eq.1.16, r^{th} moments of AG3PSD are given as;

$$\begin{aligned}
E(X^r) &= \frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}\int_0^\infty x^r(1+\alpha\beta x+\alpha\beta x^2)e^{-\theta x}dx \\
&= \frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}\left[\int_0^\infty x^r e^{-\theta x}dx+\alpha\beta\int_0^\infty x^{r+1}e^{-\theta x}dx+\alpha\beta\int_0^\infty x^{r+2}e\right. \\
&= \frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}\left[\frac{\Gamma(r+1)}{\theta^{r+1}}+\frac{\alpha\beta\Gamma(r+2)}{\theta^{r+2}}+\frac{\alpha\beta\Gamma r+3}{\theta^{r+3}}\right] \\
&= \frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}\left[\frac{r\Gamma r}{\theta^{r+1}}+\frac{\alpha\beta(r+1)r\Gamma r}{\theta^{r+2}}+\frac{\alpha\beta(r+2)(r+1)r\Gamma r}{\theta^{r+3}}\right] \\
&= \frac{\theta^3 r\Gamma r}{\theta^2+\alpha\beta\theta+2\alpha\beta}\left[\frac{1}{\theta}+\frac{\alpha\beta(r+1)}{\theta^2}+\frac{\alpha\beta(r+2)(r+1)}{\theta^3}\right] \\
&= \frac{r!\left[\theta^2+\alpha\beta\theta(r+1)+\alpha\beta(r+1)(r+2)\right]}{\theta^r(\theta^2+\alpha\beta\theta+2\alpha\beta)}; \quad r=1,2,3,\cdots
\end{aligned}$$

further by definition Eq.1.17, the mgf is given as;

$$
\begin{aligned}
m_x(t) &= \frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}\int_0^\infty e^{tx}(1+\alpha\beta x+\alpha\beta x^2)dx \\
&= \frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}\left[\frac{1}{\theta-t}+\frac{\alpha\beta}{(\theta-t)^2}+\frac{2\alpha\beta}{(\theta-t)^3}\right] \\
&= \frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}\left[\frac{1}{\theta}\sum_{k=0}^{\infty}\Big(\frac{t}{\theta}\Big)^k+\frac{\alpha\beta}{\theta^2}\sum_{k=0}^{\infty}\binom{k+1}{k}\Big(\frac{t}{\theta}\Big)^k+\frac{\alpha\beta}{\theta^3}\sum_{k=0}^{\infty}\Big(\right. \\
&= \sum_{k=0}^{\infty}\frac{\theta^2+\alpha\theta(k+1)+\alpha\beta(k+1)(k+2)}{\theta^2+\alpha\beta\theta+2\alpha\beta}\Big(\frac{t}{\theta}\Big)^k
\end{aligned}
$$

The r^{th} moments of a generalized three parameter Sujatha distribution Eq.7.51 are obtained as a coefficient of $\frac{t^r}{r!}$ of the moment generating function $m_x(t)$ as shown below;

$$
\mu_r^1 = \frac{r!\Big[\theta^2+\alpha\beta\theta(r+1)+\alpha\beta(r+1)(r+2)\Big]}{\theta^r(\theta^2+\alpha\beta\theta+2\alpha\beta)}; \quad r = 1,2,3,\cdots
$$

□

We obtain the four none centralized moments of AG3PSD Eq.7.51 by putting $r = 1,2,3$ and 4 in Eq.7.55 as shown below;

$$
\mu_1^1 = \frac{\theta^2+2\alpha\beta\theta+6\alpha\beta}{\theta(\theta^2+\alpha\beta\theta+2\alpha\beta)}, \quad \mu_2^1 = \frac{2(\theta^2+3\alpha\beta\theta+12\alpha\beta)}{\theta^2(\theta^2+\alpha\beta\theta+2\alpha\beta)}
$$

$$
\mu_3^1 = \frac{6(\theta^2+4\alpha\beta\theta+20\alpha\beta)}{\theta^3(\theta^2+\alpha\beta\theta+2\alpha\beta)}, \quad \mu_4^1 = \frac{24(\theta^2+5\alpha\beta\theta+30\alpha\beta)}{\theta^4(\theta^2+\alpha\beta\theta+2\alpha\beta)}
$$

The centralized moments of AG3PSD are derived as;

$$\mu_1 = \mu_1^1$$
$$\mu_2 = \mu_2^1 - [\mu_1^1]^2$$
$$\mu_2 = \frac{\theta^4 + 4\alpha\beta\theta^3 + 16\alpha\beta\theta^2 + 36\alpha^2\beta^2\theta + 2\alpha^2\beta^2\theta^2 + 108\alpha^2\beta^2}{\theta^2(\theta^2 + \alpha\beta\theta + 2\alpha\beta)^2}$$

The following relations are applied to obtain 3^{rd} and 4^{th} centralized moments of a proposed generalized three parameter Sujatha distribution Eq.7.51.

$$\mu_3 = \mu_3^1 - 3[\mu_1^1\mu_2^1] + 2[\mu_1^1]^3$$
$$\mu_4 = \mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2$$

Proposition 7.4.5. *Other related measures ofAG3PSD Eq.7.51 such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) are stated as;*

$$C.v = \frac{\sqrt{\theta^4 + 4\alpha\beta\theta^3 + 16\alpha\beta\theta^2 + 36\alpha^2\beta^2\theta + 2\alpha^2\beta^2\theta^2 + 108\alpha^2\beta^2}}{\theta^2 + 2\alpha\beta\theta + 6\alpha\beta} \tag{7.56}$$

$$\upsilon_1 = \frac{\mu_3^1 - 3[\mu_1^1\mu_2^1] + 2[\mu_1^1]^3}{(\mu_2^1 - [\mu_1^1]^2)^{\frac{3}{2}}} \tag{7.57}$$

$$\upsilon_2 = \frac{\mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2}{(\mu_2^1 - [\mu_1^1]^2)^2} \tag{7.58}$$

$$\upsilon_3 = \frac{\theta^4 + 4\alpha\beta\theta^3 + 16\alpha\beta\theta^2 + 36\alpha^2\beta^2\theta + 2\alpha^2\beta^2\theta^2 + 108\alpha^2\beta^2}{\theta(\theta^2 + \alpha\beta\theta + 2\alpha\beta)(\theta^2 + 2\alpha\beta\theta + 6\alpha\beta)} \tag{7.59}$$

Proof. To begin with, variation coefficient Eq.7.56 is given as;

$$C.v = \frac{\sigma}{\mu_1^1}$$
$$= \frac{\sqrt{\theta^4+4\alpha\beta\theta^3+16\alpha\beta\theta^2+36\alpha^2\beta^2\theta+2\alpha^2\beta^2\theta^2+108\alpha^2\beta^2}}{\theta^2(\theta^2+\alpha\beta\theta+2\alpha\beta)^2} \frac{\theta(\theta^2 \cdots}{\theta^2 \cdots}$$
$$= \frac{\sqrt{\theta^4+4\alpha\beta\theta^3+16\alpha\beta\theta^2+36\alpha^2\beta^2\theta+2\alpha^2\beta^2\theta^2+108\alpha^2\beta^2}}{\theta^2+2\alpha\beta\theta+6\alpha\beta}$$

The following relation is applied to derive skewness coefficient of AG3PSD.

$$\upsilon_1 = \frac{\mu_3}{[\mu_2]^{\frac{3}{2}}} = \frac{\mu_3^1 - 3[\mu_1^1\mu_2^1] + 2[\mu_1^1]^3}{[\mu_2^1 - [\mu_1^1]^2]^{\frac{3}{2}}}$$

The following expression is applied to derive coefficient of kurtosis of a generalized three parameter Sujatha distribution Eq.7.51.

$$\upsilon_2 = \frac{\mu_4}{[\mu_2]^2} = \frac{\mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2}{[\mu_2^1 - [\mu_1^1]^2]^2}$$

index of dispersion presented in Eq,7.59 is obtained as;

$$\upsilon_3 = \frac{\sigma^2}{\mu_1^1}$$
$$= \frac{\theta^4+4\alpha\beta\theta^3+16\alpha\beta\theta^2+36\alpha^2\beta^2\theta+2\alpha^2\beta^2\theta^2+108\alpha^2\beta^2}{\theta^2(\theta^2+\alpha\beta\theta+2\alpha\beta)^2} \frac{\theta(\theta^2+ \cdots}{\theta^2+2 \cdots}$$
$$= \frac{\theta^4+4\alpha\beta\theta^3+16\alpha\beta\theta^2+36\alpha^2\beta^2\theta+2\alpha^2\beta^2\theta^2+108\alpha^2\beta^2}{\theta(\theta^2+\alpha\beta\theta+2\alpha\beta)(\theta^2+2\alpha\beta\theta+6\alpha\beta)}$$

□

7.4.4 Excess Loss Distribution

Proposition 7.4.6. *In this section we define probability density function of excess function $f_l(x;\alpha,\beta,\theta)$, mean residual lifetime $(m(x))$,equilibrium distribution $f_e(x;\alpha,\beta,\theta)$, survival function based on the equilibrium distribution $S_e(x;\alpha,\beta,\theta)$ and hazard function based on the equilibrium distribution $h_e(x;\alpha,\beta,\theta)$ of a generalized three parameter Sujatha distribution Eq.7.51 as;*

$$f_l(x;\alpha,\beta,\theta) = \frac{\theta^3(1+\alpha\beta(x+x^2))e^{-(x-z)\theta}}{\theta^2+\alpha\beta(\theta+2+\theta^2 z+\theta^2 z^2+2\theta z)}; \quad x > z \tag{7.60}$$

$$m(x) = \frac{\theta^2+\alpha\beta(\theta+6+\theta^2 x+\theta^2 x^2+4\theta x)}{\theta(\theta^2+\alpha\beta(\theta+\theta^2 x+\theta^2 x^2+2\theta x))} \tag{7.61}$$

$$f_e(x;\alpha,\beta,\theta) = \frac{\theta(\theta^2+\alpha\beta(\theta+2+\theta^2 x+\theta^2 x^2+2\theta x))e^{-\theta x}}{\theta^2+2\alpha\beta\theta+6\alpha\beta} \tag{7.62}$$

$$S_e(x;\alpha,\beta,\theta) = \frac{(\theta^2+\alpha\beta(2\theta+6+\theta^2 x+\theta^2 x^2+4\theta x))e^{-\theta x}}{\theta^2+2\alpha\beta\theta+6\alpha\beta} \tag{7.63}$$

$$h_e(x;\alpha,\beta,\theta) = \frac{\theta(\theta^2+\alpha\beta(\theta+\theta^2 x+\theta^2 x^2+2\theta x))}{\theta^2+\alpha\beta(\theta+6+\theta^2 x+\theta^2 x^2+4\theta x)} \tag{7.64}$$

Proof. By definition Eq.1.10, the pdf of excess loss distribution Eq.7.60 is obtained as;

$$f_l(x;\alpha,\beta,\theta) = \frac{\frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}\left[1+\alpha\beta x+\alpha\beta x^2\right]e^{-\theta x}}{\left[\frac{\theta^2+\alpha\beta(\theta+2+\theta^2 z+\theta^2 z^2+2\theta z)}{\theta^2+\alpha\beta\theta+2\alpha\beta}\right]e^{-\theta z}}$$

$$= \frac{\theta^3(1+\alpha\beta(x+x^2))e^{-(x-z)\theta}}{\theta^2+\alpha\beta(\theta+2+\theta^2 z+\theta^2 z^2+2\theta z)}; \quad x > z$$

Mean residual lifetime (MRL) further the mean residual lifetime Eq.7.61 is obtained by use of the relation Eq.1.11 as;

$$m(x) = \frac{\theta^2+\alpha\beta\theta+2\alpha\beta}{(\theta^2+\alpha\beta(\theta+2+\theta^2x+\theta^2x^2+2\theta x))e^{-\theta x}}\int_x^\infty \frac{\theta^2+\alpha\beta(\theta+2+\theta^2t+}{\theta^2+\alpha\beta\theta+}$$

$$= \frac{1}{(\theta^2+\alpha\beta(\theta+2+\theta^2x+\theta^2x^2+2\theta x))e^{-\theta x}}\int_x^\infty [\theta^2+\alpha\beta(\theta+\theta^2t+\theta^2t$$

$$= \frac{1}{(\theta^2+\alpha\beta(\theta+2+\theta^2x+\theta^2x^2+2\theta x))e^{-\theta x}}I_1$$

$$I_1 = \int_x^\infty [\theta^2+\alpha\beta(\theta+\theta^2t+\theta^2t^2+2\theta t)e^{-\theta t}]dt$$

$$u = \theta^2+\alpha\beta(\theta+\theta^2t+\theta^2t^2+2\theta t) \Longrightarrow \alpha\beta\theta^2+2\alpha\beta\theta^2t+2\alpha\beta\theta$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_1 = -(\theta^2+\alpha\beta(\theta+\theta^2t+\theta^2t^2+2\theta t))\frac{-e^{\theta t}}{\theta}+\int_x^\infty(\alpha\beta\theta+2\alpha\beta\theta t+2\alpha\beta)$$

$$I_1 = -(\theta^2+\alpha\beta(\theta+\theta^2t+\theta^2t^2+2\theta t))\frac{-e^{\theta t}}{\theta}+I_2$$

$$I_2 = \int_x^\infty(\alpha\beta\theta+2\alpha\beta\theta t+2\alpha\beta)e^{-\theta t}dt$$

$$u = (\alpha\beta\theta+2\alpha\beta\theta t+2\alpha\beta) \Longrightarrow du = 2\alpha\beta\theta$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_2 = -(\alpha\beta\theta+2\alpha\beta\theta t+2\alpha\beta)\frac{e^{-\theta t}}{\theta}+2\alpha\beta\int_x^\infty e^{-\theta t}dt$$

$$I_2 = -(\alpha\beta\theta+2\alpha\beta\theta t+2\alpha\beta)\frac{e^{-\theta t}}{\theta}+2\alpha\beta * I_3$$

$$I_3 = \int_x^\infty e^{-\theta t}dt \Longrightarrow \frac{-e^{-\theta t}}{\theta}$$

From I_1, I_2 and I_3 we have the following;

$$\begin{aligned}
I_1 &= -(\theta^2+\alpha\beta(2+\theta^2 t+\theta^2 t^2+2\theta t))\frac{e^{-\theta t}}{\theta}-(\alpha\beta(\theta+2\theta t+2))\frac{e^{-\theta t}}{\theta} \\
&= \left[\frac{-e^{-\theta t}}{\theta}\left[\theta^2+\alpha\beta(2\theta+6+\theta^2 t+\theta^2 t^2+4\theta t)\right]\right]_x^{\infty} \\
I_1 &= \frac{e^{-\theta x}}{\theta}\left[\theta^2+\alpha\beta(2\theta+6+\theta^2 x+\theta^2 x^2+4\theta x)\right] \\
&= \frac{1}{(\theta^2+\alpha\beta(\theta+2+\theta^2 x+\theta^2 x^2+2\theta x))e^{-\theta x}} I_1 \\
m(x) &= \frac{\theta^2+\alpha\beta(\theta+6+\theta^2 x+\theta^2 x^2+4\theta x)}{\theta(\theta^2+\alpha\beta(\theta+\theta^2 x+\theta^2 x^2+2\theta x))}
\end{aligned}$$

by definition Eq.1.12, equilibrium distribution Eq.7.62 is given as;

$$\begin{aligned}
f_e(x;\alpha,\beta,\theta) &= \frac{(\theta^2+\alpha\beta(\theta+2+\theta^2 x+\theta^2 x^2+2\theta x))e^{-\theta x}}{\theta^2+\alpha\beta\theta+2\alpha\beta}\frac{\theta(\theta^2+\alpha\beta\theta+}{\theta^2+2\alpha\beta\theta+} \\
&= \frac{\theta(\theta^2+\alpha\beta(\theta+2+\theta^2 x+\theta^2 x^2+2\theta x))e^{-\theta x}}{\theta^2+2\alpha\beta\theta+6\alpha\beta}
\end{aligned}$$

using the relation Eq.1.13, $S_e(x;\alpha,\beta,\theta)$ in Eq.7.63 is obtained as;

$$\begin{aligned}
\int_x^{\infty} S(t;\alpha,\beta,\theta)dt &= \frac{(\theta^2+\alpha\beta(2\theta+6+\theta^2 x+\theta^2 x^2+4\theta x))e^{-\theta x}}{\theta(\theta^2+\alpha\beta\theta+2\alpha\beta)} \\
&= \frac{(\theta^2+\alpha\beta(2\theta+6+\theta^2 x+\theta^2 x^2+4\theta x))e^{-\theta x}}{\theta(\theta^2+\alpha\beta\theta+2\alpha\beta)}\frac{\theta(\theta^2+}{\theta^2+2} \\
S_e(x;\alpha,\beta,\theta) &= \frac{(\theta^2+\alpha\beta(2\theta+6+\theta^2 x+\theta^2 x^2+4\theta x))e^{-\theta x}}{\theta^2+2\alpha\beta\theta+6\alpha\beta}
\end{aligned}$$

lastly, hazard function based on equilibrium distribution Eq.7.64 is obtained by use of Eq.1.14 as;

$$h_e(x;\alpha,\beta,\theta) = \frac{\frac{\theta(\theta^2+\alpha\beta(\theta+2+\theta^2x+\theta^2x^2+2\theta x))e^{-\theta x}}{\theta^2+2\alpha\beta\theta+6\alpha\beta}}{\left[\frac{(\theta^2+\alpha\beta(2\theta+6+\theta^2x+\theta^2x^2+4\theta x))e^{-\theta x}}{\theta^2+2\alpha\beta\theta+6\alpha\beta}\right]}$$

$$= \frac{\theta(\theta^2+\alpha\beta(\theta+\theta^2x+\theta^2x^2+2\theta x))}{\theta^2+\alpha\beta(\theta+6+\theta^2x+\theta^2x^2+4\theta x)}$$

□

7.4.5 Estimation of a generalized three parameter Sujatha distribution

In this section, we discuss estimation methods such as MOME and MLE.

MOME

MOME of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\theta}$ of α, β and θ respectively in AG3PSD Eq.7.51 are obtained as;

$$\frac{\mu_2^1}{[\mu_1^1]^2} = k(constant)$$

$$\frac{2(\theta^2+3\alpha\beta\theta+12\alpha\beta)}{\theta^2(\theta^2+\alpha\beta\theta+2\alpha\beta)}\left[\frac{\theta(\theta^2+\alpha\beta\theta+2\alpha\beta)}{\theta^2+2\alpha\beta\theta+6\alpha\beta}\right]^2 = k$$

$$\frac{2(\theta^2+3\alpha\beta\theta+12\alpha\beta)(\theta^2+\alpha\beta\theta+2\alpha\beta)}{(\theta^2+2\alpha\beta\theta+6\alpha\beta)^2} = k$$

$$2(\theta^2+3\alpha\beta\theta+12\alpha\beta)(\theta^2+\alpha\beta\theta+2\alpha\beta) = k(\theta^2+2\alpha\beta\theta+6\alpha\beta)^2$$

$$[6\theta^2+24\theta+24-4k\theta^2-24\theta k-36k](\alpha\beta)^2+[8\theta^3+16\theta^2-4\theta^3k-12\theta^2k](\alpha\beta$$

The value of b is obtained by applying quadratic formula on the following expression.

$$[6\theta^2+24\theta+24-4k\theta^2-24\theta k-36k]b^2+[8\theta^3+16\theta^2-4\theta^3k-12\theta^2k]b+$$

The population mean and sample mean are replaced with sample mean and parameter estimates respectively as;

$$\mu_1^1=\frac{\theta^2+2\alpha\beta\theta+6\alpha\beta}{\theta(\theta^2+\alpha\beta\theta+2\alpha\beta)}$$

$$\bar{x}=\frac{\hat{\theta}^2+2\hat{\alpha}\hat{\beta}\hat{\theta}+6\hat{\alpha}\hat{\beta}}{\hat{\theta}(\hat{\theta}^2+\hat{\alpha}\hat{\beta}\hat{\theta}+2\hat{\alpha}\hat{\beta})}$$

$$\bar{x}=\frac{\hat{\theta}^2}{\hat{\theta}(\hat{\theta}^2+\hat{\alpha}\hat{\beta}\hat{\theta}+2\hat{\alpha}\hat{\beta})}\bar{x}_1+\frac{\hat{\alpha}\hat{\beta}\hat{\theta}}{\hat{\theta}(\hat{\theta}^2+\hat{\alpha}\hat{\beta}\hat{\theta}+2\hat{\alpha}\hat{\beta})}\bar{x}_2+\frac{\hat{\alpha}\hat{\beta}}{\hat{\theta}(\hat{\theta}^2+\hat{\alpha}\hat{\beta}\hat{\theta}+}$$

The values $\bar{x}$, $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ are mean for AG3PSD, Gamma $(1,\theta)$, Gamma $(2,\theta)$ and Gamma $(3,\theta)$ respectively.

$$\bar{x}=\frac{\hat{\theta}^2}{\hat{\theta}^2+\hat{\alpha}\hat{\beta}\hat{\theta}+2\hat{\alpha}\hat{\beta}}\left(\frac{1}{\hat{\theta}}\right)+\frac{\hat{\alpha}\hat{\beta}\hat{\theta}}{\hat{\theta}^2+\hat{\alpha}\hat{\beta}\hat{\theta}+2\hat{\alpha}\hat{\beta}}\left(\frac{2}{\hat{\theta}}\right)+\frac{\hat{\alpha}\hat{\beta}}{\hat{\theta}^2+\hat{\alpha}\hat{\beta}\hat{\theta}+2\hat{\alpha}\hat{\beta}}$$

Setting $b=\alpha\beta$, the MOME estimates of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\theta}$ of α, β and θ respectively are obtained as;

$$\hat{\alpha}=\frac{\hat{\theta}^2-\hat{\theta}^2\bar{x}}{(\hat{\beta}\hat{\theta}\bar{x}+2\hat{\beta}\bar{x}-2\hat{\beta}\hat{\theta}-3\hat{\beta})}$$

$$\hat{\beta}=\frac{-\hat{\theta}^2\bar{x}+\hat{\theta}^2}{\hat{\alpha}\hat{\theta}\bar{x}+2\hat{\alpha}\bar{x}-\hat{\alpha}\hat{\theta}-3\hat{\alpha}}$$

$$\hat{\theta}=\frac{-(b\bar{x}-2b)\mp\sqrt{(b\bar{x}-2b)^2-4(\bar{x}-1)(2b\bar{x}-3b)}}{2(\bar{x}-1)}$$

MLE

Let $(x_1, x_2, x_3, \cdots n)$ be a random sample of size say n drawn from AG3PSD 7.51, the l function $(L(x;\alpha,\beta,\theta))$ is derived as;

$$\prod_{i=1}^{n} f(x;\alpha,\beta,\theta) = \prod_{i=1}^{n}\left[\frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}(1+\alpha\beta x_i+\alpha\beta x_i^2)e^{-\theta x_i}\right]$$
$$L(x;\alpha,\beta,\theta) = \left(\frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}\right)^n \prod_{i=1}^{n}(1+\alpha\beta x_i+\alpha\beta x_i^2)e^{-\sum_{i=1}^{n} x_i}$$

We obtain natural log function of the l function as;

$$lnL = nln\left[\frac{\theta^3}{\theta^2+\alpha\beta\theta+2\alpha\beta}\right] + \sum_{i=1}^{n} ln(1+\alpha\beta x_i+\alpha\beta x_i^2) - n\theta\bar{x}$$

where $\bar{x}$ is the sample mean. The following systems of non-linear equations are solved to obtain maximum likelihood estimates of the parameters α, β and θ as $\hat{\alpha}$, $\hat{\beta}$ and θ respectively;

$$\frac{\partial lnL}{\partial\alpha} = \frac{-n(\beta\theta+2\beta)}{\theta^2+\alpha\beta\theta+2\alpha\beta} + \sum_{i=1}^{n}\frac{\beta x_i(1+x_i)}{1+\alpha\beta x_i+\alpha\beta x_i^2} = 0 \quad (7.65)$$
$$\frac{\partial lnL}{\partial\beta} = \frac{-n(\alpha\theta+2\alpha)}{\theta^2+\alpha\beta\theta+2\alpha\beta} + \sum_{i=1}^{n}\frac{\alpha x_i(1+x_i)}{1+\alpha\beta x_i+\alpha\beta x_i^2} = 0 \quad (7.66)$$
$$\frac{\partial lnL}{\partial\theta} = \frac{n(\theta^2+2\alpha\beta\theta+6\alpha\beta)}{\theta(\theta^2+\alpha\beta\theta+2\alpha\beta)} - n\bar{x} = 0 \quad (7.67)$$

Equations 7.65, 7.66 and 7.67 can not be solved directly. Methods like fisher scoring is applied to solve the equations. The second derivative equations are applied in fisher scoring method.

The equations are;

$$\frac{\partial^2 lnL}{\partial\theta^2} = \frac{-n(\theta^4+2\alpha^2\beta^2\theta^2+18\alpha^2\beta^2\theta+20\alpha\beta\theta^2+12\alpha^2\beta^2)}{\theta^2(\theta^2+\alpha\beta\theta+2\alpha\beta)^2}$$

$$\frac{\partial^2 lnL}{\partial\theta\partial\alpha} = \frac{-n(\beta\theta^4-4\beta\theta^3)}{(\theta(\theta^2+\alpha\beta\theta+2\alpha\beta))^2} = \frac{\partial^2 lnL}{\partial\alpha\partial\theta}$$

$$\frac{\partial^2 lnL}{\partial\beta\partial\theta} = \frac{-n(\alpha\theta^2-\alpha^2\beta\theta+\alpha^2\beta^2+4\alpha\theta)}{(\theta^2+\alpha\beta\theta+2\alpha\beta)^2} = \frac{\partial^2 lnL}{\partial\theta\partial\beta}$$

$$\frac{\partial^2 lnL}{\partial\alpha^2} = \frac{n(\beta\theta+2\beta)^2}{(\theta^2+\alpha\beta\theta+2\alpha\beta)^2} - \sum_{i=1}^{n}\frac{(\beta x_i+\beta x_i^2)^2}{(1+\alpha\beta x_i+\alpha\beta x_i^2)^2}$$

$$\frac{\partial^2 lnL}{\partial\alpha\partial\beta} = \frac{-n(\theta^3+2\theta^2)}{(\theta^2+\alpha\beta\theta+2\alpha\beta)^2} + \sum_{i=1}^{n}\frac{x_i+x_i^2+\alpha\beta x_i^2-\alpha\beta x_i^3}{(1+\alpha\beta x_i+\alpha\beta x_i^2)^2} = \frac{\partial^2 lnL}{\partial\beta\partial\alpha}$$

The maximum likelihood estimates $\hat{\theta}$, $\hat{\alpha}$ and $\hat{\beta}$ of the parameters θ, α and β respectively are obtained by iteratively solving the following systems of equations.

$$\begin{bmatrix} \frac{\partial^2 lnL}{\partial\theta^2} & \frac{\partial^2 lnL}{\partial\theta\partial\alpha} & \frac{\partial^2 lnL}{\partial\theta\partial\beta} \\ \frac{\partial^2 lnL}{\partial\alpha\partial\theta} & \frac{\partial^2 lnL}{\partial\alpha^2} & \frac{\partial^2 lnL}{\partial\alpha\partial\beta} \\ \frac{\partial^2 lnL}{\partial\beta\partial\theta} & \frac{\partial^2 lnL}{\partial\beta\partial\alpha} & \frac{\partial^2 lnL}{\partial\beta^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0 \\ \hat{\beta}=\beta_0}} \begin{bmatrix} \hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0 \\ \hat{\beta}=\beta_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial lnL}{\partial\theta} \\ \frac{\partial lnL}{\partial\alpha} \\ \frac{\partial lnL}{\partial\beta} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0 \\ \hat{\beta}=\beta_0}}$$

The values θ_0, α_0 and β_0 are initial values of the parameters θ, α and β respectively. Iterative technique is applied to solve the equations until sufficiently large values of the estimates θ_0, α_0 and β_0 are obtained.

7.4.6 Application

We shall apply the data on carbon breaking stress presented in table 1 to fit a generalized three parameter Lindley distribution (AG3PLD), a generalized three parameter Aradhana distribution (AG3PAD) and a generalized three parameter Sujatha distribution (AG3PSD).

Table 5. MLE estimates, -2lnL, AIC, AICC and BIC values of fitted Three Parameter Distributions

Model	Estimates	-2lnL	AIC	AICC	BIC
AG3PLD	$\hat{\alpha}$=-0.9664 $\hat{\beta}$=2.5903 $\hat{\theta}$=0.9113	204.4509	210.4509	210.8380	217.0199
AG3PAD	$\hat{\alpha}$=30.0386 $\hat{\beta}$=29.8082 $\hat{\theta}$=1.0866	204.4656	210.4656	210.8527	217.0346
AG3PSD	$\hat{\alpha}$=27.9980 $\hat{\beta}$=28.1557 $\hat{\theta}$=0.9685	213.4691	219.4691	219.8562	226.0381

Table 5 above illustrates fitted distributions. A distribution with lower calculated values of -2lnL, BIC, AIC and AICC is considered the most flexible model over the others. Evidently, AG3PLD is the best fit than AG3PAD AND AG3PSD. In figure 19 below, fitted density of AG3PLD has the highest curve than other fits.

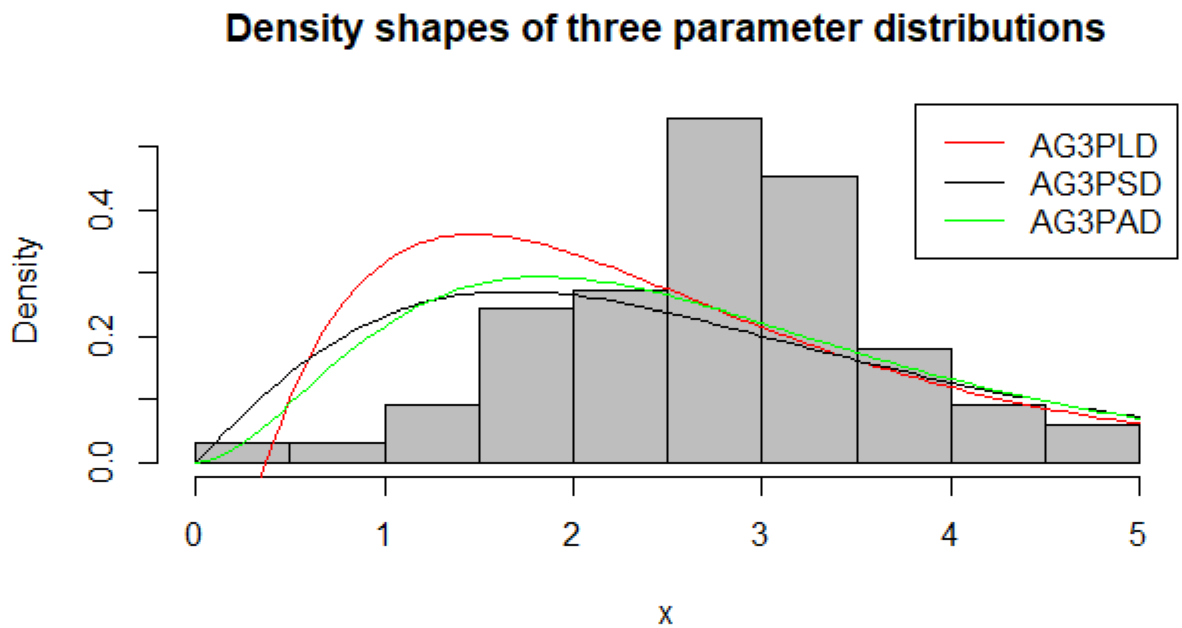

Figure 19. Estimated densities of fitted three parameter distributions

8 THREE COMPONENT FINITE GAMMA MIXTURE (A case of Aradhana distribution)

8.1 Introduction

In this chapter we consider a three component finite gamma mixture a case of Aradhana distribution. We shall construct and derive statistical properties of Aradhana distribution and its generalizations. The mixed distribution is expressed in terms of pdf and Cdf. Statistical properties such as reliability analysis, equilibrium distribution properties and moments.

8.2 One parameter Aradhana distribution

8.2.1 Construction of one parameter Aradhana distribution

Proposition 8.2.1. *Let* $\omega_1 = \frac{\theta^2}{\theta^2+2\theta+2}$ *and* $\omega_2 = \frac{2\theta}{\theta^2+2\theta+2}$ *be mixing proportions, one parameter Aradhana distribution is a finite mixed distribution of Gamma* $(1,\theta)$*, Gamma* $(2,\theta)$ *and Gamma* $(3,\theta)$*. The pdf and Cdf of one parameter Aradhana distribution are;*

$$f(x;\theta) = \frac{\theta^3}{\theta^2+2\theta+2}[1+x]^2 e^{-\theta x}; x>0, \theta>0 \tag{8.1}$$

$$F(x;\theta) = 1 - \left[1 + \frac{\theta x(2\theta+\theta x+2)}{\theta^2+2\theta+2}\right] e^{-\theta x}; x>0, \theta>0 \tag{8.2}$$

Proof. To begin with, by definition of finite mixture Eq.1.1 the pdf Eq.8.1 is obtained as;

$$
\begin{aligned}
f(x;\theta) &= \frac{\theta^2}{\theta^2+2\theta+2}\left[\theta e^{-\theta x}\right] + \frac{2\theta}{\theta^2+2\theta+2}\left[\frac{\theta^2 e^{-\theta x}x}{\Gamma 2}\right] + \frac{2}{\theta^2+2\theta+2}\Big[\\
&= \frac{\theta^3 e^{-\theta x}}{\theta^2+2\theta+2} + \frac{2\theta^3 e^{-\theta x}x}{\theta^2+2\theta+2} + \frac{2\theta^3 e^{-\theta x}x^2}{2(\theta^2+2\theta+2)} \\
&= \frac{\theta^3 e^{-\theta x}}{\theta^2+2\theta+2} + \frac{2\theta^3 e^{-\theta x}x}{\theta^2+2\theta+2} + \frac{\theta^3 e^{-\theta x}x^2}{\theta^2+2\theta+2} \\
&= \frac{\theta^3}{\theta^2+2\theta+2}\left[1+2x+x^2\right]e^{-\theta x} \\
&= \frac{\theta^3}{\theta^2+2\theta+2}[1+x]^2 e^{-\theta x}; x>0, \theta>0
\end{aligned}
$$

further Cdf Eq.8.2 is given as;

$$
\begin{aligned}
F(x;\theta) &= \frac{\theta^3}{\theta^2+2\theta+2}\int_0^\infty (1+2x+x^2)e^{-\theta x}dx \\
F(x;\theta) &= \frac{\theta^3}{\theta^2+2\theta+2} I_1 \\
I_1 &= \int_0^\infty (1+2x+x^2)e^{-\theta x}dx \\
u &= (1+2x+x^2) \Longrightarrow du = (2+2x)dx \\
dv &= e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta} \\
I_1 &= -(1+2x+x^2)\frac{e^{-\theta x}}{\theta} - \frac{1}{\theta}\int_0^\infty -(2+2x)e^{-\theta x}dx
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= -(1+2x+x^2)\frac{e^{-\theta x}}{\theta} - \frac{1}{\theta}I_2 \\
I_2 &= \int_0^\infty -(2+2x)e^{-\theta x}dx \\
u &= -(2+2x) \Longrightarrow du = -2dx \\
dv &= e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta} \\
I_2 &= (2+2x)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}\int_0^\infty e^{-\theta x}dx \\
I_2 &= (2+2x)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}I_3 \\
I_3 &= \int_0^\infty e^{-\theta x}dx \\
u &= -\theta x \Longrightarrow du = -\theta dx \Longrightarrow \frac{du}{-\theta} = dx \\
I_3 &= \int_0^\infty e^u \frac{du}{-\theta} \Longrightarrow \frac{-1}{\theta}\int_0^\infty e^u du \Longrightarrow \frac{-e^{-\theta x}}{\theta}
\end{aligned}
$$

From I_1, I_2 and I_3 we have the following;

$$
\begin{aligned}
I_1 &= -(1+2x+x^2)\frac{e^{-\theta x}}{\theta} - \frac{1}{\theta}\left[(2+2x)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}\left(\frac{-e^{-\theta x}}{\theta}\right)\right] \\
I_1 &= \frac{-e^{-\theta x}}{\theta^3}\left[\theta^2+2\theta+2+\theta x(2\theta+\theta x+2)\right] \\
&= 1 + \frac{-e^{-\theta x}}{\theta^3}\left[\theta^2+2\theta+2+\theta x(2\theta+\theta x+2)\right]\frac{\theta^3}{\theta^2+2\theta+2} \\
&= 1 - \left[\frac{\theta^2+2\theta+2+\theta x(2\theta+\theta x+2)}{\theta^2+2\theta+2}\right]e^{-\theta x} \\
F(x;\theta) &= 1 - \left[1+\frac{\theta x(2\theta+\theta x+2)}{\theta^2+2\theta+2}\right]e^{-\theta x}; x>0, \theta>0
\end{aligned}
$$

□

8.2.2 Reliability Analysis

Proposition 8.2.2. *In this section, we state survival function denoted by $S(x;\theta)$ and hazard function denoted by $h(x;\theta)$ of a one parameter Aradhana distribution Eq.8.1 as;*

$$S(x;\theta) = \left[1 + \frac{\theta x(2\theta + \theta x + 2)}{\theta^2 + 2\theta + 2}\right] e^{-\theta x}; x > 0, \theta > 0 \tag{8.3}$$

$$h(x;\theta) = \frac{\theta^3 [1+x]^2}{\theta^2 + 2\theta + 2 + \theta x(2\theta + \theta x + 2)}; x > 0, \theta > 0 \tag{8.4}$$

Proof. By definition Eq.1.8, survival function Eq.8.3 is obtained as;

$$\begin{aligned} S(x;\theta) &= 1 - \left[1 - \left[1 + \frac{\theta x(2\theta + \theta x + 2)}{\theta^2 + 2\theta + 2}\right] e^{-\theta x}\right] \\ &= \left[1 + \frac{\theta x(2\theta + \theta x + 2)}{\theta^2 + 2\theta + 2}\right] e^{-\theta x}; x > 0, \theta > 0 \end{aligned}$$

further hazard function Eq.8.4 is obtained by use of Eq.1.9 as;

$$\begin{aligned} h(x;\theta) &= \frac{\frac{\theta^3}{\theta^2+2\theta+2}[1+x]^2 e^{-\theta x}}{\left[\frac{\theta^2+2\theta+2+\theta x(2\theta+\theta x+2)}{\theta^2+2\theta+2}\right] e^{-\theta x}} \\ &= \frac{\theta^3 [1+x]^2}{\theta^2 + 2\theta + 2 + \theta x(2\theta + \theta x + 2)}; x > 0, \theta > 0 \end{aligned}$$

□

8.2.3 Moments and related measures

Proposition 8.2.3. *The r^{th} moments of a one parameter Aradhana distribution Eq.8.1 are derived using both method of mo-*

ments and moment generating function (mgf) as;

$$\mu_r^1 * = \frac{r!\left[\theta^2+2(r+1)\theta+(r+1)(r+2)\right]}{\theta^r(\theta^2+2\theta+2)}; r=1,2,3,\cdots \quad \textit{(8.5)}$$

Proof. To begin with, by definition Eq.1.16 moments of one parameter Aradhana distribution are obtained as;

$$\begin{aligned}
E(X^r) &= \frac{\theta^3}{\theta^2+2\theta+2}\int_0^\infty x^r(1+2x+x^2)e^{-\theta x}dx \\
&= \frac{\theta^3}{\theta^2+2\theta+2}\left[\int_0^\infty x^r e^{-\theta x}dx+2\int_0^\infty x^{r+1}e^{-\theta x}dx+\int_0^\infty x^{r+2}e^{-\theta x}dx\right] \\
&= \frac{\theta^3}{\theta^2+2\theta+2}\left[\frac{\Gamma(r+1)}{\theta^{r+1}}+\frac{2\Gamma r+2}{\theta^{r+2}}+\frac{\Gamma r+3}{\theta^{r+3}}\right] \\
&= \frac{\theta^3 r\Gamma r}{\theta^r(\theta^2+2\theta+2)}\left[\frac{1}{\theta}+\frac{2(r+1)}{\theta^2}+\frac{(r+2)(r+1)}{\theta^3}\right] \\
&= \frac{r\Gamma r\left[\theta^2+2(r+1)\theta+(r+1)(r+2)\right]}{\theta^r(\theta^2+2\theta+2)} \\
&= \frac{r!\left[\theta^2+2(r+1)\theta+(r+1)(r+2)\right]}{\theta^r(\theta^2+2\theta+2)}; r=1,2,3,\cdots
\end{aligned}$$

further by definition Eq.1.17, mgf of one parameter Aradhana distribution Eq.8.1 is given as;

$$\begin{aligned}
m_x(t) &= \frac{\theta^3}{\theta^2+2\theta+2}\int_0^\infty e^{tx}(1+2x+x^2)e^{-\theta x}dx \\
&= \frac{\theta^3}{\theta^2+2\theta+2}\int_0^\infty e^{-(\theta-t)x}(1+2x+x^2)dx \\
&= \frac{\theta^3}{\theta^2+2\theta+2}\left[\frac{1}{\theta-t}+\frac{2}{(\theta-t)^2}+\frac{2}{(\theta-t)^3}\right] \\
&= \frac{\theta^3}{\theta^2+2\theta+2}\left[\frac{1}{\theta}\sum_{k=0}^{\infty}\left(\frac{t}{\theta}\right)^k+\frac{2}{\theta^2}\sum_{k=0}^{\infty}\binom{k+1}{k}\left(\frac{t}{\theta}\right)^k+\frac{2}{\theta^2}\sum_{k=0}^{\infty}\left(\begin{matrix}k+ \\ k\end{matrix}\right.\right. \\
&= \sum_{k=0}^{\infty}\frac{\theta^2+2(k+1)\theta+(k+1)(k+2)}{\theta^2+2\theta+2}\left(\frac{t}{\theta}\right)^k
\end{aligned}$$

The none centralized moments of a one parameter Aradhana distribution Eq.8.1 are obtained as a coefficient of $\frac{t^r}{r!}$ of the derived moment generating function $m_x(t)$ as;

$$\mu_r^1 = \frac{r!\left[\theta^2+2(r+1)\theta+(r+1)(r+2)\right]}{\theta^r(\theta^2+2\theta+2)}; r = 1,2,3,\cdots$$

□

For the values of r as 1,2,3 and 4 in Eq.8.5 we obtain the four moments about the origin of a one parameter Aradhana distribution Eq.8.1 as;

$$\mu_1^1 = \frac{\theta^2+4\theta+6}{\theta(\theta^2+2\theta+2)}, \quad \mu_2^1 = \frac{2(\theta^2+6\theta+12)}{\theta^2(\theta^2+2\theta+2)}$$
$$\mu_3^1 = \frac{6(\theta^2+8\theta+20)}{\theta^3(\theta^2+2\theta+2)}, \quad \mu_4^1 = \frac{24(\theta^2+10\theta+30)}{\theta^4(\theta^2+2\theta+2)}$$

The none centralized moments of a One parameter Aradhana distribution are as;

$$\mu_1 = \mu_1^1$$
$$\mu_2 = \mu_2^1 - [\mu_1^1]^2 \Longrightarrow \frac{\theta^4+8\theta^3+24\theta^2+24\theta+12}{\theta^2(\theta^2+2\theta+2)^2}$$
$$\mu_3 = \mu_3^1 - 3[\mu_1^1\mu_2^1] + 2[\mu_1^1]^3$$
$$\mu_3 = \frac{2(\theta^6+12\theta^5+54\theta^4+100\theta^3+108\theta^2+72\theta+24)}{\theta^3(\theta^2+2\theta+2)^3}$$

The following relation is used to derive the expression for the 4^{th} centralized moments of one parameter Aradhana distribution Eq.8.1.

$$\mu_4 = \mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2$$

Proposition 8.2.4. *We now define Other related measures of one parameter Aradhana distribution Eq.8.1 such as variation (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) are stated as;*

$$C.v = \frac{\sqrt{\theta^4+8\theta^3+24\theta^2+24\theta+12}}{\theta^2+4\theta+6} \tag{8.6}$$
$$\upsilon_1 = \frac{2(\theta^6+12\theta^5+54\theta^4+100\theta^3+108\theta^2+72\theta+24)}{[\theta^4+8\theta^3+24\theta^2+24\theta+12]^{\frac{3}{2}}} \tag{8.7}$$
$$\upsilon_2 = \frac{\mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2}{(\mu_2^1 - [\mu_1^1]^2)^2} \tag{8.8}$$
$$\upsilon_3 = \frac{\theta^4+8\theta^3+24\theta^2+24\theta+12}{\theta(\theta^2+2\theta+2)(\theta^2+4\theta+6)} \tag{8.9}$$

Proof. To begin with, variation coefficient Eq.8.6 is obtained as;

$$C.v = \frac{\sigma}{\mu_1^1}$$
$$= \frac{\sqrt{\theta^4+8\theta^3+24\theta^2+24\theta+12}}{\theta(\theta^2+2\theta+2)}\frac{\theta(\theta^2+2\theta+2)}{\theta^2+4\theta+6} \Longrightarrow \frac{\sqrt{\theta^4+8\theta^3+2}}{\theta^2+}$$

secondly, skewness coefficient Eq.8.7 is given as;

$$\upsilon_1 = \frac{\mu^3}{[\mu_2]^{\frac{3}{2}}}$$
$$= \frac{2(\theta^6+12\theta^5+54\theta^4+100\theta^3+108\theta^2+72\theta+24)}{\theta^3(\theta^2+2\theta+2)^3}\left[\frac{\theta^2(\theta^2+2}{\theta^4+8\theta^3+24\theta}\right.$$
$$= \frac{2(\theta^6+12\theta^5+54\theta^4+100\theta^3+108\theta^2+72\theta+24)}{[\theta^4+8\theta^3+24\theta^2+24\theta+12]^{\frac{3}{2}}}$$

The following expression is applied to derive coefficient of kurtosis of a one parameter Aradhana distribution Eq.8.1.

$$\upsilon_2 = \frac{\mu_4}{[\mu_2]^2} = \frac{\mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2}{[\mu_2^1 - [\mu_1^1]^2]^2}$$

lastly, index of dispersion coefficient Eq.8.9 is given as;

$$\upsilon_3 = \frac{\sigma^2}{\mu_1^1}$$
$$= \frac{\theta^4+8\theta^3+24\theta^2+24\theta+12}{\theta^2(\theta^2+2\theta+2)^2}\frac{\theta(\theta^2+2\theta+2)}{\theta^2+4\theta+6} \Longrightarrow \frac{\theta^4+8\theta^3+24\theta^2}{\theta(\theta^2+2\theta+2)(\theta}$$

□

8.2.4 Excess Loss Distribution

Proposition 8.2.5. *In this section we state probability density function of the excess loss function $f_l(x;\theta)$, mean residual lifetime (MRL), Equilibrium distribution $f_e(x;\theta)$, survival function of equilibrium distribution $S_e(x;\theta)$ and hazard function of equilibrium distribution $h_e(x;\theta)$ of a one parameter Aradhana distribution Eq.8.1 as;*

$$f_l(x;\theta) = \frac{\theta^3 \left[1+x\right]^2 e^{-(x-z)\theta}}{\theta^2+2\theta+2+\theta z(2\theta+\theta z+2)}; \quad x > z \tag{8.10}$$

$$m(x) = \frac{\theta^2+4\theta+6+2\theta^2 x+\theta^2 x^2+4\theta x}{\theta\left[(\theta^2+2\theta+2)+\theta x(2\theta+\theta x+2)\right]} \tag{8.11}$$

$$f_e(x;\theta) = \frac{\theta(\theta^2+2\theta+2+\theta x(2\theta+\theta x+2))e^{-\theta x}}{\theta^2+4\theta+6} \tag{8.12}$$

$$S_e(x;\theta) = \frac{(\theta^2+4\theta+6+2\theta^2 x+\theta^2 x^2+4\theta x)e^{-\theta x}}{\theta^2+4\theta+6} \tag{8.13}$$

$$h_e(x;\theta) = \frac{\theta(\theta^2+2\theta+2+\theta x(2\theta+\theta x+2))}{\theta^2+4\theta+6+2\theta^2 x+\theta^2 x^2 4\theta x} \tag{8.14}$$

Proof. By definition Eq.1.10, the pdf of excess loss distribution Eq.8.10 is given as;

$$\begin{aligned} f_l(x;\theta) &= \frac{\frac{\theta^3}{\theta^2+2\theta+2}[1+x]^2 e^{-\theta x}}{\left[\frac{\theta^2+2\theta+2+\theta z(2\theta+\theta z+2)}{\theta^2+2\theta+2}\right]e^{-\theta z}} \\ &= \frac{\theta^3 \left[1+x\right]^2 e^{-(x-z)\theta}}{\theta^2+2\theta+2+\theta z(2\theta+\theta z+2)}; \quad x > z \end{aligned}$$

mean residual lifetime Eq.8.11 is obtained by use of relation Eq.1.11 as;

$$m(x) = \frac{\theta^2+2\theta+2}{[\theta^2+2\theta+2+\theta x(2\theta+\theta x+2)]e^{-\theta x}} \int_x^\infty \frac{[\theta^2+2\theta+2+\theta t(2\theta+}{\theta^2+2\theta+2}$$

$$= \frac{1}{[\theta^2+2\theta+2+\theta x(2\theta+\theta x+2)]e^{-\theta x}} \int_x^\infty [\theta^2+2\theta+2+\theta t(2\theta+$$

$$= \frac{\theta^2+2\theta+2}{[\theta^2+2\theta+2+\theta x(2\theta+\theta x+2)]e^{-\theta x}} \int_x^\infty \frac{[\theta^2+2\theta+2+\theta t(2\theta+}{\theta^2+2\theta+2}$$

$$= \frac{1}{[\theta^2+2\theta+2+\theta x(2\theta+\theta x+2)]e^{-\theta x}} I_1$$

$$I_1 = \int_x^\infty \frac{[\theta^2+2\theta+2+\theta t(2\theta+\theta t+2)]e^{-\theta t}}{\theta^2+2\theta+2}$$

$$u = (\theta^2+2\theta+2+2\theta^2 t+\theta^2 t^2+2\theta t) \Longrightarrow du = (2\theta^2+2\theta^2 t+2\theta)$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_1 = -(\theta^2+2\theta+2+2\theta^2 t+\theta^2 t^2+2\theta t)\frac{-e^{\theta t}}{\theta} + \int_x^\infty (2\theta+2\theta t+2)e^{-}$$

$$I_1 = -(\theta^2+2\theta+2+2\theta^2 t+\theta^2 t^2+2\theta t)\frac{-e^{\theta t}}{\theta} + I_2$$

$$I_2 = \int_x^\infty (2\theta+2\theta t+2)e^{-\theta t}dt$$

$$u = (2\theta+2\theta t+2) \Longrightarrow du = 2\theta dt$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_2 = -(2\theta+2\theta t+2)\frac{-e^{\theta t}}{\theta} + 2\int_x^\infty e^{-\theta t}dt$$

$$I_2 = -(2\theta+2\theta t+2)\frac{-e^{\theta t}}{\theta} + 2I_3$$

$$I_3 = \int_x^\infty e^{-\theta t}dt \Longrightarrow \frac{-e^{-\theta t}}{\theta}$$

From I_1, I_2 and I_3 we have the following;

$$I_1 = -(\theta^2+2\theta+2+2\theta^2t+\theta^2t^2+2\theta t)\frac{e^{-\theta t}}{\theta} - (2\theta+2\theta t+2)\frac{e^{-\theta t}}{\theta} - \frac{2e^{-\theta}}{\theta}$$

$$= \left[\frac{-e^{-\theta t}}{\theta}\left(\theta^2+4\theta+6+2\theta^2t+\theta^2t^2+4\theta t\right)\right]_x^\infty$$

$$I_1 = \frac{e^{-\theta x}}{\theta}\left[\theta^2+4\theta+6+2\theta^2x+\theta^2x^2+4\theta x\right]$$

$$= \frac{1}{[\theta^2+2\theta+2+\theta x(2\theta+\theta x+2)]e^{-\theta x}}\frac{e^{-\theta x}}{\theta}\left[\theta^2+4\theta+6+2\theta^2x+\theta^2x^2\right.$$

$$m(x) = \frac{\theta^2+4\theta+6+2\theta^2x+\theta^2x^2+4\theta x}{\theta\left[(\theta^2+2\theta+2)+\theta x(2\theta+\theta x+2)\right]}$$

similarly, by definition Eq.1.12 equilibrium distribution Eq.8.12 is obtained as;

$$f_e(x;\theta) = \left[\frac{\theta^2+2\theta+2+\theta x(2\theta+\theta x+2)}{\theta^2+2\theta+2}\right]e^{-\theta x}\frac{\theta(\theta^2+2\theta+2)}{\theta^2+4\theta+6}$$

$$= \frac{\theta(\theta^2+2\theta+2+\theta x(2\theta+\theta x+2))e^{-\theta x}}{\theta^2+4\theta+6}$$

by definition Eq.1.13, $S_e(x;\theta)$ in Eq.8.13 is given as;

$$\int_x^\infty S(t;\theta)dt = \frac{e^{-\theta x}(\theta^2+4\theta+6+2\theta^2x+\theta^2x^2+4\theta x)}{\theta(\theta^2+2\theta+2)}$$

$$= \frac{e^{-\theta x}(\theta^2+4\theta+6+2\theta^2x+\theta^2x^2+4\theta x)}{\theta(\theta^2+2\theta+2)}\frac{\theta(\theta^2+2\theta+2)}{\theta^2+4\theta+6}$$

$$S_e(x;\theta) = \frac{(\theta^2+4\theta+6+2\theta^2x+\theta^2x^2+4\theta x)e^{-\theta x}}{\theta^2+4\theta+6}$$

further by definition Eq.1.14, $h_e(x;\theta)$ in Eq.8.14 is given as;

$$h_e(x;\theta) = \frac{\frac{\theta(\theta^2+2\theta+2+\theta x(2\theta+\theta x+2))e^{-\theta x}}{\theta^2+4\theta+6}}{\frac{(\theta^2+4\theta+6+2\theta^2 x+\theta^2 x^2+4\theta x)e^{-\theta x}}{\theta^2+4\theta+6}}$$
$$= \frac{\theta(\theta^2+2\theta+2+\theta x(2\theta+\theta x+2))}{\theta^2+4\theta+6+2\theta^2 x+\theta^2 x^2 4\theta x}$$

□

8.3 Two parameter Aradhana distribution

8.3.1 Construction of a two parameter Aradhana distribution

Proposition 8.3.1. *Let* $\omega_1 = \frac{\theta^2}{\theta^2+2\alpha\theta+2\alpha^2}$ *and* $\omega_2 = \frac{2\alpha\theta}{\theta^2+2\alpha\theta+2\alpha^2}$ *be mixing probabilities, AG2PAD is a three component finite mixed distribution of Gamma* $(1,\theta)$*, Gamma* $(2,\theta)$ *and Gamma* $(3,\theta)$*. The pdf and Cdf of AG2PAD are;*

$$f(x;\alpha,\theta) = \frac{\theta^3}{\theta^2+2\alpha\theta+2\alpha^2}\left[1+\alpha x\right]^2 e^{-\theta x}; x>0, \alpha>0, \theta>0 \tag{8.15}$$

$$F(x;\alpha,\theta) = 1-\left[1+\frac{\alpha\theta x(2\theta+\alpha\theta x+2\alpha)}{\theta^2+2\alpha\theta+2\alpha^2}\right]e^{-\theta x}; x>0, \alpha>0, \theta>0 \tag{8.16}$$

Proof. To begin with, by definition of a finite mixture Eq.1.1, pdf Eq.8.15 is constructed as;

$$
\begin{aligned}
f(x;\alpha,\theta) &= \frac{\theta^2}{\theta^2+2\alpha\theta+2\alpha^2}\left[\theta e^{-\theta x}\right] + \frac{2\alpha\theta}{\theta^2+2\alpha\theta+2\alpha^2}\left[\frac{\theta^2 e^{-\theta x} x}{\Gamma 2}\right] + \frac{2}{\theta^2+2\alpha \cdots} \\
&= \frac{\theta^3 e^{-\theta x}}{\theta^2+2\alpha\theta+2\alpha^2} + \frac{2\alpha\theta^3 e^{-\theta x} x}{\theta^2+2\alpha\theta+2\alpha^2} + \frac{2\alpha^2\theta^3 e^{-\theta x} x^2}{2(\theta^2+2\alpha\theta+2\alpha^2)} \\
&= \frac{\theta^3}{\theta^2+2\alpha\theta+2\alpha^2}\left[1+2\alpha x+\alpha^2 x^2\right] e^{-\theta x}
\end{aligned}
$$

Taking the part;

$$\left[1+2\alpha x+\alpha^2 x^2\right]$$

We have the following;

$$
\begin{aligned}
&= 1+\alpha x+\alpha x+\alpha^2 x^2 \\
&= 1(1+\alpha x)+\alpha x(1+\alpha x) \Longrightarrow (1+\alpha x)(1+\alpha x) \Longrightarrow \left[1+\alpha x\right]^2
\end{aligned}
$$

$$f(x;\alpha,\theta) = \frac{\theta^3}{\theta^2+2\alpha\theta+2\alpha^2}\left[1+\alpha x\right]^2 e^{-\theta x}; x>0, \alpha>0, \theta>0$$

Remark 8.3.2. *A generalized two parameter Aradhana distribution Eq.8.15 is nested with two distributions. To begin with, putting $\alpha = 0$, a generalized two parameter Aradhana distribution reduces to exponential distribution Eq.2.17. Similarly, putting $\alpha = 1$ a generalized two parameter Aradhana distribution reduces to a one parameter Aradhana distribution Eq.8.1.*

further Cdf Eq.8.16 is obtained as;

$$F(x;\alpha,\theta) = \frac{\theta^3}{\theta^2+2\alpha\theta+2\alpha^2}\int_0^{\infty}(1+2\alpha x+\alpha^2x^2)e^{-\theta x}dx$$

$$F(x;\alpha,\theta) = \frac{\theta^3}{\theta^2+2\alpha\theta+2\alpha^2}I_1$$

$$I_1 = \int_0^{\infty}(1+2\alpha x+\alpha^2x^2)e^{-\theta x}dx$$

$$u = (1+2\alpha x+\alpha^2x^2) \Longrightarrow du = (2\alpha+2\alpha^2x^2)$$

$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$

$$I_1 = -(1+2\alpha x+\alpha^2x^2)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}\int_0^{\infty} -(\alpha+\alpha^2x)e^{-\theta x}dx$$

$$I_1 = -(1+2\alpha x+\alpha^2x^2)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}I_2$$

$$I_2 = \int_0^{\infty} -(\alpha+\alpha^2x)e^{-\theta x}dx$$

$$u = -(\alpha+\alpha^2x) \Longrightarrow du = -\alpha^2dx$$

$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$

$$I_2 = (\alpha+\alpha^2x)\frac{e^{-\theta x}}{\theta} - \frac{\alpha^2}{\theta}\int_0^{\infty} e^{-\theta x}dx$$

$$I_2 = (\alpha+\alpha^2x)\frac{e^{-\theta x}}{\theta} - \frac{\alpha^2}{\theta}I_3$$

$$I_3 = \int_0^{\infty} e^{-\theta x}dx$$

Let $u=-\theta x$ we have;

$$du=-\theta dx \Longrightarrow \frac{du}{-\theta}=dx$$
$$I_3=\int_0^{\infty} e^u \frac{du}{-\theta} \Longrightarrow \frac{-1}{\theta}\int_0^{\infty} e^u du \Longrightarrow \frac{-1}{\theta}e^{-\theta x}$$

Combining I_1, I_2 and I_3 we have the following;

$$I_1=-(1+2\alpha x+\alpha^2x^2)\frac{e^{-\theta x}}{\theta}-\frac{2}{\theta}\left[(\alpha+\alpha^2x)\frac{e^{-\theta x}}{\theta}-\frac{\alpha^2}{\theta}\left(\frac{-e^{-\theta x}}{\theta}\right)\right]$$
$$I_1=\frac{-e^{-\theta x}}{\theta}\left[\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha)\right]$$
$$F(x;\alpha,\theta)=1+\frac{-e^{-\theta x}}{\theta}\left[\theta^3+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha)\right]\frac{\theta^2}{\theta^2+2\alpha\theta+}$$
$$=1-\left[\frac{\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha)}{\theta^2+2\alpha\theta+2\alpha^2}\right]e^{-\theta x}$$
$$=1-\left[1+\frac{\alpha\theta x(2\theta+\alpha\theta x+2\alpha)}{\theta^2+2\alpha\theta+2\alpha^2}\right]e^{-\theta x};x>0,\alpha>0,\theta>0$$

□

8.3.2 Reliability Analysis

Proposition 8.3.3. *In this section, we define survival function $S(x;\alpha,\theta)$ and hazard function $h(x;\alpha,\theta)$ of a generalized two* *parameter Aradhana distribution Eq.8.15 as;*

$$S(x;\alpha,\theta)=\left[1+\frac{\alpha\theta x(2\theta+\alpha\theta x+2\alpha)}{\theta^2+2\alpha\theta+2\alpha^2}\right]e^{-\theta x}; x>0,\alpha>0,\theta>0 \tag{8.17}$$

$$h(x;\alpha,\theta)=\frac{\theta^3(1+\alpha x)^2}{\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha)}; x>0,\alpha>0,\theta> \tag{8.18}$$

Proof. By definition Eq.1.8, the survival function Eq.8.17 is obtained as;

$$\begin{aligned}S(x;\alpha,\theta)&=1-\left[1-\left[1+\frac{\alpha\theta x(2\theta+\alpha\theta x+2\alpha)}{\theta^2+2\alpha\theta+2\alpha^2}\right]e^{-\theta x}\right]\\&=\left[1+\frac{\alpha\theta x(2\theta+\alpha\theta x+2\alpha)}{\theta^2+2\alpha\theta+2\alpha^2}\right]e^{-\theta x}; x>0,\alpha>0,\theta>0\end{aligned}$$

similarly, hazard function Eq.8.18 is obtained by use of the relation Eq.1.9 as;

$$\begin{aligned}h(x;\alpha,\theta)&=\frac{\frac{\theta^3}{\theta^2+2\alpha\theta+2\alpha^2}\left[1+\alpha x\right]^2e^{-\theta x}}{\left[\frac{\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha)}{\theta^2+2\alpha\theta+2\alpha^2}\right]e^{-\theta x}}\\&=\frac{\theta^3(1+\alpha x)^2}{\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha)}; x>0,\alpha>0,\theta>\end{aligned}$$

□

8.3.3 Moments and related measures

Proposition 8.3.4. *The r raw moments of a generalized two parameter Aradhana distribution Eq.8.15 are derived using both* *method of moments and moment generating function (mgf). We define the r raw moments as;*

$$\mu_r^1 * = \frac{r!\left[\theta^2 + 2\alpha\theta(r+1) + \alpha^2(r+1)(r+2)\right]}{\theta^r(\theta^2 + 2\alpha\theta + 2\alpha^2)}; \quad r = 1, 2, 3, \cdots \tag{8.19}$$

Proof. By definition Eq.1.16, the moments of AG2PAD are derived as;

$$\begin{aligned}
E(X^r) &= \frac{\theta^3}{\theta^2 + 2\alpha\theta + 2\alpha^2} \int_0^\infty x^r(1 + 2\alpha x + \alpha^2 x^2)e^{-\theta x}dx \\
&= \frac{\theta^3}{\theta^2 + 2\alpha\theta + 2\alpha^2} \left[\int_0^\infty x^r e^{-\theta x}dx + 2\alpha \int_0^\infty x^{r+1}e^{-\theta x}dx + \alpha^2 \int_0^\infty x^{r+2}e^{-\theta \right. \\
&= \frac{\theta^3}{\theta^2 + 2\alpha\theta + 2\alpha^2} \left[\frac{\Gamma(r+1)}{\theta^{r+1}} + \frac{2\alpha\Gamma(r+2)}{\theta^{r+2}} + \frac{\alpha^2\Gamma r+3}{\theta^{(r+3)}} \right] \\
&= \frac{\theta^3 r\Gamma r}{\theta^r(\theta^2 + 2\alpha\theta + 2\alpha^2)} \left[\frac{1}{\theta} + \frac{2\alpha(r+1)}{\theta^2} + \frac{\alpha^2(r+2)(r+1)}{\theta^3} \right] \\
&= \frac{r!\left[\theta^2 + 2\alpha\theta(r+1) + \alpha^2(r+1)(r+2)\right]}{\theta^r(\theta^2 + 2\alpha\theta + 2\alpha^2)}; \quad r = 1, 2, 3, \cdots
\end{aligned}$$

similarly by definition Eq.1.17, mgf of AG2PAD is given as;

$$
\begin{aligned}
m_x(t) &= \frac{\theta^3}{\theta^2+2\alpha\theta+2\alpha^2}\int_0^\infty e^{tx}(1+2\alpha x+\alpha^2x^2)e^{-\theta x}dx \\
&= \frac{\theta^3}{\theta^2+2\alpha\theta+2\alpha^2}\int_0^\infty e^{-(\theta-t)x}(1+2\alpha x+\alpha^2x^2)dx \\
&= \frac{\theta^3}{\theta^2+2\alpha\theta+2\alpha^2}\left[\frac{1}{(\theta-t)}+\frac{2\alpha}{(\theta-t)^2}+\frac{\alpha^2}{(\theta-t)^3}\right] \\
&= \frac{\theta^3}{\theta^2+2\alpha\theta+2\alpha^2}\left[\frac{1}{\theta}\sum_{k=0}^{\infty}(\frac{t}{\theta})^k+\frac{2\alpha}{\theta^2}\sum_{k=0}^{\infty}\binom{k+1}{k}(\frac{t}{\theta})^k+\frac{\alpha^2}{\theta^3}\sum_{k=0}^{\infty}\Big(k\right. \\
&= \sum_{k=0}^{\infty}\frac{\theta^2+2\alpha\theta(k+1)+\alpha^2(k+1)(k+2)}{\theta^2+2\alpha\theta+2\alpha^2}\left(\frac{t}{\theta}\right)^k
\end{aligned}
$$

The raw moments of a generalized two parameter Aradhana distribution Eq.8.15 are obtained as a coefficient of $\frac{t^r}{r!}$ of the moment generating function as;

$$
\mu_r^1 = \frac{r!\left[\theta^2+2\alpha\theta(r+1)+\alpha^2(r+1)(r+2)\right]}{\theta^r(\theta^2+2\alpha\theta+2\alpha^2)}; \quad r=1,2,3,\cdots
$$

□

For the values of r as 1,2,3 and 4 in Eq.8.19 we obtain the four moments about the origin of a generalized two parameter Aradhana distribution as;

$$
\mu_1^1 = \frac{\theta^2+4\alpha\theta+6\alpha^2}{\theta(\theta^2+2\alpha\theta+2\alpha^2)}, \quad \mu_2^1 = \frac{2(\theta^2+6\alpha\theta+12\alpha^2)}{\theta^2(\theta^2+2\alpha\theta+2\alpha^2)}
$$

$$
\mu_3^1 = \frac{6(\theta^2+8\alpha\theta+20\alpha^2)}{\theta^3(\theta^2+2\alpha\theta+2\alpha^2)}, \quad \mu_4^1 = \frac{24(\theta^2+10\alpha\theta+30\alpha^2)}{\theta^4(\theta^2+2\alpha\theta+2\alpha^2)}
$$

The centralized moments of a generalized two parameter Aradhana distribution are obtained as;

$$\mu_1 = \mu_1^1$$
$$\mu_2 = \frac{\theta^4 + 8\alpha\theta^3 + 24\alpha^2\theta^2 + 24\alpha^3\theta + 12\alpha^4}{\theta^2(\theta^2 + 2\alpha\theta + 2\alpha^2)^2}$$
$$\mu_3 = \frac{2(\theta^6 + 12\theta^5\alpha + 54\theta^4\alpha^2 + 100\alpha^3\theta^3 + 108\alpha^4\theta^2 + 72\alpha^3\theta + 24\alpha^6)}{\theta^3(\theta^2 + 2\alpha\theta + 2\alpha^2)^3}$$

The following relation is used to derive the expression for the 4^{th} centralized moments of a generalized two parameter Aradhana distribution Eq.8.15.

$$\mu_4 = \mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2$$

Proposition 8.3.5. *We now define Other related measures of a generalized two parameter Aradhana distribution Eq.8.15 such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) are stated as;*

$$C.v = \frac{\sqrt{\theta^4 + 8\alpha\theta^3 + 24\alpha^2\theta^2 + 24\alpha^3\theta + 12\alpha^4}}{\theta^2 + 4\alpha\theta + 6\alpha^2} \tag{8.20}$$
$$\upsilon_1 = \frac{2(\theta^6 + 12\theta^5\alpha + 54\theta^4\alpha^2 + 100\alpha^3\theta^3 + 108\alpha^4\theta^2 + 72\alpha^3\theta + 24\alpha^6)}{(\theta^4 + 8\alpha\theta^3 + 24\alpha^2\theta^2 + 24\alpha^3\theta + 12\alpha^4)^{\frac{3}{2}}} \tag{8.21}$$
$$\upsilon_2 = \frac{\mu_4}{[\mu_2]^2} = \frac{\mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2}{[\mu_2^1 - [\mu_1^1]^2]^2} \tag{8.22}$$
$$\upsilon_3 = \frac{\theta^4 + 8\alpha\theta^3 + 24\alpha^2\theta^2 + 24\alpha^3\theta + 12\alpha^4}{\theta(\theta^2 + 2\alpha\theta + 2\alpha^2)(\theta^2 + 4\alpha\theta + 6\alpha^2)} \tag{8.23}$$

Proof. To begin with, coefficient of variation Eq.8.20 is obtained as;

$$
\begin{aligned}
C.v &= \frac{\sigma}{\mu_1^1} \\
&= \frac{\sqrt{\theta^4+8\alpha\theta^3+24\alpha^2\theta^2+24\alpha^3\theta+12\alpha^4}}{\theta(\theta^2+2\alpha\theta+2\alpha^2)}\frac{\theta(\theta^2+2\alpha\theta+2\alpha^2)}{\theta^2+4\alpha\theta+6\alpha^2} \\
&= \frac{\sqrt{\theta^4+8\alpha\theta^3+24\alpha^2\theta^2+24\alpha^3\theta+12\alpha^4}}{\theta^2+4\alpha\theta+6\alpha^2}
\end{aligned}
$$

secondly, skewness coefficient Eq.8.21 is given as;

$$
\begin{aligned}
\upsilon_1 &= \frac{\mu_3}{[\mu_2]^2} \\
&= \frac{\theta^4+8\alpha\theta^3+24\alpha^2\theta^2+24\alpha^3\theta+12\alpha^4}{\theta(\theta^2+2\alpha\theta+2\alpha^2)(\theta^2+4\alpha\theta+6\alpha^2)}\left[\frac{\theta^2(\theta^2+2\alpha\theta+2\alpha}{\theta^4+8\alpha\theta^3+24\alpha^2\theta^2+24\alpha}\right. \\
&= \frac{2(\theta^6+12\theta^5\alpha+54\theta^4\alpha^2+100\alpha^3\theta^3+108\alpha^4\theta^2+72\alpha^3\theta+24\alpha^6)}{(\theta^4+8\alpha\theta^3+24\alpha^2\theta^2+24\alpha^3\theta+12\alpha^4)^{\frac{3}{2}}}
\end{aligned}
$$

The following expression is applied to derive coefficient of kurtosis of a generalized two parameter Aradhana distribution Eq.8.15.

$$
\upsilon_2 = \frac{\mu_4}{[\mu_2]^2} = \frac{\mu_4^1-4[\mu_3^1\mu_1^1]+6[\mu_1^1\mu_2^1]-3[\mu_1^1]^2}{[\mu_2^1-[\mu_1^1]^2]^2}
$$

lastly, index of dispersion value Eq.8.23 is given as;

$$
\begin{aligned}
\upsilon_3 &= \frac{\sigma^2}{\mu_1^1} \\
&= \frac{\theta^4+8\alpha\theta^3+24\alpha^2\theta^2+24\alpha^3\theta+12\alpha^4}{\theta^2(\theta^2+2\alpha\theta+2\alpha^2)^2}\frac{\theta(\theta^2+2\alpha\theta+2\alpha^2)}{\theta^2+4\alpha\theta+6\alpha^2} \\
&= \frac{\theta^4+8\alpha\theta^3+24\alpha^2\theta^2+24\alpha^3\theta+12\alpha^4}{\theta(\theta^2+2\alpha\theta+2\alpha^2)(\theta^2+4\alpha\theta+6\alpha^2)}
\end{aligned}
$$

□

8.3.4 Excess Loss Distribution

Proposition 8.3.6. *In this section we state probability density function of excess loss function $f_l(x;\alpha,\theta)$, mean residual life-time (MRL), equilibrium distribution $f_e(x;\alpha,\theta)$, survival function based on the equilibrium distribution $S_e(x;\alpha,\theta)$ and hazard function of the equilibrium distribution $h_e(x;\alpha,\theta)$ as;*

$$f_l(x;\alpha,\theta) = \frac{\theta^3(1+\alpha x)^2 e^{-(x-z)\theta}}{\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta z(2\theta+\alpha\theta z+2\alpha)}; \quad x > z \tag{8.24}$$

$$m(x) = \frac{\theta^2+4\alpha\theta+6\alpha^2+2\alpha\theta^2 x+\alpha^2\theta^2 x^2+4\alpha^2\theta x}{\theta(\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha))} \tag{8.25}$$

$$f_e(x;\alpha,\theta) = \frac{\theta(\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha))e^{-\theta x}}{\theta^2+4\alpha\theta+6\alpha^2} \tag{8.26}$$

$$S_e(x;\alpha,\theta) = \frac{e^{-\theta x}(\theta^2+4\alpha\theta+6\alpha^2+2\alpha\theta^2 x+\alpha^2\theta^2 x+4\alpha^2\theta x)}{\theta^2+4\alpha\theta+6\alpha^2} \tag{8.27}$$

$$h_e(x;\alpha,\theta) = \frac{\theta(\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha))}{\theta^2+4\alpha\theta+6\alpha^2+2\alpha\theta^2 x+\alpha^2\theta^2 x+4\alpha^2\theta x} \tag{8.28}$$

Proof. To begin with, pdf of excess loss distribution Eq.8.24 is obtained by use of Eq.1.10 as;

$$
\begin{aligned}
f_l(x;\alpha,\theta) &= \frac{\frac{\theta^3}{\theta^2+2\alpha\theta+2\alpha^2}\left[1+\alpha x\right]^2 e^{-\theta x}}{\left[\frac{\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta z(2\theta+\alpha\theta z+2\alpha)}{\theta^2+2\alpha\theta+2\alpha^2}\right]e^{-\theta z}} \\
&= \frac{\theta^3(1+\alpha x)^2 e^{-(x-z)\theta}}{\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta z(2\theta+\alpha\theta z+2\alpha)}; \quad x>z
\end{aligned}
$$

further mean residual lifetime Eq.8.25 is obtained by use of Eq.1.11 as;

$$m(x) = \frac{\theta^2+2\alpha\theta+2\alpha^2}{\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha)e^{-\theta x}}\int_x^\infty \frac{[\theta^2+2\alpha\theta+2\alpha^2+\alpha}{\alpha^2+}$$

$$= \frac{1}{\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha)e^{-\theta x}}\int_x^\infty [\theta^2+2\alpha\theta+2\alpha^2+\alpha$$

$$= \frac{1}{\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha)e^{-\theta x}} I_1$$

$$I_1 = \int_x^\infty [\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta t(2\theta+\alpha\theta t+2\alpha)]e^{-\theta t}dt$$

$$u = (\theta^2+2\alpha\theta+2\alpha^2+2\alpha\theta^2 t+\alpha^2\theta^2 t^2+2\alpha^2\theta t) \Longrightarrow du = (2\alpha\theta^2+2\alpha^2\theta$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_1 = -(\theta^2+2\alpha\theta+2\alpha^2+2\alpha\theta^2 t+\alpha^2\theta^2 t^2+2\alpha^2\theta t)\frac{e^{-\theta t}}{\theta} + 2\int_x^\infty (\alpha\theta+\alpha^2\theta$$

$$I_1 = -(\theta^2+2\alpha\theta+2\alpha^2+2\alpha\theta^2 t+\alpha^2\theta^2 t^2+2\alpha^2\theta t)\frac{e^{-\theta t}}{\theta} + 2I_2$$

$$I_2 = \int_x^\infty (\alpha\theta+\alpha^2\theta t+\alpha^2)e^{-\theta t}dt$$

$$u = (\alpha\theta+\alpha^2\theta t+\alpha^2) \Longrightarrow du = \alpha^2\theta dt$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_2 = -(\alpha\theta+\alpha^2\theta t+\alpha^2)\frac{e^{-\theta t}}{\theta} + \alpha^2\int_x^\infty e^{-\theta t}dt$$

$$I_2 = -(\alpha\theta+\alpha^2\theta t+\alpha^2)\frac{e^{-\theta t}}{\theta} + \alpha^2 I_3$$

$$I_3 = \int_x^\infty e^{-\theta t}dt \Longrightarrow \frac{-e^{-\theta t}}{\theta}$$

From I_1, I_2 and I_3 we have the following;

$$I_1 = -(\theta^2+2\alpha\theta+2\alpha^2+2\alpha\theta^2t+\alpha^2\theta^2t^2+2\alpha^2\theta t)\frac{e^{-\theta t}}{\theta}+2\Big[-(\alpha\theta$$

$$I_1 = \left[\frac{-e^{-\theta t}}{\theta}\left[\theta^2+4\alpha\theta+6\alpha^2+2\alpha\theta^2t+\alpha^2\theta^2t^2+4\alpha^2\theta t\right]\right]_x^\infty$$

$$I_1 = \frac{e^{-\theta x}}{\theta}\left[\theta^2+4\alpha\theta+6\alpha^2+2\alpha\theta^2x+\alpha^2\theta^2x^2+4\alpha^2\theta x\right]$$

$$= \frac{1}{\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha)e^{-\theta x}}I_1$$

$$m(x) = \frac{\theta^2+4\alpha\theta+6\alpha^2+2\alpha\theta^2x+\alpha^2\theta^2x^2+4\alpha^2\theta x}{\theta(\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha))}$$

By definition Eq.1.12, equilibrium distribution Eq.8.26 is obtained as;

$$f_e(x;\alpha,\theta) = \frac{(\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha))e^{-\theta x}}{\theta^2+2\alpha\theta+2\alpha^2}\frac{\theta(\theta^2+2\alpha\theta}{\theta^2+4\alpha\theta}$$

$$= \frac{\theta(\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha))e^{-\theta x}}{\theta^2+4\alpha\theta+6\alpha^2}$$

further by definition Eq.1.13, $S_e(x;\alpha,\theta)$ in Eq.8.27 is given as;

$$\int_x^\infty S(t;\alpha,\theta)dt = \frac{e^{-\theta x}(\theta^2+4\alpha\theta+6\alpha^2+2\alpha\theta^2x+\alpha^2\theta^2x+4\alpha^2\theta x)}{\theta(\theta^2+2\alpha\theta+2\alpha^2)}$$

$$= \frac{e^{-\theta x}(\theta^2+4\alpha\theta+6\alpha^2+2\alpha\theta^2x+\alpha^2\theta^2x+4\alpha^2\theta x)}{\theta(\theta^2+2\alpha\theta+2\alpha^2)}\frac{\theta(\theta}{\theta^2}$$

$$S_e(x;\alpha,\theta) = \frac{e^{-\theta x}(\theta^2+4\alpha\theta+6\alpha^2+2\alpha\theta^2x+\alpha^2\theta^2x+4\alpha^2\theta x)}{\theta^2+4\alpha\theta+6\alpha^2}$$

lastly, hazard function based on equilibrium distribution Eq.8.28 is obtained by use of eq.1.14 as;

$$h_e(x;\alpha,\theta) = \frac{\frac{\theta(\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha))e^{-\theta x}}{\theta^2+4\alpha\theta+6\alpha^2}}{\left[\frac{e^{-\theta x}(\theta^2+4\alpha\theta+6\alpha^2+2\alpha\theta^2 x+\alpha^2\theta^2 x+4\alpha^2\theta x)}{\theta^2+4\alpha\theta+6\alpha^2}\right]}$$

$$= \frac{\theta(\theta^2+2\alpha\theta+2\alpha^2+\alpha\theta x(2\theta+\alpha\theta x+2\alpha))}{\theta^2+4\alpha\theta+6\alpha^2+2\alpha\theta^2 x+\alpha^2\theta^2 x+4\alpha^2\theta x}$$

□

8.4 Quasi Aradhana distribution

8.4.1 Construction of a Quasi Aradhana distribution

Proposition 8.4.1. *Let* $\omega_1 = \frac{\alpha^2}{\alpha^2+2\alpha+2}$ *and* $\omega_2 = \frac{2\alpha}{\alpha^2+2\alpha+2}$ *be mixing proportions, Quasi Aradhana distribution is constructed as a three component finite mixed distribution of Gamma* $(1,\theta)$*, Gamma* $(2,\theta)$ *and Gamma* $(3,\theta)$*. The pdf and Cdf of a QAD are defined as;*

$$f(x;\alpha,\theta) = \frac{\theta}{\alpha^2+2\alpha+2}\left[\alpha+\theta x\right]^2 e^{-\theta x}; x>0, \alpha>0, \theta>0 \tag{8.29}$$

$$F(x;\alpha,\theta) = 1-\left[1+\frac{2\alpha\theta x+2\theta x+\theta^2 x^2}{\alpha^2+2\alpha+2}\right]e^{-\theta x}; x>0, \alpha>0, \theta>0 \tag{8.30}$$

Proof. By definition of a finite mixed distribution Eq.1.1, pdf Eq.8.29 is obtained as;

$$
\begin{aligned}
f(x;\alpha,\theta) &= \frac{\alpha^2}{\alpha^2+2\alpha+2}\left[\theta e^{-\theta x}\right] + \frac{2\alpha}{\alpha^2+2\alpha+2}\left[\frac{\theta^2 e^{-\theta x} x}{\Gamma 2}\right] + \frac{2}{\alpha^2+2\alpha+} \\
&= \frac{\alpha^2\theta e^{-\theta x}}{\alpha^2+2\alpha+2} + \frac{2\alpha\theta^2 e^{-\theta x} x}{\alpha^2+2\alpha+2} + \frac{2\theta^3 e^{-\theta x} x^2}{2(\alpha^2+2\alpha+2)} \\
&= \frac{\theta}{\alpha^2+2\alpha+2}\left[\alpha^2+2\alpha\theta x+\theta^2 x^2\right] e^{-\theta x}
\end{aligned}
$$

Taking the part;

$$\alpha^2+2\alpha\theta x+\theta^2 x^2$$

we have the following

$$= \alpha^2+2\alpha\theta x+\theta^2 x^2 \Longrightarrow (\alpha+\theta x)(\alpha+\theta x) \Longrightarrow (\alpha+\theta x)^2$$

$$f(x;\alpha,\theta) = \frac{\theta}{\alpha^2+2\alpha+2}\left[\alpha+\theta x\right]^2 e^{-\theta x}; x>0, \alpha>0, \theta>0$$

Remark 8.4.2. *Quasi Aradhana distribution Eq.8.29 is nested with two distributions that are one parameter Aradhana distribution and gamma distribution. To begin with, putting* $\alpha=\theta$ *Quasi Aradhana distribution Eq.8.29 reduces to one parameter Aradhana distribution in Eq.8.1. Secondly, putting* $\alpha=0$ *Quasi Aradhana distribution Eq.8.29 reduces to a gamma* $(3,\theta)$ *distribution Eq8.31.*

$$f(x;\theta) = \frac{\theta^3 e^{-\theta x} x^2}{\Gamma 3}; \quad x>0, \theta>0 \tag{8.31}$$

further Cdf Eq.8.30 is obtained as;

$$F(x;\alpha,\theta) = \frac{\theta}{\alpha^2+2\alpha+2}\int_0^\infty(\alpha^2+2\alpha\theta x+\theta^2x^2)e^{-\theta x}dx$$

$$F(x;\alpha,\theta) = \frac{\theta}{\alpha^2+2\alpha+2}I_1$$

$$I_1 = \int_0^\infty(\alpha^2+2\alpha\theta x+\theta^2x^2)e^{-\theta x}dx$$

$$u = (\alpha^2+2\alpha\theta x+\theta^2x^2) \Longrightarrow du = (2\alpha\theta+2\theta^2x)dx$$

$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$

$$I_1 = -(\alpha^2+2\alpha\theta x+\theta^2x^2)\frac{e^{-\theta x}}{\theta} - 2\int_0^\infty -(\alpha+\theta x)e^{-\theta x}dx$$

$$I_1 = -(\alpha^2+2\alpha\theta x+\theta^2x^2)\frac{e^{-\theta x}}{\theta} - 2I_2$$

$$I_2 = \int_0^\infty -(\alpha+\theta x)e^{-\theta x}dx$$

$$u = -(\alpha+\theta x) \Longrightarrow du = -\theta x$$

$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$

$$I_2 = (\alpha+\theta x)\frac{e^{-\theta x}}{\theta} - \int_0^\infty e^{-\theta x}dx$$

$$I_2 = (\alpha+\theta x)\frac{e^{-\theta x}}{\theta} - I_3$$

$$I_3 = \int_0^\infty e^{-\theta x}dx$$

$$u = -\theta x \Longrightarrow du = -\theta dx \Longrightarrow \frac{du}{-\theta} = dx$$

$$I_3 = \int_0^\infty e^u\frac{du}{-\theta} \Longrightarrow \frac{-1}{\theta}\int_0^\infty e^u du \Longrightarrow \frac{-1}{\theta}e^{-\theta x}$$

From I_1, I_2 and I_3 we have the following;

$$I_1 = -(\alpha^2+2\alpha\theta x+\theta^2x^2)\frac{e^{-\theta x}}{\theta} - 2\left[(\alpha+\theta x)\frac{e^{-\theta x}}{\theta} - \left(\frac{-1}{\theta}e^{-\theta x}\right)\right.$$

$$I_1 = \frac{-e^{-\theta x}}{\theta}\left[\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2x^2\right]$$

$$= 1+\frac{-e^{-\theta x}}{\theta}\left[\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2x^2\right]\frac{\theta}{\alpha^2+2\alpha+2}$$

$$F(x;\alpha,\theta) = 1-\left[1+\frac{2\alpha\theta x+2\theta x+\theta^2x^2}{\alpha^2+2\alpha+2}\right]e^{-\theta x}; x>0, \alpha>0, \theta>0$$

□

8.4.2 Reliability Analysis

Proposition 8.4.3. *In this section we state survival function denoted as $S(x;\alpha,\theta)$ and hazard function denoted as $h(x;\alpha,\theta)$ of a Quasi Aradhana distribution Eq.8.29 as;*

$$S(x;\alpha,\theta) = \left[1+\frac{2\alpha\theta x+2\theta x+\theta^2x^2}{\alpha^2+2\alpha+2}\right]e^{-\theta x}; x>0, \alpha>0, \theta>0 \tag{8.32}$$

$$h(x;\alpha,\theta) = \frac{\theta(\alpha+\theta x)^2}{\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2x^2}; x>0, \alpha>0, \theta>0 \tag{8.33}$$

Proof. By definition Eq.1.8, $S(x;\alpha,\theta)$ in Eq.8.32 is obtained as;

$$S(x;\alpha,\theta) = 1-\left[1-\left[1+\frac{2\alpha\theta x+2\theta x+\theta^2x^2}{\alpha^2+2\alpha+2}\right]e^{-\theta x}\right]$$

$$= \left[1+\frac{2\alpha\theta x+2\theta x+\theta^2x^2}{\alpha^2+2\alpha+2}\right]e^{-\theta x}; x>0, \alpha>0, \theta>0$$

further by definition Eq.1.9, $h(x;\alpha,\theta)$ in Eq.8.33 is given as;

$$h(x;\alpha,\theta)=\frac{\frac{\theta}{\alpha^2+2\alpha+2}\left[\alpha+\theta x\right]^2e^{-\theta x}}{\left[\frac{\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2x^2}{\alpha^2+2\alpha+2}\right]e^{-\theta x}}$$

$$=\frac{\theta(\alpha+\theta x)^2}{\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2x^2};x>0,\alpha>0,\theta>0$$

□

8.4.3 Moments and related measures

Proposition 8.4.4. *The r^{th} moments of a Quasi Aradhana Eq.8.29 are derived using the method of moments and moment generating function technique as;*

$$\mu_r^1*=\frac{r!\left[\alpha^2+2\alpha(r+1)+(r+1)(r+2)\right]}{\theta^2(\alpha^2+2\alpha+2)};r=1,2,3,\cdots \tag{8.34}$$

`Proof.` By definition Eq.1.16, r^{th} moments of a QAD are obtained as;

$$E(X^r) = \frac{\theta}{\alpha^2+2\alpha+2}\int_0^\infty x^r(\alpha^2+2\alpha\theta x+\theta^2x^2)e^{-\theta x}dx$$

$$= \frac{\theta}{\alpha^2+2\alpha+2}\left[\alpha^2\int_0^\infty x^r e^{-\theta x}dx + 2\alpha\theta\int_0^\infty x^{r+1}e^{-\theta x}dx + \theta^2\int_0^\infty x^{r+}\right.$$

$$= \frac{\theta}{\alpha^2+2\alpha+2}\left[\frac{\alpha^2\Gamma(r+1)}{\theta^{r+1}} + \frac{2\alpha\theta\Gamma(r+2)}{\theta^{r+2}} + \frac{\theta^3\Gamma(r+3)}{\theta^{r+3}}\right]$$

$$= \frac{r\Gamma r\left[\alpha^2+2\alpha(r+1)+(r+1)(r+2)\right]}{\theta^r(\alpha^2+2\alpha+2)}$$

$$= \frac{r!\left[\alpha^2+2\alpha(r+1)+(r+1)(r+2)\right]}{\theta^2(\alpha^2+2\alpha+2)}; r = 1,2,3,\cdots$$

similarly, by definition Eq.1.17, mgf of QAD is given as;

$$m_x(t) = \frac{\theta}{\alpha^2+2\alpha+2}\int_0^\infty e^{tx}(\alpha^2+2\alpha\theta x+\theta^2x^2)e^{-\theta x}dx$$

$$= \frac{\theta}{\alpha^2+2\alpha+2}\int_0^\infty e^{tx}e^{-(\theta-t)x}(\alpha^2+2\alpha\theta x+\theta^2x^2)dx$$

$$= \frac{\theta}{\alpha^2+2\alpha+2}\left[\frac{\alpha^2}{(\theta-t)} + \frac{2\alpha\theta}{(\theta-t)^2} + \frac{\theta^2}{(\theta-t)^3}\right]$$

$$= \frac{\theta}{\alpha^2+2\alpha+2}\left[\frac{\alpha^2}{\theta}\sum_{k=0}^\infty\left(\frac{t}{\theta}\right)^k + \frac{2\alpha\theta}{\theta^2}\sum_{k=0}^\infty\binom{k+1}{k}\left(\frac{t}{\theta}\right)^k + \frac{\theta^2}{\theta^3}\sum_{k=0}^\infty\right.$$

$$= \sum_{k=0}^\infty \frac{\alpha^2+2\alpha(k+1)+(k+1)(k+2)}{\alpha^2+2\alpha+2}\left(\frac{t}{\theta}\right)^k$$

The r^{th} moments of a Quasi Aradhana distribution are obtained as;

$$\mu_r^1 = \frac{r!\left[\alpha^2 + 2\alpha(r+1) + (r+1)(r+2)\right]}{\theta^2(\alpha^2+2\alpha+2)}; r = 1,2,3,\cdots$$

□

For the values of r as 1,2,3 and 4 in Eq.8.34 we obtain the four none centralized moments QAD as;

$$\mu_1^1 = \frac{\alpha^2+4\alpha+6}{\theta(\alpha^2+2\alpha+2)}, \quad \mu_2^1 = \frac{2(\alpha^2+6\alpha+12)}{\theta^2(\alpha^2+2\alpha+2)}$$
$$\mu_3^1 = \frac{6(\alpha^2+8\alpha+20)}{\theta^3(\alpha^2+2\alpha+2)}, \quad \mu_4^1 = \frac{24(\alpha^2+10\alpha+30)}{\theta^4(\alpha^2+2\alpha+2)}$$

We obtain the centralized moments of QAD as;

$$\mu_1 = \mu_1^1$$
$$\mu_2 = \mu_2^1 - [\mu_1^1]^2 \Longrightarrow \frac{\alpha^4+8\alpha^3+24\alpha^2+24\alpha+12}{\theta^2(\alpha^2+2\alpha+2)^2}$$
$$\mu_3 = \frac{2\alpha^6+24\alpha^5+108\alpha^4+200\alpha^3+216\alpha^2+144\alpha+48}{\theta^3(\alpha^2+2\alpha+2)^3}$$
$$\mu_4 = \frac{9\alpha^8+144\alpha^7+912\alpha^6+2832\alpha^5+5448\alpha^4+6912\alpha^3+5760\alpha^2+2880\alpha}{\theta^4(\alpha^2+2\alpha+2)^4}$$

Proposition 8.4.5. *We now define Other related measures of a Quasi Aradhana distribution Eq.8.29 such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and dispersion index (υ_3) are* *stated as;*

$$C.v=\frac{\sqrt{\alpha^4+8\alpha^3+24\alpha^2+24\alpha+12}}{\alpha^2+4\alpha+6} \tag{8.35}$$

$$\upsilon_1=\frac{2\alpha^6+24\alpha^5+108\alpha^4+200\alpha^3+216\alpha^2+144\alpha+48}{[\alpha^4+8\alpha^3+24\alpha^2+24\alpha+12]^{\frac{3}{2}}} \tag{8.36}$$

$$\upsilon_2=\frac{9\alpha^8+144\alpha^7+912\alpha^6+2832\alpha^5+5448\alpha^4+6912\alpha^3+5760\alpha^2+\cdots}{[\alpha^4+8\alpha^3+24\alpha^2+24\alpha+12]^2} \tag{8.37}$$

$$\upsilon_3=\frac{\alpha^4+8\alpha^3+24\alpha^2+24\alpha+12}{\theta(\alpha^2+2\alpha+2)(\alpha^2+4\alpha+6)} \tag{8.38}$$

Proof. To begin with, coefficient of variation Eq.8.35 is given as;

$$C.v=\frac{\sigma}{\mu_1^1}=\frac{\sqrt{\alpha^4+8\alpha^3+24\alpha^2+24\alpha+12}}{\theta(\alpha^2+2\alpha+2)}\frac{\theta(\alpha^2+2\alpha+2)}{\alpha^2+4\alpha+6}\implies\frac{\sqrt{\alpha^4+8\alpha^3+\cdots}}{\alpha^2+\cdots}$$

similarly, skewness value Eq.8.36 is given as;

$$\upsilon_1=\frac{\mu_3}{[\mu_2]^{\frac{3}{2}}}=\frac{2\alpha^6+24\alpha^5+108\alpha^4+200\alpha^3+216\alpha^2+144\alpha+48}{\theta^3(\alpha^2+2\alpha+2)^3}\left[\frac{\theta^2(\alpha^2+\cdots}{\alpha^4+8\alpha^3+2\cdots}\right.$$
$$=\frac{2\alpha^6+24\alpha^5+108\alpha^4+200\alpha^3+216\alpha^2+144\alpha+48}{[\alpha^4+8\alpha^3+24\alpha^2+24\alpha+12]^{\frac{3}{2}}}$$

also kurtosis coefficient Eq.8.37 is given as;

$$\upsilon_2 = \frac{\mu_4}{[\mu_2]^2}$$

$$= \frac{\frac{9\alpha^8+144\alpha^7+912\alpha^6+2832\alpha^5+5448\alpha^4+6912\alpha^3+5760\alpha^2+2880\alpha+720}{\theta^4(\alpha^2+2\alpha+2)^4}}{\left[\frac{\alpha^4+8\alpha^3+24\alpha^2+24\alpha+12}{\theta^2(\alpha^2+2\alpha+2)^2}\right]^2}$$

$$= \frac{9\alpha^8+144\alpha^7+912\alpha^6+2832\alpha^5+5448\alpha^4+6912\alpha^3+5760\alpha^2+2880\alpha}{[\alpha^4+8\alpha^3+24\alpha^2+24\alpha+12]^2}$$

lastly, index of dispersion Eq.8.38 is given as;

$$\upsilon_3 = \frac{\sigma^2}{\mu_1^1}$$

$$= \frac{\alpha^4+8\alpha^3+24\alpha^2+24\alpha+12}{\theta^2(\alpha^2+2\alpha+2)^2}\frac{\theta(\alpha^2+2\alpha+2)}{\alpha^2+4\alpha+6}$$

$$= \frac{\alpha^4+8\alpha^3+24\alpha^2+24\alpha+12}{\theta(\alpha^2+2\alpha+2)(\alpha^2+4\alpha+6)}$$

□

8.4.4 Excess Loss Distribution

Proposition 8.4.6. *The probability density function of excess loss function $f_l(x;\alpha,\theta)$, mean residual lifetime (MRL), equilibrium distribution $f_e(x;\alpha,\theta)$, survival function based on the equilibrium distribution $s_e(x;\alpha,\theta)$ and hazard function based on the* *equilibrium distribution* $h_e(x;\alpha,\theta)$ *are defined as;*

$$f_l(x;\alpha,\theta) = \frac{\theta(\alpha+\theta x)^2 e^{-\theta(x-z)}}{\alpha^2+2\alpha+2+2\alpha\theta z+2\theta z+\theta^2 z^2}; \quad x>z \tag{8.39}$$

$$m(x) = \frac{\alpha^2+4\alpha+6+2\alpha\theta x+4\theta x+\theta^2 x^2}{\theta(\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2 x^2)} \tag{8.40}$$

$$f_e(x;\alpha,\theta) = \frac{\theta(\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2 x^2)e^{-\theta x}}{\alpha^2+4\alpha+6} \tag{8.41}$$

$$s_e(x;\alpha,\theta) = \frac{\theta(\alpha^2+4\alpha+6+2\alpha\theta x+4\theta x+\theta^2 x^2)e^{-\theta x}}{\alpha^2+4\alpha+6} \tag{8.42}$$

$$h_e(x;\alpha,\theta) = \frac{\theta(\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2 x^2)}{\alpha^2+4\alpha+6+2\alpha\theta x+4\theta x+\theta^2 x^2} \tag{8.43}$$

Proof. To begin with, using Eq.1.10 the pdf of excess loss distribution Eq.8.39 is given as;

$$f_l(x;\alpha,\theta) = \frac{\frac{\theta}{\alpha^2+2\alpha+2}\left[\alpha+\theta x\right]^2 e^{-\theta x}}{\left[\frac{\alpha^2+2\alpha+2+2\alpha\theta z+2\theta z+\theta^2 z^2}{\alpha^2+2\alpha+2}\right]e^{-\theta z}}$$

$$= \frac{\theta(\alpha+\theta x)^2 e^{-\theta(x-z)}}{\alpha^2+2\alpha+2+2\alpha\theta z+2\theta z+\theta^2 z^2}; \quad x>z$$

by definition Eq.1.11, mean residual lifetime Eq.8.40 is given as;

$$m(x) = \frac{\alpha^2+2\alpha+2}{(\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2x^2)e^{-\theta x}}\int_x^\infty \frac{[\alpha^2+2\alpha+2+2\theta t+2\alpha}{\alpha^2+2\alpha+2}$$

$$= \frac{1}{(\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2x^2)e^{-\theta x}}\int_x^\infty [\alpha^2+2\alpha+2+2\theta t+2\alpha\theta$$

$$= \frac{1}{(\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2x^2)e^{-\theta x}} I_1$$

$$I_1 = \int_x^\infty [\alpha^2+2\alpha+2+2\theta t+2\alpha\theta t+\theta^2t^2]e^{-\theta t}dt$$

$$u = [\alpha^2+2\alpha+2+2\theta t+2\alpha\theta t+\theta^2t^2] \Longrightarrow du = (2\alpha\theta+2\theta+2\theta^2t)dt$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_1 = [\alpha^2+2\alpha+2+2\theta t+2\alpha\theta t+\theta^2t^2]\frac{e^{-\theta t}}{\theta}+2\int_x^\infty(\alpha+1+\theta t)e^{-\theta t}dt$$

$$I_1 = [\alpha^2+2\alpha+2+2\theta t+2\alpha\theta t+\theta^2t^2]\frac{e^{-\theta t}}{\theta}+2I_2$$

$$I_2 = \int_x^\infty (\alpha+1+\theta t)e^{-\theta t}dt$$

$$u = (\alpha+1+\theta t) \Longrightarrow du = \theta dt$$

$$dv = e^{-\theta t} \Longrightarrow v = \frac{-e^{-\theta t}}{\theta}$$

$$I_2 = -(\alpha+1+\theta t)\frac{e^{-\theta t}}{\theta}+\int_x^\infty e^{-\theta t}dt$$

$$I_2 = -(\alpha+1+\theta t)\frac{e^{-\theta t}}{\theta}+I_3$$

$$I_3 = \int_x^\infty e^{-\theta t}dt \Longrightarrow \frac{-e^{-\theta t}}{\theta}$$

From I_1, I_2 and I_3 we have the following;

$$I_1 = -[\alpha^2+2\alpha+2+2\alpha\theta t+2\theta t+\theta^2 t^2]\frac{e^{-\theta t}}{\theta}+2\Bigg[-(\alpha+1+\theta t)\frac{e^{-\theta}}{\theta}$$

$$I_1 = \left[\frac{-e^{-\theta t}}{\theta}\left[\alpha^2+4\alpha+6+2\alpha\theta t+4\theta t+\theta^2 t^2\right]\right]_x^\infty$$

$$I_1 = \frac{e^{-\theta x}}{\theta}\left[\alpha^2+4\alpha+6+2\alpha\theta x+4\theta x+\theta^2 x^2\right]$$

$$m(x) = \frac{1}{(\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2x^2)e^{-\theta x}}I_1$$

$$= \frac{\alpha^2+4\alpha+6+2\alpha\theta x+4\theta x+\theta^2x^2}{\theta(\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2x^2)}$$

further by definition Eq.1.12, equilibrium distribution Eq.8.41 is obtained as;

$$f_e(x;\alpha,\theta) = \frac{(\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2x^2)e^{-\theta x}}{\alpha^2+2\alpha+2}\frac{\theta(\alpha^2+2\alpha+2)}{\alpha^2+4\alpha+6}$$

$$= \frac{\theta(\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2x^2)e^{-\theta x}}{\alpha^2+4\alpha+6}$$

using the relation Eq.1.13, $S_e(x;\alpha,\theta)$ in Eq.8.42 is given as;

$$\int_x^\infty S(t;\alpha,\theta)dt = \frac{e^{-\theta x}(\alpha^2+4\alpha+6+2\alpha\theta x+4\theta x+\theta^2x^2)}{\theta(\alpha^2+2\alpha+2)}$$

$$= \frac{e^{-\theta x}(\alpha^2+4\alpha+6+2\alpha\theta x+4\theta x+\theta^2x^2)}{\theta(\alpha^2+2\alpha+2)}\frac{\theta(\alpha^2+2\alpha+}{\alpha^2+4\alpha+6}$$

$$S_e(x;\alpha,\theta) = \frac{\theta(\alpha^2+4\alpha+6+2\alpha\theta x+4\theta x+\theta^2x^2)e^{-\theta x}}{\alpha^2+4\alpha+6}$$

lastly by definition Eq.1.14, $h_e(x;\alpha,\theta)$ in Eq.8.43 is given as;

$$h_e(x;\alpha,\theta) = \frac{\frac{\theta(\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2x^2)e^{-\theta x}}{\alpha^2+4\alpha+6}}{\frac{\theta(\alpha^2+4\alpha+6+2\alpha\theta x+4\theta x+\theta^2x^2)e^{-\theta x}}{\alpha^2+4\alpha+6}}$$
$$= \frac{\theta(\alpha^2+2\alpha+2+2\alpha\theta x+2\theta x+\theta^2x^2)}{\alpha^2+4\alpha+6+2\alpha\theta x+4\theta x+\theta^2x^2}$$

□

8.5 Three parameter Aradhana distribution

8.5.1 Construction of a three parameter Aradhana distribution

Proposition 8.5.1. *Let* $\omega_1 = \frac{\theta^2}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}$ *and* $\omega_2 = \frac{2\alpha\beta\theta}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}$ *be mixing proportions, a generalized three parameter Aradhana distribution is a three component finite mixed distribution of Gamma* $(1,\theta)$*, Gamma* $(2,\theta)$ *and Gamma* $(3,\theta)$*. The pdf and Cdf of AG3PAD are;*

$$f(x;\alpha,\beta,\theta) = \frac{\theta^3}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\left[1+\alpha\beta x\right]^2 e^{-\theta x}; x>0,\alpha>0,\beta>0,\theta> \tag{8.44}$$

$$F(x;\alpha,\beta,\theta) = 1-\left[1+\frac{2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\right]e^{-\theta x}; x>0,\alpha>0, \tag{8.45}$$

Proof. By the definition of a finite mixed distribution Eq. 1.1, pdf Eq.8.44 is obtained as;

$$f(x;\alpha,\beta,\theta) = \frac{\theta^2}{\theta^2+2\alpha\beta\theta}\left[\theta e^{-\theta x}\right] + \frac{2\alpha\beta\theta}{\theta^2+2\alpha\beta\theta}\left[\frac{\theta^2 e^{-\theta x}x}{\Gamma 2}\right] + \frac{2\alpha^2\beta^2}{\theta^2+2\alpha\beta}$$
$$= \frac{\theta^3 e^{-\theta x}}{\theta^2+2\alpha\beta\theta} + \frac{2\alpha\beta\theta^3 e^{-\theta x}x}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2} + \frac{\alpha^2\beta^2\theta^3 e^{-\theta x}x^2}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta}$$
$$= \frac{\theta^3}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\left[1+2\alpha\beta x+\alpha^2\beta^2x^2\right]e^{-\theta x}$$

Taking the part $\left[1+2\alpha\beta x+\alpha^2\beta^2x^2\right]$ we have the following;

$$= \left[1+2\alpha\beta x+\alpha^2\beta^2x^2\right]$$
$$= 1+\alpha\beta x+\alpha\beta x+\alpha^2\beta^2x^2 \Longrightarrow 1(1+\alpha\beta x)+\alpha\beta x(1+\alpha\beta x) \Longrightarrow \Big[1+$$
$$= \frac{\theta^3}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\left[1+\alpha\beta x\right]^2 e^{-\theta x}; x>0, \alpha>0, \beta>0, \theta>0$$

Remark 8.5.2. *The modified generalized three parameter Aradhana distribution Eq.8.44 is nested with three distributions. To begin with, putting $\alpha=\beta=1$ the generalized three parameter Aradhana distribution reduces to a one parameter Aradhana distribution Eq.8.1. Secondly, putting $\beta=1$ the generalized three parameter turns to a generalized two parameter Eq.8.15. Lastly, exponential distribution Eq.2.17 is a particular case of a generalized three parameter Aradhana distribution Eq.8.44 at $\alpha=\beta=0$.*

further Cdf Eq.8.45 is obtained as;

$$F(x;\alpha,\beta,\theta) = \frac{\theta^3}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\int_0^\infty \left[1+2\alpha\beta x+\alpha^2\beta^2x^2\right]e^{-\theta x}dx$$

$$F(x;\alpha,\beta,\theta) = \frac{\theta^3}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}I_1$$

$$I_1 = \int_0^\infty \left[1+2\alpha\beta x+\alpha^2\beta^2x^2\right]e^{-\theta x}dx$$

$$u = (1+2\alpha\beta x+\alpha^2\beta^2x^2) \Longrightarrow du = (2\alpha\beta+2\alpha^2\beta^2x)dx$$

$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$

$$I_1 = -(1+2\alpha\beta x+\alpha^2\beta^2x^2)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}\int_0^\infty -(\alpha\beta+\alpha^2\beta^2x)e^{-\theta x}dx$$

$$I_1 = -(1+2\alpha\beta x+\alpha^2\beta^2x^2)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}I_2$$

$$I_2 = \int_0^\infty -(\alpha\beta+\alpha^2\beta^2x)e^{-\theta x}dx$$

$$u = -(\alpha\beta+\alpha^2\beta^2x) \Longrightarrow du = -\alpha^2\beta^2dx$$

$$dv = e^{-\theta x} \Longrightarrow v = \frac{-e^{-\theta x}}{\theta}$$

$$I_2 = (\alpha\beta+\alpha^2\beta^2x)\frac{e^{-\theta x}}{\theta} - \frac{\alpha^2\beta^2}{\theta}\int_0^\infty e^{-\theta x}dx$$

$$I_2 = (\alpha\beta+\alpha^2\beta^2x)\frac{e^{-\theta x}}{\theta} - \frac{\alpha^2\beta^2}{\theta}I_3$$

$$I_3 = \int_0^\infty e^{-\theta x}dx$$

$$u = -\theta x \Longrightarrow du = -\theta dx \Longrightarrow \frac{du}{-\theta} = dx$$

$$I_3 = \int_0^\infty e^u\frac{du}{-\theta} \Longrightarrow \frac{-1}{\theta}\int_0^\infty e^u du \Longrightarrow \frac{-e^u}{\theta} \Longrightarrow \frac{-e^{-\theta x}}{\theta}$$

From I_1, I_2 and I_3 we have the following;

$$I_1 = -(1+2\alpha\beta x+\alpha^2\beta^2x^2)\frac{e^{-\theta x}}{\theta} - \frac{2}{\theta}\Bigg[(\alpha\beta+\alpha^2\beta^2x)\frac{e^{-\theta x}}{\theta} - \frac{\alpha^2\beta^2}{\theta}\Bigg(\frac{e^{-\theta x}}{\theta}$$

$$I_1 = \frac{-e^{-\theta x}}{\theta^3}\Bigg[\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x\Bigg]$$

$$= 1+\frac{-e^{-\theta x}}{\theta^3}\Bigg[\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x\Bigg]\frac{}{\theta^2}$$

$$F(x;\alpha,\beta,\theta) = 1-\Bigg[\frac{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}$$

$$= 1-\Bigg[1+\frac{2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\Bigg]e^{-\theta x}; x>0,\alpha$$

□

8.5.2 Reliability Analysis

Proposition 8.5.3. *We define survival function denoted by $S(x;\alpha,\beta,\theta)$ and hazard function denoted by $h(x;\alpha,\beta,\theta)$ of a generalized three parameter Aradhana distribution Eq.8.44 as;*

$$S(x;\alpha,\beta,\theta) = \Bigg[1+\frac{2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\Bigg]e^{-\theta x}; x>0,\alpha>0$$

(8.46)

$$h(x;\alpha,\beta,\theta) = \frac{\theta^3\Big[1+\alpha\beta x\Big]^2e^{-\theta x}}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x};$$

(8.47)

Proof. To begin with, survival function Eq.8.46 is obtained by use of Eq.1.8 as;

$$S(x;\alpha,\beta,\theta) = 1-\left[1-\left[1+\frac{2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\right]e^{-\theta x}\right]$$
$$= \left[1+\frac{2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\right]e^{-\theta x}; x>0, \alpha>0, \beta>$$

using the relation Eq.1.9, hazard function Eq.8.47 is obtained as;

$$h(x;\alpha,\beta,\theta) = \frac{\frac{\theta^3}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\left[1+\alpha\beta x\right]^2 e^{-\theta x}}{\left[\frac{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\right]e^{-\theta x}}$$
$$= \frac{\theta^3\left[1+\alpha\beta x\right]^2 e^{-\theta x}}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x}; x>0$$

□

8.5.3 Moments and related measures

Proposition 8.5.4. *The r^{th} moments of the proposed generalized three parameter Aradhana distribution Eq.8.44 are derived using both method of moments and moment generating function (mgf) as;*

$$\mu_r^1* = \frac{r!\left[\theta^2+2\alpha\beta(r+1)\theta+\alpha^2\beta^2(r+1)(r+2)\right]}{\theta^r(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2)}; r=1,2,3,\cdots \tag{8.48}$$

Proof. By definition Eq.1.16, moments of AG3PAD are obtained as;

$$E(X^r) = \frac{\theta^3}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\int_0^\infty x^r(1+2\alpha\beta x+\alpha^2\beta^2x^2)e^{-\theta x}$$
$$= \frac{\theta^3}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\left[\int_0^\infty x^r e^{-\theta x}dx + 2\alpha\beta\int_0^\infty x^{r+1}e^{-\theta x}dx + \alpha^2\beta\right.$$
$$= \frac{\theta^3}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\left[\frac{\Gamma(r+1)}{\theta^{r+1}} + \frac{2\alpha\beta\Gamma(r+2)}{\theta^{r+2}} + \frac{\alpha^2\beta^2\Gamma(r+3)}{\theta^{r+3}}\right.$$
$$= \frac{r\Gamma r\left[\theta^2+2\alpha\beta(r+1)\theta+\alpha^2\beta^2(r+1)(r+2)\right]}{\theta^r(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2)}$$
$$= \frac{r!\left[\theta^2+2\alpha\beta(r+1)\theta+\alpha^2\beta^2(r+1)(r+2)\right]}{\theta^r(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2)}; r=1,2,3,\cdots$$

further by definition Eq.1.17, mgf of AG3PAD is given as;

$$m_x(t) = \frac{\theta^3}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\int_0^\infty e^{tx}(1+2\alpha\beta x+\alpha^2\beta^2x^2)e^{-\theta x}dx$$
$$= \frac{\theta^3}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\int_0^\infty e^{-(\theta-t)x}(1+2\alpha\beta x+\alpha^2\beta^2x^2)dx$$
$$= \frac{\theta^3}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\left[\frac{1}{(\theta-t)} + \frac{2\alpha\beta}{(\theta-t)^2} + \frac{\alpha^2\beta^2}{(\theta-t)^3}\right]$$
$$= \frac{\theta^3}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\left[\frac{1}{\theta}\sum_{k=0}^{\infty}\left(\frac{t}{\theta}\right)^k + \frac{2\alpha\beta}{\theta^2}\sum_{k=0}^{\infty}\binom{k+1}{k}\left(\frac{t}{\theta}\right)^k + \right.$$
$$m_x(t) = \sum_{k=0}^{\infty}\frac{\theta^2+2\alpha\beta(k+1)\theta+\alpha^2\beta^2(k+1)(k+2)}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\left(\frac{t}{\theta}\right)^k$$

The r^{th} moments are obtained from the derived $m_x(t)$ as;

$$\mu_r^1 = \frac{r!\left[\theta^2 + 2\alpha\beta(r+1)\theta + \alpha^2\beta^2(r+1)(r+2)\right]}{\theta^r(\theta^2 + 2\alpha\beta\theta + 2\alpha^2\beta^2)}; r = 1, 2, 3, \cdots$$

□

For the values of r as 1,2,3 and 4 in Eq.8.48 we get four moments about the origin of AG3PAD Eq.8.44 as;

$$\mu_1^1 = \frac{\theta^2 + 4\alpha\beta\theta + 6\alpha^2\beta^2}{\theta(\theta^2 + 2\alpha\beta\theta + 2\alpha^2\beta^2)}, \quad \mu_2^1 = \frac{2(\theta^2 + 6\alpha\beta\theta + 12\alpha^2\beta^2)}{\theta^2(\theta^2 + 2\alpha\beta\theta + 2\alpha^2\beta^2)}$$
$$\mu_3^1 = \frac{6(\theta^2 + 8\alpha\beta\theta + 20\alpha^2\beta^2)}{\theta^3(\theta^2 + 2\alpha\beta\theta + 2\alpha^2\beta^2)}, \quad \mu_4^1 = \frac{24(\theta^2 + 10\alpha\beta\theta + 30\alpha^2\beta^2)}{\theta^4(\theta^2 + 2\alpha\beta\theta + 2\alpha^2\beta^2)}$$

The centralized moments of AG3PQD are obtained as;

$$\mu_1 = \mu_1^1 = \frac{\theta^2 + 4\alpha\beta\theta + 6\alpha^2\beta^2}{\theta(\theta^2 + 2\alpha\beta\theta + 2\alpha^2\beta^2)}$$
$$\mu_2 = \frac{\theta^4 + 8\alpha\beta\theta^3 + 24\alpha^2\beta^2\theta^2 + 24\alpha^3\beta^3 + 12\alpha^4\beta^4}{\theta^2(\theta^2 + 2\alpha\beta\theta + 2\alpha^2\beta^2)}$$

The third and forth centralized moments are derived using the following relations;

$$\mu_3 = \mu_3^1 - 3[\mu_1^1\mu_2^1] + 2[\mu_1^1]^3$$
$$\mu_4 = \mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2$$

Proposition 8.5.5. *We now define Other related measures of a generalized three parameter Aradhana distribution Eq.8.44 such as variation coefficient (C.v), skewness (υ_1), kurtosis (υ_2) and* *dispersion index (υ_3) are stated as;*

$$C.v = \frac{\sqrt{\theta^4+8\alpha\beta\theta^3+24\alpha^2\beta^2\theta^2+24\alpha^3\beta^3+12\alpha^4\beta^4}}{\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2} \quad (8.49)$$

$$\upsilon_1 = \frac{\mu_3}{[\mu_2]^{\frac{3}{2}}} \quad (8.50)$$

$$\upsilon_2 = \frac{\mu_4}{[\mu_2]^2} \quad (8.51)$$

$$\upsilon_3 = \frac{\theta^4+8\alpha\beta\theta^3+24\alpha^2\beta^2\theta^2+24\alpha^3\beta^3+12\alpha^4\beta^4}{\theta(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2)(\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2)} \quad (8.52)$$

Proof. To begin with, coefficient of variation Eq.8.49 is obtained as;

$$\begin{aligned} C.v &= \frac{\sigma}{\mu_1^1} \\ &= \frac{\sqrt{\theta^4+8\alpha\beta\theta^3+24\alpha^2\beta^2\theta^2+24\alpha^3\beta^3+12\alpha^4\beta^4}}{\theta(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2)} \frac{\theta(\theta^2+2\alpha\beta\theta+2}{\theta^2+4\alpha\beta\theta+6} \\ &= \frac{\sqrt{\theta^4+8\alpha\beta\theta^3+24\alpha^2\beta^2\theta^2+24\alpha^3\beta^3+12\alpha^4\beta^4}}{\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2} \end{aligned}$$

the following relation is applied to derive skewness value of AG3PAD Eq.8.44.

$$\upsilon_1 = \frac{\mu^3}{[\mu_2]^{\frac{3}{2}}} = \frac{\mu_3^1 - 3[\mu_1^1\mu_2^1] + 2[\mu_1^1]^3}{[\mu_2^1 - [\mu_1^1]]^{\frac{3}{2}}}$$

the following relation is applied to derive kurtosis value of AG3PAD Eq.8.44.

$$\upsilon_2 = \frac{\mu_4}{[\mu_2]^2} = \frac{\mu_4^1 - 4[\mu_3^1\mu_1^1] + 6[\mu_1^1\mu_2^1] - 3[\mu_1^1]^2}{[\mu_2^1 - [\mu_1^1]]^2}$$

$$\upsilon_3 = \frac{\sigma^2}{\mu_1^1}$$
$$= \frac{\theta^4+8\alpha\beta\theta^3+24\alpha^2\beta^2\theta^2+24\alpha^3\beta^3+12\alpha^4\beta^4}{\theta^2(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2)}\frac{\theta(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2)}{\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2}$$
$$= \frac{\theta^4+8\alpha\beta\theta^3+24\alpha^2\beta^2\theta^2+24\alpha^3\beta^3+12\alpha^4\beta^4}{\theta(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2)(\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2)}$$

□

8.5.4 Excess Loss Distribution

Proposition 8.5.6. *In this section, we define probability function of the excess loss function $f_l(x;\alpha,\beta,\theta)$, mean residual lifetime (MRL), equilibrium distribution $f_e(x;\alpha,\beta,\theta)$, survival function based on the equilibrium distribution $S_e(x;\alpha,\beta,\theta)$ and hazard function based on the equilibrium distribution $h_e(x;\alpha,\beta,\theta)$*

as;

$$f_l(x;\alpha,\beta,\theta) = \frac{\theta^3\left[1+\alpha\beta x\right]^2 e^{-\theta(x-z)}}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2 z+\alpha^2\beta^2\theta^2 z^2+2\alpha^2\beta^2\theta z};$$

(8.53)

$$m(x) = \frac{\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2+2\alpha\beta\theta^2 x+\alpha^2\beta^2\theta^2 x^2+4\alpha^2\beta^2\theta x}{\theta(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2 x+\alpha^2\beta^2\theta^2 x^2+2\alpha^2\beta^2\theta}$$

(8.54)

$$f_e(x;\alpha,\beta,\theta) = \frac{\theta\left[\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2 x+\alpha^2\beta^2\theta^2 x^2+2\alpha^2\beta^2\theta\right.}{\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2}$$

(8.55)

$$S_e(x;\alpha,\beta,\theta) = \frac{\left[\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2+2\alpha\beta^2\theta^2 x+\alpha^2\beta^2\theta^2 x^2+4\alpha^2\beta^2\theta\right.}{\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2}$$

(8.56)

$$h_e(x;\alpha,\beta,\theta) = \frac{\theta\left[\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2 x+\alpha^2\beta^2\theta^2 x^2+2\alpha^2\beta^2\theta\right.}{\left[\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2+2\alpha\beta^2\theta^2 x+\alpha^2\beta^2\theta^2 x^2+4\alpha^2\beta^2\theta\right.}$$

(8.57)

Proof. Using the relation Eq.1.10, the pdf of excess loss distribution Eq.8.53 is obtained as;

$$f_l(x;\alpha,\beta,\theta)=\frac{\frac{\theta^3}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\left[1+\alpha\beta x\right]^2e^{-\theta x}}{\left[\frac{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2z+\alpha^2\beta^2\theta^2z^2+2\alpha^2\beta^2\theta z}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\right]e^{-\theta z}}$$

$$=\frac{\theta^3\left[1+\alpha\beta x\right]^2e^{-\theta(x-z)}}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2z+\alpha^2\beta^2\theta^2z^2+2\alpha^2\beta^2\theta z};x>z$$

further mean residual lifetime Eq.1.11 is obtained by use of Eq.1.11 as;

$$m(x)=\frac{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}{(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x)e^{-\theta x}}I_1$$

$$I_1=\int_x^\infty\frac{(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2t+\alpha^2\beta^2\theta^2t^2+2\alpha^2\beta^2\theta t)e^{-\theta t}}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}dt$$

Taking the numerator part $\int_x^\infty(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2t+\alpha^2\beta^2\theta^2t^2+2\alpha^2\beta^2\theta t)e^{-\theta t}$ we have the following;

$$I_2=\int_x^\infty(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2t+\alpha^2\beta^2\theta^2t^2+2\alpha^2\beta^2\theta t)e^{-\theta t}dt$$

$$u=(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2t+\alpha^2\beta^2\theta^2t^2+2\alpha^2\beta^2\theta t)\implies(2\alpha\beta$$

$$dv=e^{-\theta t}\implies v=\frac{-e^{-\theta t}}{\theta}$$

$$I_2=-(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2t+\alpha^2\beta^2\theta^2t^2+2\alpha^2\beta^2\theta t)\frac{e^{-\theta t}}{\theta}+2$$

$$I_3=\int_x^\infty(\alpha\beta\theta+\alpha^2\beta^2\theta t+\alpha^2\beta^2)e^{-\theta t}dt$$

$$u=(\alpha\beta\theta+\alpha^2\beta^2\theta t+\alpha^2\beta^2)\implies du=(\alpha^2\beta^2\theta)dt$$

$$dv=e^{-\theta t}\implies v=\frac{-e^{-\theta t}}{\theta}$$

$$I_3=-(\alpha\beta\theta+\alpha^2\beta^2\theta t+\alpha^2\beta^2)\frac{e^{-\theta t}}{\theta}+\alpha^2\beta^2\int_x^\infty e^{-\theta t}dt$$

$$I_3=-(\alpha\beta\theta+\alpha^2\beta^2\theta t+\alpha^2\beta^2)\frac{e^{-\theta t}}{\theta}+\alpha^2\beta^2I_4$$

$$I_4=\int_x^\infty e^{-\theta t}dt\implies\frac{-e^{-\theta t}}{\theta}$$

From I_2, I_3 , I_4 we have the following;

$$I_2 = \left[\frac{-e^{-\theta t}}{\theta}\left[\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2+2\alpha\beta\theta^2+\alpha\beta^2\theta^2t^2+4\alpha^2\beta^2\theta t\right]\right]_x^\infty$$

$$I_2 = \frac{e^{-\theta x}}{\theta}\left[\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2+2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+4\alpha^2\beta^2\theta x\right]$$

$$I_1 = \frac{I_2}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}$$

$$= \frac{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}{(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x)e^{-\theta x}}I_1$$

$$m(x) = \frac{\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2+2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+4\alpha^2\beta^2\theta x}{\theta(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x)}$$

further using relation Eq. 1.12, equilibrium distribution Eq.8.55 is obtained as;

$$f_e(x;\alpha,\beta,\theta) = \left[\frac{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x}{\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2}\right]e^{-\theta$$

$$= \frac{\theta\left[\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x\right]e^{-}}{\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2}$$

using the relation Eq.1.13, $S_e(x;\alpha,\beta,\theta)$ in Eq.8.56 is obtained as;

$$\int_x^\infty S(t;\alpha,\beta,\theta)dt = \frac{\left[\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2+2\alpha\beta^2\theta^2x+\alpha^2\beta^2\theta^2x^2+4\alpha^2\beta^2\theta\right.}{\theta(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2)}$$

$$= \frac{\left[\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2+2\alpha\beta^2\theta^2x+\alpha^2\beta^2\theta^2x^2+4\alpha^2\beta^2\theta x\right]e^{-\theta x}}{\theta(\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2)}\frac{\theta(\theta^2+2\alpha}{\theta^2+4\alpha\beta}$$

$$S_e(x;\alpha,\beta,\theta) = \frac{\left[\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2+2\alpha\beta^2\theta^2x+\alpha^2\beta^2\theta^2x^2+4\alpha^2\beta^2\theta\right.}{\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2}$$

lastly, using the relation Eq.1.14 hazard function based on the equilibrium distribution Eq.8.57 is obtained as;

$$h_e(x;\alpha,\beta,\theta) = \frac{\frac{\theta\left[\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta x\right]e^{-\theta x}}{\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2}}{\frac{\left[\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2+2\alpha\beta^2\theta^2x+\alpha^2\beta^2\theta^2x^2+4\alpha^2\beta^2\theta x\right]e^{-\theta x}}{\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2}}$$

$$= \frac{\theta\left[\theta^2+2\alpha\beta\theta+2\alpha^2\beta^2+2\alpha\beta\theta^2x+\alpha^2\beta^2\theta^2x^2+2\alpha^2\beta^2\theta\right.}{\left[\theta^2+4\alpha\beta\theta+6\alpha^2\beta^2+2\alpha\beta^2\theta^2x+\alpha^2\beta^2\theta^2x^2+4\alpha^2\beta^2\theta\right.}$$

□

9 Conclusion

In the research project we have constructed finite gamma mixtures up to 3 components and their generalizations. We expressed the constructed distributions in terms of pdfs, Cdfs, S(x) and h(x). Both statistical and mathematical properties of the constructed distributions were studied. The properties include reliability analysis, moments and excess loss distribution properties. Generalization of the constructed distributions enhanced model flexibility in modeling lifetime data. The finite gamma mixed distributions were fitted to a lifetime data (breaking stress data of 66 carbon fibers in GPa). The modes of estimation are MLE and MOME. On to the one parameter fitted distributions, Suja provided a better fit than Rama, Aradhana, Sujatha, Akash, Shanker, Lindley and exponential distribution. The density curve of Suja in figure 5 was highest than the rest. Hence it was best fitted distribution. Based on the fitted two parameter distributions, QSD was a better fit than AG2PAD, QAD, AG2PSD, AG2PLD AND G2PSD. Evidently, in figure 10 QSD demonstrated to have the highest fitted density curve. Therefore, it was considered to be the best fit among the two parameter distributions compared. We fitted and compared goodness of fit measures of AG3PLD, AG3PAD and AG3PSD. AG3PLD provided a better fit that AG3PAD AND AG3PSD. Based on the density shapes of the fitted three parameter distributions fitted in figure 19, AG3PLD had the highest curve. Therefore, AG3PLD was a better fit that other fitted three parameter distributions. We were able to note that model generalization improved goodness of fit measures.

9.1 Future Research

In the context of finite gamma mixtures, we can recommend future research work on length-Biased Weighted finite mixed gamma distributions.

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