UNIVERSITY OF NAIROBI
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Construction of a Subclass of Balanced Asymmetrical Factorial Designs

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Declaration

This is my original work and has never been presented for any academic award in any other learning institution.

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Dedication

For Laban Maina Wanyoike and Mary Wairimu Maina for their encouragement and moral support when it was getting done.

Abstract

The main focus of this research is to construct balanced asymmetrical factorial designs in which main effects and higher order interactions are estimated with high efficiencies if not full efficiencies. The specific objectives in this work is to illustrate straight forward procedures for constructing balanced arrays/resolvable balanced incomplete block designs and hence balanced asymmetrical factorial designs.

The available literature has given methods of calculating efficiencies for balanced asymmetrical factorial designs. These methods are not clear and have used the traditional approaches. Therefore in this work we have made a contribution in which we have given a direct method that uses Kronecker product of matrices to evaluate such efficiencies.

Another major contribution is the use of Resolvable balanced incomplete block designs in construction of balanced asymmetrical factorial designs

A notable contribution in this research work is in the construction of transitive arrays which are extensively used in the construction of balanced asymmetrical factorial designs by the use of Latin squares. In literature, such arrays have been constructed by using $t - ply$ transitive groups

An additional contribution in this work is in the construction of resolvable balanced incomplete block designs (BIBD's) that have block size $k = 3$ More specifically we have used the geometry of chords constructed in a circle. We have used resolvable BIBD's of block size $k = 3$ to construct many more balanced asymmetrical factorial designs

This research work has come up with a noble method of constructing balanced arrays/resolvable BIBD's which have been used to construct a wide range of balanced asymmetrical factorial designs.

This research work is however based on the construction of balanced asymmetrical factorial designs that are connected, so the results that we have illustrated in this thesis are not valid in the disconnected case. This calls for suitable modifications of these results to make them applicable to the disconnected case.

In this thesis we have restricted our considerations of balanced asymmetrical factorial designs to one way designs only. These concepts can also be extended to two way designs i.e. designs with rows and columns as blocks

Abbreviations and Notations

BLUE	Best Linear Unbiased Estimator
BAFD	balanced Asymmetrical Factorial Design
BIBD	Balanced Incomplete Block Design
BFD	Balanced Facorial Design
SUB	Symmetrical Unequal Block
GD	Group Divisible
PB	Pairwise Balanced
PBIB	Partially Balanced Incomplete Block
PBAF	Partially Balanced Assymetrical Factorial
BAF	Balanced Asymetrical Factorial
OFS	Orthogonal Factorial Structure
EGD	Extended Group Divisible
PBIBD	Partially Balanced Incomplete Block Design
C - Matrix	Design Matrix
$\otimes$	Kronecker or Tensor Product
$\times$	Symbolic Direct Product
$\mathcal{R}(.)$	Row Space of a Matrix
SS	Sum of Squares
ANOVA	analysis of variance
OA[b,k,v,t;λ]	Orthogonal array with b assemblies, k constraints, v symbols and strength t with index λ
$BA[b, k, v, t]$	Balanced Array with b assemblies, k constraints, v symbols and strength t
$TA[b, k, v, t; \lambda]$	Transitive Array with b assemblies, k constraints, v symbols, and strength t with index λ
$GF(s)$	Galois Field with s elements
$D(r, c, s)$	Difference Scheme with r constraints, c assemblies and s levels with index λ
H_n	Hadarmard matrix of order n
$F(k, s, t)$	Minimal number of runs N in any $OA[N, k, s, t]$
$f(k, s, t)$	Maximal number of runs N in any $OA[N, k, s, t]$
$(GF(s), +)$	Galois Field with s elements and a binary operation $+$
$(GF(s), -)$	Galois Field with s elements and a binary operation $\times$
$BA(T)[k, s, \lambda]$	A Balanced Array with $(ks-1)s\lambda$ assemblies, ks constraints, s symbols and strength 2
$BIBD(s, N, k)$	Balanced Incomplete Block Design with s symbols, N assemblies, and k constraints
FD	Factorial Design
$\binom{n}{r}$	nC$_r$
$DF_\lambda(k, v)$	Difference Family with a constant block size k and v symbols with index λ
Ω	A set of none null binary tuples of the same order
Ω^*	A set of all possible binary tuples of the same order

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Chapter 1

Introduction

This chapter reviews earlier work, gives literature review or work done by previous authors in balanced asymmetrical factorial designs and on the construction of some arrays useful in the construction of these designs. The chapter also gives background information, statement of the problem and states the main objective of the study and the specific objectives

1.1 Background Information

One major area of statistics is balanced factorial designs: their construction, properties, their efficiencies and applications. There are various methods of constructing these balanced factorial designs. There are those that are based on orthogonal arrays, balanced arrays, curvilinear spaces or hyper surfaces and truncated $EG(m, s)$ e.t.c.

In this work we wish to construct efficient balanced asymmetrical factorial designs using transitive arrays, balanced arrays and balanced Incomplete block designs that are resolvable.

Historical background of balanced asymmetrical factorial designs goes back to 1937 when Yates used trial and hit methods to obtain confounded plans for experiments of the type $3^m \times 2^n$ where n and m are any positive integers. This work considers many other confounded plans of experiments of the type: $s_1 \times s_2, s_1 \times s_2 \times \ldots \times \times s_m$ where $s_i, i = 1, 2, \ldots, m$ are prime numbers.

1.2 Statement of the Problem

Nair and Rao (1948) and Lewis and Tuck. (1985), Gupta (1987*b*), Suen and Chakravati (1986) are among those who reviewed work on balanced asymmetrical factorial designs. However their ways of constructing balanced arrays/resolvable balanced incomplete block designs were not very clear.

In construction of balanced arrays/resolvable balanced incomplete block designs and hence BAFD's a number of issues arose and need to be addressed.

Thus some of these issues have formed part of the problem statement described below inform of questions

i Only a few BAFD's are obtained using balanced arrays/resolvable balanced incomplete block designs that have been obtained using earlier methods. The question therefore is, is there a direct method that can be used to construct balanced arrays/resolvable balanced incomplete block designs so as to obtain more BAFD's explicitly?

ii Is there a direct method that can be used to calculate efficiencies of BAFD's without necessarily having to use the traditional method of evaluating efficiencies?

1.3 Objectives

1.3.1 Main Objective

Our main objective is to construct a subclass of Balanced Asymmetrical Factorial Designs.

1.3.2 Specific Objectives

a. To construct balanced asymmetrical factorial designs via

  i Transitive arrays

  ii Balanced arrays

  iii Resolvable incomplete block designs

b. To evaluate the efficiencies of the designs constructed in *a* above

1.4 Literature Review

In a factorial design, an interaction will be said to be balanced if either

a.) All treatment contrasts belonging to the same interaction are estimable and the best linear unbiased estimators (BLUE's) of all normalised contrasts belonging to that same interaction have the same variance; or

b.) No contrast belonging to a given interaction is estimable. The trivial solution (b) has been included mainly for Mathematical completeness; this situation will never arise if in particular, the design is connected.

An interpretation of balance which is useful in practice is as follows:

> In a factorial design, an interaction is balanced in the sense of (a) above if and only if all treatment contrasts belonging to this interaction are estimable and the BLUE's of every two mutually orthogonal contrasts belonging to the said interaction are uncorrelated.

Yates (1937*a*), by trial and hit methods obtained confounded plans for experiments of the type $3^m \times 2^n$, where m and n are any positive integers.

Li (1944) also employed similar methods and obtained plans for confounding in asymmetrical factorial experiments 4×2^2, 5×2^2, $4\times3\times2$, $4^2\times2$, 4×3^2, $4^2\times3$ and $4^2\times2$.

It was however the work of Nair and Rao (1941, 1942, 1948) which yielded a number of useful plans. They developed a set of sufficient combinatorial conditions for this purpose.

Bose and Kishen (1940) developed the method of finite geometries for solving the problem of confounding in the general symmetrical factorial design s^m, where s is a prime positive integer or a power of a prime and m any positive integer.

Kishen and Srivastava (1959*a*,*b*) constructed confounded balanced asymmetrical factorial designs (BAFDs) of the type $s_1 \times s_2 \times \cdots \times s_m$, where s_1 is a prime positive integer or a power of a prime, m any positive integer, the s_i's $(i = 1, 2, \ldots, m)$ are not all equal $(s_i \leq s_1, \forall i = 2, 3, \ldots, m)$.

Shah (1958, 1960*a*) gave the exact definition of a balanced factorial experiment.

To overcome the drawbacks of using finite geometries, Kishen and Srivastava (1959*c*,*b*) proposed the utilisation of Balanced Incomplete Block (BIB) designs.

Kishen (1960), Kishen and Tyagi (1961, 1964*a*), Tyagi (1971*b*, 1972) developed the method of constructing confounding plans for BAF designs with the help of Incomplete Block designs. They not only made extensive use of BIB but fully used the balancing properties of Symmetrical Unequal Block (SUB) designs developed by Bose and Kishen (1940) and pairwise balanced (PB) designs.

Tyagi (1972) has also used the GD designs to construct BAF designs.

The use of SUB and PB designs in getting confounded plans for BAF designs has led to considerable economy in the experimental resources. There are however, situations where the experimental resources are very scarce and only two or three replications are desired. To meet such contingencies, Kishen and Tyagi (1961) suggested the use of partially balanced incomplete block (PBIB) designs as developed by Bose and Nair (1939)

Kishen and Tyagi (1964*b*) discussed in detail the construction of partially balanced asymmetrical factorial (PBAF) designs of the type $q \times 2^2$ and $q \times 3^2$ associated to PBIB designs and those derivable by the use of pseudofactors. Thus, the use PBIB has enabled them to obtain 12×2^2 PBAF design in 3 replications only, and a large number of $q \times 3^2$ $(q = 4, 6, 8, 9, 10, 12)$ PBAF designs in 2 replications only.

The reduction in the number of replications was achieved by the use of PB or PBIB designs in the case of asymmetrical designs of the class $q \times 2^2$ and $q \times 3^2$. However the problem of having blocks of smaller size covering a wide class of BAF designs with varying block sizes was still there. This has been tackled by Tyagi (1971*b*) by use of group divisible (G.D) designs with parameters $v, b, r, k, \lambda_1, \lambda_2, m, n$ for constructing BAF designs of the type $m \times n$ in blocks of size k. As a number of the GD designs are available in two or three replications, the corresponding BAF designs are also obtained with three replications. Some more confounded plans associated to hierarchical group divisible designs have also been obtained.

Das (1960*b*) used the method of fractional replication of symmetrical factorial designs to obtain BAFDs. He also gave the criteria of choosing suitable interactions confounded in each replication so that the complete design becomes balanced and is easily amenable to statistical analysis.

Das and Rao (1967) gave an alternative series of confounded asymmetrical factorial design with factors at two and three levels which confound interactions involving linear and quadratic components of the factors.

Banerjee and Das (1969) have also developed further methods for constructing BAF designs derivable from the 2^n series.

In order to obtain confounded asymmetrical factorial designs with the smallest number of replications which can provide mutually orthogonal estimates of all effects, Banerjee (1968) advanced a method of construction of such designs by linking the main effect and interaction contrasts of suitable groups of factors, each at two levels, forming a symmetrical design. This method provides BAF designs even in one replication.

The problem of using a small number of treatment combinations as possible was solved by Bohra (1970) who obtained $q \times 2^2$ designs in four blocks of $(q+1)$ plots each in a single replication.

Puri and Nigam (1976, 1978), Nigam et al. (1988), Gupta (1988), Gupta and Mkerjee (1989) and Mukerjee and Wu (2006*a*) made use of cyclic or generalised cyclic designs and Kronecker-type products in construction of BAFDs. These designs have orthogonal factorial structure (OFS) and if appropriately used, are capable of ensuring high efficiencies.

Parsad et al. (2007) generated a large number of extended group divisible (EGD) designs for BAF experiments. These designs have orthogonal factorial structure (OFS) with balance. They also generated a catalogue of designs giving efficiency of factorial effects. But all these designs do not ensure that main effects would be estimated with full precision, although their efficiencies are too high.

Parsad et al. (2007) gave a series of EGD designs with three factors in which it is possible to generate BAF designs that estimate main effects with full efficiency. In most of these designs, however, the levels of the first factor are two.

Parsad et al. (2007) gave a method of running a BAF experiment in an incomplete block design. An alternative approach to generate BAF designs with OFS is to employ Kronecker-type products of unstructured block designs popularly known as varietal designs. These alternative approaches were developed by David and Wolock (1965), Dean and John (1975), John (1966, 1973, 1987), Gupta (1983, 1985, 1987*a*), and Mukerjee (1980*b*,*a*), Mukerjee and Wu (2006*b*), Gupta and Mkerjee (1989).

Gupta et al. (2011) proposed a method of construction of resolvable block designs for BAF experiments. These designs have orthogonal factorial structure, have balance, estimate all main effects with full efficiency and have control over the interaction.

Sreenath (2011*b*) gave a general method of obtaining block designs for asymmetrical confounded factorial experiments using block designs for symmetrical factorial experiments. The effects of confounded interactions of the symmetrical factorial, in the context of association schemes and also on the context of connectivity of the asymmetrical factorial are also discussed.

Using orthogonal arrays of strength 2, Nair and Rao (1948) gave methods for constructing *EGD* designs for an $s_1 \times s_2$ experiment in block size s_1 or s_2. These authors also indicated how, starting from these one can construct *EGD* designs involving more than 2 factors in simple cases. Following Nair and Rao (1948), several authors considered various methods for constructing these designs. Thomson and Dick (1951), starting from a basic $s_1 \times s_2$ design in block size s_2 ($s_2 < s_1, s_1$ being a prime number or a power of a prime), obtained three factor designs with the same block size, and number of levels being s_1, s_2 or factors of s_2.

Rao (1956) constructed some series of designs from orthogonal Latin squares for $s_1 \times s_2$ experiments in blocks of size s_1 and $s_2 - 1$ replications. He also gave some designs for $2 \times {s_2}^2$ experiments. Kishen (1958) has given balanced designs with *OFS* of type $s_1 \times 2^2$ and $s_1 \times {s_2}^2$. Kramer and Bradley (1957) and Zelen (1958) used group divisible incomplete block designs which have *EGD* scheme for two factors.

Kishen and Srivastava (1959*a*) gave some general methods for constructing balanced designs with *OFS* for asymmetrical factorial experiments. They extended the methods of finite geometries of Bose and Kishen (1940) by using curvilinear spaces or hypersurfaces and truncating the $EG(m, s)$ suitably and by using vectors in Galois fields. They illustrated their methods by constructing the following series of designs: (i)${s_1}^2 \times s_2$ design in blocks of size $s_1 s_2$, balanced in $(s_1 - 1)$ replications $(s_1 > s_2)$, (ii)$s_1 \times s_2 \times s_3$ design in blocks of size $s_2 s_3$, balanced in $(s_1 - 1)$ replications $(s_1 \geq s_2 s_3)$, (iii) designs for experiments where the number of levels is a prime number or a power of a prime number; (iv)$s_1 \times s_2 \ldots \times s_m$ design in blocks of size $s_1 s_3 s_4 \ldots s_m$ where s_2 is a factor of $s_1 s_3 s_4 \ldots s_m$ and is a prime number or a power of a prime number ($s_2{}^2 \geq s_i, i \neq 2$). Several other series of designs were also given by them. Das (1960*a*) gave a method of construction for asymmetrical factorials by linking them with the fractions of suitable Symmetrical factorials. Das and Giri (1986) have discussed this method in details and also gave several examples. Tharthare (1965) gave a class of balanced designs with OFS for $s_1 \times s_2{}^m$ experiments. Similar designs were obtained by Kishen and Tyagi (1964*a*) using pairwise balanced designs of Bose et al. (1960). Shah (1960*b*) gave a method of construction for $s_1 + s_2$ factor experiment using balanced designs with OFS in s_1 and $s_2 + 1$ factors respectively. Shah (1960*b*), Kishen and Tyagi (1963) constructed a 5×2^2 design in 10 blocks of size 2 each. An alternative design for this experiment can also be obtained by using the method of Tharthare (1965). Muller (1966) developed designs for $s_1 \times s_2 \times \ldots \times s_m$ experiments where m_1 is a prime or a power of a prime number. His procedure is to replace each factor except the first one by one or more pseudofactors each at s_1 levels. He also considered the use of balanced incomplete block designs for the construction of $s_1 \times s_2$ balanced factorials with OFS, when $m_1 > m_2$. Further construction procedures were suggetsed by Tyagi (1971*a*), Aggarwal (1974).

Among the more recent authors, Lewis and Tuck. (1985) gave some designs with block size 2 while Gupta (1987*a*) presented an algorithm for obtaining a class of EGD designs.

Suen and Chakravati (1986) constructed several series of two factor balanced designs with OFS using balanced arrays of strength 2. Gupta et al. (2011) purposed unified methods of construction of resolvable block designs for factorial experiments. These designs have orthogonal factorial structure, have balance, estimate all main effects with full efficiency and have control over interaction efficiencies.

Sreenath (2011*a*) proposed a general method for obtaining block designs for asymmetrical confounded factorial experiments, using block designs for symmetrical factorial experiments.

Rajarathinam et al. (2014) utilized the methods used in constructing variance balanced designs in order to construct variance balanced block designs that are highly efficient. They proposed two methods for their construction. Specifically they used incidence matrices and also 2^m symmetrical factorial designs.

Ghosh et al. (2018) also extended on the work of Rajarathinam et al. (2014) in the construction of variance balanced block designs using methods used in construction of variance balanced designs i.e. designs which are balanced and in which the BLUE's of every two mutually orthogonal contrasts are uncorrelated

The purpose of this thesis is to use the methods used in construction of variance balanced designs in order to construct variance balanced asymmetrical factorial designs that posses an additional property known as orthogonal factorial structure(OFS). The methods used will involve utilization of well known arrays viz orthogonal arrays, balanced arrays and transitive arrays. Specifically we shall construct variance balanced asymmetrical factorial designs with OFS where two or more factors are involved and in which main effects and lower order interactions are estimated with high efficiencies. However, we shall restrict designs to those with the same replication.

Chapter 2

Treatment Contrasts (TC) and their Relevance in defining Balance and Orthogonal Factorial Structure(BOFS)

This chapter shows the relevance of treatment contrasts in defining balance and orthogonal factorial structure

2.1 Treatment Contrasts

1. There are v treatments, each replicated r times.

2. There are b blocks, each having k plots.

3. No treatment occurs more than once in a block.

The fixed effect model is assumed:

$$y_{ij} = \mu + \Psi_i + \beta_j + \varepsilon_{ij} \qquad \text{where } i = 1, \ldots, v, j = 1, \ldots, b \qquad (2.1.1)$$

y_{ij} is the yield of the i^{th} treatment applied to the j^{th} block, μ is the overall effect, β_j is the effect of the j^{th} block, Ψ_i is the effect of the i^{th} treatment, ε_{ij} is the experimental error. ε_{ij}'s are independent normal distributions with mean 0 and variance σ^2.

An experiment involving $m \geq 2$ factors $F_1, F_2, \ldots, F_m$ that appear at $s_1, \ldots, s_m (\geq 2)$ levels is called an $s_1 \times \cdots \times s_m$ factorial experiment (or an $s_1 \times \cdots \times s_m$ factorial for brevity).

In particular, if $s_1 = \cdots = s_m$, it is called symmetrical s^m factorial; otherwise it is called an asymmetrical factorial.

For $1 \leq i \leq m$, the s_i levels of the i^{th} factor F_i are denoted by s_i symbols. Suppose that these levels are coded as $0, 1, \ldots, s_i - 1$, then a typical treatment combination, i.e. a combination of the levels of the m factors will be represented by an ordered $m-$ tuple $j_i, \ldots, j_m$ where $j_i \in \{0, 1, \ldots, s_i - 1\}$, $1 \leq i \leq m$ clearly, altogether there are $\prod_{i=1}^{m} s_i$ treatment combinations.

For example, if there are three factors at two, three and three levels respectively, then $m = 3$, $s_1 = 2$, $s_2 = 3$ and $s_3 = 3$, and there are 18 treatment combinations, namely,

$$\begin{gathered} 000, 001, 002, 010, 011, 012, 020, 021, 022, \\ 100, 101, 102, 110, 111, 112, 120, 121, 122 \end{gathered} \tag{2.1.2}$$

Let $\Psi(j_1, \ldots, j_m)$ denote the treatment effect corresponding to a treatment combination $j_1 \ldots j_m$. These treatment effects are unknown parameters in the context of a factorial experiment; a linear parametric function

$$\sum_{j_1=0}^{s_1-1} \cdots \sum_{j_m=0}^{s_m-1} \ell(j_1 \cdots j_m) \Psi(j_1 \cdots j_m) \tag{2.1.3}$$

where $\ell(j_1 \cdots j_m)$ are real numbers, not all zero, such that

$$\sum_{j_1=0}^{s_1-1} \cdots \sum_{j_m=0}^{s_m-1} \ell(j_1 \cdots j_m) = 0 \qquad (2.1.4)$$

is called a treatment contrast.

In factorial experiments, we are concerned with special type of treatment contrasts, namely those belonging to factorial effects.

To motivate the ideas, consider a 3×4 BAF experiment. Then there are two factors F_1 and F_2. The first factor is at three levels 0, 1, 2 while the second factor is at four levels 0, 1, 2, 3. We have twelve treatment combinations given by 00, 01, 02, 03, 10, 11, 12, 13, 20, 21, 22, 23.

The effect of changing the factor F_1 from a given level to a different level with factor 2 held fixed at level 0 is clearly given by

$$L(F_1|F_2 = 0) = \Psi(10) - \Psi(00) + \Psi(20) - \Psi(00) + \Psi(20) - \Psi(10) \quad (2.1.5)$$

The effect of changing the factor F_1 from a given level to a different level with factor F_2 held fixed at level 1 is clearly given by

$$L(F_1|F_2 = 1) = \Psi(11) - \Psi(01) + \Psi(21) - \Psi(01) + \Psi(21) - \Psi(11) \quad (2.1.6)$$

It also follows that the effect of changing the factor F_1 from a given level to a different level with factor F_2 held fixed at level 2 is clearly given by

$$L(F_1|F_2 = 2) = \Psi(12) - \Psi(02) + \Psi(22) - \Psi(02) + \Psi(22) - \Psi(12) \quad (2.1.7)$$

and that the effect of changing the factor F_1 with factor F_2 held fixed at level 3 is given by

$$L(F_1|F_2=3)=\Psi(13)-\Psi(03)+\Psi(23)-\Psi(03)+\Psi(23)-\Psi(13) \quad (2.1.8)$$

Thus the main effect of F_1 is measured by the arithmetic mean of the 12 quantities (2.1.5), (2.1.6), (2.1.7) and (2.1.8) respectively which is given by

$$\begin{aligned}
L(F_1) = {} & \frac{1}{12}\{\Psi(10)-\Psi(00)+\Psi(20)-\Psi(00)+\Psi(20)-\Psi(10)+\Psi(11) \\
& -\Psi(01)+\Psi(21)-\Psi(01)+\Psi(21)-\Psi(11)+\Psi(12)-\Psi(02) \\
& +\Psi(22)-\Psi(02)+\Psi(22)-\Psi(12)+\Psi(13)-\Psi(03)+\Psi(23) \\
& -\Psi(03)+\Psi(23)-\Psi(13)\} \\
& =\frac{1}{12}\{-2\Psi(00)+2\Psi(20)-2\Psi(01)+2\Psi(21)-2\Psi(02) \\
& +2\Psi(22)-2\Psi(03)+2\Psi(23)\}
\end{aligned} \quad (2.1.9)$$

$$\begin{aligned}
= {} & -\frac{1}{6}\Psi(00)-\frac{1}{6}\Psi(01)-\frac{1}{6}\Psi(02)-\frac{1}{6}\Psi(03) \\
& +0\Psi(10)+0\Psi(11)+0\Psi(12)+0\Psi(13)+\frac{1}{6}\Psi(20) \quad (2.1.10) \\
& +\frac{1}{6}\Psi(21)+\frac{1}{6}\Psi(22)+\frac{1}{6}\Psi(23)
\end{aligned}$$

$$= \sum_{j_1=0}^{s_1-1}\sum_{j_2=0}^{s_2-1}\ell(j_1j_2)\Psi(j_1j_2) \quad (2.1.11)$$

is a linear parametric function where $\ell(j_1 j_2)$ are real numbers not all zero such that

$$\begin{aligned}
&\ell(00) + \ell(01) + \ell(02) + \ell(03) + \ell(10) + \ell(11) + \\
&\ell(12) + \ell(13) + \ell(20) + \ell(21) + \ell(22) + \ell(23) \\
&= \sum_{j_1=0}^{s_1-1} \sum_{j_2=0}^{s_2-1} \ell(j_1 j_2) \\
&= -\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} + 0 + 0 + 0 + 0 + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\
&= 0 \qquad (2.1.12)
\end{aligned}$$

where

$$\begin{aligned}
\text{where} \quad & \ell(00) = \ell(01) = \ell(02) = \ell(03) = -\frac{1}{6} \\
\text{and} \quad & \ell(10) = \ell(11) = \ell(12) = \ell(13) = 0 \\
\text{while} \quad & \ell(20) = \ell(21) = \ell(22) = \ell(23) = +\frac{1}{6} \qquad (2.1.13)
\end{aligned}$$

clearly $\ell(00), \ell(01), \ell(02), \ell(03), \ell(10), \ell(11), \ell(12),$ $\ell(13), \ell(20), \ell(21), \ell(22), \ell(23)$, add up to zero, i.e. satisfy (2.1.4) thus $\ell(F_1)$ is a treatment contrast that measures the main effect of F_1.

Now, the effect of changing factor 2 from a given level to a different level, given that factor 1 is fixed at level 0 is given by

$$\begin{aligned}
L(F_2|F_1 = 0) = & \Psi(00) - \Psi(01) + \Psi(00) - \Psi(02) + \Psi(00) - \Psi(03) \\
& + \Psi(01) - \Psi(02) + \Psi(01) - \Psi(03) + \Psi(02) - \Psi(03) \qquad (2.1.14)
\end{aligned}$$

Similarly, the effect of changing factor 2 from a given level to a different level, given that factor 1 is fixed at level 1 is given by

$$\begin{aligned} L(F_2|F_1 = 1) =& \Psi(10) - \Psi(11) + \Psi(10) - \Psi(12) + \Psi(10) - \Psi(13) \\ &+ \Psi(11) - \Psi(12) + \Psi(11) - \Psi(13) + \Psi(12) - \Psi(13) \end{aligned} \tag{2.1.15}$$

and that the effect of changing factor 2 from a given level to a different level, given that factor 1 is fixed at level 2 is given by

$$\begin{aligned} L(F_2|F_1 = 2) =& \Psi(20) - \Psi(21) + \Psi(20) - \Psi(22) + \Psi(20) - \Psi(23) \\ &+ \Psi(21) - \Psi(22) + \Psi(21) - \Psi(23) + \Psi(22) - \Psi(23) \end{aligned} \tag{2.1.16}$$

Hence the main effect of F_2 is measured by the arithmetic mean of the 18 quantities in (2.1.14),(2.1.15),(2.1.16) given by

$$
\begin{aligned}
L(F_2) =& \frac{1}{18}\{\Psi(00) - \Psi(01) + \Psi(10) - \Psi(11) + \Psi(20) - \Psi(21) \\
& + \Psi(00) - \Psi(02) + \Psi(10) - \Psi(12) + \Psi(20) - \Psi(22) \\
& + \Psi(00) - \Psi(03) + \Psi(10) - \Psi(13) + \Psi(20) - \Psi(23) \\
& + \Psi(01) - \Psi(02) + \Psi(11) - \Psi(12) + \Psi(21) - \Psi(22) \\
& + \Psi(01) - \Psi(03) + \Psi(11) - \Psi(13) + \Psi(21) - \Psi(23) \\
& + \Psi(02) - \Psi(03) + \Psi(12) - \Psi(13) + \Psi(22) - \Psi(23)\} \\
=& \frac{1}{18}\{3\Psi(00) + 3\Psi(10) + 3\Psi(20) + \Psi(01) + \Psi(11) + \Psi(21) \\
& - \Psi(02) - \Psi(12) - \Psi(22) - 3\Psi(03) - 3\Psi(13) - 3\Psi(23)\} \\
=& \frac{1}{6}\Psi(00) + \frac{1}{18}\Psi(01) - \frac{1}{18}\Psi(02) - \frac{1}{6}\Psi(03) \\
& \quad + \frac{1}{6}\Psi(10) + \frac{1}{18}\Psi(11) \\
& - \frac{1}{18}\Psi(12) - \frac{1}{6}\Psi(13) + \frac{1}{6}\Psi(20) \\
& + \frac{1}{18}\Psi(21) - \frac{1}{18}\Psi(22) - \frac{1}{6}\Psi(23) \\
=& \sum_{j_1=0}^{s_1-1} \sum_{j_2=0}^{s_2-1} \ell(j_1 j_2)\Psi(j_1 j_2)
\end{aligned} \tag{2.1.17}
$$

this is a linear parametric function where $\ell(j_1 j_2)$ are real numbers not all zero such that

$$\begin{aligned}
&\ell(00)+\ell(01)+\ell(02)+\ell(03)+\ell(10)+\ell(11)+\\
&\ell(12)+\ell(13)+\ell(20)+\ell(21)+\ell(22)+\ell(23)\\
=&\sum_{j_1=0}^{s_1-1}\sum_{j_2=0}^{s_2-1}\ell(j_1 j_2)\\
=&\frac{1}{6}+\frac{1}{18}-\frac{1}{18}-\frac{1}{6}+\frac{1}{6}\\
&+\frac{1}{18}-\frac{1}{18}-\frac{1}{6}+\frac{1}{6}+\frac{1}{18}-\frac{1}{18}-\frac{1}{6}\\
=&0
\end{aligned} \tag{2.1.18}$$

where

$$\begin{aligned}
\text{where}\quad & \ell(00)=\ell(10)=\ell(20)=\frac{1}{6}\\
\text{and}\quad & \ell(01)=\ell(11)=\ell(21)=\frac{1}{18}\\
\text{and}\quad & \ell(02)=\ell(12)=\ell(22)=-\frac{1}{18}\\
\text{and}\quad & \ell(03)=\ell(13)=\ell(23)=-\frac{1}{6}
\end{aligned} \tag{2.1.19}$$

The $\ell(i,j)$ satisfy(2.1.4) and thus $L(F_2)$ is a treatment contrast that measures the main effect of F_2.

Next, consider the interaction between F_1 and F_2. This is measured by the influence of the level where F_2 is held fixed on the effect of a level change of F_1. Thus

$$
\begin{aligned}
L^*(F_1F_1) =& L(F_1|F_2=0) - L(F_1|F_2=1) \\
& + L(F_1|F_2=0) - L(F_1|F_2=2) \\
& + L(F_1|F_2=0) - L(F_1|F_2=3) \\
& + L(F_1|F_2=1) - L(F_1|F_2=2) \\
& + L(F_1|F_2=1) - L(F_1|F_2=3) \\
& + L(F_1|F_2=2) - L(F_1|F_2=3) \\
=& \Psi(10) - \Psi(00) + \Psi(20) - \Psi(00) + \Psi(20) - \Psi(10) \\
& - \Psi(11) + \Psi(01) - \Psi(21) + \Psi(01) - \Psi(21) + \Psi(11) \\
& + \Psi(10) - \Psi(00) + \Psi(20) - \Psi(00) + \Psi(20) - \Psi(10) \\
& - \Psi(12) + \Psi(02) - \Psi(22) + \Psi(02) - \Psi(22) + \Psi(12) \\
& + \Psi(10) - \Psi(00) + \Psi(20) - \Psi(00) + \Psi(20) - \Psi(10) \\
& - \Psi(13) + \Psi(03) - \Psi(23) + \Psi(03) - \Psi(23) + \Psi(13) \\
& + \Psi(11) - \Psi(01) + \Psi(21) - \Psi(01) + \Psi(21) - \Psi(11) \\
& - \Psi(12) + \Psi(02) - \Psi(22) + \Psi(02) - \Psi(22) + \Psi(12) \\
& + \Psi(11) - \Psi(01) + \Psi(21) - \Psi(01) + \Psi(21) - \Psi(11) \\
& - \Psi(13) + \Psi(03) - \Psi(23) + \Psi(03) - \Psi(23) + \Psi(13) \\
& + \Psi(12) - \Psi(02) + \Psi(22) - \Psi(02) + \Psi(22) - \Psi(12) \\
& - \Psi(13) + \Psi(03) - \Psi(23) + \Psi(03) - \Psi(23) + \Psi(13)
\end{aligned} \tag{2.1.20}
$$

So the interaction between F_1 and F_2 is measured by the arithmetic mean of the 36 quantities above i.e.

$$
\begin{aligned}
L(F_1F_2) =& \frac{1}{36}L^*(F_1F_2) \\
=& \frac{1}{36}\{-2\Psi(00) + 2\Psi(20) + 2\Psi(01) - 2\Psi(21) \\
& - 2\Psi(00) + 2\Psi(20) + 2\Psi(02) - 2\Psi(22) \\
& - 2\Psi(00) + 2\Psi(20) + 2\Psi(03) - 2\Psi(23) \\
& - 2\Psi(01) + 2\Psi(21) + 2\Psi(02) - 2\Psi(22) \\
& - 2\Psi(01) + 2\Psi(21) + 2\Psi(03) - 2\Psi(23) \\
& - 2\Psi(02) + 2\Psi(22) + 2\Psi(03) - 2\Psi(23)\} \\
=& \frac{1}{36}\{-6\Psi(00) - 2\Psi(01) + 2\Psi(02) \\
& + 6\Psi(03) + 0\Psi(10) \\
& + 0\Psi(11) + 0\Psi(12) + 0\Psi(13) + 6\Psi(20) \\
& + 2\Psi(21) - 2\Psi(22) - 6\Psi(23)\} \\
=& -\frac{1}{6}\Psi(00) - \frac{1}{18}\Psi(01) + \frac{1}{18}\Psi(02) + \frac{1}{6}\Psi(03) \\
& + 0\Psi(10) + 0\Psi(11) + 0\Psi(12) \\
& + 0\Psi(13) + \frac{1}{6}\Psi(20) + \frac{1}{18}\Psi(21) \\
& - \frac{1}{18}\Psi(22) - \frac{1}{6}\Psi(23) \\
=& \sum_{j_1=0}^{s_1-1} \sum_{j_2=0}^{s_2-1} \ell(j_1j_2)\Psi(j_1j_2) \qquad (2.1.21)
\end{aligned}
$$

which is also a linear parametric function where $\ell(j_1j_2)$ are real numbers not all zero such that

$$\begin{aligned}
&\ell(00)+\ell(01)+\ell(02)+\ell(03)+\ell(10)+\ell(11)+\ell(12)+\\
&\quad\ell(13)+\ell(20)+\ell(21)+\ell(22)+\ell(23)\\
&=\sum_{j_1=0}^{s_1-1}\sum_{j_2=0}^{s_2-1}\ell(j_1j_2)\\
&=-\frac{1}{6}-\frac{1}{18}+\frac{1}{18}+\frac{1}{6}+0+0+0+0+\frac{1}{6}+\frac{1}{18}-\frac{1}{18}-\frac{1}{6}\\
&=0 \qquad (2.1.22)
\end{aligned}$$

where

$$\begin{aligned}
\ell(00)=&\ell(23)=-\frac{1}{6},\\
\ell(01)=&\ell(22)=-\frac{1}{18},\\
\ell(02)=&\ell(21)=\frac{1}{18},\\
\ell(03)=&\ell(20)=\frac{1}{6},\\
\ell(10)=&\ell(11)=\ell(12)=\ell(13)=0 \qquad (2.1.23)
\end{aligned}$$

Hence the $\ell(j_1j_2)$'s satisfy(2.1.4) thus $L(F_1F_2)$ is a treatment contrast that measures the interaction effect of F_1F_2.

Definition 2.1.1.

A treatment contrast

$$\sum_{j_1=0}^{s_1-1} \cdots \sum_{j_m=0}^{s_m-1} \ell(j_1 \cdots j_m)\Psi(j_1 \cdots j_m)$$

belongs to the factorial effect

$$F_{i_1} \ldots F_{i_g} \qquad (1 \le i_1 < \ldots < i_g \le m\,;\, 1 \le g \le m)$$

if

a) $\ell(j_1 \ldots j_m)$ depends only on $j_{i_1} \ldots j_{i_g}$,

b) writing $\ell(j_1 \ldots j_m) = \bar{\ell}(j_{i_1} \ldots j_{i_g})$, in view of (a) above, the sum of $\bar{\ell}(j_{i_1} \ldots j_{i_g})$ separately over each of the arguments $j_{i_1} \ldots j_{i_g}$ is zero.

A factorial effect $F_{i_1} \ldots F_{i_g}$, as defined above, will be called a main effect if it involves exactly one factor (i.e., $g = 1$) and an interaction if it involves more than one factor (i.e., $g > 1$). Clearly, there are m main effects and $\binom{m}{g}$ g-factor interactions. Thus, the total number of factorial effects in $s_1 \times \cdots \times s_m$ factorial is

$$\binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{m} = 2^m - 1.$$

The order of a factorial effect is the number of factors that it involves. For example, a main effect is of order 1, a two factor interaction is of order 2 and so on. Now, to motivate the ideas further, for a 3×4 BAFD taking $g = 1$ and $i_1 = 1$ in this definition, a treatment contrast belongs to the main effect of F_1 provided that it is of the form

$$\sum_{j_1=0}^{s_1-1} \cdots \sum_{j_m=0}^{s_m-1} \bar{l}(j_1)\Psi(j_1 \cdots j_m) \tag{2.1.24}$$

where

$$\sum_{j_1=0}^{s_1-1} \bar{l}(j_1) = 0 \tag{2.1.25}$$

note that (2.1.24) and(2.1.25) correspond to the requirements (a) and (b) respectively of the definition 2.1.1. Consider now the contrast $L(F_1)$ given in (2.1.10) and observe that it can be expressed as

$$\begin{aligned} L(F_1) =& -\frac{1}{6}[\Psi(00) + \Psi(01) + \Psi(02) + \Psi(03)] \\ &+ 0[\Psi(10) + \Psi(11) + \Psi(12) + \Psi(13)] + \frac{1}{6}[\Psi(20) \\ &+ \Psi(21) + \Psi(22) + \Psi(23)] \end{aligned}$$

hence in compatibility with definition 2.1.1 (a), the coefficients of $\Psi(j_1j_2)$ in $L(F_1)$ depends only on j_1. In other words, $L(F_1)$ is of the form (2.1.24), since the coefficient of $\Psi(j_1j_2)$ in $L(F_1)$ depends only on j_1. We can write

$$\bar{l}(0) = -\frac{1}{6}, \ \bar{l}(1) = 0, \text{ and } \bar{l}(2) = \frac{1}{6}$$

obviously

$$\bar{l}(0) + \bar{l}(1) + \bar{l}(2) = -\frac{1}{6} + 0 + \frac{1}{6} = 0$$

as it should in view of(2.1.25).

Similarly, $L(F_2)$ given in (2.1.17) can be expressed as

$$\begin{aligned} L(F_2) =& \frac{1}{6}[\Psi(00) + \Psi(10) + \Psi(20)] + \frac{1}{18}[\Psi(01) + \Psi(11) + \Psi(21)] \\ &- \frac{1}{18}[\Psi(02) + \Psi(12) + \Psi(22)] - \frac{1}{6}[\Psi(03) + \Psi(13) + \Psi(23)] \end{aligned}$$

hence in compatibility with definition 2.1.1 (a), the coefficients of $\Psi(j_1j_2)$

in $L(F_2)$ depends only on j_2. In other words, $L(F_2)$ is of the form (2.1.24) with

$$\bar{l}(0) = \frac{1}{6}, \bar{l}(1) = \frac{1}{18}, \bar{l}(2) = -\frac{1}{18} \text{ and } \bar{l}(3) = -\frac{1}{6}$$

obviously

$$\bar{l}(0) + \bar{l}(1) + \bar{l}(2) + \bar{l}(3) = \frac{1}{6} + \frac{1}{18} - \frac{1}{18} - \frac{1}{6} = 0$$

as it should in view of(2.1.25).

Turning to the case of the two factor interaction F_1F_2, we take $g = 2$, $l_1 = 1$ and $l_2 = 2$ in definition 2.1.1. The treatment contrast F_1F_2 provided it is of the form

$$\sum_{j_1=0}^{s_1-1} \cdots \sum_{j_m=0}^{s_m-1} \ell(j_1 \cdots j_m)\Psi(j_1 \cdots j_m) = \sum_{j_1=0}^{s_1-1} \sum_{j_2=0}^{s_2-1} \bar{\ell}(j_1 j_2)\Psi(j_1 j_2) \qquad (2.1.26)$$

where

$$\sum_{j_1=0}^{s_1-1} \bar{l}(j_1 j_2) = 0 \qquad \text{for each } j_2 (0 \leq j_2 \leq s_2 - 1) \qquad (2.1.27)$$

and

$$\sum_{j_2=0}^{s_2-1} \bar{l}(j_1 j_2) = 0 \qquad \text{for each } j_1 (0 \leq j_1 \leq s_1 - 1) \qquad (2.1.28)$$

As before (2.1.26) is dictated by requirement (a) of definition 2.1.1 where (2.1.27) and (2.1.28) are dictated by requirement (b). The contrast $L(F_1F_2)$ defined in (2.1.21) can be expressed as

$$L(F_1F_2) = -\frac{1}{6}[\Psi(00) + \Psi(23)] - \frac{1}{18}[\Psi(01) + \Psi(22)] + \frac{1}{6}[\Psi(03) + \Psi(20)] \\ + 0[\Psi(10) + \Psi(11) + \Psi(12) + \Psi(13)] + \frac{1}{18}[\Psi(02) + \Psi(21)]$$

where

$$\bar{l}(00) = \bar{l}(23) = -\frac{1}{6}, \bar{l}(01) = \bar{l}(22) = -\frac{1}{18}, \bar{l}(03) = \bar{l}(20) = \frac{1}{6},$$
$$\bar{l}(01) = \bar{l}(11) = \bar{l}(12) = \bar{l}(13) = 0, \text{ and } \bar{l}(02) = \bar{l}(21) = \frac{1}{18}$$

obviously the sum of these coefficients is equal to zero.

2.2 Kronecker Product Formulation For Factorial Effects.

Continuing with an $s_1 \times \cdots \times s_m$ factorial, we now discuss some basic properties of treatment contrasts belonging to factorial effects. An alternative formulation for such contrasts which is equivalent to definition 2.1.1 but involves Kronecker products of matrices, will be helpful in this context. This formulation was introduced formally by Kurkjian and Zelen (1962, 1963). Some of their ideas were inherent in Zelen (1958) and Shah (1958).

The definition and few elementary properties of the Kronecker product matrices are given here; more details are available in Rao (1973).

If $B_1 = ((b_{ij}^{(1)}))$ and B_2 are matrices of orders $p_1 \times q_1$ and $p_2 \times q_2$ respectively, then the Kronecker product of B_1 and B_2, denoted by $B_1 \otimes B_2$, is a $(p_1 p_2) \times (q_1 q_2)$ matrix defined by $B_1 \otimes B_2 = ((b_{ij}^{(1)} B_2))$ in the partitioned form.

Similarly, the Kronecker product of three matrices B_1, B_2 and B_3 is defined as $B_1 \otimes B_2 \otimes B_3 = B_1 \otimes (B_2 \otimes B_3) = (B_1 \otimes B_2) \otimes B_3$, and so on. The following properties of Kronecker products will be useful in the sequel:

i. For any m matrices $B_1, \ldots, B_m$,

$$(B_1 \otimes \cdots \otimes B_m)' = B_1' \otimes \cdots \otimes B_m'$$

where the prime denotes transpose.

ii. For any m matrices $B_1, \ldots, B_m$,

$$\text{rank}(B_1 \otimes \cdots \otimes B_m) = \prod_{i=1}^{m} \text{rank}(B_i)$$

iii. For any $2m$ matrices $B_{11}, \ldots, B_{1m}, B_{21}, \ldots, B_{2m}$,

$$(B_{11} \otimes \cdots \otimes B_{1m})(B_{21} \otimes \cdots \otimes B_{2m}) = (B_{11}B_{21}) \otimes \cdots \otimes (B_{1m}B_{2m})$$

provided that the ordinary product $B_{1i}B_{2i}$ is well defined for every $i \, (1 \leq i \leq m)$.

We are now in a position to proceed with Kronecker product formulation for treatment contrasts belonging to factorial effects in an $s_1 \times \cdots \times s_m$ factorial. We first write

$$v = \prod_{i=1}^{m} s_i$$

to denote the total number of treatment combinations. Without loss of generality, we assume that the v treatment combinations are arranged lexicographically . For example, if $m = 2$, they are arranged as $00, 01, \ldots, 0\bar{s_2}, 10, 11, \ldots, 1\bar{s_2}, \ldots, \bar{s_1}0, \bar{s_1}1, \ldots, \bar{s_1}\bar{s_2}$ where $\bar{s_1} = s_1 - 1$ and $\bar{s_2} = s_2 - 1$. Let Ψ be a column vector, of order v, with elements given by the treatment effects $\Psi(j_1 \ldots j_m)(0 \leq j_i \leq s_i - 1, \; 1 \leq i \leq m)$, which are lexicographically arranged. Any treatment contrast can then be expressed as $\ell' \Psi$, where ℓ is a non-null $v \times 1$ vector whose elements add up to zero.

Observe that a typical factorial effect $F_{i_1}...F_{i_g}$ can be denoted by $F(y)$, where $y = y_1 \cdots y_m$ is a binary $m-$tuple such that

$$y_i = \begin{cases} 1, & \text{if } i \in \{i_1, \ldots, i_g\} \\ 0, & \text{otherwise} \end{cases} \tag{2.2.1}$$

This establishes a one-to-one correspondence between the set of the $2^m - 1$ factorial effects and the set Ω of the $2^m - 1$ non-null binary $m-$tuples. For example, with $m = 3$, the main effect of F_2 can be denoted by $F(010)$, the interaction F_1F_3 by $F(101)$, and so on. We need some more notation.

For $1 \leq i \leq m$, let 1_i be the $s_i \times 1$ vector with all elements unity, I_i, the identity matrix of order s_i, and M_i an $(s_i - 1) \times s_i$ matrix such that

$$\text{rank}(M_i) = s_i - 1, \; M_i 1_i = \mathbf{0} \tag{2.2.2}$$

These equations do not specify M_i uniquely, but the present discussion does not depend on the specific choice of M_i as long as it satisfies the conditions(2.2.2). For any $y = y_1 \ldots y_m \in \Omega$, the set of non-null binary m-tuples, define

$$M(y) = M_1^{y_1} \otimes \cdots \otimes M_m^{y_m} \tag{2.2.3}$$

where, for $1 \leq i \leq m$,

$$M_i^{y_i} = \begin{cases} 1_i', & \text{if } y_i = 0 \\ M_i, & \text{if } y_i = 1 \end{cases} \tag{2.2.4}$$

It is not hard to see that $M(y)$ involves $m(y)$ rows and v columns, where

$$m(y) = \prod_{i=1}^{m} (s_i - 1)^{y_i} \tag{2.2.5}$$

we state the following theorems without proof:

Theorem 2.2.1.

For any $y = y_1 \ldots y_m \in \Omega$, a treatment contrast $\ell' \Psi$ belongs to the factorial effect $F(y)$ if and only if $\ell' \in \mathcal{R}[M(y)]$ where $\mathcal{R}[M(y)]$ stands for the row space of $M(y)$.

Note that by (2.2.2) and (2.2.4), $M_i^{y_i}$ has full row rank for each i $(1 \leq i \leq m)$. Hence by (2.2.3), $M(y)$ has full row rank for every $y \in \Omega$. Since $M(y)$ has $m(y)$ rows, the following result is evident from Theorem 2.2.1.

Theorem 2.2.2.

For any $y = y_1 \ldots y_m \in \Omega$ the maximal number of linearly independent treatment contrasts belonging to factorial effect $F(y)$ is $m(y)$. Furthermore, the $m(y)$ elements of $M(y)\Psi$ represents a maximal set of linearly independent treatment contrasts belonging to $F(y)$.

The concept of orthogonality of treatment contrasts plays a crucial role in factorial experiments. Two treatment contrasts $\ell^{(1)'}\Psi$ and $\ell^{(2)'}\Psi$ are said to be orthogonal if

$$\ell^{(1)'}\ell^{(2)} = 0 \tag{2.2.6}$$

For example from (2.1.10),(2.1.17), and (2.1.21) any two of the contrasts $L(F_1)$, $L(F_2)$, and $L(F_1F_2)$ are orthogonal to each other since

$$L(F_1) : \ell^{(1)'} = \left[-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, 0, 0, 0, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right]$$

$$L(F_2) : \ell^{(2)'} = \left[\frac{1}{6}, \frac{1}{18}, -\frac{1}{18}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{18}, -\frac{1}{18}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{18}, -\frac{1}{18}, -\frac{1}{6}\right]$$

$$L(F_1F_2) : \ell^{(3)'} = \left[-\frac{1}{6}, -\frac{1}{18}, \frac{1}{18}, \frac{1}{6}, 0, 0, 0, 0, \frac{1}{6}, \frac{1}{18}, -\frac{1}{18}, -\frac{1}{6}\right]$$

The contrasts are mutually orthogonal since they belong to different factorial effects, this is actually a consequence of a more general result as presented below

Theorem 2.2.3.

Any two treatment contrasts belonging to different factorial effects are orthogonal

Proof. In view of (2.2.6) and Theorem 2.2.1 it is enough to show that

$$M(y)M(z)' = 0 \tag{2.2.7}$$

whenever $y = y_1 \ldots y_m$ and $z = z_1 \ldots z_m$ are distinct members of Ω. Now, by (2.2.3)

$$M(y)M(z)' = (M_1^{y_1}(M_1^{z_1})') \otimes \cdots \otimes (M_m^{y_m}(M_m^{z_m})') \tag{2.2.8}$$

If y and z are distinct members of Ω, then $y_i \neq z_i$ for some i. Without loss of generality, let $y_1 \neq z_1$ and suppose $y_1 = 1, z_1 = 0$. then by (2.2.2) and (2.2.4), $M_1^{y_1}(M_1^{z_1})' = 0$ and (2.2.7) follows from (2.2.8). □

Theorems 2.2.2 and 2.2.3 together have an interesting implication. Since a typical treatment contrast is of the form $\ell'\Psi$, where ℓ is a none-null $v\times 1$ vector whose elements add up to zero, clearly the maximal number of linearly independent treatment contrasts (belonging to factorial effects or not) is $v-1$ by equation (2.2.5),

$$v - 1 = \prod_{i=1}^{m} s_i - 1 = \prod_{i=1}^{m}(s_i - 1 + 1) - 1 = \sum_{y\in\Omega} m(y).$$

hence, in view of theorems 2.2.2 and 2.2.3 we reach the satisfying conclusions that treatment contrasts belonging to a factorial effects together span all treatments.

Theorem 2.2.2, in conjunction with equations (2.2.3) and (2.2.4) helps in explicitly describing treatment contrasts belonging to various factorial effects in any given context, to motivate the ideas consider a 2×3×3 BAFD whose treatment combinations have already been given in equation (2.1.2). Here $m = 3$, following equation (2.1.2) the vector Ψ, with lexicographically arranged elements $\Psi(j_1 j_2 j_3)$, is given by
$\Psi = (\Psi(000), \Psi(001), \ldots, \Psi(121), \Psi(122))^{'}$.
Since $s_1 = 2$, $s_2 = s_3 = 3$, we have $1_1 = (1,1)^{'}$, $1_2 = 1_3 = (1,1,1)^{'}$. Also, following equation (2.2.2), one can take

$$M_1 = (-1, 1), M_2 = M_3 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

by (2.2.3) and(2.2.4).
$M(100) = M_1 \otimes 1_2^{'} \otimes 1_3^{'}$,
$M(001) = 1_1^{'} \otimes 1_2^{'} \otimes M_3$,
$M(101) = M_1 \otimes 1_2^{'} \otimes M_3$,
$M(111) = M_1 \otimes M_2 \otimes M_3$,
$M(010) = 1_1^{'} \otimes M_2 \otimes 1_3^{'}$,
$M(110) = M_1 \otimes M_2 \otimes 1_3^{'}$,
$M(011) = 1_1^{'} \otimes M_2 \otimes M_3$

where the matrices M_i and the vectors 1_i are stated above.

By theorem 2.2.2, the elements of $M(100)\Psi$, $M(010)\Psi$, and $M(001)\Psi$ represents maximal sets of linearly independent treatment contrasts belonging to factorial effects $F(100)$, $F(010)$, and $F(001)$ i.e. the main effects of F_1, F_2 and F_3 respectively. Similarly, the elements of $M(110)\Psi, M(101)\Psi, M(011)\Psi$ and $M(111)\Psi$ represents maximal sets of linearly independent treatment contrasts belonging to interactions F_1F_2, F_1F_3, F_2F_3 and $F_1F_2F_3$ respectively.

Lemma 2.2.1.

For any $g(1 \leq g \leq m)$, the row spaces of the matrices $M_1 \otimes \cdots \otimes M_g$ and

$$H_g = \begin{bmatrix} 1_1^{'} \otimes I_2 \otimes & \cdots & \otimes I_g \\ & \vdots & \\ I_1 \otimes & \cdots & \otimes I_{g-1} \otimes 1_g^{'} \end{bmatrix} \tag{2.2.9}$$

are orthogonal compliments of each other.

Proof. To motivate or to give flavour of the basic idea of proof without making the notation too complex, we prove the lemma for $g = 3$ though at the expense of heavier notation, the lemma can be proved similarly for any g. By (2.2.9)

$$H_3 = \begin{bmatrix} 1_1^{'} \otimes I_2 \otimes I_3 \\ I_1 \otimes 1_2^{'} \otimes I_3 \\ I_1 \otimes I_2 \otimes 1_3^{'} \end{bmatrix}$$

for $1 \leq i \leq 3$, let

$$\bar{M}_i = \begin{bmatrix} 1_i^{'} \\ M_i \end{bmatrix} \tag{2.2.10}$$

by (2.2.2), $\bar{M}_i$ is non-singular for every i. Hence, premultiplying H_3 by the non-singular matrix diag$(\bar{M}_2 \otimes \bar{M}_3, \bar{M}_1 \otimes \bar{M}_3, \bar{M}_1 \otimes \bar{M}_2)$ yields

$$\mathcal{R}(H_3) = \mathcal{R}\begin{bmatrix} 1_1^{'} \otimes \bar{M}_2 \otimes \bar{M}_3 \\ \bar{M}_1 \otimes 1_2^{'} \otimes \bar{M}_3 \\ \bar{M}_1 \otimes \bar{M}_2 \otimes 1_3^{'} \end{bmatrix} \tag{2.2.11}$$

where as before $\mathcal{R}(\cdot)$ stands for row space of a matrix. But by(2.2.10),

$$1_1^{'} \otimes \bar{M}_2 \otimes \bar{M}_3 = \begin{bmatrix} 1_1^{'} \otimes 1_2^{'} \otimes 1_3^{'} \\ 1_1^{'} \otimes 1_2^{'} \otimes M_3 \\ 1_1^{'} \otimes M_2 \otimes 1_3^{'} \\ 1_1^{'} \otimes M_2 \otimes M_3 \end{bmatrix}$$

on the basis of similar considerations for $\bar{M}_1 \otimes 1_2^{'} \otimes \bar{M}_3$ and $\bar{M}_1 \otimes \bar{M}_2 \otimes 1_3^{'}$.

It follows from(2.2.11) that

$$\mathcal{R}(H_3) = \mathcal{R}(\tilde{M}), \tag{2.2.12}$$

where

$$\tilde{M} = \begin{bmatrix} 1_1^{'} \otimes 1_2^{'} \otimes 1_3^{'} \\ 1_1^{'} \otimes 1_2^{'} \otimes M_3 \\ 1_1^{'} \otimes M_2 \otimes 1_3^{'} \\ 1_1^{'} \otimes M_2 \otimes M_3 \\ M_1 \otimes 1_2^{'} \otimes 1_3^{'} \\ M_1 \otimes 1_2^{'} \otimes M_3 \\ M_1 \otimes M_2 \otimes 1_3^{'} \end{bmatrix} \tag{2.2.13}$$

Now, by(2.2.10),

$$\begin{bmatrix} \tilde{M} \\ M_1 \otimes M_2 \otimes M_3 \end{bmatrix} = \bar{M}_1 \otimes \bar{M}_2 \otimes \bar{M}_3$$

is non-singular, while by (2.2.2) and (2.2.13)

$$\tilde{M}(M_1 \otimes M_2 \otimes M_3)^{'} = \mathbf{0}$$

hence the row spaces of $\tilde{M}$ and $M_1 \otimes M_2 \otimes M_3$ are orthogonal compliments of each other. Therefore, by (2.2.12), the row spaces of H_3 and $M_1 \otimes M_2 \otimes M_3$ are also orthogonal compliments of each other. □

2.3 Elements and Operations of Calculus for Factorial Arrangements

The calculus for factorial arrangements provides a very powerful tool for expressing the notation in a very compact and convenient form. This calculus was introduced by Kurkjian and Zelen (1962, 1963) although it appears that some of their ideas were also inherent in Zelen (1958) and Shah (1958).

Let $\underline{a}_i = (0, 1, \ldots, m_i - 1)^{'}$, $1 \leq i \leq m$. The v treatment combinations will be considered in the lexicographic order given by $\underline{a}_1 \times \underline{a}_2 \times \cdots \times \underline{a}_m$, where $\times$ denotes symbolic direct product as defined by Shah (1958). Let Ψ be a $v \times 1$ vector, with elements given by the $\Psi(j_1, j_2, \ldots, j_m)$'s arranged lexicographically. For example, if $m = 2, s_1 = 2, s_2 = 3$, then

$$\underline{a}_1 = (0, 1)^{'}, \qquad \underline{a}_2 = (0, 1, 2)^{'}$$

then

$$\underline{a}_1 \times \underline{a}_2 = (00, 01, 02, 10, 11, 12)^{'}$$

and

$$\underline{\Psi} = (\Psi(0,0), \Psi(0,1), \Psi(0,2), \Psi(1,0), \Psi(1,1), \Psi(1,2))^{'}$$

A typical treatment contrast is of the form $\underline{\ell}^{'}\underline{\Psi}$, where the $v{\times}1$ coefficient vector $\underline{\ell}$ is non-null and the sum of elements of $\underline{\ell}$ equal to zero. Such a contrast will be said to be normalised if $\underline{\ell}^{'}\underline{\ell} = 1$. Two treatment contrasts $\underline{\ell}_1^{'}\underline{\Psi}$ and $\underline{\ell}_2^{'}\underline{\Psi}$ will be called mutually orthogonal if $\underline{\ell}_1^{'}\underline{\ell}_2 = 0$. A set of treatment contrasts will be called orthonormal if the contrasts in the set are all normalised and mutually orthogonal.

Let Ω be the set of all m-component non-null binary vectors. It is easy to see that there is a one to one correspondence between Ω and the set of all interactions, in the sense that a typical interaction

$$F_{i_1}F_{i_2}\cdots F_{i_g} \qquad (1 \le i_1 < i_2 < \cdots < i_g \le m, 1 \le g \le m)$$

corresponds to the element $y = (y_1, \ldots, y_m)$ of Ω such that $y_{i_1} = y_{i_2} = \cdots = y_{i_g} = 1$ and $y_u = 0$.

For $u \neq i_1, i_2, \ldots, i_g$. Thus the $2^m - 1$ interactions may be denoted by $F(y)$, $y{\in}\Omega$. For expample, if $m = 2$, then the main effects F_1, F_2 and the 2-factor interaction F_1F_2 may be denoted by $F(10), F(01)$ and $F(11)$ respectively. The treatment contrasts belonging to the interactions may be conveniently represented making use of Kronecker products, as indicated below.

For each $y = (y_1, \ldots, y_m) \in \Omega$, let

$$M^y = M_1^{y_1} \otimes M_2^{y_2} \otimes \cdots \otimes M_m^{y_m} = \otimes_{i=1}^{m} M_i^{y_i} \tag{2.3.1}$$

where $\otimes$ denotes the Kronecker product and for $1 \leq i \leq m$,

$$M_i^{y_i} = \begin{cases} I_i - s_i^{-1} J_i, & \text{if} \quad y_i = 1 \\ s_i^{-1} J_i, & \text{if} \quad y_i = 0 \end{cases} \tag{2.3.2}$$

here I_i is an identity matrix and J_i is a matrix of 1's both of order $s_i \times s_i$.

Lemma 2.3.1.

For each $y \in \Omega$, the elements of $M^y \Psi$ represents a complete set of treatment contrasts belonging to the interaction $F(y)$.

Proof. The proof may now be completed by observing that in view of (2.3.1) and (2.3.2)

$$\text{rank}(M^y) = \prod_{i=1}^{m} \text{rank}(M_i^{y_i}) = \prod_{i=1}^{m} (s_i - 1)^{y_i},$$

which is the same as the maximum number of linearly independent contrasts belonging to $F(y)$. □

Lemma 2.3.2.

Treatment contrasts belonging to any two distinct interactions are mutually orthogonal.

Proof. By lemma 2.3.1, it is enough to show that for every $y = (y_1, \ldots, y_m)$ and $x = (x_1, \ldots, x_m) \in \Omega$, $x \neq y$,

$$M^x M^y = 0 \tag{2.3.3}$$

Now, by (2.3.1) and the standard rules for operations with Kronecker products,

$$M^x M^y = \otimes_{i=1}^m (M_i^{x_i} M_i^{y_i}) \tag{2.3.4}$$

Now, $x \neq y$, implies that $x_i \neq y_i$ for at least one i, and for this i, by (2.3.2) $(M_i^{x_i} M_i^{y_i}) = 0$. Hence (2.3.3) follows from(2.3.4). □

Lemma 2.3.1 gives a representation for the treatment contrasts belonging to the different interactions. Another equivalent representation in terms of orthonormal contrasts is often helpful. For $1 \leq i \leq m$, let 1_i be $s_i \times 1$ vector with all elements unity and p_i be an $(s_i - 1) \times s_i$ matrix such that $s_i \times s_i$ matrix $(s_i^{-\frac{1}{2}} 1_i, P_i^{'})^{'}$ is orthogonal. For example, if $m = 2$, $s_1 = 2$, $s_2 = 3$ i.e. we have a 2×3 asymmetrical factorial then one may take

$$P_1 = \begin{bmatrix} +\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$
$$P_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

For each $y = (y_1, \ldots, y_m) \in \Omega$, let

$$P^y = \otimes_{i=1}^m P_i^{y_i} \tag{2.3.5}$$

where, for $1 \leq i \leq m$

$$P_i^{y_i} = \begin{cases} P_i & \text{if } y_i = 1 \\ s_i^{-\frac{1}{2}} 1_i^{'} & \text{if } y_i = 0 \end{cases} \tag{2.3.6}$$

By equations (2.3.2) and (2.3.6), the relation $P_i^{y_i'} P_i^{y_i} = M_i^{y_i}$ holds for every i, whether y_i equals 0 or 1. Hence, by (2.3.1) and (2.3.5), for every $y \in \Omega$

$$P^{y'} P^{y} = M^{y} \tag{2.3.7}$$

Analogously to (2.3.3), it may be seen that for each
$x, y \in \Omega,\ x \neq y,$

$$P^{x} P^{x'} = I,\ \ P^{x} P^{y'} = 0 \tag{2.3.8}$$

where I is the identity matrix of appropriate order.

Lemma 2.3.3.

For each $y \in \Omega$, the elements of $P^y \Psi$ represents a complete set of orthonormal contrasts belonging to the interaction F^y.

In the sequel, the representations as given in lemmas 2.3.1 and 2.3.3 above, will be found to be useful.

2.4 Orthogonal Factorial Structure and Balance

Consider an arrangement of the $v = \prod s_i$ treatment combinations in a block design involving b blocks of sizes $k_1, k_2, \ldots, k_b$, the j^{th} treatment combination being replicated r_j times. The design will be called proper if $k_1, k_2, \ldots, k_b$ are all equal and called equi-replicate if the r_j's are all equal. The $v \times b$ matrix $N = ((n_{jh}))$ will be termed the incidence matrix of the design where $n_{jh} (\geq 0)$ is the number of times the j^{th} treatment combination occurs in the h^{th} block.

Let

$\underset{\sim}{r} = (r_1, r_2, \ldots, r_v)'$,

$\underset{\sim}{k} = (k_1, k_2, \ldots, k_b)^{'}$,

$r^{\delta} = \text{Diag}(r_1, r_2, \ldots, r_v)$,

r^{δ} is a $v \times v$ diagonal matrix whose main diagonal elements are $r_1, r_2 \ldots r_v$

$k^{\delta} = \text{Diag}(k_1, k_2, \ldots, k_b)$. k^{δ} is a $b \times b$ diagonal matrix whose main diagonal elements are $k_1, k_2 \ldots k_b$

The fixed effects intrablock model with independent homoscedastic errors (The constant error variance being, say σ^2) and no block versus treatment interaction will be assumed. Then it is well known (see e.g. Raghavarao (1971)) that the intrablock reduced normal equations for the vector of treatment effects $\underset{\sim}{\Psi}$ are given by

$$C\underset{\sim}{\Psi} = \underset{\sim}{Q} \tag{2.4.1}$$

where

$$C = r^{\delta} - Nk^{-\delta}N^{'} \tag{2.4.2}$$

is the usual C-matrix of the design, $k^{-\delta} = (k^{\delta})^{-1}$ and $\underset{\sim}{Q}$ is the vector of adjusted treatment totals.

From equation (2.4.2), C is a symmetric matrix with all row sums zero. Hence $\text{rank}(C) \leq v - 1$. A design is called connected if $\text{rank}(C) = v - 1$. A treatment contrast $\underset{\sim}{\ell}^{'}\underset{\sim}{\Psi}$ is estimable if $\underset{\sim}{\ell}^{'} \in \mathcal{R}(C)$, where for any matrix A, $\mathcal{R}(A)$ stands for it's row space. Clearly, for an estimable treatment contrast $\underset{\sim}{\ell}^{'}\underset{\sim}{\Psi}$, there exists a $v \times 1$ vector $\underset{\sim}{\ell}^{*}$ such that $\underset{\sim}{\ell}^{'} = \underset{\sim}{\ell}^{*'}C$. The best linear unbiased estimator (BLUE) of $\underset{\sim}{\ell}^{'}\underset{\sim}{\Psi}$ is given by $\underset{\sim}{\ell}^{'}\hat{\underset{\sim}{\Psi}} = \underset{\sim}{\ell}^{*'}\underset{\sim}{Q}$. All treatment contrasts are estimable if and only if the design is connected.

In analysing the results of a factorial design, the experimenter is primarily interested in drawing conclusion on the contrasts belonging to different interactions. A great simplification occurs in interpreting the results of analysis if the design has orthogonal factorial structure. In the sense of definition 2.4.1 given below. Another use of OFS is realized while constructing confidence intervals for estimable contrasts belonging to different interactions. A discussion on the importance of orthogonal estimation of parameters in constructing confidence intervals can be found in Box and Draper (1987).

Definition 2.4.1.

A factorial design will be said to have orthogonal factorial structure (OFS) if the BLUEs of estimable treatment contrasts belonging to distinct interactions are mutually orthogonal, i.e. uncorrelated.

In other words, by lemmas 2.3.1 and 2.3.3, OFS holds for each $x, y, \in \Omega$, $x \neq y$ if the BLUE of every estimable linear combination of the elements of $M^x \underline{\Psi}$ (or $P^x \underline{\Psi}$) is uncorrelated with the BLUE of every estimable linear combination of the elements of $M^y \underline{\Psi}$ or $P^y \underline{\Psi}$. When this is realised, in the connected case, the adjusted treatment sum of squares (SS) can be split up orthogonally into components due to the different interactions and, as such, these components may be shown in the same analysis of variance (ANOVA) table. The same can be done also in the disconnected case, provided some further conditions hold. Incidentally, in lemma 2.3.2, it was shown that contrasts belonging to distinct interactions are mutually orthogonal. The OFS calls for a reflection of this property in terms of the BLUEs of such contrasts.

Another important and useful concept in the context of factorial design is that of balance. A definition of balance along the line of Shah (1958) is as follows:

Definition 2.4.2.

In a factorial design, an interaction F^y, $y \in \Omega$, will be said to be balanced if either

(a) all treatment contrasts belonging to F^y are estimable and the BLUEs of all normalised contrasts belonging to F^y have the same variance or;

(b) No contrast belonging to F^y is estimable.

A factorial design will be called balanced if F^y is balanced for every $y \in \Omega$ in definition 2.4.2, the trivial situation (b) has been included mainly for mathematical completeness; this situation will never arise if, in particular, the design is connected. The following lemma provides an interpretation for balance which is useful in practice.

Lemma 2.4.1.

In a factorial design, an interaction F^y is balanced in the sense (a) of definition 2.4.2 if and only if all treatment contrasts belonging to F^y are estimable and the BLUEs of every two mutually orthogonal contrasts belonging to F^y are uncorrelated.

Proof. Only if: Suppose that the interaction F^y is balanced in the sense of (a) of definition 2.4.2. Let $\underline{\ell}_1^{'}\underline{\Psi}$ and $\underline{\ell}_2^{'}\Psi$ be two mutually orthogonal treatment contrasts belonging to F^y. Define $\underline{\varepsilon}_i = (\underline{\ell}_i^{'}\underline{\ell}_i)^{-\frac{1}{2}}\underline{\ell}_i, (i = 1, 2)$ and $\underline{\varepsilon} = \frac{1}{\sqrt{2}}(\underline{\varepsilon}_1 + \underline{\varepsilon}_2)$, and note that $\underline{\varepsilon}_1^{'}\underline{\Psi}$, $\underline{\varepsilon}_2^{'}\underline{\Psi}$ and $\underline{\varepsilon}^{'}\underline{\Psi}$ are normalised contrasts belonging to F^y. Since F^y is balanced in the sense of (a) of definition 2.4.2

$$Var(\underline{\varepsilon}^{'}\hat{\underline{\Psi}}) = Var(\underline{\varepsilon}_1^{'}\hat{\underline{\Psi}}) = Var(\underline{\varepsilon}_2^{'}\hat{\underline{\Psi}}) \qquad (2.4.3)$$

again

$$
\begin{aligned}
Var(\underline{\varepsilon}'\underline{\hat{\Psi}}) &= \frac{1}{2}Var(\underline{\varepsilon}_1'\underline{\hat{\Psi}} + \underline{\varepsilon}_2'\underline{\hat{\Psi}}) \\
&= \frac{1}{2}\{Var(\underline{\varepsilon}_1'\underline{\hat{\Psi}}) + Var(\underline{\varepsilon}_2'\underline{\hat{\Psi}}) \\
&+ 2Cov(\underline{\varepsilon}_1'\underline{\hat{\Psi}}, \underline{\varepsilon}_2'\underline{\hat{\Psi}})\}
\end{aligned}
$$

which together with (2.4.3) yields $Cov(\underline{\varepsilon}_1'\underline{\hat{\Psi}}, \underline{\varepsilon}_2'\underline{\hat{\Psi}}) = 0$ and hence, $Cov(\underline{\ell}_1'\underline{\hat{\Psi}}, \underline{\ell}_2'\underline{\hat{\Psi}}) = 0$, as desired.

<u>If:</u> Let the conditions stated in the lemma hold. Consider any two distinct contrasts which are normalised $\underline{\varepsilon}_1'\underline{\hat{\Psi}}$ and $\underline{\varepsilon}_2'\underline{\hat{\Psi}}$ belonging to F^y. If $\underline{\varepsilon}_1 = -\underline{\varepsilon}_2$ then trivially $Var(\underline{\varepsilon}_1'\underline{\hat{\Psi}}) = Var(\underline{\varepsilon}_2'\underline{\hat{\Psi}})$, otherwise, $(\underline{\varepsilon}_1 + \underline{\varepsilon}_2)'\underline{\Psi}$ and $(\underline{\varepsilon}_1 - \underline{\varepsilon}_2)'\underline{\Psi}$ are mutually orthogonal treatment contrasts belonging to F^y, and hence under the condition stated in the lemma
$Cov\{(\underline{\varepsilon}_1 + \underline{\varepsilon}_2)'\underline{\hat{\Psi}}, (\underline{\varepsilon}_1 - \underline{\varepsilon}_2)'\underline{\hat{\Psi}}\} = 0$, which yields $Var(\underline{\varepsilon}_1'\underline{\hat{\Psi}}) = Var(\underline{\varepsilon}_2'\underline{\hat{\Psi}})$. □

The following corollary is an immediate consequence of lemmas 2.3.3 and 2.4.1

Corollary 2.4.1

In a factorial design, an interaction F^y is balanced in the sense (a) of definition 2.4.2 if and only if all treatment contrasts belonging to F^y are estimable and the dispersion matrix of $P^y\underline{\hat{\Psi}}$ is proportional to the identity matrix.

2.5 Balanced Arrays

The concept of orthogonal arrays was first introduced by Rao (1946). They play a vital role in the construction of asymmetrical confounded factorial experiments.Rao (1946), Bose and Bush (1952),Bush (1952),Plackett (1946) and Addelman and Kempthorne (1961) have constructed useful orthogonal arrays. Several authors(Masuyama (1969);Xu (1979);Jungnickel (1979), Xiang (1983)) studied the construction of difference schemes that are used in construction of orthogonal arrays. The recursive construction of orthogonal arrays that makes use of a series of difference schemes is based on the work of several authors, including Shrikhande (1964),Masuyama (1969),Xu (1979) and Mukhopadhyay (1981). Further, information about difference schemes can be found in Beth et al. (1986),Butson (1962, 1963),Dawson (1985) and Jungnickel (1979), Launey (1986), Drake (1979), Dulmage et al. (1961), Jungnickel (1979, 1992) and Seberry (1980). Addelman and Kempthorne (1961), Wang and Wu. (1991), Hedayat et al. (1992), Dey (1985), Wu et al. (1992) used expansive replacement method in construction of mixed orthogonal arrays. While Wang and Wu. (1991), Hedayat et al. (1992), Mendeli (1995), Wang and Wu. (1991), Dey and Midha (1996) constructed mixed orthogonal arrays using difference schemes.

Chakravati (1956) introduced the concept of partially balanced arrays, which generalize the concept of orthogonal arrays. He (1961) constructed partially balanced arrays from tactical configuration and pairwise partially balanced designs. Srivastava and Chopra (1975) made contributions to the theory and construction of partially balanced arrays, renaming them balanced arrays. For some constructions of balanced arrays reference may be made to Chakravati (1961), Srivastava and Chopra (1972), Rafter and Seiden (1974), Sinha and Nigam (1983), and Saha and Samanta (1985), Niishi (1981), Srivastava (1990), Raktoe et al. (1980).

Definition 2.5.1.

A $k \times b$ array with entries from a set of v symbols is called an orthogonal array of strength t if each $t \times b$ subarray of A contains all possible v^t column vectors with the same frequency $\lambda = \frac{b}{v^t}$. It is denoted $OA(b, k, v, t; \lambda)$; the number λ is called the index of the array. The numbers b and k are known as the number of assemblies and constraints of the orthogonal array respectively.

Example 2.5.1. *OA(8,4,2,3;1)*

0	1	1	1	1	0	0	0
1	0	1	1	0	1	0	0
1	1	0	1	0	0	1	0
1	1	1	0	0	0	0	1

Definition 2.5.2.

Let A be a $k \times b$ array with entries from a set of v symbols. Consider the v^t ordered t-tuples $(x_1, \ldots, x_t)$ that can be formed from a t-rowed subarray of A, and let there be associated a non-negative integer $\lambda(x_1, \ldots, x_t)$ that is invariant under permutations of $x_1, \ldots, x_t$. If for any t-rowed subarray of A the v^t ordered t-tuples $(x_1, \ldots, x_t)$, each occur $\lambda(x_1, \ldots, x_t)$ times as a column, then A is said to ba a balanced array of strength t. It is denoted by BA(b, k, v, t) and the numbers $\lambda(x_1, \ldots, x_t)$ are called the index parameters of the array.

Clearly a BA(b, k, v, t) with $\lambda(x_1, \ldots, x_t) = \lambda$ for all t-tuples $(x_1, \ldots, x_t)$ is simply an orthogonal array OA$(b, k, v, t; \lambda)$.

Example 2.5.2. *BA(10,5,2,2)*

0	1	0	1	0	1	0	1	0	1
1	1	1	0	1	1	0	0	0	0
0	0	1	1	1	0	0	0	1	1
1	1	0	0	0	0	1	0	1	1
0	0	0	0	1	1	1	1	1	0

$\lambda(0,0) = \lambda(1,1) = 2$ and $\lambda(0,1) = \lambda(1,0) = 3$

Definition 2.5.3.

An $\text{OA}(\lambda s^2, m, s, 2)$, where $\lambda = \alpha\beta$ is said to be β-resolvable if it is the juxtaposition of αs different $\text{OA}(\beta s, m, s, 1)$. A 1-resolvable array is said to be completely resolvable.

For example if $m = 3$, $s = 3$, $\beta = 1$, $\lambda = 1$ it implies that if $\alpha = 1$ then the OA$[1{\times}3{\times}3, 3, 2; 1]$ = OA[9,3,3,2;1] is $\beta = 1$ resolvable if it is the juxtaposition of $\alpha s = (1)(3) = 3$ different orthogonal arrays OA$[1{\times}3, 3, 1; \lambda]$ = OA[3,3,3,1;λ] which includes

$$OA_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} OA_2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} OA_3 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

The $\beta = 1$ resolvable (completely resolvable) OA[9,3,3,2;1] is however

0	0	0	1	1	1	2	2	2
0	1	2	1	2	0	2	0	1
0	2	1	1	0	2	2	1	0

Definition 2.5.4.

An $\mathrm{OA}(\lambda s^2, m, s, 2)$ is said to be partly resolvable if there exists s assemblies which form an $\mathrm{OA}(s, m, s, 1)$.

A completely resolvable orthogonal array is certainly partly resolvable.

Example 2.5.3. *The completely resolvable orthogonal array OA[9,3,3,2;1] is partly resolvable since we can obtain s=3 assemblies which form an OA(3,3,3;1) that includes*

$$OA_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} OA_2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} OA_3 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

The following example gives a partly resolvable orthogonal array which is not completely resolvable.

Example 2.5.4. *A Partly resolvable OA(12,6,2,2)*

0	1	0	0	0	0	0	1	1	1	1	1
0	1	0	0	1	1	1	0	0	0	1	1
0	1	1	0	0	1	1	1	1	0	0	0
0	1	0	1	1	0	1	0	1	1	0	0
0	1	1	1	1	0	0	1	0	0	0	1
0	1	1	1	0	1	0	0	0	1	1	0

This orthogonal array is partly resolvable since it is the juxtaposition of $\frac{12}{6\times 2} = 1$ orthogonal arrays such that each factor occurs in each of these arrays 6 times at each level

Theorem 2.5.1.

If λ and s are both powers of the same prime p, a completely resolvable $\mathrm{OA}(\lambda s^2, \lambda s, s, 2)$ can always be constructed.

Addelman and Kempthorne (1961) gave a method of constructing an OA($2s^n$, $2(s^n-1)$/(s-1)-1, s, 2) but only for the case $n=2$. Mukhopadhyay (1981) observed that the arrays for $n \geq 3$ can be obtained recursively from those of $n=2$.

Definition 2.5.5.

A transitive array TA$(b,k,v,t;\lambda)$ is a $k \times b$ array of v symbols such that for any choice of t rows, the $\frac{v!}{(v-t)!}$ ordered t-tuples of distinct symbols each occur λ times as a column.

Example 2.5.5. *TA(12,4,4,2;1)*

0	1	2	3	0	1	2	3	0	1	2	3
1	0	3	2	2	3	0	1	3	2	1	0
2	3	0	1	3	2	1	0	1	0	3	2
3	2	1	0	1	0	3	2	2	3	0	1

Bose et al. (1960) constructed TA($v(v-1)$,k,v,2;1) from a set of $k-2$ mutually orthogonal latin squares of order v. Suen (1982) constructed TA($v(v-1)$,v,v,2;1) from doubly transitive groups of order v if v is even. Morgan and Chakravati (1988) showed that b must be a multiple of $2\binom{v}{2} = v(v-1)$ in construction of transitive arrays.

Chapter 3

Methodology of evaluating efficiencies of Balanced Asymmetrical Factorial Designs BAFD's

In this chapter balanced confounded asymmetrical factorial designs with orthogonal factorial structure are considered. Algebraic characterization for balance with orthogonal factorial structure is given and a methodology of evaluating efficiencies of such designs is given

3.1 Algebraic Characterization for Balance With Orthogonal Factorial Structure

This chapter considers asymmetrical factorial designs which are balanced and have orthogonal factorial structure (OFS). Such designs have been termed balanced asymmetrical factorial experiments by Shah (1958, 1960*a*). They are known as balanced confounded asymmetrical designs according to the nomenclature of Nair and Rao (1948). The main result, namely theorem 3.1.1 gives an algebraic characterization for balance with orthogonal factorial structure. For equireplicate and proper designs, the efficiency part of this result was proved by Kurkjian and Zelen (1963) while the necessity part was proved by Kshirsagar (1966). Gupta (1983) considered extensions to designs that are not necessarily equireplicate or proper. The following definitions and lemmas will be helpful

Let $F_1, F_2, \ldots, F_m$ be m factors at $s_1, s_2, \ldots, s_m$ levels respectively and N be the incidence matrix of a BAFD

Definition 3.1.1.

Suppose we have a $C-$ matrix of the design in $v(= s_1 s_2 \ldots s_m)$ treatment combinations, then the design is said to possess property A if

$$C = \sum_{y \in \Omega^*} g(y)(J_1 - I_1)^{y_1} \otimes (J_2 - I_2)^{y_2} \otimes \ldots \otimes (J_m - I_m)^{y_m} \quad (3.1.1)$$

where $g(y)'s$ are constants depending on $y_i's$ and $y_i = 0$ or 1
and $(J_i - I_i)^{y_i} = J_i - I_i$ if $y_i = 1$
while $(J_i - I_i)^{y_i} = I_i$ if $y_i = 0$

The element which is in the $(x_1, x_2, \ldots, x_m)^{th}$ row and $(y_1, y_2, \ldots, y_m)^{th}$ column of the matrix (the treatments are in lexicographic order) is 1 if $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_m)$ are $(y_1, y_2, \ldots, y_m)^{th}$ associates, and 0 otherwise.

Two treatments which are $(y_1, y_2, \ldots, y_m)^{th}$ associates occur together in $\lambda y_1 y_2 \ldots y_m$ blocks; hence we have the following lemma

Lemma 3.1.1.

Let N be the incidence matrix of a BAFD; then

$$NN' = \sum_{y \in \Omega^*} \lambda y_1 y_2 \ldots y_m (J_1 - I_1)^{y_1} \otimes (J_2 - I_2)^{y_2} \otimes \ldots \otimes (J_m - I_m)^{y_m} \tag{3.1.2}$$

where $\lambda_{000\ldots0}$ is defined to be r

Further let $J_i = s_i \otimes s_i$ to be a matrix with all elements equal to 1

Let Ω^* be the set of all $m-$ component binary vectors, that is

$\Omega^* = \Omega \cup \{(0, 0, \ldots, 0)\}$ where Ω is defined in chapter 2

for $y = (y_1, y_2, \ldots, y_m) \in \Omega^*$ let

$$Z^y = \otimes_{i=1}^m Z_i^{y_i} \tag{3.1.3}$$

where for $1 \leq i \leq m$,

$$\begin{aligned} Z_i^{y_i} &= I_i \quad \text{if} \quad y_i = 1 \\ &= J_i \quad \text{if} \quad y_i = 0 \end{aligned} \tag{3.1.4}$$

Definition 3.1.2.

A $v \times v$ matrix G where $v = \Pi s_i$ will be said to have property A if it is of the form

$G = \sum_{y \in \Omega^*} h(y) Z^y$

where $h(y), y \in \Omega^*$, are real numbers

$$\text{Let} \quad M^{000\ldots0} = \otimes_{i=1}^m (s_i^{-1} J_i), \tag{3.1.5}$$

Which together with equation (2.3.1) and (2.3.2), define M^y for every $y \in \Omega^*$. Also let the $(v-1) \times v$ matrix P be defined as

$$P = (\ldots, P^{y'}, \ldots)', \tag{3.1.6}$$

where P^y is included in P for every $y \in \Omega$. For example if $m = 2$ then $P = (P^{01'}, P^{10'}, P^{11'})'$

Lemma 3.1.2.

a. For each $y \in \Omega^*, Z^y$ can be expressed as a linear combination of $M^y, y \in \Omega^*$.

b. Conversely, for each $y \in \Omega^*, M^y$ can be expressed as a linear combination of $Z^y, y \in \Omega^*$

Proof.

$$M^y = M_1^{y_1} \otimes M_2^{y_2} \otimes \cdots \otimes M_m^{y_m}$$

where

$$\mathrm{M}_i^{y_i} = \begin{cases} I_i - s_i^{-1} J_i, & \text{if } y_i = 1 \\ s_i^{-1} J_i, & \text{if } y_i = 0 \end{cases}$$

Now

$$Z^y = Z_1^{y_1} \otimes Z_2^{y_2} \otimes \cdots \otimes Z_m^{y_m}$$

where

$$\mathrm{Z}_i^{y_i} == \begin{cases} I_i, & \text{if } y_i = 1 \\ J_i, & \text{if } y_i = 0 \end{cases}$$

Thus if $y_i = 1$

$$M_i^{y_i} = Z_i^1 - s_i^{-1} z_i^0$$

and if $y_i = 0$

$$M_i^{y_i} = s_i^{-1} = s_i^{-1} z_i^0$$

Hence M^y can be expressed as a linear combination of Z^y and conversely Z^y can be expressed as a linear combination of M^y □

Lemma 3.1.3.

$P'P = I - v^{-1}J$

where I is an identity matrix and J is a matrix of $1's$ both of order $v \times v$

Proof. It may be seen from equations (2.3.5), (2.3.6), (2.3.8) and (3.1.6) that in the v dimensional Euclidian space, the rows of P form an orthonomal basis of the orthocompliment of the space of vectors having all elements equal. Hence the lemma follows □

Example 3.1.1. *Consider a* 2×3 *BAFD where* $m = 2$, $s_1 = 2, s_2 = 3$ $b = 4, r = 2, k = 3$ *and*
$\lambda_{01} = \lambda_{11} = 1, \lambda_{10} = 0$

If we take

$$P_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{bmatrix} \text{ then}$$

$$P^{01} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & \frac{2}{\sqrt{12}} \end{bmatrix}$$

$$P^{10} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$P^{11} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & \frac{2}{\sqrt{12}} \end{bmatrix}$$

but

$$P = \left[P^{01'}, P^{10'}, P^{11'}\right]' \text{ is a } (v-1) \times v = 5 \times 6 \text{ matrix}$$

hence

$$P' = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} & \frac{1}{2} & \frac{1}{\sqrt{12}} \\ -\frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} & -\frac{1}{2} & \frac{1}{\sqrt{12}} \\ 0 & -\frac{2}{\sqrt{12}} & \frac{1}{\sqrt{6}} & 0 & -\frac{2}{\sqrt{12}} \\ -\frac{1}{2} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{12}} \\ \frac{1}{2} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{1}{\sqrt{12}} \\ 0 & \frac{2}{\sqrt{12}} & -\frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{12}} \end{bmatrix}$$

$$\text{hence } P = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & \frac{2}{\sqrt{12}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & \frac{2}{\sqrt{12}} \end{bmatrix}$$

and

$$P'P = \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{5}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{5}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{5}{6} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{5}{6} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= I - v^{-1}J$$

where I *is an identity matrix and* J *is a matrix of* $1's$ *both of order* $v \times v = 6 \times 6$

and

$$PP' = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & \frac{2}{\sqrt{12}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & \frac{2}{\sqrt{12}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} & \frac{1}{2} & \frac{1}{\sqrt{12}} \\ -\frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} & -\frac{1}{2} & \frac{1}{\sqrt{12}} \\ 0 & -\frac{2}{\sqrt{12}} & \frac{1}{\sqrt{6}} & 0 & -\frac{2}{\sqrt{12}} \\ -\frac{1}{2} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{\sqrt{12}} \\ \frac{1}{2} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{12}} \\ 0 & \frac{2}{\sqrt{12}} & -\frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{12}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I$$

is an identity matrix which is $((v-1) \times (v-1) = 5 \times 5))$

Lemma 3.1.4.

For a connected factorial design

(a) $PCP^{'}$ is positive definite $(p.d)$

(b) $P^yCP^{y'}$ is pd for every $y \in \Omega$

Proof. Since C is non-negative definite $(n.n.d)$, for every $v-1$ component vector $\underset{\sim}{\text{U}}$,
$\underset{\sim}{\text{U}}^{'}PCP^{'}\underset{\sim}{\text{U}} \geq 0$

Furthermore as the design is connected equality holds in the above only if the elements of $P^{'}\underset{\sim}{\text{U}}$ are all equal i.e. only if

$$P^{'}\underset{\sim}{\text{U}} = u_o(\otimes_{i=1}^{m}\underset{\sim}{1}_i) \tag{3.1.7}$$

For some constant u_0. By equations (2.3.6), (2.3.8), (3.1.6), $P^{'}P$ equals an identity matrix and

$P(\otimes_{i=1}^{m}\underset{\sim}{1}_i) = \underset{\sim}{0}$

Hence on premultiplication by P

$$\begin{aligned} PP^{'}\underset{\sim}{\text{U}} &= Pu_o(\otimes_{i=1}^{m}\underset{\sim}{1}_i) \\ &= u_oP(\otimes_{i=1}^{m}\underset{\sim}{1}_i) \end{aligned}$$

which yields $\underset{\sim}{\text{U}} = 0$ This proves (a). The proof of (b) now follows noting that for each $y \in \Omega$, $P^yCP^{y'}$ is a principal submatrix of $PCP^{'}$. $\square$

Lemma 3.1.5.

For a connected factorial design to be balanced with OFS, it is necessary and sufficient that the C matrix of the design be of the form

$$C = \sum_{y \in \Omega} \rho(y) M^y \tag{3.1.8}$$

where $\rho(y), y \in \Omega$, are real numbers

Proof. **Sufficiency**

Let C be of the form (3.1.8) then by equation (2.3.7), (2.3.8), it is easy to see that for every $y, z \in \Omega, y \neq z$,

$$P^y C = \rho(y) P^y \tag{3.1.9}$$

$$P^y C P^{y'} = \rho(y) I^{(y)} \tag{3.1.10}$$

$$P^y C P^{z'} = \mathbf{0} \tag{3.1.11}$$

where $I^{(y)}$ is the identity matrix of order $\prod (s_i - 1)^{y_i} (= \alpha(y)\ say)$ □

Example 3.1.2. *Suppose* $y = (10) and z = (01)$ *in example 3.1.1 then*

$$P^{10}C = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} \end{bmatrix}$$

$= \frac{4}{3} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$

$= \frac{4}{3} P^{10} = \rho(y) P^y$

hence

$$P^{10}CP^{10'} = \frac{4}{3}\begin{bmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}}\end{bmatrix}$$

i.e.

$P^{10}CP^{10'} = \frac{4}{3}\left[\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\right] = \frac{4}{3}[\frac{6}{6}] = \frac{4}{3}.1 = \rho(y)I^{(y)}$

where $I^{(y)}$ *is an identity matrix of order* $\prod_{i=1}^{m}(s_i-1)^{yi}$ *which in this case*

$\prod_{i=1}^{m}(s_i-1)^{y_i} = (2-1)^1(3-1)^0$

$= (1)(1)$

$= (1)$

and $P^yCP^{z'} = P^{10}CP^{01'}$

$$= \frac{4}{3}\begin{bmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{\sqrt{12}}\\ -\frac{1}{2} & \frac{1}{\sqrt{12}}\\ 0 & -\frac{2}{\sqrt{12}}\\ -\frac{1}{2} & -\frac{1}{\sqrt{12}}\\ \frac{1}{2} & -\frac{1}{\sqrt{12}}\\ 0 & \frac{2}{\sqrt{12}}\end{bmatrix}$$

$= \frac{4}{3}\begin{bmatrix}0 & 0\end{bmatrix}$

$= \begin{bmatrix}0 & 0\end{bmatrix}$

$= \boldsymbol{0}$

Example 3.1.3. *For a* 2×3 *BAFD where* $m=2, s_1=2, s_2=3,$ $b=4, r=2, k=3, \lambda_{01}=\lambda_{11}=1, \lambda_{10}=0$ *in example 3.1.1 we can obtain* *matrix C of the design as follows*

$$\begin{aligned}
NN^{'} &= \sum_{y\in\Omega^*} \lambda_y (J_1 - I_1)^{y_1} \otimes (J_2 - I_2)^{y_2} \otimes \ldots \otimes (J_m - I_m)^{y_m} \\
&= \lambda_{00}(J_1 - I_1)^0 \otimes (J_2 - I_2)^0 + \lambda_{01}(J_1 - I_1)^0 \otimes (J_2 - I_2)^1 \\
&+ \lambda_{10}(J_1 - I_1)^1 \otimes (J_2 - I_2)^0 + \lambda_{11}(J_1 - I_1)^1 \otimes (J_2 - I_2)^1 \\
&= 2(J_1 - I_1)^0 \otimes (J_2 - I_2)^0 + \lambda_{01}(J_1 - I_1)^0 \otimes (J_2 - I_2)^1 \\
&+ \lambda_{10}(J_1 - I_1)^1 \otimes (J_2 - I_2)^0 + \lambda_{11}(J_1 - I_1)^1 \otimes (J_2 - I_2)^1 \\
&= 2I_1 \otimes I_2 + \lambda_{01} I_1 \otimes (J_2 - I_2) + \lambda_{10}(J_1 - I_1) \otimes I_2 \\
&+ \lambda_{11}(J_1 - I_1) \otimes (J_2 - I_2)
\end{aligned}$$

but in this design we have that $\lambda_{01} = \lambda_{11} = 1, \lambda_{10} = 0$ hence

$$\begin{aligned}
NN^{'} &= 2I_1 \otimes I_2 + (1)I_1 \otimes (J_2 - I_2) + (J_1 - I_1) \otimes (J_2 - I_2) \\
&= 2I_1 \otimes I_2 + I_1 \otimes J_2 - I_1 \otimes I_2 + J_1 \otimes J_2 - J_1 \otimes I_2 \\
&- I_1 \otimes J_2 + I_1 \otimes I_2 \\
&= 2I_1 \otimes I_2 + 0I_1 \otimes J_2 + J_1 \otimes J_2 - J_1 \otimes I_2
\end{aligned}$$

where as usual I_1, I_2 are 2×2 and 3×3 identity matrices and J_1, J_2 are 2×2 and 3×3 matrices of all $1's$. By (3.1.3) *and* (3.1.4),
$NN^{'} = 2Z^{11} + 0Z^{10} - Z^{01} + Z^{00}$
which shows that $NN^{'}$ has property A

Furthermore

$$\begin{aligned} C &= r(\otimes_{i=1}^{m} I_i) - k^{-1}NN' \\ &= 2(I_1 \otimes I_2) - \frac{1}{3}\big[2Z^{11} + 0Z^{10} - Z^{01} + Z^{00}\big] \\ &= 2Z^{11} - \frac{2}{3}Z^{11} + 0Z^{10} + \frac{1}{3}Z^{01} - \frac{1}{3}Z^{00} \\ &= \frac{4}{3}Z^{11} + \frac{1}{3}Z^{01} - \frac{1}{3}Z^{00} \end{aligned}$$

where by (3.1.3) and (3.1.4)

$$C = \tfrac{4}{3}Z^{11} + \tfrac{1}{3}Z^{01} - \tfrac{1}{3}Z^{00}$$

$$= \begin{bmatrix} \frac{4}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{3} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix} +$$

$$\begin{bmatrix} \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{3} & \frac{-1}{3} & \frac{-1}{3} & 0 & \frac{-1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{4}{3} & \frac{-1}{3} & \frac{-1}{3} & 0 & \frac{-1}{3} \\ \frac{-1}{3} & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{3} & \frac{-1}{3} & 0 \\ 0 & \frac{-1}{3} & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & 0 & \frac{-1}{3} & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{-1}{3} & 0 & \frac{-1}{3} & \frac{-1}{3} & \frac{4}{3} \end{bmatrix}$$

By Lemma 3.1.4 (b) and (3.1.10) $\rho(y) > 0$; hence by (3.1.9) $P^y = \{\rho(y)\}^{-1}P^yC$and from the reduced normal equations $C\underset{\sim}{\Psi} = \underset{\sim}{Q}$ it follows that the BLUE of $P^y\underset{\sim}{\Psi}$ is given by

$$P^y\hat{\underset{\sim}{\Psi}} = \{\rho(y)\}^{-1}P^y\underset{\sim}{Q}, y \in \Omega \tag{3.1.12}$$

It is well known that the dispersion of $\underset{\sim}{Q}$ is given by

$$Disp(\underset{\sim}{Q}) = \sigma^2 C \tag{3.1.13}$$

σ^2 being the constant error variance.

Hence (3.1.11) and (3.1.12) for every $y, z \in \Omega, y \neq z$

$$\begin{aligned} Cov(P^y\hat{\underset{\sim}{\Psi}}, P^z\hat{\underset{\sim}{\Psi}}) &= \{\rho(y)\rho(z)\}^{-1}Cov(P^y\underset{\sim}{Q}, P^z\underset{\sim}{Q}) \\ &= \sigma^2\{\rho(y)\rho(z)\}^{-1}P^yCP^{z'} \\ &= 0 \end{aligned}$$

which shows that the design has OFS. Also by (3.1.10), (3.1.12) and (3.1.13) for every $y \in \Omega$

$$Disp(p^y\hat{\underset{\sim}{\Psi}}) = \sigma^2\{\rho(y)\}^{-1}I^{(y)} \tag{3.1.14}$$

So by corollary 2.4.1 the design is balanced.

Proof. **necessity**

Let P be as in (3.1.6) since C has all row and column sums equal to zero, by lemma 3.1.3,

$$P^{'}PC = C = CP^{'}P \tag{3.1.15}$$

hence

$PC = PCP^{'}P.$

But by lemma 3.1.4 (a) $PCP^{'}$ is $p.d.$ Therefore, $P = (PCP^{'})^{-1}$PC, and from the reduced normal equations $C\underline{\Psi} = \underline{Q}$, the BLUE of $P\underline{\Psi}$ is given by $P\hat{\underline{\Psi}} = (PCP^{'})^{-1}P\underline{Q}$ by (3.1.13)

$$Disp\left[P\hat{\underline{\Psi}}\right] = \sigma^2(PCP^{'})^{-1} \tag{3.1.16}$$

Suppose now the design is balanced and has OFS. Since the design has OFS, $cov(P^x\hat{\underline{\Psi}}, P^y\hat{\underline{\Psi}}) = 0$. For every $x, y(x \neq y) \in \Omega$. Hence all of diagonals blocks in $Disp(P\hat{\underline{\Psi}})$ must vanish so that by (3.1.16),

$$(PCP^{'})^{-1} = Diag(\ldots, A_x, \ldots), x \in \Omega \tag{3.1.17}$$

where $Disp(P\hat{\underline{\Psi}}) = \sigma^2 A_x, x \in \Omega$

Since the design is balanced, by corollary 2.4.1 for every $x \in \Omega$, A_x must be proportional to the identity matrix.

Let $A_x = a_x I^{(x)}$ where $a_x > 0$ this together with (3.1.16) yields

$PCP^{'} = Diag(\ldots, a_x^{-1}I^{(x)}, \ldots)$

Pre and post multiplying the above by $P^{'}$ and P respectively, one obtains $C = \sum_{x\in\Omega} a_x^{-1} P^{x\prime} P^x = \sum_{x\in\Omega} a_x^{-1} M^x$, by (2.3.7), (3.1.6) and (3.1.15), hence the necessity of the stated conditions follows □

Now

$C = \sum_{x\in\Omega} a_x^{-1} P^{x'} P^x$ where $a_x^{-1} = \rho(x)$ hence

$$
\begin{aligned}
C &= \sum_{x\in\Omega} \rho(x) P^{x'} P^x \\
&= \sum_{x\in\Omega} \rho(x) M^x
\end{aligned}
$$

hence the necessity of the stated condition follows

Example 3.1.4. *For the* 2×3 *BAFD in example 3.1.3*

$$
C = \begin{bmatrix}
\frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\
0 & -\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \\
-\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{4}{3}
\end{bmatrix}
$$

$$
P = \begin{bmatrix}
\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & \frac{2}{\sqrt{12}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\
\frac{1}{2} & \frac{-1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & -\frac{1}{\sqrt{12}} & \frac{2}{\sqrt{12}}
\end{bmatrix}
$$

Hence

$$
\begin{aligned}
PCP^{'} &= \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{3} \end{bmatrix} \\
&= Diag(\ldots, a_x^{-1} I^x, \ldots) \\
&= Diag(a_{(01)}^{-1} I^{(01)}, a_{(10)}^{-1} I^{(10)}, a_{(11)}^{-1} I^{(11)}) \\
&= Diag(2I^{(01)}, \frac{4}{3} I^{(10)}, \frac{4}{3} I^{(11)})
\end{aligned}
$$

where $I^{(01)}$ *is of order* $(s_1 - 1)^0 (s_2 - 1)^1 = (2-1)^0 (3-1)^1 = 2$
and $I^{(10)}$ *is of order* $(s_1 - 1)^1 (s_2 - 1)^0 = (2-1)^1 (3-1)^0 = 1$
and $I^{(11)}$ *is of order* $(s_1 - 1)^1 (s_2 - 1)^1 = (2-1)^1 (3-1)^1 = 2$

It can also be verified that

$$
\begin{aligned}
C &= \sum_{x\in\Omega} a_x^{-1} M^x \\
&= 2\begin{bmatrix}
\tfrac{1}{3} & -\tfrac{1}{6} & -\tfrac{1}{6} & \tfrac{1}{3} & -\tfrac{1}{6} & -\tfrac{1}{6} \\
-\tfrac{1}{6} & \tfrac{1}{3} & -\tfrac{1}{6} & -\tfrac{1}{6} & \tfrac{1}{3} & -\tfrac{1}{6} \\
-\tfrac{1}{6} & -\tfrac{1}{6} & \tfrac{1}{3} & -\tfrac{1}{6} & -\tfrac{1}{6} & \tfrac{1}{3} \\
\tfrac{1}{3} & -\tfrac{1}{6} & -\tfrac{1}{6} & \tfrac{1}{3} & -\tfrac{1}{6} & -\tfrac{1}{6} \\
-\tfrac{1}{6} & \tfrac{1}{3} & -\tfrac{1}{6} & -\tfrac{1}{6} & \tfrac{1}{3} & -\tfrac{1}{6} \\
-\tfrac{1}{6} & -\tfrac{1}{6} & \tfrac{1}{3} & -\tfrac{1}{6} & -\tfrac{1}{6} & \tfrac{1}{3}
\end{bmatrix}
+\frac{4}{3}\begin{bmatrix}
\tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} \\
\tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} \\
\tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} \\
\tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} \\
\tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} \\
\tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{6}
\end{bmatrix} \\
&+\frac{4}{3}\begin{bmatrix}
\tfrac{1}{3} & -\tfrac{1}{6} & -\tfrac{1}{6} & -\tfrac{1}{3} & \tfrac{1}{6} & \tfrac{1}{6} \\
-\tfrac{1}{6} & \tfrac{1}{3} & -\tfrac{1}{6} & \tfrac{1}{6} & -\tfrac{1}{3} & \tfrac{1}{6} \\
-\tfrac{1}{6} & -\tfrac{1}{6} & \tfrac{1}{3} & \tfrac{1}{6} & \tfrac{1}{6} & -\tfrac{1}{3} \\
-\tfrac{1}{3} & \tfrac{1}{6} & \tfrac{1}{6} & \tfrac{1}{3} & -\tfrac{1}{6} & -\tfrac{1}{6} \\
\tfrac{1}{6} & -\tfrac{1}{3} & \tfrac{1}{6} & -\tfrac{1}{6} & \tfrac{1}{3} & -\tfrac{1}{6} \\
\tfrac{1}{6} & \tfrac{1}{6} & -\tfrac{1}{3} & -\tfrac{1}{6} & -\tfrac{1}{6} & \tfrac{1}{3}
\end{bmatrix}
=\begin{bmatrix}
\tfrac{4}{3} & -\tfrac{1}{3} & -\tfrac{1}{3} & 0 & -\tfrac{1}{3} & -\tfrac{1}{3} \\
-\tfrac{1}{3} & \tfrac{4}{3} & -\tfrac{1}{3} & -\tfrac{1}{3} & 0 & -\tfrac{1}{3} \\
-\tfrac{1}{3} & -\tfrac{1}{3} & \tfrac{4}{3} & -\tfrac{1}{3} & -\tfrac{1}{3} & 0 \\
0 & -\tfrac{1}{3} & -\tfrac{1}{3} & \tfrac{4}{3} & -\tfrac{1}{3} & -\tfrac{1}{3} \\
-\tfrac{1}{3} & 0 & -\tfrac{1}{3} & -\tfrac{1}{3} & \tfrac{4}{3} & -\tfrac{1}{3} \\
-\tfrac{1}{3} & -\tfrac{1}{3} & 0 & -\tfrac{1}{3} & -\tfrac{1}{3} & \tfrac{4}{3}
\end{bmatrix} \\
&= 2P^{01'}P^{01} + \frac{4}{3}P^{10'}P^{10} + \frac{4}{3}P^{11'}P^{11} \\
&= \rho(0,1)P^{01'}P^{01} + \rho(1,0)P^{10'}P^{10} + \rho(1,1)P^{11'}P^{11} \\
&= \sum_{x\in\Omega} \rho(x) P^{x'}P^x \\
&= \sum_{x\in\Omega} \rho(x) M^x \\
&= C.
\end{aligned}
$$

where $\rho(0,1) = a_{01}^{-1} = 2, \rho(1,0) = a_{10}^{-1} = \frac{4}{3}$ *and* $\rho(1,1) = a_{11}^{-1} = \frac{4}{3}$

Theorem 3.1.1.

For a connected factorial design to be balanced with OFS, it is necessary and sufficient that the C matrix of the design has property A

Proof. **Necessity**

This follows from lemma 3.1.2 (b) and the necessity part of lemma 3.1.5 □

Proof. **Sufficiency**

Let the C matrix have property A. Then by lemma 3.1.2 (a), it is possible to express the C matrix as

$$C = \sum_{x \in \Omega^*} \rho(x) M^x = \sum_{x \in \Omega} \rho(x) M^x + \rho(0, 0, \ldots, 0) M^{00\ldots0} \quad (3.1.18)$$

where $\rho(x), x \in \Omega^*$ are constants

By (2.3.1), (2.3.2), and (3.1.5) for every $y \in \Omega$

$M^y M^{00\ldots0} = 0$

also

$M^{00\ldots0} M^{00\ldots0} = M^{00\ldots0} (\neq 0)$

and

$CM^{00\ldots0} = 0$

as each row sum of C equals *zero*.

Hence post multiplying equation (3.1.18) by $M^{00\ldots0}$,

it follows that $\rho(0, 0, \ldots, 0) = 0$. The sufficiency of the stated condition now follows from (3.1.18) and sufficiency part of lemma 3.1.5. □

To motivate the ideas further using the BAFD in example 3.1.3

$$M^{00} = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

$$M^{01} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

Then

$M^y M^{00} = 0$ since for example if $y = 01$ then

$$M^{01}M^{00} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & \frac{-1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & \frac{-1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & \frac{-1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & \frac{-1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \mathbf{0}$$

Also $M^{00}M^{00} = M^{00}$ since

$$M^{00}M^{00} = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

$$= M^{00}$$

Similarly

$$CM^{00} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \mathbf{0}$$

Theorem 3.1.1 provides a characterization for balance with OFS in terms property A of the C matrix in the connected case. Hereafter, a design will be said to have property A if its C matrix has property A. As indicated below, one can work out very simple formulae for the analysis of such designs.

Consider a connected design with property A. Then by Lemma 3.1.5 and theorem 3.1.1 the BLUE of $P^y \underset{\sim}{\Psi}$ and the dispersion matrix of this BLUE are given by (3.1.12) and (3.1.14) respectively. Hence by (2.3.7) SS due to interaction $F^y = SS$ due to $P^y \hat{\underset{\sim}{\Psi}}$

$$= (P^y \hat{\underset{\sim}{\Psi}})^{'} \left[\frac{Disp(P^y \hat{\underset{\sim}{\Psi}})}{\sigma^2} \right]^{-1} (P^y \hat{\underset{\sim}{\Psi}})$$

$$= (P^y \hat{\underset{\sim}{\Psi}})^{'} \left[\frac{Cov(P^y \hat{\underset{\sim}{\Psi}}, P^y \hat{\underset{\sim}{\Psi}})}{\sigma^2} \right]^{-1} (P^y \hat{\underset{\sim}{\Psi}})$$

$$= \{\rho(y)\}^{-1} \underset{\sim}{Q}^{'} P^{y'} P^y \underset{\sim}{Q}$$

$$= \{\rho(y)\}^{-1} \underset{\sim}{Q}^{'} M^y \underset{\sim}{Q}, \in \Omega \qquad (3.1.19)$$

Since the BLUE of $l^{'}$ is $l^{'*}$ and BLUE of $\hat{\underset{\sim}{\Psi}}$ is $\underset{\sim}{Q}$ The formulae (3.1.19) is extremely simple in the sense no matrix inversion is required.

Designs which are equireplicate or proper deserves some attention. For an equireplicate design with common replication number r,
$C = r(\otimes_{i=1}^{m} I_i) - N k^{-\delta} N^{'}$,
and by definition 3.1.1, C has property A if and only if NN' has property A. Hence our next result follows as a consequence of theorem 3.1.1

Theorem 3.1.2.

(a). For a connected, equireplicate factorial design to be balanced with OFS, it is necessary and suffient that the matrix $Nk^{-\delta}N'$ has property A

(b). For a connected, equireplicate, proper factorial design to be balanced with OFS, it is necessary and sufficient that the matrix NN' has property A

For connected equireplicate designs with property A and a common replication number r, it is possible to give a simple formulae for interaction efficiences. Recall that for the kind of design under consideration, the dispersion matrix $p^y\hat{\underline{\Psi}}$ is given by (3.1.14). On the other hand, it is readily seen that for a randomized (complete) block design with the same number of replicates, one would have obtained

$$Disp(P^y\hat{\underline{\Psi}}) = \sigma^2 r^{-1} I^{(y)} \tag{3.1.20}$$

A comparison of (3.1.14) and (3.1.20) shows that the efficiency with respect to interaction F^y in the design under consideration is given by

$$E(y) = \frac{\sigma^2 r^{-1} I^{(y)}}{\sigma^2\{\rho(y)\}^{-1} I^{(y)}} = \frac{\rho(y)}{r} \quad y \in \Omega \tag{3.1.21}$$

To motivate the ideas further using BAFD in example 3.1.3 we have

$\rho(0,1) = 2, \rho(1,0) = \frac{4}{3} \quad \rho(1,1) = \frac{4}{3}$

So

$E(0,1) = \frac{\rho(0,1)}{r} = \frac{2.0}{2.0} = 1$ and

$E(1,0) = \frac{\rho(1,0)}{r} = \frac{4}{3(2)} = \frac{2}{3}$

and

$E(1,1) = \frac{\rho(1,1)}{r} = \frac{4}{3(2)} = \frac{2}{3}$

Example 3.1.5. *Consider a 3×4 factorial arranged in twelve blocks as shown below*

$\longleftarrow$ *Blocks* $\longrightarrow$

00	*00*	*00*	*01*	*01*	*01*	*02*	*02*	*02*	*03*	*03*	*03*
11	*12*	*13*	*10*	*12*	*13*	*10*	*11*	*13*	*10*	*11*	*12*
22	*23*	*21*	*23*	*20*	*22*	*21*	*23*	*20*	*22*	*20*	*21*

The design is connected, proper with constant block size 3, and equireplicate with common replication number $r = 3$. It may be seen, by explicit computation, that for the above design

$$\begin{aligned} NN^{'} &= \sum_{y \in \Omega^*} \lambda_y (J_1 - I_1)^{y_1} \otimes (J_2 - I_2)^{y_2} \otimes \ldots \otimes (J_m - I_m)^{y_m} \\ &= \lambda_{00}(J_1 - I_1)^0 \otimes (J_2 - I_2)^0 + \lambda_{01}(J_1 - I_1)^0 \otimes (J_2 - I_2)^1 \\ &+ \lambda_{10}(J_1 - I_1)^1 \otimes (J_2 - I_2)^0 + \lambda_{11}(J_1 - I_1)^1 \otimes (J_2 - I_2)^1 \\ &= 3I_1 \otimes I_2 + \lambda_{01} I_1 \otimes (J_2 - I_2) + \lambda_{10}(J_1 - I_1) \otimes I_2 \\ &+ \lambda_{11}(J_1 - I_1) \otimes (J_2 - I_2) \end{aligned}$$

but from the design it can be verified that $\lambda_{01} = \lambda_{10} = 0$ while $\lambda_{11} = 1$ hence

$$
\begin{aligned}
NN^{'} &= 3I_1 \otimes I_2 + 1(J_1 - I_1) \otimes (J_2 - I_2) \\
&= 3I_1 \otimes I_2 + J_1 \otimes J_2 - J_1 \otimes I_2 - I_1 \otimes J_2 + I_1 \otimes I_2 \\
&= 4I_1 \otimes I_2 - I_1 \otimes J_2 - J_1 \otimes I_2 + J_1 \otimes J_2
\end{aligned}
$$

where as usual I_1 and I_2 are 3×3 and 4×4 identity matrices and J_1 and J_2 are 3×3 and 4×4 matrices of all 1's

By (3.1.3) and (3.1.4)
$NN' = 4Z^{11} - Z^{10} - Z^{01} + Z^{00}$
which shows that NN' has property A. Hence by theorem 3.1.2 (b) the design is balanced and has OFS. Furthermore

$$
\begin{aligned}
C &= 3(I_1 \otimes I_2) - \frac{1}{3}NN' \\
&= r(\otimes_{i=1}^{m} I_i) - k^{-1}NN' \\
&= \frac{5}{3}Z^{11} + \frac{1}{3}Z^{10} + \frac{1}{3}Z^{01} - \frac{1}{3}Z^{00}
\end{aligned}
$$

which also shows that C has the property A

Suppose $P_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{bmatrix}$
and P_2 well chosen by (2.3.5), (2.3.6), (3.1.3), (3.1.4)

$$
P^{10} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}
$$

and $Z^{11} = I_v$ *an identity matrix of order 12*

hence

$$P^{10}Z^{11}P^{10'} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I^{(10)} \text{ where } I^{(y)} \text{ is identity matrix of order}$$

$$\begin{aligned} \prod (s_i - 1)^{y_i} &= (s_1 - 1)^1 (s_2 - 1)^0 \\ &= (3-1)^1 (4-1)^0 \\ &= 2 \end{aligned}$$

Similarly $P^{10}Z^{10}P^{10'} = 4I^{(10)}$,

$P^{10}Z^{01}P^{10'} = P^{10}Z^{00}P^{10'} = 0$

It also follows that

$$Z^{10} = \begin{bmatrix} 1&1&1&1&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1 \end{bmatrix}$$

$$Z^{01} = \begin{bmatrix} 1&0&0&0&1&0&0&0&1&0&0&0 \\ 0&1&0&0&0&1&0&0&0&1&0&0 \\ 0&0&1&0&0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1&0&0&0&1 \\ 1&0&0&0&1&0&0&0&1&0&0&0 \\ 0&1&0&0&0&1&0&0&0&1&0&0 \\ 0&0&1&0&0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1&0&0&0&1 \\ 1&0&0&0&1&0&0&0&1&0&0&0 \\ 0&1&0&0&0&1&0&0&0&1&0&0 \\ 0&0&1&0&0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1&0&0&0&1 \end{bmatrix}$$

and

$$Z^{00} = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1 \end{bmatrix}$$

hence

$C = \frac{5}{3}Z^{11} + \frac{1}{3}Z^{10} + \frac{1}{3}Z^{01} - \frac{1}{3}Z^{00}$ *that is,*

$$C = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 2 & 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 2 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 2 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 2 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 2 & 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 2 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 2 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 2 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 2 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 2 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

hence

$$P^{10}CP^{10'} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 3I^{(10)}$$

where $I^{(y)}$ *is the Identity matrix of order* $\prod (s_i - 1)^{y_i}$

$$\begin{aligned} &= (s_1 - 1)^1 (s_2 - 1)^0 \\ &= (3-1)^1 (4-1)^0 \\ &= 2. \end{aligned}$$

Similarly it may be seen that

$P^{01}CP^{01'} = \frac{8}{3}I^{(01)}, P^{11}CP^{11'} = \frac{5}{3}I^{(11)}$.

A comparison with (3.1.10) *shows that* $\rho(1,0) = 3, \rho(0,1) = \frac{8}{3}$ *and* $\rho(1,1) = \frac{5}{3}$. *Hence by* (3.1.21) *the interaction efficiencies in the design under consideration are given by*

$E\,[1,0] = \frac{\rho(1,0)}{r} = \frac{3}{3} = 1.0$

$E\,[0,1] = \frac{\rho(0,1)}{r} = \frac{8}{3(3)} = \frac{8}{9}$

and

$E[1,1] = \frac{\rho(1,1)}{r} = \frac{5}{3(3)} = \frac{5}{9}$

The characterization for balance with OFS, as given in theorem 3.1.1 has immediate applicability in the actual construction of designs. It is possible to derive another characterization, mainly of theoretical interest, in terms of eigenvalues and eigenvectors for the C matrix.

Theorem 3.1.3.

For connected factorial design to be blanced with OFS, it is necessary and sufficient that every $x \in \Omega$, the columns of $P^{x'}$ represents an orthonomal system of eigenvectors corresponding to the same eigenvalue of C.

Proof. This is an immediate consequence of (2.3.7), (2.3.8) and lemma 3.1.5 □

Theorem 3.1.4.

The eigenvalues of NN' of a BAFD are $g(y_1, y_2, \ldots, y_m)$'s with corresponding eigenvectors given by the columns of $p^{y'}$
where $y = (y_1, y_2, \ldots, y_m) \in \Omega$

It should be noted that the multiplicity of $g(y_1, y_2, \ldots, y_m)$ is $\prod_{i=1}^{m}(s_i - 1)^{y_i}$.
Since $C = r(\otimes_{i=1}^{m} I_i) - k^{-1}NN'$,
The columns of $P^{y'}$ $y \in \Omega$ are also the eigenvectors of C with corresponding eigenvalues

$$\rho(y) = r - \frac{1}{k} g(y_1, y_2, \ldots, y_m) \tag{3.1.22}$$

$$= r - \frac{1}{k} g(y), \qquad y \in \Omega \tag{3.1.23}$$

Let $E(y)$ denote the interaction efficiences, then

Corollary 3.1.1.

$E(y) = 1 - \frac{1}{rk} g(y)$ and $E(y) = 1$ if and only if $g(y) = 0$

3.2 A Combinatorial Charaterization

Nair and Rao (1948) and Shah (1960*a*,*b*) defined a $2^m - 1$ class associate scheme for m factor experiments. Their association scheme has been referred to as the extended group divisible(EGD) scheme by Hinkelmann and Kempthorne (1963) and binary numbers association by Paik and Federer (1973). In this association scheme, two distinct treatment combinations are defined as x^{th} associates, $x \in \Omega$ where $x_i = 0$ if the i^{th} factor occurs at the same level in both the treatment combinations and $x_i = 1$ otherwise. The number of x^{th} associates of any treatment is given by $\alpha(x) = \prod(s_i - 1)^{x_i}$

For $x \in \Omega$ let B^x be a $v \times v$ matrix such that its $(J, J')^{th}$ element equal to 1 if the J^{th} and J'^{th} treatment combinations are x^{th} associates and zero otherwise. Then B^x, $x \in \Omega$, defines the $2^m - 1$ association matrices of the EGD association scheme. Using the method of induction, Gupta (1988) verified that

$$B^x = \otimes_{i=1}^m B_i^{x_i} \tag{3.2.1}$$

where for $1 \leq i \leq m$

$$B_i^{x_i} = J_i - I_i \quad \text{if} \quad x_i = 1$$
$$= I_i \quad \text{if} \quad x_i = 0 \tag{3.2.2}$$

Definition 3.2.1.

n arrangement of the $v = \prod s_i$ treatment combinations in b blocks each of size k will be called an Extended Group Divisible (EGD) design if

(i) The design is binary in the sense that each treatment combination occurs at most once in each block.

(ii) Each treatment combination occurs in exactly r blocks and

(iii) Every two distict treatment combinations, which are $x - th$ associates of each other occur together in $\lambda(x)$ blocks, $x \in \Omega$

For example it may be seen that the design in example 3.1.5 is an EGD design with parameters $m = 2, s_1 = 3, s_2 = 4, b = 12, r = k = 3, \lambda_{01} = \lambda_{10} = 0, \lambda_{11} = 1$.
It is readily seen e.g. Raghavarao (1971) that for an EGD design

$$NN' = rI + \sum_{x \in \Omega} \lambda(x) B^x \tag{3.2.3}$$

where I is the $v \times v$ identity matrix.

By (3.1.3), (3.1.4), (3.2.2), for each $x \in \Omega, B^x$ can be expressed as a linear combination of $Z^y, y \in \Omega^*$. Also, $I = Z^{11...1}$. Hence by (3.2.3), for an

EGD design, the matrix NN' has property A. Conversely, for each $x \in \Omega^*$, Z^x can be expressed as a linear combination of the I and $B^y, y \in \Omega$. Consequently if the NN' matrix of a binary proper design has property A, then the design must be an EGD design. We thus have the following result that was proved by Paik and Federer (1973) using an alternative argument.

Theorem 3.2.1.

A binary proper design is an EGD design if and only if NN' matrix has property A

Proof. Combining theorems 3.1.2 (b), 3.2.1, one obtains the result. □

Theorem 3.2.2.

For a connected, equireplicate, proper, binary factorial design to be balanced with OFS, it is necessary and sufficient that the design is an EGD design.

Theorem 3.2.2 presents a combinatorial characterization for balance with OFS in the case of connected, equireplicate, proper, binary designs. Such a characterization is of much help in the actual construction of the designs. The 'Sufficiency' part of this theorem was proved by Nair and Rao (1948), while the proof for 'Necessity' part is due to Shah (1958, 1960a)and Kshirsagar (1966).

It is of interest to find explicit formulae for interaction efficiences in an EGD design. To that effect note that by (3.2.3), for an EGD design

$$\begin{aligned}
C &= rI - \frac{1}{k}NN' \\
&= rI - \frac{1}{k}\left[rI + \sum_{x\in\Omega}\lambda(x)B^x\right] \\
&= rI - \frac{1}{k}rI + \frac{1}{k}\sum_{x\in\Omega}\lambda(x)B^x \\
&= rI(1-\frac{1}{k}) - \frac{1}{k}\sum_{x\in\Omega}\lambda(x)B^x \\
&= rI(\frac{k-1}{k}) - \frac{1}{k}\sum_{x\in\Omega}\lambda(x)B^x \\
&= rIk^{-1}(k-1) - k^{-1}\sum_{x\in\Omega}\lambda(x)B^x \\
&= k^{-1}\left[r(k-1)I - \sum_{x\in\Omega}\lambda(x)B^x\right] \qquad (3.2.4)
\end{aligned}$$

From (2.3.5), (2.3.6), (3.2.1), (3.2.2), it may be seen after a little algebra that for every $x, y \in \Omega$

$P^yB^xP^{y'} = [\prod_{i=1}^{m}\{(1-y_i)s_i - 1\}^{x_i}]\, I^{(y)}$

hence by (3.2.4), $P^yCP^{y'} = \rho(y)I^{(y)}$

where

$$\rho(y) = k^{-1}\left[r(k-1)I - \sum_{x\in\Omega}\lambda(x)\{\prod_{i=1}^{m}((1-y_i)s_i-1)^{x_i}\}\right] \qquad (3.2.5)$$

From (3.1.21), (3.2.5), the following result is evident

Theorem 3.2.3.

For a connected EGD design, with parameters as stated above, the efficiency with respect to the interaction F^y is given by

$E(y) = k^{-1}\left[(k-1) - r^{-1}\sum_{x\in\Omega}\lambda(x)\{\prod_{i=1}^{m}((1-y_i)s_i-1)^{x_i}\}\right]$

Chapter 4

Construction of balanced Arrays

In this chapter we have given a method of constructing transitive arrays of strength t by using orthogonal arrays of strength t. A method of constructing orthogonal arrays using difference schemes is also given

4.1 Construction of transitive arrays

Transitive arrays are defined in section 2.5 and transitive arrays of strength two are useful in the construction of two-factor BAFDs. Therefore, we are especially interested in constructing the transitive arrays of strength two. Very important arrangements of arrays are defined by Rao (1961), these are orthogonal arrays of type 1 and type 2, later renamed as transitive arrays and semi-balanced arrays respectively. Bose et al. (1960), constructed $TA[v(v-1), k, v, 2 : 1]$ from pairwise balanced designs while Suen (1982) constructed $TA[v(v-1), v, v, 2 : 1]$ from doubly transitive groups of order v. In this section, we shall give a method of constructing transitive arrays of strength t by using orthogonal Arrays of strength t.

Theorem 4.1.1.

If $v = s$ is a prime number or a power of a prime number, then a $TA[\lambda s(s-1)...(s-t+1), s, s, t]$ can always be constructed.

Proof. Construct an orthogonal array $OA[s^t, s, s, t]$ by using a suitable method. From this orthogonal array, delete all columns whose elements are not distinct if s is odd in order to obtain a reduced orthogonal array. The required transitive array is the reduced orthogonal array. However, if s is even, the required transitive array is obtained by appending the reduced orthogonal array to the transpose of each of its columns. We would like to construct $TA[\lambda s(s-1), s, s, 2]$ hence, by theorem 4.1.1, we shall be interested in constructing $OA[s^2, s, s, 2]$ first before we construct the required transitive arrays. Usually λ is required to be as small as possible so that the size of the transitive array would not be too large. However, if λ is not restricted to be too small, we can always construct a $TA[\lambda s(s-1), s, s, 2]$ for any $s \geq 2$. □

A few examples which illustrate the applications of theorem 4.1.1 are given below.

Corollary 4.1.1.

If s is a prime power, then there exists a $TA[s(s-1), s, s, 2]$

Proof. If we delete s columns whose elements are not distinct from the orthogonal array $OA[s^2, s, s, 2]$, the reduced orthogonal array will be $TA[s(s-1), s, s, 2]$ by theorem 4.1.1 □

Example 4.1.1. *For s=5, we can construct a* $TA(20, 5, 5, 2)$ *where* $\lambda = 1$ *i.e.* $TA(20, 5, 5, 2; 1)$. *This is obtained by first constructing* $OA[25, 5, 5, 2]$ *and deleting five assemblies with elements that are not distinct.*

1	2	3	4	0	2	3	4	0	1	3	4	0	1	2	4	0	1	2	3
2	3	4	0	1	4	0	1	2	3	1	2	3	4	0	3	4	0	1	2
3	4	0	1	2	1	2	3	4	0	4	0	1	2	3	2	3	4	0	1
4	0	1	2	3	3	4	0	1	2	2	3	4	0	1	1	2	3	4	0
0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4

TABLE 4.1: TA[20,5,5,2;1]

Example 4.1.2. *For s=3, we can construct a* $TA[18, 3, 3, 2]$*, where* $\lambda = 3$ *this is obtained by first constructing an* $OA[27, 3, 3, 2]$ *and then deleting nine assemblies with elements which are not distinct.*

1	2	0	2	0	1	1	2	0	2	0	1	1	2	0	2	0	1
2	0	1	1	2	0	2	0	1	1	2	0	2	0	1	1	2	0
0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2

TABLE 4.2: TA[18,3,3,2;3]

The $TA[s(s-1), s, s, 2]'s$ constructed in corollary 4.1.1 are completely resolvable.

Theorem 4.1.2.

The existence of a resolvable $TA[s(s-1), s-1, s, 2 : \lambda]$ is equivalent to the existence of $s-1$ mutually orthogonal latin squares of order s and hence also equivalent to the existence of an $OA[s^2, s-1, s, 2 : \lambda]$

Example 4.1.3. *For* $s = 7$*, we can construct a* $TA[42, 6, 7, 2 : 1]$*. This transitive array is completely resolvable and it is equivalent to* $OA[49, 6, 7, 2]$*. The transitive array* $TA[42, 6, 7, 2 : 1]$ *is given below.*

1	2	3	4	5	6	0	2	3	4	5	6	0	1	3	4	5	6	0	1	2	4	5	6
2	3	4	5	6	0	1	4	5	6	0	1	2	3	6	0	1	2	3	4	5	1	2	3
3	4	5	6	0	1	2	6	0	1	2	3	4	5	2	3	4	5	6	0	1	5	6	0
4	5	6	0	1	2	3	1	2	3	4	5	6	0	5	6	0	1	2	3	4	2	3	4
5	6	0	1	2	3	4	3	4	5	6	0	1	2	1	2	3	4	5	6	0	6	0	1
6	0	1	2	3	4	5	5	6	0	1	2	3	4	4	5	6	0	1	2	3	3	4	5

Continuation...

0	1	2	3	5	6	0	1	2	3	4	6	0	1	2	2	4	5
4	5	6	0	3	4	5	6	0	1	2	5	6	0	1	3	3	4
1	2	3	4	1	2	3	4	5	6	0	4	5	6	0	1	2	3
5	6	0	1	6	0	1	2	3	4	5	3	4	5	6	0	1	2
2	3	4	5	4	5	6	0	1	2	3	2	3	4	5	6	0	1
6	0	1	2	2	3	4	5	6	0	1	1	2	3	4	5	6	0

TABLE 4.3: TA[42,6,7,2:1] - continued

Theorem 4.1.3.

The existence of a $TA[s(s-1), s-2, s, 2]$ is equivalent to the existence of $s-2$ mutually orthogonal latin squares and hence also equivalent to $OA[s^2, s-2, s, 2]$. The said mutually orthogonal latin squares are of order s and all have different symbols in the diagonal.

Example 4.1.4. *For $s = 5$, we can construct a $TA[20, 3, 5, 2; 1]$ by first constructing an $OA[25, 3, 5, 2]$ and then deleting five assemblies with elements that are not distinct.*

1	2	3	4	0	2	3	4	0	1	3	4	0	1	2	4	0	1	2	3
2	3	4	0	1	4	0	1	2	3	1	2	3	4	0	3	4	0	1	2
3	4	0	1	2	1	2	3	4	0	4	0	1	2	3	2	3	4	0	1

TABLE 4.4: TA[20,3,5,2;1]

Example 4.1.5. *For $s = 4$, we can construct a $TA[24, 4, 4, 3; 1]$ by first constructing an $OA[32, 4, 4, 3]$ and then deleting eight assemblies which contain elements that are not distinct.*

3	0	1	0	2	0	0	1	2	1	3	1	1	2	3	2	0	2	2	3	0	3	1	3
1	2	2	3	3	1	2	3	3	0	0	2	3	0	0	1	1	3	0	1	1	2	2	0
2	1	3	2	1	3	3	2	0	3	2	0	0	3	1	0	3	1	1	0	2	1	0	2
0	3	0	1	0	2	1	0	1	2	1	3	2	1	2	3	2	0	3	2	3	0	3	1

TABLE 4.5: TA[24,4,4,3;1]

4.2 Difference Schemes

In this section, $(\mathcal{A}, +)$,or simply $\mathcal{A}$, will denote a finite abelian group with a binary operation $+$. $\mathcal{A}$ will have a cardinality denoted by s, its identity element will be 0 and the inverse of Θ will be $-\Theta$. The pair $(\mathcal{A}, +)$ will be taken to be the additive group associated with the Galois fields with s elements. We shall then have the following definition of a difference scheme based on $(\mathcal{A}, +)$.

Definition 4.2.1.

An $r \times c$ array D with entries from $\mathcal{A}$ is called a **difference scheme** based on $(\mathcal{A}, +)$ if it has the property that for all i and j with $1 \leq i$, $j \leq c$, the vector difference between the i^{th} and j^{th} columns contains every element of $\mathcal{A}$ equally often if $i \neq j$

Necessarily r is a multiple of s, say, $r = \lambda s$, where λ is the number of times each element of $\mathcal{A}$ occurs in the difference of two columns. We will denote such an array by $D(r, c, s)$ and refer to it as a difference scheme with s levels and index λ.

Example 4.2.1. *Any orthogonal array* $OA(N, k, s, t)$, *with* $t \geq 2$ *may be regarded as a difference scheme* $D(N, k, s)$, *simply by taking the levels to be integers modulo* s

Example 4.2.2. *Let* $(\mathcal{A}, +)$ *be the additive group associated with the Field GF(s), whose elements, we denote by* $[\alpha_0, \alpha_1, \cdots, \alpha_{s-1}]$. *Let* D *be the* $s \times s$ *multiplication table of this field. (Thus the table contains a row and column of zeros corresponding to the multiplication by* 0*).Then* D *is a difference scheme* $D(s, s, s)$.

Example 4.2.3. *By Juxtaposing difference schemes $D(r_1, c, s)$ and $D(r_2, c, s)$ we obtain a difference scheme $D(r_1 + r_2, c, s)$. By taking the component-wise products of the rows of different schemes $D(r_1, c, s_1)$ and $D(r_2, c, s_2)$ in all possible ways we obtain a Difference scheme $D(r_1 r_2, c, s_1 s_2)$.*

Theorem 4.2.1.

A difference scheme $D(p^m, p^m, p^n)$ exists for any prime p and integers $m \geq n \geq 1$.

Proof. Let the elements of $GF(p^m)$ be represented by polynomials

$$b_0 + b_1 x + \cdots + b_{n-1} x^{n-1} + \cdots + b_{m-1} x^{m-1} \tag{4.2.1}$$

where $b_0, \cdots, b_{m-1} \in GF(p)$. We may regard $GF(p^n)$ as the additive subgroup of $GF(p^m)$ by identifying it's elements with the subset of $GF(p^m)$ consisting of elements of the form $b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}$. Note that here we are only using the additive structure of $GF(p^n)$. Now let D^* be the $p^m \times p^m$ multiplication table of $GF(p^m)$. Map every entry $b_0 + b_1 x + \cdots + b_{m-1} x^{m-1}$ in this table to $b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}$. Let D be the array obtained this way, and view it's entries as elements of $GF(p^n)$. Then D is the desired difference scheme. □

Example 4.2.4. *We illustrate the construction for the case $p = 3, m = 2, n = 1$, this will result to a difference scheme $D(9, 9, 3)$. In this special case the field $GF(p^n)$ in the construction is actually the subfield of $GF(p^m)$ and the multiplication of elements of $GF(p^n)$ is the same in both fields.*

Table 4.6 is a multiplication table for $GF(3^2)$, based on the irreducible polynomial $f(x) = x^2 + x + 2$, we represent the nine elements of $GF(3^2)$ in condensed notation writing 0 as 00, 1 as 10, $1 + 2x$ as 01 and so on.

(*)	00	10	20	01	11	21	02	12	22
00	00	00	00	00	00	00	00	00	00
10	00	10	20	01	11	21	02	12	22
20	00	20	10	02	22	12	01	21	11
01	00	01	02	12	10	11	21	22	20
11	00	11	22	10	21	02	20	01	12
21	00	21	12	11	02	20	22	10	01
02	00	02	01	21	20	22	12	11	10
12	00	12	21	22	01	10	11	20	02
22	00	22	11	20	12	01	10	02	21

TABLE 4.6: Multiplication table for $GF(3^2)$

Upon applying the map: For $m = 2 : b_0 + b_1x$ is the polynomial and for $n = 1 : b_0$ is the polynomial hence we apply the map. $b_0 + b_1x \mapsto b_0$ to the entries of this table to obtain the Difference Scheme $D(9, 9, 3)$ based on $(GF(3), +)$ which is exhibited in Table 4.7

0	0	0	0	0	0	0	0	0
0	1	2	0	1	2	0	1	2
0	2	1	0	2	1	0	2	1
0	0	0	1	1	1	2	2	2
0	1	2	1	2	0	2	0	1
0	2	1	1	0	2	2	1	0
0	0	0	2	2	2	1	1	1
0	1	2	2	0	1	1	2	0
0	2	1	2	1	0	1	0	2

TABLE 4.7: A difference scheme based on $(GF(3), +)$

4.3 Orthogonal Arrays Via Difference Schemes

The procedure that converts a difference scheme into an orthogonal array can be illustrated as follows If D is a difference scheme based on $(\mathcal{A}, +)$, where $\mathcal{A} = \left\{\theta_0, ..., \theta_{s-1}\right\}$ We will use D_i to denote the array obtained from D by adding θ_i to each of its entries. Obviously D_i is a difference scheme with the same parameters as D. We just juxtapose the D_i's to obtain an orthogonal array of strength two. We refer to this process as developing the difference scheme into an orthogonal array.

Lemma 4.3.1.

If D is a difference scheme $D(r, c, s)$ then

$$A = \begin{bmatrix} D_0 \\ D_1 \\ . \\ . \\ . \\ D_{s-1} \end{bmatrix}$$

is an orthogonal array$OA(rs, c, s, 2)$

Proof.

Select two factors from A, say F_1 and F_2, $F_1 \neq F_2$and two elements from $\mathcal{A}$, say θ and $\theta^{'}$. If c_1 and c_2 denote the columns of D corresponding to the factors F_1 and F_2 respectively, then we know that λ entries in $c_1 - c_2$ are equal to θ -$\theta^{'}$. For each occurrence of θ - $\theta^{'}$ in c_1 - c_2 there is a unique row in a unique D_i in which F_1 is at level θ and F_2 is at level $\theta^{'}$. Since these are the only runs with factor F_1 at level θ and factor F_2 at level $\theta^{'}$, we conclude that there are indeed λ such runs in A. □

Example 4.3.1. *Construct a difference scheme* $D[9, 9, 9]$ *and use it to construct* $OA[81, 9, 9, 2]$

0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8
0	2	3	4	5	6	7	8	1
0	3	4	5	6	7	8	1	2
0	4	5	6	7	8	1	2	3
0	5	6	7	8	1	2	3	4
0	6	7	8	1	2	3	4	5
0	7	8	1	2	3	4	5	6
0	8	1	2	3	4	5	6	7

TABLE 4.8: The Difference Scheme D[9,9,9]

0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2
0	1	2	3	4	5	6	7	8	1	2	0	4	5	3	7	8	6	2	0	1	5	3	4
0	1	2	3	4	5	6	7	8	2	0	1	5	3	4	8	6	7	1	2	0	4	5	3
0	1	2	3	4	5	6	7	8	3	4	5	6	7	8	0	1	2	6	7	8	0	1	2
0	1	2	3	4	5	6	7	8	4	5	3	7	8	6	1	2	0	8	6	7	2	0	1
0	1	2	3	4	5	6	7	8	5	3	4	8	6	7	2	0	1	7	8	6	1	2	0
0	1	2	3	4	5	6	7	8	6	7	8	0	1	2	3	4	5	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8	7	8	6	1	2	0	4	5	3	5	3	4	8	6	7
0	1	2	3	4	5	6	7	8	8	6	7	2	0	1	5	3	4	4	5	3	7	8	6

2	2	2	3	3	3	3	3	3	3	3	3	4	4	4	4	4	4	4	4	4	5	5	5
8	6	7	3	4	5	6	7	8	0	1	2	4	5	3	7	8	6	1	2	0	5	3	4
7	8	6	6	7	8	0	1	2	3	4	5	8	6	7	2	0	1	5	3	4	7	8	6
3	4	5	4	5	3	7	8	6	1	2	0	7	8	6	1	2	0	4	5	3	1	2	0
5	3	4	7	8	6	1	2	0	4	5	3	2	0	1	5	3	4	8	6	7	3	4	5
4	5	3	1	2	0	4	5	3	7	8	6	3	4	5	6	7	8	0	1	2	8	6	7
0	1	2	8	6	7	2	0	1	5	3	4	5	3	4	8	6	7	2	0	1	2	0	1
2	0	1	2	0	1	5	3	4	8	6	7	6	7	8	0	1	2	3	4	5	4	5	3
1	2	0	5	3	4	8	6	7	2	0	1	1	2	0	4	5	3	7	8	6	6	7	8

5	5	5	5	5	5	6	6	6	6	6	6	6	6	6	7	7	7	7	7	7	7	7	7
8	6	7	2	0	1	6	7	8	0	1	2	3	4	5	7	8	6	1	2	0	4	5	3
1	2	0	4	5	3	3	4	5	6	7	8	0	1	2	5	3	4	8	6	7	2	0	1
4	5	3	7	8	6	8	6	7	2	0	1	5	3	4	2	0	1	5	3	4	8	6	7
6	7	8	0	1	2	5	3	4	8	6	7	2	0	1	6	7	8	0	1	2	3	4	5
2	0	1	5	3	4	2	0	1	5	3	4	8	6	7	4	5	3	7	8	6	1	2	0
5	3	4	8	6	7	4	5	3	7	8	6	1	2	0	1	2	0	4	5	3	7	8	6
7	8	6	1	2	0	1	2	0	4	5	3	7	8	6	8	6	7	2	0	1	5	3	4
0	1	2	3	4	5	7	8	6	1	2	0	4	5	3	3	4	5	6	7	8	0	1	2

8	8	8	8	8	8	8	8	8
8	6	7	2	0	1	5	3	4
4	5	3	7	8	6	1	2	0
5	3	4	8	6	7	2	0	1
1	2	0	4	5	3	7	8	6
6	7	8	0	1	2	3	4	5
7	8	6	1	2	0	4	5	3
3	4	5	6	7	8	0	1	2
2	0	1	5	3	4	8	6	7

TABLE 4.9: An $OA[81, 9, 9, 2]\lambda = 1$

Definition 4.3.1.

An orthogonal array $OA[N, k, s, 2]$ is said to be a -resolvable if it is statistically equivalent to the juxtaposition of $\frac{N}{as}$ arrays such that each factor occurs in each of these arrays a times at each level. A 1 -resolvable orthogonal array is also called **completely resolvable**, otherwise it is called **Partly resolvable.**

Example 4.3.2. *The* $OA[9, 3, 3, 2]$ *shown in table 4.10 is completely resolvable.*

0	0	0
0	1	2
0	2	1
1	1	1
1	2	0
1	0	2
2	2	2
2	0	1
2	1	0

TABLE 4.10: A completely resolvable $OA[9, 3, 3, 2]$

The $\frac{N}{as} = 3$ arrays in which each of the factors occur once at each level and whose Juxtaposition is statistically equivalent to the $OA[9, 3, 3, 2]$ are

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

Theorem 4.3.1.

The existence of both an a- resolvable $OA[N, k_1, s, 2]$ and an $OA[\frac{N}{as}, k_2, s, 2]$ implies the existence of an $OA[N, k_1 + k_2, s, 2]$. Furthermore, if the $OA[\frac{N}{as}, k_2, s, 2]$ is b - resolvable, there is an (abs) - resolvable $OA[N, k_1 + k_2, s, 2]$

Proof. Let $A = \left[A_1^{\intercal}, ..., A_u^{\intercal}\right]^{\intercal}$ be the a - resolvable $OA[N, k_1, s, 2]$ where $u = \frac{N}{as}$ and in each $A_1, A_2, ..., A_u$ every factor occurs a times at each level. Let B be the $OA[\frac{N}{as}, k_2, s, 2]$ if B is b - resolvable we take it to be $B = \left[B_1^{\intercal}, ..., B_v^{\intercal}\right]^{\intercal}$ where $v = \frac{N}{abs^2}$ and in each B_i every factor occurs b - times at each level for $i = 1, 2, ..., v$.

Now let C be the $N \times [k_1 + k_2]$ array formed by following each run in A_i by the i^{th} run in B for $i = 1, ..., u.C$ is an $OA[N, k_1 + k_2, s, 2]$ which is (abs) -

resolvable since B is b - resolvable.

□

Corollary 4.3.1.

Suppose a divides $\frac{N}{s^2}$, then, an a -resolvable $OA[N, k, s, 2]$ can always be extended to an (as) resolvable $OA[N, k+1, s, 2]$. In particular any completely resolvable $OA[N, k, s, 2]$ can be extended to an s-resolvable $OA[N, k+1, s, 2]$.

Theorem 4.3.2.

An orthogonal array $OA[N, k, s, 2]$ that is obtained by developing a difference scheme is completely resolvable.

Proof. Let the orthogonal array be as in Lemma 4.3.1, with $N = rs$ and $k = c$. Let A_i consist of the s-runs obtained by taking the i^{th} row of each of $D_0, D_1, \cdots, D_{s-1}$, for $i = 1, \cdots, r$. Thus, A_i consists of the runs formed by taking the i^{th} row in D and adding $\theta_0 1_c^\intercal, \cdots, \theta_{s-1} 1_c^\intercal$ to it in turn. Every factor occurs once at each level in each A_i and the Juxtaposition of the $A_i's$ is statistically equivalent to the $OA[rs, c, s, 2]$ that we obtain by developing difference schemes. □

Corollary 4.3.2.

A difference scheme $D(r, c, s)$ can be used to construct an orthogonal array $OA[rs, c+1, s, 2]$

Example 4.3.3. *Let D be the difference scheme $D(s, s, s)$ according to corollary 4.3.2 D can be used to construct an $OA[s^2, s+1, s, 2]$ of index unity. Suppose now $s = 4$, as usual we denote the elements of $GF(4)$ by $0, 1, 2, 3$. The Difference Scheme $D(4, 4, 4)$ is given by the following table.*

0	0	0	0
0	1	2	3
0	2	3	1
0	3	1	2

TABLE 4.11: A difference scheme $D(4, 4, 4)$

0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
0	1	2	3	1	0	3	2	2	3	0	1	3	2	1	0
0	2	3	1	1	3	2	0	2	0	1	3	3	1	0	2
0	3	1	2	1	2	0	3	2	1	3	0	3	0	2	1

TABLE 4.12: A completely resolvable $OA[16, 4, 4, 2]$ obtained by developing a difference scheme $D(4, 4, 4)$ in table 4.11

0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
0	1	2	3	1	0	3	2	2	3	0	1	3	2	1	0
0	2	3	1	1	3	2	0	2	0	1	3	3	1	0	2
0	3	1	2	1	2	0	3	2	1	3	0	3	0	2	1
0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3

TABLE 4.13: An $OA[16, 5, 4, 2]$ obtained by appending a symbol to each of the runs in the $OA[16, 4, 4, 2]$ of table 4.12

Example 4.3.4. *Let D be the difference scheme $D(9, 9, 3)$ exhibited in table 4.7 by developing this, we obtain a completely resolvable $OA[27, 9, 3, 2]$ exhibited in table 4.14. From example 4.3.2 we also know that a completely resolvable $OA[9, 3, 3, 2]$ exists. From theorem 4.3.1, we can then deduce the existence of a 3 - resolvable $OA[27, 12, 3, 2]$. By applying corollary 4.3.1 we obtain an $OA[27, 13, 3, 2]$.*

0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
0	1	2	0	1	2	0	1	2	1	2	0	1	2	0	1	2	0	2	0	1	2	0	1	2	0	1
0	2	1	0	2	1	0	2	1	1	0	2	1	0	2	1	0	2	2	1	0	2	1	0	2	1	0
0	0	0	1	1	1	2	2	2	1	1	1	2	2	2	0	0	0	2	2	2	0	0	0	1	1	1
0	1	2	1	2	0	2	0	1	1	2	0	2	0	1	0	1	2	2	0	1	0	1	2	1	2	0
0	2	1	1	0	2	2	1	0	1	0	2	2	1	0	0	2	1	2	1	0	0	2	1	1	0	2
0	0	0	2	2	2	1	1	1	1	1	1	0	0	0	2	2	2	2	2	2	1	1	1	0	0	0
0	1	2	2	0	1	1	2	0	1	2	0	0	1	2	2	0	1	2	0	1	1	2	0	0	1	2
0	2	1	2	1	0	1	0	2	1	0	2	0	2	1	2	1	0	2	1	0	1	0	2	0	2	1

TABLE 4.14: An $OA(27, 9, 3, 2)\lambda = 3$

0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
0	1	2	0	1	2	0	1	2	1	2	0	1	2	0	1	2	0	2	0	1	2	0	1	2	0	1
0	2	1	0	2	1	0	2	1	1	0	2	1	0	2	1	0	2	2	1	0	2	1	0	2	1	0
0	0	0	1	1	1	2	2	2	1	1	1	2	2	2	0	0	0	2	2	2	0	0	0	1	1	1
0	1	2	1	2	0	2	0	1	1	2	0	2	0	1	0	1	2	2	0	1	0	1	2	1	2	0
0	2	1	1	0	2	2	1	0	1	0	2	2	1	0	0	2	1	2	1	0	0	2	1	1	0	2
0	0	0	2	2	2	1	1	1	1	1	1	0	0	0	2	2	2	2	2	2	1	1	1	0	0	0
0	1	2	2	0	1	1	2	0	1	2	0	0	1	2	2	0	1	2	0	1	1	2	0	0	1	2
0	2	1	2	1	0	1	0	2	1	0	2	0	2	1	2	1	0	2	1	0	1	0	2	0	2	1
0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2
0	1	2	1	2	0	2	0	1	0	1	2	1	2	0	2	0	1	0	1	2	1	2	0	2	0	1
0	2	1	1	0	2	2	1	0	0	2	1	1	0	2	2	1	0	0	2	1	1	0	2	2	1	0
0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2

TABLE 4.15: An $OA(27, 13, 3, 2)\lambda = 3$

Definition 4.3.2.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be respectively $m \times n$ and $u \times v$ matrices with entries from an abelian group $\mathcal{A}$ with binary operation $*$ (usually addition or multiplication). Their tensor or kronecker product denoted by $A \otimes B$, is the $mu \times nv$ matrix

$$A \otimes B = \begin{bmatrix} a_{11} * B & . & . & . & a_{1n} * B \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ a_{m1} * B & . & . & . & a_{mn} * B \end{bmatrix}$$

where $a_{ij} * B$ stands for the $u \times v$ matrix with entries $a_{ij} * brs(1 \leq r \leq u, 1 \leq s \leq v)$. In this chapter,$*$ will denote addition. Using this definition, we may write the array A in lemma 4.3.1 as

$$A = E \otimes D \tag{4.3.1}$$

where $E = \left[\theta_0, \theta_1, ..., \theta_{s-1}\right]^{\intercal}$ however there are other choices of E in equation (4.3.1) for which A is also an orthogonal array of strength 2.

Lemma 4.3.2.

If D is a difference scheme $D(r, c, s)$ and B is an $OA[N, k, s, 2]$ both based on the abelian group $\mathcal{A}$, then the array $A = B \otimes D$ is an orthogonal array $OA[Nr, kc, s, 2]$

4.4 Bose and Bush Recursive Construction

The construction to be discussed in this section due to Bose and Bush (1952) is in the spirit of example (4.3.4). It allows us to construct orthogonal arrays of strength two with a large number of factors, possibly the maximal number, provided that the number of symbols s and the index λ are powers of the same prime p. In example 4.3.4, we took $s = \lambda = p = 3$ and obtained an orthogonal array with maximal number of factors. i.e $f(27, 3, 2) = 13$

Theorem 4.4.1.

Let $s = p^v$ and $\lambda = p^u$, where p is prime and u and v are integers with $u \geq 0, v \geq 1$.Let $d = \left[\frac{u}{v}\right]$

Then there exists an $OA\left[\lambda s^2, \frac{\lambda s^{d+1}-1}{s^d - s^{d-1}} + 1, s, 2\right]$

Proof. Let $D(i)$ be the difference scheme $D(\lambda s^{1-i}, \lambda s^{1-i}, s)$, For $i = 0, 1, ..., d$. Since s and λs^{1-i} are powers of the same prime and $\lambda s^{1-i} \geq s$, these difference schemes can be constructed as in theorem 4.2.1. Develop the Difference scheme $D(0)$ to obtain a completely resolvable $OA(\lambda s^2, \lambda s, s, 2)$.If $d = 0$, add one more factor to this orthogonal array as in corollary 4.3.1 to obtain the desired $OA(\lambda s^2, \lambda s{+}1, s, 2)$. If $d \geq 1$, use $D(1)$ to construct a completely resolvable $OA(\lambda s, \lambda, s, 2)$. The completely resolvable $OA(\lambda s^2, \lambda s, s, 2)$ and the completely resolvable $OA(\lambda s, \lambda, s, 2)$ can be used as in theorem 4.3.1 to obtain an s resolvable $OA(\lambda s^2, \lambda s + \lambda, s, 2)$.If $d = 1$, we can again add one more factor as in corollary 4.3.1 to obtain the desired $OA[\lambda s^2, \lambda s + \lambda + 1, s, 2]$. If $d \geq 2$, we can use the s resolvable $OA[\lambda s^2, \lambda s + \lambda, s, 2]$ and the completely resolvable $OA[\lambda, \frac{\lambda}{s}, s, 2]$ which can be obtained from $D(2)$ to obtain an s^2-resolvable $OA[\lambda s^2, \lambda s + \lambda + \frac{\lambda}{s}, s, 2]$ using the method described in theorem 4.3.1

If we continue in this way using all the $D(i)'s$ after d applications of theorem 4.3.1 and one application of corollary 4.3.1 we obtain an array with $\lambda s + \lambda + \frac{\lambda}{s} + \cdots \frac{\lambda}{s^{d-1}} + 1$ factors $= \frac{\lambda(s^{d+1}-1)}{(s^d - s^{d-1})} + 1$ factors.

$$\lambda s + \lambda + \frac{\lambda}{s} + \cdots \frac{\lambda}{s^{d-1}} + 1$$
$$= \lambda\left[s + 1 + \frac{1}{s} + \cdots + \frac{1}{s^{d-1}}\right] + 1$$
$$= \lambda\left[S_n = \frac{a[1-r^n]}{1-r}\right] + 1$$
$$= \lambda\left[\frac{s(1-(\frac{1}{s})^n)}{1-(\frac{1}{s})}\right] + 1$$
$$= \lambda\left[s\{\frac{\frac{1}{1}-\frac{1}{s^{d-1}}}{1-\frac{1}{s}}\}\right] + 1$$

$= \lambda \left[s\{\frac{s^{d-1}-1}{\frac{s^{d-1}}{\frac{s-1}{s}}}\} \right] + 1$
$= \lambda \left[s\{\frac{s^{d-1}-1}{s^{d-1}}.\frac{s}{s-1}\} \right] + 1$
$= \lambda\{\frac{s^{d+1}-s^2}{s^d-s^{d-1}}\} + 1$
$= \lambda\{\frac{s^{d+1}-1}{s^d-s^{d-1}}\} + 1$

since $s^2 = s(mods - 1)$

This is the required or desired orthogonal array. □

Example 4.4.1. *The construction of an* $OA(27, 13, 3, 2)$ *in example 4.3.4 follows the recipe of theorem 4.4.1. Since* λ *is a power of* s *(infact* $\lambda = s^1$*), we know that this array has maximal number of factors*

The levels of the first nine factors are obtained by developing $D(0)$*, a difference scheme* $D(9, 9, 3)$*.*

The levels of the next three factors are obtained by appending the appropriate runs of an $OA[9, 3, 3, 2]$ *which was obtained by developing* $D(1)$*, a difference scheme* $D(3, 3, 3)$*.*

The levels of the last factor are obtained by appending appropriate runs of an $OA[3, 1, 3, 1]$ *which was obtained by developing* $D(2)$*, a difference scheme* $D(1, 1, 3)$ *or by using corollary 4.3.1, using the* 3*- resolvability of the orthogonal array formed by the first twelve factors.*

Example 4.4.2. *Let* $p = 2$*,* $s = 2^v$ *and* $\lambda = 2s^{n-2}$*. This corresponds to an orthogonal array that was mentioned following the method of the Addelman and Kempthorne (1961). When* s *is a power of* 2 *we shall give a general construction for an* $OA[2s^n, \frac{2(s^n-1)}{(s-1)} - 1, s, 2]$*,* $n \geq 2$ *using* *theorem 4.4.1*

First consider the case when $v \geq 2$ *and* $u = 1 + (n-2)log_2 s$ *then*

$\lambda = 2s^{n-2} = 2 \cdot (2^v)^{\frac{u-1}{v}} = 2 \cdot 2^{u-1} = 2^u$

$d = \frac{u}{v}$ *but* $u = (n-2)v + 1$

Thus $d = \frac{u}{v} = \frac{(n-2)v+1}{v} = n - 2 + \frac{1}{v}$

hence

$$\lambda \frac{s^{d+1}-1}{s^d - s^{d-1}} + 1 = \lambda \left[\frac{s^{n-2+\frac{1}{v}+1} - (s^{\frac{1}{v}})^0}{s^{n-2+\frac{1}{v}} - s^{n-2+\frac{1}{v}-1}} \right] + 1$$

$$= \frac{\lambda s^{\frac{1}{v}}[s^{n-1} - s^0]}{s^{\frac{1}{v}}[s^{n-2} - s^{n-3}]} + 1$$

$$= \frac{\lambda[s^{n-1}-1]}{s^{n-2}-s^{n-3}} + 1, \left\{ \begin{array}{c} but\ \lambda = 2^u \quad and \quad u = 1 + (n-2)log_2 s \\ so\ \lambda = 2^u = 2^{1+(n-2)log_2 s} \\ = 2.2^{(n-2)log_2 s} \\ but \\ 2^{(n-2)log_2 s} = x \\ \Rightarrow (n-2)log_2 s log_2 2 = log_2 x \\ or \\ (n-2)log_2 s = log_2 x \\ or \\ log_2 x = log_2 s^{n-2} \\ \Rightarrow x = s^{n-2} \\ hence \\ \lambda = 2 \cdot s^{n-2} \end{array} \right\}$$

$$= \frac{2s^{n-2}[s^{n-1}-1]}{s^{n-3}[s-1]} + 1$$

$$= \frac{2s[s^{n-1}-1]}{s-1} + 1$$

$= \frac{2s[s^{n-1}-1]+[s-1]}{s-1}$

$= \frac{2s^n-2s+s-1}{s-1}$

$= \frac{2s^n-s-1}{s-1}$

$= \frac{2s^n-s-2+1}{s-1}$

$= \frac{2s^n-2-s+1}{s-1}$

$= \frac{2[s^n-1]-[s-1]}{s-1}$

$= \frac{2[s^n-1]}{s-1} - 1$

$= \frac{2[s^n-1]}{s-1} - 1$

thus the parameters above yields

$$OA\left[2s^n, \tfrac{2[s^n-1]}{s-1} - 1, s, 2\right] \tag{4.4.1}$$

Since

$$\lambda s^2 = 2 \cdot s^{n-2} \cdot s^2 = \text{ Number of Assemblies}$$

$$= 2 \cdot s^n \cdot s^{-2} \cdot s^2$$

$$= 2 \cdot s^n \cdot s^0$$

$$= 2 \cdot s^n$$

If $v = 1$, *then* $s = 2^v = 2^1 = 2$

$\Rightarrow d = \frac{u}{v} = \frac{u}{1} = d = 1 + (n-2)log_2 2$

$= 1 + (n-2)$

$= n - 1$

Theorem 4.4.1 yields an array

$$OA[2^{n+1}, 2^{n+1} - 3, 2, 2]$$

Example 4.4.3. *An orthogonal array with parameters in theorem 4.4.1 in the case* $s = 4$ *and* $n = 2$ *can be constructed using theorem 4.4.1*

Since $s = 4, n = 2$

$\Rightarrow 4 = 2^v \ , \Rightarrow 2^2 = 2^v$

and $v = 2$

but $u = 1 + (n-2)log_2 s$

i.e $u = 1 + (2-2)log_2 4$

$= 1$

hence

$d = \frac{u}{v} = \frac{1}{2} \notin \mathbb{N}$ *and* $d < 1$

$d = \frac{1}{2}, (0, \frac{1}{2}, 1)$

so we need only $D(0)$, *a difference scheme* $D(8, 8, 4)$ *to construct this array.*

The Difference Scheme

$$D(0) = D(8, 8, 4) = D(2^3, 2^3, 2^2)$$

can be obtained by starting with multiplication table of $GF(2^3)$ *using the irreducible polynomial* $f(x) = x^3 + x + 1$ *over* $GF(2)$*. The entries* $b_0 + b_1x + b_2x^2$ *in this table will then be mapped to* $b_0 + b_1x$ *and these images will be considered as the elements of* $GF(2^2)$ *with* x *written as* 2 *and* $x+1$ *as* 3.

0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3
0	1	2	3	0	1	2	3	1	0	3	2	1	0	3	2	2	3	0	1	2	3	0	1	3	2	1	0	3	2	1	0
0	2	0	2	3	1	3	1	1	3	1	3	2	0	2	0	2	0	2	0	1	3	1	3	3	1	3	1	0	2	0	2
0	3	2	1	3	0	1	2	1	2	3	0	2	1	0	3	2	1	0	3	1	2	3	0	3	0	1	2	0	3	2	1
0	0	3	3	2	2	1	1	1	1	2	2	3	3	0	0	2	2	1	1	0	0	3	3	3	3	0	0	1	1	2	2
0	1	1	0	2	3	3	2	1	0	0	1	3	2	2	3	2	3	3	2	0	1	1	0	3	2	2	3	1	0	0	1
0	2	3	1	1	3	2	0	1	3	2	0	0	2	3	1	2	0	1	3	3	1	0	2	3	1	0	2	2	0	1	3
0	3	1	2	1	2	0	3	1	2	0	3	0	3	1	2	2	1	3	0	3	0	2	1	3	0	2	1	2	1	3	0
0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3

TABLE 4.16: An $OA[32, 9, 4, 2]$

Parameters of this orthogonal array

Number of assemblies $= 32$

Number of constraints $= 9$

Number of symbols $= 4$

Strength $= 2$

$\lambda = 2$

4.5 Difference Schemes of Index 2

Several authors Masuyama (1969), Xu (1979), Jungnickel (1979), Xiang (1983) have studied the construction of index 2 difference schemes

$D(2s, 2s, s)$ where s is a power of a prime. The construction given below is a modification of Xu (1979), Jungnickel (1979).

If s is a power of 2, we have already seen that a difference scheme $D(2s, 2s, s)$ exists (in theorem 4.2.1) and in example 4.4.2 we have shown that an $OA[2s^2, 2s+1, s, 2]$ can be obtained. We will therefore restrict our consideration to the case when s is the power of an odd prime. We will write the elements of $GF(s)$ as $k_0, k_1, ..., k_{s-1}$, where $k_0 = 0$ and $k_i = k^i, i = 1, ..., s-1$, for a primitive element k. In particular $k_{s-1} = k^{s-1} = 1$.

Theorem 4.5.1.

If s is a power of an odd prime then there exists a difference scheme $D(2s, 2s, s)$ and an orthogonal array $OA(2s^2, 2s+1, s, 2)$.

Proof. We construct four $s \times s$ matrices

$A = (a_{ij}),\ B = (b_{ij}),\ C = (c_{ij}),\ F = (f_{ij})$

$0 \leq i,\ j \leq s-1$ whose entries are given by

$$\left\{\begin{array}{c} a_{ij} = k_i k_j \\ b_{ij} = k_i k_j + h k_j^2 \\ c_{ij} = k_i k_j + m k_i^2 \\ f_{ij} = n k_i k_j + g k_j^2 + e k_i^2 \end{array}\right\} \tag{4.5.1}$$

Where h, m, n, g, e are elements of $GF(s)$ that satisfy the conditions

$$n = 1 + 4he = e/m = n^2 - 4ge, \tag{4.5.2}$$

In particular we may take

$$n = k, h = \frac{1}{2}, m = \frac{k-1}{2k}, g = \frac{k}{2} \text{ and e} = \frac{k-1}{2}, \tag{4.5.3}$$

Then

$$D = \begin{bmatrix} A & C \\ B & F \end{bmatrix} \tag{4.5.4}$$

is a difference scheme $D(2s, 2s, s)$ based on the additive group $GF(s)$ □

Example 4.5.1. *We use theorem 4.5.1 to construct a difference scheme* $D(10, 10, 5)$*. We work modulo* 5*, taking* $k = 2$ *as the primitive element of* $GF(5)$ *and use* (4.5.3) *to obtain the difference scheme exhibited in table 4.17 and use this to construct an orthogonal array* $OA[50, 11, 5, 2]$ *in table 4.18*

0	0	0	0	0	0	0	0	0	0
0	4	3	1	2	1	0	4	2	3
0	3	1	2	4	4	2	0	1	3
0	1	2	4	3	1	2	3	0	4
0	2	4	3	1	4	1	3	2	0
0	2	3	2	3	0	4	1	4	1
0	1	1	3	0	2	4	4	3	2
0	0	4	4	2	3	3	1	1	2
0	3	0	1	1	2	3	2	4	4
0	4	2	0	4	3	1	2	3	1

TABLE 4.17: A difference scheme $D(10, 10, 5)$

where

$$n = k = 2$$

$$h = \frac{1}{2}$$

$$m = \frac{2-1}{(2)(2)} = \frac{1}{4} = \frac{k-1}{2k}$$

$$g = \frac{k}{2} = \frac{2}{2} = 1$$

$$e = \frac{k-1}{2} = \frac{2-1}{2} = \frac{1}{2}$$

We can check :

$$n = 1 + 4he = \frac{e}{m} = n^2 - 4ge$$

$$= 1 + 4 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2^2 - 4 \cdot 1 \cdot \frac{1}{2} = 2$$

0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2
0	4	3	1	2	1	0	4	2	3	1	0	4	2	3	2	1	0	3	4	2	1	0	3	4
0	3	1	2	4	4	2	0	1	3	1	4	2	3	0	0	3	1	2	4	2	0	3	4	1
0	1	2	4	3	1	2	3	0	4	1	2	3	0	4	2	3	4	1	0	2	3	4	1	0
0	2	4	3	1	4	1	3	2	0	1	3	0	4	2	0	2	4	3	1	2	4	1	0	3
0	2	3	2	3	0	4	1	4	1	1	3	4	3	4	1	0	2	0	2	2	4	0	4	0
0	1	1	3	0	2	4	4	3	2	1	2	2	4	1	3	0	0	4	3	2	3	3	0	2
0	0	4	4	2	3	3	1	1	2	1	1	0	0	3	4	4	2	2	3	2	2	1	1	4
0	3	0	1	1	2	3	2	4	4	1	4	1	2	2	3	4	3	0	0	2	0	2	3	3
0	4	2	0	4	3	1	2	3	1	1	0	3	1	0	4	2	3	4	2	2	1	4	2	1
0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4

2	2	2	2	2	3	3	3	3	3	3	3	3	3	3	4	4	4	4	4	4	4	4	4	4
3	2	1	4	0	3	2	1	4	0	4	3	2	0	1	4	3	2	0	1	0	4	3	1	2
1	4	2	3	0	3	1	4	0	2	2	0	3	4	1	4	2	0	1	3	3	1	4	0	2
3	4	0	2	1	3	4	0	2	1	3	0	1	3	2	4	0	1	3	2	0	1	2	4	3
1	3	0	4	2	3	0	2	1	4	2	4	1	0	3	4	1	3	2	0	3	0	2	1	4
2	1	3	1	3	3	0	1	0	1	3	2	4	2	4	4	1	2	1	2	4	3	0	3	0
4	1	1	0	4	3	4	4	1	3	0	2	2	1	0	4	0	0	2	4	1	3	3	2	1
0	0	3	3	4	3	3	2	2	0	1	1	4	4	0	4	4	3	3	1	2	2	0	0	1
4	0	4	1	1	3	1	3	4	4	0	1	0	2	2	4	2	4	0	0	1	2	1	3	3
0	3	4	0	3	3	2	0	3	2	1	4	0	1	4	4	3	1	4	3	2	0	1	2	0
0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4

TABLE 4.18: $OA[50, 11, 5, 2] : \lambda = 2$

Parameters of this array

- Number of assemblies = 50
- Number of constraints = 11
- Number of symbols = 4
- Strength =2
- $\lambda = 2$

Example 4.5.2. *Table 4.19 shows a difference scheme* $D(6, 6, 3)$ *constructed in a similar way from* $GF(3)$

0	0	0	0	0	0
0	1	2	1	2	0
0	2	1	1	0	2
0	2	2	0	1	1
0	0	1	2	2	1
0	1	0	2	1	2

TABLE 4.19: A difference Scheme $D(6, 6, 3)$

Theorem 4.5.1 provides a simple way of constructing an $OA(2s^2, 2s + 1, s, 2)$, there is an analogous construction for orthogonal arrays $OA(2s^n, \frac{2(s^n-1)}{(s-1)} - 1, s, 2)$, with $n \geq 3$ that is also very simple.

The recursive construction that we give makes use of a series of difference schemes. This is based on the work of several authors, including Shrikhande (1964), Masuyama (1969), Xu (1979), Mukhopadhyay (1981).

The following lemma, due to Shrikhande (1964), is a convenient tool for recursively constructing difference schemes.

Lemma 4.5.1.

The tensor product of difference schemes $D(r_1, c_1, s)$ and $D(r_2, c_2, s)$ based on the abelian group $\mathcal{A}$ is a difference scheme $D(r_1 r_2, c_1 c_2, s)$ based on $\mathcal{A}$

Proof. Let $D = (d_{ij})$ be a difference scheme $D(r_1, c_1, s)$ and $D' = (d'_{lm})$ a difference scheme $D(r_2, c_2, s)$. The entries of $c_1 - c_2$ where c_1 and c_2 are distinct columns of $D \otimes D'$ are of the form

$d_{ij} + d'_{lm} - d_{ij'} - d'_{lm'}, i = 1, ..., r_1$

$l = 1, ..., r_2$ for fixed $j, j' \in \left\{1, ..., c_1\right\}$,

$m, m' \in [1, ..., c_2]$

and $(j, m) \neq (j', m')$. If $j \neq j'$ then for any fixed l we see that

$$d_{ij} + d'_{lm} - d_{ij'} - d'_{lm'}, i = \ell, ..., r_1,$$

contains every element of $\mathcal{A}$ equally often because D is a difference scheme. If $j = j'$ then $m \neq m'$ and for every fixed i we see that

$$d_{ij} + d'_{lm} - d_{ij'} - d'_{lm'} = d'_{lm} - d'_{lm'} l = 1, ..., r_2,$$

Contains every elements of $\mathcal{A}$ equally often, because D is a difference scheme. This shows that $D \otimes D'$ is a difference scheme. □

Corollary 4.5.1.

For any $n \geq 1$ and prime power s, there exists a difference scheme $D(2s^n, 2s^n, s)$ based on the additive group associated with $GF(s)$.

Proof. If $s = 2^v, v \geq 1$ then we know that from the proof of theorem 4.2.1 that there exists a difference scheme $D(2s, 2s, s)$ based on the additive group $GF(s)$. If s is a power of an odd prime the existence of such a difference scheme was established in the proof of theorem 4.5.1. From the proof of theorem 4.2.1 we also know that there exists a difference scheme $D(s, s, s)$ based on the additive group of $GF(s)$. The desired result now follows by repeatedly using these difference schemes in Lemma 4.5.1

This corollary enables us to establish the claim that the family of orthogonal arrays $OA(2s^n, \frac{2(s^n-1)}{(s-1)} - 1, s, 2)$ can also be obtained via the use of difference schemes if $n \geq 3$. We already know that this is true if s is a power of 2 from example 4.4.2. Now we need to distinguish between odd and even values of s. □

Theorem 4.5.2.

If s is a power of a prime and $n \geq 2$, then an orthogonal array

$$OA[2s^n, \frac{2(s^n-1)}{(s-1)} - 1, s, 2]$$

can be obtained by using difference schemes.

Proof. From corollary 4.5.1 and theorem 4.3.2 we know how to construct a completely resolvable $OA[2s^m, 2s^{m-1}, s, 2]$ say $A(m)$, For $m \geq 2$ applying theorem 4.3.1 to $A(3)$ and $A(2)$ results in an s resolvable $OA[2s^3, 2s^2 + 2s, s, 2]$ say $A(2,3)$. Applying theorem 4.3.1 to $A(4)$ and $A(2,3)$ results in an s^2 resolvable $OA[2s^4, 2s^3 + 2s^2 + 2s, s, 2]$ say $A(2,3,4)$. Continuing this way we eventually obtain an s^{n-2} resolvable $OA[2s^n, 2s^{n-1} + 2s^{n-2} + \cdots + 2s, s, 2]$ say $A(2, 3, \cdots, n)$. As in corollary 4.3.1 we can add one more factor but $2s^{n-1} + 2s^{n-2} + \cdots + 2s + 1 = 2[s^{n-1} + s^{n-2} + \cdots + s] + 1 = \frac{2(s^n-1)}{(s-1)} - 1$

Hence this gives the desired orthogonal array. □

4.6 Orthogonal Arrays and Hadamard Matrices

Hadamard matrices are square matrices with $+1's$ and $-1's$ as the only elements and whose rows are orthogonal. The study of two level orthogonal arrays of strength 2 and 3 is essentially equivalent to the study of these matrices.

Definition 4.6.1.

A hadamard matrix of order n is an $n \times n$ matrix H_n of $+1's$ and $-1's$ whose rows are orthogonal, that is, which satisfies

$$H_n H_n^{\intercal} = nI_n \tag{4.6.1}$$

For example, here are hadamard matrices of order $1, 2$ and 4.

$$H_1 = [1], H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{4.6.2}$$

These matrices are named after a French mathematician Jacques hadamard. In Hadamard (1893) he showed that if $A = (a_{ij})$ is an $n \times n$ matrix with $|a_{ij}| \leq 1$ then

$$|detA| \leq n^{\frac{n}{2}} \tag{4.6.3}$$

Hadamard matrices may be regarded as the special class of difference schemes $D(r, c, s)$ with $s = 2$, $r = c$ and index $\lambda = \frac{c}{2}$.

4.6.1 Basic Properties of Hadamard Matrices

Suppose H_n is a hadamard matrix of order n.Then $H_n^{-1} = n^{-1} H_n^{\intercal}$, so

$$H_n^{\intercal} H_n = nI_n \tag{4.6.4}$$

which implies that the columns of H_n are orthogonal. If H_n satisfies (4.6.1) or (4.6.4), then so does any matrix obtained from H_n by permuting its rows (or columns) and negating any of its rows or columns. All the matrices obtained in this way are said to be isomorphic or equivalent. By transformations of this kind, we can always arrange the first row and column of H_n to have entirely of $+1s$. Such a hadamard matrix is said to be Normalized.

Lemma 4.6.1.

Let H_n be a Normalized hadamard matrix of order $n, n > 2$ let $u = (u_1, ..., u_n)$ and $v = (v_1, ..., v_n)$ be any two distinct rows of H_n, not including the first. Then

a). There are $\frac{n}{2}$ coordinates i with $u_i = +1$ and $\frac{n}{2}$ with $u_i = -1$;

b). There are $\frac{n}{4}$ coordinates with $u_i = v_i = +1$, $\frac{n}{4}$ with $u_i = +1$,$v_i = -1$, $\frac{n}{4}$ with $u_i = -1$, $v_i = +1$, and $\frac{n}{4}$ with $u_i = v_i = -1$;

c). Similar results hold for the columns of H_n

Proof. These are the immediate consequences of orthogonal relations (4.6.1) and (4.6.3). □

Corollary 4.6.1.

If a hadamard matrix H_n exists then n is $1, 2$ or a multiple of 4. The hadamard conjencture is that the converse to this corollary holds.

4.6.2 The Connection between hadamard matrices and orthogonal arrays

Rao (1946) proved $f(4\lambda, 2, 2) \leq 4\lambda - 1$ and $f(8\lambda, 2, 3) \leq 4\lambda$. As we shall now show equality holds in these two bounds if and only if there exists a hadamard matrix of order 4λ

Theorem 4.6.2.1.

Orthogonal arrays $OA[4\lambda, 4\lambda - 1, 2, 2]$ and $OA[8\lambda, 4\lambda, 2, 3]$ exists if and only if there exists a hadamard matrix of order 4λ.

Proof. An $OA[4\lambda, 4\lambda - 1, 2, 2]$ exists if and only if an $OA[8\lambda, 4\lambda, 2, 3]$ exists as shown by Hedayat et al. (1997). Suppose $H_{4\lambda}$ is a normalized hadamard matrix by Lemma 4.6.1 the matrix obtained by omitting the first column of $H_{4\lambda}$ is an $OA[4\lambda, 4\lambda - 1, 2, 2]$. Conversely let A be an $OA[4\lambda, 4\lambda - 1, 2, 2]$ in which the levels are $+1$ and -1. It follows from the definition of an orthogonal array that the matrix formed by adding the initial column of $+1$ to A satisfies equation (4.6.1).

It can aslo be shown that $F(4\lambda - 1 - \mu, 2, 2) \geq 4\lambda$ and $F(4\lambda - \mu, 2, 3) \geq 8\lambda$ for $\lambda \geq 1, 0 \leq \mu \leq 3$. These two inequalities become equalities if and only if a hadamard matrix $H_{4\lambda}$ exists.

In view of theorem 4.6.2.1 orthogonal arrays with parameters $OA(4\lambda, 4\lambda - 1, 2, 2)$ and $OA(8\lambda, 4\lambda, 2, 3)$ are called **hadamard arrays.** □

Theorem 4.6.2.2.

A hadamard matrix H_n exists if and only if a difference scheme $D(n, n, 2)$ exists

Proof. By lemma 4.6.1 any two distinct columns of a hadamard matrix H_n must agree in $n/2$ places and disagree in $n/2$ places. The component-wise product of these columns therefore contains $n/2$ $+1's$ and $n/2$ $-1's$ and so H_n is a difference scheme $D(n, n, 2)$ based on the multiplicative group $[+1, -1]$. □

4.6.3 Construction of hadamard matrices

Several classical methods for constructing hadamard matrices are available in literature but in this thesis we shall only construct **the Sylvester type hadamard matrices** using the tensor product construction of definition 4.3.2. This method can enable one to construct hadamard matrices of most orders up to 200.

Theorem 4.6.3.1.

The tensor product of $H_a \otimes H_b$ of hadamard matrices of order a and b is a hadamard matrix of order ab.

Proof. This is an immediate consequence of lemma 4.5.1 and theorem 4.6.2.2 So once we have a hadamard matrix of order b we can immediately obtain matrices of order $2b, 4b, 8b, 16b, ...$ by repeatedly tensoring with the matrix

$$H_2 = \begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix}$$

The hadamard matrices of order 2^m obtained in this way by starting from $b = 1$ and $H_1 = [1]$ are called sylvester type matrices.

Using lemma 4.3.1 the difference scheme $D(8, 8, 2)$ can be used to construct an orthogonal array $OA(16, 8, 2, 2)$ □

+	+	+	+	+	+	+	+
+	-	+	-	+	-	+	-
+	+	-	-	+	+	-	-
+	-	-	+	+	-	-	+
+	+	+	+	-	-	-	-
+	-	+	-	-	+	-	+
+	+	-	-	-	-	+	+
+	-	-	+	-	+	+	-

TABLE 4.20: Sylvester type hadamard matrix of order 8

0	0	0	0	0	0	0	0
0	1	0	1	0	1	0	1
0	0	1	1	0	0	1	1
0	1	1	0	0	1	1	0
0	0	0	0	1	1	1	1
0	1	0	1	1	0	1	0
0	0	1	1	1	1	0	0
0	1	1	0	1	0	0	1

TABLE 4.21: Difference Scheme $D(8, 8, 2)$

0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0
0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
0	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1
0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
0	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1
0	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1
0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0

TABLE 4.22: $OA[16, 8, 2, 2]\lambda = 4$

4.7 Orthogonal Arrays Of Strength $t > 2$

In order to obtain orthogonal arrays of higher strength by developing a difference scheme, the scheme should posses Additional regularity property. We call such schemes "difference schemes of strength t". The case $t = 2$ will correspond to ordinary difference schemes. These notions were first formulated by Seiden (1954).

Let $\mathcal{A}$ be an abelian group of order s. By $\mathcal{A}^t$, for $t \geq 1$ we will denote the abelian group of order s^t consisting of $t-$tuples of elements from $\mathcal{A}$ with the usual vector addition as the binary operation. Further, let

$$\mathcal{A}_0^t = \Big\{(x_1, ..., x_t) : x_1 = ... = x_t \in \mathcal{A}\Big\}$$

Then $\mathcal{A}_0^t$ is a subgroup of $\mathcal{A}^t$ of order s, and we will denote its cosets by $\mathcal{A}_i^t, i = 1, ..., s^{t-1} - 1$

Definition 4.7.1.

An $r \times c$ array D based on $\mathcal{A}$ is a difference scheme of strength t if for every $r \times t$ subarray each set $\mathcal{A}_i^t, i = 0, 1, ..., s^{t-1} - 1$, is represented equally often when the rows of the subarray are viewed as elements of $\mathcal{A}^t$. It follows that r is a multiple of s^{t-1}, say $r = \lambda s^{t-1}$. We denote such an array by $D_t(r, c, s)$. For $t = 2$ this definition is equivalent to definition 4.2.1. Developing a difference scheme of strength t results in an orthogonal array of strength t, to which under certain conditions at least one additional factor can be added.

Theorem 4.7.1.

A difference scheme $D_t(r, c, s)$ of strength t can be used to construct an $OA[rs, c, s, t]$. If the difference scheme itself is already an orthogonal array of strength $t-1$ or if it can be written as the Juxtaposition of s Difference schemes $D_{t-1}(\frac{r}{s}, c, s)$ then it can be used to construct an $OA[rs, c+1, s, t]$. Note that corollary 4.3.2 is a special case of this theorem, since any difference scheme $D_2(r, c, s)$ is the Juxtaposition of s difference schemes $D_1(\frac{r}{s}, c, s)$

Example 4.7.1. *Let D be an $OA[4\lambda, k, s, 2]$ over $(GF(2), +), k \geq 3$. Then D is also a difference scheme of strength 3. So we can use it to construct an $OA[8\lambda, k + 1, 2, 3]$*

Example 4.7.2. *An* $OA(8,7,2,2)$ *which is also a difference scheme* $D_3^T(8,7,2)$ *of strength* 3 *over* $(GF(2),+)$ *is shown below.*

0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	0	1	1
0	1	1	1	1	0	0
1	0	0	1	1	0	1
1	0	1	1	0	1	0
1	1	0	0	1	1	0
1	1	1	0	0	0	1

TABLE 4.23: Difference Scheme $D_3^T(8,7,2)$ of Strength 3

The resulting $OA(16,8,2,3)$ is shown in table 4.24

0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0
0	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1
0	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1
0	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1
0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

TABLE 4.24: $OA[16,8,2,3]$ $\lambda = 2$

4.8 The Product Of Balanced Arrays

Bush (1952) provided the following theorem for the product of orthogonal arrays

Theorem 4.8.1.

The existence of $OA[N_i, k_i, s_i, t]$ for $i = 1, 2, ..., m$ implies the existence of an $OA[N, k, s, t]$ where $N = N_1N_2...N_m$, $s = s_1s_2...s_m$ and $k = min(k_1, k_2, ..., k_m)$.
The product of orthogonal arrays can generate orthogonal arrays from several known orthogonal arrays. The procedure can be similarly used to determine the product of balanced arrays.

Theorem 4.8.2.

The existence of $BA[N_i, k_i, s_i, t]$ for $i = 1, 2, ..., m$ implies the existence of a $BA[N, k, s, t]$ where $N = N_1N_2...N_m$, $s = s_1s_2...s_m$ and $k = min(k_1, k_2, ..., k_m)$.
If the symbols of the $BA[N, k, s, t]$ are denoted by ordered k tuples then the parameters are

$$\lambda((a_{11}, a_{21}, ..., a_{m1})(a_{12}, a_{22}, ..., a_{m2})...(a_{1t}, a_{2t}, ..., a_{mt}))$$

$$= \lambda(a_{11}, a_{12}, ..., a_{1t})\lambda(a_{21}, a_{22}, ..., a_{2t})...\lambda(a_{m1}, a_{m2}, ..., a_{mt})$$

Proof. Let the $BA(N_1, k_1, s_1, t)$ be denoted by the $k_1 \times N_1$ matrix $A = (aij)$ and the $BA[N_2, k_2, s_2, t]$ be denoted by $k_2 \times N_2$ matrix $B = (bij)$. Let A_1 and B_1 denote the first k rows of A and B respectively. Then form the $k \times N_1N_2$ matrix.

$$(a_{11}, b_{11})...(a_{11}, b_{1N2})...(a_{1N1}, b_{11})...(a_{1N1}, b_{1N2})$$

$$(a_{21}, b_{21})...(a_{21}, b_{2N2})...(a_{2N1}, b_{21})...(a_{2N1}, b_{2N2})$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(a_{k1}, b_{k1}) \cdots (a_{k1}, b_{kN2}) \cdots (a_{kN1}, b_{k1}) \cdots (a_{kN1}, b_{kN2})$$

This is a $BA[N_1N_2, k, s_1s_2, t]$ with parameters $\lambda((a_1, b_1), ..., (a_t, b_t))$

$$= \lambda(a_1, ...a_t)\lambda(b_1, ..., b_t).$$

From this array by following the same procedure with $BA(N_3, k_3, s_3, t)$ we get a $BA(N_1N_2N_3, k, s_1s_2s_3, t)$. Continuing this procedure,we finally get a $BA[N, k, s, t]$. □

Example 4.8.1. *The product of the following balanced arrays*

BA[2,2,2,2]

0	1
1	0

BA[6,2,3,2]

0	1	2	0	1	2
1	2	0	2	0	1

$\lambda(0, 1) = 1 = \lambda(1, 0)$ $\quad$ $\lambda(0, 0) = \lambda(1, 1) = \lambda(2, 2) = 0$

$\lambda(0, 0) = \lambda(1, 1) = 0$ $\quad$ $\lambda(0, 1) = \lambda(0, 2) = \lambda(1, 0) = \lambda(2, 0)$

$$= \lambda(1, 2) = \lambda(2, 1) = 1$$

is a $BA[12, 2, 6, 2]$

00	01	02	00	01	02	10	11	12	10	11	12
11	12	10	12	10	11	01	02	00	02	00	01

with parameters

$$\lambda(a_1b_1, a_2b_2) = \begin{Bmatrix} 0 & if & a_1 = a_2 & or & b_1 = b_2 \\ 1 & & Otherwise & & \end{Bmatrix}$$

4.9 Construction Of Some Balanced Arrays of strength $t = 2$

In this section we are interested in construction of balanced arrays of strength $t = 2$ with parameters $\lambda(x, y) = \lambda_1$ or λ_2 according as $x = y$ or not. In particular we are interested in the $BA[(ks-1)s\lambda, ks, s, 2]$ with parameters $\lambda(x, y) = (k-1)\lambda$ or $(k\lambda)$ according as $x = y$ or Not. For brevity we shall call it the balanced array of type T with index λ and denote it by $BA[T][k, s, \lambda]$.

It is clear that a $BA[T][1, s, \lambda]$
$= BA[\lambda s(s-1), s, s, 2]$

$= TA[\lambda s(s-1), s, s, 2]$.

In constructing a $BA[T][k, s, \lambda]$ for any given k and s we would like λ to be as small as possible so that the size of the balanced array is not too large. However if there is no restriction on λ, we can always construct a $BA[T][k, s, \lambda]$ for any k and s.

Theorem 4.9.1.

For all k and s,there always exists a $BA[T][k, s, \lambda]$ for some λ.

Proof. For all k and s, there exists a $TA[(ks-1)ksn, ks, ks, 2]$ for some n from the discussion in section 4.1. Let the symbols of the transitive array be denoted by $[0, 1, ..., ks-1]$. If we replace each symbol in the transitive array by $x(modk)$. Then the transitive array becomes a $BA[(ks-1)ksn, ks, s, 2]$ with parameters $\lambda(x, y) = (k-1)kn$ or k^2n according as $x = y$ or not, which is a $BA[T][ks, s, kn]$. The method of construction in theorem 4.9.1 does not usually provide balanced arrays with a small number of assemblies as we desire. □

Example 4.9.1. *Suppose* $k = 2$,$s = 2$, *and* $n = 1$ *then we can construct a* $TA[(ks-1)ksn, ks, ks, 2] = TA[12, 4, 4, 2]$

3	1	0	2	1	3	2	0	2	0	1	3
1	3	2	0	2	0	1	3	3	1	0	2
2	0	1	3	3	1	0	2	1	3	2	0
0	2	3	1	0	2	3	1	0	2	3	1

Table 4.25: $TA[12, 4, 4, 2]$

replacing every symbol in $TA[12, 4, 4, 2]$ by $x(mod2)$, we shall have a $BA[12, 4, 2, 2] = BA(T)[2, 2, 2]$

1	1	0	0	1	1	0	0	0	0	1	1
1	1	0	0	0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0	1	1	0	0
0	0	1	1	0	0	1	1	0	0	1	1

Table 4.26: BA[12,4,2,2]

parameters of $BA[12, 4, 4, 2]$ are

$\lambda(0, 0) = \lambda(1, 1) = 2 = (k-1)kn$ and
$\lambda(1, 0) = \lambda(0, 1) = 4 = k^2n.$

We shall now discuss methods of constructing balanced arrays of type T with index unity.

Theorem 4.9.2.

The existence of a partly resolvable (Definition 4.3.1) $OA[ks^2, ks, s, 2]$is equivalent to the existence of a $BA[T][k, s, 1]$

Proof. If a partly resolvable $OA[ks^2, ks, s, 2]$ exists then there exists s assemblies which form $OA[s, ks, s, 1]$. We can permute the symbols of the orthogonal array in each row such that these s, assemblies are of the form $(i, i, ..., i)'$ for $i = 0, 1, ..., s-1$. Deleting these assemblies we obtain a $BA[T][k, s, 1]$ □

Example 4.9.2. *Suppose* $k = 2$ *and that* $s = 5$, *we can construct* $OA[50, 10, 5, 2]$ *by developing a difference scheme* $D(10, 10, 5)$ *as in example* 4.5.1 *and also a* $BA[45, 10, 5, 2]$. *The Difference Scheme* $D(10, 10, 5)$ *is*

0	0	0	0	0	0	0	0	0	0
0	4	3	1	2	1	0	4	2	3
0	3	1	2	4	4	2	0	1	3
0	1	2	4	3	1	2	3	0	4
0	2	4	3	1	4	1	3	2	0
0	2	3	2	3	0	4	1	4	1
0	1	1	3	0	2	4	4	3	2
0	0	4	4	2	3	3	1	1	2
0	3	0	1	1	2	3	2	4	4
0	4	2	0	4	3	1	2	3	1

TABLE 4.27: Table $D(10, 10, 5)$

0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2
4	3	1	2	1	0	4	2	3	0	4	2	3	2	1	0	3	4	1	0	3	4	3
3	1	2	4	4	2	0	1	3	4	2	3	0	0	3	1	2	4	0	3	4	1	1
1	2	4	3	1	2	3	0	4	2	3	0	4	2	3	4	1	0	3	4	1	0	3
2	4	3	1	4	1	3	2	0	3	0	4	2	0	2	4	3	1	4	1	0	3	1
2	3	2	3	0	4	1	4	1	3	4	3	4	1	0	2	0	2	4	0	4	0	2
1	1	3	0	2	4	4	3	2	2	2	4	1	3	0	0	4	3	3	3	0	2	4
0	4	4	2	3	3	1	1	2	1	0	0	3	4	4	2	2	3	2	1	1	4	0
3	0	1	1	2	3	2	4	4	4	1	2	2	3	4	3	0	0	0	2	3	3	4
4	2	0	4	3	1	2	3	1	0	3	1	0	4	2	3	4	2	1	4	2	1	0

2	2	2	2	3	3	3	3	3	3	3	3	3	4	4	4	4	4	4	4	4	4
2	1	4	0	2	1	4	0	4	3	2	0	1	3	2	0	1	0	4	3	1	2
4	2	3	0	1	4	0	2	2	0	3	4	1	2	0	1	3	3	1	4	0	2
4	0	2	1	4	0	2	1	3	0	1	3	2	0	1	3	2	0	1	2	4	3
3	0	4	2	0	2	1	4	2	4	1	0	3	1	3	2	0	3	0	2	1	4
1	3	1	3	0	1	0	1	3	2	4	2	4	1	2	1	2	4	3	0	3	0
1	1	0	4	0	4	1	3	0	2	2	1	0	0	0	2	4	1	3	3	2	1
0	3	3	4	3	2	2	0	1	1	4	4	0	4	3	3	1	2	2	0	0	1
0	4	1	1	1	3	4	4	0	1	0	2	2	2	4	0	0	1	2	1	3	3
3	4	0	3	2	0	3	2	1	4	0	1	4	3	1	4	3	2	0	1	2	0

TABLE 4.28: Table $BA[45, 10, 5, 2]$

$$\left\{\begin{array}{c}
.\lambda(0,0)=\lambda(1,1)=\lambda(2,2)=\lambda(3,3)=\lambda(4,4)=1 \\
.\lambda(0,1)=\lambda(0,2)=\lambda(0,3)=\lambda(0,4) \\
=\lambda(1,0)=\lambda(1,1)=\lambda(1,2)=\lambda(1,3)=\lambda(1,4) \\
=\lambda(2,0)=\lambda(2,1)=\lambda(2,2)=\lambda(2,3)=\lambda(2,4) \\
=\lambda(3,0)=\lambda(3,1)=\lambda(3,2)=\lambda(3,3)=\lambda(3,4) \\
=\lambda(4,0)=\lambda(4,1)=\lambda(4,2)=\lambda(4,3)=\lambda(4,4) \\
=2
\end{array}\right\}$$

TABLE 4.29: Parameters of $BA[45, 10, 5, 2]$

Corollary 4.9.1.

Suppose there exists a $BA[T][k, s, 1]$ we can obtain a partly resolvable $OA[ks^2, ks, s, 2]$ by adding s assemblies of the form $(i, i, ..., i)'(i = 0, 1, ..., s-1)$.

We first give a method of constructing $BA(T)[k, 2, 1]$ by using hadamard matrices.

Corollary 4.9.2.

If a hadamard matrix of order $4k$ exists, then a $BA(T)[k, 2, 1]$ exists, and can always be constructed.

Proof. If a hadamard matrix of order $4k$ exists, we can arrange its elements such that all the elements in the first column and the first row are $+1$. All other columns must then contain $2k(+1's)$ and $2k(-1's)$. Deleting $2k$ rows whose second column is 1. We obtain $OA[4k, 2k, 2, 2]$ with all the elements equal to +1 in the first column and equal to −1 in the second column. By theorem 4.9.2 we can construct a $BA(T)[k, 2, 1]$ since the $OA[4k, 2k, 2, 2]$ is partly resolvable. □

Example 4.9.3. *Using the Sylvester type hadamard matrix of order* 8 *in table 4.20* $k = 4$*, that leads to an* $OA[16, 8, 2, 2]$ *in table 4.22. Using theorem 4.9.2 we obtain* $BA[T][4, 2, 1]$.

0	0	0	0	0	0	0	1	1	1	1	1	1	1
1	0	1	0	1	0	1	0	1	0	1	0	1	0
0	1	1	0	0	1	1	1	0	0	1	1	0	0
1	1	0	0	1	1	0	0	0	1	1	0	0	1
0	0	0	1	1	1	1	1	1	1	0	0	0	0
1	0	1	1	0	1	0	0	1	0	0	1	0	1
0	1	1	1	1	0	0	1	0	0	0	0	1	1
1	1	0	1	0	0	1	0	0	1	0	1	1	0

TABLE 4.30: Table $BA[14, 8, 2, 2] = BA[(T)[4, 2, 1]]$

Parameters of $BA(T)[4, 2, 1]$

- $\lambda(0, 0) = \lambda(1, 1) = 3$
- $\lambda(0, 1) = \lambda(1, 0) = 4$

If the symbols of the $BA(T)[k, 2, 1]$ are denoted by 0 and 1, the balanced array becomes the incidence matrix of a balanced incomplete block design with $2k$ treatments, $4k - 2$ blocks of k plots each and any two treatments occur together in $k - 1$ blocks. Thus we state the following theorem without proof;

Theorem 4.9.3.

The existence of a $BA(T)[k, 2, 1]$ is equivalent to the existence of a BIBD$[2k, 2k - 2, k]$.

The corollary below follows from theorem 4.9.3

Corollary 4.9.3.

If a hadamard matrix of order $4k$ exists, then a BIBD$[2k, 2k - 2, k]$ can always be constructed since it is well known that hadamard matrices of order $4k$ exist for all $k \leq 25$ we can always construct a $BA(T)[k, 2, 1]$ for $k = 1, 2, ..., 25$.

Corollary 4.9.4.

If k and s are both powers of the same prime p a $BA(T)[k, s, 1]$ can always be constructed.

Proof. By theorem 4.4.1 we can construct a completely resolvable orthogonal array $OA[\lambda s^2, \lambda(s+1)+1, s, 2]$ by deleting any $\lambda+1$ constraints(factors) we obtain $OA[\lambda s^2, \lambda s, s, 2]$. Then theorem 4.9.2 is applied. □

Example 4.9.4. *For $k = 3$ and $s = 3$ we can construct a $BA(T)[3, 3, 1]$ by first constructing a completely resolvable $OA[27, 9, 3, 2]$ which is exhibited in table 4.14. Applying theorem 4.9.2, we obtain $BA(T)[3, 3, 1]$*

0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2
1	2	0	1	2	0	1	2	2	0	1	2	0	1	2	0	0	1	2	0	1	2	0	1
2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0
0	0	1	1	1	2	2	2	1	1	2	2	2	0	0	0	2	2	0	0	0	1	1	1
1	2	1	2	0	2	0	1	2	0	2	0	1	0	1	2	0	1	0	1	2	1	2	0
2	1	1	0	2	2	1	0	0	2	2	1	0	0	2	1	1	0	0	2	1	1	0	2
0	0	2	2	2	1	1	1	1	1	0	0	0	2	2	2	2	2	1	1	1	0	0	0
1	2	2	0	1	1	2	0	2	0	0	1	2	2	0	1	0	1	1	2	0	0	1	2
2	1	2	1	0	1	0	2	0	2	0	2	1	2	1	0	1	0	1	0	2	0	2	1

TABLE 4.31: Table $BA(T)[3, 3, 1] = BA[24, 9, 3, 2]$

Parameters are;

$\lambda(0, 0) = \lambda(1, 1) = \lambda(2, 2) = 2$

$\lambda(0, 1) = \lambda(1, 0) = \lambda(0, 2) = \lambda(2, 0) = \lambda(1, 2) = \lambda(2, 1) = 3$

Corollary 4.9.5.

If $s = p^n, k = 2s^l$ where p is an odd prime, $n \geq 1$ and $l \geq 0$, then a $BA(T)[k, s, 1]$ can always be constructed.

Proof. By using theorem 4.5.1, we can construct $OA[ks^2, ks, s, 2]$ by developing a difference scheme $D(2s, 2s, s)$. We then apply theorem 4.9.2 to construct a $BA(T)(k, s, \lambda)$ □

Example 4.9.5. *For $s = 3$ and $k = 2$ implies $3 = 3^1, k = 2.3^0 \mapsto n = 1$ and $l = 0$ We can therefore construct*

$$OA[2.3^2, 2.3, 3, 2] = OA[18, 6, 3, 2]$$

by developing a difference scheme $D(2s, 2s, s) = D(6, 6, 3)$ which is exhibited in table 4.19

0	0	0	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2
0	1	2	1	2	0	1	2	0	2	0	1	2	0	1	0	1	2
0	2	1	1	0	2	1	0	2	2	1	0	2	1	0	0	2	1
0	2	2	0	1	1	1	0	0	1	2	2	2	1	1	2	0	0
0	0	1	2	2	1	1	1	2	0	0	2	2	2	0	1	1	0
0	1	0	2	1	2	1	2	1	0	2	0	2	0	2	1	0	1

Table 4.32: Table $OA[18, 6, 3, 2]$

0	0	0	0	0	1	1	1	1	1	2	2	2	2	2
1	2	1	2	0	2	0	2	0	1	0	1	0	1	2
2	1	1	0	2	0	2	2	1	0	1	0	0	2	1
2	2	0	1	1	0	0	1	2	2	1	1	2	0	0
0	1	2	2	1	1	2	0	0	2	2	0	1	1	0
1	0	2	1	2	2	1	0	2	0	0	2	1	0	1

Table 4.33: Table $BA(T)[2, 3, 1] = BA[15, 6, 3, 2]$

Applying theorem 4.9.2 to this orthogonal array we obtain $BA(T)[2, 3, 1]$

Parameters of $BA(t)[2, 3, 1]$

- $\lambda(0, 0) = \lambda(1, 1) = \lambda(2, 2) = 1$
- $\lambda(0, 1) = \lambda(1, 0) = \lambda(0, 2) = \lambda(2, 0) = \lambda(1, 2) = \lambda(2, 1) = 2$

The method of Difference Schemes used in the construction of orthogonal arrays can also be used to construct the type of balanced arrays discussed in this section.

Theorem 4.9.4.

Let M be a module of s elements. It is possible to choose k rows and N columns ($N = \lambda_1 + \lambda_2(s-1)$, λ_1 and λ_2 integers)

$$\begin{array}{cccccc} a_{11} & a_{12} & . & . & . & a_{1N} \\ a_{21} & a_{22} & . & . & . & a_{2N} \\ . & . & . & . & . & . \\ a_{k1} & a_{k2} & . & . & . & a_{kN} \end{array}$$

with elements belonging to M such that among the differences of the corresponding elements of any two rows, the element 0 occurs λ_1 times and the other non zero elements occur λ_2 times, then by adding the elements of the module to the elements in the above array and reducing mod s, we can generate Ns columns: this constitutes a $BA[N, k, s, 2]$ with parameters $\lambda(x, y) = \lambda_1$ or λ_2 according as $x = y$ or $x \neq y$.

The balanced arrays that can be constructed by theorem 4.9.4 are completely resolvable. We shall give the following four examples to illustrate the application of theorem 4.9.4.

Example 4.9.6. *Let* $M = [0, 1, 2]$. *Among the differences of the corresponding elements of any two rows of the following array* 0 *occurs once wheres* 1 *and* 2 *occur three times each.*

$$\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 & 0 & 1 & 2 \\ 2 & 1 & 2 & 1 & 0 & 2 & 1 \end{array}$$

hence we can construct a $BA[21, 3, 3, 2]$ *shown in table 4.34 below*

0	0	0	0	0	0	0	1	1	1	1	1	1	1	2	2	2	2	2	2	2
1	2	1	2	0	1	2	2	0	2	0	1	2	0	0	1	0	1	2	0	1
2	1	2	1	0	2	1	0	2	0	2	1	0	2	1	0	1	0	2	1	0

TABLE 4.34: Table BA[21,3,3,2]

Parameters of $BA[21, 3, 3, 2]$ are

- $\lambda(0,0) = \lambda(1,1) = \lambda(2,2) = 1$

  $\lambda(0,1) = \lambda(1,0) = \lambda(0,2) = \lambda(2,0) = \lambda(1,2) = \lambda(2,1) = 3$

Example 4.9.7. *Let $M = [0, 1]$. Among the differences of the corresponding elements of any two rows of the following array* 0 *occurs twice wheres* 1 *occurs four times*

0	0	0	0	0	0
1	1	0	1	0	1
0	0	1	1	1	1
1	1	1	0	1	0

hence we can construct a $BA[12, 4, 2, 2]$ shown in table 4.35 below

0	0	0	0	0	0	1	1	1	1	1	1
1	1	0	1	0	1	0	0	1	0	1	0
0	0	1	1	1	1	1	1	0	0	0	0
1	1	1	0	1	0	0	0	0	1	0	1

TABLE 4.35: Table $BA[12, 4, 2, 2]$

Parameters of BA[12,4,2,2]

- $\lambda(0,0) = \lambda(1,1) = 2$

  $\lambda(0,1) = \lambda(1,0) = 4$

Example 4.9.8. *Let $M = [0, 1, 2]$. Among the Differences of corresponding elements of any two rows of the following array,*0 *occurs* 6 *times wheres* 1 *and* 2 *each occur* 8 *times.*

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	2	0	1	0	0	2	1	2	1	2	1	1	0	2	0	0	1	2	1	2
1	1	2	2	0	1	0	0	2	1	2	2	2	1	1	0	2	0	0	1	2	1
2	1	1	2	2	0	1	0	0	2	1	1	2	2	1	1	0	2	0	0	1	2
1	2	1	1	2	2	0	1	0	0	2	2	1	2	2	1	1	0	2	0	0	1
2	1	2	1	1	2	2	0	1	0	0	1	2	1	2	2	1	1	0	2	0	0
0	2	1	2	1	1	2	2	0	1	0	0	1	2	1	2	2	1	1	0	2	0
0	0	2	1	2	1	1	2	2	0	1	0	0	1	2	1	2	2	1	1	0	2
1	0	0	2	1	2	1	1	2	2	0	2	0	0	1	2	1	2	2	1	1	0
0	1	0	0	2	1	2	1	1	2	2	0	2	0	0	1	2	1	2	2	1	1
2	0	1	0	0	2	1	2	1	1	2	1	0	2	0	0	1	2	1	2	2	1
2	2	0	1	0	0	2	1	2	1	1	1	1	0	2	0	0	1	2	1	2	2

hence we can construct a $BA[66, 12, 3, 2]$ *with parameters* $\lambda(x, y) = 6$ *or* 8 *according as* $x = y$ *or not. i.e* $BA(T)[4, 3, 2]$.

Example 4.9.9. *Let* $M = [0, 1, 2, 3]$*. Among the differences of the corresponding elements of any two rows of the following array,*0 *occurs* 4 *times, wheres* $1, 2$ *and* 3 *occur* 6 *times each.*

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	2	0	2	1	1	3	2	3	1	0	3	2	0	2	3	3	1	2	1
3	3	0	1	2	0	2	1	1	3	2	1	1	0	3	2	0	2	3	3	1	2
2	3	3	0	1	2	0	2	1	1	3	2	1	1	0	3	2	0	2	3	3	1
3	2	3	3	0	1	2	0	2	1	1	1	2	1	1	0	3	2	0	2	3	3
1	3	2	3	3	0	1	2	0	2	1	3	1	2	1	1	0	3	2	0	2	3
1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3	2	0	2
2	1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3	2	0
0	2	1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3	2
2	0	2	1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3
1	2	0	2	1	1	3	2	3	3	0	3	2	0	2	3	3	1	2	1	1	0
0	1	2	0	2	1	1	3	2	3	3	0	3	2	0	2	3	3	1	2	1	1

hence we can construct a $BA[88, 12, 4, 2]$ *with parameters* $\lambda(x, y) = 4$ *or* 6 *according as* $x = y$ *or not,i.e* $BA(T)[3, 4, 2]$

Efforts have been made to reduce the number of assemblies in example 4.9.8 and 4.9.9 by a half, i.e to construct a $BA(T)[4, 3, 1]$ and $BA(T)[3, 4, 1]$ but without success. Examples 4.9.3, 4.9.4, 4.9.5 can also be constructed by using theorem 4.9.4 but certainly there are balanced arrays which can be constructed by corollary 4.9.3 and cannot be constructed using theorem4.9.4. For example, a $BA(T)[3, 2, 1]$ which can be constructed using corollary 4.9.3, is not completely resolvable. Therefore it cannot be constructed using theorem 4.9.4. However, all balanced arrays that can be constructed using corollary 4.9.4 can also be constructed using theorem 4.9.4 since the orthogonal arrays used in corollary 4.9.4 are constructed using the method of Differences.

4.10 Balanced Arrays of Strength $t > 2$

Using theorem 4.7.1 and theorem 4.9.2 we can construct balanced arrays of strength $t > 2$.

Example 4.10.1. *We can apply theorem 4.9.2 in the* $OA[16, 8, 2, 2]$ *that appears in table 4.22 to obtain* $BA[14, 8, 2, 3]$ *shown below;*

0	0	0	1	1	1	1	1	1	1	0	0	0	0
0	1	1	0	0	1	1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0	1	0	1	0	1	0
0	1	1	1	1	0	0	1	0	0	0	0	1	1
1	0	1	1	0	1	0	0	1	0	0	1	0	1
1	1	0	0	1	1	0	0	0	1	1	0	0	1
1	1	0	1	0	0	1	0	0	1	0	1	1	0
0	0	0	0	0	0	0	1	1	1	1	1	1	1

TABLE 4.36: Table $BA[14, 8, 2, 3]$

parameters of $BA[14, 8, 2, 3]$ *are:*

$\lambda(x, y, z) = 1$ *or* 2 *according as*

$x = y = z$ *or not.*

Chapter 5

Two Factor BAFD'S

This chapter will focus on some methods of constructing balanced asymmetrical factorial designs (BAFD's) consisting of two factors only

5.1 $s_1 \times s_2$ BAFD'S with block size $s_1 (s_1 \leq s_2)$

We shall discuss the construction of two factor BAFD's. Construction of more than two factor BAFD's using two factor BAFD's will be discussed in the next chapter. We are only interested in BAFD's in which the main effects are estimated with high efficiencies. These designs can usually be constructed using arrays discussed in the previous chapter.

Let F_1 and F_2 be the two factors in a BAFD at s_1 and s_2 levels respectively. We assume that $s_1 \leq s_2$ without loss of generality. Let N denote the incidence matrix of BAFD. By equations (3.1.3) and (3.1.4), the eigenvalues of $NN^{\intercal}$ are obtained using the following

$$g[y_1, y_2] = r + \sum_{x \in \Omega} \lambda(x_1, x_2)\{\prod_{i=1}^{2}[(1 - y_i)s_i - 1]^{x_i}\}$$

where $y_i = 0$ or 1 and $x_i = 0$ or 1

hence

$$g(1,0) = r + (s_2 - 1)\lambda_{01} - \lambda_{10} - (s_2 - 1)\lambda_{11} \tag{5.1.1}$$

$$g(0,1) = r - \lambda_{01} + (s_1 - 1)\lambda_{10} - (s_1 - 1)\lambda_{11} \tag{5.1.2}$$

$$g(1,1) = r - \lambda_{01} - \lambda_{10} + \lambda_{11} \tag{5.1.3}$$

using the equality due to Nair and Rao (1948) and corollary 3.1.1

$$\sum(\theta(k_1) \cdot \theta(k_2) \cdots \theta(k_m) \cdot \lambda_{k_1 k_2 \cdots k_m}) = r(k-1) \tag{5.1.4}$$

where $\theta(k_t) = 1$ or $(s_t - 1)$ according as $k_t = 0$ or 1 and $\sum$ denotes the summation over $2^m - 1$ terms. We then obtain

$$r(k-1) = (s_1 - 1)\lambda_{10} + (s_2 - 1)\lambda_{01} + (s_1 - 1)(s_2 - 1)\lambda_{11} \tag{5.1.5}$$

we can now derive the efficiencies of the main effects as follows;-

$$E[0,1] = 1 - \frac{1}{rk} g[0,1]$$

$$= 1 - \left[\frac{r - \lambda_{01} + (s_1 - 1)\lambda_{10} - (s_1 - 1)\lambda_{11}}{rk}\right]$$

$$= \frac{kr - r + \lambda_{01} - (s_1 - 1)\lambda_{10} + (s_1 - 1)\lambda_{11}}{kr}$$

$$= \frac{\begin{array}{c}(s_1 - 1)\lambda_{10} + (s_2 - 1)\lambda_{01} + (s_1 - 1)(s_2 - 1)\lambda_{11} \\ +\lambda_{01} - (s_1 - 1)\lambda_{10} + (s_1 - 1)\lambda_{11}\end{array}}{kr}$$

$$= \frac{(s_2 - 1)\lambda_{01} + (s_1 - 1)(s_2 - 1)\lambda_{11} + \lambda_{01} + (s_1 - 1)\lambda_{11}}{kr}$$

$$= \frac{[s_2 - 1 + 1]\lambda_{01} + [s_1 - 1]\lambda_{11}[s_2 - 1 + 1]}{kr}$$

$$= \frac{s_2[\lambda_{01} + (s_1 - 1)\lambda_{11}]}{kr}$$

$$= \frac{s_2[\lambda_{01} + (s_1 - 1)\lambda_{11}]}{kr[k-1]}[k-1]$$

$$= \frac{s_2[\lambda_{01} + (s_1 - 1)\lambda_{11}]}{r[k-1]} \frac{[k-1]}{k}$$

$$= \frac{[k-1]s_2}{k} \frac{[\lambda_{01} + (s_1 - 1)\lambda_{11}]}{r[k-1]}$$

$$= \frac{[k-1]s_2}{k} \frac{[\lambda_{01} + (s_1 - 1)\lambda_{11}]}{[s_1 - 1]\lambda_{10} + [s_2 - 1]\lambda_{01} + [s_1 - 1][s_2 - 1]\lambda_{11}} \tag{5.1.6}$$

using equation (5.1.5)

Similarly

$$E[1,0] = 1 - \frac{1}{rk} g[1,0]$$

$$= \frac{[k-1]s_1}{k} \frac{[\lambda_{10} + (s_2 - 1)\lambda_{11}]}{[s_2 - 1]\lambda_{01} + [s_1 - 1]\lambda_{10} + [s_1 - 1][s_2 - 1]\lambda_{11}} \tag{5.1.7}$$

If the main effect of F_1 are estimated with full efficiency i.e $E[1,0] = 1$ then the block size k must be a multiple of s_1. We shall assume that $k = s_1$ throughout this section. For $k = s_1$ equation (5.1.7) becomes

$$E[1,0] = (s_1 - 1)\frac{\lambda_{10} + (s_2 - 1)\lambda_{11}}{(s_2 - 1)\lambda_{01} + (s_1 - 1)\lambda_{10} + (s_1 - 1)(s_2 - 1)\lambda_{11}} \tag{5.1.8}$$

$E[1,0] = 1$ if and only if $\lambda_{01} = 0$; that is two treatments at the same level of F_1 never occur together in the same block.

Theorem 5.1.1.

In an $s_1 \times s_2$ BAFD with block size s_1, the main effetcs are estimated with full efficiency if and only if $\lambda_{01} = 0$. This design is equivalent to a $BA[\lambda_{10}s_2 + \lambda_{11}s_2(s_2 - 1), s_1, s_2, 2]$ with parameters $\lambda(x, y) = \lambda_{10}$ or λ_{11} according as $x = y$ or not.

- *Proof.*

The first part of the theorem has has been shown: we only need to prove the latter part. Suppose such a balanced array exists: if we identify columns, rows and symbols with blocks, the levels of F_1, and the levels of F_2 respectively then it is the specified BAFD.

In proving theorem 5.1.1, we don't really use the condition $s_1 \leq s_2$;hence the theorem is true for all s_1 and s_2.

For $k = s_1$ and $\lambda_{01} = 0$ in equation (5.1.6), we have

$$\begin{aligned} E[0,1] &= \frac{[s_1 - 1]s_2[0 + (s_1 - 1)\lambda_{11}]}{s_1 \left\{ (s_1 - 1)\lambda_{10} + 0 + (s_1 - 1)(s_2 - 1)\lambda_{11} \right\}} \\ &= \frac{(s_1 - 1)^2 s_2 \lambda_{11}}{s_1 \left\{ (s_1 - 1)\lambda_{10} + (s_1 - 1)(s_2 - 1)\lambda_{11} \right\}} \\ &= \frac{(s_1 - 1)^2 s_2 \lambda_{11}}{s_1 (s_1 - 1) \left\{ \lambda_{10} + (s_2 - 1)\lambda_{11} \right\}} \\ &= \frac{(s_1 - 1) s_2 \lambda_{11}}{s_1 \left\{ (s_2 - 1)\lambda_{11} + \lambda_{10} \right\}} \end{aligned} \tag{5.1.9}$$

$E[0,1]$ has the maximum value of $\frac{(s_1-1)s_2}{s_1(s_2-1)}$ when $\lambda_{10} = 0$ □

Theorem 5.1.2.

In an $s_1 \times s_2$ BAFD with block size $s_1(s_1 \leq s_2)$, if the main effects of F_1 are estimated with full efficiency and the main effects F_2 are estimated with maximum efficiency $\frac{(s_1-1)s_2}{s_1(s_2-1)}$ then the BAFD has parameters $\lambda_{10} = \lambda_{01} = 0$ and $\lambda_{11} \neq 0$. This design is equivalent to a $TA[\lambda_{11}s_2(s_2-1), s_1, s_2, 2]$.

Since $\lambda_{10} = 0$ means that two treatments at the same level of F_2 do not occur together in the same block, which implies $s_2 \geq k = s_1$ we do not need $s_1 \leq s_2$ in the construction of the designs in theorem 5.1.2.

The construction of $TA[s_2(s_2-1)\lambda_{11}, s_1, s_2, 2]$ has been discussed in section 4.1. Deleting any $(s_2 - s_1)$ constraints from a $TA[s_2(s_2-1)\lambda_{11}, s_2, s_2, 2]$ we obtain a $TA[s_2(s_2-1)\lambda_{11}, s_1, s_2, 2]$. If we restrict $\lambda_{11} = 1$ then the existence of a $TA[s_2(s_2-1), s_1, s_2, 2]$ is equivalent to the existence of $s_1 - 1$ mutually orthogonal latin squares of order s_2 or $s_1 - 2$ mutually orthogonal latin squares of order s_2 with different elements in the diagonal.

Example 5.1.1. *A* 3×4 *BAFD with* $b = 12$*,* $k = 3$*,* $r = 3$*,* $\lambda_{01} = \lambda_{10} = 0$ *and* $\lambda_{11} = 1$ *can be constructed from a* $TA[12, 3, 4, 2]$

Blocks	1	2	3	4	5	6	7	8	9	10	11	12
levels of F_1	Levels	of	F_2									
0	0	1	2	3	0	1	2	3	0	1	2	3
1	1	0	3	2	2	3	0	1	3	2	1	0
2	2	3	0	1	3	2	1	0	1	0	3	2

TABLE 5.1: Table of a 3×4 BAFD

In this design,

$E[1,0] = 1$*,* $E[0,1] = \frac{8}{9}$ *and* $E[1,1] = \frac{5}{9}$

Example 5.1.2. *A* 3×6 *BAFD with* $b = 30, k = 3, r = 5$ $\lambda_{01} = \lambda_{10} = 0$ *and* $\lambda_{11} = 1$ *can be constructed from a* $TA[30, 3, 6, 2]$.

The efficiencies are

$E[1, 0] = 1.0$,$E[0, 1] = \frac{4}{5}$ *and* $E[1, 1] = \frac{3}{5}$

Example 5.1.3. *A* 7×20 *BAFD with* $b = 80$, $k = 7$, $r = 4$, $\lambda_{01} = \lambda_{10} = 0$, $\lambda_{11} = 1$, *can be constructed from a* $TA[80, 7, 20, 2]$

The efficiencies are;

$E[1, 0] = 1.0$, $E[0, 1] = \frac{120}{133}$ *and* $E[1, 1] = \frac{23}{28}$

Example 5.1.4. *A* 12×15 *BAFD with* $b = 630, k = 12, r = 42, \lambda_{01} = \lambda_{10} = 0$, $\lambda_{11} = 3$, *can be constructed from a* $TA[630, 12, 15, 2]$

the efficiencies are;

$E[1, 0] = 1.0, E[0, 1] = \frac{165}{168}$ *and* $E[1, 1] = \frac{51}{56}$

Corollary 5.1.1.

In an s^2 symmetrical FD with block size s and if all the main effects are estimated with full efficiency then the FD has parameters $\lambda_1 = 0$ and $\lambda_2 \neq 0$.This design is equivalent to a

$$TA[\lambda_2 s(s-1), s, s, 2]$$

.

Example 5.1.5. *If* s *is a prime power, then there exists a* $TA[s(s-1), s, s, 2]$ *by corollary 4.1.1. Hence we can always construct an* s^2 *symmetrical* FD *with* $r = s-1, b = s(s-1), k = s, \lambda_1 = 0, E_1 = 1$ *and* $E_2 = \frac{s-2}{s-1}$ *assuming that* $\lambda_2 = 1$

Example 5.1.6. *A* 6^2 *symmetrical* FD *with* $r = 10, b = 60, k = 6, \lambda_1 = 0, \lambda_2 = 2$ *can be constructed from a* $TA[60, 6, 6, 2]$. *The efficiencies are:* $E_1 = 1$ *and* $E_2 = \frac{4}{5}$

Example 5.1.7. *A* 21×21 *FD can be constructed from a* $TA[2,100,21,21,2]$ *with* $\lambda_1 = 0, \lambda_2 = 5, b = 2,100, k = 21$ *and* $r = 100$.

The efficiencies are $E_1 = 1$ *and* $E_2 = \frac{19}{20}$.

Similarly we can construct $10^2, 12^2, 14^2$ *BAFD'S by using* $TA[360,10,10,2]$, $TA[660,12,12,2]$ *and* $TA[1092,14,14,2]$ *respectively.*

5.2 $s_1 \times s_2$ BAFD's with blocksize $s_2(s_1 < s_2)$

If the main effects of F_2 are estimated with full efficiency then the block size k must be a multiple of s_2. Assume that $k = s_2$ throughout this section. By theorem 5.1.1, $E[0,1] = 1$ if and only if $\lambda_{10} = 0$. Furthermore the design is equivalent to a $BA[\lambda_{01}s_1 + \lambda_{11}s_1(s_1-1), s_2, s_1, 2]$ with parameters $\lambda(x,y) = \lambda_{01}$ or λ_{11} according as $x = y$ or not, if we identify the columns, rows, and symbols of the balanced array with blocks, the levels of F_2 and the levels of F_1 of the design.

Example 5.2.1. *A* 2×3 *BAFD with* $b = 4, k = 3, r = 2, \lambda_{10} = 0$ *and* $\lambda_{01} = \lambda_{11} = 1$ *can be constructed from the* $OA[4,3,2,2]$

Blocks	1	2	3	4
Levels of F_2	levels	of	F_1	
0	0	0	1	1
1	0	1	0	1
2	0	1	1	0

Table 5.2: Table of a 2×3 BAFD

In this design, the efficiencies are: $E[0,1] = 1$ *and* $E[1,0] = E[1,1] = \frac{2}{3}$

Example 5.2.2. *A* 11×12 *BAFD with* $b = 121, k = 12, r = 11, \lambda_{10} = 0$ *and* $\lambda_{01} = \lambda_{11} = 1$ *can be constructed from an* $OA[121,12,11,2]$

The efficiencies are;

$E[0,1] = 1$ *and*

$E[1,0] = \frac{11}{12}$

$E(1,1) = \frac{11}{12}$

Example 5.2.3. *A* 3×13 *BAFD with* $b = 27, k = 13, r = 9, \lambda_{10} = 0, \lambda_{01} = \lambda_{11} = 3$ *can be constructed from an* $OA[27, 13, 3, 2]$

$E[0,1] = 1.0$

$E[1,0] = \frac{12}{13}$

$E[1,1] = \frac{12}{13}$

Example 5.2.4. *A* 4×9 *BAFD with* $b = 32, k = 9, r = 8, \lambda_{10} = 0, \lambda_{01} = \lambda_{11} = 2$ *can be constructed from an* $OA[32, 9, 4, 2]$

$E[0,1] = 1.0$

$E[1,0] = \frac{8}{9}$

$E[1,1] = \frac{8}{9}$

Example 5.2.5. *A* 5×11 *BAFD with* $b = 50, k = 11, r = 10, \lambda_{10} = 0, \lambda_{01} = \lambda_{11} = 2$ *can be constructed from an* $OA[50, 11, 5, 2]$

$E[0,1] = 1.0$

$E[1,0] = \frac{10}{11}$

$E[1,1] = \frac{10}{11}$

Example 5.2.6. *A* 2×8 *BAFD with* $b = 16, k = 8, r = 8, \lambda_{10} = 0, \lambda_{01} = \lambda_{11} = 4$ *can be constructed from an* $OA[16, 8, 2, 2]$

$E[0,1] = 1.0$

$E[1,0] = \frac{7}{8}$

$E[1,1] = \frac{7}{8}$

For $\lambda_{10} = 0$ and $k = s_2$ equation (5.1.7) becomes

$$\begin{aligned}
E[1,0] &= \frac{(k-1)s_1}{k} \frac{\lambda_{10} + (s_2-1)\lambda_{11}}{(s_2-1)\lambda_{01} + (s_1-1)\lambda_{10} + (s_1-1)(s_2-1)\lambda_{11}} \\
&= \frac{(s_2-1)s_1}{s_2} \frac{(s_2-1)\lambda_{11}}{(s_2-1)\lambda_{01} + (s_1-1)(s_2-1)\lambda_{11}} \\
&= \frac{s_1\lambda_{11}(s_2-1)^2}{s_2(s_2-1)[\lambda_{01} + (s_1-1)\lambda_{11}]} \\
&= \frac{s_1\lambda_{11}(s_2-1)}{s_2[\lambda_{01} + (s_1-1)\lambda_{11}]} \\
&= \frac{(s_2-1)s_1\lambda_{11}}{s_2[\lambda_{01} + (s_1-1)\lambda_{11}]} \\
&= \frac{(s_2-1)s_1}{s_2[(s_1-1) + \frac{\lambda_{01}}{\lambda_{11}}]} \\
&= \frac{(s_2-1)s_1}{s_2} \frac{1}{(s_1-1) + \frac{\lambda_{01}}{\lambda_{11}}}
\end{aligned} \tag{5.2.1}$$

Note that $\lambda_{01} \neq 0$, since $k = s_2 > s_1$ implies that at least two treatments in a given block have the same level of F_1. To maximize $E(1,0)$, it is required that $\frac{\lambda_{01}}{\lambda_{11}}$ be as small as possible.

Theorem 5.2.1.

In an $s_1 \times s_2 (s_1 \leq s_2)$ BAFD with block size s_2 and $\lambda_{10} = 0$ the following inequality holds:

$$\frac{\lambda_{01}}{\lambda_{11}} \geq \frac{s_2 - s_1}{s_2} \tag{5.2.2}$$

when the equality holds, $E(1,0) = 1.0$ and $E(1,1) = \frac{s_2-2}{s_2-1}$

Proof. $g(0,1) = 0$ in this BAFD since the main effect of F_2 is estimated with full efficiency. By equation (5.1.2) we have

$$r = \lambda_{01} + (s_1-1)\lambda_{11} \tag{5.2.3}$$

If we substitute r in equation (5.1.1) in equation (5.2.3), we have

$$\begin{aligned} g(1,0) &= \lambda_{01} + (s_1-1)\lambda_{11} + (s_2-1)\lambda_{01} - (s_2-1)\lambda_{11} \\ &= s_2\lambda_{01} + [s_1 - 1 - s_2 + 1]\lambda_{11} \\ &= s_2\lambda_{01} - (s_2 - s_1)\lambda_{11} \end{aligned} \tag{5.2.4}$$

but $g(1,0) \geq 0$, since $g(1,0)$ is an eigenvalue of the none negative definite matrix NN'. Therefore we have equation (5.2.2) and the equality holds if and only if $g(1,0) = 0$. i.e $E(1,0) = 1$.

i.e

$$\begin{aligned} g(1,1) &= r - \lambda_{01} - \lambda_{10} + \lambda_{11} \\ &= \lambda_{01} + (s_1-1)\lambda_{11} - \lambda_{01} - 0 + \lambda_{11} \\ &= (s_1-1)\lambda_{11} + \lambda_{11} \end{aligned}$$

but

$$\begin{aligned} E[1,1] &= 1 - \frac{1}{rk}[g(1,1)] \\ &= 1 - \frac{[s_1-1]s_2 + s_2}{s_2[\lambda_{01} + (s_1-1)\lambda_{11}]} \end{aligned}$$

if $\frac{\lambda_{01}}{\lambda_{11}} = \frac{s_2 - s_1}{s_2}$ and using equations (5.1.5) and (5.2.3)

$$\begin{aligned}
&= 1 - \frac{(s_1 - 1)s_2 + s_2}{s_2[s_2 - s_1 + (s_1 - 1)s_2]} \\
&= \frac{s_2[s_2 - s_1 + (s_1 - 1)s_2] + -[(s_1 - 1)s_2 + s_2]}{s_2[s_2 - s_1 + (s_1 - 1)s_2]} \\
&= \frac{s_2[s_1 s_2 - s_1] - s_1 s_2}{s_2[s_1 s_2 - s_1]} \\
&= \frac{s_2 s_1 s_2 - s_1 s_2 - s_1 s_2}{s_2[s_1 s_2 - s_1]} \\
&= \frac{s_2 - 2}{s_2 - 1} \quad \text{as} \quad \text{required.}
\end{aligned}$$

Since a necessary condition for $E[1, 0] = 1$ is that block size k must be a multiple of s_1 we must assume that $s_2 = ms_1 (= k)$ for some integer m in order to construct a BAFD such that all main effects are estimated with full efficiency. When $s_2 = ms_1$ equation (5.2.2) becomes

$$\frac{\lambda_{01}}{\lambda_{11}} \geq \frac{m - 1}{m} \tag{5.2.5}$$

□

Corollary 5.2.1.

In an $s_1 \times s_2$ BAFD with block size $s_2(> s_1)$ the main effects of F_1 and F_2 are estimated with full efficiency if and only if $s_2 = ms_1, \lambda_{10} = 0$ and $\frac{\lambda_{01}}{\lambda_{11}} = \frac{m-1}{m}$ for some m. The design is equivalent to a $BA[(ms_1 - 1)s_1\lambda, ms_1, s_1, 2]$ with parameters $\lambda(x, y) = (m - 1)\lambda$ or $m\lambda$ according as $x = y$ or not, i.e. a $BA(T)(m, s_1, \lambda)$.

By theorem 4.9.1 for any given m and s_1 we can always construct a $BA(T)(m, s_1, \lambda)$ for some λ. Thus we can always construct an $ms_1 \times s_1$ BAFD such that all main effects are estimated with full efficiency, but a large replication may be needed. The construction of a $BA(T)(m, s_1, 1)$ for some m and s_1 are discussed in corollary 4.9.2, 4.9.4, and 4.9.5. In example 4.9.8 and 4.9.9 we also gave a $BA(T)[4, 3, 2]$ and a $BA(T)(3, 4, 2)$.

Example 5.2.7. *A* 2×4 *BAFD with* $b = 6, k = 4, r = 3, \lambda_{10} = 0,$ *can be constructed from a* $BA(T)(2, 2, 1) = BA[6, 4, 2, 2]$ *with* $\lambda(x, y) = 1$ *or* 2 *according as* $x = y$ *or not*

Blocks	1	2	3	4	5	6
Levels of F_2	Levels	of	F_1			
0	1	0	1	0	1	0
1	0	1	1	0	0	1
2	1	0	0	1	0	1
3	0	1	0	1	1	0

TABLE 5.3: Table of a 2×4 BAFD

In this design, the efficiencies are: $E[0, 1] = 1,$ $E[1, 0] = 1$ *and* $E[1, 1] = \frac{2}{3}$

Example 5.2.8. *A* 7×42 *BAFD with* $b = 287, k = 42, r = 41, \lambda_{10} = 0, \lambda_{01} = 5, \lambda_{11} = 6$ *can be constructed from a* $BA(T)[6, 7, 1] = BA[287, 42, 7, 2]$

with $\lambda(x, y) = 5$ *or* 6 *according as* $x = y$ *or Not.*

The efficiencies of this designs are: $E[0, 1] = 1.0$*,* $E[1, 0] = 1.0$ *and* $E[1, 1] = \frac{40}{41}$

5.3 $s_1 \times s_2$ BAFD's with block size a common multiple of s_1 and s_2

In an $s_1 \times s_2$ BAFD with block size s_2, if s_2 is not a multiple of s_1, then the main effect of F_1 cannot be estimated with full efficiency. To estimate all main effects with full efficiency, the block size k must be a common multiple of s_1 and s_2. Let $s_1 = ps$ and $s_2 = qs$ where $s > 1$. A method is given below to construct $s_1 \times s_2$ BAFD with block size pqs such that all the main effects are estimated with full efficiency.

Theorem 5.3.1.

If there exists a resolvable BIBD with qs treatments and block size q, then there exists a $ps \times qs$ BAFD with block size pqs such that all main effects are estimated with full efficiency.

Proof. Construct a $BA(T)(p, s, n)$ for some integer n by theorem 4.9.1. In the resolvable BIBD, there being s blocks in each replication , we can number the block in each replication by $0, 1, \cdots, s-1$. Replacing each symbol in the balanced array by a group of symbols which represents blocks in the BIBD for each replication, we obtain a $pqs \times [ps-1]snr'$ matrix, where r' is the number of replications in the BIBD. Assign i^{th} level of F_1 to the rows from the $(i_q + 1)^{th}$ to the $(i + 1)^{th}$, where $i = 0, 1, \cdots, ps-1$.Identifying columns and symbols with blocks and the levels of F_2, we get a $ps \times qs$ design with block size pqs.

We shall show that all the main effects of the design constructed above are estimated with full efficiency.Let $\lambda^{'}$ be the number of blocks in which two treatments occur together in the BIBD, then $(qs-1)\lambda^{'} = (q-1)r^{'}$. Assume that $r^{'} = (qs-1)m$ and $\lambda^{'} = (q-1)m$, where m need not be an integer. Let $\lambda_{01}, \lambda_{10}, \lambda_{11}$ denote the parameters and r denote the number of replications in the $ps \times qs$ design, then through inspection we have

$$\lambda(x,y) = (ps-1)^{x+1}(qs-1)^{y+1}(p-1)^{x}(q-1)^{y}mn + (xy)(pq)(s-1)^{xy}mn$$

$$x, y = 0 \text{ or } 1 \text{ in mod } 2 \quad (5.3.1)$$

so

$$\left\{\begin{array}{c} \lambda_{01} = (ps-1)(q-1)mn \\ \lambda_{10} = (qs-1)(p-1)mn \\ \lambda_{11} = (p-1)(q-1)mn + pq(s-1)mn \\ \lambda_{00} = r = (ps-1)(qs-1)mn \end{array}\right\} \quad (5.3.2)$$

if we substitute the parameters of the equations (5.1.1), (5.1.2) and (5.1.3) in equations (5.3.2) and corollary 3.1.1 we get

$$E[0,1] = E[1,0] = 1 \quad \text{and} \quad E[1,1] = -\frac{s-1}{(ps-1)(qs-1)} + 1$$

Given any q and s, there always exists a resolvable BIBD with qs treatments and block size q if the number of replications is allowed to be large.

The irreducible BIBD of qs treatments with block size q in which each of the $\binom{qs}{q}$ possible $q-$ element combinations form a block is resolvable with parameters

$$v = qs, \quad b = \binom{qs}{q}, \quad r = \binom{qs-1}{q-1}, \quad k = q, \quad \lambda = \binom{qs-2}{q-2} \tag{5.3.3}$$

□

Definition 5.3.1.

Suppose $(\mathcal{F}, \mathcal{A})$ is a (v, k, λ)-BIBD, a parallel class in $(\mathcal{F}, \mathcal{A})$ is a subset of disjoint blocks from $\mathcal{A}$ whose union is $\mathcal{F}$. A partition of $\mathcal{A}$ into r parallel classes is called a **resolution**; and $(\mathcal{F}, \mathcal{A})$ is said to be a resolvable BIBD if $\mathcal{A}$ has at least one resolution. We say that $\mathcal{F}$ is a finite set of points called **treatments**.Where

$$\mathcal{F} = \left\{0, 1, 2, \cdots, v-1\right\}$$

Several methods are used to construct resolvable balanced incomplete block designs. These methods include:

To construct a resolvable BIBD with block size $k = 3$ and a finite number of symbols $V = v$ one can use the methods of one step cycles, two step cycles or three steps cycles. The method of one step cycles is applicable when

$$v = 2y + 1 = 24m + 3$$

or when

$$v = 2y + 1 = 24m + 9 \tag{5.3.4}$$

we may denote the element 0 by k and the others by $1, 2, 3, \cdots, 2y$ place k at the center of the circle and the other elements $1, 2, 3, \cdots, 2y$ at equidistant intervals on their circumference. The companions of k are to be different on each parallel class. If we suppose that on the first parallel class they are 1 and $y+1$ on the second 2 and $2+y$ and so on, then the diameters through k will give for each parallel class a triplet in which k appears. On each parallel class we have to find $\frac{2(y-1)}{3}$ other triplets satisfying the conditions of the problem. Every triplet formed from the remaining $2y-2$ elements will be represented by an inscribed triangle joining the points representing these elements. The sides of the triangles are the chords joining these $2y-2$ points. The sides of the triangles so represented are denoted by the letters p, q, r. The term, triad or grouping denotes any of p, q, r which determines the dimensions of an inscribed triangle. If p, q, r are proportional to the smaller arcs subtended then $p+q=r$ or $p+q+r=2y$. If $\frac{(y-1)}{3}$ scalene triangles can be inscribed in the circle so that to each triangle corresponds an equal complimentary triangle having its equal sides parallel to those of the first and its vertices at free points then the system of $\frac{2(y-1)}{3}$ triangles with the corresponding diameter will give an arrangement for one parallel class. If the system is permuted cyclically$(y-1)$ times we get arrangements for the other $(y-1)$ parallel classes.

The method of two step cycles is applicable when $v=12m+3$. When v is of this form and m is odd we cannot get sets of complimentary triangles as required. Hence to apply a similar method we have to find $\frac{2(y-1)}{3}$ different dissimilar inscribed triangles having no vertex in common and satisfying the conditions $p+q=r$ or $p+q+r=2y$. These solutions are also central. In the first part of this solution $\frac{v}{3}$ of these triangles must be selected to form an arrangement of the first parallel class. By rotating this arrangement two steps at a time we obtain triples for $\frac{v}{3}$ parallel classes in all.

The method of three step cycles applies if

$$\begin{aligned} v &= 18m + 3 \text{ or} \\ v &= 18m + 9 \text{ or} \\ v &= 18m + 15 \end{aligned} \tag{5.3.5}$$

It gives a solution for every value of v except $v = 15$. In this, method we may with equal propriety represent all the elements by symbols placed at equidistant intervals round the circumference of a circle. Such solutions are termed as none central. The symbols may be $1, 2, 3, \cdots, v$, or letters $a_1, b_1, c_1, a_2, b_2, c_2, \cdots$ any triplet will be represented by a triangle whose sides are chords of a circle. The arrangement of any parallel class is to include all the elements and therefore the triangles representing the triplets for a given parallel class are $\frac{v}{3}$ in number, and so each element appears in only one triplet, thus no two triangles can have a common vertex. The complete three steps solution will require the determination of a system of $\frac{(n-1)}{2}$ inscribed triangles. In the first part of the solution, $\frac{v}{3}$ of these triangles must be selected to form an arrangement for the first parallel class, so that by rotating this arrangement three steps at a time we obtain triplets for $\frac{v}{3}$ parallel classes in all.

If $q = 4t - 1$ is a prime power, then there exists a resolvable balanced incomplete block design with block size $k = 2t$ and number of symbols $v = 4t$ and $\lambda = 2t - 1$.
As for the point set we take $\mathcal{F} = GF(q) \cup \left\{\infty\right\}$. Developing the parallel classes.

$$H_0^2 \cup \left\{0\right\} \quad \text{and} \quad H_1^2 \cup \left\{\infty\right\} \tag{5.3.6}$$

over $GF(q)$ produces the required blocks and resolution.

The multiplicative cosets $H_0^e, H_1^e, H_2^e, \cdots$, H_{e-1}^e are defined by

$$H_m^e = \left\{x^t : t \equiv m \ mod \ e\right\}$$

where x denotes a primitive element of $GF(q)$.

If $q = 4t + 1$ is a prime power, then there exists a resolvable balanced incomplete block design with block size $k = 2t + 1$ and number of symbols $v = 4t + 2$ and $\lambda = 4t$. As for the point set we take $\mathcal{F} = GF(q) \cup \left\{\infty\right\}$. Developing the parallel classes.

$$\begin{matrix} H_0^2 \cup \left\{0\right\} \\ H_1^2 \cup \left\{\infty\right\} \end{matrix} \text{and} \begin{matrix} H_0^2 \cup \left\{\infty\right\} \\ H_1^2 \cup \left\{0\right\} \end{matrix} \tag{5.3.7}$$

over $GF(q)$ produces the required blocks and resolution.

Let $\lambda \leq k - 1$. Suppose there is a difference family

$$DF_\lambda(k, v) \left\{A_0, A_1, A_2, A_3, \cdots, A_{s-1}\right\}$$

over a ring R whose base blocks are mutually disjoint. If there is a set of k distinct units $\left\{u_0, u_1, u_2, \cdots, u_{k-1}\right\}$ whose differences are all units of R, then there exists a Resolvable balanced incomplete block design with block size $K = k$ and number of symbols $V = kv$, where s represents the number of base blocks and that

$$B_j^j = A_j \times \left\{i\right\} = \left\{(a_1{}^j, i), (a_2{}^j, i), \cdots, (a_k{}^j, i)\right\}; \ i \in I_k, \ j \in I_s \tag{5.3.8}$$

In order to get further blocks we put

$$C_x = \Big\{(u_0, 0), (u_1, 1), (u_2, 2), \cdots, (u_{k-1}, k-1)\Big\} \cdot x \qquad x \in R \tag{5.3.9}$$

where $(u, i) \cdot x$ means (ux, i). We must partition the blocks into $r = \frac{\lambda(kv-1)}{(k-1)}$ parallel classes. The first parallel class P_0 will take all the blocks $u_i {B_j}^i$ where $i \in I_k$, $j \in I_s$ and the blocks C_x, where x is distinct from all ${a^j}_i$

$$i \in I_k, \quad j \in I_s$$

Other classes are given by $P_g = \mathcal{T}_g P_0$ where $\mathcal{T}_g : (x, i) \mapsto (x + g, i), g \in \mathbb{R}$. that is

$$P_g = \Big\{\mathcal{T}_g(B) : B \in P_0\Big\} \tag{5.3.10}$$

We can still construct more parallel classes.
Let $Q_x = \Big\{\mathcal{T}_g C_x : g \in R\Big\}$ with

$$x \in \bigcup_{0 \le j \le s-1} A_j \ \ and \ \ R_x = \Big\{\mathcal{T}_g C_x : g \in R\Big\} \ with \ \ x \in R \backslash \bigcup_{o \le j \le s-1} A_j \tag{5.3.11}$$

Both Q_x and R_x are parallel classes. We take each parallel class Q_x λ times and each class R_x $\lambda - 1$ times.

If v is even and $v \geq 4$ a resolvable balanced incomplete block design with block size $k = 2$ and number of symbols $V = v$ can be constructed as follows

We take the point set $\mathcal{F}$ to be $\mathcal{F} = \mathbb{Z}_{v-1} \cup \Big\{\infty\Big\}$.
For $j \in \mathbb{Z}_{v-1}$, define

$$\pi_j = \Big\{\Big\{\infty, j\Big\}\Big\} \bigcup \Big\{\Big\{i + j \ \ mod(v-1), j - i \ \ mod(v-1)\Big\}\Big\} :$$

$$1 \leq i \leq \frac{v-2}{2} \tag{5.3.12}$$

π_j is a parallel class and each pair of points occurs in exactly one π_j

To construct a resolvable incomplete block design with block size $k = q$ and number of symbols $V = q^2$ where q is a prime number, we can construct an affine plane of order q. This is done by defining $P = \mathbb{F}_q \times \mathbb{F}_q$.

For any $a, b \in \mathbb{F}_q$, we define a block

$$L_{a,b} = \left\{(x, y) \in P : y = ax + b\right\} \tag{5.3.13}$$

and for any $c \in \mathbb{F}_q$, we define

$$L_{\infty,c} = \left\{(c, y) : y \in \mathbb{F}_q\right\} \tag{5.3.14}$$

Finally we define

$$L = \left\{L_{a,b} : a, b \in \mathbb{F}_q\right\} \bigcup \left\{L_{\infty,c} : c \in \mathbb{F}_q\right\} \tag{5.3.15}$$

Then (P, L) is the required affine plane of order q and it is also the required resolvable balanced incomplete block design.

Definition 5.3.2.

For $0 \leq d \leq m$, we define the Gaussian Coefficient $\begin{bmatrix} m \\ d \end{bmatrix}_q$ as follows

$$\begin{bmatrix} m \\ d \end{bmatrix}_q = \begin{Bmatrix} \frac{(q^m-1)(q^{m-1}-1)\cdots(q^{m-d+1}-1)}{(q^d-1)(q^{d-1}-1)\cdots(q-1)} & if \quad d \neq 0 \\ 1 & if \quad d = 0 \end{Bmatrix} \tag{5.3.16}$$

To construct a resolvable balanced incomplete block design

$$(q^m, b, r, q^d, \lambda)$$

where $m \geq 2$, $1 \leq d \leq m-1$

$$b = \begin{bmatrix} m \\ d \end{bmatrix}_q^{q^{m-d}}, \quad r = \begin{bmatrix} m \\ d \end{bmatrix}_q, \quad \textit{and} \quad \lambda = \begin{bmatrix} m-1 \\ d-1 \end{bmatrix}_q \tag{5.3.17}$$

we use equations (5.3.13) and equation (5.3.14) and (5.3.15). However, in some cases it might not be possible to construct resolvable balanced incomplete block designs with the given properties in (5.3.17). Like for example there exists $(8, 4, 3)-$BIBDs that are not resolvable.

To construct a Resolvable balanced incomplete block design

$$(q^d, q^{d-1}, \lambda)$$

where q is a prime power and $m \geq 2$ with $\lambda = \frac{q^{d-1}-1}{q-1}$

We first construct a symmetrical BIBD

$$\left(\tfrac{q^{d+1}-1}{q-1}, \tfrac{q^d-1}{q-1}, \tfrac{q^{d-1}-1}{q-1}\right) \tag{5.3.18}$$

From this symmetrical BIBD, we obtain a quasi-residual BIBD. The quasiresidual BIBD is an affine resolvable BIBD with parameters

$$v = q^d, \quad k = q^{d-1}, \quad \lambda = \frac{q^{d-1}-1}{q-1} \tag{5.3.19}$$

Example 5.3.1. *A* 4×6 *BAFD with block size* 12 *can be constructed using* $BA[10, 6, 2, 2]$ *and a resolvable BIBD with 4 treatments and block* *size* 2 *as shown below*

Consider the following BIBD with 4 *treatments and block size* 2 *where* $X_0, X_1, Y_0, Y_1, Z_0, Z_1$ *represents the blocks.*

X_0	X_1	Y_0	Y_1	Z_0	Z_1
0	2	0	1	0	1
1	3	2	3	3	2

TABLE 5.4: Table of BIBD[4,6,2]

Also consider the $BA(T)(3,2,1)$ *given below*

0	0	0	0	0	1	1	1	1	1
0	0	1	1	1	0	0	0	1	1
1	0	0	1	1	1	1	0	0	0
0	1	1	0	1	0	1	1	0	0
1	1	1	0	0	1	0	0	0	1
1	1	0	1	0	0	0	1	1	0

TABLE 5.5: Table of BA(T)[3,2,1]

By theorem 5.3.1 we can construct a 4×6 *BAFD with* $k = 12, r = \lambda_{00} = 15, b = 30, \lambda_{10} = 5, \lambda_{01} = 6, \lambda_{11} = 8$ *with* $E[1,0] = 1, E[0,1] = 1, E[1,1] = \frac{14}{15}$

Blocks	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*
levels of F_2	*Levels*	*of*	F_1																											
0	X_0	X_0	X_0	X_0	X_0	X_1	X_1	X_1	X_1	X_1	Y_0	Y_0	Y_0	Y_0	Y_0	Y_1	Y_1	Y_1	Y_1	Y_1	Z_0	Z_0	Z_0	Z_0	Z_0	Z_1	Z_1	Z_1	Z_1	Z_1
1	X_0	X_0	X_1	X_1	X_1	X_0	X_0	X_0	X_1	X_1	Y_0	Y_0	Y_1	Y_1	Y_1	Y_0	Y_0	Y_0	Y_1	Y_1	Z_0	Z_0	Z_1	Z_1	Z_1	Z_0	Z_0	Z_0	Z_1	Z_1
2	X_1	X_0	X_0	X_1	X_1	X_1	X_1	X_0	X_0	X_0	Y_1	Y_0	Y_0	Y_1	Y_1	Y_1	Y_1	Y_0	Y_0	Y_0	Z_1	Z_0	Z_0	Z_1	Z_1	Z_1	Z_1	Z_0	Z_0	Z_0
3	X_0	X_1	X_1	X_0	X_1	X_0	X_1	X_1	X_0	X_0	Y_0	Y_1	Y_1	Y_0	Y_1	Y_0	Y_1	Y_1	Y_0	Y_0	Z_0	Z_1	Z_1	Z_0	Z_1	Z_0	Z_1	Z_1	Z_0	Z_0
4	X_1	X_1	X_1	X_0	X_0	X_1	X_0	X_0	X_0	X_1	Y_1	Y_1	Y_1	Y_0	Y_0	Y_1	Y_0	Y_0	Y_0	Y_1	Z_1	Z_1	Z_1	Z_0	Z_0	Z_1	Z_0	Z_0	Z_0	Z_1
5	X_1	X_1	X_0	X_1	X_0	X_0	X_0	X_1	X_1	X_0	Y_1	Y_1	Y_0	Y_1	Y_0	Y_0	Y_0	Y_1	Y_1	Y_0	Z_1	Z_1	Z_0	Z_1	Z_0	Z_0	Z_0	Z_1	Z_1	Z_0

Table 5.6: Table of a 4×6 BAFD

Example 5.3.2. *A* 6×8 *BAFD with parameters* $k = 24, r = \lambda_{00} = 35, b = 70, \lambda_{01} = 15, \lambda_{10} = 14, \lambda_{11} = 18$ *and efficiencies* $E[0,1] = 1, E[1,0] = 1, E[1,1] = \frac{34}{35}$ *can be constructed by using both the* $BA(T)(3,2,1)$ *given in example 5.3.1 and a resolvable BIBD with 8 treatments and block of size 4 given below that has been constructed by developing parallel classes as in equation (5.3.6). The base blocks will be* $H_0{}^2 \cup \left\{0\right\} = \left\{1,2,4,0\right\}$ *and* $H_1{}^2 \cup \left\{\infty\right\} = \left\{3,5,6,\infty\right\}$ *The resolvable BIBD will be given as follows after replacing* ∞ *with* 7*.*

1	3	2	4	3	5	4	6	5	0	6	1	0	2
2	5	3	6	4	0	5	1	6	2	0	3	1	4
4	6	5	0	6	1	0	2	1	3	2	4	3	5
0	7	1	7	2	7	3	7	4	7	5	7	6	7

Table 5.7: Table of BIBD[8,14,4]

Example 5.3.3. *A* 6×9 *BAFD with parameters* $k = 18, r = \lambda_{00} = 20, b = 60, \lambda_{01} = 5, \lambda_{10} = 4, \lambda_{11} = 7$ *and efficiencies* $E[0,1] = 1$*,* $E[1,0] = 1$*,* $E[1,1] = \frac{19}{20}$ *can be constructed by using both the* $BA(T)(2,3,1)$ *given in example 4.9.5 and the resolvable BIBD with* 9 *treatments and block size* 3 *given below. This has been constructed as in equations (5.3.13), (5.3.14) and (5.3.15). That is, by constructing an affine plane of order* 3*.*

0	1	2	0	1	2	0	1	2	0	3	6
3	4	5	4	5	3	5	3	4	1	4	7
6	7	8	8	6	7	7	8	6	2	5	8

Table 5.8: Table of a BIBD[9,12,3]

Example 5.3.4. *A* 10×15 *BAFD with parameters* $k = 30, r = \lambda_{00} = 63, b = 315, \lambda_{01} = 9, \lambda_{10} = 7, \lambda_{11} = 13$ *and efficiencies* $E[0,1] = 1$*,* $E[1,0] = 1$*,* $E[1,1] = \frac{61}{63}$ *can be constructed by using both the* $BA(T)[2,5,1]$ *given in example 4.9.2 and a resolvable BIBD with* 15 *treatments and block size* 3 *given below which was constructed by using the method of two step* *cycles where the first parallel class gives a set of triplets*

$$(k \cdot 1 \cdot 2), (3 \cdot 7 \cdot 10), (4 \cdot 5 \cdot 13), (6 \cdot 9 \cdot 11), (8, \cdot 12, \cdot 14).$$

From which by a cyclical two step permutation we get solution

0	3	4	6	8	0	5	6	8	10
1	7	5	9	12	3	9	7	11	14
2	10	13	11	14	4	12	1	13	2
0	7	8	10	12	0	9	10	12	14
5	11	9	13	2	7	13	11	1	4
6	14	3	1	4	8	2	5	3	6
0	13	14	2	4	0	1	2	4	6
11	3	1	5	8	13	5	3	7	10
12	6	9	7	10	14	8	11	9	12
0	11	12	14	2					
9	1	13	3	6					
10	4	7	5	8					

TABLE 5.9: Table of BIBD[15,35,3]

Example 5.3.5. *A* 6×9 *BAFD with parameters* $k = 27, r = \lambda_{00} = 20, b = 96, \lambda_{01} = 5, \lambda_{10} = 4, \lambda_{11} = 7$ *and efficiencies* $E[0,1] = 1$*,* $E[1,0] = 1$*,* $E[1,1] = \frac{29}{30}$ *can be constructed by using both the* $BA(T)[3,3,1]$ *given in example 4.9.4 and a resolvable BIBD with* 9 *treatments and block size* 3 *given below which was constructed by using the method of one step cycles where the first parallel class gives a set of triplets*

$$(k \cdot 1 \cdot 5), (3 \cdot 4 \cdot 6), (7 \cdot 8 \cdot 2)$$

from which by a one cyclical one step permutation we get the solution

0	3	7	0	4	8	0	5	1	0	6	2
1	4	8	2	5	1	3	6	2	4	7	3
5	6	2	6	7	3	7	8	4	8	1	5

Table 5.10: Table of BIBD[9,12,3]

Example 5.3.6. *An* 8×14 *BAFD with parameters* $k = 28, r = \lambda_{00} = 91, b = 312, \lambda_{01} = 42, \lambda_{10} = 39, \lambda_{11} = 46$ *and efficiencies* $E[1,0] = 1.0, E[0,1] = 1.0, E[1,1] = \frac{89}{91}$ *can be constructed by using both the* $BA(T)[4,2,1]$ *which can be constructed by using theorem 4.9.1 and a resolvable BIBD with* 14 *treatments and block size* 7 *given below which was constructed by developing parallel classes as in equation 5.3.7. The base blocks will be* ${H_0}^2 \cup \left\{0\right\} = \left\{1,4,3,12,9,10,0\right\}, {H_1}^2 \cup \left\{\infty\right\} = \left\{2,8,6,11,5,7,\infty\right\}$ *and* ${H_0}^2 \cup \left\{\infty\right\} = \left\{1,4,3,12,9,10,\infty\right\}, {H_1}^2 \cup \left\{0\right\} = \left\{2,8,6,11,5,7,0\right\}$ *The resolvable BIBD is given as follows after replacing* ∞ *with* 13*.*

1	2	2	3	3	4	4	5	5	6	6	7
4	8	5	9	6	10	7	11	8	12	9	0
3	6	4	7	5	8	6	9	7	10	8	11
12	11	0	12	1	0	2	1	3	2	4	3
9	5	10	6	11	7	12	8	0	9	1	10
10	7	11	8	12	9	0	10	1	11	2	12
0	13	1	13	2	13	3	13	4	13	5	13
7	8	8	9	9	10	10	11	11	12	12	0
10	1	11	2	12	3	0	4	1	5	2	6
9	12	10	0	11	1	12	2	0	3	1	4
5	4	6	5	7	6	8	7	9	8	10	9
2	11	3	12	4	0	5	1	6	2	7	3
3	0	4	1	5	2	6	3	7	4	8	5
6	13	7	13	8	13	9	13	10	13	11	13
0	1	1	2	2	3	3	4	4	5	5	6
3	7	4	8	5	9	6	10	7	11	8	12
2	5	3	6	4	7	5	8	6	9	7	10
11	10	12	11	0	12	1	0	2	1	3	2
8	4	9	5	10	6	11	7	12	8	0	9
9	6	10	7	11	8	12	9	0	10	1	11
12	13	13	0	13	1	13	2	13	3	13	4
6	7	7	8	8	9	9	10	10	11	11	12
9	0	10	1	11	2	12	3	0	4	1	5
8	11	9	12	10	0	11	1	12	2	0	3
4	3	5	4	6	5	7	6	8	7	9	8
1	10	2	11	3	12	4	0	5	1	6	2
2	12	3	0	4	1	5	2	6	3	7	4
13	5	13	6	13	7	13	8	13	9	13	10
12	0	0	1								
2	6	3	7								
1	4	2	5								
10	9	11	10								
7	3	8	4								
8	5	9	6								
13	11	13	12								

Table 5.11: Table of a BIBD(14,52,7)

Example 5.3.7. *A* 22×55 *BAFD with parameters* $k = 110, r = \lambda_{00} = 567, b = 6,237$ *,*$\lambda_{01} = 42, \lambda_{10} = 27, \lambda_{11} = 52$ *and efficiencies* $E[0,1] = 1$*,* $E[1,0] = 1$*,* $E[1,1] = \frac{562}{567}$ *can be constructed by using both the* $BA(T)[2,11,1]$ *which can be constructed by using theorem 4.9.2 and a resolvable BIBD with* 55 *treatments and block size* 5 *given below which was constructed by developing parallel classes as in equations* (5.3.9)*,* (5.3.10)*,* (5.3.11)*. Thus we develop parallel classes* P_g*,* Q_x*,* R_x*.* $R = \left\{0, 1, 2, \cdots, 10\right\}$*. To construct a resolvable balanced incomplete block design with block size* 5 *and number of treatments* 55*,* $k = 5$*, suppose* $\lambda = 2 \leq k - 1$ *then* $s = \frac{\lambda(v-1)}{k(k-1)} = 1$ *base block hence* P_g *parallel classes:* $P_0, P_1, P_2, \cdots P_{10}$ *[*11 *blocks].* Q_x *parallel classes:* Q_0, Q_1, Q_2, Q_3, Q_4 *each of which is taken* $\lambda = 2$ *times.* R_x *parallel classes:* $R_0, R_1, R_2, R_3, R_4, R_5$ *each of which is taken* $\lambda - 1 = 2 - 1 = 1$ *times. Thus total number of parallel classes* $= 11 + 2 \times 5 + 6 \times 1 = 27$*. As an example the parallel classes* P_0, Q_1, R_0 *are given by*

P_0

5	11	17	23	29	10	21	32	43	54	0
15	31	47	8	24	30	6	37	13	44	1
20	41	7	28	49	35	16	52	33	14	2
25	51	22	48	19	40	26	12	53	39	3
45	36	27	18	9	50	46	42	38	34	4

Q_1

5	10	15	20	25	30	35	40	45	50	0
11	16	21	26	31	36	41	46	51	1	6
17	22	27	32	37	42	47	52	2	7	12
23	28	33	38	43	48	53	3	8	13	18
29	34	39	44	49	54	4	9	14	19	24

R_0

0	5	10	15	20	25	30	35	40	45	50
1	6	11	16	21	26	31	36	41	46	51
2	7	12	17	22	27	32	37	42	47	52
3	8	13	18	23	28	33	38	43	48	53
4	9	14	19	24	29	34	39	44	49	54

TABLE 5.12: Table of Parallel classesP_o, Q_1, R_o

Chapter 6

Multifactor BAFD'S

This chapter gives methods of constructing multifactor BAFD's

6.1 S^m symmetrical BFD's with block size s

The s^m symmetrical balanced factorial design (BFD) has been shown by Shah (1958) to be equivalent to a PBIB with a hypercubic association scheme.We shall consider the construction of such designs with block size s in this section, we have

$$r(s-1) = \sum_{i=1}^{m} \binom{m}{i} (s-1)^i \lambda_i \tag{6.1.1}$$

hence

$$r = \sum_{i=1}^{m} \binom{m}{i} (s-1)^{i-1} \lambda_i \tag{6.1.2}$$

r is completely determined by the values of $\lambda_i' s$. When s is a prime power Suen (1982) showed that there exists an s^m symmetrical BFD with block size s for any given $\lambda_1, \lambda_2, \ldots, \lambda_m$.

Lemma 6.1.1.

If s is a prime power, then given j $(1 \leq j \leq m)$ there exists an s^m symmetrical balanced factorial design with block size s and parameters $\lambda_j = 1,\ \lambda_i = 0 \quad \text{for} \quad \text{all} \quad i \neq j$.

The efficiencies of the symmetrical balanced factorial design constructed in Lemma 6.1.1 can be calculated by equation (3.2.5) and theorem 3.2.3 and hence

$$E_i = 1 - \frac{1}{s} - \frac{P_j(i;m,s)}{\binom{m}{j}(s-1)^{j-1}s}; \quad i = 1, 2, \ldots, m \tag{6.1.3}$$

In particular when $j = m$, $P_m(i; m, s)$

$$= (-1)^i (s-1)^{m-i}$$

and equation (6.1.3) becomes (6.1.4)

$$E_i = 1 - \frac{1}{s} - \frac{(-1)^i}{(s-1)^{i-1}s} \quad i = 1, 2, \ldots, m. \tag{6.1.4}$$

This balanced design has been constructed by Bose(1947);the main effects are estimated with full efficiency since $E_1 = 1$ in equation (6.1.4)

Theorem 6.1.1.

If s is a prime power, then for any given $\lambda_1, \lambda_2, \ldots, \lambda_m$ there exists an s^m symmetrical balanced factorial design with block size s and parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$.

Proof. Let D_j denote the design constructed in Lemma 6.1.1. The symmetrical balanced factorial design consists of λ_j D_j's for $j = 1, 2, \ldots, m$ has parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$.

Now consider the case when s is not a prime power. In an s^2 symmetrical balanced factorial design with block size s if we can construct a design with $\lambda_1 = 0$ and $\lambda_2 \neq 0$ then the design is equivalent to a $TA[\lambda_2 s(s-1), s, s, 2]$ by corollary 5.1.1 and the main effects are estimated with full efficiency.

In the case of s^m symmetrical balanced factorial designs with block size s, if we can construct a design with parameters $\lambda_m \neq 0$, and $\lambda_i = 0$ for $i = 1, 2, \ldots, m-1$, then the main effects are estimated with full efficiency. If a $TA[\lambda s(s-1), s, s, 2]$ exists,we can multiply (see theorem 4.8.2) $m-1$ such transitive arrays to get a $BA[\lambda s(s-1)^{m-1}, s, s, 2]$ with parameters $\lambda[(x_1, \ldots, x_{m-1}), (y_1 \ldots, y_{m-1})] = \lambda^{m-1}$ if $x_i \neq y_i$ for all $i = 1, 2, \ldots, m-1$ and $\lambda[(x_1, \ldots, x_{m-1}), (y_1 \ldots, y_{m-1})] = 0$ otherwise. Identifying rows with the levels of F_1, symbols with the levels of $F_2, \ldots, F_m$, and columns with blocks, we obtain an s^m symmetrical balanced factorial design in $[\lambda s(s-1)]^{m-1}$ blocks of s plots each with parameters $\lambda_m = \lambda^{m-1}$ and $\lambda_i = 0$ for $i = 1, \ldots, m-1$.Thus we have the following theorem □

Theorem 6.1.2.

The existence of a $TA[\lambda s(s-1), s, s, 2]$ implies the existence of an s^m symmetrical balanced factorial design with $b = [\lambda s(s-1)]^{m-1}, k = s, r = [\lambda(s-1)]^{m-1}$, $\lambda_m = \lambda^{m-1}$ and $\lambda_i = 0$, for $i = 1, \ldots, m-1$.

6.2 Methods of Constructing Multifactor BAFDS.

The methods are in form of theorems:

Theorem 6.2.1.

If there exists a $BA[N_i, s_m, s_i, 2](i = 1, \ldots, m-1)$ with parameters $\lambda_i(x, y) = \mu_0^i$ or μ_1^i according as $x = y$ or not then there exists an $s_1 \times s_2 \times \ldots \times s_m$ BAFD with $k = s_m$, $b = N_1 \ldots N_{m-1}$, $\lambda_{\alpha_1\alpha_2\ldots\alpha_{m-1}0=0}$,

$$\lambda_{\alpha_1\alpha_2\ldots\alpha_{m-1}1} = \mu_{\alpha_1}^1 \mu_{\alpha_2}^2 \ldots \mu_{\alpha_{m-1}}^{m-1}$$

where $\alpha_i = 0$ or 1.

Proof. Multiply the $m-1$ balanced arrays to obtain a $BA[N_1N_2\ldots N_{m-1}$, s_m, $s_1s_2\ldots s_{m-1}, 2]$ with parameters $\lambda[(x_1, x_2, \ldots, x_{m-1})$, $(y_1$, $y_2, \ldots$, $y_{m-1})] = \mu_{\alpha_1}^1 \mu^2\alpha_2 \ldots \mu_{\alpha_{m-1}}^{m-1}$ where $\alpha_i = 0$ or 1 according as $x = y$ or not. Identifying the symbols with the levels of F_1, $F_2, \ldots, F_{m-1}$, rows with the levels of F_m and columns with blocks, we obtain an $s_1 \times s_2 \times \ldots \times s_m$ BAFD with the specified parameters. The method used in theorem 6.2.1 can usually produce efficient BAFDS if we use balanced arrays corresponding to efficient two factor BAFDS. While applying this method, the block size remains the same but the number of blocks increases very rapidly. Hence this method is used when the number of assemblies in the balanced arrays are not too large. □

Example 6.2.1. *Consider the product of the* $OA[4, 3, 2, 2]$ *in Example 5.2.1 and the* $TA[6, 3, 3, 2]$ *given below*

0	1	2	0	1	2
1	2	0	2	0	1
2	0	1	1	2	0

Table 6.1: TA[6,3,3,2]

00	01	02	01	02	00	01	02	00	00	01	02
01	02	00	00	01	11	12	10	02	12	10	11
02	00	01	02	00	12	10	11	01	11	12	10

10	11	12	10	11	12	10	11	12	10	11	12
01	02	00	02	00	01	11	12	10	12	10	11
12	10	11	11	12	10	02	00	01	01	02	00

Table 6.2: BA[24,3,6,2]

which is a $BA[24, 3, 6, 2]$ with parameters $\lambda[(x_1, x_2), (y_1, y_2)] = 0$ or 1 according as $x_2 = y_2$ or not.

By theorem 6.2.1 this corresponds to a $2 \times 3 \times 3$ BAFD with $k = 3, b = 24, r = 4, \lambda(0, 1, 1) = \lambda(1, 1, 1) = 1$

$$\begin{aligned}
\lambda(0,0,1) &= \lambda(0,1,0) = \lambda(1,0,0) = \lambda(1,0,1) \\
&= \lambda(1,1,0) = 0 \\
& E[0,1,0] = E[0,0,1] = 1, \\
E[1,0,0] &= E[1,1,0] = E[1,0,1] = E[1,1,1] = \frac{2}{3} \\
E[0,1,1] &= \frac{1}{2}.
\end{aligned}$$

Example 6.2.2. *The product of $BA(T)(3, 2, 1)$ in example 5.3.1 and a $BAT(2, 3, 1)$ in example 4.9.5 generates a $2 \times 3 \times 6$ BAFD with*
$r = 25, b = 150, k = 6, \lambda(0, 1, 0) = \lambda(1, 0, 0) = \lambda(1, 1, 0) = 0,$
$\lambda(0, 0, 1) = 2, \lambda(0, 1, 1) = 4, \lambda(1, 0, 1) = 3$ *and* $\lambda(1, 1, 1) = 6.$
The efficiencies are

$$\begin{aligned}
E(0,0,1) &= E(0,1,0) = E(1,0,0) = 1.0 \\
E(0,1,1) &= E(1,0,1) = E(1,1,0) = \frac{4}{5} \\
& \text{and} \\
E(1,1,1) &= \frac{21}{25}.
\end{aligned}$$

We can also obtain an efficient $2 \times 3 \times 6$ BAFD by collapsing the first factor of the 6^2 symmetrical balanced factorial design in example 5.1.2 into two factors one at 2 levels and the other at 3 levels. The BAFD has parameters $r = 10, b = 60, k = 6, \lambda(0, 0, 1) = \lambda(0, 1, 0) = \lambda(1, 0, 0) = \lambda(0, 1, 0) = 0$ *and* $\lambda(0, 1, 1) = \lambda(1, 0, 1) = \lambda(1, 1, 1) = 2$

The efficiencies are

$$E(0,0,1) = E(0,1,0) = E(1,0,0) = E(1,1,0) = 1.0$$

and

$$E(0,1,1) = E(1,0,1) = E(1,1,1) = \frac{4}{5}.$$

all the main effects are also estimated with full efficiency like in example 6.2.2 but we only need 10 replications in this design.

Example 6.2.3. *The product of the* $BA(T)[4,3,2]$ *in example 4.9.8 and the* $BA(T)[3,4,2]$ *in example 4.9.9 generates a* $3 \times 4 \times 12$ *BAFD with* $r = 484, b = 5808, k = 12, \lambda(0,1,0) = \lambda(1,0,0) = \lambda(1,1,0) = 0,$ $\lambda(0,0,1) = 24, \lambda(0,1,1) = 36, \lambda(1,0,1) = 32$ *and* $\lambda(1,1,1) = 48$
The efficiencies are

$$\begin{aligned} E[0,0,1] &= E[0,1,0] = E[1,0,0] = 1.0 \\ E[0,1,1] &= E[1,0,1] = E[1,1,0] = \frac{10}{11} \\ &\text{and} \\ E[1,1,1] &= \frac{111}{121} \end{aligned}$$

The second method of constructing multifactor BAFD's we shall discuss was suggested by Yates (1937*b*) and employed by Nair and Rao (1941), Li (1944), Kishen (1958).The general form with exact conditions for validity was proved by Shah (1960*a*).This method replaces different levels of a factor in one design by distinct sets of treatment combinations forming the blocks of another design.

Assume that there exists a BAFD with m factors $F_1, F_2, \ldots, F_m$ at $s_1, s_2, \ldots, s_m$ levels respectively,each of the $v*(= s_1 s_2 \ldots s_m)$ treatments replicated $r*$ times in b^* blocks of k^* plots each,with the incidence matrix.

$$N^* = [A_1^*|A_2^*|\dots|A_b^**] \tag{6.2.1}$$

Further assume that $b^* = pq$, and the pq blocks can be divided into p groups of q blocks each, such that the design consisting of p blocks formed by adding together all the blocks of a group is a BAFD.The incidence matrix is

$$N_{pq}^* = \left[\sum_{j=1}^{q} A^*{}_j |\sum_{j=1}^{q} A^*{}_{j+q}|\dots|\sum_{j=1}^{q} A^*{}_{pq-q+j}\right] \tag{6.2.2}$$

for a resolvable design N^*, the corresponding $N^*{}_{pq}$ exists with $p = r^*$. The following theorem was proven by Shah (1960*a*).

Theorem 6.2.2.

Let there be a BAFD with the incidence matrix N in $n+1$ factors $F_0, F_{m+1}, \dots, F_{m+n}$ at $q, s_{m+1}, \dots, s_{m+n}$ levels respectively in b blocks of k plots each.Also let there be two BAFDs with incidence matrices N^* and $N^*{}_{pq}$ as given by equations (6.2.1) and (6.2.2) respectively.If the level $j-1$ of the factor F_0 is replaced by the block $A_{iq+j}(j = 1, 2, \dots, q)$ in each of the treatments of N ,then the design obtained by adjoining the p designs so formed (for $i = 0, 1, 2, \dots, p-1$) is a BAFD in $m+n$ factors in bp blocks of kk^* plots each.

This method generates an $m+n$ factor BAFD from an $n+1$ factor BAFD and an m factor BAFD.Thus from the two two-factor BAFD's we can generate a three-factor BAFD. If the two-factor BAFD's are efficient, then three-factor BAFD is also efficient. We can therefore construct efficient multi-factor BAFD's step by step from efficient two-factor BAFD's. While applying this method, the number of blocks does not increase so quickly as in the first method, but the block size does increase. It can be seen that the theorem 5.3.1 is a consequence of theorem 6.2.2 if we let $m = n = 1$ in theorem 6.2.2.

Example 6.2.4. *Let N be the incidence matrix of the 3×6 BAFD constructed by identifying rows,columns and symbols, with the levels of the second factor, the blocks ,and the levels of the first factors respectively in the $BA(T)(2,3,1)$ given in example 4.9.5. Let N^* be the incidence matrix of the resolvable 3^2 symmetrical balanced factorial design given below*

x_0	x_1	x_2	y_0	y_1	y_2
00	01	02	00	01	02
11	12	10	12	10	11
22	20	21	21	22	20

Table 6.3: 3^2 Symmetrical BFD

where $x_0, x_1, x_2, y_0, y_1, y_2$ represents blocks.Then by theorem 6.2.2we can construct a $3^2 \times 6$ BAFD with $r = 10$, $b = 30$, $\lambda(2,0) = 5$, $\lambda(0,1) = 2$,$\lambda(2,1) = 3$, $\lambda(1,1) = 4$, $\lambda(1,0) = 0$, $E[2,1] = \frac{9}{10}$ and all main effects and first order interactions are estimated with full efficiency.The BAFD is given below.

Blocks	1	2	3	4	5	6	7	8	9	10
Levels of F_3	Levels	of	F_1	and	F_2					
0	x_0	x_0	x_0	x_0	x_0	x_1	x_1	x_1	x_1	x_1
1	x_1	x_2	x_1	x_2	x_0	x_2	x_0	x_2	x_0	x_1
2	x_2	x_1	x_1	x_0	x_2	x_0	x_2	x_2	x_1	x_0
3	x_2	x_2	x_0	x_1	x_1	x_0	x_0	x_1	x_2	x_2
4	x_0	x_1	x_2	x_2	x_1	x_1	x_2	x_0	x_0	x_2
5	x_1	x_0	x_2	x_1	x_2	x_2	x_1	x_0	x_2	x_0

Blocks	11	12	13	14	15	16	17	18	19	20
Levels of F_3	Levels	of	F_1	and	F_2					
0	x_2	x_2	x_2	x_2	x_2	y_0	y_0	y_0	y_0	y_0
1	x_0	x_1	x_0	x_1	x_2	y_1	y_2	y_1	y_2	y_0
2	x_1	x_0	x_0	x_2	x_1	y_2	y_1	y_1	y_0	y_2
3	x_1	x_1	x_2	x_0	x_0	y_2	y_2	y_0	y_1	y_1
4	x_2	x_0	x_1	x_1	x_0	y_0	y_1	y_2	y_2	y_1
5	x_0	x_2	x_1	x_0	x_1	y_1	y_0	y_2	y_1	y_2

Blocks	21	22	23	24	25	26	27	28	29	30
Levels of F_3	Levels	of	F_1	and	F_2					
0	y_1	y_1	y_1	y_1	y_1	y_2	y_2	y_2	y_2	y_2
1	y_2	y_0	y_2	y_0	y_1	y_0	y_1	y_0	y_1	y_2
2	y_0	y_2	y_2	y_1	y_0	y_1	y_0	y_0	y_2	y_1
3	y_0	y_0	y_1	y_2	y_2	y_1	y_1	y_2	y_0	y_0
4	y_1	y_2	y_0	y_0	y_2	y_2	y_0	y_1	y_1	y_0
5	y_2	y_1	y_0	y_2	y_0	y_0	y_2	y_1	y_0	y_1

TABLE 6.4: $3^2 \times 6$ BAFD

Example 6.2.5. *Let N be the incidence matrix of the 3×6 BAFD constructed by identifying rows, columns, and symbols, with the levels of the second factor, the blocks and the levels of the first factor respectively in the $BA(T)(2, 3, 1)$given in example 4.9.5. Let N^* be the incidence matrix of the resolvable 2×3 BAFD given below.*

x_0	x_1	x_2	y_0	y_1	y_2
00	01	02	00	01	02
11	12	10	12	10	11

TABLE 6.5: Resolvable 2×3 BAFD

where $x_0, x_1, x_2, y_0, y_1, y_2$ represents blocks. Then by theorem 6.2.2, we can construct a $2 \times 3 \times 6$ BAFD with $r = 10, k = 12, b = 30$

$$\begin{aligned}
\lambda(0,0,1) &= 2, \lambda(0,1,0) = 0 \\
\lambda(0,1,1) &= 4, \lambda(1,0,0) = 0 \\
\lambda(1,0,1) &= 4, \lambda(1,1,0) = 5 \\
\lambda(1,1,1) &= 3
\end{aligned}$$

and efficiencies

$$
\begin{aligned}
E[0,0,1] &= E[0,1,0] = E[1,0,0] = 1.00 \\
&\textit{and} \\
E[1,1,0] &= E[1,0,1] = 1.00 \\
E[0,1,1] &= \frac{19}{20} \\
E[1,1,1] &= \frac{17}{20}
\end{aligned}
$$

The BAFD is given below

Blocks	1	2	3	4	5	6	7	8	9	10
Levels of F_3	Levels	of	F_1	and	F_2					
0	x_0	x_0	x_0	x_0	x_0	x_1	x_1	x_1	x_1	x_1
1	x_1	x_2	x_1	x_2	x_0	x_2	x_0	x_2	x_0	x_1
2	x_2	x_1	x_1	x_0	x_2	x_0	x_2	x_2	x_1	x_0
3	x_2	x_2	x_0	x_1	x_1	x_0	x_0	x_1	x_2	x_2
4	x_0	x_1	x_2	x_2	x_1	x_1	x_2	x_0	x_0	x_2
5	x_1	x_0	x_2	x_1	x_2	x_2	x_1	x_0	x_2	x_0

Blocks	11	12	13	14	15	16	17	18	19	20
Levels of F_3	Levels	of	F_1	and	F_2					
0	x_2	x_2	x_2	x_2	x_2	y_0	y_0	y_0	y_0	y_0
1	x_0	x_1	x_0	x_1	x_2	y_1	y_2	y_1	y_2	y_0
2	x_1	x_0	x_0	x_2	x_1	y_2	y_1	y_1	y_0	y_2
3	x_1	x_1	x_2	x_0	x_0	y_2	y_2	y_0	y_1	y_1
4	x_2	x_0	x_1	x_1	x_0	y_0	y_1	y_2	y_2	y_1
5	x_0	x_2	x_1	x_0	x_1	y_1	y_0	y_2	y_1	y_2

Blocks	21	22	23	24	25	26	27	28	29	30
Levels of F_3	Levels	of	F_1	and	F_2					
0	y_1	y_1	y_1	y_1	y_1	y_2	y_2	y_2	y_2	y_2
1	y_2	y_0	y_2	y_0	y_1	y_0	y_1	y_0	y_1	y_2
2	y_0	y_2	y_2	y_1	y_0	y_1	y_0	y_0	y_2	y_1
3	y_0	y_0	y_1	y_2	y_2	y_1	y_1	y_2	y_0	y_0
4	y_1	y_2	y_0	y_0	y_2	y_2	y_0	y_1	y_1	y_0
5	y_2	y_1	y_0	y_2	y_0	y_0	y_2	y_1	y_0	y_1

TABLE 6.6: $2 \times 3 \times 6$ BAFD

6.3 Examples of multifactor BAFDS

In this section, we shall use the methods discussed in the preceding section and some known balanced factorial designs to construct examples of multifactor BAFDS. We are especially interested in BAFDS of which the main effects and lower order interactions can be estimated with high efficiencies.

TYPE I

If there exists $TA[s_i(s_i-1), s_m, s_i, 2]$ for $i = 1, 2, \ldots, m-1$ then by theorem 6.2.1 we can construct an $s_1 \times s_2 \times \ldots s_m$ BAFD with $k = s_m$, $b = \prod_{i=1}^{m-1} s_i(s_i-1)$, $r = \prod_{i=1}^{m-1}(s_i-1)$, $\lambda(1,1,\ldots,1) = 1$ and other $\lambda' s$ being 0. By theorem 3.1.4 the eigenvalues of NN^T of a BAFD are given by

$$\begin{aligned} g(y_1, y_2, \ldots, y_m) &= rk - k\rho(y_1, y_2, \ldots, y_m) \\ &= rk - \left\{ r(k-1) - \textstyle\sum_{x\in\Omega} \lambda(x) \left\{ \prod_{i=1}^m \left[(1-y_i)s_i - 1 \right]^{x_i} \right\} \right\} \end{aligned} \tag{6.3.1}$$

$$= r + \sum_{x\in\Omega} \lambda(x) \left\{ \prod_{i=1}^m \left[(1-y_i)s_i - 1 \right]^{x_i} \right\} \tag{6.3.2}$$

hence

$$\begin{aligned} E[y_1, y_2, \ldots, y_m] &= 1 - \frac{1}{rk} g[y_1, y_2, \ldots, y_m] \\ &= 1 - \frac{1}{rk} \left[r + \textstyle\sum_{x\in\Omega} \lambda(x) \left\{ \prod_{i=1}^m \left[(1-y_i)s_i - 1 \right]^{x_i} \right\} \right] \\ &= 1 - \frac{1}{k} - \frac{1}{rk}(1) \left\{ \prod_{i=1}^m \left[(1-y_i)s_i - 1 \right]^{x_i} \right\} \end{aligned} \tag{6.3.3}$$

$$= \frac{-1}{s_m} + 1 - \frac{\prod_{i=1}^m [(1-y_i)s_i - 1]^{x_i}}{s_m \prod_{i=1}^{m-1}(s_i - 1)}$$

$$= 1 - \frac{1}{s_m} - \frac{\prod_{i=1}^{m}[(1-y_i)s_i - 1]^{x_i}}{s_m \prod_{i=1}^{m-1}(s_i - 1)} \tag{6.3.4}$$

Let $y_j = 1$ and $y_i = 0$ for $i \neq j$, equation (6.3.4) becomes $E[0, 0, \ldots, 1_j, \ldots, 0, \ldots, 0]$

$$\begin{aligned}
&= 1 - \frac{1}{s_m} - \frac{\left\{\begin{matrix}\Big\{(1-0)s_1 - 1\Big\}^1 \Big\{(1-0)s_2 - 1\Big\}^1 \ldots \\ \Big\{(1-1)s_j - 1\Big\}^1 \ldots \Big\{(1-0)s_m - 1\Big\}^1\end{matrix}\right\}}{s_m \prod_{i-1}^{m-1}(s_i - 1)} \\
&= 1 - \frac{1}{s_m} - \frac{\Big\{(s_1 - 1)(s_2 - 1)\ldots(-1)\ldots(s_m - 1)\Big\}}{s_m \prod_{i-1}^{m-1}(s_i - 1)} \\
&= 1 - \frac{1}{s_m} - \frac{(s_1 - 1)(s_2 - 1)\ldots(-1)\ldots(s_{m-1} - 1)(s_m - 1)}{s_m(s_1 - 1)(s_2 - 1)\ldots(s_j - 1)\ldots(s_{m-1} - 1)} \\
&= 1 - \frac{1}{s_m} - \frac{-(1)(s_m - 1)}{s_m(s_j - 1)} \\
&= 1 - \frac{1}{s_m} + \frac{s_m - 1}{s_m(s_j - 1)} \\
&= 1 + \frac{s_m - s_j}{s_m(s_j - 1)} \\
&= 1 - \frac{s_j - s_m}{s_m(s_j - 1)}
\end{aligned} \tag{6.3.5}$$

This is the efficiency of the main effects of the factor F_j and it is 1 when $j = m$. Hence the main effects of F_m are estimated with full efficiency. In general, let $y_{ji} = 1$ for $i = 1, \ldots, q (q \leq m)$ and other $y's$ be 0, then equation (6.3.5) is

$$
\begin{aligned}
E[y_1, y_2, \dots, y_m] &= E[1, 1, \dots, 1_q, 0, 0, \dots, 0] \\
&= 1 - \frac{1}{s_m} - \frac{\prod_{i=1}^{m}[(1-y_i)s_i - 1]^{x_i}}{s_m \prod_{i=1}^{m-1}(s_i - 1)} \\
&= 1 - \frac{1}{s_m} - \frac{\begin{bmatrix} \left\{(1-1)s_1 - 1\right\}^1 \left\{(1-1)s_2 - 1\right\}^1 \\ \dots \left\{(1-1)s_q - 1\right\}^1 \left\{(1-0)s_{q+1} - 1\right\}^1 \dots \left\{(1-0)s_m - 1\right\}^1 \end{bmatrix}}{s_m \left\{(s_1 - 1)(s_2 - 1)\dots(s_q - 1)(s_{q+1} - 1)\dots(s_{m-1} - 1)\right\}} \\
&= 1 - \frac{1}{s_m} - \frac{(-1)^1(-1)^1\dots(-1)^1(s_{q+1} - 1)\dots(s_{m-1} - 1)(s_m - 1)}{s_m(s_1 - 1)(s_2 - 1)(s_3 - 1)\dots(s_q - 1)(s_{q+1} - 1)\dots(s_{m-1} - 1)} \\
&= 1 - \frac{1}{s_m} - \frac{(-1)^q(s_m - 1)}{s_m \prod_{i=1}^{q}(s_{ji} - 1)} \\
&= 1 - \frac{1}{s_m} - \frac{(-1)^q(s_m - 1)}{s_m \prod_{i=1}^{q}(s_{ji} - 1)}
\end{aligned}
\tag{6.3.6}
$$

which is efficiency of the $(q-1)^{th}$ order interaction between $F_{j1}, F_{j2}, \dots F_{jq}$.

Example 6.3.1. *For any given* $s_i \geq 3, (i = 1, 2, 3, \dots, m-1)$. *The* $TA[s_i(s_i - 1), 3, s_i, 2]'s$ *always exists. Hence we can always construct an* $s_1 \times s_2 \times \dots \times s_{m-1} \times 3$ *BAFD with* $k = 3, b = \prod_{i=1}^{m-1} s_i(s_i - 1), r = \prod_{i=1}^{m-1}(s_i - 1), \lambda(1, 1, \dots 1) = 1$ *and all other* $\lambda's$ *being* 0.

Example 6.3.2. *Using the* $TA[20, 3, 5, 2]$ *in example 4.1.4 and the* $TA[6, 3, 3, 2]$, *we can construct a* $5 \times 3 \times 3$ *BAFD using theorem 6.2.1.The parameters of this BAFD are* $k = 3, b = 120, r = 8, \lambda(0, 0, 1) = \lambda(0, 1, 0) = \lambda(0, 1, 1) = \lambda(1, 0, 0) = \lambda(1, 0, 1) = \lambda(1, 1, 0) = 0, \lambda(1, 1, 1) = 1.0$ *The efficiencies of this design are*

$$
\begin{aligned}
E[0, 0, 1] &= E[0, 1, 0] = 1.00 \\
E[1, 0, 0] &= \frac{5}{6} \\
E[0, 1, 1] &= \frac{1}{2} \\
E[1, 0, 1] &= E[1, 1, 0] = \frac{7}{12} \\
E[1, 1, 1] &= \frac{17}{24}
\end{aligned}
$$

Other examples of this type include $5 \times 4 \times 4$,$5 \times 5 \times 4$, $7 \times 5 \times 4$, $7 \times 5 \times 4$,$7 \times 7 \times 5$,$\ldots$, *BAFD's and so on.*

TYPE II

Let $s_m = n_1 s_1 = n_2 s_2 = \ldots = n_{m-1} s_{m-1}$ and there exists $BA(T)[n_i, s_i, 1]$ for $i = 1, 2, \ldots, m-1$. By theorem 6.2.1 there exists an $s_1 \times s_2 \times \ldots \times s_m$ BAFD with $k = s_m$, $b = (s_m - 1)^{m-1} s_1 s_2 \ldots s_{m-1}$, $r = [s_m - 1]^{m-1}$, $\lambda(y_1, y_2, \ldots y_{m-1}, 0) = 0$ and $\lambda(y_1, y_2, \ldots y_{m-1}, 1) = \prod_{i=1}^{m-1} n_i^{x_i}(n_i - 1)^{1-x_i}$. By theorem 3.1.4 the eigenvalues of the NN^T of the BAFD are given by

$$g[y_1, y_2, \ldots, y_m] = r + \sum_{x \in \Omega} \lambda(x) \left\{ \prod_{i=1}^{m} [(1 - y_i) s_i - 1]^{x_i} \right\} \tag{6.3.7}$$

$$= r + \sum_{x \in \Omega} \prod_{i-1}^{m-1} n_i^{x_i}(n_i - 1)^{1-x_i} \left\{ \prod_{i=1}^{m} [(1 - y_i) s_i - 1]^{x_i} \right\} \tag{6.3.8}$$

Let $y_m = 1$, $y_i = 0$ for $i = 1, 2, \ldots, m-1$ in equation (6.3.8)

$$g(0,0,\ldots,0,1) = r +$$
$$\left\{\begin{array}{c} \lambda^1(x_1,x_2,\ldots,x_{m-1},1)[\Big\{(1-0)s_1-1\Big\}^{x_1}\Big\{(1-0)s_2-1\Big\}^{x_2}\ldots \\ \Big\{(1-1)s_m-1\Big\}^{1}] + \lambda^2(x_1,x_2,\ldots \\ x_{m-1},1)[\Big\{(1-0)s_1-1\Big\}^{x_1}\Big\{(1-0)s_2-1\Big\}^{x_2}\ldots \\ \Big\{(1-1)s_m-1\Big\}^{1}] + \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots + \lambda^{m-1}(x_1,x_2,\ldots,x_{m-1},1) \\ [\Big\{(1-0)s_1-1\Big\}^{x_1}\Big\{(1-0)s_2-1\Big\}^{x_2}\ldots \\ \Big\{(1-1)s_m-1\Big\}^{1}] \end{array}\right\} \quad (6.3.9)$$

where $\lambda^h(x_1,x_2,\ldots x_{m-1},1)$, $h = 1, 2, \ldots, m-1$ is the h^{th} distinct term of $\lambda(x_1,x_2,\ldots,x_{m-1},1)$ and hence we have $2^m - 1$ distinct terms of $\lambda(x_1,x_2,\ldots,x_{m-1},1)$.

After expanding equation (6.3.9) we have

$$\begin{aligned} & g(0,0,\ldots,0,1) \\ &= r + \Big\{-s_m^{m-1} + (m-1)s_m^{m-2} - \tfrac{(m-1)(m-2)}{2}s_m^{m-3} + \ldots - s_m^0(-1)^{m-1}\Big\} \\ &= r - \Big\{s_m^{m-1} - (m-1)s_m^{m-2} + \tfrac{(m-1)(m-2)}{2}s_m^{m-3} - \ldots + s_m^0(-1)^{m-1}\Big\} \\ &= r - \left\{\begin{array}{c} {}^{m-1}c_0 s_m^{m-1}(-1)^0 +^{m-1} c_1 s_m^{m-2}(-1)^1 +^{m-1} c_2 s_m^{m-3}(-1)^2 + \ldots + \\ {}^{m-1}c_{m-1}s_m^0(-1)^{m-1} \end{array}\right\} \\ &= r - (s_m - 1)^{m-1} = r - r = 0 \end{aligned}$$

similarly we can show that $g(1,0,0,\ldots,0) = g(0,1,0,\ldots,0) = \ldots = g(0,0,0,\ldots,1,0) = 0$. Thus all main effects are estimated with full efficiency. Let $y_{m-1} = y_m = 1$ and $y_i = 0$ for $i = 1,2,\ldots m-2$ then (6.3.8) becomes

$$g(0,0,\ldots,0,1,1) = r +$$
$$\left\{\begin{array}{c} \lambda^1(x_1,x_2,\ldots,x_{m-1},1)[\left\{(1-0)s_1-1\right\}^{x_1}\left\{(1-0)s_2-1\right\}^{x_2}\ldots(a) \\ \left\{(1-1)s_m-1\right\}^1] + \lambda^2(x_1,x_2,\ldots x_{m-1},1)[\left\{\begin{array}{c}(1-0\\ )s_1-1\end{array}\right\}^{x_1}\left\{(1-0)s_2-1\right\}^{x_2} \\ \ldots(a)\left\{(1-1)s_m-1\right\}^1] + \ldots \\ +\lambda^{2m-1}(x_1,x_2,\ldots,x_{m-1},1)\left\{(1-0)s_1-1\right\}^{x_1}\left\{(1-0)s_2-1\right\}^{x_2}\ldots \\ \left\{(1-1)s_{m-1}-1\right\}^{x_m-1}\left\{(1-1)s_m-1\right\}^1 \end{array}\right\} \tag{6.3.10}$$

(In the above equation (6.3.10), let $\left\{(1-1)s_{m-1}-1\right\}^{x_m-1}$ be a value represented by (a))

where $\lambda^h(x_1,x_2,\ldots,x_{m-1},1), h = 1,2,\ldots,2^m-1$ is the h^{th} distinct term of $\lambda(x_1,x_2,\ldots,x_{m-1},1)$ and hence we have 2^m-1 distinct terms of $\lambda(x_1,x_2,\ldots,x_{m-1},1)$.

After expanding equation (6.3.10) we have

$$g(0,0,\ldots,0,1,1)$$

$$= r + \left\{s_m^{m-2} - (m-2)s_m^{m-3} + \tfrac{(m-2)(m-3)}{2}s_m^{m-4} - \ldots + s_m^0(-1)^{m-2}\right\}$$

$$= r + \left\{\begin{array}{c} {}^{m-2}c_0 s_m^{m-2}(-1)^0 + {}^{m-2}c_1 s_m^{m-3}(-1)^1 + {}^{m-2}c_2 s_m^{m-4}(-1)^2 + \ldots + \\ {}^{m-2}c_{m-2}s_m^0(-1)^{m-2} \end{array}\right\}$$

$$= r + (s_m-1)^{m-2}$$
$$= (s_m-1)^{m-1} + (s_m-1)^{m-2}$$

In general, when $\sum_{i=1}^m y_i = q$ equation (6.3.8) is

$$g(y_1,y_2,\ldots,y_m) = (s_m-1)^{m-1} + (-1)^q(s_m-1)^{m-q} \tag{6.3.11}$$

hence by corollary 3.1.1

$$
\begin{aligned}
E[y_1, y_2, \ldots, y_m] &= 1 - \frac{g[y_1, y_2, \ldots, y_m]}{rk} \\
&= 1 - \frac{(s_m - 1)^{m-1} + (-1)^q (s_m - 1)^{m-q}}{(s_m - 1)^{m-1} s_m} \\
&= 1 - \frac{1}{s_m} - \frac{(-1)^q}{s_m (s_m - 1)^{q-1}} \qquad (6.3.12)
\end{aligned}
$$

where $\sum_{i=1}^{m} y_i = q$.

It can be seen that equation (6.3.12) is the same as the equation (6.1.4) with $s_m = s$, hence the efficiencies are equal to those of s_m^m symmetrical balanced factorial design in lemma 6.1.1 with $j = m$.

Example 6.3.3. *A BA(T)(3,2,1) given in example 5.3.1 can be used to construct a* $2^2 \times 6$ *BAFD with* $k = 6$*,* $b = 100$*,* $r = 25$*,* $\lambda(1,0) = \lambda(2,0) = 0$*,* $\lambda(0,1) = 4$*,* $\lambda(1,1) = 6$*,* $\lambda(2,1) = 9$

The efficiencies are;

$$
\begin{aligned}
E[0,1] &= E[1,0] = 1.0 \\
E[1,1] &= E[2,0] = \frac{4}{5} \\
E[2,1] &= \frac{21}{25}
\end{aligned}
$$

Example 6.3.4. *A BA(T)(2,3,1) given in example 4.9.5 can be used to construct a* $3^3 \times 6$ *BAFD with*

$k = 6, b = 3,375, r = 125, \lambda(1,0) = \lambda(2,0) = \lambda(3,0) = 0,$

$\lambda(0,1) = 1, \lambda(1,1) = 2, \lambda(2,1) = 4, \lambda(3,1) = 8$

The efficiencies are;

$$E[0,1] = E[1,0] = 1.0$$
$$E[1,1] = E[2,1] = \frac{4}{5}$$
$$E[2,1] = \frac{21}{25}$$
$$E[3,1] = \frac{104}{125}$$

Example 6.3.5. *Example 6.2.2 is also of this type. Other examples include* $2 \times 2 \times 4$, $2 \times 4 \times 4$, $3 \times 3 \times 6$, $2 \times 5 \times 10$, $\ldots$ *and so on.*

The following example is also a $2^2 \times 6$ BAFD with only 5 replications;the main effects are estimated with full efficiencies and some interactions are not estimable.

Example 6.3.6. *A* $2^2 \times 6$ *BAFD with* $k = 6$, $b = 20$, $r = 5$, $\lambda(1,1) = \lambda(1,0) = \lambda(2,0) = 0$ *and* $\lambda(0,1) = 2$, $\lambda(2,1) = 3$
The efficiencies are;

$$E[0,1] = E[1,0] = E[2,1] = 1.0$$

and

$$E[1,1] = \frac{4}{5}, E[2,0] = 0$$

can be constructed using theorem 6.2.2 and by letting N *be the incidence matrix of the* 2×6 *BAFD that was corresponding to a* $BA(T)[3,2,1]$ *which was given in example 5.3.1. In this case, we shall let* N^* *be the incidence matrix of the following* 2^2 *design with block size* 1*.*

00	11	01	10

TABLE 6.7: resolvable 2^2 Symmetrical design

and we shall let N^*_{22} be the following 2^2 balanced factorial design with interaction confounded

00	01
11	10

TABLE 6.8: 2^2 balanced factorial design with interactions confounded

Applying theorem 6.2.2 we get the following $2^2 \times 6$ BAFD.

Blocks	1	2	3	4	5	6	7	8	9	10
Levels of F_3	Levels	of	F_1	and	F_2					
0	00	00	00	00	00	11	11	11	11	11
1	00	00	11	11	11	00	00	00	11	11
2	11	00	00	11	11	11	11	00	00	00
3	00	11	11	00	11	00	11	11	00	00
4	11	11	11	00	00	11	00	00	00	11
5	11	11	00	11	00	00	00	11	11	00

Blocks	11	12	13	14	15	16	17	18	19	20
Levels of F_3	Levels	of	F_1	and	F_2					
0	01	01	01	01	01	10	10	10	10	10
1	01	01	10	10	10	01	01	01	10	10
2	10	01	01	10	10	10	10	01	01	01
3	01	10	10	01	10	01	10	10	01	01
4	10	10	10	01	01	10	01	01	01	10
5	10	10	01	10	01	01	01	10	10	01

TABLE 6.9: $2^2 \times 6$ BAFD

Example 6.3.7. *A* $2^2 \times 4$ *BAFD with* $k = 4$, $b = 24$, $r = 6$, $\lambda(1,1) = \lambda(1,0) = \lambda(2,0) = 0$ *and* $\lambda(0,1) = 2$,$\lambda(2,1) = 4$ *and efficiencies are;*

$$E[0,1] = E[1,0] = E[2,1] = 1.0$$

and

$$E[1,1] = \frac{2}{3}, E[2,0] = 0$$

can be constructed by letting N *be the incidence matrix of the* 2×4 *BAFD that corresponds to the* $BA[12,4,2,2]$ *in example 4.9.7. In this case, we shall let* N^* *be the incidence matrix of the following* 2^2 *design with block size* 1

00	11	01	10

TABLE 6.10: 2^2 design

and we shall let N^*_{22} *be the following* 2^2 *balanced factorial design with interaction confounded*

00	01
11	10

TABLE 6.11: 2^2 BFD with interactions confounded

Applying theorem 6.2.2 we get the following $2^2 \times 4$ *BAFD.*

Blocks	1	2	3	4	5	6	7	8	9	10	11	12
Levels of F_3	Levels	of	F_1	and	F_2							
0	00	00	00	00	00	00	11	11	11	11	11	11
1	11	11	00	11	00	11	00	00	11	00	11	00
2	00	00	11	11	11	11	11	11	00	00	00	00
3	11	11	11	00	11	00	00	00	00	11	00	11

Blocks	13	14	15	16	17	18	19	20	21	22	23	24
Levels of F_3	Levels	of	F_1	and	F_2							
0	01	01	01	01	01	01	10	10	10	10	10	10
1	10	10	01	10	01	10	01	01	10	01	10	01
2	01	01	10	10	10	10	10	10	01	01	01	01
3	10	10	10	01	10	01	01	01	01	10	01	10

TABLE 6.12: $2^2 \times 4$ BAFD

Example 6.3.8. *A* $3^2 \times 9$ *BAFD with* $k = 9$*,* $b = 144$*,* $r = 16$*,* $\lambda(1,1) = \lambda(1,0) = \lambda(2,0) = 0$ *and* $\lambda(0,1) = 4, \lambda(2,1) = 3$

and efficiencies;

$$E[0,1] = E[1,0] = 1.0$$
$$E[1,1] = \frac{7}{8}, E[2,0] = \frac{1}{2}$$
$$E[2,1] = \frac{15}{16}$$

can be constructed by letting N *be the incidence matrix of the* 3×9 *BAFD that corresponds to the* $BA(T)[3,3,1]$ *in example 4.9.4. In this case, we shall let* N^* *be the incidence matrix of the following* 3^2 *design with block size* 1*.*

00	12	21	01	10	22	02	11	20	00	11	22	02
10	21	01	12	20								

TABLE 6.13: 3^2 design with block size 1

and we shall let N_{22}^* *be the incidence matrix of the following* 3^2 *balanced factorial design with interaction confounded.*

00	01	02	00	02	01
12	10	11	11	10	12
21	22	20	22	21	20

TABLE 6.14: 3^2 BFD with interactions confounded

Applying theorem 6.2.2 we get the following $3^2 \times 9$ *BAFD.*

Blocks	1	2	3	4	5	6	7	8	9	10	11	12
Levels of F_3	Levels	of	F_1	and	F_2							
0	00	00	00	00	00	00	00	00	12	12	12	12
1	12	21	00	12	21	00	12	21	21	00	12	21
2	21	12	00	21	12	00	21	12	00	21	12	00
3	00	00	12	12	12	21	21	21	12	12	21	21
4	12	21	12	21	00	21	00	12	21	00	21	00
5	21	12	12	00	21	21	12	00	00	21	21	12
6	00	00	21	21	21	12	12	12	12	12	00	00
7	12	21	21	00	12	12	21	00	21	00	00	12
8	21	12	21	12	00	12	00	21	00	21	00	21

Blocks	13	14	15	16	17	18	19	20	21	22	23	24
Levels of F_3	Levels	of	F_1	and	F_2							
0	12	12	12	12	21	21	21	21	21	21	21	21
1	00	12	21	00	00	12	21	00	12	21	00	12
2	21	12	00	21	12	00	21	12	00	21	12	00
3	21	00	00	00	21	21	00	00	00	12	12	12
4	12	00	12	21	00	12	00	12	21	12	21	00
5	00	00	21	12	12	00	00	21	12	12	00	21
6	00	21	21	21	21	21	12	12	12	00	00	00
7	21	21	00	12	00	12	12	21	00	00	12	21
8	12	21	12	00	12	00	12	00	21	00	21	12

Blocks	25	26	27	28	29	30	31	32	33	34	35	36
Levels of F_3	Levels	of	F_1	and	F_2							
0	01	01	01	01	01	01	01	01	10	10	10	10
1	10	22	01	10	22	01	10	22	22	01	10	22
2	22	10	01	22	10	01	22	10	01	22	10	01
3	01	01	10	10	10	22	22	22	10	10	22	22
4	10	22	10	22	01	22	01	10	22	01	22	01
5	22	10	10	01	22	22	10	01	01	22	22	10
6	01	01	22	22	22	10	10	10	10	10	01	01
7	10	22	22	01	10	10	22	01	22	01	01	10
8	22	10	22	10	01	10	01	22	01	22	01	22

Blocks	37	38	39	40	41	42	43	44	45	46	47	48
Levels of F_3	Levels	of	F_1	and	F_2							
0	10	10	10	10	22	22	22	22	22	22	22	22
1	01	10	22	01	01	10	22	01	10	22	01	10
2	22	10	01	22	10	01	22	10	01	22	10	01
3	22	01	01	01	22	22	01	01	01	10	10	10
4	10	01	10	22	01	10	01	10	22	10	22	01
5	01	01	22	10	10	01	01	22	10	10	01	22
6	01	22	22	22	22	22	10	10	10	01	01	01
7	22	22	01	10	01	10	10	22	01	01	10	22
8	10	22	10	01	10	01	10	01	22	01	22	10

Blocks	49	50	51	52	53	54	55	56	57	58	59	60
Levels of F_3	Levels	of	F_1	and	F_2							
0	02	02	02	02	02	02	02	02	11	11	11	11
1	11	20	02	11	20	02	11	20	20	02	11	20
2	20	11	02	20	11	02	20	11	02	20	11	02
3	02	02	11	11	11	20	20	20	11	11	20	20
4	11	20	11	20	02	20	02	11	20	02	20	02
5	20	11	11	02	20	20	11	02	02	20	20	11
6	02	02	20	20	20	11	11	11	11	11	02	02
7	11	20	20	02	11	11	20	02	20	02	02	11
8	20	11	20	11	02	11	02	20	02	20	02	20

Blocks	61	62	63	64	65	66	67	68	69	70	71	72
Levels of F_3	Levels	of	F_1	and	F_2							
0	11	11	11	11	20	20	20	20	20	20	20	20
1	02	11	20	02	02	11	20	02	11	20	02	11
2	20	11	02	20	11	02	20	11	02	20	11	02
3	20	02	02	02	20	20	02	02	02	11	11	11
4	11	02	11	20	02	11	02	11	20	11	20	02
5	02	02	20	11	11	02	02	20	11	11	02	20
6	02	20	20	20	20	20	11	11	11	02	02	02
7	20	20	02	11	02	11	11	20	02	02	11	20
8	11	20	11	02	11	02	11	02	20	02	20	11

Blocks	73	74	75	76	77	78	79	80	81	82	83	84
Levels of F_3	Levels	of	F_1	and	F_2							
0	00	00	00	00	00	00	00	00	11	11	11	11
1	11	22	00	11	22	00	11	22	22	00	11	22
2	22	11	00	22	11	00	22	11	00	22	11	00
3	00	00	11	11	11	22	22	22	11	11	11*	22
4	11	22	11	22	00	22	00	11	22	00	22	00
5	22	11	11	00	22	22	11	00	00	22	22	11
6	00	00	22	22	22	11	11	11	11	11	22	00
7	11	22	22	00	11	11	22	00	22	00	00	11
8	22	11	22	11	00	11	00	22	00	22	00	22

Blocks	85	86	87	88	89	90	91	92	93	94	95	96
Levels of F_3	Levels	of	F_1	and	F_2							
0	11	11	11	11	22	22	22	22	22	22	22	22
1	00	11	22	00	00	11	22	00	11	22	00	11
2	22	11	00	22	11	00	22	11	00	22	11	00
3	22	00	00	00	22	22	00	00	00	11	11	11
4	11	00	11	22	00	11	00	11	22	11	22	00
5	00	00	22	11	11	00	00	22	11	11	00	22
6	00	22	22	22	22	22	11	11	11	00	00	00
7	22	22	00	11	00	11	11	22	00	00	11	22
8	11	22	11	00	11	00	11	00	22	00	22	11

Blocks	97	98	99	100	101	102	103	104	105	106	107	108
Levels of F_3	Levels	of	F_1	and	F_2							
0	02	02	02	02	02	02	02	02	10	10	10	10
1	10	21	02	10	21	02	10	21	21	02	10	21
2	21	10	02	21	10	02	21	10	02	21	10	02
3	02	02	10	10	10	21	21	21	10	10	21	21
4	10	21	10	21	02	21	02	10	21	02	21	02
5	21	10	10	02	21	21	10	02	02	21	21	10
6	02	02	21	21	21	10	10	10	10	10	02	02
7	10	21	21	02	10	10	21	02	21	02	02	10
8	21	10	21	10	02	10	02	21	02	21	02	21

Blocks	109	110	111	112	113	114	115	116	117	118	119	120
Levels of F_3	Levels	of	F_1	and	F_2							
0	10	10	10	10	21	21	21	21	21	21	21	21
1	02	10	21	02	02	10	21	02	10	21	02	10
2	21	10	02	21	10	02	21	10	02	21	10	02
3	21	02	02	02	21	21	02	02	02	10	10	10
4	10	02	10	21	02	10	02	10	21	10	21	02
5	02	02	21	10	10	02	02	21	10	10	02	21
6	02	21	21	21	21	21	10	10	10	02	02	02
7	21	21	02	10	02	10	10	21	02	02	10	21
8	10	21	10	02	10	02	10	02	21	02	21	10

Blocks	121	122	123	124	125	126	127	128	129	130	131	132
Levels of F_3	Levels	of	F_1	and	F_2							
0	01	01	01	01	01	01	01	01	12	12	12	12
1	12	20	01	12	20	01	12	20	20	01	12	20
2	20	12	01	20	12	01	20	12	01	20	12	01
3	01	01	12	12	12	20	20	20	12	12	20	20
4	12	20	12	20	01	20	01	12	20	01	20	01
5	20	12	12	01	20	20	12	01	01	20	20	12
6	01	01	20	20	20	12	12	12	12	12	01	01
7	12	20	20	01	12	12	20	01	20	01	01	12
8	20	12	20	12	01	12	01	20	01	20	01	20

Blocks	133	134	135	136	137	138	139	140	141	142	143	144
Levels of F_3	Levels	of	F_1	and	F_2							
0	12	12	12	12	20	20	20	20	20	20	20	20
1	01	12	20	01	01	12	20	01	12	20	01	12
2	20	12	01	20	12	01	20	12	01	20	12	01
3	20	01	01	01	20	20	01	01	01	12	12	12
4	12	01	12	20	01	12	01	12	20	12	20	01
5	01	01	20	12	12	01	01	20	12	12	01	20
6	01	20	20	20	20	20	12	12	12	01	01	01
7	20	20	01	12	01	12	12	20	01	01	12	20
8	12	20	12	01	12	01	12	01	20	01	20	12

TABLE 6.15: $3^2 \times 9$ BAFD

TYPE III

Let there exist a $BA(T)(n_1, s, 1)$, by corollary (5.2.1). This corresponds to $n_1 s \times s$ BAFD with $k = n_1 s$, $b = (n_1 s - 1)s$, and $\lambda(0,0) = n_1 s - 1 = r$

$$\lambda(0,1) = 0$$

$$\lambda(1,0) = n_1 - 1 \tag{6.3.13}$$

$$\lambda(1,1) = n_1$$

by equations (5.1.1), (5.1.2) and (5.1.3), the eigenvalues of NN^T are

$$g(1,0) = 0$$

$$g(0,1) = 0$$

$$g(1,1) = n_1 s \tag{6.3.14}$$

If there exists a resolvable $BA(T)(n_2, s, 1)$,then this corresponds to a resolvable $n_2 s \times s$ BAFD. By theorem 6.2.2, if we replace the levels of the second factor of the $n_1 s \times s$ BAFD by the blocks of the $n_2 s \times s$ BAFD, we get an $n_1 s \times n_2 s \times s$ BAFD with $k = n_1 n_2 s^2, b = (n_1 s - 1)(n_2 s - 1)s$, and

$$\lambda(0,0,0) = \lambda(0,0)(n_2 s - 1) = (n_1 s - 1)(n_2 s - 1)$$

$$
\begin{aligned}
\lambda(0,0,1) &= \lambda(0,1)(n_2 s-1) = 0 \\
\lambda(0,1,0) &= \lambda(0,0)(n_2-1)+\lambda(0,1)(n_2 s-n_2) = (n_1 s-1)(n_2-1) \\
\lambda(0,1,1) &= \lambda(0,0)n_2+\lambda(0,1)(n_2 s-n_2-1) = (n_1 s-1)n_2 \\
\lambda(1,0,0) &= \lambda(1,0)(n_2 s-1) = (n_1-1)(n_2 s-1) \\
\lambda(1,0,1) &= \lambda(1,1)(n_2 s-1) = n_1(n_2 s-1) \\
\lambda(1,1,0) &= \lambda(1,0)(n_2-1)+\lambda(1,1)(n_2 s-1) \\
&= (n_1-1)(n_2-1)+n_1(n_2 s-n_2) \\
\lambda(1,1,1) &= \lambda(1,0)n_2+\lambda(1,1)(n_2 s-n_2-1) \\
&= (n_1-1)n_2+n_1(n_2 s-n_2-1)
\end{aligned}
\tag{6.3.15}
$$

where $\lambda(0,0), \lambda(0,1), \lambda(1,0), \lambda(1,1)$ are given by equation (6.3.13). The eigenvalues of NN^T are

$g[y_1, y_2, y_3] = n_1 n_2 s^2$ if $y_1 = y_2 = y_3 = 1$

$$= n_1 n_2 s^2 (n_1 s-1)(n_2 s-1) \quad if \quad y_1 = y_2 = y_3 = 0 \tag{6.3.16}$$

$$= 0 \qquad\qquad Otherwise$$

hence $E[1,1,1] = 1 - \frac{1}{(n_1 s-1)(n_2 s-1)}$, and all the main effects and first order interactions are estimated with full efficiency.

If further there exists a resolvable $BA(T)(n_3, s, 1)$, we can replace the levels of the third factor of the $n_1 s \times n_2 s \times s$ BAFD by the blocks of the $n_3 s \times s$ BAFD to obtain $n_1 s \times n_2 s \times n_3 s \times s$ BAFD with $k = n_1 n_2 n_3 s^3$ such that all the main effects and interactions are estimated with full efficiency except the third order interactions, which are estimated with efficiency, $1 - \frac{1}{(n_1 s-1)(n_2 s-1)(n_3 s-1)}$.

continuing this procedure, we can get an $n_1 s \times n_2 s \times \ldots \times n_L s \times s$ BAFD with $k = s^L n_1 n_2 \ldots n_L, b = s(n_1 s-1)(n_2 s-1)\ldots(n_L s-1)$. The $\lambda' s$ can be calculated recursively by the following formulae:

Note: Replace y_i with x_i.

$$\begin{aligned}
\lambda(y_1, y_2, \ldots, y_{L-2}, 0, 0) &= \lambda(y_1, y_2, \ldots, y_{L-2}, 0)(n_L s - 1) \\
\lambda(y_1, y_2, \ldots, y_{L-2}, 0, 1) &= \lambda(y_1, y_2, \ldots, y_{L-2}, 1)(n_L s - 1) \\
\lambda(y_1, y_2, \ldots, y_{L-2}, 1, 0) &= \lambda(y_1, y_2, \ldots, y_{L-2}, 0)(n_L - 1) \\
&\quad + \lambda(y_1, y_2, \ldots, y_{L-2}, 1)(n_L s - n_L) \\
\lambda(y_1, y_2, \ldots, y_{L-2}, 1, 1) &= \lambda(y_1, y_2, \ldots, y_{L-2}, 0)(n_L) \\
&\quad + \lambda(y_1, y_2, \ldots, y_{L-2}, 1)(n_L s - n_L - 1)
\end{aligned} \tag{6.3.17}$$

we shall prove that

$$E[1, 1, \ldots, 1] = 1 - \frac{1}{(n_1 s - 1)(n_2 s - 1)\ldots(n_L s - 1)}$$

and all other efficiencies are 1. The proof is given by induction. Equation (6.3.2) can be written as

$$\begin{aligned}
&g[y_1, y_2, \ldots, y_{L+1}] \\
&= \sum_{x_1, x_2, \ldots, x_{L-1}} \left\{\Pi_{i=1}^{L-1} \left\{(1 - y_i)n_i s - 1\right\}^{x_i}\right\} \times \\
&\Big[\lambda(x_1, x_2, \ldots, x_{L-1}, 0, 0) \times \left\{(1 - y_L)n_L s - 1\right\}^{x_L} \left\{(1 - y_{L+1})s - 1\right\}^{x_{L+1}} \\
&+ \lambda(x_1, x_2, \ldots, x_{L-1}, 0, 1)\left\{(1 - y_L)n_L s - 1\right\}^{x_L} \left\{(1 - y_{L+1})s - 1\right\}^{x_{L+1}} \\
&+ \lambda(x_1, x_2, \ldots, x_{L-1}, 1, 0)\left\{(1 - y_L)n_L s - 1\right\}^{x_L} \left\{(1 - y_{L+1})s - 1\right\}^{x_{L+1}} \\
&+ \lambda(x_1, x_2, \ldots, x_{L-1}, 1, 1)\left\{(1 - y_L)n_L s - 1\right\}^{x_L} \left\{(1 - y_{L+1})s - 1\right\}^{x_{L+1}}\Big]
\end{aligned}$$

using equation (6.3.16) we have

$$
\begin{aligned}
& g[y_1, y_2, \ldots, y_{L-1}, 0, 0] \\
& = \sum_{x_1, x_2, \ldots, x_{L-1}} \left\{ \prod_{i=1}^{L-1} \left\{ (1 - y_i) n_i s - 1 \right\}^{x_i} \right\} \Big[\lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L s - 1) \\
& + \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - 1) \left\{ (1 - 0) s - 1 \right\}^1 \\
& + \left\{ \begin{array}{c} \lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L - 1) + \lambda(x_1, x_2, \ldots, x_{L-1}, 1) \\ (n_L s - n_L) \end{array} \right\} \left\{ n_L s - 1 \right\} \\
& + \left\{ \begin{array}{c} \lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L) + \lambda(x_1, x_2, \ldots, x_{L-1}, 1) \\ (n_L s - n_L - 1) \end{array} \right\} (n_L s - 1)(s - 1) \Big] \\
& = n_L s (n_L s - 1) \sum_{x_1, x_2, \ldots, x_{L-1}} \left\{ \prod_{i=1}^{L-1} \left\{ (1 - y_i) n_i s - 1 \right\}^{x_i} \right\} \Big[\lambda(x_1, x_2, \ldots, x_{L-1}, 0) \\
& + \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(s - 1) \Big] \\
& = n_L s (n_L s - 1) g[y_1, y_2, \ldots, y_{L-1}, 0]
\end{aligned} \tag{6.3.18}
$$

$$
\begin{aligned}
& g[y_1, y_2, \ldots, y_{L-1}, 0, 1] \\
& = \sum_{x_1, x_2, \ldots, x_{L-1}} \left\{ \prod_{i=1}^{L-1} \left\{ (1 - y_i) n_i s - 1 \right\}^{x_i} \right\} \Bigg[\lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L s - 1) \\
& + \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - 1) \left\{ (1 - 1)s - 1 \right\}^1 \\
& + \Big[\lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L - 1) + \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - n_L) \Big] \\
& \times \left\{ (1 - 0) n_L s - 1 \right\}^1 \\
& + \Big[\lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L) + \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - n_L - 1) \Big] \\
& \times \left\{ (1 - 0) n_L s - 1 \right\}^1 \left\{ (1 - 1)s - 1 \right\} \Bigg] \\
& = (n_L s - 1) \sum_{x_1, x_2, \ldots, x_{L-1}} \left\{ \prod_{i=1}^{L-1} \left\{ (1 - y_i) n_i s - 1 \right\}^{x_i} \right\} \Bigg[\lambda(x_1, x_2, \ldots, x_{L-1}, 0) \\
& - \lambda(x_1, x_2, \ldots, x_{L-1}, 1) + \lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L - 1) \\
& + \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - n_L) \\
& - \lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L) - \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - n_L - 1) \Bigg] \\
& = (n_L s - 1)(0) = 0
\end{aligned}
\tag{6.3.19}
$$

$$
\begin{aligned}
&g[y_1, y_2, \ldots, y_{L-1}, 1, 0] \\
&= \sum_{x_1,x_2,\ldots,x_{L-1}} \left\{\prod_{i=1}^{L-1} \left\{(1-y_i)n_i s - 1\right\}^{x_i}\right\} \Bigg[\lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L s - 1) \\
&+ \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - 1) \left\{(1-0)s - 1\right\}^1 \\
&+ \left[\lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L - 1) + \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - n_L)\right] \\
&\times \left\{(1-1)s - 1\right\}^1 \\
&+ \left[\lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L) + \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - n_L - 1)\right] \\
&\times \left\{(1-1)n_L s - 1\right\}^1 \left\{(1-0)s - 1\right\}^1 \Bigg] \\
&= \sum_{x_1,x_2,\ldots,x_{L-1}} \left\{\prod_{i=1}^{L-1} \left\{(1-y_i)n_i s - 1\right\}^{x_i}\right\} \Bigg[\lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L s - 1) \\
&+ \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - 1)(s - 1) \\
&- \lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L - 1) - \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - n_L) \\
&- \lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L)(s - 1) - \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - n_L - 1)(s - 1)\Bigg] \\
&= \sum_{x_1,x_2,\ldots,x_{L-1}} \left\{\prod_{i=1}^{L-1} \left\{(1-y_i)n_i s - 1\right\}^{x_i}\right\} \\
&\times \left[k_0 n_L s - k_0 + k_1 n_L s^2 - k_1 n_L s - k_1 s + k_1 - k_0 n_L + k_0 - k_1 n_L s + k_1 n_L\right] \\
&+ \left[-k_0 n_L s + k_0 n_L - k_1 n_L s^2 + k_1 n_L s + k_1 n_L s - k_1 n_L + K_1 s - k_1\right] \\
&= 0
\end{aligned}
\tag{6.3.20}
$$

where $k_0 = \lambda(x_1, x_2, \ldots, x_{L-1}, 0)$ and $k_1 = \lambda(x_1, x_2, \ldots, x_{L-1}, 1)$

$$
\begin{aligned}
&g[y_1, y_2, \ldots, y_{L-1}, 1, 1] \\
&= \sum_{x_1, x_2, \ldots, x_{L-1}} \left\{ \prod_{i=1}^{L-1} \left\{ (1-y_i) n_i s - 1 \right\}^{x_i} \right\} \Bigg[\lambda(x_1, x_2, \ldots, x_{L-1}, 0, 0) \\
&+ \lambda(x_1, x_2, \ldots, x_{L-1}, 0, 1) \left\{ (1-y_{L+1}) s - 1 \right\}^{x_{L+1}} \\
&+ \lambda(x_1, x_2, \ldots, x_{L-1}, 1, 0) \left\{ (1-y_L) n_L s - 1 \right\}^{x_L} \\
&+ \lambda(x_1, x_2, \ldots, x_{L-1}, 1, 1) \left\{ (1-y_L) n_L s - 1 \right\}^{x_L} \left\{ (1-y_{L+1}) s - 1 \right\}^{x_{L+1}} \Bigg] \\
&= \sum_{x_1, x_2, \ldots, x_{L-1}} \left\{ \prod_{i=1}^{L-1} \left\{ (1-y_i) n_i s - 1 \right\}^{x_i} \right\} \Bigg[\lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L s - 1) \\
&+ \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - 1) \left\{ (1-1) s - 1 \right\}^{1} \\
&+ \Big[\lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L - 1) + \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - n_L) \Big] \times \\
&\left\{ (1-1) n_L s - 1 \right\}^{1} \\
&+ \Big[\lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L) + \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - n_L - 1) \Big] \\
&\times \left\{ (1-1) n_L s - 1 \right\}^{1} \left\{ (1-1) s - 1 \right\}^{1} \Bigg] \\
&= \sum_{x_1, x_2, \ldots, x_{L-1}} \left\{ \prod_{i=1}^{L-1} \left\{ (1-y_i) n_i s - 1 \right\}^{x_i} \right\} \Bigg[\lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L s - 1) \\
&- \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - 1) \\
&- \lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L - 1) - \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - n_L) \\
&+ \Big[\lambda(x_1, x_2, \ldots, x_{L-1}, 0)(n_L) + \lambda(x_1, x_2, \ldots, x_{L-1}, 1)(n_L s - n_L - 1) \Big] \Bigg] \\
&= \sum_{x_1, x_2, \ldots, x_{L-1}} \left\{ \prod_{i=1}^{L-1} \left\{ (1-y_i) n_i s - 1 \right\}^{x_i} \right\} \\
&\times \begin{bmatrix} k_0 n_L s - k_0 - k_1 n_L s + k_1 - k_0 n_L + k_0 - k_1 n_L s \\ + k_1 n_L + k_0 n_L + k_1 n_L s - k_1 n_L - k_1 \end{bmatrix} \\
&= \sum_{x_1, x_2, \ldots, x_{L-1}} \left\{ \prod_{i=1}^{L-1} \left\{ (1-y_i) n_i s - 1 \right\}^{x_i} \right\} \Big[k_0 n_L s - k_1 n_L s \Big] \\
&= n_L s \sum_{x_1, x_2, \ldots, x_{L-1}} \left\{ \prod_{i=1}^{L-1} \left\{ (1-y_i) n_i s - 1 \right\}^{x_i} \right\} \Big[k_0 - k_1 \Big] \\
&= n_L s \sum_{x_1, x_2, \ldots, x_{L-1}} \left\{ \prod_{i=1}^{L-1} \left\{ (1-y_i) n_i s - 1 \right\}^{x_i} \right\} \begin{bmatrix} \lambda(x_1, x_2, \ldots, x_{L-1}, 0) \\ -\lambda(x_1, x_2, \ldots, x_{L-1}, 1) \end{bmatrix} \\
&= n_L s g[y_1, y_2, \ldots, y_{L-1}, 1]
\end{aligned} \tag{6.3.21}
$$

where $k_0 = \lambda(x_1, x_2, \ldots, x_{L-1}, 0)$ and $k_1 = \lambda(x_1, x_2, \ldots, x_{L-1}, 1)$.

By the recursive formulae (6.3.18), (6.3.19), (6.3.20), (6.3.21) and the initial values (6.3.16) we have

$$\begin{aligned} g(y_1, y_2, \ldots, y_{L+1}) &= s^L \prod_{i=1}^{L} n_i \quad if \quad y_1 = y_2 = \ldots = y_{L+1} = 1 \\ &= s^L \prod_{i=1}^{L} n_i(n_i s - 1) \quad if \quad y_1 = y_2 = \ldots = y_{L+1} = 0 \\ &= 0 \qquad\qquad otherwise \end{aligned} \tag{6.3.22}$$

hence the efficiencies are

$$E[y_1, y_2, \ldots, y_{L+1}] = 1 - \frac{1}{\prod_{i=1}^{L}(n_i s - 1)} \tag{6.3.23}$$

If $y_1 = y_2 = \ldots = y_{L+1} = 1$ and 0 otherwise. From the discussion of the type III designs, we state the following theorem;

Theorem 6.3.1.

If there exists a $BA(T)(n_1, s, 1)$, and a resolvable $BA(T)(n_i, s, 1)$ for $i = 2, 3, \ldots, l$ then we can always construct an $n_1 s \times n_2 s \times \ldots n_L s$ BAFD with

$$k = s^L \prod_{i=1}^{L} ni, b = s \prod_{i=1}^{L} (n_i s - 1)$$

and

$$r = \prod_{i=1}^{L} (n_i s - 1)$$

such that

$$E[1, 1, \ldots, 1] = \frac{-1}{r} + 1$$

and all other efficiencies are 1.0.

Proof. If there exist a $BA(T)(n_1, s, 1)$ by corollary (5.2.1) this corresponds to $n_1 s \times s$ BAFD with $k = n_1 s, b = (n_1 s - 1)s$ If there exist a $BA(T)(n_2, s, 1)$ by theorem (6.2.2) we can replace levels of the second factor in $n_1 s \times s$ by blocks of $n_2 s \times s$ to obtain

$k = (n_1 s)(n_2 s) = s^2 n_1 n_2, b = (n_1 s - 1)(n_2 s - 1)s$
Continuing with this procedure
If there exist a $BA(T)(n_L, s, 1)$ this corresponds to $n_L s \times s$ BAFD. If we replace the levels of L^{th} factor in $n_1 s \times n_2 s \times \cdots \times n_{L-1} s \times s$ by using blocks of $n_L s$ we obtain $n_1 s \times n_2 s \times \cdots n_{L-1} s \times n_L s$ BAFD with $k = s^L \prod_{i=1}^{L} n_i$, $b = (n_1 - 1)(n_2 - 1) \cdots (n_L s - 1)s = s \prod_{i=1}^{L} (n_i s - 1)$ and by using equations (6.3.22) and (6.3.23) it follows that $E[1, 1, \cdots, 1] = -\frac{1}{r} + 1$ and all other efficiencies are 1.00. □

Example 6.3.9. *A $BA(T)(3, 2, 1)$ is given in example 5.3.1 and a resolvable $BA(T)(2, 2, 1)$ given in example 5.2.7 which is equivalent to the following 4×2 resolvable BAFD.*

x_0	x_1	y_0	y_1	z_0	z_1
00	01	00	01	00	01
10	11	11	10	11	10
21	20	20	21	21	20
31	30	31	30	30	31

TABLE 6.16: 4×2 Resolvable BAFD

where x_0, x_1, y_0, y_1, z_0, z_1 represent the blocks can be used to construct a $6 \times 4 \times 2$ BAFD with $k = 24$, $b = 30$, $r = 15$, $\lambda(0, 0, 1) = 0$, $\lambda(0, 1, 0) = 5$, $\lambda(0, 1, 1) = 10$, $\lambda(1, 0, 0) = 6$, $\lambda(1, 0, 1) = 9$, $\lambda(1, 1, 0) = 8$, $\lambda(1, 1, 1) = 7$.

The efficiencies are $E(1, 1, 1) = \frac{14}{15}$, and all other efficiencies are 1.0. The design can be expressed as the same table in example 5.3.1 the differences are the rows representing the levels of the first factor and the x_0, x_1, y_0, y_1, z_0, z_1 representing the blocks as shown above.

Example 6.3.10. *A $BA(T)(2,2,2)$ given in example 4.9.7 and a resolvable $BA(T)(2,2,1)$ given in example 5.2.7 which is equivalent to the following 4×2 resolvable BAFD.*

x_0	x_1	y_0	y_1	z_0	z_1
00	01	00	01	00	01
10	11	11	10	11	10
21	20	20	21	21	20
31	30	31	30	30	31

Table 6.17: 4×2 Resolvable BAFD

where $x_0, x_1, y_0, y_1, z_0, z_1$ represent the blocks can be used to construct a $4 \times 4 \times 2$ BAFD with different parameters as the ones given in theorem 6.3.1 and hence with different values of λ as the ones given in equation (6.3.15). For this design $k = 16$, $b = 36$, $r = 18$, $\lambda(1,0) = 6$, $\lambda(1,1) = 12$, $\lambda(2,0) = 10$, $\lambda(2,1) = 8$ and the efficiencies are

$$E[1,0] = E[1,1] = E[2,0] = E[0,1] = 1.00 \quad \textit{and}$$
$$E[2,1] = \frac{8}{9} \approx 1 - \frac{1}{r}.$$

The $4 \times 4 \times 2$ BAFD is given below

Blocks	1	2	3	4	5	6	7	8	9	10	11	12
Levels of F_1	Levels	of	F_2	and	F_3							
0	x_0	x_0	x_0	x_0	x_0	x_0	x_1	x_1	x_1	x_1	x_1	x_1
1	x_1	x_1	x_0	x_1	x_0	x_1	x_0	x_0	x_1	x_0	x_1	x_0
2	x_0	x_0	x_1	x_1	x_1	x_1	x_1	x_1	x_0	x_0	x_0	x_0
3	x_1	x_1	x_1	x_0	x_1	x_0	x_0	x_0	x_0	x_1	x_0	x_1

Block	13	14	15	16	17	18	19	20	21	22	23	24
Levels of F_1	Levels	of	F_2	and	F_3							
0	y_0	y_0	y_0	y_0	y_0	y_0	y_1	y_1	y_1	y_1	y_1	y_1
1	y_1	y_1	y_0	y_1	y_0	y_1	y_0	y_0	y_1	y_0	y_1	y_0
2	y_0	y_0	y_1	y_1	y_1	y_1	y_1	y_1	y_0	y_0	y_0	y_0
3	y_1	y_1	y_1	y_0	y_1	y_0	y_0	y_0	y_0	y_1	y_0	y_1

Block	25	26	27	28	29	30	31	32	33	34	35	36
Levels of F_1	Levels	of	F_2	and	F_3							
0	z_0	z_0	z_0	z_0	z_0	z_0	z_1	z_1	z_1	z_1	z_1	z_1
1	z_1	z_1	z_0	z_1	z_0	z_1	z_0	z_0	z_1	z_0	z_1	z_0
2	z_0	z_0	z_1	z_1	z_1	z_1	z_1	z_1	z_0	z_0	z_0	z_0
3	z_1	z_1	z_1	z_0	z_1	z_0	z_0	z_0	z_0	z_1	z_0	z_1

TABLE 6.18: $4 \times 4 \times 2$ BAFD

Other BAFDs that can be constructed by using theorem 6.3.1 include $4 \times 4 \times 2, 6 \times 6 \times 3, 6 \times 3 \times 3, 6 \times 9 \times 3, 8 \times 4 \times 4 \ldots$e.t.c

Corollary 6.3.1.

If s is a prime power,then there exists a $(2s)^L \times s^m (m \geq 1)$ BAFD with $k = 2^L s^{L+m-1}$, $r = (2s-1)^L(s-1)^{m-1}$, $b = (2s-1)^L(s-1)^{m-1}$, $E(L,m) = 1 - \frac{1}{r}$, and all other efficiencies are 1.

Proof. This is a consequence of theorem 6.3.1 since a resolvable $BA(T)(2,s,1)$ and a $BA(T)(1,s,1)$i.e a $TA[s(s-1), s, s, 2]$ exists for s a prime power.

If in addition to the conditions in theorem 6.3.1, there exists a resolvable BIBD with $n_{L+1}s$ treatments and block size n_{L+1}, then we can replace the levels of the last factor of the $n_1 s \times n_2 s \times \ldots \times n_L s \times s$ BAFD by the blocks of the BIBD to get an $n_1 s \times n_2 s \times \ldots n_L s \times n_{L+1} s$ BAFD with block size $n_1 \ldots n_L n_{L+1} s^L$. All the main effects and interactions are estimated with full efficiency except the L^{th} order interactions. □

TYPE IV

If there exists a $BA[Ps^2 - s, u, s, 2]$ with parameters $\lambda(x, y) = p - 1$ or p according as $x = y$ or not and a resolvable $BA[qs^2 - s, t, s, 2]$ with parameters $\lambda(x, y) = q - 1$ or q according as $x = y$ or not, then similar to theorem 6.3.1, we can construct a $u \times t \times s$ BAFD with $k = ut$,

$$\begin{aligned} r &= (ps-1)(qs-1), b = s(ps-1)(qs-1), \\ \lambda(0,0,1) &= 0, \lambda(0,1,0) = (ps-1)(q-1) \\ \lambda(0,1,1) &= (ps-1)q, \lambda(1,0,0) = (p-1)(qs-1), \\ \lambda(1,0,1) &= p(qs-1), \lambda(1,1,0) = (p-1)(q-1) + p(qs-q) \\ &= pqs - p - q + 1 \\ \lambda(1,1,1) &= (p-1)q + p(qs-q-1) \\ &= pqs - p - q. \end{aligned}$$

The efficiencies are given below assuming $ps = u$ and $qs = t$.

$$E[1,0,0] = \tfrac{1}{ut}\left\{ut - 1 - \tfrac{1}{(u-1)(t-1)}\right\} \times$$
$$\left\{\begin{array}{c} (ps-1)(q-1)\left\{(1-1)u-1\right\}^0\left\{(1-0)t-1\right\}^1\left\{(1-0)s-1\right\}^0 \\ +(ps-1)q\left\{(1-1)u-1\right\}^0\left\{(1-0)t-1\right\}^1\left\{(1-0)s-1\right\}^1 \\ +(p-1)(qs-1)\left\{(1-1)u-1\right\}^1\left\{(1-0)t-1\right\}^0\left\{(1-0)s-1\right\}^0 \\ +p(qs-1)\left\{(1-1)u-1\right\}^1\left\{(1-0)t-1\right\}^0\left\{(1-0)s-1\right\}^1 \\ +(pqs-p-q+1)\left\{(1-1)u-1\right\}^1\left\{(1-0)t-1\right\}^1\left\{(1-0)s-1\right\}^0 \\ +(pqs-p-q)\left\{(1-1)u-1\right\}^1\left\{(1-0)t-1\right\}^1\left\{(1-0)s-1\right\}^1 \end{array}\right\}$$

$$= \frac{1}{ut}\left\{ut - 1 - \frac{1}{(u-1)(t-1)}\right\} \left\{ \begin{array}{c} (ps-1)(q-1)(1)(t-1)(1) \\ +(ps-1)q(1)(t-1)(s-1) \\ +(p-1)(qs-1)(-1)(1)(1) \\ +p(qs-1)(-1)(1)(s-1) \\ +(pqs-p-q+1)(-1)(t-1)(1) \\ +(pqs-p-q)(-1)(t-1)(s-1) \end{array} \right\}$$

$$\begin{aligned}
\text{so} \quad \text{sum} &= \sum \lambda(x) \left\{ \prod_{i=1}^{m} \left\{(1-y_i)s_i - 1\right\}^{x_i} \right\} \\
&= -pqs^2 + ps + qs - 1 \\
&= -\left\{pqs^2 - ps - qs + 1\right\} \\
&= -[ps-1][qs-1] = -(u-1)(t-1)
\end{aligned}$$

Thus

$$\begin{aligned}
E[1,0,0] &= \frac{1}{ut}\left\{ut - 1 - \frac{1}{(u-1)(t-1)}\left\{sum\right\}\right\} \\
&= \frac{1}{ut}\left\{ut - 1 - \frac{\left\{-(u-1)(t-1)\right\}}{(u-1)(t-1)}\right\} \\
&= \frac{1}{ut}\left\{ut - 1 + \frac{(u-1)(t-1)}{(u-1)(t-1)}\right\} \\
&= \frac{1}{ut}\left\{ut - 1 + 1\right\} = \frac{1}{ut}\left\{ut\right\} = \frac{ut}{ut} \\
&= 1.00
\end{aligned}$$

$$E[0,1,0] = \frac{1}{ut}\left\{ut - 1 - \frac{1}{(u-1)(t-1)}\right\} \times$$

$$\left\{\begin{array}{c}(ps-1)(q-1)\Big\{(1-0)u-1\Big\}^0\Big\{(1-1)t-1\Big\}^1\Big\{(1-0)s-1\Big\}^0\\ +(ps-1)q\Big\{(1-0)u-1\Big\}^0\Big\{(1-1)t-1\Big\}^1\Big\{(1-0)s-1\Big\}^1\\ +(p-1)(qs-1)\Big\{(1-0)u-1\Big\}^1\Big\{(1-1)t-1\Big\}^0\Big\{(1-0)s-1\Big\}^0\\ +p(qs-1)\Big\{(1-0)u-1\Big\}^1\Big\{(1-1)t-1\Big\}^0\Big\{(1-0)s-1\Big\}^1\\ +(pqs-p-q+1)\Big\{(1-0)u-1\Big\}^1\Big\{(1-1)t-1\Big\}^1\Big\{(1-0)s-1\Big\}^0\\ +(pqs-p-q)\Big\{(1-0)u-1\Big\}^1\Big\{(1-1)t-1\Big\}^1\Big\{(1-0)s-1\Big\}^1\end{array}\right\}$$

$$=\tfrac{1}{ut}\Big\{ut-1-\tfrac{1}{(u-1)(t-1)}\Big\}\left\{\begin{array}{c}(ps-1)(q-1)(1)(-1)(1)\\ +(ps-1)q(1)(-1)(s-1)\\ +(p-1)(qs-1)(u-1)(1)(1)\\ +p(qs-1)(u-1)(1)(s-1)\\ +(pqs-p-q+1)(u-1)(-1)(1)\\ +(pqs-p-q)(u-1)(-1)(s-1)\end{array}\right\}$$

$$\begin{aligned}\text{and}\quad \text{sum} &= \sum\Big\{\lambda(x)\textstyle\prod_{i=1}^{m}\Big\{(1-y_i)s_i-1\Big\}^{x_i}\Big\}\\ &= -pqs^2+ps+qs-1\\ &= -[ps-1][qs-1] = -(u-1)(t-1)\end{aligned}$$

Thus

$$\begin{aligned}E[0,1,0] &= \frac{1}{ut}\Big\{ut-1-\tfrac{1}{(u-1)(t-1)}\Big\{sum\Big\}\Big\}\\ &= \frac{1}{ut}\Big\{ut-1-\tfrac{1(-[u-1][t-1])}{(u-1)(t-1)}\Big\}\\ &= \frac{1}{ut}\Big\{ut-1+\tfrac{(u-1)(t-1)}{(u-1)(t-1)}\Big\}\\ &= \frac{1}{ut}\Big\{ut-1+1\Big\}\\ &= \frac{1}{ut}\Big\{ut\Big\}\\ &= \frac{ut}{ut} = 1.00\end{aligned}$$

$$E[0,0,1] = \frac{1}{ut}\left\{ut - 1 - \frac{1}{(u-1)(t-1)}\right\} \times$$

$$\left\{\begin{array}{c} (ps-1)(q-1)\left\{(1-0)u-1\right\}^0\left\{(1-0)t-1\right\}^1\left\{(1-1)s-1\right\}^0 \\ +(ps-1)q\left\{(1-0)u-1\right\}^0\left\{(1-0)t-1\right\}^1\left\{(1-1)s-1\right\}^1 \\ +(p-1)(qs-1)\left\{(1-0)u-1\right\}^1\left\{(1-0)t-1\right\}^0\left\{(1-1)s-1\right\}^0 \\ +p(qs-1)\left\{(1-0)u-1\right\}^1\left\{(1-0)t-1\right\}^0\left\{(1-1)s-1\right\}^1 \\ +(pqs-p-q+1)\left\{(1-0)u-1\right\}^1\left\{(1-0)t-1\right\}^1\left\{(1-1)s-1\right\}^0 \\ +(pqs-p-q)\left\{(1-0)u-1\right\}^1\left\{(1-0)t-1\right\}^1\left\{(1-1)s-1\right\}^1 \end{array}\right\}$$

$$= \frac{1}{ut}\left\{ut - 1 - \frac{1}{(u-1)(t-1)}\right\}\left\{\begin{array}{c} (ps-1)(q-1)(1)(t-1)(1) \\ +(ps-1)q(1)(t-1)(-1) \\ +(p-1)(qs-1)(u-1)(1)(1) \\ +p(qs-1)(u-1)(1)(-1) \\ +(pqs-p-q+1)(u-1)(t-1)(1) \\ +(pqs-p-q)(u-1)(t-1)(-1) \end{array}\right\}$$

$$\begin{aligned}
\text{so sum } &= \sum \lambda(x) \left\{ \Pi_{i=1}^{m} \left\{ (1-y_i)s_i - 1 \right\}^{x_i} \right\} \\
&= ps - pst - qsu + qs - 1 + ut
\end{aligned}$$

Thus

$$\begin{aligned}
E[0,0,1] &= \frac{1}{ut} \left\{ ut - 1 - \tfrac{1}{(u-1)(t-1)} \left\{ sum \right\} \right\} \\
&= \frac{ut}{ut} - \frac{1}{ut} - \frac{sum}{(u-1)(t-1)ut} \\
&= 1 - \frac{1}{ut} - \frac{sum}{(u-1)(t-1)ut} \\
&= 1 - \left\{ \tfrac{1}{ut} + \tfrac{sum}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{(u-1)(t-1)+sum}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-u-t+1+ps-pst-qsu+qs-1+ut}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-pst-qsu+ut}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-qsu-pst+ut}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{u(t-qs)-t(ps-u)}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{u(t-t)-t(u-u)}{(u-1)(t-1)ut} \right\} = 1 - \left\{ \tfrac{u(0)-t(0)}{(u-1)(t-1)ut} \right\} \\
&= 1.00
\end{aligned}$$

$$E[0,1,1] = \tfrac{1}{ut} \left\{ ut - 1 - \tfrac{1}{(u-1)(t-1)} \right\} \times$$

$$\left\{\begin{array}{c} (ps-1)(q-1)\left\{(1-0)u-1\right\}^{0}\left\{(1-1)t-1\right\}^{1}\left\{(1-1)s-1\right\}^{0} \\ +(ps-1)q\left\{(1-0)u-1\right\}^{0}\left\{(1-1)t-1\right\}^{1}\left\{(1-1)s-1\right\}^{1} \\ +(p-1)(qs-1)\left\{(1-0)u-1\right\}^{1}\left\{(1-1)t-1\right\}^{0}\left\{(1-1)s-1\right\}^{0} \\ +p(qs-1)\left\{(1-0)u-1\right\}^{1}\left\{(1-1)t-1\right\}^{0}\left\{(1-1)s-1\right\}^{1} \\ +(pqs-p-q+1)\left\{(1-0)u-1\right\}^{1}\left\{(1-1)t-1\right\}^{1}\left\{(1-1)s-1\right\}^{0} \\ +(pqs-p-q)\left\{(1-0)u-1\right\}^{1}\left\{(1-1)t-1\right\}^{1}\left\{(1-1)s-1\right\}^{1} \end{array}\right\}$$

$$= \frac{1}{ut}\left\{ut-1-\frac{1}{(u-1)(t-1)}\right\}\left\{\begin{array}{c} (ps-1)(q-1)(1)(-1)(1) \\ +(ps-1)q(1)(-1)(-1) \\ +(p-1)(qs-1)(u-1)(1)(1) \\ +p(qs-1)(u-1)(1)(-1) \\ +(pqs-p-q+1)(u-1)(-1)(1) \\ +(pqs-p-q)(u-1)(-1)(-1) \end{array}\right\}$$

$$
\begin{aligned}
\text{sum} &= \sum \lambda(x) \left\{ \prod_{i=1}^{m} \left\{ (1-y_i)s_i - 1 \right\}^{x_i} \right\} \\
&= ps - qsu + qs - 1 \\
\text{Thus } E[0,1,1] &= \frac{1}{ut} \left\{ ut - 1 - \tfrac{1}{(u-1)(t-1)} \left\{ sum \right\} \right\} \\
&= \frac{ut}{ut} - \frac{1}{ut} - \frac{sum}{(u-1)(t-1)ut} \\
&= 1 - \frac{1}{ut} - \frac{sum}{(u-1)(t-1)ut} \\
&= 1 - \left\{ \tfrac{1}{ut} + \tfrac{sum}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{(u-1)(t-1)+sum}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-u-t+1+sum}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-u-t+1+ps-qsu+qs-1}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-qsu+ps-u+qs-t}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{u(t-qs)+(u-u)+(t-t)}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{u(t-t)+0+0}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{u(0)+0+0}{(u-1)(t-1)ut} \right\} = 1 - \left\{ \tfrac{0+0+0}{(u-1)(t-1)ut} \right\} \\
&= 1 - 0 = 1.00
\end{aligned}
$$

$$
E[1,0,1] = \tfrac{1}{ut} \left\{ ut - 1 - \tfrac{1}{(u-1)(t-1)} \right\} \times
$$

$$\left\{\begin{array}{c} (ps-1)(q-1)\left\{(1-1)u-1\right\}^{0}\left\{(1-0)t-1\right\}^{1}\left\{(1-1)s-1\right\}^{0} \\ +(ps-1)q\left\{(1-1)u-1\right\}^{0}\left\{(1-0)t-1\right\}^{1}\left\{(1-1)s-1\right\}^{1} \\ +(p-1)(qs-1)\left\{(1-1)u-1\right\}^{1}\left\{(1-0)t-1\right\}^{0}\left\{(1-1)s-1\right\}^{0} \\ +p(qs-1)\left\{(1-1)u-1\right\}^{1}\left\{(1-0)t-1\right\}^{0}\left\{(1-1)s-1\right\}^{1} \\ +(pqs-p-q+1)\left\{(1-1)u-1\right\}^{1}\left\{(1-0)t-1\right\}^{1}\left\{(1-1)s-1\right\}^{0} \\ +(pqs-p-q)\left\{(1-1)u-1\right\}^{1}\left\{(1-0)t-1\right\}^{1}\left\{(1-1)s-1\right\}^{1} \end{array}\right\}$$

$$= \tfrac{1}{ut}\left\{ut-1-\tfrac{1}{(u-1)(t-1)}\right\}\left\{\begin{array}{c} (ps-1)(q-1)(1)(t-1)(1) \\ +(ps-1)q(1)(t-1)(-1) \\ +(p-1)(qs-1)(-1)(1)(1) \\ +p(qs-1)(-1)(1)(-1) \\ +(pqs-p-q+1)(-1)(t-1)(1) \\ +(pqs-p-q)(-1)(t-1)(-1) \end{array}\right\}$$

$$
\begin{aligned}
\text{sum} &= \sum \lambda(x) \left\{ \prod_{i=1}^{m} \left\{ (1-y_i)s_i - 1 \right\}^{x_i} \right\} \\
&= -pst + ps + qs - 1
\end{aligned}
$$

hence

$$
\begin{aligned}
E[1,0,1] &= \frac{1}{ut}\left\{ ut - 1 - \tfrac{1}{(u-1)(t-1)} \left\{ sum \right\} \right\} \\
&= \frac{1}{ut}\left\{ ut - 1 - \tfrac{sum}{(u-1)(t-1)} \right\} \\
&= \frac{ut}{ut} - \frac{1}{ut} - \frac{sum}{(u-1)(t-1)ut} \\
&= 1 - \frac{1}{ut} - \frac{sum}{(u-1)(t-1)ut} \\
&= 1 - \left\{ \tfrac{1}{ut} + \tfrac{sum}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{(u-1)(t-1)+sum}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-u-t+1+sum}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-u-t+1-pst+ps+qs-1}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-pst+(ps-u)+(qs-t)}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{t(u-ps)+(u-u)+(t-t)}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{t(u-u)+0+0}{(u-1)(t-1)ut} \right\} = 1 - \left\{ \tfrac{t(0)+0+0}{(u-1)(t-1)ut} \right\} \\
&= 1 - \frac{0}{(u-1)(t-1)ut} = 1 - 0 = 1.00
\end{aligned}
$$

$$E[1,1,0] = \frac{1}{ut}\left\{ut - 1 - \frac{1}{(u-1)(t-1)}\right\} \times$$

$$\left\{\begin{array}{c}
(ps-1)(q-1)\left\{(1-1)u-1\right\}^0\left\{(1-1)t-1\right\}^1\left\{(1-0)s-1\right\}^0 \\
+(ps-1)q\left\{(1-1)u-1\right\}^0\left\{(1-1)t-1\right\}^1\left\{(1-0)s-1\right\}^1 \\
+(p-1)(qs-1)\left\{(1-1)u-1\right\}^1\left\{(1-1)t-1\right\}^0\left\{(1-0)s-1\right\}^0 \\
+p(qs-1)\left\{(1-1)u-1\right\}^1\left\{(1-1)t-1\right\}^0\left\{(1-0)s-1\right\}^1 \\
+(pqs-p-q+1)\left\{(1-1)u-1\right\}^1\left\{(1-1)t-1\right\}^1\left\{(1-0)s-1\right\}^0 \\
+(pqs-p-q)\left\{(1-1)u-1\right\}^1\left\{(1-1)t-1\right\}^1\left\{(1-0)s-1\right\}^1
\end{array}\right\}$$

$$= \frac{1}{ut}\left\{ut - 1 - \frac{1}{(u-1)(t-1)}\right\}\left\{\begin{array}{c}
(ps-1)(q-1)(1)(-1)(1) \\
+(ps-1)q(1)(-1)(s-1) \\
+(p-1)(qs-1)(-1)(1)(1) \\
+p(qs-1)(-1)(1)(s-1) \\
+(pqs-p-q+1)(-1)(-1)(1) \\
+(pqs-p-q)(-1)(-1)(s-1)
\end{array}\right\}$$

$$\begin{aligned}
\text{sum} &= \sum \lambda(x) \left\{ \prod_{i=1}^{m} \left\{ (1-y_i)s_i - 1 \right\}^{x_i} \right\} \\
&= qs - 1 - pqs^2 + ps \\
\text{Thus} \quad E[1,1,0] &= \frac{1}{ut} \left\{ ut - 1 - \tfrac{1}{(u-1)(t-1)} \left\{ sum \right\} \right\} \\
&= \frac{ut}{ut} - \frac{1}{ut} - \frac{sum}{(u-1)(t-1)ut} \\
&= 1 - \frac{1}{ut} - \frac{sum}{(u-1)(t-1)ut} \\
&= 1 - \left\{ \tfrac{1}{ut} + \tfrac{sum}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{(u-1)(t-1)+sum}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-u-t+1+sum}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-u-t+1+qs-1-pqs^2+ps}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-u-t+u+t-pqs^2}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-pqs^2}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-(ps)(qs)}{(u-1)(t-1)ut} \right\} = 1 - \left\{ \tfrac{ut-(u)(t)}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-ut}{(u-1)(t-1)ut} \right\} = 1 - \left\{ \tfrac{0}{(u-1)(t-1)ut} \right\} \\
&= 1 - 0 = 1.00
\end{aligned}$$

$$E[1,1,1] = \frac{1}{ut}\left\{ut - 1 - \frac{1}{(u-1)(t-1)}\right\} \times$$

$$\left\{\begin{array}{c}
(ps-1)(q-1)\left\{(1-1)u-1\right\}^{0}\left\{(1-1)t-1\right\}^{1}\left\{(1-1)s-1\right\}^{0} \\
+(ps-1)q\left\{(1-1)u-1\right\}^{0}\left\{(1-1)t-1\right\}^{1}\left\{(1-1)s-1\right\}^{1} \\
+(p-1)(qs-1)\left\{(1-1)u-1\right\}^{1}\left\{(1-1)t-1\right\}^{0}\left\{(1-1)s-1\right\}^{0} \\
+p(qs-1)\left\{(1-1)u-1\right\}^{1}\left\{(1-1)t-1\right\}^{0}\left\{(1-1)s-1\right\}^{1} \\
+(pqs-p-q+1)\left\{(1-1)u-1\right\}^{1}\left\{(1-1)t-1\right\}^{1}\left\{(1-1)s-1\right\}^{0} \\
+(pqs-p-q)\left\{(1-1)u-1\right\}^{1}\left\{(1-1)t-1\right\}^{1}\left\{(1-1)s-1\right\}^{1}
\end{array}\right\}$$

$$= \frac{1}{ut}\left\{ut - 1 - \frac{1}{(u-1)(t-1)}\right\}\left\{\begin{array}{c}
(ps-1)(q-1)(1)(-1)(1) \\
+(ps-1)q(1)(-1)(-1) \\
+(p-1)(qs-1)(-1)(1)(1) \\
+p(qs-1)(-1)(1)(-1) \\
+(pqs-p-q+1)(-1)(-1)(1) \\
+(pqs-p-q)(-1)(-1)(-1)
\end{array}\right\}$$

$$
\begin{aligned}
\text{sum} &= \sum \lambda(x) \left\{ \Pi_{i=1}^{m} \left\{ (1-y_i)s_i - 1 \right\}^{x_i} \right\} \\
&= ps + qs - 1
\end{aligned}
$$

Thus

$$
\begin{aligned}
E[1,1,1] &= \frac{1}{ut} \left\{ ut - 1 - \tfrac{1}{(u-1)(t-1)} \left\{ sum \right\} \right\} \\
&= \frac{1}{ut} \left\{ ut - 1 - \tfrac{sum}{(u-1)(t-1)} \right\} \\
&= \frac{ut}{ut} - \frac{1}{ut} - \frac{sum}{(u-1)(t-1)ut} \\
&= 1 - \frac{1}{ut} - \frac{sum}{(u-1)(t-1)ut} \\
&= 1 - \left\{ \tfrac{1}{ut} + \tfrac{sum}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{(u-1)(t-1)+sum}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut-u-t+1+ps+qs-1}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut+ps+qs-u-t}{(u-1)(t-1)ut} \right\} \\
&= 1 - \left\{ \tfrac{ut+u+t-u-t}{(u-1)(t-1)ut} \right\} \\
&= 1 - \frac{ut}{(u-1)(t-1)ut} \\
&= 1 - \frac{(ps)(qs)}{(ps-1)(qs-1)ut} \\
&= 1 - \frac{pqs^2}{(ps-1)(qs-1)ut}
\end{aligned}
$$

Usually the $BA[ps^2 - s, u, s, 2]$ can be obtained by deleting $ps - u$ constraints in a $BA(T)(p, s, 1)$ if it exists; Similarly,the resolvable $BA[qs^2 - s, t, s, 2]$ can be obtained by deleting $qs - t$ constraints in a resolvable $BA(T)(q, s, 1)$.Other methods of constructing these type of balanced arrays are still to be developed.

If $ps \neq u$ and $qs \neq t$ then

$$\begin{aligned}
\cdot E[1,0,0] &= \frac{1}{ut}\left\{ut - 1 - \tfrac{1}{(ps-1)(qs-1)}\left\{sum\right\}\right\} \\
&= \frac{ut}{ut} - \frac{1}{ut} - \frac{sum}{(ps-1)(qs-1)ut} \\
&= 1 - \frac{1}{ut} - \frac{sum}{(ps-1)(qs-1)ut} \\
&= 1 - \left\{\tfrac{1}{ut} + \tfrac{sum}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{(ps-1)(qs-1)+sum}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{(ps-1)(qs-1)+-(ps-1)(qs-1)}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \frac{0}{(ps-1)(qs-1)ut} = 1 - 0 \\
&= 1.00
\end{aligned}$$

$$\begin{aligned}
\cdot E[0,1,0] &= \frac{1}{ut}\left\{ut - 1 - \tfrac{1}{(ps-1)(qs-1)}\left\{sum\right\}\right\} \\
&= \frac{1}{ut}\left\{ut - 1 - \tfrac{1}{(ps-1)(qs-1)}\left\{-(ps-1)(qs-1)\right\}\right\} \\
&= \frac{1}{ut}\left\{ut - 1 + \tfrac{(ps-1)(qs-1)}{(ps-1)(qs-1)}\right\} \\
&= \frac{1}{ut}\left\{ut - 1 + 1\right\} = \frac{1}{ut}\left\{ut\right\} = \frac{ut}{ut} \\
&= 1.00
\end{aligned}$$

$$
\begin{aligned}
\cdot E[0,0,1] &= \frac{1}{ut}\left\{ut - 1 - \tfrac{1}{(ps-1)(qs-1)}\left\{sum\right\}\right\} \\
&= \frac{ut}{ut} - \frac{1}{ut} - \frac{sum}{(ps-1)(qs-1)ut} \\
&= 1 - \frac{1}{ut} - \frac{sum}{(ps-1)(qs-1)ut} \\
&= 1 - \left\{\tfrac{1}{ut} + \tfrac{sum}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{(ps-1)(qs-1)+sum}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{pqs^2-ps-qs+1+ps-pst-qsu+qs-1+ut}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{pqs^2-pst-qsu+ut}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{pqs^2-qsu+ut-pst}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{qs(ps-u)+t(u-ps)}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{qs(ps-u)-t(ps-u)}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{(ps-u)(qs-t)}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \frac{(ps-u)(qs-t)}{(ps-1)(qs-1)ut}
\end{aligned}
$$

$$
\begin{aligned}
\cdot E[0,1,1] &= \frac{1}{ut}\left\{ut - 1 - \tfrac{1}{(ps-1)(qs-1)}\left\{sum\right\}\right\} \\
&= \frac{ut}{ut} - \frac{1}{ut} - \frac{sum}{(ps-1)(qs-1)ut} \\
&= 1 - \frac{1}{ut} - \frac{sum}{(ps-1)(qs-1)ut} \\
&= 1 - \left\{\tfrac{1}{ut} + \tfrac{sum}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{(ps-1)(qs-1)+sum}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{pqs^2-ps-qs+1+sum}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{pqs^2-ps-qs+1+ps-qsu+qs-1}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{pqs^2-qsu}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \frac{qs(ps-u)}{(ps-1)(qs-1)ut}
\end{aligned}
$$

$$
\begin{aligned}
.\cdot E[1,0,1] &= \frac{1}{ut}\left\{ut - 1 - \tfrac{1}{(ps-1)(qs-1)}\left\{sum\right\}\right\} \\
&= \frac{ut}{ut} - \frac{1}{ut} - \frac{sum}{(ps-1)(qs-1)ut} \\
&= 1 - \frac{1}{ut} - \frac{sum}{(ps-1)(qs-1)ut} \\
&= 1 - \left\{\tfrac{1}{ut} + \tfrac{sum}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{(ps-1)(qs-1)+sum}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{pqs^2-ps-qs+1+ps-pst+qs-1}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{pqs^2-pst}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \frac{ps(qs-t)}{(ps-1)(qs-1)ut}
\end{aligned}
$$

$$
\begin{aligned}
\cdot E[1,1,0] &= \frac{1}{ut}\left\{ut - 1 - \tfrac{1}{(ps-1)(qs-1)}\left\{sum\right\}\right\} \\
&= \frac{ut}{ut} - \frac{1}{ut} - \frac{sum}{(ps-1)(qs-1)ut} \\
&= 1 - \frac{1}{ut} - \frac{sum}{(ps-1)(qs-1)ut} \\
&= 1 - \left\{\tfrac{1}{ut} + \tfrac{sum}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{(ps-1)(qs-1)+sum}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{pqs^2-ps-qs+1+qs-1-pqs^2+ps}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \frac{0}{(ps-1)(qs-1)ut} \\
&= 1 - 0 = 1.00
\end{aligned}
$$

$$
\begin{aligned}
\cdot E[1,1,1] &= \frac{1}{ut}\left\{ut - 1 - \tfrac{1}{(ps-1)(qs-1)}\left\{sum\right\}\right\} \\
&= \frac{ut}{ut} - \frac{1}{ut} - \frac{sum}{(ps-1)(qs-1)ut} \\
&= 1 - \frac{1}{ut} - \frac{sum}{(ps-1)(qs-1)ut} \\
&= 1 - \left\{\tfrac{1}{ut} + \tfrac{sum}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{(ps-1)(qs-1)+sum}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{pqs^2-ps-qs+1+ps+qs-1}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \left\{\tfrac{pqs^2}{(ps-1)(qs-1)ut}\right\} \\
&= 1 - \frac{pqs^2}{(ps-1)(qs-1)ut}
\end{aligned}
$$

Example 6.3.11. *As in example 6.3.9 if we use the* $BA[10,5,3,2]$ *obtained by deleting a constraint in* $BA(T)(3,2,1)$*, and other procedures* *being the same, then we get a* $5 \times 4 \times 2$ *BAFD with* $k = 20$, $b = 30$, $r = 15$ *and* $\lambda(0,0,1) = 0$, $\lambda(0,1,0) = 5$, $\lambda(0,1,1) = 10$, $\lambda(1,0,0) = 6$, $\lambda(1,0,1) = 9$, $\lambda(1,1,0) = 8$, $\lambda(1,1,1) = 7$

The efficiencies are as follows

$$\begin{aligned} E[0,0,1] &= E[0,1,0] = E[1,0,0] = E[1,1,0] \\ &= 1.00 \\ E[0,1,1] &= \frac{74}{75}, E[1,1,1] = \frac{23}{25}. \end{aligned}$$

The $5 \times 4 \times 2$ *BAFD is given below.*

Blocks	1	2	3	4	5	6	7	8	9	10
Levels of F_1	Levels	of	F_2	and	F_3					
0	x_0	x_0	x_0	x_0	x_0	x_1	x_1	x_1	x_1	x_1
1	x_0	x_0	x_1	x_1	x_1	x_0	x_0	x_0	x_1	x_1
2	x_1	x_0	x_0	x_1	x_1	x_1	x_1	x_0	x_0	x_0
3	x_0	x_1	x_1	x_0	x_1	x_0	x_1	x_1	x_0	x_0
4	x_1	x_1	x_1	x_0	x_0	x_1	x_0	x_0	x_0	x_1

Block	11	12	13	14	15	16	17	18	19	20
Levels of F_1	Levels	of	F_2	and	F_3					
0	y_0	y_0	y_0	y_0	y_0	y_1	y_1	y_1	y_1	y_1
1	y_0	y_0	y_1	y_1	y_1	y_0	y_0	y_0	y_1	y_1
2	y_1	y_0	y_0	y_1	y_1	y_1	y_1	y_0	y_0	y_0
3	y_0	y_1	y_1	y_0	y_1	y_0	y_1	y_1	y_0	y_0
4	y_1	y_1	y_1	y_0	y_0	y_1	y_0	y_0	y_0	y_1

Block	21	22	23	24	25	26	27	28	29	30
Levels of F_1	Levels	of	F_2	and	F_3					
0	z_0	z_0	z_0	z_0	z_0	z_1	z_1	z_1	z_1	z_1
1	z_0	z_0	z_1	z_1	z_1	z_0	z_0	z_0	z_1	z_1
2	z_1	z_0	z_0	z_1	z_1	z_1	z_1	z_0	z_0	z_0
3	z_0	z_1	z_1	z_0	z_1	z_0	z_1	z_1	z_0	z_0
4	z_1	z_1	z_1	z_0	z_0	z_1	z_0	z_0	z_0	z_1

TABLE 6.19: $5 \times 4 \times 2$ BAFD

Example 6.3.12. *As in example 6.3.9, if we use a* $BA[24, 7, 3, 2]$ *obtained by deleting two constraints in* $BA(T)[3, 3, 1]$ *which is in example*

4.9.4, and also a resolvable $BA(T)(1, 3, 1)$.

$$= BA[(s-1)s, s, s, 2]$$

$$= TA[s(s-1), s, s, 2]$$

$$= TA[(6, 3, 3, 2]$$

which can be constructed by using corollary 4.1.1. We can construct a $7 \times 3 \times 3$ *BAFD.The* $TA[(6, 3, 3, 2]$ *is given by*

0	1	2	0	1	2
1	2	0	2	0	1
2	0	1	1	2	0

Table 6.20: TA[6,3,3,2]

and it is equivalent to the following resolvable BAFD.

x_0	x_1	x_2	y_0	y_1	y_2
00	01	02	00	01	02
11	12	10	12	10	11
22	20	21	21	22	20

Table 6.21: 3^2 Resolvable BFD

where $x_0, x_1, x_2, y_0, y_1, y_2$ *represents the blocks.The parameters of the* $7 \times 3 \times 3$ *BAFD are* $k = 21, r = 16, b = 48,$

$$\lambda(0,1) = 0,$$

$$\lambda(0,2) = 8,$$

$$\lambda(1,0) = 4,$$

$$\lambda(1,1) = 6,$$

$$\lambda(1,2) = 5$$

with efficiencies

$$E[0,1] = E[1,0] = E[1,1] = 1.00$$

$$E[0,2] = \frac{55}{56}$$

$$E[1,2] = \frac{103}{112}$$

The $7 \times 3 \times 3$ *BAFD is given below.*

Blocks	1	2	3	4	5	6	7	8	9	10
Levels of F_1	Levels	of	F_2	and	F_3					
0	x_0	x_0	x_0	x_0	x_0	x_0	x_0	x_0	x_1	x_1
1	x_1	x_2	x_0	x_1	x_2	x_0	x_1	x_2	x_2	x_0
2	x_2	x_1	x_0	x_2	x_1	x_0	x_2	x_1	x_0	x_2
3	x_0	x_0	x_1	x_1	x_1	x_2	x_2	x_2	x_1	x_1
4	x_1	x_2	x_1	x_2	x_0	x_2	x_0	x_1	x_2	x_0
5	x_2	x_1	x_1	x_0	x_2	x_2	x_1	x_0	x_0	x_2
6	x_0	x_0	x_2	x_2	x_2	x_1	x_1	x_1	x_1	x_1

Blocks	11	12	13	14	15	16	17	18	19	20
Levels of F_1	Levels	of	F_2	and	F_3					
0	x_1	x_1	x_1	x_1	x_1	x_1	x_2	x_2	x_2	x_2
1	x_1	x_2	x_0	x_1	x_2	x_0	x_0	x_1	x_2	x_0
2	x_1	x_0	x_2	x_1	x_0	x_2	x_1	x_0	x_2	x_1
3	x_2	x_2	x_2	x_0	x_0	x_0	x_2	x_2	x_0	x_0
4	x_2	x_0	x_1	x_0	x_1	x_2	x_0	x_1	x_0	x_1
5	x_2	x_1	x_0	x_0	x_2	x_1	x_1	x_0	x_0	x_2
6	x_0	x_0	x_0	x_2	x_2	x_2	x_2	x_2	x_1	x_1

Blocks	21	22	23	24	25	26	27	28	29	30
Levels of F_1	Levels	of	F_2	and	F_3					
0	x_2	x_2	x_2	x_2	y_0	y_0	y_0	y_0	y_0	y_0
1	x_1	x_2	x_0	x_1	y_1	y_2	y_0	y_1	y_2	y_0
2	x_0	x_2	x_1	x_0	y_2	y_1	y_0	y_2	y_1	y_0
3	x_0	x_1	x_1	x_1	y_0	y_0	y_1	y_1	y_1	y_2
4	x_2	x_1	x_2	x_0	y_1	y_2	y_1	y_2	y_0	y_2
5	x_1	x_1	x_0	x_2	y_2	y_1	y_1	y_0	y_2	y_2
6	x_1	x_0	x_0	x_0	y_0	y_0	y_2	y_2	y_2	y_1

Blocks	31	32	33	34	35	36	37	38	39	40
Levels of F_1	Levels	of	F_2	and	F_3					
0	y_0	y_0	y_1	y_1	y_1	y_1	y_1	y_1	y_1	y_1
1	y_1	y_2	y_2	y_0	y_1	y_2	y_0	y_1	y_2	y_0
2	y_2	y_1	y_0	y_2	y_1	y_0	y_2	y_1	y_0	y_2
3	y_2	y_2	y_1	y_1	y_2	y_2	y_2	y_0	y_0	y_0
4	y_0	y_1	y_2	y_0	y_2	y_0	y_1	y_0	y_1	y_2
5	y_1	y_0	y_0	y_2	y_2	y_1	y_0	y_0	y_2	y_1
6	y_1	y_1	y_1	y_1	y_0	y_0	y_0	y_2	y_2	y_2

Blocks	41	42	43	44	45	46	47	48
Levels of F_1	Levels	of	F_2	and	F_3			
0	y_2	y_2	y_2	y_2	y_2	y_2	y_2	y_2
1	y_0	y_1	y_2	y_0	y_1	y_2	y_0	y_1
2	y_1	y_0	y_2	y_1	y_0	y_2	y_1	y_0
3	y_2	y_2	y_0	y_0	y_0	y_1	y_1	y_1
4	y_0	y_1	y_0	y_1	y_2	y_1	y_2	y_0
5	y_1	y_0	y_0	y_2	y_1	y_1	y_0	y_2
6	y_2	y_2	y_1	y_1	y_1	y_0	y_0	y_0

TABLE 6.22: $7 \times 3 \times 3$ BAFD

Other examples include BAFD's $4 \times 3 \times 2$, $6 \times 5 \times 3$, $5 \times 3 \times 3$, $6 \times 8 \times 3$, $7 \times 4 \times 4$, … *and so on.*

Chapter 7

Summary,Conclusions, Contributions and Recommendations

This chapter covers, summary, conclusions, contributions and recommendations

7.1 Summary, Results and Challenges

The objective of this research was to construct efficient balanced asymmetrical factorial designs via three methods namely, balanced arrays, transitive arrays, and resolvable balanced incomplete block designs.

Balanced Arrays

By using balanced arrays, efficient balanced asymmetrical factorial designs were constructed. Balanced arrays were achieved by using galois fields. The construction of balanced arrays was challenging.

Transitive Arrays

By using transitive arrays, efficient balanced asymmetrical factorial designs were constructed. Transitive arrays were achieved using $s-1$ mutually orthogonal latin squares of order s. The construction of transitive arrays was however not challenging.

Resolvable Balanced Incomplete Block Designs

By using balanced arrays and resolvable balanced incomplete block designs or transitive arrays and resolvable balanced incomplete block designs, two factor and multifactor balanced asymmetrical factorial designs were constructed. The construction of resolvable balanced incomplete block designs was achieved by using galois fields and also by using geometry of chords constructed inside circles. Their construction was however not as challenging.

7.2 Conclusions

The results presented in this thesis relate to connected factorial designs.The disconnected case posses special problems. In particular then, the conclusions of lemma 3.1.4, which is helpful in providing subsequent results, no more remain valid.Of course one may work with generalized inverses of matrices but even then, some special considerations are required.As a matter of fact, the results proved in chapter three, at least in their present forms, do not remain valid in the disconnected case.The following example illustrates the point.

Consider a disconnected 2^3 design in two blocks as shown below.

BLOCK I:	000,	100,	010,	001
BLOCKII:	110,	101,	011,	111

TABLE 7.1: Disconnected 2^3 Design

Clearly, each interaction is represented by a single contrast.It may be seen from the elementary considerations that the contrasts belonging to interactions $F(1,1,0), F(1,0,1), F(0,1,1)$ are estimable while those belonging to $F(1,0,0), F(0,1,0), F(0,0,1), F(1,1,1)$ are not estimable.Moreover, the BLUE's of the contrasts belonging to $F(1,1,0), F(1,0,1), F(0,1,1)$ may be seen to be mutually orthogonal, i.e Uncorrelated. Hence the design has OFS. Also trivially, the design is balanced since each interaction is represented by a single contrast. Thus the design is balanced and has OFS. However, the C-matrix is not of the form, (3.1.8). In order to appreciate this point, note that if the C-matrix be of the form, (3.1.8), then by (2.3.7), (2.3.8), one must have $M^y C = CM^y$ for every $y \in \Omega$. For this design, explicit computation shows that, in particular $M(0,0,1) = M^{001}$ does not commute with C. The above example demonstrates that the necessity part of lemma 3.1.5 does not necessarily remain valid. Similarly, it may be shown that the necessity part of theorems 3.1.1, 3.1.2, 3.1.3, 3.2.2 may not remain valid for the disconnected designs. In chapter four, we have shown the usefulness of difference schemes in constructing orthogonal arrays of any strength. We have shown that several families of arrays, often with maximal number of factors can be constructed in this way. Difference Schemes are therefore an important tool to consider in cases where the maximal number factors has not been determined. Another advantage is that an orthogonal array obtained in this way has a concise description since $D(r, c, s)$ yields an $OA[rs, c+1, s, 2]$. Although constructing a new difference scheme is probably easier than the direct construction of the corresponding orthogonal array, it remains a very challenging problem. No general algorithm is known. However, it seems likely that the group-theoretic approach of L. and P. (1986, 1987, 1990) and Kreher (1990) (see also Kreher and Stinson (1998)) could successfully be applied here. An $n \times n$ array based on s symbols is called a Frequency square or an F-Square if each symbol appears $\frac{n}{s}$ times in each row and in each column. Some orthogonal arrays in chapter four can be constructed by using F-Squares and pairwise orthogonal F-Squares or by using Latin Squares and Pairwise Orthogonal Latin Squares. However, although the connections between orthogonal arrays and pairwise orthogonal F-Squares are fascinating, it is debatable how important these connections are for the construction of new orthogonal arrays. Orthogonal arrays constructed by using various types of combinatorial structures are typically larger than the combinatorial structures used to construct them and hence the constructions of orthogonal arrays using these structures is perceived to be a more tractable problem than the direct construction of orthogonal arrays. This reduction in complexity is no longer apparent when constructing orthogonal arrays using F-Squares. After all, each of the F-Square presents the levels for a factor in an orthogonal array merely displaying them in a square instead of a vector.

7.3 Recommendation

Since the results in this thesis relate to connected BAFD's, this calls for suitable modifications of these results to make them applicable to the disconnected case. Efforts have been made to reduce the number of assemblies in example 4.9.8 and 4.9.9 by half i.e., to construct a $BA(T)[4, 3, 1]$ and $BA(T)[3, 4, 1]$ but without success. In chapter four, examples 4.9.3, 4.9.4, 4.9.5 can also be constructed by using theorem 4.9.4, but certainly there are balanced arrays which can be constructed by corollary 4.9.3 and cannot be constructed by theorem 4.9.4. For example, a $BA(T)[3, 2, 1]$ which can be constructed by corollary 4.9.3 is not completely resolvable. Therefore it cannot be constructed by theorem 4.9.4. However, all balanced arrays that can be constructed by corollary 4.9.4 can also be constructed by theorem 4.9.4 since the orthogonal arrays used in corollary 4.9.4 are constructed by the method of differences.

In the previous chapters, we restricted our consideration of BAFD's to one-way designs only.These concepts can also be extended to two way designs i.e. designs with rows and columns as blocks. Designs with two-way elimination of heterogeneity are designs that satisfy the following conditions:

(a) Each treatment is replicated the same number of times, say r

(b) There are $k-$rows and $b-$columns.At a given row and column, there exists u plots.

(c) Estimates of contrasts belonging to different interactions are uncorrelated with each other.

(d) All the normalized contrasts belonging to the same interaction are estimated with the same variance.

In designs with two-way elimination of heterogeneity, we do not need the condition that each treatment occur at most once in each row or each column. In most cases, $u = 1$, but for generality, we do not make this assumption.

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