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ON LATTICES OF SOME SPECIAL SUBSPACES OF SOME OPERATORS IN HILBERT SPACES

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Martin Mugi

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Research Report in Mathematics, Number 02, 2021

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Master Thesis

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Abstract

In this project, we investigate lattices of subspaces (like invariant, reducing and hyper-invariant subspaces among others) of some operators in Hilbert spaces and also give a detailed information on equivalence of operators and show how lattices of equivalent operators relate to each other.

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Declaration and Approval

I, the undersigned, declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

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In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

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Dedication

This project is dedicated to my grandparents Mr and Mrs. Herman Njoroge

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Martin Mugi

Nairobi, 2021.

1 PRELIMINARIES

1.1 Introduction

This chapter outlines the notations, terminologies and definitions that will be used throughout this work and also give some basic theory on some operators in Hilbert space.

1.1.1 Notations, Terminologies and Definitions

Notations

H: Hilbert Space over the complex numbers $\mathbb{C}$

$B(H)$: Banach algebra of bounded linear operators on$H$

T: A bounded linear operator

T^* : The adjoint of T

$\|T\|$: The operator norm of T

$\|x\|$: The norm of a vector x

$<x,y>$: The inner product of x and y on a Hilbert Space H

$Ran(T)$: The range of an operator T

$Ker(T)$: The kernel of an operator T

$M \oplus N$: The direct sum of the Subspace$M$ and N

$\{T\}'$: The commutant of T

P_M: the orthogonal projection of H onto M.

Terminologies and Definitions

In this project, H and Kwill denote complex Hilbert spaces which may be finite or infinite dimensional and $B(H)$ will denote the Banach algebra of bounded linear operators.
$B(H,K)$ will denote the set of bounded linear operators from H to K and equipped with the norm.
By an operator, we mean a bounded (i.e continuous) linear transformation with domain H and range a subset of H.
we denote the identity operator by I.
We denote by Ma linear manifold in H as a subset of H which is closed under vector addition and under multiplication by complex numbers.
The spectrum of an operator $T \in B(H)$(denoted by $\sigma(T)$) is defined as $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not invertible $\}$.
If $T \in B(H)$, then T^* denotes the adjoint of T while $Ker(T)$, $Ran(T)$, $\bar{M}$ and $M^{\perp}$ stands for the kernel of T, range of T, closure of M and orthogonal complement of M.
Let $A, B \in B(H)$ be operators. The commutant of A and B is given by $[A,B] = AB - BA$.
Definition 1.1.1 A subspace $M \subseteq H$ is said to be invariant under an operator $T \in B(H)$ if $TM \subseteq M$.
Remark 1.1.2 For an invariant subspace $M \subseteq H$ under T, we say that M is $T-$invariant.
Definition 1.1.3 A subspace $M \subseteq H$ is said to be a reducing subspace of $T \in B(H)$ if it is invariant under both T and T^* (equivalently if both M and $M^{\perp}$ are invariant under T).
Definition 1.1.4 The commutant of $T \in B(H)$ is the set of all operators that commute with T.
Remark 1.1.5 We denote the commutant of $T \in B(H)$ by $\{T\}'$ and define it as $\{T\}' = (S \in B(H) : ST = TS)$.
Definition 1.1.6 The double commutant of $T \in B(H)$,denoted by $\{T\}''$ is defined by $\{T\}'' = \{A \in B(H) : AS = SA, S \in \{T\}'\}$.
Definition 1.1.7 A subspace $M \subseteq H$ is said to be a hyperinvariant subspace for $T \in B(H)$ if $SM \subseteq M$ for each $S \in \{T\}'$.
Definition 1.1.8 A subspace $M \subseteq H$ is said to be a hyper-reducing subspace for $T \in B(H)$ if M reduces every operator in the commutant of T.
Remark 1.1.9 We denote the collection of all hyper-reducing subspaces for $T \in B(H)$ by $HyperRed(T)$.
Definition 1.1.10 Let A be a nonempty set. A binary operation $*$ is called associative if $\forall a, b \in A, (a*b)*c = a*(b*c)$ and commutative if $a*b = b*a$.
Definition 1.1.11 Let X be a non-empty set. We define an equivalence relation over X as a relation that satisfies the conditions below(we denote a relation by $\sim$)
(i)$x \sim x$ (Reflexive)
(ii) if $x \sim y$, then $y \sim x$ (Symmetric)
(iii) if $x \sim y$ and $y \sim z$, then $x \sim z$(Transitivity) $\forall x, y, z \in X$.

By relaxing the conditions of an equivalence relation, we have a partial order and is defined as follows

Definition 1.1.12 Let X be a non-empty set. Let $\leq$ be a binary relation on X satisfying $\forall x, y, z \in X$

(i) $x \leq x$ Reflexive

(ii) if $x \leq y$ and $y \leq x$, then $x = y$ Antisymmetric

(iii) if $x \leq y$ and $y \leq z$, then $x \leq z$ Transitive

Remark 1.1.13 The Set $(X, \leq)$ which satisfies the conditions of definition 1.1.12 is called a partially ordered set or a poset.

Definition 1.1.14 A hasse diagram is a graphical representation of a poset.

Definition 1.1.15 Let X be a partially ordered set with elements $a, b \in X$. If we have that for every pair of elements $a, b \in X$, either $a \leq b$ or $b \leq a$, then X is said to be totally ordered and is called a chain.

Remark 1.1.16 A chain is a totally ordered poset.

The figure below is a hasse diagram representation of a chain.

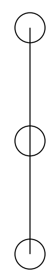

Figure 1. A chain

Definition 1.1.17 Let X be a set and $A \subseteq X$ be a nonempty subset of X. An element $x \in X$ is called an upper bound for A if $a \leq x, \forall a \in A$.

Remark 1.1.18 x is called the least upper bound(denoted by $lub(A)$ or $sup(A)$) for A if x is the smallest upper bound for A.

Definition 1.1.19 Let X be a set and $A \subseteq X$be a nonempty subset of X. An element $x \in (X)$ is called a lower bound for A if $x \leq a, \forall a \in A$. It is called the greatest lower bound if x is the largest lower bound and is denoted by $glb(A)$ or $inf(A)$.

Definition 1.1.20 Let$A$ be a set and $x, y \in A$. We denote the supremum of the pair (x, y), called a join of x and y, by $x \vee y$ and the infimum, which is also called the meet, by $x \wedge y$.

Remark 1.1.21 A set which has both the supremum and the infimum is said to have both the join and the meet.

Remark 1.1.22 Let A be a set and $a, b, c \in A$. To say that c is the supremum of the pair $\{a, b\}$, we write $c = a \vee b$ and to say c is the infimum of the pair we write $c = a \wedge b$.

Definition 1.1.23 Let $(X, \subseteq)$ be a poset and $x, y \in X$ with $x \neq y$. An element $x \in X$ is called an immediate predecessor of element $y \in X$ if $x < y$ and there does not exists $z \in X$ such that $x < z < y$. y is an immediate successor of x if $x < y$ and there does not exists $z \in X$ such that $x < z < y$.

Definition 1.1.24 Let $(X, \subseteq)$ be a poset. An element $x \in X$ is said to be maximal if there does not exist $y \in X$ such that $x < y$ and is said to be minimal if there does not exists a $y \in X$ such that $y < x$.
Remark 1.1.25 From the above results, we note the following:
Let A be a nonempty set and $a, b \in A$, then
(i) The greatest lower bound ($glb(A)$) of A may not belong to A and this is also true for the least upper bound ($lub(A)$) of A.
(ii) An element may have more than one immediate predecessor or more than one immediate successor.
(iii) minimal or maximal elements of A belong to A but they are not necessarily unique.
(iv) The least element of A is a lower bound of A that also belong to A and the greatest element of A is an upper bound of A that also belong to A.
Definition 1.1.26 Let X be a non-empty set. A semi-lattice is a partial order$(X, \leq)$ in which every pair of elements $x, y \in X$ has a least upper bound say z (i.e $z = x \vee y$ for every $x, y \in X$).
Definition 1.1.27 A lattice, L, is a partially ordered set in which every pair of elements $x, y \in L$ has a least upper bound and a greatest lower bound.
Definition 1.1.28 A subspace lattice is a family of subspaces of H which is closed under the formation of arbitrary intersections and arbitrary linear spans and which contains the zero subspace $\{0\}$ and H.
Remark 1.1.29 The subspace lattice of all invariant, reducing and hyperinvariant subspaces of $T \in B(H)$ is denoted by Lat (T), Red(T) and HyperLat(T) respectively.
Definition 1.1.30 A lattice L of subspaces of H is said to be trivial if $L = \{\{0\}, \{H\}\}$.
Definition 1.1.31 An operator $T \in B(H)$ is said to be:
Self-adjoint or Hermitian if $T^* = T$
Unitary if $T^*T = TT^* = I$
Normal if $T^*T = TT^*$
an isometry if $T^*T = I$
a co-isometry if $TT^* = I$
a partial isometry if $T = TT^*T$
quasinormal if $T(T^*T) = (T^*T)T$
hyponormal if $T^*T \geq TT^*$
left shift operator if $Tx = y$ where $x = (x_1, x_2 \cdots)$ and $y = (x_2, x_3 \cdots)$
right shift operator if $Tx = y$, where $x = (x_1, x_2 \cdots)$ and $y = (0, x_1, x_2 \cdots)$.

1.2 Some Bounded Operators in Hilbert Spaces

We recall that an operator is a continuous linear transformation between normed spaces over the same field. In this section, we define some bounded operators in Hilbert spaces and show how they are related.

Definition 1.2.1 Let $T \in B(H,K)$ be an operator. The adjoint of the operator T denoted by, T^* is the unique mapping of K into H, such that

$$< Tx, y > = < x, T^* y > \forall x \in H, y \in K$$

.
We note that T^* is bounded.
Proposition 1.2.2 Let $T, S \in B(H,K)$ be operators and $\lambda \in \mathbb{C}$ be a scalar. The following properties hold true in general:
(i) $(S+T)^* = S^* + T^*$
(ii) $(\lambda T)^* = \bar{\lambda} T^*$
(iii) $(ST)^* = S^* T^*$
(iv) $I^* = I$
(v) $(T^*)^* = T$
(vi) $\|T^*T\| = \|T\|^2$.
Definition 1.2.3 An operator $A \in B(H)$ is said to be positive if A is self-adjoint and $< Ax, x > \ \geq 0 \ \forall x \in H$.
We note that for a self adjoint operator A, then A^2 is positive and for any operator $A \in B(H)$, both A^*A and AA^* are positive.
Proposition 1.2.4 If $A, B \in B(H)$ are positive and for $\alpha \geq 0$, then $A+B$ is positive and so is αA.
Definition 1.2.5 An operator $P \in B(H)$ is said to be idempotent if $P^2 = P$.
Remark 1.2.6 If $P \in B(H)$ is idempotent, then $Ran(P) = Ker(I-P)$so that $Ran(P)$ is a subspace of H.
Definition 1.2.7 An operator $T \in B(H)$ is called a projection if $T^2 = T$.
Definition 1.2.8 An operator $T \in B(H)$ is called an orthogonal projection if T is idempotent and self-adjoint(i.e if $T^2 = T$ and $T^* = T$).
Remark 1.2.9 Both the projection and orthogonal projection operators are positive and self-adjoint.
Definition 1.2.10 An operator $T \in B(H,K)$ is said to be invertible if it has an inverse and the range $Ran(T) = K$, and such an inverse must be bounded.
Proposition 1.2.11 Let H be a Hilbert space and $T \in B(H)$ be an operator. Then T is invertible if and only if T^* is invertible.
Proposition 1.2.13 A unitary operator is an invertible isometry.
Definition 1.2.14 An operator $T \in B(H)$ is normal if it commutes with its adjoint (i.e $T^*T = TT^*$ or $0 = T^*T - TT^*$).
Definition1.2.15 Let $T \in B(H)$ be an operator and $\lambda \in \mathbb{C}$. Then T is said to be hyponormal if $TT^* \leq T^*T$ (i.e$(\lambda I - T)(\lambda I - T)^* \leq (\lambda I - T)^*(\lambda I - T)$).
Definition 1.2.16 An operator$T \in B(H)$ is said to be cohyponormal if its adjoint is hyponormal (i,e if $T^*T \geq TT^*$) for $\lambda \in \mathbb{C}$.
Remark 1.2.17 $T \in B(H)$ is normal if and only if it is both hyponormal and cohyponormal.

Definition 1.2.18 An operator $T \in B(H)$ is said to be semi-normal if T is either a hyponormal or cohyponormal operator.
Theorem 1.2.19 Let T be an operator. Then the following assertions are equivalent
(a) T is normal
(b) $\|T^*x\| = \|Tx\|$ for every $x \in H$
(c) T^n is normal for every integer $n \geq 1$
(d) $\|T^{*n}x\| = \|T^n x\|$ for every $x \in H$ and every integer $n \geq 1$.
Remark 1.2.20 Every self adjoint and unitary operator is normal.
Proposition 1.2.21 An operator $P \in B(H)$ is an orthogonal projection if an only if it is a normal projection.
Definition 1.2.21 An operator $T \in B(H,K)$ is compact if for every bounded sequence $\{x_n\} \in H$, the sequence $\{Tx_n\} \in K$ has a convergent subsequence .
Proposition 1.2.22 let T be a unitary operator on H. Then T is compact if and only if H has a finite dimension.
Definition 1.2.23 Two operators $A,B \in B(H,K)$ are said to be similar if there exists an invertible operator $S \in B(H,K)$ such that $SA = BS$ or equivalently, $A = S^{-1}BS$.
Definition 1.2.24[15] Two operators $A,B \in B(H,K)$ are said to be almost similar if there exists an invertibe operator $S \in B(H,K)$ such that $A^*A = S^{-1}(B^*B)S$ and $A^* + A = S^{-1}(B^* + B)S$.
Definition 1.2.25 Two operators $A,B \in B(H,K)$ are unitarily equivalent if there exists a unitary operator $U \in B_+(H,K)$(Banach algebra of all invertible operators in $B(H)$) such that $UA = BU$(i.e $A = U^*BU$, equivalently, $A = U^{-1}BU$).
Definition 1.2.26 Two operators $A,B \in B(H)$ are said to be almost unitarily equivalent if there exists a unitary operator $U \in B_+(H,K)$ such that $A^*A = U^*(B^*B)U$ and $A^* + A = U^*(B^* + B)U$.
Proposition 1.2.27 If $A,B \in B(H)$ are unitarily equivalent, then they are almost similar.
Definition 1.2.28 An operator $X \in B(H,K)$ is said to be a quasi- affinity or quasi-invertible if it is injective and has a dense range.
i,e$N(T) = \{0\}$ and $\overline{R(T)} = K$
Equivalently,$N(T) = \{0\}$ and $N(T^*) = 0$.
Proposition 1.2.29 An operator $T \in B(H)$ is quasi-invertible if and only if T^* is quasi-invertible.
Definition 1.2.30 Two operators $A,B \in B(H)$ are said to be quasiaffine transforms of each other if there exists a quasi-affinity $X \in B(H,K)$ such that $AX = BX$.
Definition 1.2.31 Two operators are said to be quasisimilar if there exists quasi-affinities $X \in B(H,K)$ and $Y \in B(K,H)$ such that $XA = BX$ and $AY = YB$.
Definition 1.2.32 Two operators $A,B \in B(H,K)$ are said to be metrically equivalent if $\|Ax\| = \|Bx\|$ or equivalently $|<Ax,Ax>|^{\frac{1}{2}} = |<Bx,Bx>|^{\frac{1}{2}}$.
Definition 1.2.33 Two operators $A,B \in B(H,K)$ are said to be $\alpha-$metrically equivalent if there exists an $\alpha > 0$ such that $A^*A = \alpha^2 B^*B$.
Definition 1.2.34 [31] An operator $T \in B(H)$ is said to be quasi-unitary if $T^*T = TT^* = T^* + T$.

Proposition 1.2.35 If $T \in B(H)$ is quasi-unitary, then T^* is also quasi-unitary.
Definition 1.2.36 Two operators $A, B \in B(H)$ are said to be absolutely equivalent if both the absolute values of the operators are unitarily equivalent (i.e $|A| = U|B|U^*$).
Definition 1.2.37[25] Two operators $A, B \in B(H)$ are said to be nearly-equivalent if there exists unitary operator $U \in B(H)$ such that $A^*A = UB^*BU^*$.
Definition 1.2.38 An operator $T \in B(H)$ is said to be reducible if it has nontrivial reducing subspace.
Definition 1.2.39 An operator $T \in B(H)$ is said to be reductive if all its invariant subspaces reduce it.

2 LITERATURE REVIEW

In this chapter, a brief historical development of lattices is given by showing how the lattice theory has grown as a 'younger brother' to group theory to an independent branch of mathematics over time.
Also, we give the invariant subspace problem and try to show how numerous mathematicians have tried to solve it.

2.1 Development of lattice theory

The first step on the journey towards the concept of lattice theory was taken by George Boole[6] in 1848 who tried to formalize propositional logic in the style of algebra and presented the list of laws which are satisfied by various algebras.
However, it was the American philosopher and mathematician Charles S. Peirce (1880)who introduced the notion of lattice theory in [32]by introducing the use of meet and join in posets.
Later on, many mathematicians across the world, like the German mathematician Ernst Schroder(1887), contributed highly in the lattice theory by finding some flaws in the work of Boole and Pierce and summarized and extended the works of Boole and Pierce by proving proving that the distributive law in the set of rules for Boole's Calculus is independent of others .
Schroder was thus able to distinguish two lattice structures: Identity calculus as the Boolean algebra and the logical calculus with groups as a more general system which did not satisfy the distributive law.
Schroder is highly regarded in the mathematical world for the advancement of lattice theory and for improving on Boole's and Pierce's work. For instance, Pierce had thought that the distributive law in the set of rules for Boole's calculus is dependent of others.
In the late 19th and early 20th Centuries, Dedekind introduced the modular law in [10] and studied the structures with modular property and non- modular structures in[11] and was able to arrive at even more general structures than Schroder because modularity is a weakened form of distributivity.
It was not until mid-1930s that lattice theory got its major boost when Garret Birkhoff (1940) started the general development of lattice theory in [5] by introducing the notion of a complete lattice.

The field of lattice theory has gone and is still going through various stages of development since its limelight.
Today, the field has found many important applications and thus its position in mathematics cannot be overlooked .

2.2 Invariant subspace problem

The research on invariant subspaces was first introduced to the mathematical world by the famous mathematician and computer scientist J.von Neumann[23]in 1935 who initiated the research on the invariant Subspace problem and asks the question: does every operator have a non-trivial invariant Subspace?.
A related question is the hyper-invariant subspace problem:does every operator that is not a complex multiple of the identity operator, I, have a nontrivial hyperinvariant subspace?.
Since then, many great mathematicians have tried to come up with solutions to the these questions.
In 1954, Aronszajn and Smith [2]proved that every compact operator on Banach space has a nontrivial invariant closed subspace.
In 1996, Bernstein and Robinson[4] proved that every polynomially compact operator has a non-trivial invariant closed subspace.
However, it was not until 1973 that the great physicist and mathematician V. Lomonosov astounded the mathematical world by proving in [20] that every bounded linear operator on a Banach space which commute with a nonzero compact operator has a non-trivial invariant closed subspace.
In 1978, S. Brown [8] proved that every subnormal operator has a nontrivial invariant subspace and in 1987 proved in [7] that every hyponormal operator with thick spectrum has a nontrivial invariant closed subspace.
In 1988, S. Brown, B. Chevreau and C. Pearchy [9]proved that every contraction operator on a Hilbert space with spectrum containing the unit circle has a nontrivial invariant closed subspace.
In 2005, Liu[19] proved that the converse of proposition of the famous Lomonosov theorem[20] is true and obtained some new necessary and conditions for the invariant closed subspace.
In 2004, Ambrozie and Muller [1] proved that every polynomially bounded operator on a Hilbert space such as the spectrum of T containing the unit circle has a non-trivial invariant closed subspace.
In 2020, Nzimbi et al. [24] introduced the concept of metric equivalence of operators and was able to explain metric equivalence relation and closely relations on some classes of operators. The author also gave some conditions under which metric equivalence of operators implies unitary equivalence of operators.

However, despite of all these research and advancement of the invariant subspace problem since its birth by Neumann [23], very little in the literature about relationship between invariant, reducing and hyper-invariant subspaces is published today.

In this Project, we set to investigate some of these relations. For instance, given two operators say, $A, B \in B(H)$, when is
(i).Lat(A)=Lat(B)
(ii). Lat$(A) \equiv$Lat(B)
(iii). Hyperlat$(A) =$Hyperlat(B)
(iv).Hyperlat$(A) \equiv$Hyperlat(B)
(v). Red$(A) =$Red(B)?.
That is, for the above properties to hold, what is the relationship between the operators A and B and what do they have in common?.

3 ON ELEMENTARY LATTICE THEORY

In this chapter, we give some basic information on lattice theory and show subspace lattices of some operators in Hilbert spaces and also invariant subspaces.

3.1 LATTICES

We recall that a semi lattice is a partial order $(X,\leq)$ in which every pair $x,y\in X$ has a least upper bound $x\vee y$.
In figure (2), the first figure is a meet semilattice while the second one fails on several accounts.

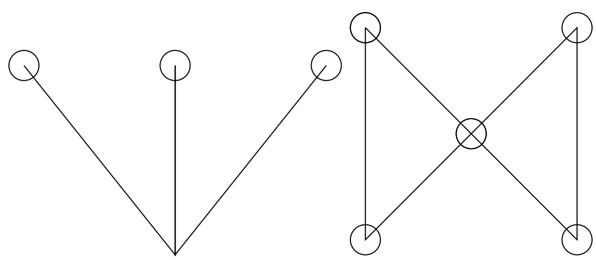

Figure 2. Semilattice

Clearly, from the definition of a semi lattice, we note the following result.
Proposition 3.1.1 Every chain is a semi lattice.
Example 3.1.2 The power set of a set, ordered by inclusion is a semi lattice since it forms an upper semi lattice with $\vee s$ as union and lower semi lattice with $\wedge s$ as intersection.
The notion of a semi lattice is a weaker form of a lattice. This is clearly illustrated in the definition of a lattice.
Definition 3.1.3 A Lattice L is a partially ordered set in which every pair of elements has a least upper bound and a greatest lower bound. That is, every elements $x,y\in L$ has a join $x\vee y$ and a meet $x\wedge y$.
A lattice is therefore both a meet semi lattice and a join semi lattice.
Proposition 3.1.4 Let L be a lattice. Then the following are true
(i) (**Idempotence**) :$a\vee a=a, \qquad a\wedge a=a.$
(ii) (**Commutativity**): $a\vee b=b\vee a, \qquad a\wedge b=b\wedge a.$
(iii)(**Associativity**) :$a\vee(b\vee c)=(a\vee b)\vee c, \qquad a\wedge(b\wedge c)=(a\wedge b)\wedge c.$
(iv) $a\leq b \iff a\vee b=b, \qquad a\leq b \iff a\wedge b=a.$
(v) (**Absorption**) :$a\vee(a\wedge b)=a=a\wedge(a\vee b).$
(vi) (**Isotonicity**): $b\leq c \implies a\vee b\leq a\vee c, \qquad b\leq c \implies a\wedge b\leq a\wedge c.$
(vii) $a\leq b, c\leq d, \implies a\vee c\leq b\vee d, \qquad a\leq b, c\leq d, \implies a\wedge c\leq b\wedge d.$

(viii) (**Distributive inequality**): $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c), a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$.
(ix) (**Modularity**) :$a \leq c \iff a \vee (b \wedge c) \leq (a \vee b) \wedge c, \forall a,b,c,d \in L$.

One of the beauty of lattices is that apart from being a poset, a lattice is an algebra thus connecting two major branches of mathematics that of functional Analysis and Algebra.
Definition 3.1.5 A lattice L is an algebra satisfying for all $x,y,z \in L$
(i) $x \vee x = x, \quad x \wedge x = x$
(ii) $x \vee y = y \vee x, \quad x \wedge y = y \wedge x$
(iii)$x \vee (y \vee z) = (x \vee y) \vee z, \quad x \wedge (y \wedge z) = (x \wedge y) \wedge z$
(iv) $x \vee (x \wedge y) \simeq L, x \wedge (x \vee y) \simeq L$.
Remark 3.1.6 We have noted that the notion of a semi lattice is a weaker form of a lattice but this is not to be confused with a sub lattice as we see in the following result.
Definition 3.1.7 Let L be a lattice and K be a subset of L. Then K is a sub lattice if for $x,y \in K$, we have that $x \vee y, x \wedge y \in K$.
Clearly, we see that a sub lattice is closed under $\vee$ and $\wedge$ of L unlike a semi lattice which is closed under $\vee$ or $\wedge$.
Example 3.1.8 Let X be any set and $P(X)$ be its power set. Then $(P(X), \vee, \wedge)$ is a lattice in which join and meet are union of sets and intersection of sets respectively.
Example 3.1.9 Let W be a family of subsets of X such that $A \cup B$ and $A \cap B$ are in W for $A,B \in W$. Then W is a sub lattice of $(P(X), \vee, \wedge)$.
Figure (3) is a hasse diagram of two figures showing a lattice (first figure) and the second one which not a lattice since $f \vee g$ does not exists

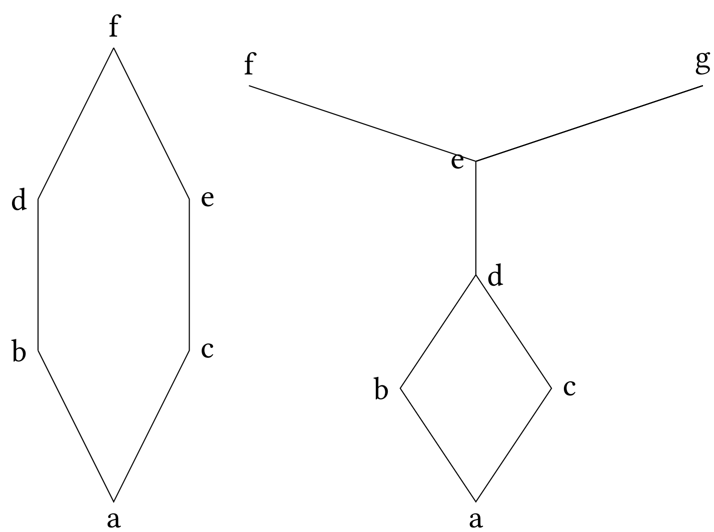

Figure 3. Lattice

Proposition 3.1.10 Every interval $[a,b]$ of a lattice is a sub lattice.
Proof
The proof is trivial since for every pair of elements $x,y \in [a,b], xy, x+y \in [a,b]$ which shows that $[a,b]$ contains the meet xy and the join $x+y$ hence $[a,b]$ is a sub lattice.

Corollary 3.1.11 Let L be a lattice. Then any interval of L is also a lattice.
Proof
Let $[a,b]$ be any interval of L. Then for any $x,y \in [a,b], x \vee y \leq b$ since b is an upper bound for x and y.
Clearly, $x \vee y \geq x \geq a$, so $x \vee y \in [a,b]$.
A similar argument shows that $x \wedge y \in [a,b]$. Thus the result follows.
Theorem 3.1.12 Every chain is a lattice.
Proof
Let L be a chain. It suffices to show that every pair of elements $a,b \in L$ has a greatest lower bound and least upper bound. Now, for any pair of elements $a,b \in L$, suppose that $a \leq b$.
By the reflexitivity of $\leq$, it follows that $a \leq a$ and thus a is a lower bound of the set $\{a,b\}$.
To show that it is the greatest lower bound, suppose $c \in L$ is another lower bound of $\{a,b\}$. Then from definition, we have that $c \leq a$ which means a is the greatest lower bound of $\{,b\}$.
A similar argument shows that b is the least upper bound of $\{a,b\}$. Which shows that L is a lattice.

3.2 Types of Lattices

Here, we give some types of lattices and some of their basic properties.
Definition 3.2.1 (Finite lattice) A lattice L is called finite length if all chains in L are finite.
Proposition 3.2.2 A finite lattice has a least element and a greatest element.
Proof
It suffices to show that a finite lattice has a zero and a unit element.
Let L be a lattice and let $(a_1 \cdots a_n) \in L$ Define $x_0 = a_1, x_1 = x_0 a_1, x_2 = x_1 a_2 \cdots x_n = x_{n-1} a_n$ and $y_0 = a_1, y_1 = y_0 + a_1, \cdots y_n = y_{n-1} + a_n$
By the meet definition, $x_n \leq x_{n-1} \leq \cdots x_1 = x_0$ and by the join definition$y_n \geq y_{n-1} \geq \cdots \geq y_1 = y_0$
Thus for any element $V_k (k = 1 \cdots n)$,

$$x_n \leq x_k = x_{k-1} \leq V_k \leq y_{k-1} + V_k = y_k \leq y_n$$

$\therefore x_n = 0$ and $y_n = 1$
Thus L has zero and a unitary element and hence finite.
Definition 3.2.3 (Bounded lattice) A lattice L is said to be bounded if it has a greatest element 1 and a least element 0.
Remark 3.2.4 The following result shows some basic properties of a bounded lattice.

Proposition 3.2.5 Let L be a bounded lattice with a minimum and maximum elements 0 and 1 respectively, and let x be any element in L. Then
(i). $0 \vee x = x = x \vee 0, \qquad 1 \wedge x = x = x \wedge 1.$
(ii). $0 \wedge x = 0 = x \wedge 0, \qquad 1 \vee x = 1 = x \vee 1.$
Theorem 3.2.6 A finite lattice is bounded.
Proof
Let $L = \{a_1 \cdots a_n\}$ be a finite lattice.Then L is clearly bounded.
Figure (5) shows a bounded lattice $L = \{1,2\}$

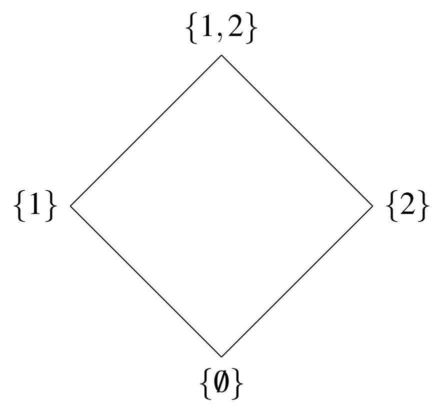

Figure 4. Bounded lattice

Definition 3.2.7 (Complete lattice) A semi lattice L is said to be a meet complete semi lattice if every subset $A \subset L$ has an infimum while L is said to be a join complete semi lattice if every subset $A \subset L$ has a supremum.
Thus a lattice is said to be complete if it is both meet complete and join complete
Clearly, we note that a complete lattice has a top and bottom element(i.e $0 = sup\emptyset$ and $1 = sup\emptyset$) and hence it follows that a finite lattice is complete.
Definition 3.2.8 (Distributive lattice) A lattice that satisfies
$(x \wedge y) \vee (x \wedge y) = (x \wedge (y \vee z)$ or
$(x \vee y) \wedge (x \vee y) = x \vee (y \wedge z)$ is called distributive.
Definition 3.2.9 (Modular lattice) A lattice that satisfies

$$x \geq z \implies (x \wedge y) \vee z = x \wedge (y \vee z)$$

is called modular. From Definitions 3.2.8 and 3.2.9, it easily follows that every distributive lattice is modular.
Proposition 3.2.10 Every chain is a distributive lattice.
Definition 3.2.11 Let L be a bounded lattice with a minimum and maximum elements 0 and 1. A complement of an element x is an element $z$such that $x \wedge z = 0$ and $x \vee z = 1$ for $x, z \in L$.
Definition 3.2.12 (Complemented lattice) A bounded lattice L is said to be complemented if $\forall a \in L, \exists b \in L$ such that b is a complement of a.

Definition 3.2.13 (Boolean lattice) A lattice is Boolean if it is distributive, has a zero and unity and each element has a unique complement. Thus,a Boolean lattice is therefore a complemented distributive lattice.
Definition 3.2.14 (Isomorphic lattices) Let L and K be lattices. A map $f: L \to K$ is said to be a lattice homomorphism if

$$\forall a, b \in L, f(a \vee b) = f(a) \vee f(b)$$

and

$$f(a \wedge b) = f(a) \wedge f(b)$$

.
A bijective lattice homomorphism is a lattice isomorphism.
Thus,two lattices L and K are isomorphic if there exist an isomorphic map between them . We denote isomorphic lattices L and$K$ by $L \equiv K$.
Remark 3.2.15 We note that order relations are preserved under lattice homomorphisms as shown below.
Proposition 3.2.15 Let $(L_1, \vee_1, \wedge_1)$ and $(L_2, \vee_2, \wedge_2)$ be lattices and $\leq_1$ and $\leq_2$ be partial orders on L_1 and L_2 respectively. Let $f: L_1 \to L_2$ be lattice homomorphism. If $a, b \in L_1$, then $a \leq_1 b \iff a \vee_1 b = b$ and so
$f(b) = f(a \vee_1 b)$
$= f(a) \vee_2 f(b)$
$\iff f(a) \leq_2 f(b)$.
Thus $a \leq_1 b \iff f(a) \leq_2 f(b)$.

3.3 Invariant subspaces on Hilbert spaces

We recall from remark 1.1.30 that the the subspace lattice of all invariant subspaces is denoted by $Lat(T)$. In the following result, we show that $Lat(T)$ is a lattice.
Theorem 3.3.1 [13] The set of all invariant subspaces of $T \in B(H)$ is a lattice.
Proof
Let $Lat(T)$ denote the set of all invariant subspaces of $T \in B(H)$ in H.Let $M, N \in Lat(T)$, and suppose $x \in M \cap N$,then by the invariance of property of T, we have that $Tx \in M$ and that $Tx \in N$ so that $M \cap N$ is T invariant.
Suppose $x \in M + N$, then by definition we have that $M + N = \{m + n : m \in M, n \in N\}$.
Thus, $x \in M + N$ implies $x = x_1 + x_2$ where $x_1 \in M$ and $x_2 \in N$
Then $Tx = Tx_1 + Tx_2 \in M + N$ which implies that $M + N$ is T invariant
Since $\{0\}$ and H belong to $Lat(T)$, we find that $Lat(T)$ preserves both the meet and join properties and hence its a lattice.
Corollary 3.3.2 The set of all hyperinvariant subspaces is a lattice.
Remark 3.3.3 [21] From Corollary 3.3.2 and the fact that T commutes with itself , we have that the following results, $HyperLat(T) \subseteq Lat(T)$.

Corollary 3.3.4 The set of all reducing and hyper-reducing subspaces of $T \in B(H)$ are lattices. Thus we have the following inequalities :
$Red(T) \subseteq Lat(T)$ and $HyperRed(T) \subseteq Lat(T)$.
Example 3.3.5

Let $T \in B(H)$ be an operator on $H = \mathbb{R}^2$ represented as $T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

and let M=Span$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

A simple computation shows that Lat(T)=$\left\{ \{0\}, M, \mathbb{R}^2 \right\}$ and
Red(T)=$\left\{ \{0\}, \mathbb{R}^2 \right\}$.
Remark 3.3.6 From Example 3.3.5, we can clearly see that $Red(T) \subseteq Lat(T)$.

We call $a \subseteq B(H)$ a subalgebra of $B(H)$ if a is closed under scalar multiplication, addition and composition. If a is also closed under taking adjoint, we call it a $^*-$subalgebra of $B(H)$. If the identity operator I belongs to the subalgebra a, we say that a is a unital subalgebra of $B(H)$.
Theorem 3.3.7 Let $T, S \in B(H)$. If $Lat(T) = Lat(S)$, then $HyperLat(T) = HyperLat(S)$.
Proof
This follows easily from the definition.
We note that the converse of Theorem 3.3.7 need not hold in general. To see this, let

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$S = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

on the Hilbert space $H = \mathbb{R}^2$
A simple computyation shows that
Lat$T = \left\{ \{0\}, \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle, \mathbb{R}^2 \right\} \neq \left\{ \{0\}, \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle, \mathbb{R}^2 \right\}$ =Lat$S$
Another computation shows that $\{T\}' = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{11} \end{bmatrix} : a_{11}, a_{12} \in \mathbb{R} \right\}$ and $\{S\}' = \left\{ \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix} : b_{11}, b_{22} \in \mathbb{R} \right\}$
We note that $\{T\}' \cap \{S\}' = \left\{ \begin{bmatrix} a_{11} & 0 \\ 0 & a_{11} \end{bmatrix} : a_{11} \in \mathbb{R} \right\}$

which is the set of scalar operators.
Clearly, the commutant of T consists of operators similar to scalar operators. This result is true for isometries and co-isometries. Another computation shows that

$$\text{HyperLat}(T) = \left\{ \{0\}, \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle, \mathbb{R}^2 \right\} \neq \left\{ \{0\}, \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle, \mathbb{R}^2 \right\} = \text{HyperLat}(S).$$

However, it is clear that the subspace $M = \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle \in \text{HyperLat}(S)$ if $a_{12} = 0$
This happens if and only if $\{T\}' = \{\alpha I : \alpha \in \mathbb{C}\}$.
This extra condition implies that
HyperLat(T) =HyperLat(S).
Theorem 3.3.7 can be relaxed as follows.
Corollary 3.3.8 If $T, S \in B(H)$ such that $Lat(T)$ is isomorphic to $Lat(S)$, $(i.eLat(T) \equiv Lat(S))$, then $HyperLat(T) \equiv HyperLat(S)$.
Theorem 3.3.9 (von Neumann Double Commutant Theorem) Let H be a Hilbert space and $a \subseteq B(H)$ be a unital self-adjoint $^*-$subalgebra of $B(H)$.
Then the following conditions are equivalent
(i) $a = \{a\}''$
(ii) a is closed with respect to the weak topology on $B(H)$
(iii) a is closed with respect to the strong topology (SOT) on $B(H)$.
If a unital $^*-$subalgebra a on $B(H)$ satisfies either of the three equivalent conditions above, then we say that it is a von Neumann algebra.
Theorem 3.3.9 simply asserts that the double commutant $a = \{a\}''$ of a unital self-adjoint subalgebra a of $B(H)$ is always strongly closed (and hence weakly closed). That is, a is strongly (and hence weakly) dense in $a = \{a\}''$
Equivalently, it says that the strongly closed unital self adjoint subalgebra of $B(H)$ are always their own double commutant.
For convenience, we take a von Neumann algebra as a $^*-$subalgebra a of $B(H)$ satisfying $a = \{a\}''$. A von Neumann algebra is a unital weakly closed and contains an abundance of projections. If a is a von Neumann algebra, then a is generated by the projections in a.
Let $T \in B(H)$. We define $W^*(T)$ to be the von Neumann algebra generated by $\{I, T\}$. Note that $W^*(T) = \{T\}'' \cup \{\alpha I : \alpha \in \mathbb{C}\}$.
From the Double Commutant Theorem, if $T = T^*$, then $\{T\}'' = W^*(T)$ and $\{T\}'$ is a von Neumann algebra and is therefore generated by its projections.
Since the projections in $\{T\}'$ are also in $\{T^*\}'$, it follows that the Double Commutant Theorem has the following reformulation.
$W^*(T) = \{T : PT = TP, \forall \text{ projection } P \in \{T\}'\}$.
Corollary 3.3.10 Let $T \in B(H)$. Then $Lat(T)$ =Lat$(W^*(T))$.
Proof
Since $T \in W^*(T)$, trivially Lat$(W^*(T)) \subseteq$ Lat(T).
On the other hand, $W^*(T)$ consists of polynomials in I and T and hence Lat$(T) \subseteq$Lat$(W^*(T))$.

Combining these two inclusions, equality follows.
Corollary 3.3.11 [26] Let $T \in B(H)$. Then $HyperLat(T) = Lat\{T\}'$.
Theorem 3.3.12 Let $A, B \in B(H)$. If $A \in W^*(B)$, then $Lat(B) \subseteq Lat(A)$.
Proof
Now, from the hypothesis, that is , $A \in W^*(B)$ and the fact that $HyperLat(T) \subseteq Lat(T)$ for an operator $T \in B(H)$, we have that $QP_M = P_M Q$ where we recall that P_M is the orthogonal projection of H onto M and $Q \in \{W^*(B)\} = \{B\}' \cap \{B^*\}'$ is an orthogonal projection in $\{B\}'$ and $M \in HyperLat(B)$ and thus we have that $P_M A P_M = P_M A$.
Now we note that $P_M \in W^*(B)$ and hence $M \in HyperLat(B) \subseteq Lat(B) \implies M \in Lat(A)$. Thus $M \in Lat(B) \implies M \in Lat(A)$ and hence it follows that $Lat(B) \subseteq Lat(A)$.
Corollary 3.3.13 Let $A, B \in B(H)$. If $A \in W^*(B)$, then $HyperLat(B) \subseteq HyperLat(A)$.

In the following result, we show the relationship between $HyperLat(T)$ and $Red(T)$ for a unitary operator $T \in B(H)$. To set the pace for this, we consider the following lemma;
Lemma 3.3.14 Let $T \in B(H)$ be an operator and $M \subseteq H$ be a subspace of H. The following statements are equivalent
(i) M reduces T
(ii) $M \in Lat(T) \cap Lat(T^*)$
(iii) $P_M \in \{T\}'$.
Lemma 3.3.15 Let $A, B \in B(H)$ be operators such that B is normal. If $AB = BA$, then $AB^* = B^*A$.
Theorem 3.3.16 Let $T \in B(H)$ be a unitary operator. A Subspace $M \in H$ is hyperivariant if an only if M reduces T.
Proof
Let $M \in HyperLat(T)$. Then $AP_M = P_M = P_M A P_M$ for every $A \in \{T\}'$. Now, since T is unitary and hence normal, then by lemma 3.3.14 we have that $A^* \in \{T\}'$.
Thus $A^* P_M = P_M A^*$ and thus $AP_M = P_M A P_M = P_M A$. Therefore, by lemma 3.3.13, we have that M reduces T.
Conversely, suppose that Mreduces T. Then, $AP_M = P_M A$ and hence $AM = AP_M H = P_M AH \subseteq P_M H = M$ which shows that M is invariant under A.
Corollary 3.3.17 Let $T \in B(H)$ be a unitary operator and $M \subseteq H$ be a subspace of H. Then $HyperLat(T) = Red(T)$.
From Proposition 1.2.13, Theorem 3.3.16 can be relaxed as follows.
Theorem 3.3.18 Let $T \in B(H)$ be an isometry. If $M \in H$ is such that $TM = M$,then M reduces T.
Proof
If $TM = M$, then $T^*M = T^*TM = M$.
Corollary 3.3.19 Let $T \in B(H)$ be an isometry. If $M \subseteq H$ is such that $TM = M$, then $Red(T) = Lat(T)$.

3.4 Reductive operators

We recall that an operator $T \in B(H)$ is reductive if all its invariant subspaces reduce it and that T is said to be reducible if it has a nontrivial reducing subspace. From these definitions, we note that for any operator$T \in B(H)$, Red(T)=Red(T^*).
Given the above definitions, we arrive at the following theorem
Theorem 3.4.1 A Subspace M reduces an operator T if and only if $M \in Lat(T) \cap Lat(T^*)$.
Proof
Follows easily from the definition.
Corollary 3.4.2 Let $T \in B(H)$ be an operator, then $Red(T) = Lat(T) \cap Lat(T^*)$.
Remark 3.4.3 Let $T \in B(H)$ be a self-adjoint operator, then from Corollary 3.4.2 we have that Lat $(T) \subseteq$ Red(T). Since we have already shown that Red$(T) \subseteq$ Lat (T) for a general case, the following is an immediate result of Corollary 3.4.2.
Theorem 3.4.4 Let $T \in B(H)$ be a self-adjoint operator. Then we get Red(T)=Lat(T).
Theorem 3.4.5 An operator $T \in B(H)$ is reductive if and only if Lat (T)=Red(T).
Proof
$\Longrightarrow$ if$T$ is reductive, then we have that lat (T)$\subseteq$Red(T) =Red(T^*)
But Red(T) $\subseteq$Lat (T) always
thus Red(T) =Lat(T)
$\Longleftarrow$ Suppose Lat (T) =Red(T)
Then lat (T)=Red(T^*)
=Lat $(T^*) \cap Lat(T) \subseteq Lat(T^*)$
$\Longrightarrow$ Lat$(T) \subseteq$ Lat(T^*)
which implies that T is reductive.
Corollary 3.4.6 Let $T \in B(H)$.If Lat $(T) \subseteq$ Lat (T^*),then T is reductive.
Remark 3.4.7 [26]The class of reducible operators contains the class of reductive operators.
However, an operator may be reducible but fail to be reductive.
The following example gives an illustration of Remark 3.4.7.
Example 3.4.8

Let $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus 1$ and hence T is reducible

A simple computation shows that

Lat (T)= $\left\{ \{0\}, \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle, \mathbb{R}^3 \right\}$

Lat(T^*)= $\left\{ \{0\}, \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle, \mathbb{R}^3 \right\}$

Thus Lat (T) $\neq$Lat (T^*) which implies that T is not reductive.

Thus ,Reductive$\subseteq$Reducible.

We note that every self adjoint operator(and by extension ,normal operators on a finite dimensional Hilbert space) is reductive.

Theorem 3.4.9 Let $T \in B(H)$ be a normal operator. Then Lat(T)$\equiv$Lat(T^*).

Example 3.4.10 Let

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus 1$$

and hence T is reductive.

A simple computation shows that

$$Lat(T) = \left\{ \{0\}, \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle, \mathbb{R}^3 \right\}$$

while

$$Lat(T^*) = \left\{ \{0\}, \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle, \mathbb{R}^3 \right\}$$

Thus $Lat(T) \neq Lat(T^*)$ and thus T is not reductive.

A simple computation shows that T is not normal and we thus have that the class of reductive operators contains the class of normal operators. Thus the following inclusion holds true

Normal$\subseteq$Reductive$\subseteq$Reducible.

We note that the inclusion above is strict, that is, not every reductive operator is normal.

The following result gives the condition under which a normal operator is reducible.

Theorem 3.4.11 A reductive operator is normal if and only if it has a nontrivial subspace.

Example 3.4.12 Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

acting on $\mathbb{R}^2$

A simple computation shows that

Lat A=$\left\{ \{0\}, span \begin{bmatrix} 1 \\ 0 \end{bmatrix}, Span \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbb{R}^2 \right\}$=Red$(A)$

and Lat (B)=$\left\{ \{0\}, Span \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbb{R}^2 \right\} \neq \{\{0\}, \mathbb{R}^2\}$=Red$(B)$

Thus A is reductive while B is not since not every invariant subspace of B reduces B. Another computation shows that

$$\{B\}' = \left\{ X : X = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

and

$$\{A\}' = \left\{ Y : Y = \begin{bmatrix} \alpha & \beta \\ \gamma & \lambda \end{bmatrix} : \alpha, \beta, \gamma, \lambda \in \mathbb{R} \right\}$$

hence

HyperLat$(A) = \{\{0\}, \mathbb{R}^2\}$ and

HyperLat(B) =Lat(B).

Theorem 3.4.13 [22] If A is a reductive operator, then A can be written as a direct sum $A = A_1 \oplus A_2$ where A_1 is normal, A_2 is reductive and all invariant subspaces of A_2 are hyper-invariant.

Theorem 3.4.14 [12] Suppose A is a reductive operator such that $A = A_1 \oplus A_2$, then

HyperLat(A_1)$\oplus$HyperLat(A_2) and

Lat(A)=HyperLat(A).

Thus we conclude that for A reductive operator and completely non- normal , then Lat (A)=Lat($\{A\}'$).

Corollary 3.4.15 If A is reductive operator, then every hyperinvariant subspace of A is hyperreducing.

Proof

See [22]

Corollary 3.4.16 If A is a reductive operator, then $HyperLat(A) \subseteq hyperRed(A)$.

Remark 3.4.17 Corollary 3.4.16 says that if A is a reductive operator, then $Lat(\{A\}') = Lat(\{A^*\})$.

Theorem 3.4.18 If $T \in B(H)$ is an invertible reductive operator, then T^{-1} is also reducible.

Proof
Since T is reducible, then from theorem 3.4.14 we have that T can be expressed as

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} = T_1 \oplus T_2$$

with respect to the direct sum decomposition $H = M \oplus M^{\perp}$ where M is a subspace that reduces T and thus invertibility of T implies that of T_1 and T_2.
Thus,

$$T^{-1} = \begin{bmatrix} T^{-1}{}_1 & 0 \\ 0 & T^{-1}{}_2 \end{bmatrix} = T^{-1}{}_1 \oplus T^{-1}{}_2$$

with respect to the direct sum decomposition $H = M \oplus M^{\perp}$.
Corollary 3.4.19 Let $T \in B(H)$ be invertible. If a subspace $M \subseteq H$ reduces T, then M reduces T^{-1}.
Proof
Since M reduces T, we have that $TP_M = P_M T$ where P_M is the orthogonal projection of H onto M . We therefore have by the proof of theorem(above), $T^{-1}P_M = P_M T^{-1}$ and hence our result follows.
Remark 3.4.20 From the result of Theorem 3.4.18 and Corollary 3.4.19, it follows that for an invertible operator $T \in B(H)$, then $Red(T) = Red(T^{-1})$.
Theorem 3.4.21 Let $T \in B(H)$ be an operator and $M \subseteq H$. Then the following are true for any integer $n > 1$
(i)If $M \in lat(T)$ then $M \in Lat(T^n)$.
(ii) If $M \in Red(T)$ then $M \in Red(T^n)$.
(iii) If $M \in HyperLat(T)$ then $M \in HyperLat(T^n)$.

3.5 Hyper-invariant and Hyper-reducing Subspaces

We recall that a subspace $M \subseteq H$ is said to be hyper-reducing for $T \in B(H)$ if M reduces every operator in the communtant of T and M is said to be hyperinvariant for T if it is invariant under any operator that commutes with T.
In this section, we investigate hyper-invariant and hyper-reducing subspaces of some operators in Hilbert space.
Theorem 3.5.1 Let $T \in B(H)$. Then HyperRed(T)=Lat($\{T\}'$)∩Lat($\{T\}'$).
Proof
HyperRed(T)=$\{M \subseteq H : M \in Red(\{T\}'\}$
=$\{M \subseteq H : SM \subseteq M, S^*M \subseteq M, S \in \{T\}'\}$
$= \{M \subseteq H : M \in Lat(S) \cap Lat(S^*), S \in \{T\}'\}$
$= Lat(\{T\}') \cap Lat(\{T^*\})$.
Theorem 3.5.2 If $T \in B(H)$. Then HyperRed(T)=HyperLat(T)∩ HyperLat(T^*).

Proof

Follows easily from Theorem 3.5.1 and the fact that Lat($\{T\}'$) =HyperLat(T) and Lat($\{T^*\}'$) =HyperLat(T^*), for any operator $T \in B(H)$.

Corollary 3.5.3 If $T \in B(H)$ is self adjoint. Then HyperRed(T)=HyperLat(T).

Theorem 3.5.4 [3] Let $T \in B(H)$ be normal. Then $HyperLat(T) = \{M \subseteq H : P_M \in W^*(T)\}$.

Corollary 3.5.5 Let $T \in B(H)$ be normal. Then every hyperinvariant subspace of T is hyperinvariant for T^*.

Corollary 3.5.5 says HyperLat(T) $\subseteq$HyperLat(T^*) for any normal operator $T \in B(H)$.

The converse of Corollary 3.5.5 is true. This leads to the following result.

Corollary 3.5.6 If T$\in B(H)$ is normal, HyperLat (T) =HyperLat(T^*).

Proof

Since T is normal if an only if T^* is normal, the result follows from the fact that $T^* \in T'$ if and only if $T \in T^{*\prime}$.

4 EQUIVALENCE OF OPERATORS AND SUBSPACE LATTICES

In this chapter, equivalence of operators is discussed and we also show how lattices of some equivalent operators relate to each other.

4.1 Unitary and similarity of operators

We recall that two operators $A, B \in B(H)$ are said to be unitarily equivalent if there exists a unitary operator $U \in B(H)$ such that $UA = BU$ (or equivalently $A = U^*BU$) and are similar if there exists an invertible operator $U \in B(H)$ such that $UA = BU$.

Theorem 4.1.1 If $T \in B(H)$ is a normal operator and $S \in B(H)$ is unitarily equivalent to T, then S is normal.

Proof

Suppose S is unitary equivalent to T. Then by definition, there exists a unitary operator $U \in B(H)$ such that $S = U^*TU$. Taking adjoints both sides, we have
$S^* = UT^*U^*$ and hence

$$S^*S = (U^*T^*U)(U^*TU) = U^*T^*TU = SU^*T^*U = SU^*US^* = SS^*.$$

which proves the normality of S.

Theorem 4.1.2 If $S, T \in B(H)$ are similar, then S^* and T^* are similar.

Corollary 4.1.3 If $S, T \in B(H)$ are unitary equivalent, then S and T are similar.

Remark 4.1.4 The converse of corollary 4.1.3 is not true in general. The following result shows the conditions under which it is true.

Proposition 4.1.5 If $S, T \in B(H)$ are similar and A is self-adjoint, then S and T are unitarily equivalent.

Proposition 4.1.6 If $T, S \in B(H)$ are similar normal operators, then S and T are unitarily equivalent.

Proof

Follows easily from Proposition 4.1.5.

Corollary 4.1.7 If $S, T \in B(H)$ are normal operators, then S is unitarily equivalent to T if and only if S is similar to T.

Theorem 4.1.8 Similarity of operators preserve non trivial invariant and non trivial hyper invariant subspaces.

Proof

We prove the case for invariance since the proof for hyperinvariance can be proved similarly.

Suppose $A, B \in B(H)$ are similar. Then there exists an invertible operator $X \in B(H$ such that $A = X^{-1}BX$.

Suppose M is a non trivial $A-$invariant subspace. Then $BXM = XAM \subseteq XM$.

Since M is non-trivial and X is invertible, we conclude that XM is a non-trivial invariant subspace for B.

Thus M is $A-$invariant if and only if M is $B-$invariant.

Remark 4.1.9 Similarv operators need not have isomorphic invariant(hyper invariant) lattices.

Example 4.1.10 Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

A simple computation shows that A and B are similar. However, another computation shows that

$Lat(A) = \left\{ \{0\}, \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle, \mathbb{R}^2 \right\}$ and Lat(B) $\left\{ \{0\}, \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle, \mathbb{R}^2 \right\}$.

Thus, Lat(A) and Lat(B) are not isomorphic.

Let $M =$Span$\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$ and $N =$Span $\left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$, figure 5 is a hasse diagram of Lat(A) and Lat(B) respectively.

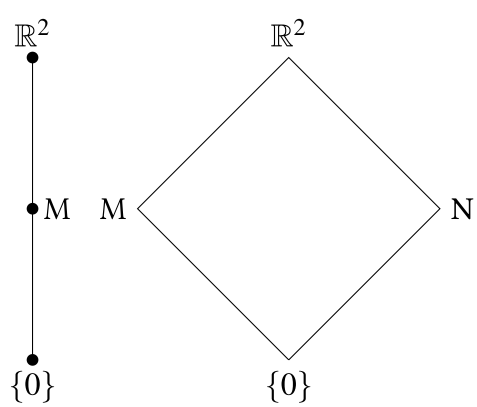

Figure 5. Lat(A) and Lat(B) respectively

4.2 Almost similarity of operators

We recall that two operators $A,B \in B(H)$ are said to be almost similar(denoted by $\overset{\text{a.s}}{\sim}$) if there exists an invertible operator $N \in B(H)$ such that $A^*A = N^{-1}(B^*B)N$ and $A^* + A = N^{-1}(B^* + B)N$.
The concept of almost similarity was introduced by Jibril in [15] and studied in [27].
The following result shows that unitarily equivalence implies almost similarity.
Proposition 4.2.1 [27] If $A,B \in B(H)$ are unitarily equivalent, then A is almost similar to B.
Proof
Suppose A and B are unitarily equivalent. Then there exists a unitary operator $U \in B(H)$ such that $A = U^*BU$.
By taking adjoints both sides, we get
$A^* = U^*B^*U$.
Thus, $A^*A = U^*B^*UU^*BU + U^*B^*BU = U^{-1}B^*BU$ and
$A^* + A = U^*B^*U + U^*BU = U^*(B^* + B)U = U^{-1}(B^* + B)U$.
Hence $A \overset{\text{a.s}}{\sim} B$.
Remark 4.2.2 The converse of proposition 4.2.1 is not generally true. The following result gives the conditions under which the converse of proposition 4.2.1 holds true.
Lemma 4.2.3 If $A,B \in B(H)$ are almost similar and B is hermitian, then A is hermitian.
Theorem 4.2.4 If $A,B \in B(H)$ are almost similar and A is hermitian,then A and B are unitarily equivalent.
proof
Suppose $A \overset{\text{a.s}}{\sim} B$. Then by definition, there exists an invertible operator $N \in B(H)$ such that
$A^* + A = N^{-1}(B^* + B)N$. Now since A is hermitian, it follows that B is also hermitian by Lemma 4.2.3, and hence
$A = N^{-1}BN$. Which implies that A and B are hermitian and hence normal(Remark 1.2.27) and hence are unitarily equivalent by Proposition 4.1.5.

We now show how the three types of operator equivalences discussed so far in this research are related.
Theorem 4.2.5 Let P and Q be orthogonal projections on a Hilbert Space H. Then the following statements are equivalent
(a) P and Q are almost similar
(b) P and Q are similar
(c) P and Q are unitarily equivalent
Proof
$(a) \implies (b)$
Suppose N is an invertible operator.Then Since P and Q are almost similar, we have $P^*P = N^{-1}(Q^*Q)N$ and

$P^* + P = N^{-1}(Q^* + Q)N$
Since P and Q are orthogonal projections, by their idempotent and self-adjoint properties, we have
$P^*P = N^{-1}(Q^*Q)N \implies P = N^{-1}QN$
and $P^* + P = N^{-1}(Q^* + Q)N \implies 2p = N^{-1}2QN$
$\implies P = N^{-1}QN$
$\implies$ P and Q are similar
$(b) \implies (a)$
Suppose P and Q are similar, then $P = N^{-1}QN$ for some invertible operator N.
Then by the idempotence of P and Q
$P = N^{-1}QN \implies P^2 = N^{-1}Q^2N$
since P and Q self adjoint, we have
$P^*P = N^{-1}(Q^*Q)N$ and the other equality is
$P^* + P = 2P = 2N^{-1}(Q^* + Q)N$
$\implies P^* + P = N^{-1}(Q^* + Q)N$
$\implies$ P and Q are almost Similar
$(b) \implies (c)$
follows easily from Proposition 1.2.27 and Proposition 4.1.6
$(c) \implies (a)$
follows easily from Proposition 4.2.2.

The following result shows that almost similarity of operators have isomorphic invariant subspace lattices.
Theorem 4.2.6 If P and Q are almost similar projections, then Lat$(P) \equiv$ Lat(Q).
Proof
Follows easily from Theorems 4.1.8 and 4.2.5

4.3 Quasisimilarity of operators

We recall that two operators are $A, B \in B(H)$ are said to be quasisimilar (denoted by $A \sim B$) if they are quasiaffine transforms of each other. That is, there exists quasiaffinities $X \in B(H, K)$ and $Y \in B(K, H)$ such that $XA = BX$ and $AY = YB$.
The concept of quasisimilarity was introduced by Sz-Nagy and Foias [33] which is a weaker form of similarity.
The following result result is the case when quasisimilarity implies almost similarity.
Proposition 4.3.1 [34] If $A, B \in B(H)$ are operators and H is a finite dimensional Hilbert space are quasisimilar, then A and B are almost similar.
Proof
Suppose A and B are quasisimilar. Then there exists quasiaffinities $X \in B(H, K)$ and $Y \in B(K, H)$ such that $XA = BX$ and $BY = YA$.

Since in finite dimensional Hilbert spaces similarity is the same as quasisimikarity, we assume without loss of generality that $X = Y$ is unitary. Then we have $X^*X = XX^* = I = X^* = X^{-1}$.
But $A = X^{-1}BX$ which implies that
$A^* = X^*B^*(X^{-1})^* = X^*B^*X$
Now, $A^*A = (X^*B^*X)(X^{-1}BX) = X^*B^*BX = X^{-1}B^*BX$ and
$A^* + A = (X^*B^*X) + (X^{-1}BX) = X^*(B^* + B)X = X^{-1}(B^* + B)X$ and our result follows.
Remark 4.3.2 We recall that quasisimilarity is a weaker form of similarity. The following result shows that similar operators are quasisimilar.
Proposition 4.3.3 If $T, S \in B(H)$ are similar operators, then they are quasisimilar.
Proof
If Suppose $S, T \in B(H)$ are similar. Then, by definition, there exists a quasi-invertible operator $X \in B(K, H)$ such that $XT = SX$ thus $X^{-1}S = TX^{-1}$, where $X^{-1} \in B(K, H)$ which implies that S and T are quasisimilar.
Theorem 4.3.4 [26] Let H and K be Hilbert spaces and let $T \in B(H)$ be quasisimilar to a unitary operator $U \in B(K)$ and $M \subseteq K$
If $M \in Hyperlat(T)$, then $M \in Hyperlat(U)$.
Proof
Suppose that $T \in B(H)$ is quasisimilar to a unitary operator $U \in B(K)$. Then $TX = XU$ and $UY = YT$ for some quasiaffinities $X \in B(H, K)$ and $Y \in B(K, H)$.
If $A \in \{T\}'$, then
$U(YAX) = (UY)(AX) = (YT)(AX)$
$= Y(TAX)$
$= (YA)(TX)$
$= (YA)(XU)$
$(YAX)U$
which implies that $YAX \in \{U\}'$ for every $A \in \{T\}'$.
Thus, M is invariant for YAX and hence for any operator that commutes with U.
Corollary 4.3.5 If $T \in B(H)$ is quasisimilar to a unitary operator $U \in B(K)$, then $Hyperlat(T) \subseteq Red(U)$.
Proof
Follows from the fact that for a unitary operator U, $Hyperlat(U) = Red(U)$.
Remark 4.3.6 From Corollary 4.1.3 and Proposition 4.3.3, we note that unitarily equivalent implies similarity which in turn implies quasisimilarity.

4.4 Metric equivalence of operators

We recall that two operators $A, B \in B(H)$ are said to be metrically equivalent if $|Ax\| = |Bx\|$ (or equivalently, $|< Ax, Ax >|^{\frac{1}{2}} = |< Bx, Bx >|$.
The following result shows that the concept of metric equivalence is stronger than unitary equivalence.

Theorem 4.4.1 Let $S,T \in B(H)$. If S and T are unitarily equivalent, then they are metrically equivalent.
Remark 4.4.2 The converse of Theorem 4.4.1 is not generally true as illustrated in the following example.
Example 4.4.3 Let

$$S = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$T = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

be operators on $H = \mathbb{C}^2$
A simple computation shows that S and T are metrically equivalent but not unitarily equivalent.
Another computation shows that Lat$(S) = \left\{\{0\}, \mathbb{C}^2\right\}$ =Lat(T)
In the following result, we show the conditions under which the converse of Theorem 4.4.1 holds true.
Theorem 4.4.4 Let $S,T \in B(H)$. If S and T are metrically equivalent projections, then they are unitarily equivalent.
Corollary 4.4.5 Let $S,T \in B(H)$ be metrically equivalent operators. Then S and T are unitarily equivalent if and only if they are projections.
Proposition 4.4.6 Let $A \in B(H)$ be an operator. The following statements are equivalent
(i). A is unitarily equivalent to an isometry
(ii). A is metrically equivalent to a unitary operator.
Proof
$(a) \implies (b)$
Suppose V is an isometry and A is unitarily equivalent to V.
Then $A = U^*VU$, for some unitary operator U. Then $A^*A = U^*VVU = U^*U = I$.
$(b) \implies (a)$
Suppose A is metrically equivalent to a unitary operator U.
Then $A^*A = U^*U = I$
Then A is isometric and thus unitarily equivalent to an isometry.
Remark 4.4.7 Metric equivalence of operators does not preserve self-adjointness of operators. That is, metric equivalence of S and T does not imply metric equivalence of S^* and T^*.
Example 4.4.8 Let

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

be operators on H.
A simple computation shows that S and T are metrically equivalent but T is not self adjoint.
Theorem 4.4.9 [28] Metrically equivalent of self adjoint operators on a finite dimensional Hilbert space have a common non-trivial invariant subspace.
Proof
Let $A,B \in B(H)$ be self-adjoint and M,N be non-trivial $A-$invariant and $B-$ invariant subspaces respectively.
Then
$A^*AM \subseteq M \subseteq B^*BN \subseteq N$ and
$B^*BN \subseteq N \subseteq A^*AM \subseteq M$ which implies that $M = N$.
Example 4.4.10 The operator

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

on $H = \mathbb{R}^3$ are metrically equivalent.
A simple computation shows that M=Span$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a non-trivial common invariant subspace for A and B.
Proposition 4.4.11 Two positive operators $A,B \in B(H)$ are metrically equivalent if and only if $A = B$.
Since orthogonal projections are positive, Proposition 4.4.11 can further be specified as follows.
Proposition 4.4.12 Two orthogonal projections $P,Q \in B(H)$ are metrically equivalent if and only if $P = Q$.
Proof
$\Longrightarrow$ Suppose P and Q are metrically equivalent. Then $P^*P = Q^*Q$(by definition)
By self adjoint property of P and Q,we have $P^2 = Q^2$
which implies P=Q
$\Longleftarrow$ suppose P=Q

By their idempotent property,we have $P^2 = Q^2$
$\Longrightarrow$ P.P=Q.Q
But since they are self adjoint
$P^*P = Q^*Q$
$\Longrightarrow$ metrically equivalent.
Remark 4.4.13 Two metrically equivalent projections have equal invariant subspace lattices.
Theorem 4.4.14 If P and Q are metrically equivalent projections, then Lat(P) =Lat(Q).
Proof
Follows immediately from proposition 4.4.12.
Theorem 4.4.15 If P and Q are orthogonal projections on a Hilbert space, then
(a) If P and Q are both trivial, then
Lat(P)$\equiv$ Lat(Q)
(b) if atleast P or Q is non trivial, then
Lat(P)=Lat(Q).
Corollary 4.4.16 If $A, B \in B(H)$ are metrically equivalent positive operators, then Lat(A)=Lat(B).
Remark 4.4.17 Metric equivalence of operators need not preserve reducibility of operators.
Example 4.4.18 The operators

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

acting on $H = \mathbb{R}^3$ are metrically equivalent
A simple computation shows that A is reducible while B is irreducible.
Remark 4.4.19 It is clear that the concept of metric equivalence is stronger than unitary equivalence, similarity and almost similarity in regard of invariant subspace lattice.

4.5 Unitarily quasi-equivalence of opertors

We recall that two operators $A, B \in B(H)$ are unitarily quasi-equivalent(denoted by $A \overset{\text{u.q.e}}{\sim} B$) if there exists a unitary operator $U \in B(H)$ such that
$A^*A = UB^*BU^*$ and $AA^* = UBB^*U^*$
The notion of unitarily quasi-equivalence was introduced by Mahmoud[18] and was investigated by Othman[30] under the near equivalence relation.
Theorem 4.5.1 Unitary quasi-equivalence is an equivalence relation.

Proof
See [29]
Remark 4.5.2 Unitary quasi-equivalence of operators is weaker than unitary equivalence of operators. We show this in the following result.
Theorem 4.5.3 If $S,T \in B(H)$ are unitarily equivalent operators, then they are unitarily quasi-equivalent.
Proof
Suppose $S,T \in B(H)$ are unitarily equivalent. Then there exists a unitary operator $U \in B(H)$ such that $S = UTU^*$.
Then $S^*S = UT^*U^*UTU^* = UT^*TU$ and
$SS^* = UTU^*UB^*U^* = UBB^*U^*$.
Thus $S \overset{\text{u.q.e}}{\sim} T$.
Remark 4.5.4 The following example shows that the converse of Theorem 4.5.3 is not generally true.
Example 4.5.5 Consider operators

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

A simple computation shows that S and T are unitarily quasi-equivalent but not unitarily equivalent.
We now show some conditions under which the converse of Theorem 4.5.3 holds true.
Theorem 4.5.6 If $S,T \in B(H)$ are unitarily quasi-equivalent projections, then S and T are unitarily equivalent.
Corollary 4.5.7 Two projections P and Q are unitarily equivalent if and only if they are unitarily quasi-equivalent.
Remark 4.5.8 We recall that two operators $S,T \in B(H)$ are said to be almost unitarily equivalent if there exists a unitary operator $U \in B(H)$ such that
$A^*A = U^*(B^*B)U$ and $A^* + A = U^*(B^* + B)U$.
In the result below, we show how almost unitarily equivalence, unitarily equivalence and unitarily quasi-equivalence are related.
Corollary 4.5.9 For orthogonal projection operators $P,Q \in B(H)$ the following assertions are equivalent
(i) P and Q are almost unitarily equivalent
(ii) P and Q are unitarily equivalent
(iii) P and Q are unitarily quasi-equivalent.

5 CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

In our study ,we have been able to show the following:
(i) if P and Q are metrically equivalent projections, then Lat(P) =Lat(Q).
(ii) If A and B are metrically equivalent positive operators, then Lat(A) =Lat(B).
(iii) An operator $T \in B(H)$ is reductive if and only if Lat(T) =Red(T).
Thus, the following result hold:
Reductive$\subseteq$ Reducible
(iv) For a self adjoint operator T$\in$ B(H),then HyperLat(T) =HyperRed(T).
(v) If P and Q are almost similar projections,then Lat$(P) \equiv$Lat(Q).
(vi) If T is self-adjoint, Red(T) =Lat(T).
We have also shown that for a unitary operator $T \in B(H)$, HyperLat(T) =Red(T). It has also been shown that the following inclusion holds
Normal$\subseteq$Reductive$\subseteq$Reducible

5.1.1 Recommendations

Lattice theory has grown a considerably great deal since its birth both in its applications and its own intrinsic questions.
It has become a topic of active research in many mathematical areas like in growth and development of functional analysis and other modern areas such as in computer science where lattices are used as an algorithmic tool to solve a wide variety of problems such as in crytography and cryptanalysis.
The notion of equivalence of operators is very useful in solving classical moment problems and the preconditioned iterative solution of linear system that arise from the discretization of uniformly elliptical differential equations.
Equivalence of operators is also applied in the characterization of time delay systems (TDS) in many control processes which are equivalent to a delay-free system (DFS), or to a system with a reduced number of delays.
In particular, metric equivalence of operators is very useful when it comes to the solution of the operator interpolation problem with norm constraint: $Ax = B$ and $\|x\| \leq 1$.

In our research, we were able to show that for a reductive operator $T \in B(H)$, Lat$(T) \subseteq$Lat(T^*) and also if T is reductive, then Lat$\{T\}'$ =Lat$\{T^*\}$.

However, we have failed to show a scenario when Lat(T) =Lat(T^*) for an operator $T \in B(H)$. Hence, a further research on this problem is highly recommended.

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