UNIVERSITY OF NAIROBI

GENERALIZATIONS OF LOGISTIC DISTRIBUTION

BY

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A Thesis Submitted in Fulfillment of the Requirements for Registration of the Degree of Doctor of Philosophy in Mathematical Statistics of the University of Nairobi

GENERALIZATIONS OF LOGISTIC DISTRIBUTION

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PhD Thesis

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Abstract

In this work we determine generalizations of the logistic distribution using the methods of construction of the logistic distribution. The generalized logistic distributions are of type I,II,III and IV. The methods of cons considered are: the difference of two standard Gumbel random variables, the Burr differential equation, transformations and mixtures. Also, the generalized logistic distributions have been considered using various transformations, the Burr differential equation, mixtures of Gumbel,beta I and beta II distributions. GLIV, extended GLIV and Exponential generalized beta II distributions have been obtained whence with their special cases.

A new distribution, the "extended standard logistic" has been introduced as a result of the generalizations. We also show the application of the cdf of the logistic distribution in determining the probability of default in logistic regression using data from a money lending company in Kenya - Mobipesa Ltd.

Additionally, generalized logistic distributions based on beta I and beta II distributions have been constructed. Special cases of the extended generalized Logistic type IV have also been obtained.

We further determine the discrete and continuous mixtures of minimum and maximum order statistic distributions from the standard logistic and exponentiated logistic distributions. The mixing distributions used are zero truncated Poisson, binomial, negative binomial, geometric and the logarithmic series distribution. The minimum and maximum order statistics distributions have been constructed alongside their hazard and survival functions.

We also construct continuous mixtures of the logistic distribution with scale and location parameters. The mixing distributions used are; logistic, exponential, gamma I, gamma II, inverse gamma, half logistic and reciprocal inverse Gaussian. The mixed distributions have been expressed in terms of the modified Bessel function of the third kind. A new distribution, the Logistic Inverse Gaussian distribution has been introduced. Its properties like the log-likelihood function, moments and expected maximum algorithm have been obtained.

Since we have quite a number of generalized logistic distributions and their special cases, the current study has not exhausted all the mixing distributions proposed by Nadarajah and Kotz (2004), however, the rest can be employed in a similar manner.

Declaration and Approval

I the undersigned declare that this thesis is my original work and to the best of my knowledge, has not been submitted for the award of a degree in any other university or institution of higher learning.

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In my capacity as a supervisor of the candidate's dissertation, I certify that this thesis has my approval for submission.

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Dedication

For my wife Cate and lovely family who stood by my side in every circumstance of my academic traversals.

Contents

Abbreviations and Notations

Abbreviations and notations for specific charpters can be found in those charpters. Abbreviations and notations generally used are given below.

cdf	Cummulative distribution function
pmf	probability mass function
RHS	Right Hand Side
LHS	Left Hand Side
CRB	Credit Reference Bureau
IV	Information Value
$P.D$	Probability of Default
$W.O.E$	Weight of Evidence
$A.P.I$	Application Programming Interface
$k_v(w)$	Modified Bessel function of the third kind
$BGLIV$	Beta Generalized Logistic type Four
GLI,$GLII$,$GLIII$,$GLIV$	Generalized logistics distributions types I,II,III and IV
$G4BII$	Generalized four parameter Beta 2 II distribution
$EGBII$	Exponential Generalized Beta 2 distribution
SEV	Smallest Extreme Value
LEV	Largest Extreme Value
MGF	Moment Generating Function
pgf	Probability Generating Function
LGD	Loss Given Default
LAD	Loss At Default

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1 GENERAL INTRODUCTION

The main objective of this charpter is to outline the background information of the logistic distribution, define terminologies to be used in the research, state various notations, outline the research problem, objectives of the study, literature review, the mathematical tools to be used in the research and the significance of the study respectively.

1.1 Background Information

In recent times, the focus of constructing distributions,has shifted to generalizations, that is, adding more parameters to a distribution under consideration for flexibility in its shape, location, scale and tail(s).

The logistic probability distribution originated as a model for growth of a population. It has no shape parameter, this means that the logistic distribution function has only one shape, the bell shape and this does not change.

In practical applications it cannot be distinguished from the normal. It is therefore one of the most important statistical distributions because of its simplicity and significance as a growth curve. The use of the logistic curve for economic and demographic purposes was very popular from the end of the nineteenth century onwards.

Various methods have been developed for constructing distributions; among them are the generator approach.

The concept of "Exponentiated Generator Approach" was given by Burr (1942); while the beta-generator was given by Eugene et al (2002) and Jones (2004). In this case, if one of the parameters in a distribution is varying then we have the notion of mixtures which also takes care of flexibility.

The pioneers of mixtures are Greenwood and Yule (1920) who mixed a Poisson distribution with a gamma distribution. Adamidis and Loukas (1998) introduced lifetime distributions based on the discrete mixtures of order statistics. They considered the minimum order statistics from an exponential distribution as the conditional distribution and shifted geometric distribution as the mixing distribution.

The general purpose of this research falls under mixtures of generalized distributions and their special cases. The mixtures of Logistic distributions cannot be studied without a sound knowledge of associated univariate distributions. Therefore we shall pay special attention to the univariate standard Logistic distribution.

1.2 Definitions and Terminologies.

1.2.1 Beta Generator

The beta generator was First introduced by Eugene et al. (2002) through its cumulative distribution. Beta generated distributions may be characterized by their pdf. It has been used to generate the generalized beta-generated distributions. The distributions generated have very flexible tails and tractable properties. The generator enables the resulting distributions to gain a modest degree of kurtosis. Beta generated distributions with more parameters were studied by Nadarajah and Kotz (2004, 2005).

1.2.2 Logistic distribution

The logistic distribution is a continuous probability density function that is symmetric and unimodal; It's characterized by two main parameters, the location μ and the scale σ. It has a shape similar to the normal distribution though it has heavier tails (higher kurtosis).

1.2.3 Beta distribution

The beta distribution is a continuous probability distribution which models events constrained to take place within an interval defined by a minimum and a maximum value.

1.2.4 Mixtures

A mixture is a distribution obtained by combining two or more distributions. A mixture or mixed distribution also arises when the pdf or the pmf of a random variable depends on a parameter. Mixing one distribution with another to get a new distribution is one way of constructing probability distributions. Finite, discrete and continuous mixed distributions are the three types of mixtures.

1.2.5 Generalized distributions.

The term 'Generalized' used in this research has the notion of adding more parameters to the existing distribution in order to give a more universal distribution. GLI,$GLII$,$GLIII$ and $GLIV$ stand for generalized logistic distribution types I,II,III and IV respectively.

1.2.6 Special Functions

The special functions used in this research are outlined as follows.

The Beta function

The Beta function denoted by B(a,b) is defined as;

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt,\ a>0, b>0 \tag{1.1}$$

The Gamma function

The Gamma function denoted by $\Gamma(\alpha)$ is defined as;

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt,\ \alpha>0 \tag{1.2}$$

Modified Bessel Function of the third Kind

The Modified Bessel Function of the third kind denoted by $K_v(w)$ is defined as;

$$K_v(w) = \frac{1}{2}\int_0^\infty x^{v-1}\exp\Big(-\frac{w}{2}(x+\frac{1}{x})\Big)dx,\ -\infty<x<\infty \tag{1.3}$$

1.2.7 Probability density function pdf, Probability Mass function pmf and Cumulative distribution function cdf

Consider a random variable X. A real valued non-negative function $f(x)$, that is, $f(x)\geq 0$ such that $\int_{-\infty}^{\infty} f(x)dx = 1$ for continuous X then $f(x)$ is a pdf and such that $\sum_x f(x) = 1$ then $f(x)$ is a pmf. In general both pdf and pmf are referred to as probability distributions. The cummulative distribution function (cdf) denoted by $F(x)$.

$$F(x) = \begin{cases} \int_{-\infty}^{x} f(t)dt, & \textit{continous X} \\ \sum_{t=-\infty}^{x} f(t), & \textit{discrete X} \end{cases} \tag{1.4}$$

In particular, for $f(x) = \frac{dF(x)}{dx}$ for continuous X. Moreover, for discrete X,

$$f(x) = F(x) - \lim_{x \to \bar{x}} F(t)$$

1.2.8 Exponentiated Generator

The power of a cumulative distribution function can be referred to as an exponentiated generated distribution. Thus,let

$$F(x) = \Big(G(x)\Big)^{\alpha},\ \alpha > 0 \tag{1.5}$$

where $G(x)$ is the old or parent or baseline cdf. Then $F(x)$ is the exponentiated distribution function. The corresponding pdf is given by;

$$f(x) = \alpha\Big(G(x)\Big)^{\alpha-1} g(x),\ \alpha > 0 \tag{1.6}$$

where $g(x)$ is the old pdf.

1.2.9 Order Statistics

Ordered observations of an independent identically distributed sample are known as order statistics. Let $X_1, X_2, ..., X_n$ be a random sample of size n from a population having $\mathrm{pdf}\ f(x)$ and $\mathrm{cdf}\ F(x)$. Then, $X_1 \leq X_2 \leq, ..., \leq X_n$ denotes order statistics for continuous X. Also, let $X_1 < X_2 <, ..., < X_n$ where X_i denotes i^{th} order statistic. The sample observations can be arranged in ascending order of magnitude and written in the symbol form as $X_1, X_2, ..., X_n$ where the numbers $i = 1, 2, ..., n$ in parenthesis indicate the rank of the observation in the sample so. X_i is the i^{th}order statistic, X_1 is the first order statistic, X_n is the n^{th}order statistic, similarly $f_{1:n}$ is the pdf of the i^{th} order statistic and $F_{1:n}$ is the cdf of the i^{th} order statistic.

1.2.10 Beta Generator Approach

Let

$$w(t) = \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)},\ 0 < t < 1,\ a > 0,\ b > 0 \tag{1.7}$$

which is a classical beta pdf; also called beta 1 pdf or beta pdf of the first kind; where $B(a,b)$ is a beta function with parameters a and b. The cdf of the beta distribution is given by;

$$W(x) = \int_0^x w(t)dt = \frac{1}{B(a,b)} \int_0^{G(x)} t^{a-1}(1-t)^{b-1}dt,\ 0 < x < 1,\ a > 0,\ b > 0 \tag{1.8}$$

The beta generated distribution is defined in terms of cdf as

$$F(x) = W\Big(G(x)\Big) = \int_0^{G(x)} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt \;\; 0 < t < 1, \;\; a > 0, \;\; b > 0 \tag{1.9}$$

Where $0 \leq G(x) \leq 1$ is the cdf of any distribution for $-\infty < x < \infty$

The corresponding pdf is

$$f(x) = \frac{\Big(G(x)\Big)^{a-1}\Big(1 - G(x)\Big)^{b-1} g(x)}{B(a,b)} \tag{1.10}$$

We refer to $G(x)$ and $g(x)$ to as old/parent cdf and pdf respectively.

1.2.11 The Method of Mixtures

Mixtures can be obtained by taking different distributions with $\mathrm{pdf}'s$ or $pmf's$, $f_1x, f_2x, \ldots, f_kx$ with weights $\omega_1, \omega_2, \ldots, \omega_k$. The new density or mass function is

$$f(x) = \sum_{j=1}^{k} \omega_j f_j(x)$$

for $\omega_j > 0$ and $\sum_{j=1}^{k} \omega_j = 1$ which is called a finite mixture.

Suppose the distribution of a random variable X depends on a parameter θ, then the mixture or mixed distribution becomes;

$$f(x) = \int_{-\infty}^{\infty} f(x|\theta) g(\theta) d\theta \tag{1.11}$$

Where $f(x|\theta)$ is a conditional pdf or pmf and $g(\theta)$ is a continuous mixing distribution. Then $f(x)$ is a continuous mixture. If $g(\theta)$ is discrete, then we have;

$$f(x) = \sum_{\theta} f(x|\theta) g(\theta) \tag{1.12}$$

In this case $f(x)$ is a discrete or countable mixture.

1.2.12 Credit scoring

Credit scoring is a term commonly used in the finance industry. It refers to the process that financial institutions use to determine the credit worthiness of a borrower. Credit scoring models are used to predict probability of default $P.D$ of a borrower. The models give financial institutions a chance to determine the risk they are willing to take when determining the credit worth of a borrower.

Predicting probability of default is a classification problem and hence classification models are mostly used. A classification model is designed to use observed values to come up with conclusions. These models use single or multiple inputs to predict single or multiple outcomes.

The inputs (explanatory variables) can be about a borrowers financial history, their personal status or social life. The financial institutions decide the variables to use (or which they can get) in their models. Its also crucial to ensure that the model robust and adequacy checks are done to keep the model properly interpreted and fitted.

1.2.13 Logistic Regression

One of the classification models used in credit scoring is logistic regression. This is a type of generalized linear model mainly used to approximate probability that a binary response occurs based on the given explanatory variables. Logistic regression is used in different industries for purposes like marketing, biomedical studies, financial applications and many others.

1.2.14 Stepwise Logistic regression

This is the process of automatically selecting a reduced number of predictor variables for building the best performing logistic regression model. Stepwise selection reduces the complexity of the model without compromising its accuracy.

1.2.15 Model Robustness

This is the ability of a model to remain stable even when there are external disturbances. This implies that the output forecasts are consistently accurate even when an assumption or input variables are drastically changed due to unforeseen circumstances.

1.2.16 Model Adequacy

The fit of a regression model shows how adequate it is. Globally- to understand this term, there is an assumption that the borrower borrows from the primary financial institution (Mobipesa- the institution under study) and from other institutions. To give the borrower a loan, Mobipesa needs to know the borrowers behavior when borrowing from other institutions. The behavior is obtained from CRB. Hence CRB gives the borrowers behavior in the entire financial industry hence the term "globally".

1.2.17 My-Sector /Other Sectors

CRB gives data in two parts, my sector and other-sectors. In this case, Mobipesa is a digital lending firm hence "my sector" implies similar financial institutions i.e digital lending institutions that give short-term loans. "Other-sectors" implies banks and other financial institutions giving long-term loans.

1.2.18 Credit Risk

This is the probability that someone who has borrowed money will not repay it all. When someone fails to pay a loan, it is said to be in default. Sometimes it is the calculated risk difference between lending individuals and government bonds.

1.2.19 Probability of default

The likelihood that an individual will default on a loan is called the probability of default. The cdf of the logistic gives the probability of default. The estimates are the coefficients. The probability of default is a credit risk which gives a gauge of the probability of a borrowers will and identity unfitness to meet the financial commitments. Evaluating the P.D enables a firm to manage credit exposure.

1.2.20 Bad Customers/Good customers

Bad are customers who defaulted on paying a loan while good customers are customers who paid back the loan.

1.2.21 Expected Loss

This is the amount a firm loses as a result of loan default. It has three components.

(i) Probability of default (PD)

This is the likelihood that a borrower will fail to pay back a debt. The cdf of the logistic gives the probability of default

$$p = \frac{1}{1+e^{-y}} \tag{1.13}$$

where

$$y = \beta_0 + \beta_1 x_1 + ... + \beta_i x_i$$

(ii) Exposure at default (EAD)

This is the predicted amount a lender is likely to lose when a borrower defaults on a loan.

(iii) Loss given default (LGD) This is the amount of money a lender loses when a borrower defaults on payment of a loan.

$$\text{Expected loss} = PD * EAD * LGD$$

1.2.22 Bin

To bin is to categorize/group variables in a way that summarizes as much information as possible in order to build models that eliminate weak variables or those that do not conform to good business logic. Bins are a form of data partitioning. Binning is a powerful process because it eliminates outliers because they can be contained in the smallest or the largest group, missing values are assigned to their own group, grouping can reflect non linear relationships since there is no need to worry about linearity assumptions.

1.2.23 Weight of evidence (WOE)

WOE defines the relationship between a binary response variable and an explanatory variable. Its calculated from the basic odds ratio. This is the difference between the distribution of good and bad characteristics of an individual borrower. WOE tells the predictive power of an independent variable in relation to the dependent variable.

Its a measure of separation of good and bad customers. The computer package being used in building the model for credit scoring automatically defines the bins or they are automatically defined by an expert based on certain characteristics or their distribution. From raw data bins are calculated and a weight of evidence is assigned to each bin.

$$WOE = \ln\left(\frac{\text{Distribution of good}}{\text{Distribution of bad}}\right) \tag{1.14}$$

Positive WOE means that distribution of good customers is greater than distribution of bad customers while a negative WOE means that the distribution of good is less than the distribution of bad. Weight of evidence helps in eliminating outliers through grouping, it also helps in handling missing values as they can be binned separately . A new customer is assigned the weight of evidence of the bin in which they fall.

1.2.24 Information Value

Denoted by IV. This is a technique for selecting important variables in a predictive model. It helps to rank variables on the basis of their importance. It's a measure of the "strength" of a grouping for separating bad and good risk.

$$IV = \sum(\text{Distribution Good}_i - \text{Distribution Bad}_i) * WOE_i \tag{1.15}$$

1.2.25 Odds

Odds are a measure of the likelihood that something will occur. Usually expressed as a ratio. They represent the relative frequency with which different outcomes occur. Odds are directly related to probabilities and can be translated back using the translation in equation 1.2

The transformation from probability to odds is a monotonic transformation, meaning the odds increase as the probability increases and vice versa. Probability ranges from 0 to 1. Odds range from 0 to positive infinity.

The odds of probability of a success are defined as the ratio of the probability of success over that of failure. The transformation from odds to log of odds is the log transformation.

$$\text{Odds} = \frac{P.D}{1 - P.D} \tag{1.16}$$

1.2.26 Points to double Odds.

Often abbreviated as PDO. This refers to the increase in in score points that results in the score that corresponds to twice the odds. The PDO accepts integers greater than or equal to 1. The default value is 20. Its the source of the $\ln 2$. So the interpretation of $\frac{20}{\ln 2}$ is that for a 20-point increase in the score, the odds double. For example how many points does the score change if the odds increase from 100: 1 to 200:1

1.2.27 Application Programming Interface

Denoted by $A.P.I.$ This is a communication protocol between different parts of a computer program. It specifies how software components should interact. It defines the kinds of calls or requests that can be made, how to make them, the data formats that should be used and the conventions to follow.

1.2.28 Credit Modelling in R

The R-Script shows the analysis of customers data, logistic regression and conversion to scores that are used to grade customers.

1.3 Research Problem

Adamidis and Loukas (1998) introduced lifetime distributions based on the discrete mixtures of distributions of order statistics. They considered the minimum order statistic from an exponential distribution as the conditional distribution and shifted geometric distribution as the mixing distribution.

Nadarajah and Kotz (2004) constructed the extended generalized logistic type IV from the difference of two independent Gumbel random variables, but did not give explicit results. Moreover, Nadarajah and Kotz (2006) constructed logistic mixtures but did not obtain moments and posterior distributions.

Morais (2009) constructed discrete mixtures of minimum order statistics from exponential, Weibull and Pareto distributions. She did not consider discrete mixtures of maximum order statistics or continuous mixtures of both minimum and maximum order statistics. Motivated by this work, one realizes that mixtures of order statistics from logistic distribution don't seem to have been studied.

A lot of beta- generated distributions have been constructed after the work of Eugene (2002) and Jones (2004). However, mixtures of beta-generated distributions and their special cases don't seem to have been studied apart from the work of Nadarajah and Kotz (2006) mentioned above.

Olapade (2004,2005) introduced introduced the extended generalized logistic type I and II, Wu, Hung (2000) introduced the extended generalized logistic type IV. However, the extended standard logistic distribution has not been studied.

In this work we shall determine generations of the logistic distributions using the methods of construction of the logistic distribution. We shall also determine the discrete and continuous mixtures of order statistics arising zero truncated mixing distributions. The standard logistic distribution shall also be applied to data alongside the generalized logistic distributions.

1.3.1 Mathematical Formulation of the Research Problem

The cdf and pdf of the standard logistic distribution

Let the cdf and pdf of the standard logistic distribution be given by;

$$G(x) = \frac{1}{1+e^{-x}}, \ -\infty < x < \infty \tag{1.17}$$

$$g(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \ -\infty < x < \infty \tag{1.18}$$

Then the cdf and pdf of the beta-logistic shall be stated as follows;

$$F(x) = \frac{1}{B(a,b)} \int_0^{(1+e^{-x})^{-1}} t^{a-1}(1-t)^{b-1}dt$$

$$f(x) = \frac{e^{-bx}}{B(a,b)\left(1+e^{-x}\right)^{a+b}}, \ a>0, \ b>0, \ -\infty < x < \infty \tag{1.19}$$

Which is the Beta-Generalized logistic type IV, $BGLIV$.

Mixtures of Logistic Distribution

For the standard logistic we shall introduce a new parameter σ in the standard logistic to obtain the distribution:

$$f(x/\sigma) = \frac{1}{\sigma}\frac{e^{-x/\sigma}}{(1+e^{-x/\sigma})^2} \tag{1.20}$$

and the mixture or mixed distribution is

$$\begin{aligned} f(x) &= \int_0^\infty f(x|\sigma)g(\sigma)d\sigma \\ &= \int_0^\infty \frac{1}{\sigma}\frac{e^{-x/\sigma}}{(1+e^{-x/\sigma})^2}g(\sigma)d\sigma \end{aligned} \tag{1.21}$$

Discrete mixtures for minimum order statistics

For the discrete case, P_n is a zero truncated power series distribution.
Let

$$f_{1:n}(x) = \sum_{n=1}^{\infty} f_{1:n}(x|n)P_n$$

Then

$$f(x_{1:n}) = \sum_{n=1}^{\infty} \frac{ne^{-n(\frac{x-\mu}{\sigma})}}{\sigma\left(1+e^{-\frac{x-\mu}{\sigma}}\right)^{n+1}}P_n \tag{1.22}$$

Discrete mixtures for maximum order statistic

$$\begin{aligned} f_{n:n}(x) &= \sum_{n=1}^{\infty} f_{n:n}(x|n)P_n \\ &= \sum_{n=1}^{\infty}\left(F(x)\right)^n P_n \\ &= \sum_{n=1}^{\infty}(1+e^{-x})^n P_n \end{aligned} \tag{1.23}$$

P_n is a zero truncated pmf.

Continuous mixtures for n^{th} order statistics

$$f(x) = \int_0^\infty n e^{-n(\frac{x-\mu}{\sigma})} g(\lambda) d\lambda \tag{1.24}$$

Type I Exponentiated Distribution

When $b = 1, \quad a > 0$

$$\begin{aligned} F(x) =& \Big(G(x)\Big)^a \\ f(x) =& a\Big(G(x)\Big)^{a-1} g(x) \end{aligned} \tag{1.25}$$

Type II Exponentiated Distribution

When $a = 1, b > 0$

$$\begin{aligned} F(x) &= 1 - \Big(1 - G(x)\Big)^b \\ f(x) &= b\Big(1 - G(x)\Big)^{b-1} g(x) \end{aligned} \tag{1.26}$$

The i^{th} order statistics distribution

When $a = i, b = n - i + 1$

$$f_{i:n}(x) = \frac{\Big(G(x)\Big)^{i-1}\Big(1 - G(x)\Big)^{n-i}}{B(i, n-i+1)} g(x) \tag{1.27}$$

In particular, the minimum order statistic is given by:

$$f_{i:n}(x) = n\Big(1 - G(X)\Big)^{n-1} g(x) \tag{1.28}$$

and the maximum order statistic is given by:

$$f_{n:n}(x) = n\Big(G(X)\Big)^{n-1} g(x) \tag{1.29}$$

Continuous and Discrete mixtures

Using equation 1.10 and 1.11 for the continuous and discrete case respectively, we define $f(x)$ is the mixed distribution or mixture

$f(x|\theta$ is the conditional distributional
$g(\theta)$ is the mixing distribution.

1.4 Objectives of the study

The main objective of this study is to determine the generalizations of the logistic distribution and their mixtures, and the specific objectives are:

(i) To construct the standard logistic using; transformations, mixtures, the Burr differential equation and the difference of two independent Gumbel random variables.

(ii) Calculate credit scores using the cumulative distribution function of the logistic in logistic regression.

(iii) To construct the generalized logistic distributions of type I, II, III, IV and their extended versions using the methods of constructing the standard logistic .

(iv) To construct discrete and continuous mixtures from minimum and maximum order statistics of the standard logistic distribution using zero truncated Poisson, binomial, negative binomial, geometric and logarithmic distributions.

(v) To construct the continuous mixtures of logistic with scale and location parameters based on modified Bessel functions of the third kind alongside their moments.

1.5 Literature review

Various methods of constructing logistic distribution and its generalization have been developed. A review of these follows under different sub- headings .

1.5.1 Construction of the Logistic distribution and its generalizations.

Verhulst (1845) used the logistic distribution to model the growth of populations.

Burr (1942) obtained the logistic distribution by solving what is called the Burr differential equation. He applied the concept of exponentiation by solving the differential equation in cdf.

Gumbel (1944) obtained the logistic distribution by considering the distribution of the difference of two independent Gumbel random variables.

Gumbel (1944) found that the logistic distribution arises as a limiting distribution of the standard mid-range (average of the largest and smallest values) of random samples of size n from a symmetric distribution of exponential type.

Gumbel and Keeney (1950) showed that a logistic distribution is obtained as a limiting distribution of an appropriate multiple of the "the extreme quotient" that is, the largest divided by the smallest value.

Dubey (1969), Balakrishnan and Leung (1988) showed that the Logistic distribution can be obtained as a mixture of a Gumbel distribution and a one parameter Gamma mixing distribution.

Prentice (1976) and Kalefleish and Prentice (1980) studied the generalized logistic type $GLIV$.

Balakrishnan and Leung (1988) showed that If X is a random variable with type I generalized Logistic distribution, then $-X$ is a type II generalized logistic distribution.

Stefanski (1991) showed that the standard logistic distribution can be represented as a scale mixture of the standard normal distribution where the mixing density is related to Kolmogorov-Smirnov distribution.

Wu et al (2000) showed a mixture of a two- parameter generalized Gumbel with a two-parameter gamma mixing distribution yields an Extended Generalized Logistic distribution.

Eugene et al (2002)obtained exponentiated distributions as special cases of the beta generated distribution.

Olapade (2004) called it the Extended Generalized Logistic distribution.
Nadarajah and Kotz (2004) constructed a generalized logistic distribution by considering the distribution of the difference of two independent Gumbel random variables.

Villa and Escobar (2006) obtained the logistic distribution from the mixtures of Gumbel. Gupta and Kundu (2010) discussed two types of generalizations of logistic distribution. The first generalization is carried out using the basic idea of Azzalin (1985) and called it the skew logistic distribution. The second generalization was called "proportional reversed hazard logistic distribution".

1.5.2 Application of logistic to credit scoring

Credit scoring is the most recent development in the application of the logistic to data analysis. Mondal (2016) discussed response modeling and credit scoring in R using machine learning techniques. He showed the R codes for logistic regression in machine learning using German credit data. Hongri (2018) showed how to build a statistical model using a scorecard. He defined the variables to be used in a model showing their information value and weight of evidence. Dominguez (2011) showed how the cdf of the logistic distribution can be used in calculating the probability of default in logistic regression.

1.5.3 The Beta-Generated Approach

The beta- generated approach was introduced by Eugene et al (2002) and Jones (2004). Jones calls it a generalization of order statistics. The popularity of this approach can be verified by considering the number of papers that have been published as;

Beta - Normal	(Eugene et al (2002)
Beta - Log F	(Jones (2004)
Beta - Gumbel	(Nadarajah and Kotz (2004)
Beta - Fretchet	(Nadarajah and Gupta (2004)
Beta - Weibull	(Famoye et al (2005)
Beta - exponential	(Nadarajah and Kotz (2006)
Beta - Hyperbolic Secant	(Fisher and Vaughan (2007)
Beta - Gamma	(Kong et al (2007)
Beta - Pareto	(Akinsete et al (2008)
Beta - Rayleigh	(Akinsete and Lowe (2009)
Beta - beta Prime	(Morais (2009)
Bata - F	(Morais (2009)
Beta - Burr XII	(Paranaiba et al (2010)
Beta - Dagum	(Condino and Domma (2010)
Beta - Fisk (Log- Logistic)	(Paranaiba (2010)

Morais (2009) extended $GLIV$ distribution by introducing two extra shape parameters to introduce skewness and to vary tail weights. The $GLIV$ has been beta-generated to come up with a four parameter beta generalized logistic distribution.

Nassar and Elmasry (2011) studied Beta-logistic distribution which is $GLIV$ distribution. They expressed it in terms of the Gauss hyper-geometric function. They also obtained the hazard function, the mode, the median and the characteristic function.

1.5.4 Mixtures Based on order statistics from the Logistic distribution

Adamidis and Loukas (1998) introduced lifetime distributions based on the discrete mixtures of order statistics. They used the distribution of the minimum order statistic from an exponential distribution as the conditional distribution and shifted geometric distribution as the mixing distribution. The mixture is known as the exponential-geometric distribution.

Nadarajah and Kotz (2006) studied mixtures of logistic distribution. The mixing distributions used are;

i. Exponential

ii. Gamma

iii. Half Logistic

iv. Inverse Gaussian

v. Weibull

vi. Stacy

vii. Half Normal

viii. Fre'chet

ix. Two sided power

x. Beta

xi. Inverted Beta

xii. Lomax

xiii. Generalized Pareto

xiv. Burr III

xv. Burr XII

xvi. Pareto

The mixtures are expressed in terms of the modified Bessel function of the third kind; confluent hypergemetric function and generalized hypergeometric function.
Morais (2009) did not work on logistic mixtures. We can however borrow her ideas to study Logistic mixtures.
Morais (2009) obtained discrete mixtures of minimum order statistics from exponential, Weibull and Pareto distributions.
The mixing distributions are zero truncated Poisson, binomial and negative Binomial.

1.6 Research Methodology

The Mathematical tools to be used in this research are special functions; specifically, beta and gamma functions and modified Bessel functions of the third kind.

Continuous mixing distributions to be used in continuous mixtures are: exponential, gamma, inverse Gaussian, reciprocal inverse Gaussian, generalized inverse Gaussian, beta I, beta II, shifted gamma, inverse gamma and Pareto I.

For discrete mixtures , we shall use the following discrete mixing distributions; Zero-truncated Poisson,zero truncated Binomial, zero truncated negative Binomial distributions, geometric and logarithmic. We shall particularly use;

i. Beta and Gamma functions: their definitions, properties and relationships.

iii. Modified Bessel functions and their properties.

These functions have been defined in subsection $1.2.6$, we now consider some of their properties.

1.6.1 Properties of the beta and gamma functions

Beta I function (Beta function of the first kind)

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx, \quad a>0,\ b>0. \tag{1.30}$$

Beta II function(Beta function of the second kind)

Properties

$$B(a,b) = B(b,a) \tag{1.31}$$

$$B(a,b) = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}dt \tag{1.32}$$

Gamma Function

Properties

$$\begin{aligned}\Gamma(\alpha+1) &= \alpha\Gamma(\alpha),\ \alpha>0\\ \Gamma(1) &= 1\\ \Gamma(\alpha) &= (\alpha-1)!,\ \alpha = 1,2,3,...\\ \Gamma(t)\Gamma(1-t) &= \frac{\pi}{sin\pi t}\\ \Gamma(\frac{1}{2}) &= \sqrt{\pi}\end{aligned}$$

The relationship between the beta and gamma functions is given by the equation.

$$B(a,b) = \frac{\Gamma a\Gamma b}{\Gamma(a+b)} \tag{1.33}$$

1.6.2 Properties of Modified Bessel function of the third kind

$$
\begin{aligned}
K_v(w) =& K_{-v}(w) \\
K_{\frac{1}{2}}(w) =& \sqrt{(\frac{\pi}{2w})} \exp(-w) \\
K_{\frac{3}{2}}(w) =& \sqrt{(\frac{\pi}{2w})} \exp(-w)\Big(1+\frac{1}{w}\Big) \\
K_{\frac{5}{2}}(w) =& \sqrt{(\frac{\pi}{2w})} \exp(-w)\Big(1+\frac{1}{w}+\frac{1}{w^2}\Big)
\end{aligned}
$$

1.6.3 The Basel Problem

The Basel problem that has been used to simplify expressions in this research is stated as follows.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{1.34}$$

It shall be mainly applied to expansion of sequences and series in charpter two.

1.7 Significance of the Study

The results of the study will contribute to the construction of distributions through generalization. In particular, the generalization techniques used are the mixture and beta generated methods. The generalized logistic distributions shall improve flexibility when fitting data . The study has also helped in identifying a pattern in the construction of generalized logistic distributions through reviewing methods of constructing the standard logistic.

The study has also made use a practical application of the cdf of the standard logistic in credit scoring. Credit scoring is very useful in finance and for determining credit worthiness of individuals by lending institutions.

A graph of the logistic scale and location parameter, standard logistic and normal distribution indicated that the logistic distribution provided a better fit for data than the normal and the standard logistic distribution. The study has also introduced new distributions like the Logistic Inverse Gaussian and the Extended Standard Logistic. This distributions provide a basis for further investigation and research.

1.8 Outline of the Thesis

The rest of the thesis is organized as outlined below:
In chapter 2: Various methods of constructing the standard logistic have been shown explicitly. The moments have been derived alongside the first four derivatives. A practical application of the standard logistic to credit scoring has also been shown in logistic regression

In chapter 3: Various generalized logistic distributions have been constructed based on methods of constructing the standard logistic. The generalized logistic distributions are GLI,$GLII$,$GLIII$ and $GLIV$.Their extended versions have also been obtained.

In chapter 4: Generalized logistic distributions have been constructed based on the beta distributions and their generalizations. The new distribution the "extended standard logistic distribution" has also been obtained using beta distributionsand other methods of construction

Charpter 5: Discusses discrete and continuous mixtures of minimum order statistics from a standard logistic distribution. New distributions have been proposed. The truncated power series distributions have been used as mixing distributions. The shapes of the resulting distributions have been simulated

Chapter 6: Discusses the continuous mixtures of the logistic distributions and their moments. Of particular interest is the logistic inverse Gaussian distribution and its properties which has been constructed and studied in detail

Charpter 7: Gives the summary, conclusions and recommendations of the study.

2 CONSTRUCTION AND MOMENTS OF A STANDARD LOGISTIC DISTRIBUTION

2.1 Introduction

The main objective of the charpter is to construct the standard logistic distribution based on four different methods. The methods are based on;

(i) The difference of two standard Gumbel distributed random variables.

(ii) Burr differential equation.

(iii) Transformations.
and

(iv) Mixtures.

Moments of the standard logistic have also been derived by direct method and by using the moment generating function (mgf) technique.

The charpter also shows the application of the cdf of the logistic distribution to data. Logistic regression has been used to calculate the probability of default using the data.

The organization of this chapter is as follows; Section 2.2 states the results, followed by constructions based on Gumbel distribution, Burr differential equation, transformations and mixtures in sections $2.3,\ 2.4,\ 2.5$ and 2.6 respectively. The even and odd moments of the standard logistic have also been evaluated including the moment generating function.

2.2 The distribution and moments of a standard logistic random variable

Let Z be a standard logistic random variable. Then;

(i) The pdf is given by,

$$\begin{aligned} g(z) &= \frac{e^{-z}}{(1+e^{-z})^2} \\ &= \frac{e^{z}}{(1+e^{z})^2} \end{aligned} \quad \textbf{(2.1)}$$

(ii) The cdf is;

$$G(z) = \frac{1}{1+e^{-z}}, \quad -\infty < z < \infty \quad \textbf{(2.2)}$$

(iii) The survival function is

$$1 - G(z) = \frac{1}{1+e^{z}}, \quad -\infty < z < \infty \tag{2.3}$$

(iv) The hazard function is

$$h(z) = \frac{g(z)}{1-G(z)} = \frac{1}{1+e^{-z}} \tag{2.4}$$

(v)

$$E(Z) = 0 \tag{2.5}$$

(vi)

$$Var\ (Z) = E(Z^2) = \frac{\pi^2}{3} \tag{2.6}$$

(vii)

$$\mu_3 = E(Z-\mu)^3 = E(Z^3) = 0 \tag{2.7}$$

(viii)

$$\begin{aligned} \mu_4 =& E(Z-\mu)^4 \\ =& E(Z^4) \\ =& \frac{7\pi^4}{15} \end{aligned} \tag{2.8}$$

2.3 The standard logistic distribution based on the difference of two standard Gumbel random variables

Let

$$Z = X_1 - X_2$$

where X_1 and X_2 are independent random variables from a standard Gumbel distribution whose cdf is given by;

$$F(x) = \exp(-e^{-x}), \quad -\infty < x < \infty \tag{2.9}$$

The pdf is;

$$f(x) = e^{-x}\exp(-e^{-x}), \quad -\infty < x < \infty \tag{2.10}$$

Let $G(z)$ be the cdf of Z, then

$$\begin{aligned} G(z) =& \mathrm{Prob}\{Z \le z\} \\ =& \mathrm{Prob}\{X_1 - X_2 \le z\} \\ =& \mathrm{Prob}\{X_1 \le z + X_2\} \\ G(z) =& \mathrm{Prob}\{X_1 \le z + X_2,, \quad -\infty < X_2 < \infty\} \end{aligned}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z+x_2} f_1(x) f_2(x) dx_1 dx_2$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{z+X_2} f_1(x) dx_1 \right\} f_2(x) dx_2$$

$$= \int_{-\infty}^{\infty} F_1(z+x_2) f_2(x) dx_2$$

$$= \int_{-\infty}^{\infty} \exp(-e^{-(z+x_2)}) e^{-x_2} \exp(-e^{-x_2}) dx_2$$

$$= \int_{-\infty}^{\infty} e^{-x_2} \exp\{-e^{z+x_2} - e^{-x_2}\} dx_2$$

$$= \int_{-\infty}^{\infty} e^{-x_2} \exp\{-e^{x_2}(e^{-z}+1)\} dx_2$$

$$\text{Let } y = e^{-x_2} \implies dy = -e^{x_2} dx_2$$

$$G(z) = \int_{0}^{\infty} \exp\{-(e^{-z}+1)y\} dy$$

$$= \int_{0}^{\infty} y^{1-1} e^{-(e^{-z}+1)y} dy$$

$$\therefore G(z) = \frac{1}{e^{-z}+1}, -\infty < z < \infty$$

$$\implies g(z) = \frac{d}{dz} G(z) = \frac{e^{-z}}{(1+e^{-z})^2}, \ -\infty < z < \infty \qquad \textbf{(2.11)}$$

which is the standard logistic distribution as obtained by Gumbel (1944).

2.4 The standard logistic distribution based on Burr differential equation

Burr (1942) considered the differential equation;

$$y' = y(1-y)g(x,y) \qquad \textbf{(2.12)}$$

where y' is the cdf and $g(x,y)$ is a function of x and y. The cdf of standard logistic is obtained by solving the special case when $g(x,y) = 1$. Thus, $y = F(x)$ which is the cdf of standard logistic. The proof follows. Letting,

$$g(x,y) = g(x)$$

the differential equation becomes

$$
\begin{aligned}
& y' = y(1-y)g(x) \\
& \therefore \frac{1}{y(1-y)}\frac{dy}{dx} = g(x) \\
& \therefore \int \frac{dy}{y(1-y)} = \int g(x)dx \\
& \therefore \int \left(\frac{1}{y} + \frac{1}{1-y}\right) \\
& \therefore \int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \int g(x)dx \\
& \therefore \log y - \log(1-y) = \int g(x)dx \\
& \therefore \log(\frac{y}{1-y}) = \int g(x)dx \\
& \therefore \frac{y}{1-y} = e^{\int g(x)dx} \\
& y = e^{\int g(x)dx} - ye^{\int g(x)dx} \\
& \therefore y(1+e^{\int g(x)dx}) = e^{\int g(x)dx} \\
& \therefore y = \frac{e^{g(x)dx}}{1+e^{g(x)}dx} \\
& = \frac{1}{1+e^{-\int g(x)dx}} \\
& F(x) = [1+e^{-\int g(x)dx}]^{-1}
\end{aligned}
\quad \textbf{(2.13)}
$$

If $g(x) = 1$, then

$$
\begin{aligned}
F(x) =& [1+e^{-x}]^{-1} \\
=& \frac{1}{1+e^{-x}} \\
F(x) =& \ 0 \implies x = -\infty \\
F(x) =& \implies x = \infty \\
\therefore F(x) =& \frac{1}{1+e^{-x}}, \quad -\infty < x < \infty
\end{aligned}
\quad \textbf{(2.14)}
$$

2.5 The standard Logistic Distribution Based on Transformations.

2.5.1 Using Uniform distribution for the old variable.

Let

$$Z = -\ln(\frac{U}{1-U}) \tag{2.15}$$

where U has a uniform distribution in $[0,1]$

$$\begin{aligned}
&\therefore \frac{u}{1-u} = e^{-z} \\
&\therefore u = e^{-z} - ue^{-z} \\
&\therefore u(1+e^{-z}) = e^{-z} \\
&\therefore u = \frac{e^{-z}}{1+e^{-z}} = \frac{1}{1+e^{z}} \\
&\therefore \frac{dy}{dx} = \frac{e^{z}}{(1+e^{z})^2} \\
&\therefore g(z) = f(u)\left|\frac{du}{dz}\right| \\
=&1 \cdot \frac{e^{z}}{(1+e^{z})^2} \\
=&\frac{e^{z}}{(1+e^{z})^2}, \quad -\infty < z < \infty
\end{aligned}$$

Alternatively, let

$$Z = -\ln\left(\frac{e^{-X}}{1-e^{-X}}\right) \tag{2.16}$$

where X is the exponential random variable, that is, X is exponential with mean 1.

$$\begin{aligned}
\therefore \frac{e^{-x}}{1+e^{-x}} =& e^{-z} \\
\therefore e^{-x} =& e^{-z} - e^{-x}e^{-z} \\
\therefore e^{-x}(1+e^{-z}) =& e^{-z} \\
\therefore e^{-x} =& \frac{e^{-z}}{1+e^{-z}} \\
=& \frac{1}{1+e^{z}} \\
\therefore e^{x} =& 1+e^{z} \\
\therefore e^{x}\frac{dx}{dz} =& e^{z}
\end{aligned}$$

$$\begin{aligned}
\therefore \frac{dx}{dz} =& \frac{e^z}{1+e^z} \\
\therefore g(z) =& f(x)\frac{dx}{dz} \\
=& e^{-x}\frac{e^z}{1+e^z} \\
=& \frac{1}{1+e^z}\frac{e^z}{1+e^z} \\
=& \frac{e^z}{(1+e^z)^2}, \quad -\infty < z < \infty
\end{aligned}$$

2.5.2 Using the standard Laplace Distributed Random variable as the old variable

Let

$$Y = -\ln\left(\frac{\frac{1}{2}e^{-X}}{1-\frac{1}{2}e^{-X}}\right) \tag{2.17}$$

where X is a standard Laplace distributed random variable; i.e.

$$\begin{aligned}
f(x) =& \frac{1}{2}e^{-|x|}, \quad x > 0 \\
\therefore \frac{\frac{1}{2}e^{-x}}{1-\frac{1}{2}e^{-x}} &= e^{-y} \\
\therefore \frac{1}{2}e^{-x} &= e^{-y} - \frac{1}{2}e^{-x}e^{-y} \\
\therefore \frac{1}{2}e^{-x}(1+e^{-y}) &= e^{-y} \\
\therefore \frac{1}{2}e^{-x} &= \frac{e^{-y}}{1+e^{-y}} \\
e^{-x} &= \frac{2e^{-y}}{1+e^{-y}} \\
\frac{1}{e^x} &= \frac{2}{1+e^{-y}} \\
\therefore e^x &= \frac{1+e^y}{2} \\
\therefore x &= \ln\left(\frac{1+e^y}{2}\right) \\
\therefore \frac{dx}{dy} &= \frac{2e^y}{1+e^y}\frac{1}{2} = \frac{e^y}{1+e^y} \\
\therefore g(y) =& f(x)\left|\frac{dx}{dy}\right| \\
&= \frac{1}{2}e^{|x|}\frac{e^y}{1+e^y}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} e^{-|\ln(\frac{1+e^y}{2})|} \frac{e^y}{1+e^y} \\
&= \frac{1}{2} e^{\ln(\frac{1+e^y}{2})^{-1}} \frac{e^y}{1+e^y} \\
&= \frac{1}{2} \left(\frac{1+e^y}{2} \right)^{-1} \frac{e^y}{1+e^y} \\
&= \frac{1}{2} \frac{2}{1+e^y} \cdot \frac{e^y}{1+e^y} \\
&= \frac{e^y}{(1+e^y)^2}, \quad -\infty < y < \infty
\end{aligned}
$$

2.5.3 Using Pareto distributed random variable as the old variable

A Pareto I random variable has pdf given by;

$$
f(x) = \frac{\alpha \beta^\alpha}{x^{\alpha+1}}, \qquad x > \beta > 0; \quad \alpha > 0 \tag{2.18}
$$

Put $\alpha = p$ and $\beta = 1$

$$
f(x) = \frac{p}{x^{p+1}}, \quad x > 1; \quad p > 0 \tag{2.19}
$$

Now let

$$
\begin{aligned}
y =& -\ln(x^p - 1) \\
x^p - 1 =& e^{-y} \\
\therefore x =& (1+e^{-p})^{\frac{1}{p}} \\
\frac{dx}{dy} =& -\frac{1}{p}(1+e^{-y})^{\frac{1}{p}-1} e^{-y} \\
g(y) =& f(x) \left| \frac{dx}{dy} \right| \\
=& \frac{p}{x^{p+1}} \frac{1}{p} (1+e^{-y})^{\frac{1}{p}-1} e^{-y} \\
=& \frac{(1+e^{-y})^{\frac{1}{p}-1}}{(1+e^{-y})^{\frac{1}{p}(p+1)}} e^{-y} \\
\therefore g(y) =& \frac{(1+e^{-y})^{\frac{1}{p}-1}}{(1+e^{-y})^{1+\frac{1}{p}}} e^{-y} \\
=& \frac{e^{-y}}{(1+e^{-y})^2}, \quad -\infty < y < \infty
\end{aligned}
$$

2.6 Standard Logistic distribution based on mixtures

2.6.1 Gumbel mixture of standard exponential

Let

$$Y = e^{-X}$$

where Y is an exponential distribution with mean 1, that is.

$$g(y) = e^{-y}, \quad y > 0$$

Therefore the pdf of X is

$$\begin{aligned} f(x) =& g(y)\left|\frac{dy}{dx}\right| \\ =& e^{-y}e^{-x} \\ =& e^{-e^{-x}}e^{-x} \\ =& e^{-x}\exp(-e^{-x}), \quad -\infty < x < \infty \end{aligned}$$

which is called a standard Gumbel distribution or Type I extreme value distribution.

Remark 2.6.1. *The standard Gumbel distribution has no parameter.*

To introduce a parameter, let us consider an exponential distribution with mean 1, so that;

$$g(y) = \lambda e^{-\lambda y}, \quad y > 0; \lambda > 0 \tag{2.20}$$

If

$$Y = e^{-X}$$

then,

$$\begin{aligned} f(x) =& \lambda e^{\lambda y}\left| - e^{-x}\right| \\ =& \lambda e^{-\lambda e^{-x}}e^{-x} \\ =& \lambda e^{-x}\exp(-\lambda e^{-x}), \quad -\infty < x < \infty; \quad \lambda > 0 \end{aligned}$$

This is a Gumbel distribution with parameter λ.

Suppose

$$\begin{cases} f(x|\lambda) = \lambda e^{-x}\exp(-\lambda e^{-x}) \\ g(\lambda) = e^{-\lambda}, \quad \lambda > 0 \end{cases} \tag{2.21}$$

Then

$$\begin{aligned} f(x) &= \int_0^\infty f(x|\lambda)g(\lambda)d\lambda \\ &= \int_0^\infty \lambda e^{-x}\exp(-\lambda e^{-x})g(\lambda)d\lambda \end{aligned}$$

If $g(\lambda)$ is an exponential distribution with mean 1, then,

Then

$$f(x) = \int_0^\infty \lambda e^{-x}[\exp(-\lambda e^{-x})]e^{-\lambda}d\lambda \tag{2.22}$$

$$\begin{aligned} &= e^{-x}\int_0^\infty \lambda e^{-\lambda e^{-x}-\lambda}d\lambda \\ &= e^{-x}\int_0^\infty \lambda e^{-(e^{-x}+1)\lambda}d\lambda \\ &= e^{-x}\int_0^\infty \lambda e^{-(e^{-x}+1)\lambda}d\lambda \\ &= e^{-x}\int_0^\infty \lambda^{2-1}e^{-(1+e^{-x})\lambda}d\lambda \\ &= e^{-x}\frac{\Gamma(2)}{(1+e^{-x})^2} \\ &= \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty \end{aligned}$$

which is a standard logistic pdf.

Another Gumbel distribution with parameter λ is obtained by letting

$$x = -z \implies \frac{dx}{dz} = -1$$

where the pdf of X is

$$f(x) = \lambda e^{-x}\exp(-\lambda e^{-x}), \quad -\infty < x < \infty; \quad \lambda > 0 \tag{2.23}$$

Therefore the pdf of Z is

$$\begin{aligned} g(z) &= f(x)\left|\frac{dx}{dy}\right| \\ &= \lambda e^{-x}\exp(-\lambda e^{-x}) \\ &= \lambda e^{z}\exp(-\lambda e^{z}), \quad -\infty < z < \infty; \quad \lambda > 0 \end{aligned} \tag{2.24}$$

Considering

$$g(z|\lambda) = \lambda e^{z} \exp(-\lambda e^{z}) \tag{2.25}$$

Then

$$\begin{aligned} g(z) &= \int_{0}^{\infty} \lambda e^{z} \exp(-\lambda e^{z}) g(\lambda) d\lambda \\ &= \int_{0}^{\infty} \lambda e^{z} [\exp(-\lambda e^{z})] e^{-\lambda} d\lambda \\ &= e^{z} \int_{0}^{\infty} \lambda e^{-\lambda e^{z} - \lambda} d\lambda \\ \therefore g(z) &= e^{z} \int_{0}^{\infty} \lambda e^{-(1+e^{z})\lambda} d\lambda \\ &= e^{z} \frac{\Gamma(2)}{(1+e^{z})^2} \\ &= \frac{e^{z}}{(1+e^{z})^2}, \quad -\infty < z < \infty \end{aligned}$$

which is a standard logistic pdf.

2.7 Moments

The pdf of standard logistic distribution is given by

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty$$

which can also be written as

$$f(x) = \frac{e^{x}}{(1+e^{x})^2}, \quad -\infty < x < \infty$$

we shall use both expressions in determining moments.

It suffices to write

$$\begin{aligned} E(X^{2r+1}) &= \int_{-\infty}^{\infty} x^{2r+1} f(x) dx, \quad r = 0,1,2,3,... \\ &= \int_{-\infty}^{0} x^{2r+1} f(x) dx + \int_{0}^{\infty} x^{2r+1} f(x) dx, \quad r = 0,1,2,3,... \qquad (2.26) \\ &= \int_{-\infty}^{0} x^{2r+1} \frac{e^{x}}{(1+e^{x})^2} dx + \int_{0}^{\infty} x^{2r+1} \frac{e^{-x}}{(1+e^{-x})^2} dx, \quad r = 0,1,2,3,... \end{aligned}$$

Let

$$x=-y \implies dx=-dy$$

$$\therefore \int_{-\infty}^{0} x^{2r+1}\frac{e^x}{(1+e^x)^2}dx = \int_{\infty}^{0}(-y)^{2r+1}\frac{e^{-y}}{(1+e^{-y})^2}(-dy), \quad r=0,1,2,3,...$$

$$\int_{-\infty}^{0} x^{2r+1}\frac{e^x}{(1+e^x)^2}dx = \int_{0}^{\infty}(-y)^{2r+1}\frac{e^{-y}}{(1+e^{-y})^2}dy, \quad r=0,1,2,3,...$$

$$= -\int_{0}^{\infty} y^{2r+1}\frac{e^{-y}}{(1+e^{-y})^2}dy, \quad r=0,1,2,3,...$$

$$\therefore E(X^{2r+1}) = -\int_{0}^{\infty} y^{2r+1}\frac{e^{-y}}{(1+e^{-y})^2}dy + \int_{0}^{\infty}\frac{x^{2r+1}e^{-x}}{(1+e^{-x})^2}dx, \quad r=0,1,2,3,...$$

$$= \int_{0}^{\infty}\frac{x^{2r+1}e^{-x}}{(1+e^{-x})^2}dx - \int_{0}^{\infty}\frac{y^{2r+1}e^{-y}}{(1+e^{-y})^2}dy, \quad r=0,1,2,3,...$$

$$\therefore E(X^{2r+1}) = 0 \text{ for } r=0,1,2\ldots$$

$$\therefore E(X) = E(X^3) = E(X^5) = 0 \tag{2.27}$$

that is, moments of odd powers are zero. Next, let us now consider

$$E(X^{2r}) = \int_{-\infty}^{\infty} x^{2r}f(x)dx$$

$$= \int_{-\infty}^{0} x^{2r}\frac{e^x}{(1+e^x)^2}dx + \int_{0}^{\infty} x^{2r}\frac{e^x}{(1+e^x)^2}dx$$

Let

$$x=-y \implies dx=-dy$$

$$\therefore \int_{-\infty}^{0} x^{2r}\frac{e^x}{(1+e^x)^2}dx = \int_{0}^{\infty} y^{2r}\frac{e^{-y}}{(1+e^{-y})^2}dy$$

$$E(X^{2r}) = \int_{0}^{\infty}\frac{y^{2r}e^{-y}}{(1+e^{-y})^2}dy + \int_{0}^{\infty}\frac{x^{2r}e^{-x}}{(1+e^{-x})}dx$$

$$=2\int_{0}^{\infty}\frac{x^{2r}e^{-x}}{(1+e^{-x})}dx$$

$$=2\int_{0}^{\infty} x^{2r}e^{-x}(1+e^{-x})^{-2}dx$$

$$
\begin{aligned}
&=2\int_0^\infty \left\{x^{2r}e^{-X}\sum_{k=0}^{\infty}\binom{-2}{k}e^{-kx}\right\}dx \\
&=2\sum_{k=0}^{\infty}\left\{\binom{-2}{k}\int_0^\infty x^{2r}e^{-(k+1)x}\right\}dx \\
&=2\sum_{k=0}^{\infty}\left\{\binom{-2}{k}\int_0^\infty x^{(2r+1)-1}e^{-(k+1)x}\right\}dx \\
&=2\sum_{k=0}^{\infty}\left\{\binom{-2}{k}\frac{\Gamma(2r+1)}{(k+1)^{2r+1}}\right. \\
&=2(2r)!\sum_{k=0}^{\infty}\left\{\binom{-2}{k}\frac{1}{(k+1)^{2r+1}}\right. . \\
\therefore E(X^{2r}) &=2(2r)!\sum_{k=0}^{\infty}(-1)^k\binom{2+k-1}{k}\frac{1}{(k+1)^{2r+1}} \\
&=2(2r)!\sum_{k=0}^{\infty}(-1)^k\binom{k+1}{k}\frac{1}{(k+1)^{2r+1}} \\
&=2(2r)!\sum_{k=0}^{\infty}(-1)^k(k+1)\frac{1}{(k+1)^{2r+1}} \\
&=2(2r)!\sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+1)^{2r}} \qquad \textbf{(2.28)}
\end{aligned}
$$

Let

$$k+1=n \implies k=n-1$$

$$
\begin{aligned}
\therefore E(X^{2r}) &=2(2r)!\sum_{k=1}^{\infty}(-1)^{k+2}\frac{1}{(k+1)^{2r}} \\
&=2(2r)!\sum_{k=0}^{\infty}(-1)^{n+1}\frac{1}{n^{2r}} \\
E(X^{2r}) &=2(2r)!\left\{\frac{1}{1^{2r}}-\frac{1}{2^{2r}}+\frac{1}{3^{2r}}-\frac{1}{4^{2r}}\pm\dots\right\} \qquad \textbf{(2.29)}
\end{aligned}
$$

In particular,

$$
\begin{aligned}
E(X^2) &=4\left\{\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+...+(-1)^{n+1}\frac{1}{n^2}+\dots\dots\right\} \\
E(X^4) &=2(4!)\left\{\frac{1}{1^4}-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}+...+(-1)^{n+1}\frac{1}{n^4}+\dots\dots\right\} \\
&=2(4!)\frac{\pi^4}{90}=\frac{7}{15}\pi^4 \qquad \textbf{(2.30)}
\end{aligned}
$$

Proof.

$$
\begin{aligned}
E(X) &= \int\limits_{-\infty}^{\infty} x f(x) dx \\
&= \int\limits_{-\infty}^{\infty} \frac{xe^x}{(1+e^x)^2} dx \\
w &= (1+e^x)^{-1} \\
w &= (1+e^x)^{-1} \\
x &= ln(\frac{w}{1-w}) \\
&= \int\limits_0^1 ln(\frac{w}{1-w}) dw = \int\limits_0^1 ln(w) dw - \int\limits_0^1 ln(1-w) dw \\
&= \int\limits_0^1 ln(w) dw - \int\limits_0^1 ln(w) dw \\
&= 0
\end{aligned}
$$

The standard logistic distribution has a mean 0. The variance is equal to $E(X^2)$ because the mean is zero

$$
\begin{aligned}
Var(X) = E(X^2) &= \int\limits_{-\infty}^{\infty} \frac{x^2 e^{-x}}{(1+e^x)^2} dx \\
&= \int\limits_{-\infty}^{0} \frac{x^2 e^{-x}}{(1+e^x)^2} dx + \int\limits_{0}^{\infty} \frac{x^2 e^{-x}}{(1+e^x)^2} dx \\
&= 2\int_0^{\infty} \frac{x^2 e^{-x}}{(1+e^x)^2} dx \\
&= 2\int_0^{\infty} x^2 \sum_{n=1}^{\infty} (-1)^{n+1} n e^{-nx} dx \\
&= 2\sum_{n=1}^{\infty} (-1)^{n+1} n \int_0^{\infty} x^2 e^{-nx} dx
\end{aligned}
\tag{2.31}
$$

to solve the part under the integral,

$$
\begin{aligned}
&let; \\
u &= x^2 \implies du = 2x dx \\
dv &= e^{-nx} \implies v = -\frac{1}{n} e^{-nx}
\end{aligned}
$$

$$= 2\sum_{n=1}^{\infty}(-1)^{n+1}n[-\frac{x^2e^{-nx}}{n} + \frac{2}{n}\int_0^{\infty} xe^{-nx}]dx$$
$$= 4\sum_{n=1}^{\infty}(-1)^{n+1} - [\frac{xe^{-nx}}{n}.\Big|_0^{\infty} + \frac{1}{n}\int_0^{\infty} e^{-nx}]dx$$

integrating the second part of the equation by parts, let;

$$u = x, du = dx$$
$$dv = e^{-nx}dx, v = -\frac{1}{n}e^{-nx}$$
$$= 4\sum_{n=1}^{\infty}(-1)^{n+1}(\frac{1}{n})[-\frac{e^{-nx}}{n}\Big|_0^{\infty}]$$
$$= 4\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$$
$$= [\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + ...]$$
$$= 4\Big[\{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + ...\} + \frac{\pi^2}{12} - \frac{\pi^2}{12}\Big]$$
$$= 4\Big[\frac{\pi^2}{6} - \frac{\pi^2}{12}\Big]$$
$$= 4\Big[\frac{\pi^2}{12}\Big]$$
$$= \frac{\pi^2}{3}$$

Compared to the normal distribution, the variance of the logistic is different from that of the normal only by a scaling value of $\frac{\pi^2}{3}$. The expectation is equal to the location parameter μ;

$$E(X) = \mu$$

Proposition 2.7.1. *The non central and central moments are equal. Hence*

$$\mu_1 = E(X) = 0$$

as shown in 2.61 *above*

$$\mu_2 = Var(X) = E(X^2) = \frac{\pi^2}{3}$$

as shown in 2.61 *above*

$$\mu_3 = E(X - \mu_1)^3 = E(X^3) = 0$$
$$\mu_4 = E(X - \mu_1)^4 = E(X^4) = \frac{7\pi^4}{15} = 0$$

Gupta and Kundu (2010)discovered that the first four central moments of the standard logistic distribution can be given as;

$$\mu_1 = 0,\ \mu_2 = \frac{\pi^2}{3},\ \mu_3 = 0,\ \mu_4 = \frac{7\pi^4}{15}$$

Since it is symmetric about 0, the reversed hazard rate at x is the hazard rate at $-x$. Therefore the reversed hazard rate $r(x)$ and the survival rate $S(x)$; are identical moreover for $-\infty < x < \infty$

$$\begin{aligned}
Var(X) = E(X^2) =& \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} \pm \dots \\
\therefore Var(X) =& \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2}\right) - \frac{2}{2^2} - \frac{2}{4^2} - \frac{2}{6^2} - \dots \\
=& \frac{\pi^2}{6} - 2\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots\right) \\
=& \frac{\pi^2}{6} - \frac{2}{2^2}\left(\frac{\pi^2}{6}\right) \\
=& \left(1 - \frac{2}{2^2}\right)\frac{\pi^2}{6} \\
=& \frac{1}{2}\cdot\frac{\pi^2}{6} \\
Var(X) =& \frac{\pi^2}{3} \\
E[X-\mu]^4 =& E(X^4) \\
=& 2(4!)\left(\frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} \pm \dots\right) \\
=& 2(4!)\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} \pm \dots\right) - 2\times 2(4!)\left(\frac{1}{2^2} + \frac{1}{4^4} + \frac{1}{6^4} + \dots\right)2(4!) \\
=& \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} \pm \dots\right) - 2\left(\frac{1}{1^4 2^4} + \frac{1}{2^4 2^4} + \frac{1}{2^4 3 4} + \dots\right) \\
=& \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} \pm \dots\right) - \frac{2}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} \pm \dots\right) \\
=& \frac{\pi^4}{90} - \frac{2}{2^4}\frac{\mathfrak{p}^4}{90} \\
=& \left(1 - \frac{1}{2^3}\right)\frac{\pi^4}{90} \\
=& \frac{7}{8}\cdot\frac{\pi^4}{90}(2\cdot 4!) \\
=& \frac{7}{15}\pi^4
\end{aligned} \tag{2.31}$$

□

2.8 Moment Generating Function (MGF)

$$
\begin{aligned}
M_X(t) =& E[e^{tX}] \\
=& \int_{-\infty}^{\infty} \frac{e^{tx}e^{x}}{(1+e^{x})^2}dx \\
& \textit{let} \\
y =& (1+e^{x})^{-1} \\
\therefore dy =& -(1+e^{x})^{-2}e^{x}dx \\
& \textit{and} \\
1+e^{x} =& y-1 \\
e^{x} =& \frac{1}{y}-1 = \frac{1-y}{y} \\
M_X(t) =& \int_{1}^{0} \left(\frac{1-y}{y}\right)^{t}(-dy) \\
=& \int_{0}^{1} \left(\frac{1-y}{y}\right)^{t}dy \\
=& \int_{0}^{1} y^{-t}(1-y)^{t}dy \\
\therefore M_X(t) =& \int_{0}^{1} y^{1-t-1}(1-y)^{1+t-1}dy \\
=& \Gamma(1-t)\Gamma(1+t) \\
=& t\Gamma(t)\Gamma(1-t) \\
=& t\left[\frac{\pi}{\sin \pi t}\right] \\
M_X(t) =& \frac{\pi t}{\sin(\pi t)},\ -1<t<1
\end{aligned}
\tag{2.32}
$$

since $1-t>0 \implies 1>t$ and $1+t>0 \implies t<1$ and $1+t>0 \implies 1>-t \implies -1<t$ and therefore $-1<t<1$.

The first four derivatives of the mgf of a standard logistic distribution are stated by the following propositions.

Proposition 2.8.1.

$$
\begin{aligned}
M_X^{'}(t) =& \pi\left\{\frac{sin\pi t - cos\pi t}{sin^2\pi t}\right\} \\
=& -\pi[\pi t cot(\pi t)-1]cosec(\pi t)
\end{aligned}
$$

PROOF.

$$\begin{aligned}
\frac{d}{dt}\left[\frac{\pi t}{sin(\pi t)}\right] =& \pi\frac{d}{dt}\left[\frac{t}{sin\pi t}\right] \\
=& \frac{\pi\frac{d}{dt}(t)sin(\pi t) - t\frac{d}{dt}sin(\pi t)}{sin^2\pi t} \\
=& \frac{\pi sin(\pi t) - cos\pi t\frac{d}{dt}(t).t}{sin^2(\pi t)} \\
=& \frac{\pi\left[sin(\pi t) - \pi tcos(\pi t)\right]}{sin^2\pi t} \\
=& \frac{\pi\left[sin(\pi t) - \pi tcos(\pi t)\right]}{sin^2(\pi t)} \\
=& -\pi[\pi tcot(\pi t) - 1]cosec(\pi t)
\end{aligned} \tag{2.33}$$

□

Proposition 2.8.2.

$$\begin{aligned}
M''_X(t) =& \pi^2\left\{\frac{\pi t}{\sin(\pi t)} - \frac{2\cos(\pi t)[sin(\pi t) - \pi tcos(\pi t)]}{\sin^3(\pi t)}\right\} \\
=& \pi^2 cosec(\pi t)[\pi tcosec^2(\pi t) + \pi ttcot^2(\pi t) + 1) - 2cot(\pi t)]
\end{aligned}$$

PROOF.

$$\begin{aligned}
M''_X(t) =& \frac{d}{dt}\left[\pi\left\{\frac{sin\pi t - cos\pi t}{sin^2\pi t}\right\}\right] \\
=& \pi\left[\frac{\frac{d}{dt}[sin(\pi t) - \pi tcos(\pi t)]sin^2(\pi t) - \left(sin(\pi t) - \pi tcos(\pi t)\frac{d}{dt}sin^2(\pi t)\right)}{\left(sin^2(\pi t)\right)^2}\right] \\
=& \pi\left[\frac{\left(cos(\pi t)\pi t' - \pi(t'cos(\pi t) + t\frac{d}{dt}[cos(\pi t)]sin^2(\pi t) - 2sin^2(\pi t)cos(\pi t)[\pi t'sin(\pi t) - \pi tcos(\pi t)\right)}{sin^4(\pi t)}\right] \\
=& \pi\left[\frac{cos(\pi t)\pi t' - \pi(cos(\pi t) + t(-sin()\pi t)\pi t'))sin^2(\pi t) - 2sin(\pi t)cos(\pi t)\pi t'sin(\pi t) - \pi tcos(\pi t))}{sin^4(\pi t)}\right]
\end{aligned} \tag{2.34}$$

$$=\left[\frac{\pi cos(\pi t)-\pi(cos(\pi t)-t\pi t' sin(\pi t))sin^2(\pi t)-2\pi sin(\pi t)cos(\pi t)sin(\pi t)-\pi tcos(\pi t)}{sin^4(\pi t)}\right]$$
$$=\pi\left[\frac{\pi cos(\pi t)-\pi[cos(\pi t)-\pi tsin(\pi t]sin^2(\pi t))-2\pi cos(\pi t)sin(\pi t)sin(\pi t)-\pi tcos(\pi t)))}{sin^4(\pi t)}\right]$$
$$=\pi\left[\frac{sin^2(\pi t)(\pi cos(\pi t)-\pi(cos(\pi t)-tsin(\pi t)))-2\pi cos(\pi t)sin(\pi t)(sin(\pi t)-\pi tcos(\pi t)))}{sin^4(\pi t)}\right]$$
$$=\frac{\pi^3 t}{\sin(\pi t)}-\frac{2\pi\cos(\pi t)}{\sin^2(\pi t)}-\frac{2t\pi\cos^2(\pi t)}{\sin^3(\pi t)}$$
$$=\pi^2\left\{\frac{\pi t}{\sin(\pi t)}-\frac{2\cos(\pi t)[sin(\pi t)-\pi tcos(\pi t)]}{\sin^3(\pi t)}\right\}$$
$$=\pi^2 cosec(\pi t)[\pi tcosec^2(\pi t)+\pi ttcot^2(\pi t)+1)-2cot(\pi t)]$$

(2.35)

□

Proposition 2.8.3.

$$M_X'''(t)=\frac{\pi^3\left(3\sin^3(\pi t)-5\pi t\cos(\pi t)\sin^2(\pi t)+6\cos^2(\pi t)\sin(\pi t)-6\pi t\cos^3(\pi t)\right)}{\sin^4(\pi t)}$$
$$=\pi^2\csc(\pi t)\left(\pi t\csc^2(\pi t)+\pi t\cot^2(\pi t)-2\cot(\pi t)\right)$$

Proof.

$$M_X'''(t)=\frac{d}{dt}\left\{=\pi^2\left\{\frac{\pi t}{\sin(\pi t)}-\frac{2\cos(\pi t)[sin(\pi t)-\pi tcos(\pi t)]}{\sin^3(\pi t)}\right\}\right\}$$

$$=\frac{2\pi^3(\sin(\pi t)-\pi t\cos(\pi t))}{\sin^2(\pi t)}+\frac{6\pi^3\cos^2(\pi t)(\sin(\pi t)-\pi t\cos(\pi t))}{\sin^4(\pi t)}+\frac{\pi^3}{\sin(\pi t)}-\frac{3\pi^4 t\cos(\pi t)}{\sin^2(\pi t)}$$
$$=\frac{\pi^3\left(3\sin^3(\pi t)-5\pi t\cos(\pi t)\sin^2(\pi t)+6\cos^2(\pi t)\sin(\pi t)-6\pi t\cos^3(\pi t)\right)}{\sin^4(\pi t)}$$
$$=\pi^2\csc(\pi t)\left(\pi t\csc^2(\pi t)+\pi t\cot^2(\pi t)-2\cot(\pi t)\right)$$

(2.36)

□

Proposition 2.8.4.

$$M_X''''(t)=\frac{\pi^4\left(5\pi t\sin^4(\pi t)-20\cos(\pi t)\sin^3(\pi t)+\cos^2(\pi t)\sin^2(\pi t)-\cos^3(\pi t)\sin(\pi t)+24\pi t\cos^4(\pi t)\right)}{\sin^5(\pi t)}$$
$$=-\frac{\pi^4\csc^5(\pi t)(4\sin(4\pi t)-\pi t(\cos(4\pi t)+76\cos(2\pi t)+115)+88\sin(2\pi t))}{8}$$

Proof.

$$M_X''''(t) = \frac{d}{dt}\left\{\frac{\pi^3\left(3\sin^3(\pi t) - 5\pi t\cos(\pi t)\sin^2(\pi t) + 6\cos^2(\pi t)\sin(\pi t) - 6\pi t\cos^3(\pi t)\right)}{\sin^4(\pi t)}\right\}$$

$$= \frac{\pi^3\left(5\pi^2 t\sin^3(\pi t) - 8\pi\cos(\pi t)\sin^2(\pi t) + 8\pi^2 t\cos^2(\pi t)\sin(\pi t)\right)}{\sin^4(\pi t)} - \ldots$$

$$\frac{4\pi^4\cos(\pi t)\left(3\sin^3(\pi t) - 5\pi t\cos(\pi t)\sin^2(\pi t) + 6\cos^2(\pi t)\sin(\pi t) - 6\pi t\cos^3(\pi t)\right)}{\sin^5(\pi t)}$$

$$= \frac{\pi^4\left(5\pi t\sin^4(\pi t) - 20\cos(\pi t)\sin^3(\pi t) + 28\pi t\cos^2(\pi t)\sin^2(\pi t) - 24\cos^3(\pi t)\sin(\pi t) + 24\pi t\cos^4(\pi t)\right)}{\sin^5(\pi t)}$$

$$= -\frac{\pi^4\csc^5(\mathfrak{p}it)\left(4\sin(4\pi t) - \pi t\left(\cos(4\pi t) + 76\cos(2\pi t) + 115\right) + 88\sin(2\pi t)\right)}{8} \quad (2.37)$$

□

2.9 Application of the standard Logistic to Credit Scoring

2.9.1 Introduction

The use of probability distributions in data analysis has gained popularity in the recent times. Notably machine learning has become an important component of application of mathematics. In this section the cdf of the standard logistic distribution will be applied to generate the probability of default in the logistic regression.

Sample data from Mobipesa limited,a mobile lender in Nairobi has been used. The steps involved in arriving at a credit score have been clearly described. The coding has been done on the R platform attached on the various appendices.

2.9.2 Steps in calculating a credit score

(i) **Determining data quality and characteristics.**
Here the data variables must be clearly defined, the data should then be collected and cleaned. Cleaning data means removing outliers, missing variables and checking the correlation between independent variables. Highly correlated variables are discarded. Application programming interface (API) is used to help in mining data characteristics of the borrower from the lenders platform and also other sectors. Borrowers behave differently on repaying bank loans compared to mobile loans. Transactional data is mainly used in this case .

(ii) **Univariate Analysis**
This involves creating bins based on the response variables. Information value of the variables is created . This may include weight of evidence transformation, CRB score, CRB grade. Binning is a powerful process because it helps to eliminate outliers

(iii) **Logistic Stepwise Regression**
In this stage coding language is used preferably R. The R -codes are available in Appendix R_5 toR_5. All the variables must be used. The model is trained to calculate the probabilities. Probabilities of default are obtained, the smaller the PD the better. The main purpose of logistic regression is classification of individuals in different groups. It allows evaluation of multiple explanatory variables.

Dominguez (2011) suggested use of the logit transformation;

$$F(y) = \frac{1}{1+e^{-y}}$$

which is the cdf of the standard logistic distribution.
Let p be the probability of default(bad customer) and $y = \beta_0 + \beta_1 x_1 + ... + \beta_k x_k$ be the linear predictor, then:

$$y = logit(p) = ln\{\frac{p}{1-p}\}$$

and

$$p = \frac{1}{1+e^{-y}}$$

In logistic regression, $P(Y = 1|x) = p$, where x is a predictor variable. In the model, there are k predictor/explanatory variables.

(iv) **Obtaining the credit score.**
The credit score is obtained using the formula;

$$\text{Score} = \text{offset} + \text{factor} \times \ln(\text{odds})$$

$$\text{Score} = a + b \times \log\left(\frac{1 - pd_{\text{selected}}}{pd_{\text{selected}}}\right)$$

where $a =$ offset and $b =$ factor

$$\text{offset} = \text{score} - \{\text{factor} \times \ln(\text{odds})\}$$

$$factor = \frac{pdo}{\ln 2}$$ *which means for a 20-point increase in a score, the points double.*

Its usually the common default value for pdo though users can change it.

Example 2.9.1. *If a company wishes to scale a scorecard where the user wanted the odds of 60: at every 800 points and wanted the points to double every 20 points , that is* $p.d.o = 20$*; then the factor, offsets and the score would calculated as follows;*

Solution

$$Factor = \frac{20}{ln2} = 28.8539$$
$$Offset = 800 - 28.8539\ ln60$$
$$Score = 681.86$$

2.9.3 The Problem

We shall use data from Mobipesa limited, a digital lending financial institution to show logistic regression is used by the institution to reduce default rates and advise on the risk associated with lending to their borrowers. Full data set is available from the appendix

Example 2.9.2. *The data set used from Mobipesa has 33 predictor variables , 1 categorical variable and 32 numerical variables. There is a response variable for each borrower indicating their actual classification (0 = "Good customer" or 1= "Bad customer"). The table below shows all the variables in the data set. The data is for 495 individuals with their identity concealed. The variables that are defined in the table below have been obtained from the data. These variables shall form the design matrix to be used in logistic regression in arriving at the probability of default for each individual borrower.*

Table 2.1. Table of predictor Variables

S/N	Variable	Definition
1	Id	A unique identifier ton identify each borrower
2	CRB Grade	Its a grade between A and F and a special case of Y (Where CRB does not have any data on the borrower)
3	CRB Score	It is a score provided by CRB between 0 and 800.The higher, the better the borrower is.
4	Max	The maximum amount the borrower has ever borrowed globally
5	Min	The minimum amount the borrower has ever borrowed globally
6	Avg	The average amount the borrower has borrowed globally
7	Last	The last amount the borrower has borrowed globally
8	Count Non Performing Account-My sector	The number of non-performing loans within my sector
9	Count-Non Performing Account-Other sectors	The number of non- performing loans within other sectors
10	Count-Non Performing open Accounts-My sector	The number of open non-performing loans-my sector
11	Count-Non Performing open Accounts- Other sectors	The number of open non-performing loans-other sectors
12	Count-Non Performing closed Accounts-My sector	The number of closed non-performing loans-my sector
13	Count-Non Performing closed Accounts-other sectors	The number of closed non-performing loans-other sectors
14	Count- Performing Accounts-my sectors	The number of performing loans-my sector
15	Count- Performing Accounts-other sectors	The number of performing loans-other sectors
16	Count- Performing Accounts closed-my sector	The number of closed performing loans- my sector

17	Count- Performing Accounts closed-other sectors	The number of closed performing loans- other sectors
18	Count- Performing Accounts open-my sector	The number of open performing loans- my sector
19	Count- Performing Accounts open-other sectors	The number of open performing loans-other sectors
20	Credit history-My sector	The length of the borrowers credit history- my sector
21	Credit history-other sectors	The length of the borrowers credit history- other sectors
22	Count performing accounts default history-my sector	The number of performing accounts with default history-my sector
23	Count performing accounts default history-other sectors	The number of performing accounts with default history-other sectors
24	Number of loans borrowed	The total number of loans borrowed on Mobipesa platform
25	Maximum amount borrowed	The maximum amount borrowed on Mobipesa platform
26	Minimum amount borrowed	The minimum amount borrowed on Mobipesa platform
27	Average amount borrowed	The average amount borrowed on Mobipesa platform
28	Number of defaulted loans	The number of defaulted loans on Mobipesa platform
29	Maximum time to payment	The maximum time the borrower has ever taken to pay a loan on Mobipesa platform
30	Average time to payment	The average time the customer takes to pay a loan on Mobipesa platform
31	Total amount defaulted	The total amount the borrower has in default on the mobipesa platform
32	Total amount of loans	The total amount of loans the borrower has ever taken on the Mobipesa platform
33	Borrowers lifespan	The total time the borrower has been on the Mobipesa platform

34	Amountv default ratio	This is the ratio of the total amount defaulted to the total amount of loans
35	def	This is the target variable with 0 or 1 as the input

Table 2.2. Table of Variables for the first three customers

Id	Crb grade	Crb Score	Max	Min	Avg	last
1	FF	534	-	-	-	-
2	EE	604	5,000.00	1,000.00	2,000.00	5,000.00
3	AA	675	7,500.00	7,000.00	7,333.00	7500.00
Id	v1	v2	v3	v4	v5	v6
1	0	0	0	0	0	0
2	0	0	0	1	0	1
3	0	0	0	0	0	0
Id	v7	v8	v9	v10	v11	v12
1	0	2	1	0	0	1
2	0	8	7	1	0	1
3	0	5	3	0	0	2
Id	v13	v14	v15	v16	Number of loans borrowed	Maximum amount borrowed
1	0	50	0	0	13	13,000.00
2	0	40	0	2	1	1,500.00
3	0	18	0	0	1	1,500.00

Id	Minimum amount borrowed	Average amount borrowed	Number of defaulted loans	Maximum time to payment	Average time to payment	Total amount defaulted
1	1,000.00	3769.23	5.00	128.19	35.28	2,000
2	1,500.00	1,500.00	-	20.72	20.72	-
3	5,000.00	5,000.00	1.00	63.09	63.09	-
Id	Total amount of loans	Borrowers lifespan	Amount def ratio	def		
1	49,000.00	999.2743634	0.040816327	1		
1	1,500.00	185.4036111	0	0		
3	5,000.00	193.7046875	0	1		

2.9.4 Data Partitioning

The raw data shall be analyzed using R. To train the model, we have to partition the data into train and test data. The train data is usually allocated a higher value in order to improve the precision of the model. The higher the proportion of the train data, the more likely the model is to be reliable.

In this case the train data shall be 70 percent while the test data is 30 percent. The test data us to test for the model accuracy as the target variable is not defined for reliability in detecting what is default or not.

2.9.5 Bins

The next part is to bin the train data. Each variable is partitioned into different segments and the WOE and total IV calculated, we can get the variables with the highest total IV. This implies that we preselect the variables to be used.In this case we have decided to use variables whose IV is greater than 0.1. Appendix C contains the identified variables. Appendix B has full information.

2.9.6 WOE Transformation

The full train data converted to WOE is in Appendix D. We get data like the one shown below for the first three customers. The results are obtained from the R- script attached on the appendix

Table 2.3. Table showing weight of evidence transformation for the first three customers

def	Crb grade woe	Crb score woe	Max woe	Min woe	Avg woe	last woe	Count non performing account my sector woe	Count non performing account other sectors woe
1	0.34244	0.26386	0.21627	0.121483	0.242955	0.134052	0	-0.0304
1	0.34244	0.26386	0.21627	0.121483	0.242955	0.134052	0	-0.0304
0	0.34244	0.26386	0.21627	-0.0204	0.242017	0.134052	0	-0.0304
Count non performing open accounts my sector woe	Count non performing open accounts other sectors woe	Count non performing closed accounts my-sector woe	Count non performing closed accounts other sector woe	Count performing accounts my sector woe	Count performing accounts other sector woe	Count performing accounts closed my sector woe	Count performing accounts closed other sectors woe	Count performing accounts open my sector woe
0	-0.0311	0	0.00597	0.013594	0.294823	0.294326	0.01322	0.01244
0	-0.0311	0	0.00597	0.013594	0.294823	0.294326	0.01322	0.01244
0	-0.0311	0	0.00597	0.013594	0.074963	0.25354	0.01322	0.01244

Count performing open accounts open other sector woe	Credit history My sector woe	Credit history other sectors woe	Count performing accounts default history mysector woe	Count performing accounts default history other sectors woe	Number of loans borrowed woe	Maximum amount borrowed woe	Minimum amount borrowed woe	Average amount borrowed woe
0.29007	-0.01782	0.323424	0	-0.002	1.154883	1.104582	0.76811	0.991254
0.29007	-0.01782	0.323424	0	-0.002	1.154883	0.383968	-0.10197	0.332792
0.29007	-0.01782	-0.07164	0	-0.002	0.18973	0.45971	-0.10197	0.93664

2.9.7 Logistics Modeling

Next, we train the logistic model using the transformed outcomes. The following results are obtained.

Table 2.4. Table showing results of logistic modeling using train data

Term	Estimate	Std error	Statistic	p-value
(Intercept)	0.144	0.191	0.751	0.453
Crb score woe	1.283	0.522	2.458	0.014
Max- woe	0.983	0.476	2.067	0.039
Min- woe	0.947	0.42	2.258	0.024
Count performing accounts closed - my sector woe	1.005	0.527	1.906	0.057
Count performing accounts closed -Other sectors woe	-2.029	0.838	-2.422	0.015

Number of loans borrowed woe	0.973	0.317	3.073	0.002
Minimum amount borrowed woe	0.975	0.661	1.474	0.14
Average amount borrowed woe	0.951	0.371	2.564	0.01
Number of defaulted loans woe	2.582	0.58	4.451	0
Maximum time to payment woe	1.395	0.21	6.649	0
Total amount of loans borrowed woe	0.841	0.436	1.928	0.054
Borrower lifespan woe	0.661	0.411	1.608	1.108
Amount default ration woe	0.611	0.195	3.132	0.002

2.9.8 Test Data

Now, we have modeled data using the train data and have a trained.We need to use the test data to score it using the trained model. The test data needs to be transformed into WOE.

2.9.9 Train P.D

Using the trained logistic model, and the transformed test data into WOE, we can now get the probability of default for the test data. The full data PD are in Appendix F. The table below shows the calculations of probability of default for the first customer in the test data.

Table 2.5. Table showing parameter estimates for the first customer using Test data

Term	Estimate	Customer 1	
(Intercept)	0.144		0.144
Crb score woe	1.283	0.263863566	0.338536956
Max- woe	0.983	0.216276572	0.21259987
Min- woe	0.947	0.121483054	0.115044452
Count performing accounts closed - my sector woe	1.005	-0.253541028	-0.254808734
Count performing accounts closed -Other sectors woe	-2.029	-0.220882345	0.448170277
Number of loans borrowed woe	0.973	-1.283160446	-1.248515114
Minimum amount borrowed woe	0.975	-0.435862585	-0.424966021
Average amount borrowed woe	0.951	-0.936637873	-0.890742617
Number of defaulted loans woe	2.582	0.000855066	0.002207781
Maximum time to payment woe	1.395	0.74705681	1.04214425
Total amount of loans borrowed woe	0.841	-0.010803783	-0.009085982
Borrower lifespan woe	0.661	0.119414801	0.078933183
Amount default ration woe	0.611	-0.451130057	0.275640465
y			-0.722122163

To find the probability of default

$$\begin{aligned} p =& \frac{1}{1+e^{-y}} \\ =& \frac{1}{1+e^{-0.722122163}} \\ =& 0.9987622542 \end{aligned} \tag{2.38}$$

This customer has a very high likelihood of defaulting if lend cash. Therefore any loan application should be rejected. The probabilities of default for the first 50 customers are shown in the table below alongside their credit scores.

2.9.10 Scoring the PD

The next phase is to convert the probability of default obtained into a credit score. Using the formulas above the scores and probability of default can be generated. The table below gives the scores for the first thirty customers.

Table 2.6. Table of Probability of Default and Credit Scores for the first 30 customers

Borrower Id	Probability of default	Credit Score
1	0.997269968	206
2	0.9877227005	250
3	0.001884618088	558
4	0.1106696092	437
5	0.07577957773	449
6	0.4425447388	383
7	0.09346134419	442
8	0.09346134419	366
9	0.5911024803	236
10	0.9924940912	496
11	0.01559419569	563
12	0.001569612212	349
13	0.9977096842	201
14	0.9707854188	276
15	0.00327358036	542
16	0.1612004438	424
17	0.007199711476	519
18	0.0477897629	463
19	0.9619201935	283
20	0.8090821387	335
21	0.03397039292	473
22	0.412135661	387
23	0.9538852391	289
24	0.3512241865	394

Borrower Id	Probability of default	Credit Score
25	0.01379099899	500
26	0.05220895084	460
27	0.00461219678	532
28	0.4838327497	378
29	0.001530961058	564
30	0.0002904474396	612

where full set of the borrowers , their probability of default and scores is available in the appendix. From the table its evident that, the higher the probability of default the lower the credit score and the lower the probability of default, the higher the credit score. This can be be used to distinguish between good and bad borrowers and minimize the risk of loss.

2.9.11 Summary scores for test Data

The table below shows a summary of the scores calculated for the borrowers in the test data

Table 2.7. Table showing summary of scores for borrowers in the test data

def	min	max	median	mean	n
Default	137	501	334	324	68
Non default	256	560	430	428	80

Figure 1. Graph of scores

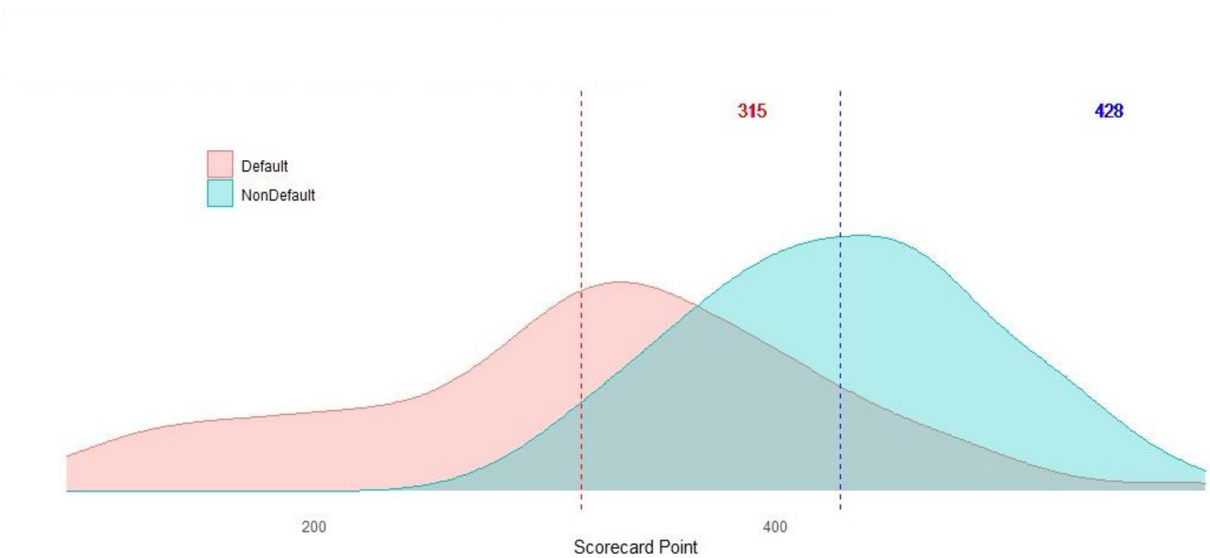
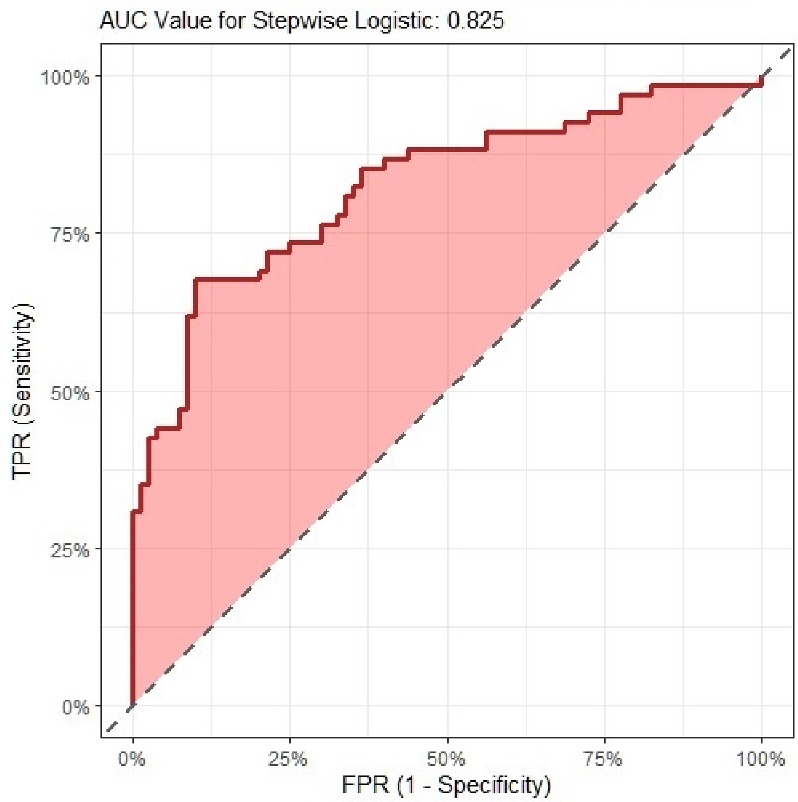


Figure 2. Graph showing model performance based on test data



2.10 Summary

In this charpter, the pdf,cdf, hazard and survival functions have been obtained. The mean, variance and moments have been calculated. Expressions for the $r^{th}$of even and odd moments have explicitly obtained. The first four central moments have also been calculated. The fourth central moments are equal to the Kurtosis. The central and non-central moments are equal because the mean of the standard logistic is zero.

The moment generating function has been calculated and its first four derivatives expressed as propositions.

Following the application of the cdf of the logistic distribution to data,its evident that the true positive rate sensitivity value of a model shows its efficiency in predicting the probability of default hence the credit score.

From figure 1, the graph of non defaulters peaks higher than that of defaulters, which means the model developed is more reliable. The mean credit score for the customers who are likely to default is 315. The mean credit score for the customers who are likely not to default is 428. The model has good performance hence gives reliable results. Using the data of 500 customers, the model developed is 82.5 percent efficient meaning that it can give reliable results in decision making about the credit worthiness of customers.

3 CONSTRUCTION OF GENERALIZED LOGISTIC DISTRIBUTIONS

3.1 Introduction

Several different forms of generalizations of the logistic distribution have proposed in literature.

The objective of this charpter which is to construct generalizations of the standard logistic distribution and their extended versions. The generalized logistic distributions considered are of type I,II,III and IV. The methods used in generalizing are transformations, difference of two independent Gumbel random variable, the Burr differential equation and mixtures.

Distributions used in each method are generalized to obtain generalized logistic distribution.

The first part of this charpter outlines the generalizations of Gumbel and Gumbel II mixtures, and Pareto I. The second and third parts show construction of the generalized logistic type IV and the extended $GLIV$ respectively. The special cases arising from $GLIV$ have also been considered

The fourth and fifth sections show the construction of generalized logistic type I and extended GLI respectively. The sixth and seventh sections show the construction of the generalized logistic type II and the extended $GLII$ respectively.

The eighth section shows the construction of the generalized logistic type III.

3.1.1 Generalizations based on Gumbel I mixtures

Generalizations based on Gumbel I mixtures shall be obtained as follows; Let

$$Y = e^{-X}$$

where Y is standard exponential.

$$\frac{dy}{dx} = -e^{-x}$$

Therefore the pdf of X is given by

$$\begin{aligned} f(x) =& g(y)\left|\frac{dy}{dx}\right| \\ =& e^{-y}e^{-x} \\ =& e^{-x}\exp(-y) \end{aligned}$$

$y = 0 \implies x = \infty$ and $y = \infty \implies x = -\infty$

Therefore

$$f(x) = e^{-x}\exp(-e^{-x}), \quad -\infty < x < \infty$$

which is a standard Gumbel I distribution or type I extreme values distribution . We shall call it Gumbel I.

Remark 3.1.1. *The standard Gumbel distribution has no parameter.*

To introduce a parameter we consider the transformation

$$Y = e^{-X}$$

where y is exponential with parameter λ

$$\begin{aligned}
\therefore f(x) =& g(y)\left|\frac{dy}{dx}\right| \\
=& \lambda e^{-\lambda y} e^{-x} \\
=& \lambda e^{-x}\exp(-\lambda y) \\
=& \lambda e^{-x}\exp(-\lambda e^{-x}), \quad \text{for} \quad -\infty < x < \infty; \quad \lambda > 0, \qquad \textbf{(3.1)}
\end{aligned}$$

This is a Gumbel I distribution with parameter λ.

If λ is a varying parameter, then we write

$$f(x|\lambda) = \lambda e^{-x}\exp(-\lambda e^{-x}) \qquad \textbf{(3.2)}$$

and the continuous Gumbel mixture is given by

$$\begin{aligned}
f(x) =& \int_0^\infty f(x|\lambda)d\lambda \\
=& \int_0^\infty \lambda e^{-x}\exp(-\lambda e^{-x})g(\lambda)d(\lambda) \qquad \textbf{(3.3)}
\end{aligned}$$

where $g(x)$ is a mixture distribution .

3.1.2 Generalized Gumbel I mixtures

We now wish to derive a generalized Gumbel I distribution as follows: Let

$$Y = e^{-X} \qquad \textbf{(3.4)}$$

where Y is a gamma distributed with two parameters α and λ; i.e.

$$g(y) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-xy} y^{\alpha-1}, \quad y > 0; \quad \alpha > 0, \quad \beta > 0.$$

Thus the pdf of X is

$$\begin{aligned}
f(x) =& \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{\alpha y} y^{\alpha-1} e^{-x} \\
=& \frac{\lambda^\alpha}{\Gamma(\alpha)} \Big[\exp(-\lambda e^{-x})\Big] e^{-x(\alpha-1)} e^{-x} \\
=& \frac{\lambda^\alpha}{\Gamma(\alpha)} \Big[\exp(-\lambda e^{-x})\Big] e^{-\alpha x} \\
=& \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\alpha x} \exp(-\alpha e^{-x}), \quad -\infty < x < \infty; \quad \alpha > 0, \; \lambda > 0
\end{aligned} \tag{3.5}$$

which is a generalized Gumbel I distribution with parameter α and λ

If λ is a varying parameter, then we write

$$f(x|\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\alpha x} \exp(-\lambda e^{-x}), \quad -\infty < x < \infty; \quad \alpha > 0, \quad \lambda > 0. \tag{3.6}$$

For a generalized Gumbel I mixture we have.

$$\begin{aligned}
f(x) =& \int_0^\infty f(x|\lambda) g(\lambda) d\lambda \\
=& \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\alpha x} \exp(-\lambda e^{-x}) g(\lambda) d\lambda
\end{aligned} \tag{3.7}$$

3.1.3 Generalized Gumbel II mixtures

Another form of Gumbel distribution is given by letting

$$X = -Z$$

where Z is a Gumbel I distribution, that is,

$$g(z) = e^{-z} \exp(-e^{-z}), \quad -\infty < x < \infty \tag{3.8}$$

Therefore the pdf of X is

$$\begin{aligned}
f(x) =& g(z) \left| \frac{dz}{dx} \right| \cdot 1 \\
=& e^{-z} \exp(-e^{-z}) \cdot 1 \\
=& e^{x} \exp(-e^{-z}) \cdot 1 \\
=& e^{x} \exp(-e^{x}), \quad -\infty < x < \infty
\end{aligned} \tag{3.9}$$

We shall call this standard Gumbel II distribution.

Let

$$Y = e^X$$

where Y is exponential with parameter λ

Then the pdf of x is

$$\begin{aligned} f(x) =& \lambda e^{-\lambda y}\left|\frac{dy}{dx}\right| \\ =& \lambda e^{-\lambda y} e^x \\ =& \lambda e^x e^x \exp(-\lambda e^x), \quad -\infty < x < \infty;\ \lambda > 0 \end{aligned} \tag{3.10}$$

3.1.4 Gumbel I- Gumbel II mixture and vice versa

Villa and Escobar (2006), name Gumbel I as the largest extreme value (LEV) distribution. Gumbel II is named the smallest extreme vales (SEV) distribution.

Introducing the location and shape parameters, we have,

$$\begin{cases} f_{LEV}(x) = \frac{1}{\sigma} e^{-(\frac{x-\mathfrak{m}}{\sigma})} \exp(-e^{-(\frac{z-\mathfrak{m}}{\sigma})}) \\ \text{and} \\ f_{SEV}(x) = 1\frac{1}{\sigma} e^{\frac{\varkappa-\mathfrak{m}}{\sigma}} \exp(e^{\frac{\varkappa-\mathfrak{m}}{\sigma}}) \end{cases} \quad \text{for} \quad -\infty < x < \infty; \quad - -\infty < \mathfrak{m} < \infty, \quad \sigma > 0. \tag{3.11}$$

3.1.5 Generalizations Based on Pareto I

The pdf of Pareto I distribution with parameter α and β shall be defined by;

$$g(y) = \frac{\alpha\beta^\alpha}{y^{\alpha+1}}, \quad y > \beta > 0; \quad \alpha > 0 \tag{3.12}$$

The cdf is

$$\begin{aligned} G(y) =& \int_\beta^y \frac{\alpha\beta^\alpha}{t^{\alpha+1}} dt \\ =& \alpha\beta^\alpha \int_\beta^y t^{-\alpha-1} dt \end{aligned}$$

$$
\begin{aligned}
&= \alpha\beta^{\alpha}\left[\frac{t^{-\alpha}}{-\alpha}\right]_{\beta}^{y} \\
&= -\beta^{\alpha}\left[t^{\alpha}\right]_{\beta}^{y} \\
&= 1-\frac{\beta^{\alpha}}{y^{\alpha}}, \quad y>\beta>0; \quad \alpha>0
\end{aligned} \tag{3.13}
$$

3.2 Generalized Logistic IV distribution (GLIV)

3.2.1 Construction based on generalized Gumbel I

$$
\begin{aligned}
g(\lambda) &= \frac{\lambda^{\beta-1}}{\Gamma(\beta)}e^{-\lambda}, \quad \lambda>0, \quad \beta>0 \\
\therefore f(x) &= \int_0^{\infty} \frac{\lambda^{\alpha}}{\Gamma(\alpha)}e^{-\alpha x}\exp(-\lambda e^{-x})\frac{\lambda^{\beta-1}}{\Gamma(\beta)}e^{-\lambda}d\lambda \\
&= \frac{e^{\alpha x}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\infty}\lambda^{(\alpha+\beta)-1}\exp(-\lambda-\lambda e^{-x})d\lambda \\
&= \frac{e^{-\alpha x}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\infty}\lambda^{(\alpha+\beta)-1}e^{-(1+e^{-x})\lambda}d\lambda \\
\therefore f(x) &= \frac{e^{-\alpha x}}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+\beta)}{(1+e^{-x})^{\alpha+\beta}} \\
&= \frac{e^{-\alpha x}}{\beta(\alpha,\beta)(1+e^{-x})^{\alpha+\beta}} \quad \text{for } -\infty<x<\infty; \quad \alpha>0, \quad \beta>0
\end{aligned} \tag{3.14}
$$

3.2.2 Construction based on the mixtures of Gumbel I and Gumbel II

Let X_1 be generalized Gumbel with parameter α and X_2 be generalized Gumbel with parameter β

Then,

$$
\begin{aligned}
g(z) &= \int_{-\infty}^{\infty} \frac{e^{-\alpha(z+x_2)}}{\Gamma(\alpha)}\exp(-e^{(z+x_2)})\frac{e^{-\beta x_2}}{\Gamma(\beta)}\exp(-e^{-(z+x_2)})dx_2 \\
&= \frac{e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\beta)}\int_{-\infty}^{\infty} e^{-(\alpha+\beta)x_2}\exp(-e^{-z}e^{-x_2}-e^{-x_2})dx_2 \\
&= \frac{e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\beta)}\int_{-\infty}^{\infty} e^{-(\alpha+\beta)x_2}\exp(-(1+e^{-z})e^{-x_2})dx_2
\end{aligned}
$$

Let $y = e^{-x_2} \implies dy = -e^{-x_2}dx_2 \implies dx_2 = -\frac{dy}{y}$

$$\therefore g(z) = \frac{e^{\alpha z}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty y^{\alpha+\beta}e^{-(1+e^{-z})y}\frac{dy}{y}$$

$$= \frac{e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty y^{\alpha+\beta-1}e^{-(1+e^{-z})y}dy$$

$$\therefore g(z) = \frac{e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+\beta)}{(1+e^{-z})^{\alpha+\beta}}$$

$$= \frac{1}{\beta(\alpha,\beta)}\frac{e^{-\alpha z}}{(1+e^{-z})^{\alpha+\beta}} \quad \text{for} \quad -\infty < z < \infty; \quad \alpha > 0,\ \beta > 0 \tag{3.15}$$

3.2.3 Construction based on transformation

A uniform distribution is a special case of beta 1 distribution.

Thus let

$$X = -\ln\frac{Y}{1-Y} \tag{3.16}$$

where Y is beta random distribution with parameter a and b

$$\therefore y = (1+e^{-x})^{-1} \tag{3.17}$$

and

$$\frac{dy}{dx} = -\frac{e^x}{(1+e^x)^2}$$

$$\therefore F(x) = g(y)\left|\frac{dy}{dx}\right|$$

$$= \frac{y^{a-1}(1-y)^{b-1}}{\beta(a,b)}\frac{e^x}{(1+e^x)^2}$$

$$= \frac{(1+e^{-x})^{-(a-1)}}{\beta(a,b)}[1-\frac{1}{1+e^x}]^{b-1}\frac{e^x}{(1+e^x)^2}$$

$$= \frac{e^{bx}}{\beta(a,b)(1+e^x)^{a+b}}$$

$$= \frac{e^{-ax}}{\beta(a,b)(1+e^{-x})^{a+b}} \quad \text{for} \ -\infty < x < \infty; \quad a > 0,\ b > 0 \tag{3.18}$$

which is generalized logistic type IV $GLIV$ distribution.

3.3 Extended Generalized Logistic IV

3.3.1 Construction based on mixtures of Generalized Gumbel I

$$
\begin{aligned}
g(\lambda) =& \frac{\Phi^{\beta}}{\Gamma(\beta)} e^{-\Phi\lambda}\lambda^{\beta-1}, \quad \lambda > 0; \quad \Phi > 0, \quad `\beta > 0 \\
\therefore f(x) =& \int_{0}^{\infty} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{\alpha x} \exp(-\lambda e^{-x}) \frac{\Phi^{\beta}}{\Gamma(\beta)} e^{\Phi\lambda}\lambda^{\beta-1} d\lambda \\
=& \frac{\Phi^{\beta} e^{-\alpha x}}{\Gamma(\alpha)\Gamma(\beta)} \int_{0}^{\infty} \lambda^{\alpha+\beta-1} \exp(-\Phi\lambda - \lambda e^{-x}) d\lambda \\
=& \frac{\Phi^{\beta} e^{-\alpha x}}{\Gamma(\alpha)\Gamma(\beta)} \int_{0}^{\infty} \lambda^{\alpha+\beta-1} e^{-(\Phi+e^{-x})\lambda} d\lambda \\
\therefore f(x) =& \frac{\Phi^{\beta} e^{-\alpha x}}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+\beta)}{(\Phi+e^{-x})^{\alpha+\beta}} \\
=& \frac{\Phi^{\beta}}{\beta(\alpha,\beta)} \frac{e^{-\alpha x}}{(\Phi+e^{-x})^{\alpha+\beta}} \quad \text{for} \quad -\infty < x < \infty; \quad \alpha > 0, \quad \beta > 0, \quad \Phi > 0. \qquad \textbf{(3.19)}
\end{aligned}
$$

Wu,Hung and Lee (2000) included the location and shape parameter δ

3.3.2 Construction based on the Mixtures of Gumbel I and Gumbel II

Let X_1 be generalized Gumbel with parameter α and X_2 be generalized Gumbel with two parameter Φ and $\mathfrak{m}$. That is;

$$
\begin{cases}
f_1(x) = \frac{e^{-\alpha x_1}}{\Gamma(\alpha)} \exp(-e^{-x_1}), -\infty < x_1 < \infty, \alpha > 0 \\
f_2(x) = \frac{\Phi^{\mathfrak{m}}}{\Gamma(\mathfrak{m})} e^{-\mathfrak{m} x_2} \exp(-\Phi e^{-x_2}), -\infty < x_2 < \infty; \quad \Phi > 0, \quad \mathfrak{m} > 0.
\end{cases} \qquad \textbf{(3.20)}
$$

Therefore,

$$
\begin{aligned}
g(z) =& \int_{-\infty}^{\infty} \frac{e^{-\alpha(z+x_2)}}{\Gamma(\alpha)} \exp(-e^{z+x_2}) \frac{\Phi^{\mathfrak{m}}}{\Gamma(\mathfrak{m})} e^{-\mathfrak{m} x_2} \exp(-\Phi e^{-x_2}) dx_2 \\
\therefore g(z) =& \frac{\Phi^{\mathfrak{m}}}{\Gamma(\alpha)\Gamma(\mathfrak{m})} \int_{-\infty}^{\infty} e^{-\alpha z} e^{-\alpha x_2} \exp(-e^{-z} e^{x_2} - \Phi e^{-x_2}) e^{-\mathfrak{m} x_2} dx_2 \\
=& \frac{\Phi^{\mathfrak{m}} e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\mathfrak{m})} \int_{-\infty}^{\infty} e^{-(\alpha+\mathfrak{m})x_2} \exp(-(\Phi + e^{-z})) e^{-x_2} dx_2
\end{aligned}
$$

Let $y = e^{-x_2} \implies dy = -e^{x_2}dx_2 \implies dx_2 = -\frac{dy}{y}$

$$
\begin{aligned}
\therefore g(z) =& \frac{\Phi^{\mathfrak{m}} e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\mathfrak{m})} \int_0^\infty y^{\alpha+\mathfrak{m}} e^{-(\Phi+e^{-z})y} \frac{dy}{y} \\
=& \frac{\Phi^{\mathfrak{m}} e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\mathfrak{m})} \int_0^\infty y^{\alpha+\mathfrak{m}-1} e^{-(\Phi+e^{-z})y} dy \\
=& \frac{\Phi^{\mathfrak{m}} e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\mathfrak{m})} \frac{\Gamma(\alpha+\mathfrak{m})}{(\Phi+e^{-z})^{\alpha+\mathfrak{m}}} \\
=& \frac{\Phi^{\mathfrak{m}} e^{-\alpha z}}{\beta(\alpha,\mathfrak{m})(\Phi+e^{-z})^{\alpha+\mathfrak{m}}} \quad \text{for} \quad -\infty < z < \infty; \quad \alpha > 0, \quad \mathfrak{m} > 0, \quad \Phi > 0
\end{aligned}
\tag{3.21}
$$

3.3.3 Special cases of the Extended Generalized Logistic IV

(i)

$$\Phi = 1 \implies f(x) = \frac{e^{\alpha x}}{\beta(\alpha,\beta)(1+e^{-x})^{\alpha+\beta}}$$

which is $GLIV$.

(ii)

$$\beta = 1 \implies f(x) = \frac{\Phi\alpha e^{-\alpha x}}{(\Phi+e^{-x})^{\alpha+1}}$$

which is extended $GLII$.

(iii)

$$\alpha = 1 \implies f(x) = \frac{\beta\Phi^{\beta} e^{-x}}{(\Phi+e^{-x})^{\beta+1}}$$

$$
\begin{aligned}
\therefore f(x) =& \frac{\beta\Phi^{\beta+1}}{\Phi} \frac{e^{-x}}{(\Phi+e^{-x})^{\alpha+1}} \\
=& \frac{\beta}{\Phi} \frac{e^{-x}}{(1+\frac{1}{\Phi}e^{-x})^{\alpha+1}} \\
=& \frac{\rho\beta e^{-x}}{(1+\rho e^{-x})^{\alpha+1}}
\end{aligned}
$$

which is extended GLI distribution.

(iv)

$$\alpha = \beta = 1 \implies f(x) = \frac{\Phi e^{-x}}{(\Phi + e^{-x})^2}$$

which is extended standard logistic distribution.

(v)

$$\alpha = \beta = \Phi = 1 \implies f(x) = \frac{e^{-x}}{(1 + e^{-x})^2}$$

which is standard logistic distribution.

3.4 Generalized Logistic I Distribution (GLI)

The generalized logistic type I can be constructed using the five methods discussed below.

3.4.1 Construction based on mixtures of Gumbel I

Let the mixing distribution be;

$$g(\lambda) = \frac{\lambda^{\alpha-1}e^{-\lambda}}{\Gamma(\lambda)}, \quad \lambda > 0; \quad \alpha > 0$$

which is a gamma mixing distribution with one parameter α

$$\begin{aligned}
\therefore f(x) &= \int_0^\infty \lambda e^{-x} \exp(-\lambda e^{-x}) \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda} d\lambda \\
&= \frac{e^{-x}}{\Gamma(\alpha)} \int_0^\infty \lambda^{(\alpha+1)-1} \exp(-\lambda - \lambda e^{-x}) d\lambda \\
&= \frac{e^{-x}}{\Gamma(\alpha)} \int_0^\infty \lambda^{(\alpha+1)-1} e^{-(1+e^{-x})\lambda} d\lambda \\
&= \frac{e^{-x}}{\Gamma(\alpha)} \frac{\Gamma(\alpha-1)}{(1+e^{-x})^{\alpha+1}} \\
&= \frac{\alpha e^{-x}}{(1+e^{-x})^{\alpha+1}}, \quad -\infty < x < \infty; \ \alpha > 0
\end{aligned} \tag{3.22}$$

3.4.2 Construction based on mixtures of Gumbell II

$$
\begin{aligned}
g(\lambda) =& e^{-\lambda}, \quad \lambda>0 \\
f(x) =& \int_0^\infty e^{\alpha x} \frac{\lambda^\alpha}{\Gamma(\alpha)} \exp(-\lambda e^x) e^{-\lambda} d\lambda \\
=& \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{\alpha x} \lambda^\alpha e^{-(1+e^x)e^{-\lambda}} d\lambda \\
=& \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(1+e^x)^{\alpha+1}} e^{\alpha x} \\
=& \frac{\alpha e^{\alpha x}}{(1+e^x)^{\alpha+1}} \\
=& \frac{\alpha e^{\alpha x} e^{-(\alpha+1)x}}{(1+e^x)^{\alpha+1} e^{-(\alpha+1)x}} \\
=& \frac{\alpha e^{-x}}{(1+e^{-x})^{\alpha+1}}, \quad -\infty < x < \infty; \ \alpha > 0
\end{aligned}
\tag{3.23}
$$

3.4.3 Construction based on the difference of two independent Gumbel random variables

Let X_1 be standard Gumbel and X_2 be Gumbel with parameter α i.e.

Let X_1 be standard Gumbel and X_2 be generalized Gumbel with one parameter α

$$
\begin{cases}
f_1(x) = e^{-x} \exp(-e^{-x}) \\
\\
f_2(x) = \frac{e^{-\alpha x_2}}{\Gamma(\alpha)} \exp(-e^{-x_2})
\end{cases}
\quad \text{for} \quad -\infty < x_2 < \infty, \ \alpha > 0
\tag{3.24}
$$

Therefore,

$$
\begin{aligned}
g(z) =& \int_{-\infty}^{\infty} e^{-(z+x_2)} \exp(-e^{-(z+x_2)}) \frac{e^{-\alpha x_2}}{\Gamma(\alpha)} \exp(-e^{-x_2}) dx_2 \\
=& \frac{e^{-z}}{\Gamma(\alpha)} \int_{-\infty}^{\infty} \exp\Big(-(e^{-z}e^{-x_2} + e^{-x_2})\Big) e^{-\alpha x_2} e^{-x_2} dx_2 \\
=& \frac{e^{-z}}{\Gamma(\alpha)} \int_{-\infty}^{\infty} y^\alpha e^{-(1+e^{-z})y} dy
\end{aligned}
$$

Let,

$$y = e^{-x_2} \implies dy = -e^{-x_2} dx_2$$

$$\therefore g(z) = \frac{e^{-z}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(1+e^{-z})^{\alpha+1}}$$

$$= \frac{\alpha e^{-z}}{(1+e^{-z})^{\alpha+1}}, \quad \alpha > 0, \ -\infty < z < \infty \tag{3.25}$$

3.4.4 Construction based on the Burr differential equation

Burr(1942) introduced a system of distribution based on a differential equation of the form;

$$y' = y(1-y)g(x,y) \tag{3.26}$$

where $y = F(x)$, a cdf of continuous random variable

$y' = f(x)$ a pdf and $g(x,y)$ is a non negative function of x and y.

The problem is to solve the above differential equation known as Burr differential equation for various cases of $g(x,y)$. Consider the case when $g(x,y) = g(x)$ Thus the differential equation becomes,

$$y' = y(1-y)g(x)$$

$$\therefore \int \frac{dy}{y(1-y)} = \int g(x)dx$$

$$\therefore \int (\frac{1}{y} + \frac{1}{1-y})dy = \int g(x)dx$$

$$\therefore \log \frac{y}{1-y} = \int g(x)dx$$

$$\therefore \frac{y}{1-y} = \exp\left\{\int g(x)dx\right\}$$

$$\therefore y = (1-y)\exp\left\{\int g(x)dx\right\}$$

$$= \exp\left\{\int g(x)dx\right\} - y\exp\left\{\int g(x)dx\right\}$$

$$\therefore = \frac{\exp\left\{\int g(x)dx\right\}}{1+\exp\left\{\int g(x)dx\right\}}$$

$$= \frac{1}{1+\exp\{-\int g(x)dx}$$

$$F(x) = [e^{\int g(x)dx} + 1]^{-1} \tag{3.27}$$

as obtained by Burr (1942). If,

$$g(x) = 1$$

then

$$F(x) = [e^{-\int dx} + 1]^{-1}$$
$$= (e^{-x} + 1)^{-1}$$
$$= \frac{1}{1 + e^{-x}}$$

$F(x) = 0 \implies x = -\infty$ and $F(x) = 1 \implies x = \infty$

$$\therefore F(x) = \frac{1}{1 + e^{-x}}, \quad -\infty < x < \infty$$

which is the cdf of standard logistic distribution

Burr (1942), further considered the power of the cdf that is:

$$G(x) = [F(x)]^{\alpha}$$
$$= (1 + e^{-x})^{-\alpha}, \quad -\infty < x < \infty, \quad \alpha > 0$$

The pdf is

$$g(x) = \alpha(1 + e^{-x})^{-\alpha-1} e^{-x}$$
$$= \frac{\alpha e^{-x}}{(1 + e^{-x})^{\alpha+1}}, \quad \text{for} \quad -\infty < x < \infty, \quad \alpha > 0$$

which is GLI distribution. It is called exponentiated logistic distribution. According to Burr system it is Burr II distribution.

3.4.5 Construction based on Exponentiated Generalized approach

Let

$$X = \ln(\frac{Y}{\beta} - 1) \quad \textbf{(3.28)}$$

where Y is Pareto I with parameter α and β

$$\therefore \frac{y}{\beta} - 1 = e^{-x}$$
$$y = \beta(1 + e^{-x})$$

and

$$\frac{dy}{dx} = -\beta e^{-x}$$

Therefore the pdf of X is

$$f(x) = \frac{\alpha\beta^{\alpha}}{y^{\alpha+1}} \left| \frac{dy}{dx} \right|$$
$$= \frac{\alpha\beta^{\alpha}}{[\beta(1 + e^{-x})]^{\alpha+1}} \beta e^{-x}$$
$$= \frac{\alpha e^{-x}}{(1 + e^{-x})^{\alpha+1}}, \quad -\infty < x < \infty; \ \alpha > 0 \quad \textbf{(3.29)}$$

3.5 Extended Generalized logistic Type I

3.5.1 Constructions based on mixtures of Gumbel I

Let the mixing distribution be;

$$g(\lambda) = \frac{\beta^{\lambda}}{\Gamma(\alpha)} e^{\beta\lambda}, \quad \lambda > 0, \quad \alpha > 0, \quad \beta > 0$$

which is a gamma mixing distribution with two parameters α and β

$$\begin{aligned}
\therefore f(x) =& \int_0^{\infty} \lambda e^{-x} \exp(-\lambda e^{-x}) \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\beta\lambda} \lambda^{\alpha-1} d\lambda \\
\therefore f(x) =& \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \lambda^{(\alpha+1)-1} e^{-(\beta+e^{-x})\lambda} d\lambda \\
=& \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-x} \frac{\Gamma(\alpha+1)}{(\beta+e^{-x})^{\alpha+1}} \\
=& \frac{\alpha\beta^{\alpha} e^{-x}}{(\beta+e^{-x})^{\alpha+1}} \\
=& \frac{\alpha}{\beta} \frac{\beta^{\alpha+1} e^{-x}}{(\beta+e^{-x})^{\alpha+1}} \\
=& \frac{\alpha}{\beta} \frac{e^{-x}}{(1+\frac{1}{\beta} e^{-x})^{\alpha+1}} \\
=& \frac{\alpha\rho e^{-x}}{(1+\rho e^{-x})^{\alpha-1}}, \quad -\infty < x < \infty, \quad \alpha > 0, \quad \frac{1}{\beta} = \rho > 0
\end{aligned} \tag{3.30}$$

which is extended Type I generalized logistic distribution as given by Olapade (2004).

3.5.2 Construction Based on mixtures of generalized Gumbel I

$$\begin{aligned}
g(\lambda) =& \beta e^{-\beta\lambda}, \quad \lambda > 0; \quad \beta > 0 \\
\therefore f(x) =& \int_0^{\infty} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} e^{-\alpha x} \exp(-\lambda e^{-x}) \beta e^{-\beta\lambda} d\lambda \\
=& \frac{\beta e^{-\alpha x}}{\Gamma(\alpha)} \int_0^{\infty} \lambda^{(\alpha+1)-1} \exp(-\beta\lambda - \lambda e^{-x}) d\lambda \\
=& \frac{\beta e^{\alpha x}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(\beta+e^{-x})^{\alpha+1}} \\
=& \frac{\beta\alpha e^{-\alpha x}}{(\beta+e^{-x})^{\alpha+1}}, \quad -\infty < x < \infty; \ \alpha > 0, \ \beta > 0
\end{aligned} \tag{3.31}$$

3.5.3 Construction based on the mixtures of Gumbel II

$$
\begin{aligned}
g(\lambda) =& \beta e^{-\beta\lambda}, \quad \lambda > 0, \quad \beta > 0 \\
\therefore f(x) =& \int_0^\infty e^{\alpha x} \frac{\lambda^\alpha}{\Gamma(\alpha)} \exp(-\lambda e^x) \beta e^{-\beta\lambda} d\lambda. \\
=& \frac{\beta}{\Gamma(\alpha)} e^{\alpha x} \frac{\Gamma(\alpha+1)}{(\beta+e^x)^{\alpha+1}} \\
=& \frac{\alpha\beta e^{\alpha x}}{(\beta+e^x)^{\alpha+1}} \\
=& \frac{\alpha\beta e^{\alpha x} e^{-(\alpha+1)x}}{(\beta+e^x)^{\alpha+1} e^{-(\alpha+1)x}} \\
=& \frac{\alpha\beta e^{-x}}{(1+\beta e^{-x})^{\alpha+1}}
\end{aligned} \tag{3.32}
$$

3.5.4 Construction based on the mixtures of Gumbel I and Gumbel II

Let X_1 be standard Gumbel and X_2 be generalized Gumbel with parameter α and λ. That is;

$$
\begin{cases}
f_1(x) = e^{-x} \exp(-e^{-x}), & -\infty < x < \infty \\
f_2(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\alpha x_2} \exp(-\lambda e^{-x_2}), & -\infty < x_2 < \infty, \quad \alpha > 0, \quad \lambda > 0
\end{cases} \tag{3.33}
$$

Therefore,

$$
\begin{aligned}
g(z) =& \int_{-\infty}^{\infty} e^{-(z+x_2)} \exp(-e^{-(z+x_2)}) \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\alpha x_2} \exp(-\lambda e^{-x_2}) dx_2 \\
\therefore g(z) =& \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-z} \int_{-\infty}^{\infty} e^{-\alpha x_2} \exp(-(\lambda+e^{-z}) e^{-x_2}) e^{-x_2} dx_2 \\
=& \frac{\lambda}{\Gamma(\alpha)} e^{-z} \int_{-\infty}^{\infty} y^\alpha e^{-(\lambda+e^{-z})y} dy \\
=& \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-z} \frac{\Gamma(\alpha+1)}{(\lambda+e^{-z})^{\alpha+1}} \\
=& \alpha\lambda^\alpha \frac{e^{-z}}{(\lambda+e^{-z})^{\alpha+1}} \\
=& \frac{\alpha\lambda^\alpha e^{-z}}{[\lambda(1+\frac{1}{\lambda}e^{-z})]^{\alpha+1}} \\
=& \frac{\alpha}{\lambda} \frac{e^{-z}}{(1+\frac{1}{\lambda}e^{-z})^{\alpha+1}} \\
=& \frac{\alpha\rho e^{-z}}{(1+\rho e^{-z})^{\alpha+1}}, \quad -\infty < z < \infty; \quad \alpha > 0, \quad \rho > 0, \; \rho = \frac{1}{\lambda}
\end{aligned} \tag{3.34}
$$

3.6 Generalized Logistic type II distribution GLII

3.6.1 Constructions based on Generalized Gumbel I

$$\begin{aligned}
g(\lambda) &= e^{-\lambda} \\
\therefore f(x) &= \frac{e^{-\alpha x}}{\Gamma(\alpha)} \int_0^\infty \lambda^\alpha \exp(-\lambda e^{-x}) \cdot e^\lambda d\lambda \\
&= \frac{e^{-\alpha x}}{\Gamma(\alpha)} \int_0^\infty \lambda^\alpha \exp(-\lambda - \lambda e^{-x}) d\lambda \\
&= \frac{e^{-\alpha x}}{\Gamma(\alpha)} \int_0^\infty \lambda^{(\alpha+1)-1} e^{-(1+e^{-x})\lambda} d\lambda \\
&= \frac{e^{-\alpha x}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(1+e^{-x})^{\alpha+1}} \\
&= \frac{\alpha e^{-\alpha x}}{(1+e^{-x})^{\alpha+1}}, \quad -\infty < x < \infty; \quad \alpha > 0
\end{aligned} \tag{3.35}$$

3.6.2 Constructions based on mixtures of generalized Gumbel II

$$\begin{aligned}
g(\lambda) &= \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda}, \quad \alpha > 0, \quad \lambda > 0 \\
\therefore f(x) &= \int_0^\infty \lambda e^x \exp(-\lambda e^x) \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda} d\lambda \\
&= \frac{e^x}{\Gamma(\alpha)} \int_0^\infty \lambda^{(\alpha+1)-1} e^{-(1+e^x)\lambda} d\lambda \\
&= \frac{e^x}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(1+e^x)^{\alpha+1}} \\
&= \frac{\alpha e^x}{(1+e^x)^{\alpha+1}} \quad -\infty < x < \infty, \quad \alpha > 0 \\
&= \frac{\alpha e^x e^{-(\alpha+1)x}}{(1+e^x)^{\alpha+1} e^{-(\alpha+1)x}} \\
&= \frac{\alpha e^{-\alpha x}}{(1+e^{-x})^{\alpha+1}}, \quad -\infty < x < \infty, \quad \alpha > 0
\end{aligned} \tag{3.36}$$

which is $GLII$ distribution according to Balalrishnan and Leung (1988)

3.6.3 Construction based on Gumbel I and Gumbel II mixtures

Let X_1 be a generalized Gumbel distributed variable with parameter α and X_2 be standard Gumbel. That is

$$\begin{cases} f_1(x) = \frac{e^{-\alpha x_1}}{\Gamma(\alpha)} \exp(-e^{-x_1}), & -\infty < x_1 < \infty, \quad \alpha > 0 \\ f(x_2) = e^{-x_2} \exp(-e^{-x_2}), & -\infty < x_2 < \infty \end{cases} \tag{3.37}$$

Therefore,

$$\begin{aligned} g(z) = & \int_{-\infty}^{\infty} \frac{e^{-\alpha(z+x_2)}}{\Gamma(\alpha)} \exp(-e^{-(z+x_2)}) e^{-x_2} \exp(-e^{-x_2}) dx_2 \\ = & \frac{e^{-\alpha z}}{\Gamma(\alpha)} \int_{-\infty}^{\infty} e^{-\alpha x_2} \exp(-e^{-z} e^{-x_2} - e^{-x_2}) e^{-x_2} dx_2 \\ = & \frac{e^{-\alpha z}}{\Gamma(\alpha)} \int_{0}^{\infty} y^{\alpha} e^{-(1+e^{-z})y} dy \\ = & \frac{e^{-\alpha z}}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(1+e^{-z})^{\alpha+1}} \\ = & \frac{\alpha e^{-\alpha z}}{(1+e^{-z})^{\alpha+1}}, \quad -\infty < z < \infty; \quad \alpha > 0 \end{aligned} \tag{3.38}$$

3.6.4 Constructions based on transformations

Constructions based on the uniform distribution

$$X = -\ln\left(\frac{U^{\frac{1}{p}}}{1 - U^{\frac{1}{p}}}\right), \quad p > 0 \tag{3.39}$$

where U has uniform distribution in$[0,1]$

Then

$$u = (1+e^x)^{-p}$$

and

$$\frac{du}{dx} = -p(1+e^x)^{-p-1} e^x$$

Therefore

$$\begin{aligned} f(x) = & 1 \cdot \left|\frac{du}{dx}\right| \\ = & \frac{pe^x}{(1+e^x)^{p+1}}, \quad -\infty < x < \infty; \quad p > 0 \end{aligned}$$

$$
\begin{aligned}
\therefore f(x) &= \frac{pe^{x}e^{-(p+1)x}}{(1+e^{x})^{p+1}e^{-(p+1)x}} \\
&= \frac{pe^{px}}{(1+e^{-x})^{p+1}}, \quad -\infty < x < \infty; \quad p > 0
\end{aligned} \tag{3.40}
$$

Construction based on the exponential

where Y is an exponential with parameter α

Then

$$
\begin{aligned}
f(x) =& g(y)\left|\frac{dy}{dx}\right| \\
=& \alpha e^{-\alpha y}\frac{e^{x}}{1+e^{x}} \\
=& \alpha(e^{y})^{-\alpha}\frac{e^{x}}{1+e^{x}} \\
=& \alpha(1+e^{x})^{-\alpha}\frac{e^{x}}{1+e^{x}} \\
=& \frac{\alpha e^{x}}{(1+e^{x})^{\alpha+1}} \\
=& \frac{\alpha e^{-\alpha x}}{(1+e^{-x})^{\alpha+1}}, \quad \text{for } -\infty < x < \infty; \quad \alpha > 0
\end{aligned} \tag{3.41}
$$

3.7 Extended Generalized Logistic type II

3.7.1 Construction based on the mixtures of Gumbel II

$$
\begin{aligned}
g(\lambda) =& * \frac{\beta^{\alpha}}{\Gamma(\alpha)}e^{-\beta\lambda}\lambda^{\alpha-1}, \quad \lambda > 0; \quad \alpha > 0, \quad \beta > 0 \\
\therefore f(x) =& * \int_{0}^{\infty} \lambda e^{x}\exp(-\lambda e^{x})\frac{\beta^{\alpha}}{\Gamma(\alpha)}e^{-\beta\lambda}\lambda^{\alpha-1}d\lambda \\
=& \frac{\beta^{\alpha}}{\Gamma(\alpha)}e^{x}\int_{0}^{\infty}\lambda^{(\alpha+1)-1}e^{-(\beta+e^{x})\lambda}d\lambda \\
=& \frac{\beta^{\alpha}}{\Gamma(\alpha)}e^{x}\frac{\Gamma(\alpha+1)}{(\beta+e^{x})^{\alpha+1}} \\
=& \alpha\beta^{\alpha}\frac{e^{x}}{(\beta+e^{x})^{\alpha+1}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha\beta^{\alpha+1}}{\beta}\frac{e^x}{(\beta+e^x)^{\alpha+1}} \\
&= \frac{\alpha}{\beta}\frac{e^x}{(1+\frac{1}{\beta}e^x)^{\alpha+1}} \\
&= \alpha\rho\frac{e^x}{(1+\rho e^x)^{\alpha+1}}, \quad \rho = \frac{1}{\beta} \\
\therefore f(x) &= \frac{\alpha\rho e^x e^{-(\alpha+1)x}}{(1+\rho e^x)^{\alpha+1} e^{-(\alpha+1)x}} \\
&= \frac{\alpha\rho e^{\alpha x}}{(\rho+e^{-x})^{\alpha+1}}
\end{aligned} \tag{3.42}
$$

3.7.2 Construction based on mixtures of Gumbel I and Gumbel II

Let X_1 be generalized Gumbel with parameter α and X_2 be Gumbel with parameter λ.

Then,

$$
\begin{cases}
f_1(x) = \frac{e^{-\alpha x_1}}{\Gamma(\alpha)}\exp(-e^{-x_1}), & -\infty < x_1 < \infty, \quad \alpha > 0 \\
f_2(x) = \lambda e^{-x_2}\exp(-\lambda e^{-x_2}), & -\infty < x_2 < \infty, \quad \lambda > 0
\end{cases}
$$

Therefore,

$$
\begin{aligned}
g(z) &= \int_{-\infty}^{\infty} \frac{e^{-\alpha(z+x_2)}}{\Gamma(\alpha)}\exp(-e^{-(z++x_2)})\cdot\lambda e^{-x_2}\exp(-\lambda e^{-x_2})dx_2 \\
&= \frac{\lambda}{\Gamma(\alpha)}e^{-\alpha z}\int_{-\infty}^{\infty} e^{-\alpha x_2}\exp(-e^{-z}e^{-x_2}-\lambda e^{-x_2})e^{-x_2}dx_2 \\
&= \frac{\lambda}{\Gamma(\alpha)}e^{-\alpha z}\int_{-\infty}^{\infty} e^{-\alpha x_2}\exp(-(\lambda+e^{-z})e^{-x_2})e^{x_2}dx_2 \\
&= \frac{\lambda}{\Gamma(\alpha)}e^{-\alpha z}\int_{-\infty}^{\infty} y^{\alpha}e^{-(\lambda+e^{-z})y}dy
\end{aligned}
$$

where $y = e^{-x_2} \Longrightarrow {\scriptstyle dy=-e^{-x_2}} dx_2$

$$
\begin{aligned}
\therefore g(z) &= \frac{\lambda}{\Gamma(\alpha)}e^{-\alpha z}\frac{\Gamma(\alpha+1)}{(\lambda+e^{-z})^{\alpha+1}} \\
&= \frac{\alpha\lambda e^{-\alpha z}}{(\lambda+e^{-z})^{\alpha+1}}, \qquad -\infty < z < \infty; \quad \alpha > 0, \quad \lambda > 0
\end{aligned} \tag{3.43}
$$

Construction based on the uniform distribution

Let

$$X = -\ln\left(\frac{\lambda Y^{\frac{1}{p}}}{1-Y^{\frac{1}{p}}}\right) \quad p>0$$

where Y has uniform distribution in $[0,1]$

$$\therefore y = \left(\frac{e^{-x}}{\lambda+e^{-x}}\right)^p$$

and

$$\frac{dy}{dx} = -p\lambda\frac{e^{-px}}{(\lambda+e^{-x})^{p+1}}$$

$$\begin{aligned}\therefore f(x) =& 1\cdot\left|\frac{dy}{dx}\right| \\ =& \frac{p\lambda e^{-px}}{(\lambda+e^{-x}])^{p+1}}, \quad -\infty<x<\infty; \quad p>0, \quad \lambda>0 \end{aligned} \tag{3.44}$$

whic is extended $GLII$ distribution according to Olapade (2005)

Construction based on the exponential

Let

$$X = -\ln\left(\frac{\lambda e^{-Y}}{1-e^{-Y}}\right) \tag{3.45}$$

where Y is an exponential with parameter α.

$$\begin{aligned}\therefore \frac{\lambda e^{-y}}{1-e^{-y}} =& e^{-x} \\ \therefore e^{-y} =& \frac{e^{-x}}{\lambda+e^{-x}} \\ \therefore y =& \ln(\lambda+e^{-x})-\ln(e^{-x}) \\ \frac{dy}{dx} =& \frac{\lambda}{\lambda+e^{-x}} \\ \therefore f(x) =& g(y)\left|\frac{dy}{dx}\right| \\ =& \alpha e^{-\alpha y}\frac{\lambda}{\lambda+e^{-x}} \\ =& \frac{\alpha\lambda}{\lambda+e^{-x}}(e^{-y})^{\alpha}\end{aligned}$$

$$=\frac{\alpha\lambda}{\lambda+e^{-x}}\cdot\frac{e^{-\alpha x}}{(\lambda+e^{-x})^{\alpha}}$$
$$=\frac{\alpha\lambda e^{-\alpha x}}{(\lambda+e^{-x})^{\alpha+1}},\quad -\infty<x<\infty;\quad \alpha>0 \tag{3.46}$$

Constructions based on Exponentiated generalized approach

The cdf of Pareto I when $\alpha=p$ and $\beta=1$ is given by;

$$G(y)=1-y^{-p}$$

Let

$$H(y)=[G(y)]^{\alpha}$$

i.e. The new cdf is the power of the old/parameter cdf.

$$\therefore H(y)=(1-y^{-p})^{\alpha}$$
$$\therefore h(y)=\alpha p(1-y^{-p})^{\alpha-1}y^{-p-1} \tag{3.47}$$

which is an exponentiated Pareto I distribution with parameter α and β

which is a $GLII$ distribution. Let

$$X=-\ln(Y^{p}-1) \tag{3.48}$$

where y is an exponentiated Pareto I distribution with parameters α and p

Then

$$y=(1+e^{-x})^{\frac{1}{p}}$$

and

$$\frac{dy}{dx}=\frac{1}{p}(1+e^{-x})^{\frac{1}{p}-1}(-e^{-x})$$
$$\therefore f(x)=h(y)\left|\frac{dy}{dx}\right|$$
$$=\alpha p\Big(1-\frac{1}{y^{p}}\Big)^{\alpha-1}\frac{1}{y^{p+1}}\cdot e^{-x}(1+e^{-x})^{\frac{1}{p}-1}\frac{1}{p}$$
$$=\alpha\left(1-\frac{1}{1+e^{-x}}\right)^{\alpha-1}\frac{e^{-x}}{(1+e^{-x})^{1+\frac{1}{p}}}(1+e^{-x})^{\frac{1}{p}-1}$$
$$=\frac{\alpha e^{-x(\alpha-1)}e^{x}}{(1+e^{-x})^{\alpha-1+1+\frac{1}{p}-\frac{1}{p}+1}}$$
$$=\frac{\alpha e^{-\alpha x}}{(1+e^{-x})^{\alpha+1}} \tag{3.49}$$

3.8 Generalized Logistic type III distribution GLIII

3.8.1 Construction based on the mixtures of Gumbel I and Gumbel II

Let X_1 and X_2 be both generalized Gumbel variable with the same parameter α

$$\begin{aligned}
\therefore g(z) &= \int_{-\infty}^{\infty} f_1(z+x_2) f_2(x) dx_2 \\
&= \int_{-\infty}^{\infty} \frac{e^{-\alpha(z+x_2)}}{\Gamma(\alpha)} \exp(-e^{-(z+x_2)}) \cdot \frac{e^{-\alpha x_2}}{\Gamma(\alpha)} \exp(-e^{-x_2}) dx_2 \\
&= \frac{e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\alpha)} \int_{-\infty}^{\infty} e^{-\alpha x_2 - \alpha x_2} \exp(-e^{-z} e^{-x_2 - e^{-x_2}}) dx_2 \\
&= \frac{e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\alpha)} \int_{-\infty}^{\infty} e^{-2\alpha x_2} \exp(-(1+e^{-z}) e^{-x_2}) dx_2
\end{aligned}$$

Let $y = e^{-x_2} dx_2 \implies dy = -e^{-x_2} dx_2 \implies -\frac{dy}{y} = dx_2$

$$\begin{aligned}
\therefore g(z) &= \frac{e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\alpha)} \int_{0}^{\infty} y^{2\alpha} e^{-(1+e^{-z})y} \frac{dy}{y} \\
&= \frac{e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\alpha)} \int_{0}^{\infty} y^{2\alpha-1} e^{-(1+e^{-z})y} dy \\
&= \frac{e^{-\alpha z}}{\Gamma(\alpha)\Gamma(\alpha)} \frac{\Gamma(2\alpha)}{(1+e^{-z})^{2\alpha}} \\
&= \frac{e^{-\alpha z}}{\beta(\alpha,\alpha)(1+e^{-z})^{2\alpha}} \quad -\infty < z < \infty, \quad \alpha > 0
\end{aligned} \tag{3.50}$$

which is $GLIII$ distribution, a result obtained by Davidson (1980).

3.8.2 Special case of GLIV

If $\beta = \alpha$, then

$$f(x) = \frac{e^{-\alpha x}}{\beta(\alpha,\alpha)(1+e^{-x})^{2\alpha}} \tag{3.51}$$

3.9 Summary

In this charpter, generalized logistic type I, II, III and IV have been explicitly obtained based on different methods of constructing the standard logistic distributions(this is a new method of generalization that has been proposed). Their extended versions have also been obtained. The results obtained are similar to what other authors obtained earlier in literature review. The new distribution introduced " extended standard logistic" has been constructed using four different methods.

Clearly, five different distributions have been obtained as special cases of the extended standard logistic type IV.

Generalized logistic type III has only been obtained from mixtures of Gumbel I and Gumbel II and as a special case of extended $GLIV$. This a distribution that needs to be further investigated.

The mixing distributions used with the method of mixtures are all continuous.

4 GENERALIZED LOGISTIC DISTRIBUTIONS BASED ON BETA DISTRIBUTIONS AND THEIR GENERALIZATIONS

4.1 Introduction

The objective of this chapter, is to obtain generalized logistic distributions based on beta distributions and their generalizations. More precisely, to use beta I and beta II distributions and their generalizations through transformations and to generalize the standard logistic distribution using the beta generated approach.

Through generalizations a new distribution" The Extended standard logistic has been proposed." It's methods of construction are also shown.

The charpter is organized as follows; the second section constructs the generalized logistic type IV and extended $GLIV$ using transformation technique based on beta I, the third section shows the construction on $GLIV$ and extended $GLIV$ using based on beta II. It also shows the construction of the four parameter generalized logistic based on beta II. Additionally, it shows construction of the generalized logistic distributions from the beta generator approach.

The fourth sections shows construction of the moment generating function for the extended standard logistic distribution. Section five gives the concluding remarks.

4.2 Generalized Logistic distribution based on beta I distribution and its generalizations

(a) Let

$$X = \ln(\frac{1-Y}{Y}) \quad \textbf{(4.1)}$$

where Y is beta I with parameters a and b
Then

$$\begin{aligned}\frac{y}{1-y} &= e^{-x}\\ y &= e^{-x}\\ \therefore y &= e^{-x} - ye^{-x}\\ \therefore y(1+e^{-x}) &= e^{-x}\\ \therefore y &= \frac{e^{-x}}{1+e^{x}},\end{aligned}$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{-(1+e^{-x})e^{-x} + e^{-x}e^{-x}}{(1+e^{-x})^2}\\ &= \frac{-e^{-x}}{(1+e^{-x})^2}\end{aligned}$$

Therefore, pdf of X is

$$\begin{aligned}f(x) &= g(y)\left|\frac{dy}{dx}\right|\\ &= \frac{y^{a-1}(1-y)^{b-1}}{\beta(a,b)}\frac{e^{-x}}{(1+e^{-x})^2}\\ &= \left(\frac{e^{-x}}{1+e^{-x}}\right)^{a-1}\left[1-\frac{e^{-x}}{1+e^{-x}}\right]^{b-1}\frac{e^{-x}}{(1+e^{-x})^2}\\ \therefore f(x) &= \frac{e^{-(a-1)x}}{\beta(a,b)(1+e^{-x})^{a-1}}\left(\frac{1}{1+e^{-x}}\right)^{b-1}\frac{e^{-x}}{(1+e^{-x})^2}\\ &= \frac{e^{-ax}}{\beta(a,b)(1+e^{-x})^{a+b}} \qquad \textbf{(4.2)}\end{aligned}$$

which is $GLIV$

(b) Let

$$X = -\ln\left(\frac{\lambda Y}{1-Y}\right),\ \lambda > 0 \qquad \textbf{(4.3)}$$

where Y is beta I with parameters a and b.

$$\begin{aligned}
&\therefore \frac{\lambda y}{1-y} = e^{-x}\\
&\therefore \lambda y = e^{-x} - y e^{-x}\\
&\therefore (\lambda + e^{-x}) y = e^{-x}\\
&\therefore y = \frac{e^{-x}}{\lambda + e^{-x}}\\
&\therefore \frac{dy}{dx} = \frac{-(\lambda + e^{-x})e^{-x} + e^{-x}e^{-x}}{(\lambda + e^{-x})^2} = \frac{-\lambda e^{-x}}{(\lambda + e^{-x})^2}
\end{aligned}$$

$$\begin{aligned}
\therefore f(x) =& \frac{y^{a-1}(1-y)^{b-1}}{\beta(a,b)} \frac{\lambda e^{-x}}{(\lambda + e^{-x})^2}\\
=& \left(\frac{e^{-x}}{\lambda + e^{-x}}\right)^{a-1} \left[1 - \frac{e^{-x}}{\lambda + e^{-x}}\right]^{b-1} \frac{\lambda e^{-x}}{(\lambda + e^{-x})^2}\\
=& \frac{\lambda e^{-ax} \lambda^{b-1}}{\beta(a,b)(\lambda + e^{-x})^{a+b}}\\
=& \frac{\lambda^b e^{-ax}}{\beta(a,b)(\lambda + e^{-x})^{a+b}}, \quad -\infty < x < \infty, \quad \lambda > 0, \quad a > 0, \quad b > 0. \qquad \textbf{(4.4)}
\end{aligned}$$

which is extended $GLIV$

4.3 Generalized logistic distributions based on beta II distribution and its generalizations

(a.) Let

$$X = -\ln Y \qquad \textbf{(4.5)}$$

where Y has beta II distribution with parameters a and b. Then,

$$\begin{aligned}
y =& e^{-x}\\
\therefore \frac{dy}{dx} =& -e^{-x}\\
\therefore f(x) =& g(y)\left|\frac{dy}{dx}\right|\\
=& \frac{y^{a-1}}{\beta(a,b)(1+y)^{a+b}} \cdot e^{-x}\\
=& \frac{e^{-x(a-1)}e^{-x}}{\beta(a,b)[1+e^{-x}]^{a+b}}\\
=& \frac{e^{-ax}}{\beta(a,b)(1+e^{-x})^{a+b}} \quad for \quad -\infty < x < \infty; \quad a > 0, \quad b > 0 \qquad \textbf{(4.6)}
\end{aligned}$$

which is a $GLIV$ distribution.

(b.) Let

$$\begin{aligned}
X &= -\ln \lambda Y \\
\therefore y &= \frac{e^x}{\lambda} \implies \frac{dy}{dx} = -\frac{1}{\lambda}e^{-x} \\
\therefore f(x) &= \frac{y^{a-1}}{\beta(a,b)[1+y]^{a+b}}\frac{1}{\lambda}e^{-x}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(e^{-x})^{a-1}}{\lambda^{a-1}\beta(a,b)}\frac{1}{[1+\frac{e^{-x}}{\lambda}]^{a+b}}\frac{1}{\lambda}e^{-x} \\
&= \frac{e^{-ax}}{\lambda^a\beta(a,b)\frac{(\lambda+e^{-x})^{a+b}}{\lambda^{a+b}}} \\
&= \frac{\lambda^b e^{-ax}}{\beta(a,b)(\lambda+e^{-x})^{a+b}} \qquad \textbf{(4.7)}
\end{aligned}$$

which is extended $GLIV$

(c.) A four parameter generalized beta II $G4BII$ distribution.
Let

$$T = \left(\frac{Y}{p}\right)^q$$

where T is a beta II distribution with two parameters a and b

Therefore the pdf of y is

$$\begin{aligned}
g(y) &= \frac{t^{a-1}}{\beta(a,b)(1+t)^{a+b}}\left|\frac{dt}{dy}\right| \\
&= \frac{t^{a-1}}{\beta(a,b)(1+t)^{a+b}} \cdot \frac{q}{p}\left(\frac{y}{p}\right)^{q-1} \\
&= \frac{\left(\frac{y}{p}\right)^{aq-q}}{\beta(a,b)[1+\left(\frac{y}{p}\right)^q]^{a+b}}\frac{q}{p}\left(\frac{y}{p}\right)^{q-1} \\
&= \frac{\left(\frac{q}{p}\right)}{\beta(a,b)}\frac{\left(\frac{y}{p}\right)^{aq-1}}{[1+\left(\frac{y}{p}\right)]^{a+b}}, \quad y>0, \quad a,b,p,q>0 \qquad \textbf{(4.8)}
\end{aligned}$$

This is $G4BII$ distribution.

Next, let

$$Y = e^{-X}$$

where y is $G4BII$ distributed.

Then the pdf of x is

$$f(x) = \frac{(\frac{q}{p})}{\beta(a,b)} \frac{\left(\frac{y}{p}\right)^{aq-1}}{\beta(a,b)[1+\left(\frac{y}{p}\right)^{p}]^{a+b}} \left[\frac{dy}{dx}\right] \tag{4.9}$$

Therefore,

$$\begin{aligned} f(x) =& \frac{q}{p} \frac{\left(\frac{e^{-x}}{p}\right)^{aq-1}}{\beta(a,b\left[1+\left(\frac{e^{-x}}{p}\right)^{q}\right]^{a+b}} \\ =& \frac{q}{p^{aq}} \frac{e^{aqx}}{\beta(a,b)[1+\left(\frac{e^{-x}}{p}\right)^{q}]^{a+b}}; \quad x>0; \quad a,b,p,q>0 \end{aligned} \tag{4.10}$$

McDonald (1984) calls this distribution exponential generalized beta II, $EGBII$

When $q=1$, then

$$\begin{aligned} f(x) =& \frac{1}{p^{a}} \frac{e^{-ax}}{\beta(a,b)[1+\frac{e^{-x}}{p}]^{a+b}} \\ =& \frac{p^{b}}{\beta(a,b)} \frac{e^{-ax}}{p+(e^{-x})^{a+b}} \end{aligned} \tag{4.11}$$

which is extended $GLIV$ distribution.

Therefore $EGBII$ is a generalization of extended $GLIV$.

4.3.1 Beta Generators Approach

A cumulative distribution function (cdf) of beta 1 random variable is given by;

$$W(y) = \int_{0}^{y} \frac{t^{a-1}(1-t)^{b-1}}{\beta(a,b)} dt \tag{4.12}$$

Since $0 \leq y \leq 1$, it can be replaced by a continuous cdf of X, say $G(x)$, for $-\infty < x\infty$

This is because $0 \leq G(x) \leq 1$

$$\begin{aligned} \therefore F(x) =& W[G(x)] = \int_{0}^{G(x)} \frac{t^{a-1}(1-t)^{b-1}}{\beta(a,b)} dt \\ \therefore f(x) =& \frac{d}{dx} \int_{0}^{G(x)} \frac{t^{a-1}(1-t)^{b-1}}{\beta(a,b)} dt \\ =& \frac{[G(x)]^{a-1}[1-G(x)]^{b-1}g(x)}{\beta(a,b)} \end{aligned} \tag{4.13}$$

where $g(x) = \frac{d}{dx}G(x)$. This concepts were introduced by Eugene et al (2000) and Jones (2004).

Special cases

(i) $b=1$

$$\begin{aligned} b=1 \implies F(x) =& \int_0^{G(x)} at^{a-1}dt \\ =& \Big(G(x)\Big)^a \end{aligned}$$

and

$$f(x) = a\Big(G(x)\Big)^{a-1} g(x), \quad a>0 \; -\infty < x < \infty$$

which is exponentiated Type I distribution.

(ii) $a=1$

$$a=1 \implies F(x) = \int_0^{G(x)} b(1-t)^{b-1}dt$$

Let $y = 1-t \implies dy = -dt$

$$\begin{aligned} F(x) =& b \int_{1-G(x)}^{1} y^{b-1}dy \\ =& 1-[1-G(x)]^b \end{aligned}$$

which is exponentiated Type II distribution.

(iii) $a=i$ and $b=n-i+1$

$$\therefore F(x) = \int_0^{G(x)} \frac{t^{i-1}(1-t)^{n-i}}{\beta(i,n-i+1)}dt$$

when $i=1$, then

$$\begin{aligned} F(x) =& \int_0^{G(x)} \frac{(1-t)^{n-1}}{\beta(1,n)} \\ =& 1-[1-G(x)]^n \end{aligned}$$

and

$$f(x) = n[1-G(x)]^{n-1}g(x)$$

which is the distribution of the minimum order statistic.

When $i = n$, then

$$F(x) = \int_0^{G(x)} \frac{t^{n-1}}{\beta(n,1)}$$
$$=[G(x)]^n$$

and

$$f(x) = n[G(x)]^{n-1} g(x)$$

which is the distribution of maximum order statistics.

For the extended standard logistic distribution

$$G(x) = \frac{\lambda}{\lambda + e^{-x}}, \quad -\infty < x < \infty; \quad \lambda > 0$$
$$\therefore F(x) = \int_{-\infty}^{\lambda(\lambda+e^{-x})^{-1}} \frac{t^{a-1}(1-t)^{b-1}}{\beta(a,b)} dt$$
$$\therefore f(x) = \frac{\left(\frac{\lambda}{\lambda+e^{-x}}\right)^{a-1}\left[1 - \frac{\lambda}{\lambda+e^{-X}}\right]^{b-1}}{\beta(a,b)} \frac{d}{dx}\lambda(\lambda + e^{-x})^{-1}$$
$$= \frac{\lambda^{a-1}[\lambda + e^{-x} - \lambda]}{\beta(a,b)(\lambda = e^{-x})^{a+b-2}} \frac{\lambda e^{-x}}{(\lambda + e^{-x})^2}$$
$$= \frac{\lambda^a e^{-bx}}{\beta(a,b)(\lambda + e^{-x})^{a+b}} \quad for \quad -\infty < x < \infty; \quad a < 0, \ b > 0, \ \lambda > 0 \qquad \textbf{(4.14)}$$

which is an extended $GLIV$

Special cases

(i) $b = 1$

$$b = 1 \implies f(x) = \frac{\lambda^a e^{-x}}{\beta(a,1)(\lambda + e^{-x})^{a+1}}$$
$$i.e. \quad f(x) = \frac{a\lambda^a e^{-x}}{(\lambda + e^{-x})^{a-1}}$$
$$= \frac{a\lambda^a e^{-x}}{(\lambda + \frac{\lambda}{\lambda} e^{-x})^{a+1}}$$
$$= \frac{\frac{a}{\lambda} e^{-x}}{(1 - \frac{1}{\lambda} e^{-x})^{a+1}}$$
$$= \frac{a\rho e^{-x}}{(1 + \rho e^{-x})^{a+1}} \quad -\infty < x < \infty, \quad a > 0, \rho > 0 \qquad \textbf{(4.15)}$$

This is an extended GLI distribution.

(ii) $a=1$

$$a=1 \implies f(x)=\frac{\lambda e^{-bx}}{\beta(1,b)(\lambda+e^{-x})^{b+1}}$$
$$=\frac{b\lambda e^{-bx}}{(\lambda+e^{-x})^{b+1}}$$

which is an extended $GLII$ distribution.

(iii) $b=a$

$$b=a \implies f(x)=\frac{\lambda^{a}e^{-ax}}{\beta(a,a)(\lambda+e^{-x})^{2a}} \tag{4.16}$$

which is an extended $GLIII$ distribution.

(iv) $a=1, \quad b=n$

$$\therefore f(x)=\frac{\lambda e^{-nx}}{\beta(1,n)(\lambda+e^{-x})^{n+1}}$$
$$=\frac{n\lambda e^{-nx}}{(\lambda+e^{-x})^{n+1}} \quad for -\infty<x<\infty, \quad \lambda>0, \quad n=1,2,3\ldots \tag{4.17}$$

This is minimum order statistics distribution of an extended standard logistic distribution.

(v) $a=n, \quad b=1$

$$\therefore f(x)=\frac{\lambda^{n}e^{-x}}{\beta(n,1)(\lambda+e^{-x})^{n+1}}$$
$$=\frac{n\lambda^{n}e^{-x}}{(\lambda+e^{-x})^{n+1}} \quad for -\infty<x<\infty, \quad \lambda>0, \quad n=1,2,3\ldots \tag{4.18}$$

which is a maximum order statistics distribution from the extended standard logistic.

(vi) Other special cases are: Standard logistic , $GLI, GLII, GLIII, GLIV$, maximum and minimum order statistic distribution of the standard logistic.

4.4 The Extended Standard Logistic Distribution

Extended Standard Logistic is a new distribution which has been introduced. It's methods of construction follow. The moment generating function of the Extended standard logistic distribution has been calculated.

4.4.1 Constructions based on Gumbel I mixtures

Let
$g(\lambda) = \beta e^{-\beta\lambda}, \quad \lambda > 0, \quad \beta > 0$
This is an exponential mixing distribution with parameter λ

$$\begin{aligned}
\therefore f(x) =& \int_0^\infty e^{-x}\lambda \exp(-\lambda e^{-x})\beta e^{-\beta\lambda} d\lambda \\
=& \beta e^{-x} \int_0^\infty \lambda \exp(-(\beta + e^{-x})\lambda) d\lambda \\
=& \beta e^{-x} \frac{\Gamma 2}{(\beta + e^{-x})^2} \\
=& \frac{\beta e^{-x}}{(\beta + e^{-x})^2}, \quad -\infty < x < \infty
\end{aligned} \tag{4.19}$$

4.4.2 Constructions based on Gumbel II mixtures

$$\begin{aligned}
g(\lambda) =& \beta e^{-\beta\lambda}, \quad \lambda > 0, \quad \beta > 0 \\
\therefore f(x) =& \int_0^\infty \lambda e^{x} \exp(-\lambda e^{x})\beta e^{-\beta} e^{-\beta\lambda} d\lambda \\
=& \beta e^{x} \int_0^\infty \lambda \exp(-(\beta + e^{x})\lambda) d\lambda \\
\therefore f(x) =& \beta e^{x} \int_0^\infty \lambda e^{-(\beta + e^{x})\lambda} d\lambda \\
=& \frac{\beta e^{x}}{(\beta + e^{x})^2} \quad -\infty < x \infty, \quad \beta > 0 \\
=& \frac{\beta e^{-x}}{(1 + \beta e^{-x})^2}
\end{aligned} \tag{4.20}$$

4.4.3 Construction based on the differences of two independent Gumbel random variables

$$\begin{cases} f(x_1) = e^{-x_1} \exp(-e^{x_1}) & -\infty < z < \infty \\ f(x_2) = \alpha e^{-x_2} \exp(-\alpha e^{-x_2}) & -\infty < z < \infty \end{cases} \tag{4.21}$$

$$
\begin{aligned}
\therefore g(z) &= \int_{-\infty}^{\infty} e^{-(z+x_2)} \exp(-e^{-(z+x_2)}) \alpha e^{-x_2} \exp(-\alpha e^{-x_2}) dx_2 \\
&= \alpha e^{-z} \int_{-\infty}^{\infty} e^{-x_2} \exp(-e^{-z} e^{-x_2} - \alpha e^{-x_2}) e^{-x_2} dx_2 \\
&= \alpha e^{-z} \int_{-\infty}^{\infty} y e^{-(\alpha + e^{-z}) y} dy
\end{aligned}
$$

where $y = e^{-x_2} \implies dy = -e^{x_2} dy$

$$
\therefore g(z) = \frac{\alpha e^{-z}}{(\alpha + e^{-z})^2}, \quad -\infty < z < \infty, \quad \alpha > 0 \tag{4.22}
$$

4.4.4 Construction based on logarithmic transformation

Let

$$
\begin{aligned}
X &= ln\left(\frac{1 - \frac{1}{2}e^{-y}}{\frac{\lambda}{2}e^{-y}}\right) \\
\therefore \frac{1 - \frac{1}{2}e^{-y}}{\frac{\lambda}{2}e^{-y}} &= e^{x} \\
\therefore 1 - \frac{1}{2}e^{-y} &= \frac{\lambda}{2} e^{x} e^{-y} \\
\therefore 1 &= \frac{1}{2} e^{-y} (1 + \lambda e^{x}) \\
\therefore e^{-y} &= \frac{2}{1 + \lambda e^{x}} \\
-y &= \ln 2 - \ln(1 + \lambda e^{x}) \\
y &= \ln(1 + \lambda e^{x}) - \ln 2 \\
\frac{dy}{dx} &= \frac{\lambda e^{x}}{1 + \lambda e^{x}} \\
\therefore f(x) &= \frac{1}{2} e^{-|y|} \frac{dy}{dx} \\
&= \frac{1}{2} \cdot \frac{2}{1 + \lambda e^{x}} \cdot \frac{\lambda e^{x}}{1 + \lambda e^{x}} \\
&= \frac{\lambda e^{x}}{(1 + \lambda e^{x})^2}, \quad -\infty < x < \infty; \quad \lambda > 0
\end{aligned}
$$

that is,

$$
f(x) = \frac{\lambda e^{-x}}{(\lambda + e^{-x})^2}, -\infty < x < \infty; \quad \lambda > 0
$$

4.4.5 MGF Extended Standard Logistic Distribution.

$$
\begin{aligned}
M_X(t) =& E[e^{tX}] \\
=& \int_{-\infty}^{\infty} e^{tx}\frac{\lambda e^{-x}}{(\lambda+e^{-x})^2}dx
\end{aligned}
$$

Let

$$
y = \frac{1}{\lambda+e^{-x}} = (\lambda+e^{-x})^{-1}
$$

$$
\begin{aligned}
\frac{dy}{dx} =& \frac{e^{-x}}{(\lambda+e^{-x})^2} \\
M_X(t) =& \int_0^{\frac{1}{\lambda}} \lambda e^{tx} dy \\
=& \lambda \int_0^{\frac{1}{\lambda}} e^{tx} dy \\
\textit{from } y =& \frac{1}{\lambda+e^{-x}} \implies \lambda+e^{-x} = \frac{1}{y} \\
\therefore e^{-x} =& \frac{1}{y} - \lambda = \frac{1-\lambda y}{y} \\
\therefore e^{x} =& \frac{y}{1+\lambda y} \\
M_X(t) =& \lambda \int_0^{\frac{1}{\lambda}} \left(\frac{y}{1-\lambda y}\right)^t dy \\
\textit{Let } z =& \lambda y \implies \frac{dz}{\lambda} = dy \\
\therefore M_X(t) =& \lambda \int_0^{1} \left(\frac{z}{\lambda(1-z)}\right)^t \frac{dz}{\lambda} \\
=& \int_0^{1} \left(\frac{z}{\lambda(1-z)}\right)^t dz \\
=& \frac{1}{\lambda^t} \int_0^{1} z^t (1-z)^{-t} dz \\
=& \frac{1}{\lambda^t} B(t+1, 1-t)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\lambda^t}\Gamma(t+1)\Gamma(1-t)\\
&= \frac{1}{\lambda^t}\Gamma(1-t)\Gamma(1+t)\\
&= \frac{1}{\lambda^t}\Gamma(t)\Gamma(1-t)\\
&= \frac{\pi t}{\lambda^t \sin \pi t}
\end{aligned}
\tag{4.23}
$$

4.5 Summary

Using beta distribution and their generalization, we have obtained $GLIV$, Extended $GLIV$ and $EGB2$ distributions.

Whence their special cases can be determined . The major contribution of this chapter is an introduction of the "Extended " standard logistic distribution and using it in beta generated approach to determine extended $GLIV$ distribution and its special cases.

$EGB2$ is a generalization of the extended $GLIV$ distribution.

5 DISCRETE MIXTURES BASED MINIMUM AND MAXIMUM ORDER STATISTICS FROM STANDARD LOGISTIC DISTRIBUTION

5.1 Introduction

The main objective of this chapter is to construct the pdf, survival functions and hazard functions of discrete mixtures based on minimum and maximum order statistics from the standard logistic distribution. The shapes of the resulting distributions thereof, have been simulated.

The mixing distributions used are the zero truncated power series distributions namely; zero -truncated Poisson, zero-truncated binomial, zero- truncated (shifted) geometric, zero-truncated negative binomial and the logarithmic series distributions.

The results shall be obtained for both minimum and maximum order statistics distributions.

David and Johnson (1952) gave the pdf of the truncated distribution. Generally they expressed the pmf of the truncated distribution as;

$$g(x) = \frac{f(x)}{1 - f(0)} \tag{5.1}$$

where $f(x)$ is the pmf of the untruncated distribution and $f(0)$ is the pmf evaluated at 0.

Generally the pdf of the mixed distribution can be given by;

$$f(x) = \sum_{\lambda} f(x|\lambda) g(\lambda) \tag{5.2}$$

where

$f(x|\lambda)$ is the conditional probability distribution,

$g(\lambda)$ is the mixing distribution and

$f(x)$ is the mixture or mixed distribution.

The problem is to find $f(x)$ for various $g(\lambda)$, which in this case are the zero- truncated distributions. The survival $S(x)$and hazard $h(x)$ functions of the mixed distributions shall also be obtained as follows;

$$S(x) = 1 - G(x) \tag{5.3}$$

where $G(x)$ is the cdf of X

$$h(x) = \frac{g(x)}{S(x)} \tag{5.4}$$

or by the pgf technique

5.1.1 Distributions of Order statistics using the beta generator

The beta generator is given by;

$$\begin{aligned} f(x) =& \frac{\left(G(x)\right)^{a-1}\left(1-G(x)\right)^{b-1}}{B(a,b)} g(x),\ a,\ b > 0;\ 0 < G(x) < 1,\ -\infty < x < \infty \\ B(a,b) =& \frac{\Gamma a \Gamma b}{\Gamma(a+b)} \end{aligned} \tag{5.5}$$

For i^{th} order statistic, $a = i$ and $b = n - r + 1$, then $i = 1$ is for the minimum and $i = n$ is for the maximum. Thus,

$$\begin{aligned} f(x) =& \frac{[G(x)]^{r-1}[1-G(x)]^{(n-r+1)-1}}{B(r, n-r+1)} g(x), \quad 0 < G(x) < 1, -\infty < x < \infty \\ =& \frac{\Gamma(n+1)}{\Gamma r \Gamma(n-r+1)} g(x)[G(x)]^{r-1}\left(1-G(x)\right)^{(n-r}, \quad 0 < G(x) < 1,\ -\infty < x < \infty \end{aligned} \tag{5.6}$$

When $r = 1$ we shall obtain the pdf of the minimum order statistics as;

$$\begin{aligned} =& \frac{\Gamma(n+1)}{\Gamma 1 \Gamma n} g(x)[1-G(x)]^{(n-1}, \qquad 0 < G(x) < 1, -\infty < x < \infty \\ =& n g(x)\left(1-G(x)\right)^{(n-1},\quad 0 < G(x) < 1,\ -\infty < x < \infty \end{aligned} \tag{5.7}$$

When $r = n$ we shall obtain the pdf of the maximum order statistics distribution as;

$$\begin{aligned} f(x) =& \frac{\left(G(x)\right)^{n-1}}{B(n,1)} g(x), \quad 0 < G(x) < 1,\ -\infty < x < \infty \\ =& n g(x)\left(G(x)\right)^{n-1} \end{aligned} \tag{5.8}$$

Also, from the beta generator, if $a = i$ and $b = n - i + 1$ where $1 \leq i \leq n$, then we shall have the i^{th} order statistics distribution given by;

$$F_{i:n}(x) = \int_0^{G(x)} \frac{t^{i-1}(1-t)^{n-i}}{B(i,n-i+1)} dt$$

and

$$f_{i:n}(x) = \frac{[G(x)]^{i-1}[1-G(x)]^{n-i}}{B(i,n-i+1)} g(x)$$

When $i = 1$, we have the minimum distribution given by:

$$F_{1:n}(x) = 1 - [1-G(x)]^n$$

and

$$f_{1:n}(x) = n[1-G(x)]^{n-1} g(x)$$

When $i = n$, we have the maximum distribution given by;

$$F_{n:n}(x) = [G(x)]^n$$

and

$$f_{n:n}(x) = n[G(x)]^{n-1} g(x)$$

5.1.2 The discrete mixture distribution

Let $X_1, X_2, ..., X_N$ be iid continuous random variables where N is also a random variable independent of $X_i's$

Suppose

$$Z = \min(X_1, X_2, ..., X_N)$$

Then the cdf of Z given $N = n$ is

$$F(z|n) = 1 - [1-G(z)]^n \tag{5.9}$$

where $G(z)$ is the cdf of the parent distribution

$$\therefore \qquad 1 - F(z|n) = [1-G(z)]^n \tag{5.10}$$

The survival function of Z given $N = n$ is

$$S(z|n) = 1 - F(z|n) \tag{5.11}$$

i.e the survival function of Z given N.

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} [1 - G(z)]^n p_n \\
&= E[1 - G(z)^N \\
&= \phi_N(1 - G(z)) \qquad \textbf{(5.12)}
\end{aligned}
$$

pgf of N at $1 - G(z)$

The survival function of the discrete mixture of minimum order statistic is the pgf of N at the survival function point of the parent distribution.

Let

$$\psi_N(s) = E(s^N)$$

the pgf of N,

$$
\begin{aligned}
\therefore \qquad S(z) &= E[1 - G(z)]^N \\
&= \psi_N[1 - G(z)]
\end{aligned}
$$

The pdf of the mixture is obtained by the formula

$$f(z) = -\frac{dS}{dz}$$

The hazard function of the mixture is

$$h(z) = \frac{f(z)}{S(z)}$$

and the cdf is

$$G(z) = \frac{1}{1 + e^{-z}} \qquad -\infty < z < \infty$$

therefore, the survival function of the discrete mixture of minimum order statistic from a standard logistic distribution is given by;

$$
\begin{aligned}
S(z) &= \psi_N(1 - G(z)) \\
&= \psi_N[1 - \frac{1}{1 + e^{-z}}] \qquad \textbf{(5.13)}
\end{aligned}
$$

5.2 Discrete Mixtures based on Minimum order statistics from Standard Logistic Distribution

5.2.1 The Minimum order statistics Distribution

Let Y be the first order statistics in a minimum sample

$$Y = \min(X_1, X_2, ..., X_n)$$

then

$$\begin{aligned} G(x) =& Prob\{X_1 \leq x\} \\ =& 1 - Prob\{X_1 > x\} \\ =& 1 - Prob\{X_1 > x, X_2 > x, ..., X_n > x\} \\ =& 1 - Prob\Big(X_i > x\Big)^n \\ =& 1 - \{1 - Prob(X_i \leq x)\}^n \\ =& 1 - \Big(1 - F(x)\Big)^n \end{aligned} \tag{5.14}$$

which is the cdf of the minimum order statistics distribution.

$$\begin{aligned} g(x) =& n\Big(1 - F(x)\Big)^{n-1})f(x) \\ G(x|n) =& 1 - \Big(1 - \frac{1}{1+e^x}\Big)^n \\ =& 1 - \Big(\frac{1+e^x - 1}{1+e^x}\Big)^n \\ =& 1 - (\frac{e^x}{1+e^x})^n \\ g(x) =& (\frac{e^x}{1+e^x})^{n-1} \frac{ne^x}{(1+e^x)^2}, \qquad -\infty < x < \infty \\ =& \frac{e^{xn}}{(1+e^x)^n} \frac{1+e^x}{e^x} \frac{ne^x}{(1+e^x)^2} \qquad -\infty < x < \infty \\ =& \frac{ne^{xn}}{(1+e^x)^{n+1}} \qquad -\infty < x < \infty \end{aligned} \tag{5.8}$$

Survival and Hazard functions of the minimum order statistics distribution for the standard logistic

$$S(x) = 1 - [1 - \frac{1}{1+e^x}]^n = (\frac{e^x}{1+e^x})^n$$

$$
\begin{aligned}
h(x) =& n(\frac{e^x}{1+e^x})^{n-1}\frac{e^x}{(1+e^x)^2}\Big/\Big(\frac{e^x}{1+e^x}\Big)^n \\
=& \frac{e^{xn}}{(1+e^x)^n}\frac{1+e^x}{e^x}\frac{e^x}{(1+e^x)^2}\times\frac{(1+e^x)^n}{e^{xn}} \\
=& \frac{e^{xn}}{(1+e^x)^n}\frac{1+e^x}{e^x}\frac{e^x}{(1+e^x)^2}\times\frac{(1+e^x)^n}{e^{xn}} \\
=& \frac{n}{1+e^x}
\end{aligned}
\tag{5.15}
$$

Mixture based on minimum order statistic from standard logistic

The pdf of the discrete mixture distribution of minimum order statistics from a standard logistic is given by :

$$
\begin{aligned}
g(x) &= \sum_{n=1}^{\infty} n(\frac{e^x}{1+e^x})^{n-1}\frac{e^x}{(1+e^x)^2}P_n \\
&= \frac{\sum_{n=1}^{\infty} n e^{xn}}{(1+e^x)^{n+2}}P_n
\end{aligned}
\tag{5.16}
$$

where P_n is the mixing distribution. Let us consider different cases of P_n

5.2.2 When P_n is Zero- Truncated Poisson distribution

Construction of the Zero- Truncated Poisson distribution

From

$$
\begin{aligned}
e^{\theta} &= \sum_{n=0}^{\infty}\frac{\theta^n}{n!} = 1+\sum_{n=0}^{\infty}\frac{\theta^n}{n!} \\
e^{\theta}-1 &= \sum_{n=0}^{\infty}\frac{\theta^n}{n!} \\
1 &= \sum_{n=0}^{\infty}\frac{\theta^n}{n!(e^{\theta})-1}
\end{aligned}
$$

therefore

$$
P_n = \frac{\theta^n}{n!(e^{\theta}-1)}, \qquad n = 1,2... \tag{5.17}
$$

is a zero truncated Poisson distribution

The Mixed distribution- Logistic- Poisson distribution

This distribution is obtained using the Poisson as the mixing distribution. It can be constructed using two different methods.

Method 1; Construction using the pgf approach:

$$\psi_N(s) = \sum_{n=1}^{\infty} p_n s^n = \frac{e^{\theta s}-1}{e^{\theta}-1} \tag{5.18}$$

therefore

$$\begin{aligned} S(z) =& \psi_N[1-G(z)] \\ =& \psi_N\Big(\frac{e^{-z}}{(1+e^{-z})}\Big) \\ =& \frac{e^{\frac{\theta e^{-z}}{1+e^{-z}}}-1}{e^{\theta}-1} \end{aligned} \tag{5.19}$$

$$\begin{aligned} \frac{ds}{dt} =& \frac{1}{e^{\theta}-1}\frac{d}{dz}\Big[e^{\frac{\theta e^{-z}}{1+e^{-z}}}-1\Big] \\ \therefore \qquad f(z) =& \frac{1}{e^{\theta}-1}\Big\{e^{\frac{\theta e^{-z}}{1+e^{-z}}}[\frac{d}{dz}\frac{\theta e^{-z}}{1+e^{-z}}]\Big\} \\ =& \frac{1}{e^{\theta}-1}e^{\frac{\theta e^{-z}}{1+e^{-z}}}\Big\{\frac{(1+e^{-z})e^{-z}-e^{-z}.e^{2}}{(1+e^{-z})2}\Big\} \\ =& \frac{1}{e^{\theta}-1}e^{\frac{\theta e^{-z}}{1+e^{-z}}}\Big\{\frac{e^{-z}}{(1+e^{-z})^2}\Big\} \\ =& \frac{\theta e^{-z}e^{\frac{\theta e^{z}}{1+e^{-z}}}}{(e^{\theta}-1)(1+e^{-z})^2} \end{aligned} \tag{5.20}$$

$$\begin{aligned} h(z) =& \frac{\theta e^{z}e^{\frac{\theta e^{-z}}{1+e^{-z}}}}{(1+e^{-z})^2}.\frac{1}{1+e^{\frac{\theta e^{-z}}{1+e^{-z}}}} \\ =& \frac{\theta e^{z}e^{\frac{\theta}{1+e^{z}}}}{(1+e^{z})^2(e^{\frac{\theta}{1+e^{z}}})} \end{aligned} \tag{5.21}$$

Method 2; Construction of the pdf using the discrete mixture approach

To obtain the discrete mixture we shall proceed as follows:

$$g(x) = \sum_{n=1}^{\infty} n(\frac{e^{x}}{1+e^{x}})^{n-1}\frac{e^{x}}{(1+e^{x})^2}\frac{\theta^n}{n!(e^{\theta}-1)}$$

$$=\frac{\theta e^x}{(1+e^x)^2(e^\theta-1)}\sum_{n=1}^{\infty}n(\frac{\theta e^x}{1+e^x})^{n-1}\frac{1}{n!}$$
$$=\frac{\theta e^x}{(1+e^x)^2(e^\theta-1)}\sum_{n=1}^{\infty}(\frac{\theta e^x}{1+e^x})^{n-1}\frac{1}{(n-1)!}$$

therefore

$$g(x)=\frac{\theta e^x}{(1+e^x)^2(e^\theta-1)}e^{\frac{\theta e^x}{1+e^x}}, \qquad -\infty<x<\infty$$
$$=\frac{\theta}{e^\theta-1}\frac{e^x}{(1+e^x)^2}e^{\frac{\theta e^x}{1+e^x}}, \qquad -\infty<x<\infty \tag{5.22}$$

which is the pdf of the minimum zero- truncated logistic-Poisson distribution.

To show that $g(x)$ is a pdf

$$\int_{-\infty}^{\infty}g(x)dx=\int_{-\infty}^{\infty}\frac{\theta}{e^\theta-1}\frac{e^x}{(1+e^x)^2}e^{\frac{\theta e^x}{1+e^x}}dx \tag{5.23}$$

Let

$$y=\frac{1}{1+e^x} \implies y+ye^x=1 \implies e^x=\frac{1-y}{y}$$
$$\frac{dy}{dx}=\frac{d}{dx}(1+e^x)^{-1}=-(1+e^x)^{-2}e^x=\frac{-e^x}{(1+e^x)^2}$$
$$x=-\infty \implies y=1; x=\infty \implies y=0$$

therefore

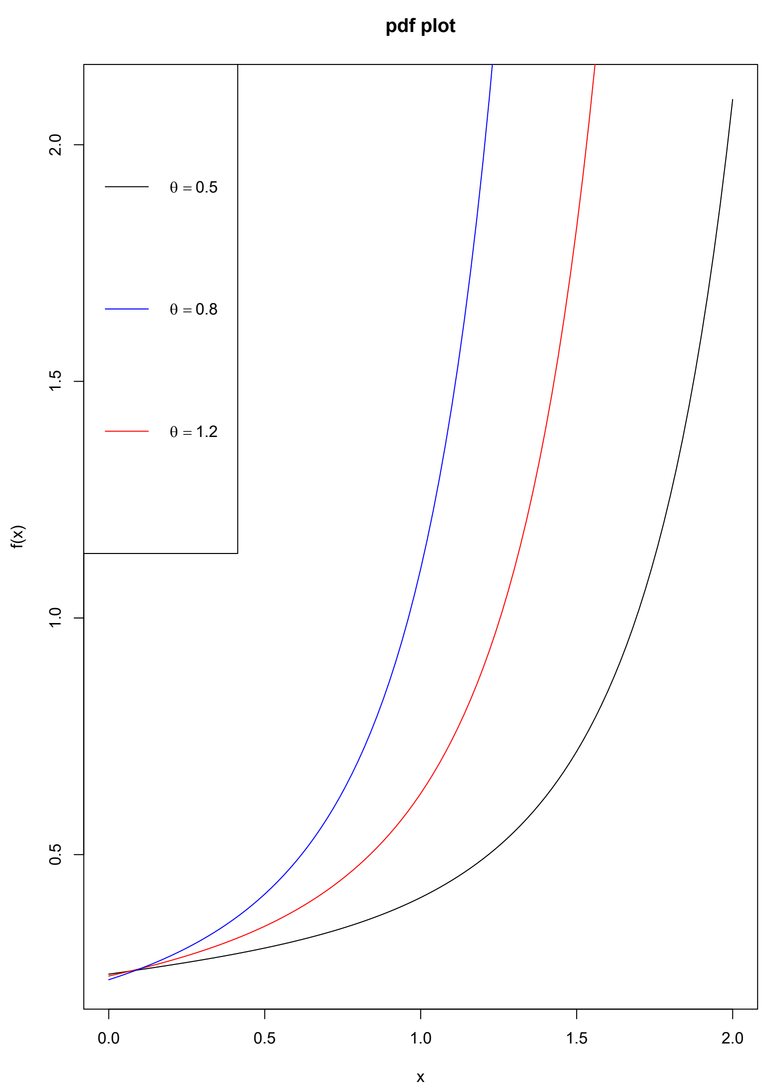
$$\int_{-\infty}^{\infty}g(x)dx=+\int_0^1\frac{\theta}{e^\theta-1}e^{\frac{\theta y(1-y)}{y}}dy$$
$$=\int_0^1\frac{\theta}{e^\theta-1}e^{\theta(1-y)}dy$$
$$=\frac{\theta}{e^\theta-1}e^\theta\int_0^1 e^{-\theta y}dy$$
$$=\frac{\theta}{e^\theta-1}e^\theta[\frac{e^{-\theta y}}{-\theta}]$$
$$=-\frac{e^\theta}{e^\theta-1}[e^{-\theta y}]$$

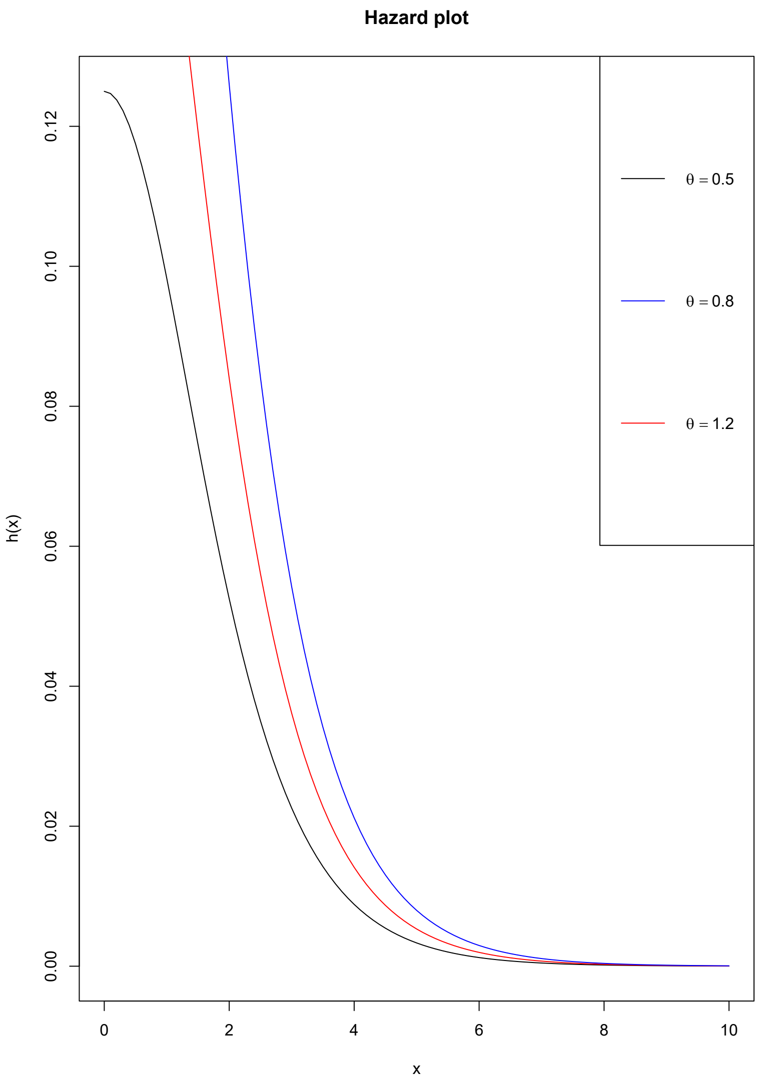
$$
\begin{aligned}
&= -\frac{e^{\theta}}{e^{\theta}-1}[e^{-\theta}-1] \\
&= -\frac{e^{\theta}}{e^{\theta}-1}[1e^{-\theta}-] \\
&= \frac{e^{\theta}-1}{e^{\theta}-1} \\
&= 1
\end{aligned} \tag{5.24}
$$

as required.

5.2.3 Graphs of the pdf and hazard of logistic Poisson distribution

The graphs below show the pdf and hazard functions of the logistic-Poisson with varying values of theta. The pdf of the logistic Poisson is an increasing function while the hazard function is a decreasing function.





5.2.4 When P_n is Zero truncated Binomial distribution

Construction of Zero- Truncated Binomial distribution

The probability distribution of the binomial is given by:

$$P(X=n)=\binom{m}{n}\theta^n(1-\theta)^{m-n}, \qquad n=0,1,2,...,m$$

$$=\sum_{k=0}^{\infty}P(X=n)=(1-\theta)^m+\sum_{k=1}^{m}\binom{m}{n}\theta^n(1-\theta)^{m-n}=1 \tag{5.25}$$

Therefore

$$P(X=n)=\frac{\binom{m}{n}\theta^n(1-\theta)^{m-n}}{1-(1-\theta)^m}, \qquad ,n=1,2,...m \tag{5.26}$$

which is the pmf of the zero-truncated binomial distribution. 5.26 can also be expressed as

$$P(X=n)=\frac{\binom{m}{n}\theta^n}{(1+\theta)^m-1}, \qquad n=1,2,...,m \tag{5.27}$$

because

$$(1+\theta)^m-1=\Big(1-(1+\theta)^m\Big)(1+\theta)^{n-m}$$

5.2.5 Logistic-Binomial Distribution

Here the mixing distribution is the the zero-truncated binomial. The mixed distribution shall be obtained using two methods as follows.

Method 1 using the pgf approach

$$\begin{aligned} p_n =& \frac{\binom{m}{n}\theta^n}{(1+\theta)^m-1}, \qquad n=1,2,...,m \\ \psi_N(s) =& \frac{(1+\theta s)^m-1}{(1+\theta)^m-1} \end{aligned} \tag{5.28}$$

therefore

$$\begin{aligned} S(z) =& \psi_N\Big(\frac{e^{-z}}{1+e^{-z}}\Big) \\ =& \frac{(1+\frac{\theta e^{-z}}{1+e^{-z}})^m-1}{(1+\theta)^m-1} \\ =& \frac{\frac{\big(1+e^{-z}+\theta e^{-z}\big)^m}{(1+e^{-z})^m}-1}{(1+\theta)^m-1} \end{aligned} \tag{5.29}$$

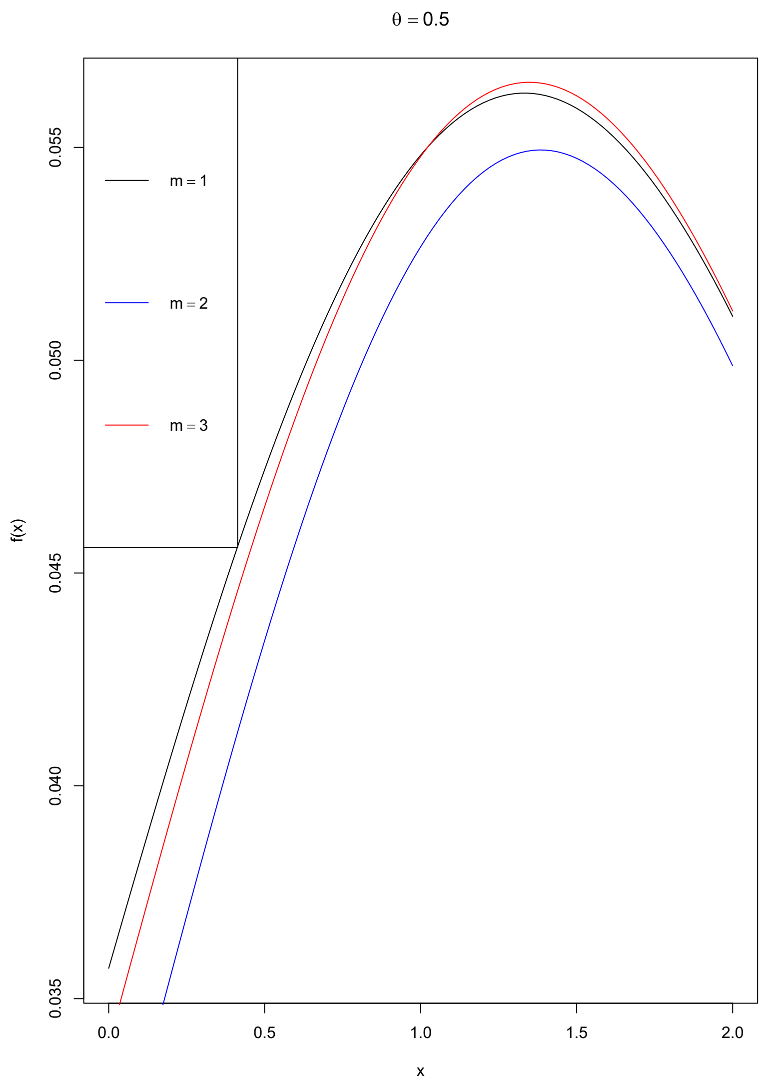
$$\begin{aligned} f(z) =& \frac{m(1+\frac{\theta e^{-z}}{1+e^{-z}})^{m-1}}{(1+\theta)^m-1}\Big\{\frac{d}{dz}\frac{\theta e^{-z}}{1+e^{-z}}\Big\} \\ =& \frac{m\theta(1+\frac{\theta e^{-z}}{1+e^{-z}})^{m-1}}{(1+\theta)^m-1}\frac{(1+e^{-z})e^{-z}-e^2e^{-z}}{(1+e^{-z})^2} \\ =& \frac{m\theta(1+\frac{\theta e^{-z}}{1+e^{-z}})^{m-1}}{(1+\theta)^m-1}\frac{e^{-z}}{(1+e^{-z})^2} \\ =& \frac{m\theta e^{-z}}{(1+e^{-z})^{m+1}}\frac{1}{(1+e^{-z}+\theta e^{-z})(1+\theta)^m-1} \end{aligned} \tag{5.30}$$

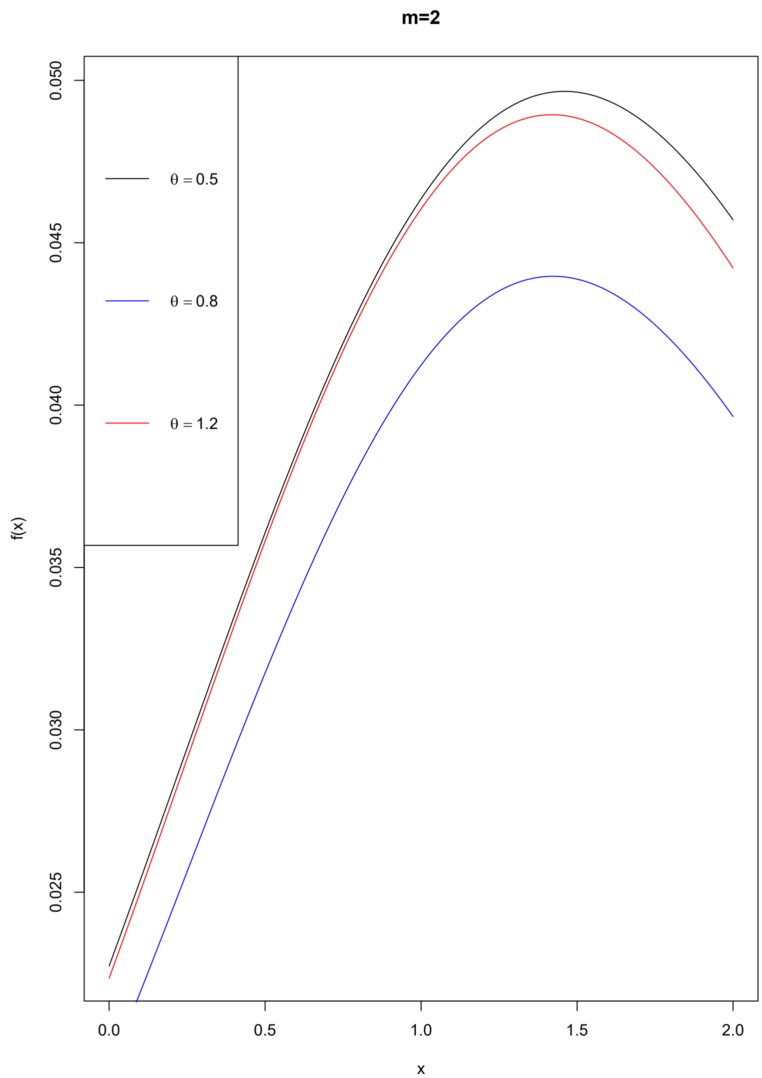
$$h(z) = \frac{m\theta(1+\frac{\theta e^{-z}}{1+e^{-z}})^{m-1}}{[(1+\frac{\theta e^{-z}}{1+e^{-z}})^{m}-1]}\frac{e^{-z}}{(1+e^{-z})^2} \tag{5.31}$$

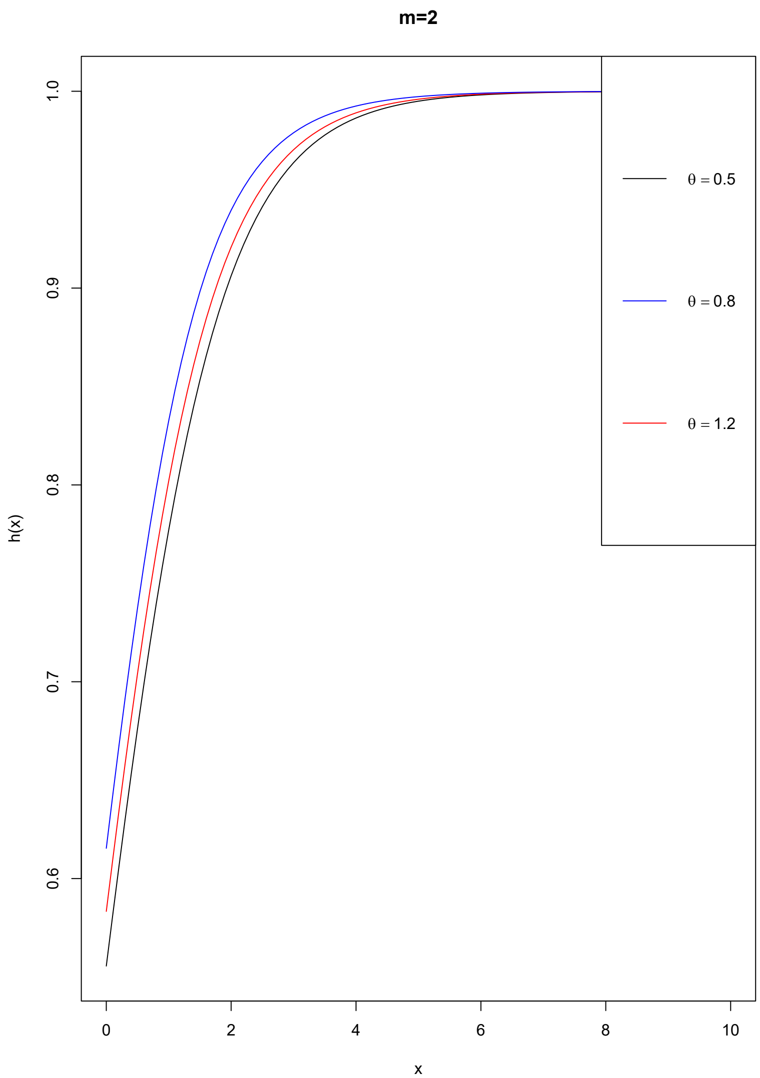
Method 2 : Using the discrete mixture approach

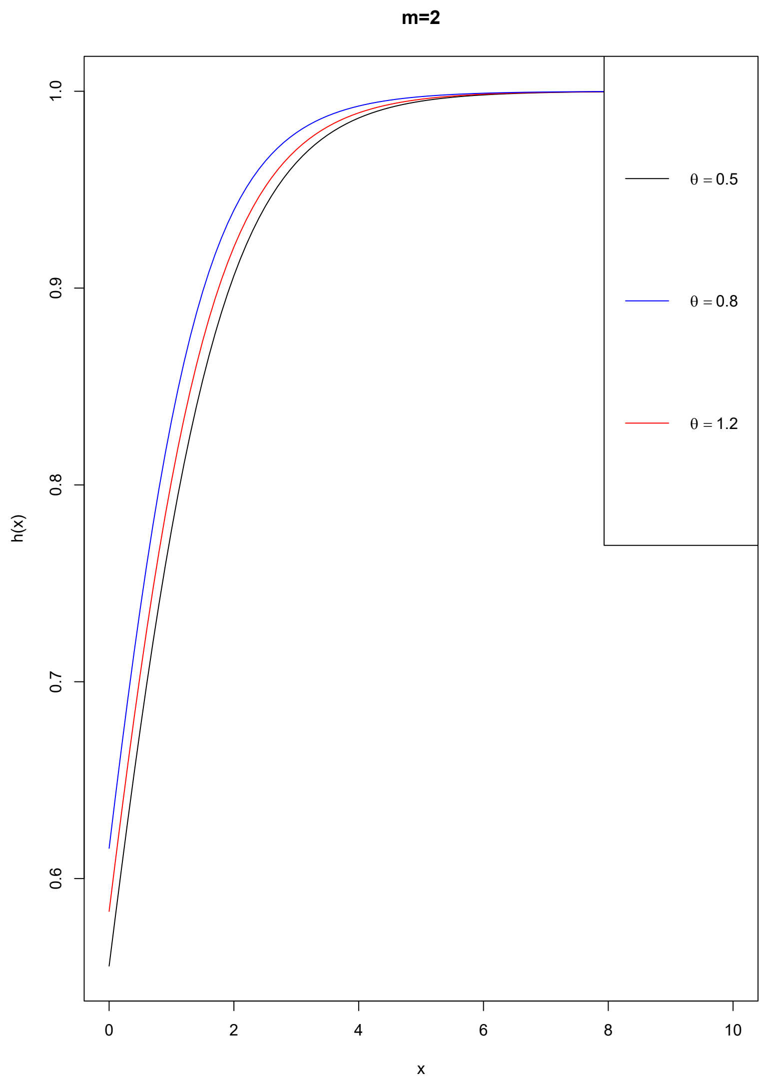
$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} n(\frac{e^x}{1+e^x})^{n-1}\frac{e^x}{(1+e^x)^2}\frac{\binom{m}{n}\theta^n}{(1+\theta)^m-1} \\ &= \frac{1}{(1+e^x)(1+\theta)^{m-1}}\sum_{n=1}^{\infty}\frac{\theta^n e^x}{(1+e^x)^n}\frac{m!}{(n-1)!(m-n)!} \\ &= \frac{1}{(1+e^x)(1+\theta)^{m-1}}\{\frac{\theta m! e^x}{(1+e^x)(m-1)!}+\frac{\theta^2 m! e^{2x}}{(1+e^x)^2(m-2)!}+\frac{\theta^3 m! e^{3x}}{(1+e^x)^3(m-3)!}+...\} \\ &= \frac{1}{(1+e^x)(1+\theta)^{m-1}}\{\frac{\theta m! e^x}{(1+e^x)(m-1)!}[1+\frac{\theta e^x}{1+e^x}+\frac{\theta^2 e^{2x}}{(1+e^x)^2}+\frac{\theta^3 e^{3x}}{(1+e^x)^3}+...\} \\ &= \frac{m\theta e^x(1+\frac{\theta e^x}{(1+e^x)^2})^{m-1}}{(1+e^x)^2(1+\theta)^m-1} \end{aligned} \tag{5.32}$$

5.2.6 Graphs of the pdf and Hazard of Logistic Binomial




m=2
θ = 0.5
θ = 0.8
θ = 1.2
f(x)
0.050
0.045
0.040
0.035
0.030
0.025
0.0
0.5
1.0
1.5
2.0
x


m=2
h(x)
x
θ = 0.5
θ = 0.8
θ = 1.2
0.6
0.7
0.8
0.9
1.0
0
2
4
6
8
10



To plot the above graphs, one parameter is fixed while the other one is varied. The graphs of the pdf are both increasing functions while the graphs for the hazard are both decreasing functions.

5.2.7 When P_n is the shifted Geometric distribution

Construction of the zero-truncated shifted Geometric distribution

The Zero-truncated geometric is also referred to as the shifted geometric distribution. The probability distribution of the geometric with parameter θ is given by:

$$P(X=\theta)=(1-\theta)\theta^{n-1}, \qquad 1,2,...,$$

The logistic -shifted geometric distribution

The logistic-shifted geometric distribution can be constructed using the two methods given below.

Method 1

$$
\begin{aligned}
g(x) &= \sum_{n=1}^{\infty} n(\frac{e^x}{1+e^x})^{n-1}\frac{e^x}{(1+e^x)^2}(1-\theta)\theta^{n-1},\\
&=\frac{e^x(1-\theta)}{(1+e^x)^2}\sum_{n=1}^{\infty} n(\frac{\theta e^x}{1+e^x})^{n-1},\\
&=\frac{e^x(1-\theta)}{(1+e^x)^2}\frac{1}{(1-\frac{\theta e^x}{1+e^x})^2}\\
&=\frac{e^x(1-\theta)}{(1+e^x)^2}\frac{(1+e^x)^2}{(1+e^x-\theta e^x)^2}\\
&=\frac{(1-\theta)e^x}{(1+e^x-\theta e^x)^2}\\
&=\frac{(1-\theta)e^x}{[1+(1-\theta)e^x]^2}, \qquad -\infty < x < \infty,
\end{aligned} \tag{5.33}
$$

Method 2

The shifted (zero-truncated) distribution has

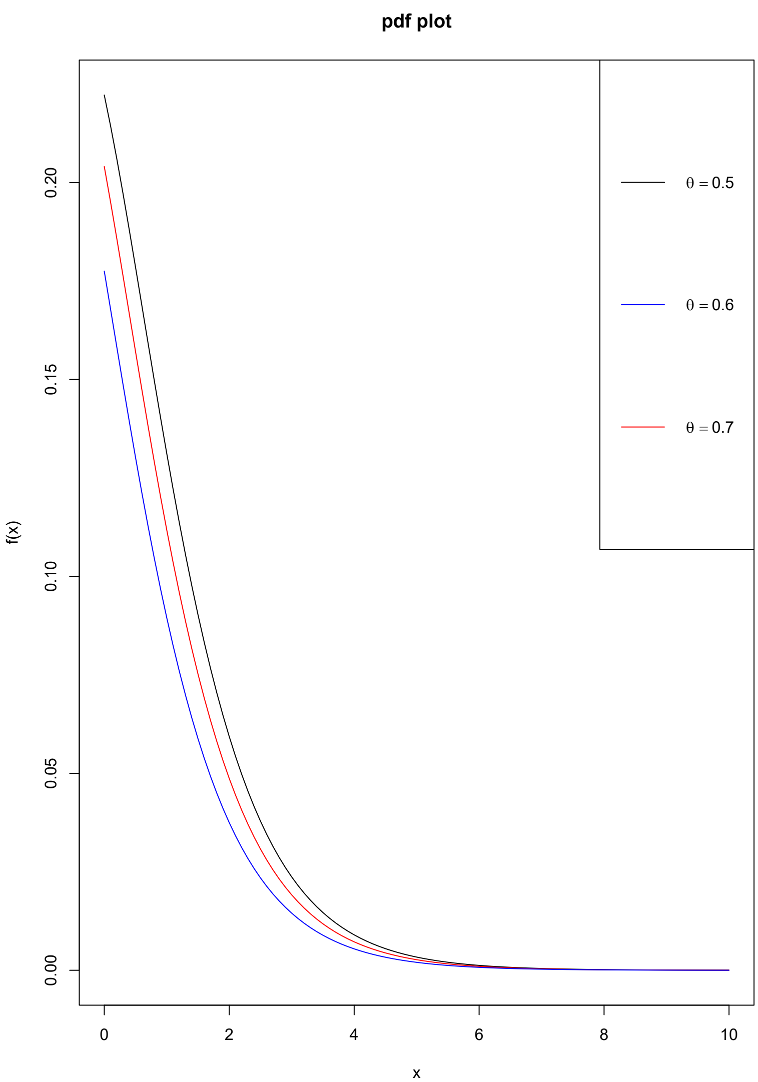
pmf:

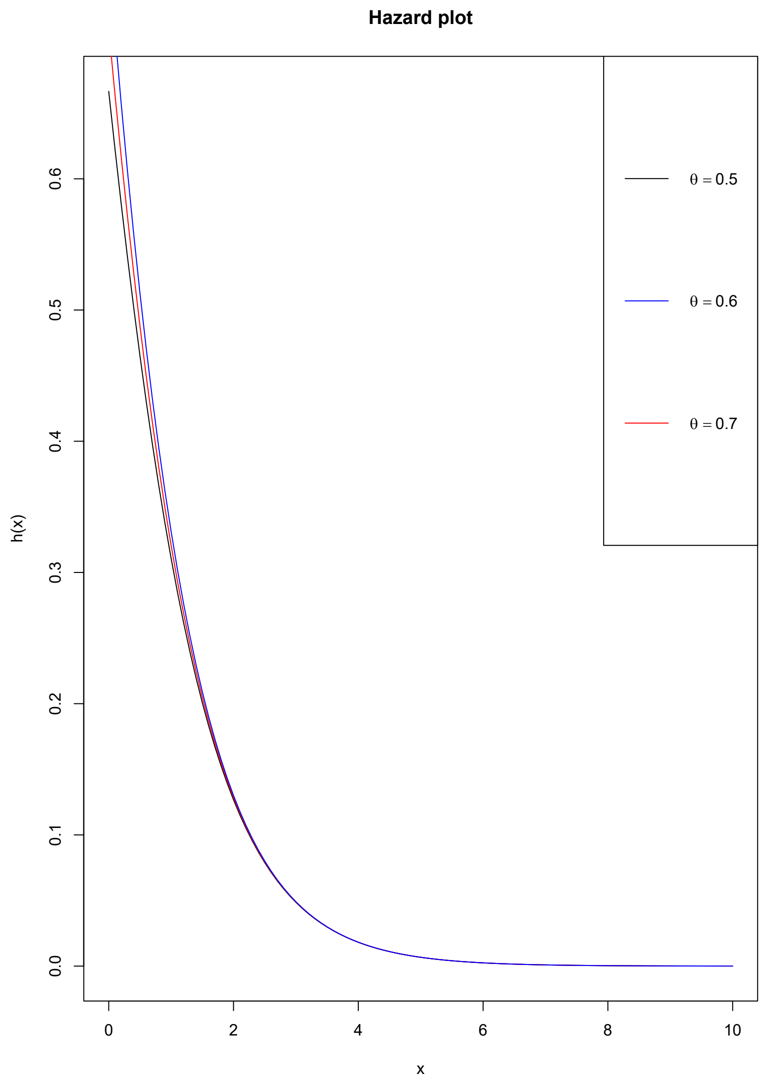
$$
\begin{aligned}
p_n &= \theta^{n-1}(1-\theta), \qquad n = 1,2,...;0<\theta<1\\
\psi_N(s) &= \frac{(1-\theta)s}{1-\theta s}\\
S(z) &= \psi_N(\frac{e^{-z}}{1+e^{-z}})\\
&= \frac{(1-\theta)e^{-z}}{1+e^{-z}-\theta e^{-z}}\\
&= \frac{(1-\theta)e^{-z}}{1+(1-\theta)e^{-z}}\\
&= \frac{1-\theta}{1-\theta+e^{z}}
\end{aligned} \tag{5.34}
$$

$$
\begin{aligned}
f(z) &= \frac{(1-\theta)e^{-z}}{(1-\theta+e^{-z})^2} \\
&= \frac{(1-\theta)e^{-z}}{(1+(1-\theta)e^{-z})^z} && \textbf{(5.35)}
\end{aligned}
$$

$$
\begin{aligned}
h(z) &= \frac{(1-\theta)e^{-z}}{[1+(1-\theta)e^{-z}]^2} \cdot \frac{1+(1-\theta)e^2}{1-\theta} \\
&= \frac{e^{-z}}{1+(1-\theta)e^{-z}} && \textbf{(5.36)}
\end{aligned}
$$

5.2.8 Graphs of the pdf and hazard functions of the logistic geometric





5.2.9 When P_n is zero-truncated negative binomial distribution

Construction of Zero- Truncated Negative binomial distribution

Let

$$(1+\theta)^{\alpha} = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!}\theta^{n}, \qquad -1 < \theta < 1 \tag{5.37}$$

Now change θ to $-\theta$ and substitute r for α

$$(1-\theta)^{-\alpha} = 1 + \sum_{n=1}^{-\alpha} \frac{(-\alpha-1)...(-\alpha-n+1)}{n!}(-\theta^{n}), \qquad -1 < \theta < 1,$$

dropping off $(-1)^n = \Big((-1)^2\Big)^n = 1^n = 1$ so that $(-\alpha)(-\alpha-1)(-\alpha-2)...(-\alpha-n+1)(-\theta)^n = \alpha(\alpha+1(\alpha+2)...(\alpha+n-1)(-1)^{2n})\theta^n = \alpha(\alpha+1(\alpha+2)...(\alpha+n-1)\theta^n$

$$= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)...(\alpha+n-1)}{n!}\theta^n,$$

rearranging

$$= 1 + \sum_{n=1}^{\infty} \frac{(\alpha+n-1(\alpha+n-2)...\alpha)}{n!}\theta^n$$
$$= \sum_{n=0}^{\infty} \binom{n+\alpha-1}{\alpha-1}\theta^n,$$

which is the negative binomial expansion

$$\begin{aligned} P(n;\theta,\alpha) =& P(Y=n) = \frac{1}{1-(1-\theta)^{\alpha}}\binom{n+\alpha-1}{\alpha-1}(1-\theta)^{\alpha}\theta^n, \\ =& P_n = \binom{\alpha+n-1}{n}\frac{\theta^n}{(1-\theta)^{-\alpha}-1}, \qquad n = 1,2,3,..., \end{aligned} \tag{5.38}$$

which is the pmf of the zero-truncated negative binomial distribution.

The logistic-negative binomial distribution

For the zero- truncated negative binomial distribution

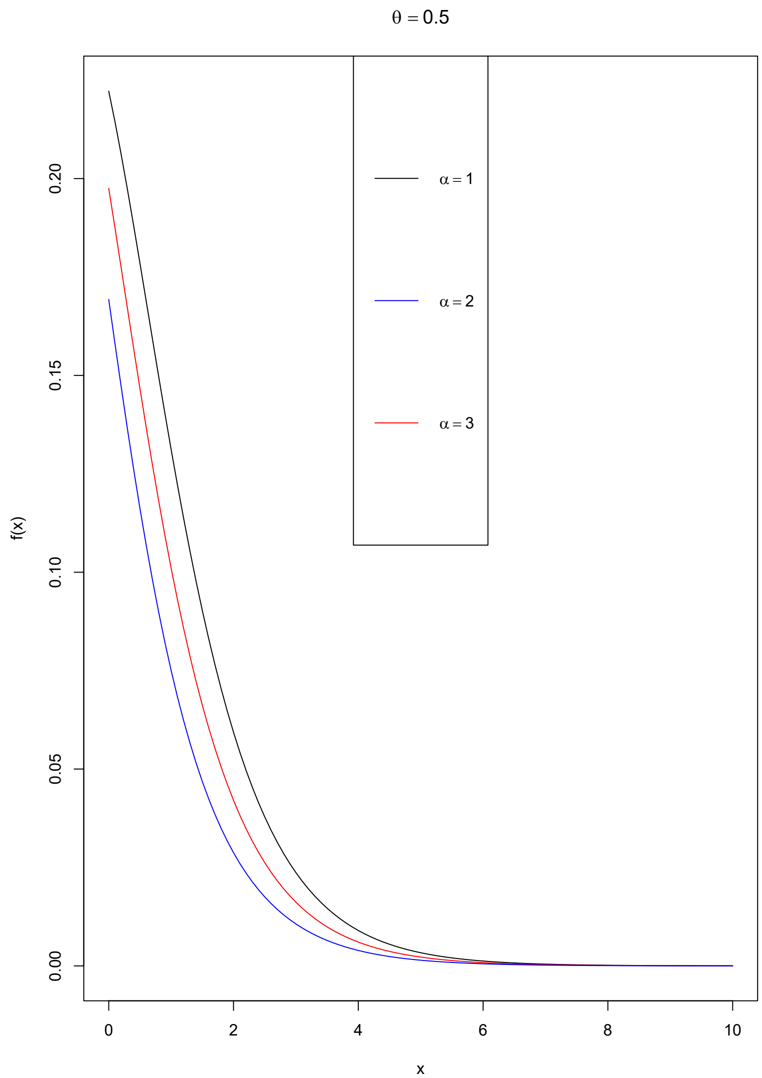
$$\begin{aligned} \psi_N(s) &= \frac{(1-\theta s)^{-\alpha}-1}{(1-\theta)^{-\alpha}-1} \\ s(z) &= \frac{(1-\frac{\theta e^z}{1+e^z})^{-\alpha}-1}{(1-\theta)^{-\alpha}-1} \\ f(z) &= \frac{1}{(1-\theta)^{-\alpha}-1}\Big\{-\alpha(1-\frac{\theta e^z}{1+e^{-z}})^{-\alpha-1}\Big\{\frac{d}{dz}(1-\frac{\theta e^{-z}}{1+e^z})\Big\} \end{aligned} \tag{5.39}$$

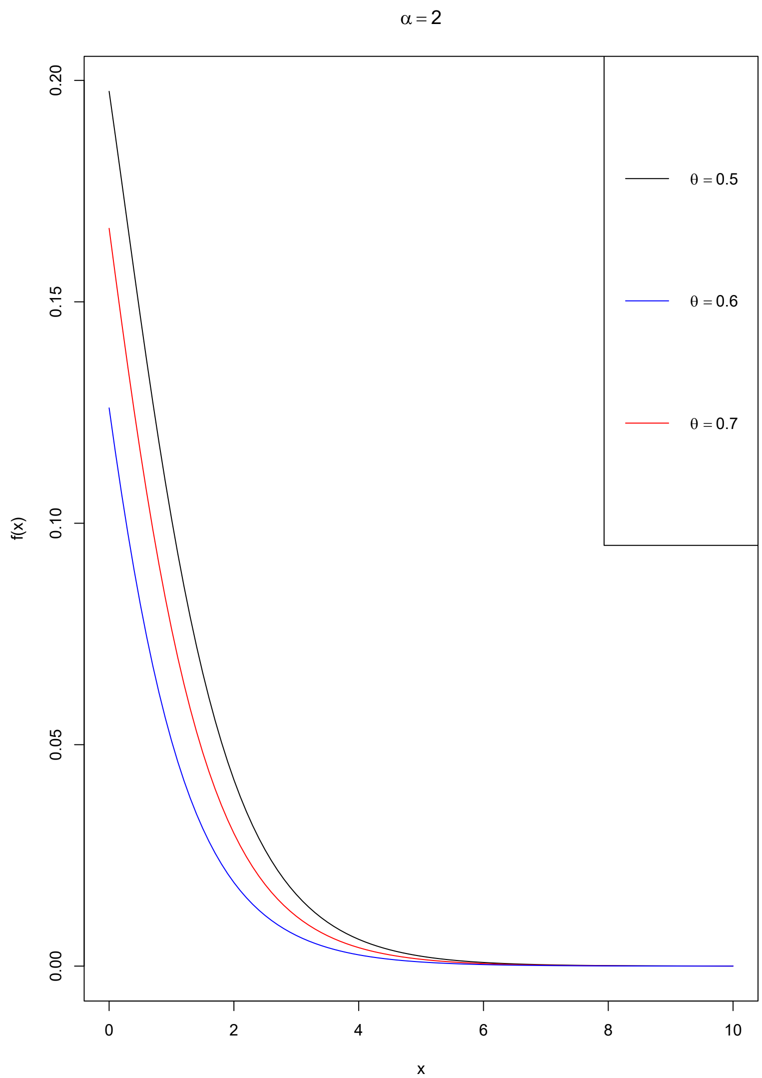
$$\begin{aligned} &= \frac{\alpha\theta}{(1-\theta)^{-\alpha}}\Big\{(1-\frac{\theta e^{-z}}{1+e^{-z}})^{-\alpha-1}\Big\}\frac{(1+e^{-z})e^{-z}-e^{-z}.e^2}{(1+e^{-z})z} \\ &= \frac{\alpha\theta}{(1-\theta)^{-\alpha}-1}\Big(1-\frac{\theta e^{-z}}{1+e^{-z}}\Big)^{-\alpha-1}\Big(\frac{e^{-z}}{(1+e^{-z})^2}\Big) \\ &= \frac{\alpha\theta}{(1-\theta)^{-\alpha}-1}\Big(\frac{1+e^{-z}}{1+e^{-z}(1-\theta)}\Big)^{\alpha+1}\frac{e^{-z}}{(1+e^{-z})^2} \end{aligned} \tag{5.40}$$

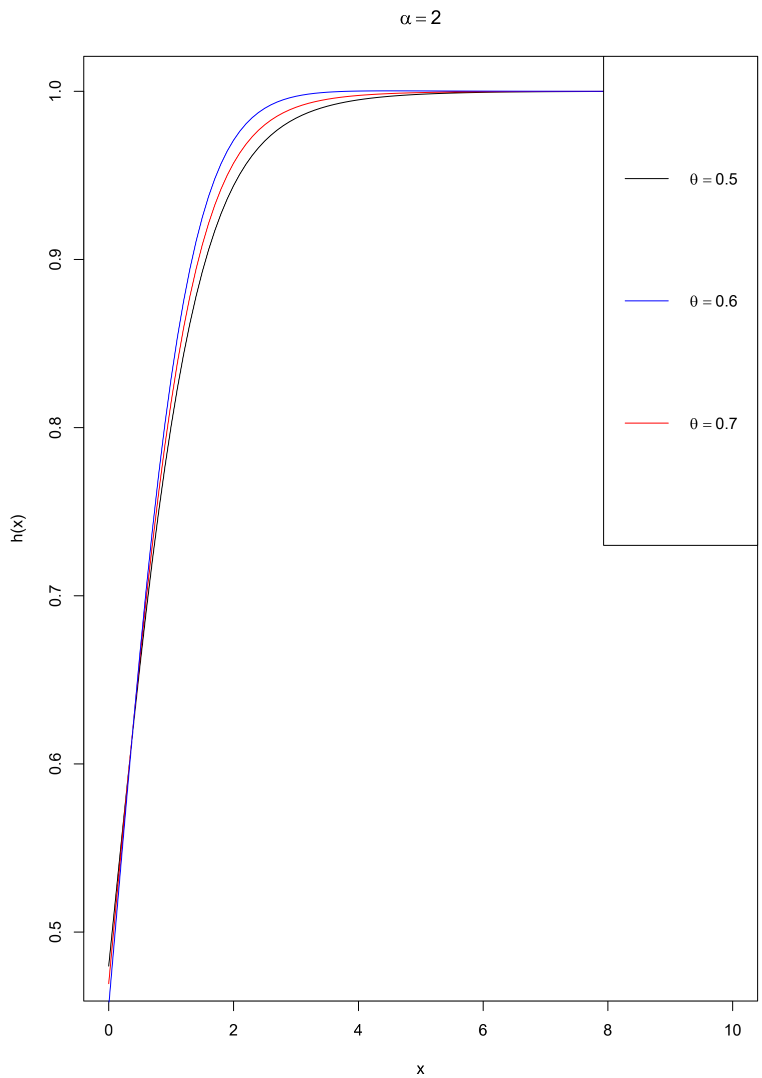
$$h(z) = \alpha\theta\Big[(1-\frac{\theta e^{-z}}{1+e^{-z}})^{-\alpha-1}\frac{e^{-z}}{(1+e^{-z})^2}\frac{1}{(1-\frac{\theta e^{-z}}{1+e^{-z}})^{-\alpha}-1}\Big] \tag{5.41}$$

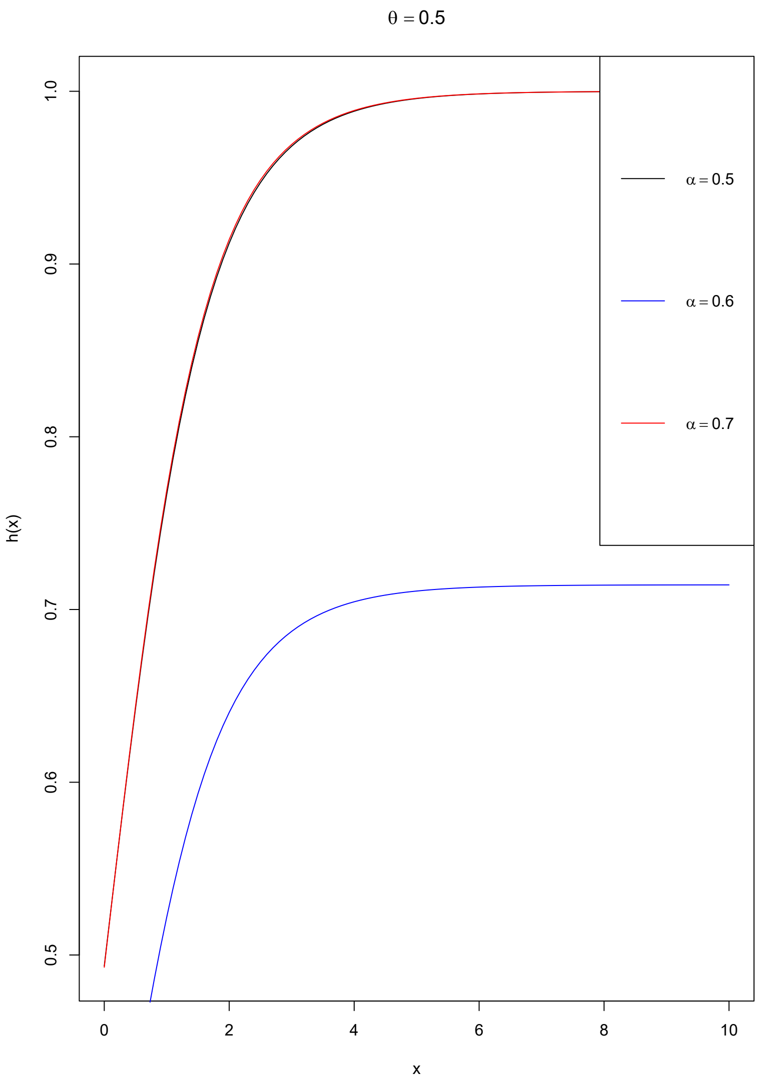
5.2.10 Graphs of pdf and hazard for the logistic negative binomial

The graphs have been obtained by fixing θ and varying α and also by fixing α and varying θ. The graphs obtained show a similar trend for the pdf and hazard functions. The pdf's are decreasing functions while the hazard functions are decreasing functions.




α = 2
f(x)
0.20
0.15
0.10
0.05
0.00
0
2
4
6
8
10
x
θ = 0.5
θ = 0.6
θ = 0.7


α = 2
h(x)
x
θ = 0.5
θ = 0.6
θ = 0.7
0.5
0.6
0.7
0.8
0.9
1.0
0
2
4
6
8
10



5.2.11 When P_n is the logarithmic distribution

Construction of the logarithmic series distribution

The probability distribution of the logarithmic with parameter θ is given by

$$P_\theta = -log(1-\theta), \tag{5.42}$$

To obtain the power series of $-log(1-\theta)$ we start by expanding $(1-\theta)^{-1}$;

$$\frac{1}{1-\theta} = 1+\theta+\theta^2+...,$$

Integrating both sides w.r.t θ

$$\int \frac{d\theta}{1-\theta} = \int [1+\theta+\theta^2+...]d\theta,$$
$$-log(1-\theta) = \theta + \frac{\theta^2}{2} + \frac{\theta^3}{3} +, ...,$$
$$= \sum_{n=1}^{\infty} \frac{\theta^n}{n},$$

therefore

$$1 = \sum_{n=1}^{\infty} \frac{\theta^n}{-nlog(1-\theta)}$$
$$P(X=n) = \frac{\theta^n}{-nlog(1-\theta)}, \qquad n = 1,2,... \tag{5.43}$$

which is the logarithmic series distribution

The logistic-logarithmic series distribution

In this case

pmf:

$$P_n = \frac{\theta^n}{-log(1-\theta)}, n = 1,2,... \tag{5.44}$$

pgf:

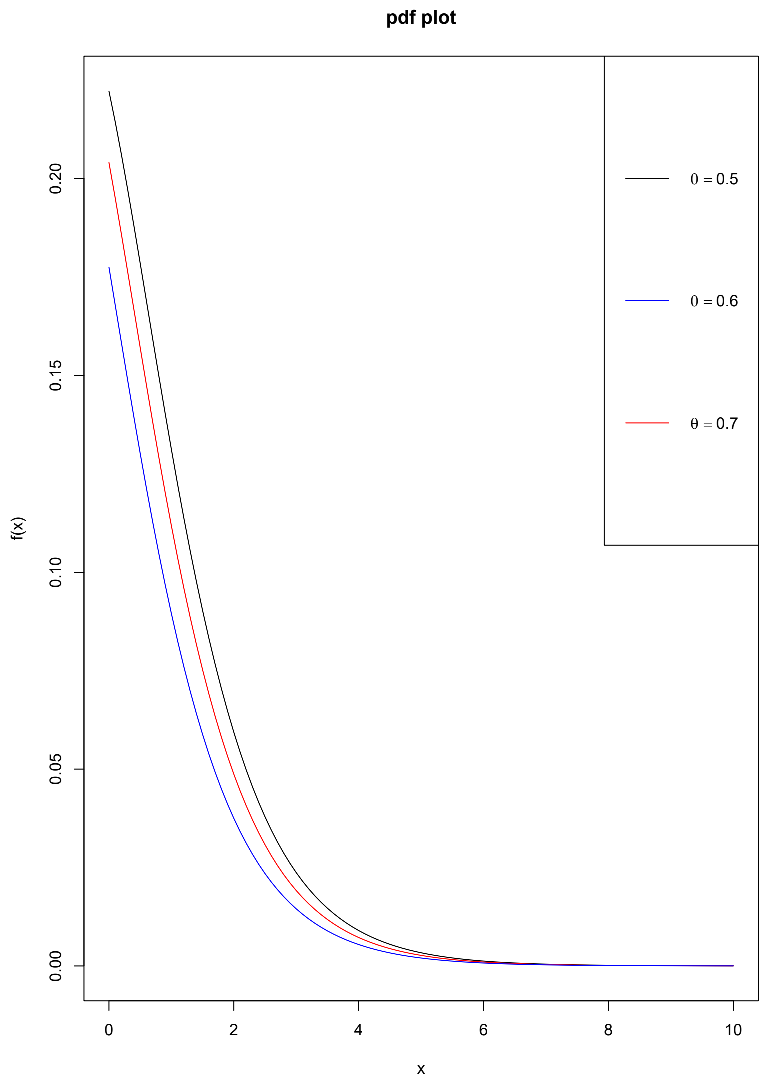
$$\psi_N(s) = \frac{log(1-\theta s)}{log(1-\theta)}$$

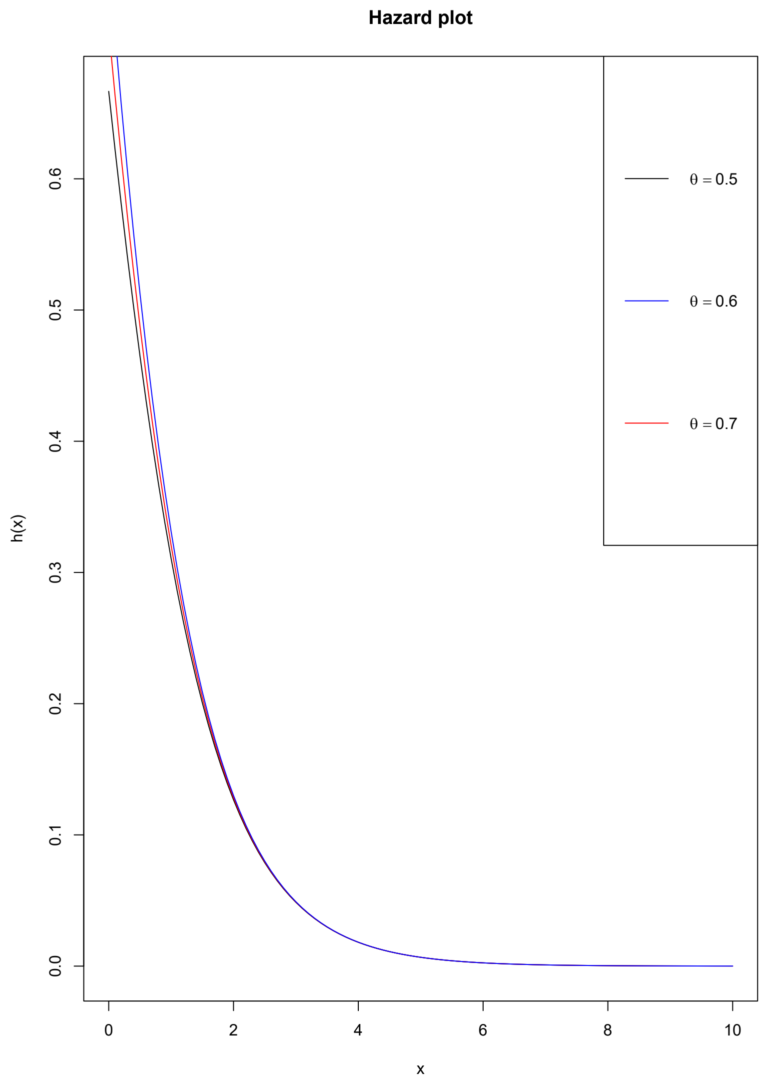
$$S(z) = \frac{log(1-\theta\frac{e^{-z}}{1+e^{z}})}{log(1-\theta)}$$
$$= \frac{log[1+(1-\theta)e^{-z}] - log(1+e^{-z})}{log(1-\theta)} \tag{5.45}$$

$$f(z) = \frac{1}{log(1-\theta)}\left\{\frac{(1-\theta)e^{-z}}{1+(1-\theta)e^{-z}} - \frac{e^{-z}}{1+e^{-z}}\right\}$$
$$= \frac{1}{log(1-\theta)}\left\{\frac{(1-\theta)(1+e^{-z})e^{-z} - e^{-z} - (1-\theta)e^{-2z}}{(1+e^{-z})[1+(1-\theta)e^{-z}]}\right\}$$
$$= \frac{1}{log(1-\theta)}\left\{\frac{(1-\theta)e^{-z} + (1-\theta)e^{-2z} - e^{-z} - (1+\theta)e^{-2z}}{(1+e^{-z})[1+(1-\theta)e^{-z}]}\right\}$$
$$= \frac{1}{log(1-\theta)}\left\{\frac{-\theta e^{-2z}}{(1+e^{-z})[1+(1-\theta)e^{-z}]}\right\}$$
$$= -\frac{\theta e^{-z}}{log(1-\theta)(1+e^{-z})[1+(1-\theta)e^{-z}]} \tag{5.46}$$

$$h(z) = \frac{\theta e^{-z}}{(1+e^{-z})[1+(1-\theta)e^{-z}]\{log(1+e^{-z}) - log[1+(1-\theta)e^{-z}]\}} \tag{5.47}$$

5.2.12 Graphs for the pdf and hazard functions for the logistic geometric distribution.





5.3 Discrete Mixtures based on Maximum Order statistics from Standard Logistic Distribution

5.3.1 Introduction

The objective of this section is to construct the pdf, survival functions and hazard functions of discrete mixtures of the maximum order statistics from the standard logistic distribution. The mixing distributions are zero truncated power series distributions i.e zero -truncated poisson, zero-truncated binomial, zero- truncated(shifted) geometric, zero-truncated negative binomial and the logarithmic series distributions. The resulting distributions shall be obtained for maximum order statistics distributions.

The pdf of the discrete mixtures of maximum order statistics from a logistic distribution is given by :

$$g(x) = \sum_{n=1}^{\infty} n\Big(\frac{e^x}{1+e^x}\Big)^n \frac{1}{1+e^x} P_n, \qquad -\infty < x < \infty \tag{5.48}$$

where P_n, the mixing distribution is a zero truncated power series distribution.

5.3.2 Distribution of maximum order statistics from logistic distribution

Let X_n denote the n^{th} observation of a random sample from such that: $X_n = \max(X_1, X_2, ..., X_n)$ sample. Let X_n be the largest X and;

$$X_n = \max(X_1, X_2, ..., X_n)$$

Let $F(x)$ be the cdf of X and $Z = \max(X_1, X_2, ..., X_n)$, then the cdf of Z is

$$\begin{aligned} G(z) =& Prob(Z < z) \\ =& Prob\Big(max(X_1, X_2, ..., X_n) < z\Big) \\ =& Prob\Big(X_1 < z, X_2 < z, ..., X_n < z\Big) \\ =& \prod_{i=1}^{n} Prob\Big(X_i < z\Big) \\ =& \prod_{i=1}^{n} F(z) \\ =& \Big(F(z)\Big)^n \end{aligned} \tag{5.49}$$

which is the cdf of the maximum order statistics distribution. This is an exponentiated distribution which can also be derived from the beta generator as ;

5.3.3 Exponentiated distribution

Eugene et al.(2000) considered a cdf of the classical beta distribution as

$$W(x) = \int_0^x \frac{t^{a-1}(1-t)^{b-1}}{\beta(a,b)} dt \tag{5.50}$$

Since $0 \le x \le 1$ Eugene et al. replaced it by a cdf $G(x)$. Thus we have

$$F(x) = W[G(x)] = \int_0^{G(x)} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt \tag{5.51}$$

Put $b=1$ we get

$$\begin{aligned} F(x) &= \int_0^{G(x)} a t^{a-1} dt \\ &= \frac{at^a}{a}\Big|_0^{G(x)} \\ &= \Big(G(x)\Big)^a \end{aligned}$$

When $a=1$, we have Making the substitution $y=1-t \implies dy=-dt$

$$F(x)=\int_0^{G(x)} b(1-t)^{b-1}dt = -\int_1^{1-G(x)} by^{b-1}(-dt)$$

The distribution of the maximum order statistics for the logistic shall be obtained as follows;

$$\begin{aligned} G(x) &= \Big[\frac{1}{1+e^x}\Big]^n \\ g(x) &= n\Big(\frac{e^x}{1+e^x}\Big)^{n-1}\frac{e^x}{(1+e^x)^2} \\ &= n\Big(\frac{e^x}{1+e^x}\Big)^{n}\frac{1}{1+e^x}, \qquad -\infty < x < \infty \\ &= \frac{ne^{xn}}{(1+e^x)^{n+1}} \qquad -\infty < x < \infty \end{aligned} \tag{5.52}$$

5.3.4 The hazard and survival functions of the maximum order statistics from a logistic distribution

The pdf and hazard functions shall be obtained as follows.

$$\begin{aligned} S(x) &= 1-\Big[\frac{1}{1+e^x}\Big]^n, \\ &= 1-\frac{1}{(1+e^x)^n} \\ &= \frac{(1+e^x)^n-1}{(1+e^x)^n} \\ h(x) &= \frac{ne^{xn}}{(1+e^x)^n(1+e^x)}.\frac{(1+e^x)^n}{[(1+e^x)^n-1]} \\ &= \frac{ne^{xn}}{(1+e^x)[(1+e^x)^n-1]}, \qquad -\infty < x < \infty \end{aligned} \tag{5.53}$$

5.3.5 The logistic Poisson distribution

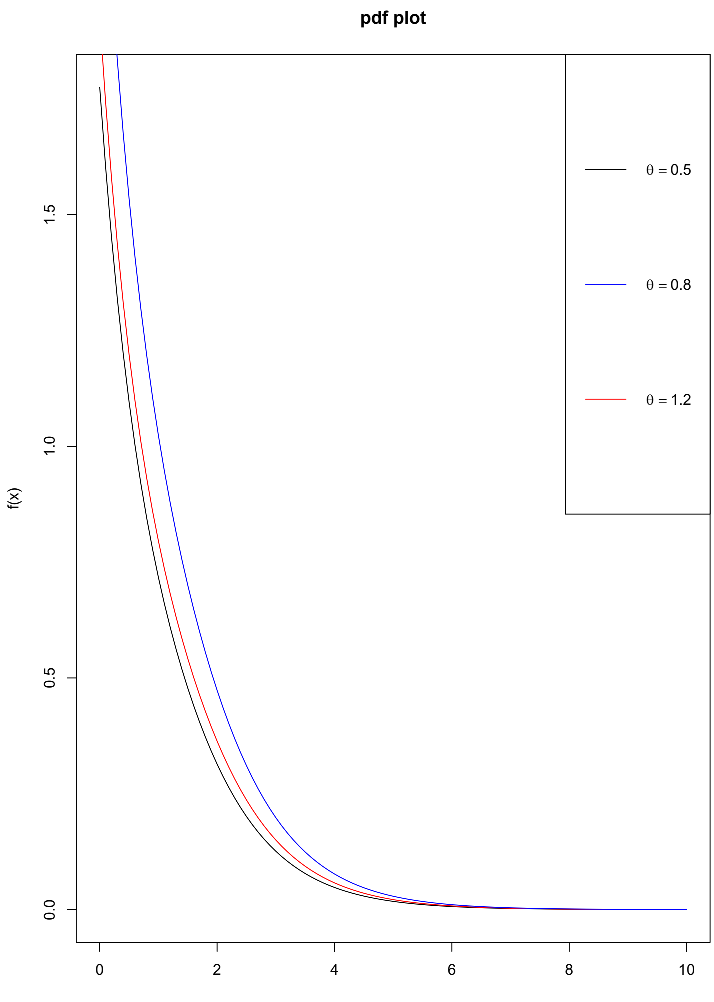
This is obtained when the mixing distribution is the zero- truncated Poisson distribution.

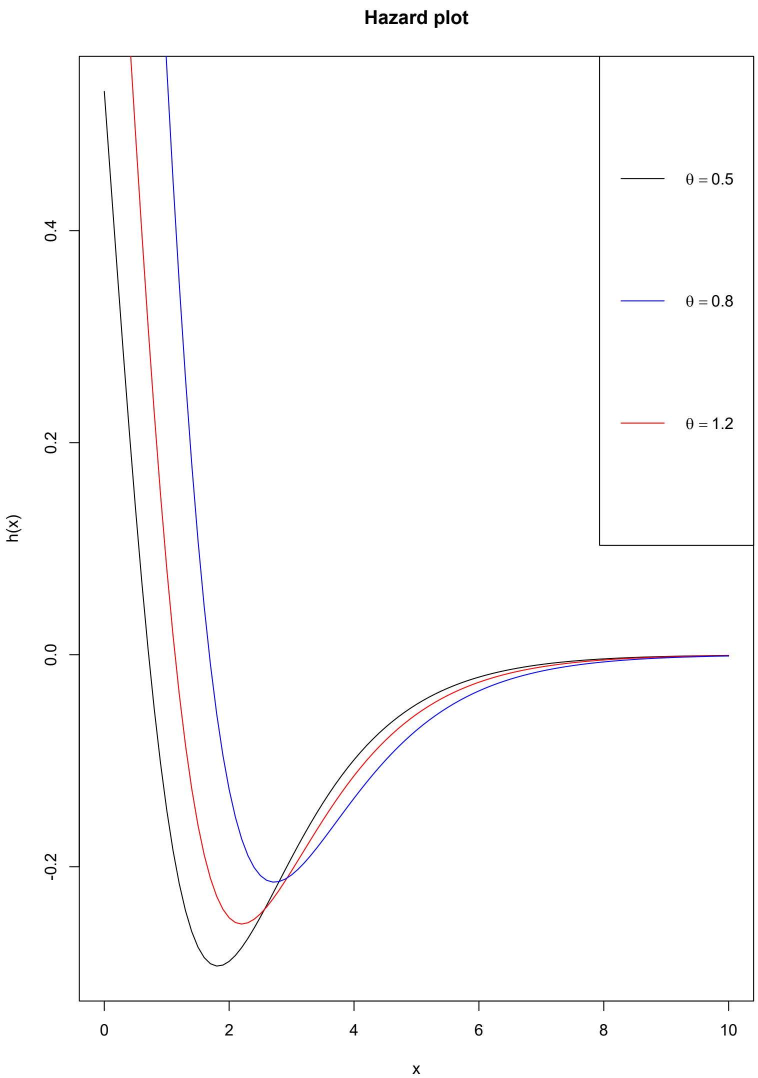
$$S(y) = \sum_{n=1}^{\infty} \frac{ne^{-y}}{(1+e^{-y})^{n+1}} \frac{\theta^n}{n!(e^\theta - 1)}, \qquad -\infty < y < \infty$$
$$= \frac{e^{-y}(e^{\frac{\theta}{1+e^{-y}}} - 1)}{(e^\theta - 1)(1+e^{-y})} \tag{5.54}$$

$$f(y) = \frac{e^{-y}(\theta - y - 1)e^{\frac{\theta - y - e^{-y}}{1+e^{-y}}} + e^{-y} + e^{\frac{\theta - y - e^{-y}}{1+e^{-y}} - y} + e^{-2y}}{(e^\theta - 1)(1+e^{-y})^2} \tag{5.55}$$

$$h(y) = \frac{(\theta - y - 1)e^{\frac{\theta - y - e^{-y}}{1+e^{-y}}} + 1 + (\theta - y - 1)e^{\frac{\theta - y - e^{-y}}{1+e^{-y}} - 1} + e^{-y}}{(1+e^{-y})(e^{\frac{\theta}{1+e^{-y}}} - 1)} \tag{5.56}$$

5.3.6 Graphs of the pdf and hazard of logistic Poisson distribution





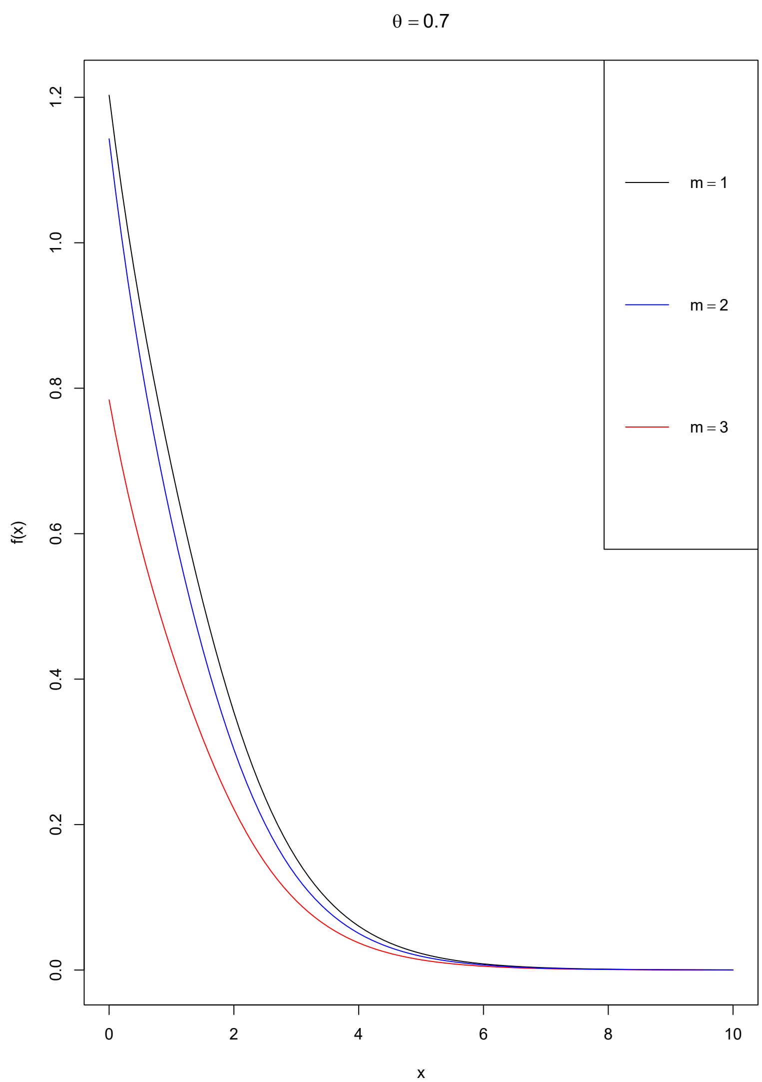
5.3.7 The logistic-binomial distribution

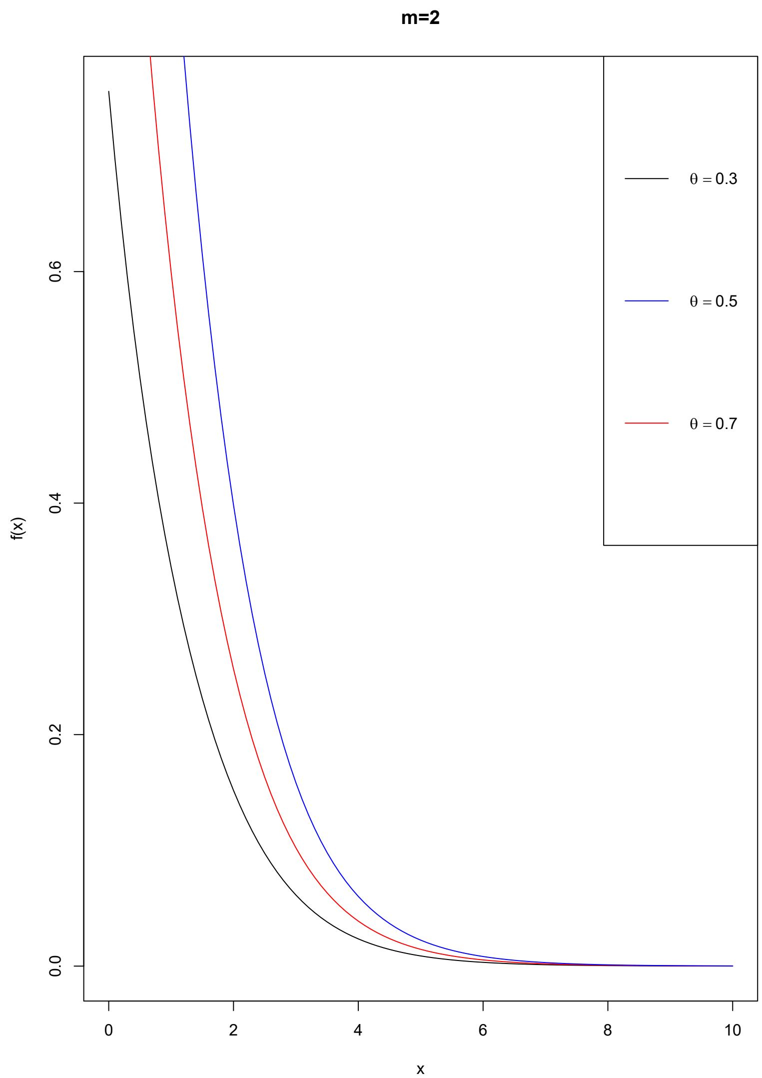
$$S(y) = \sum_{n=1}^{\infty} \frac{ne^{-y}}{(1+e^{-y})^{n+1}} \frac{\binom{m}{n}\theta^n}{(1+\theta)^m - 1}$$
$$= \frac{me^{-y}(e^{\frac{\theta}{1+e^{-y}}} - 1)}{(1+e^{-y})((1+\theta)^m - 1)} \quad (5.57)$$

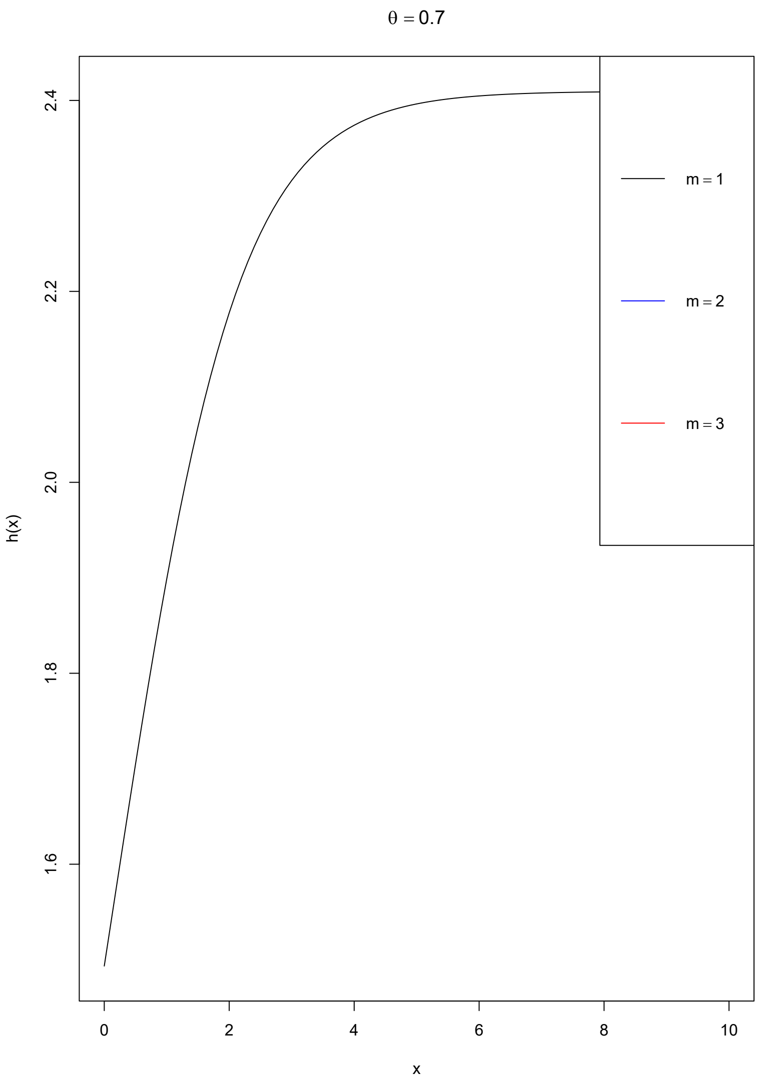
$$f(y) = -\frac{m}{(1+\theta)^m - 1} \frac{d}{dS}\Big(\frac{e^{\frac{\theta}{1+e^{-y}}} - 1}{1+e^{-y}}\Big)$$
$$= -\frac{m\theta e^{-y} e^{\frac{\theta}{1+e^{-y}}} + e^{-y}(1+e^{-y})(e^{\frac{\theta}{1+e^{-y}}} - 1)}{((1+\theta)^m - 1)(1+e^{-y})^3} \quad (5.58)$$

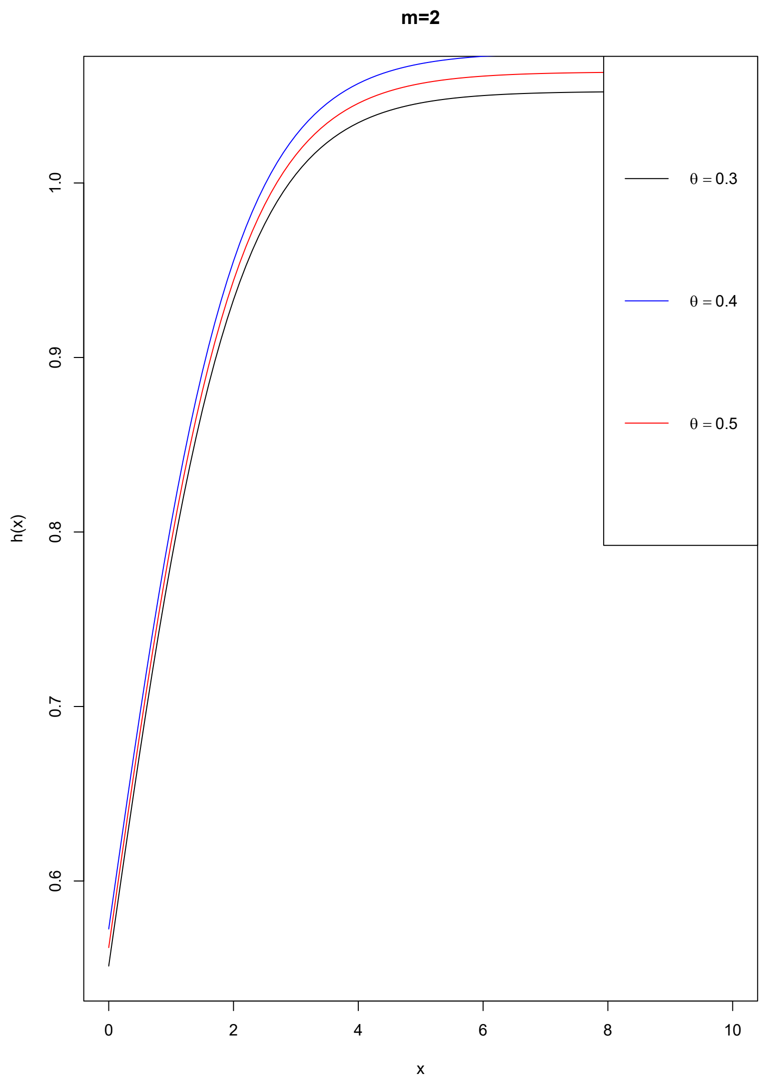
$$h(y) = \frac{\theta e^{\frac{\theta}{1+e^{-y}}} + (1+e^{-y})(e^{\frac{\theta}{1+e^{-y}}} - 1)}{(1+e^{-y})^2(e^{\frac{\theta}{1+e^{-y}}} - 1)} \tag{5.59}$$

5.3.8 Graphs of the pdf and Hazard of Logistic Binomial




m=2
θ = 0.3
θ = 0.5
θ = 0.7
f(x)
0.0
0.2
0.4
0.6
0
2
4
6
8
10
x


θ = 0.7
h(x)
x
m = 1
m = 2
m = 3
2.4
2.2
2.0
1.8
1.6
0
2
4
6
8
10



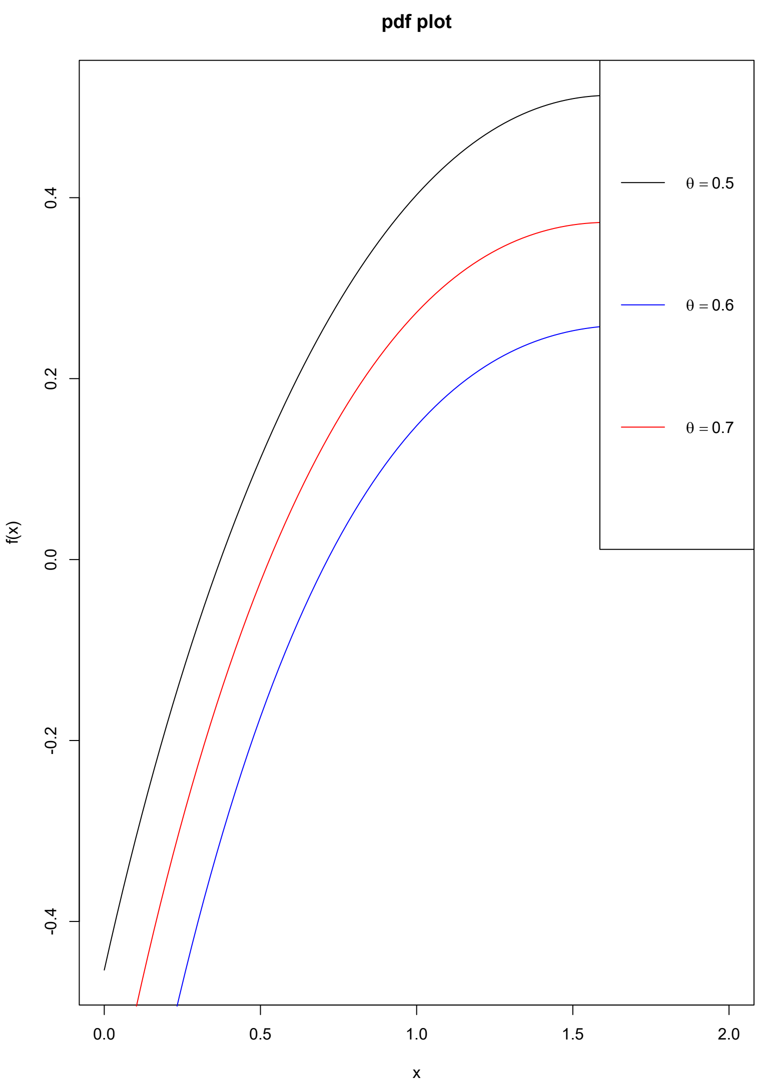
5.3.9 The logistic shifted geometric distribution

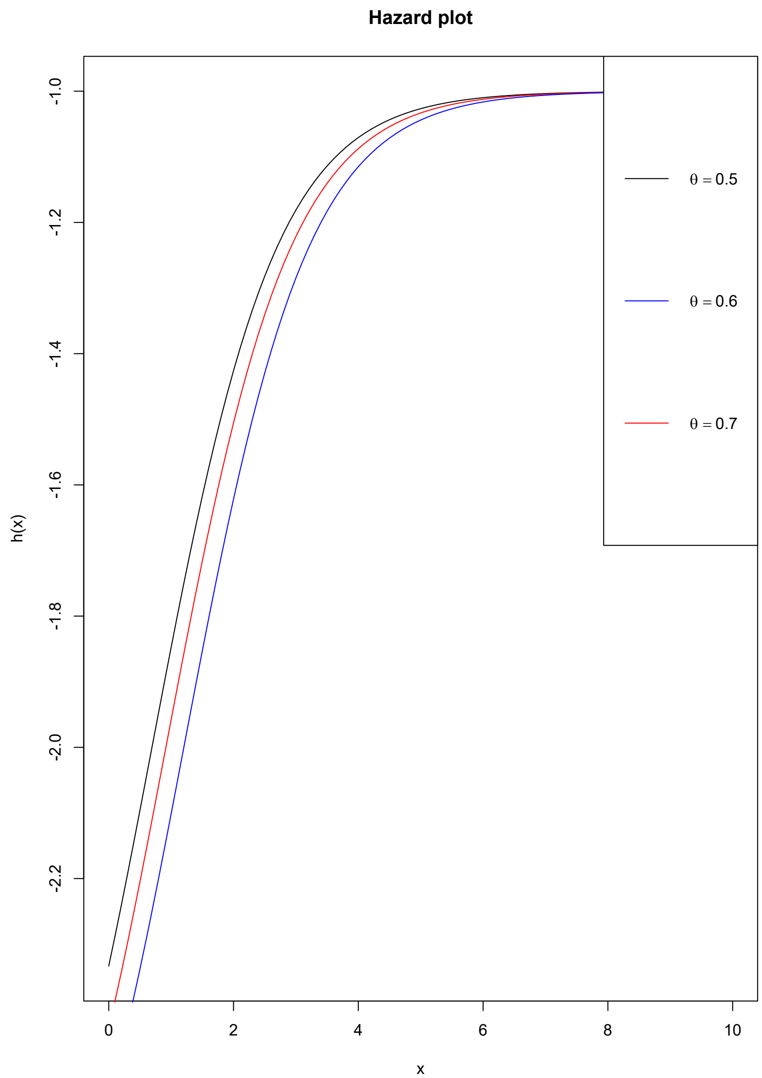
$$
\begin{aligned}
S(y) &= \sum_{n=1}^{\infty} \frac{ne^{-y}}{(1+e^{-y})^{n+1}} \theta^{n-1}(1-\theta) \\
&= \frac{(1-\theta)e^{-y}}{(1-\theta+e^{-y})^2}
\end{aligned} \tag{5.60}
$$

$$
\begin{aligned}
f(y) &= \frac{d}{dS} \frac{(1-\theta)e^{-y}}{(1-\theta+e^{-y})^2} \\
&= \frac{-e^{-y}(1-\theta-e^{-y})(1-\theta)}{(1-\theta+e^{-y})^3}
\end{aligned} \tag{5.61}
$$

$$
\begin{aligned}
h(y) &= \frac{-e^{-y}(1-\theta-e^{-y})(1-\theta)}{(1-\theta+e^{-y})^3} \times \frac{(1-\theta+e^{-y})^2}{(1-\theta)e^{-y}} \\
&= \frac{-(1-\theta-e^{-y})}{1-\theta+e^{-y}} = \frac{\theta+e^{-y}-1}{1-\theta+e^{-y}}
\end{aligned} \tag{5.62}
$$

5.3.10 Graphs of the pdf and Hazard of Logistic Geometric





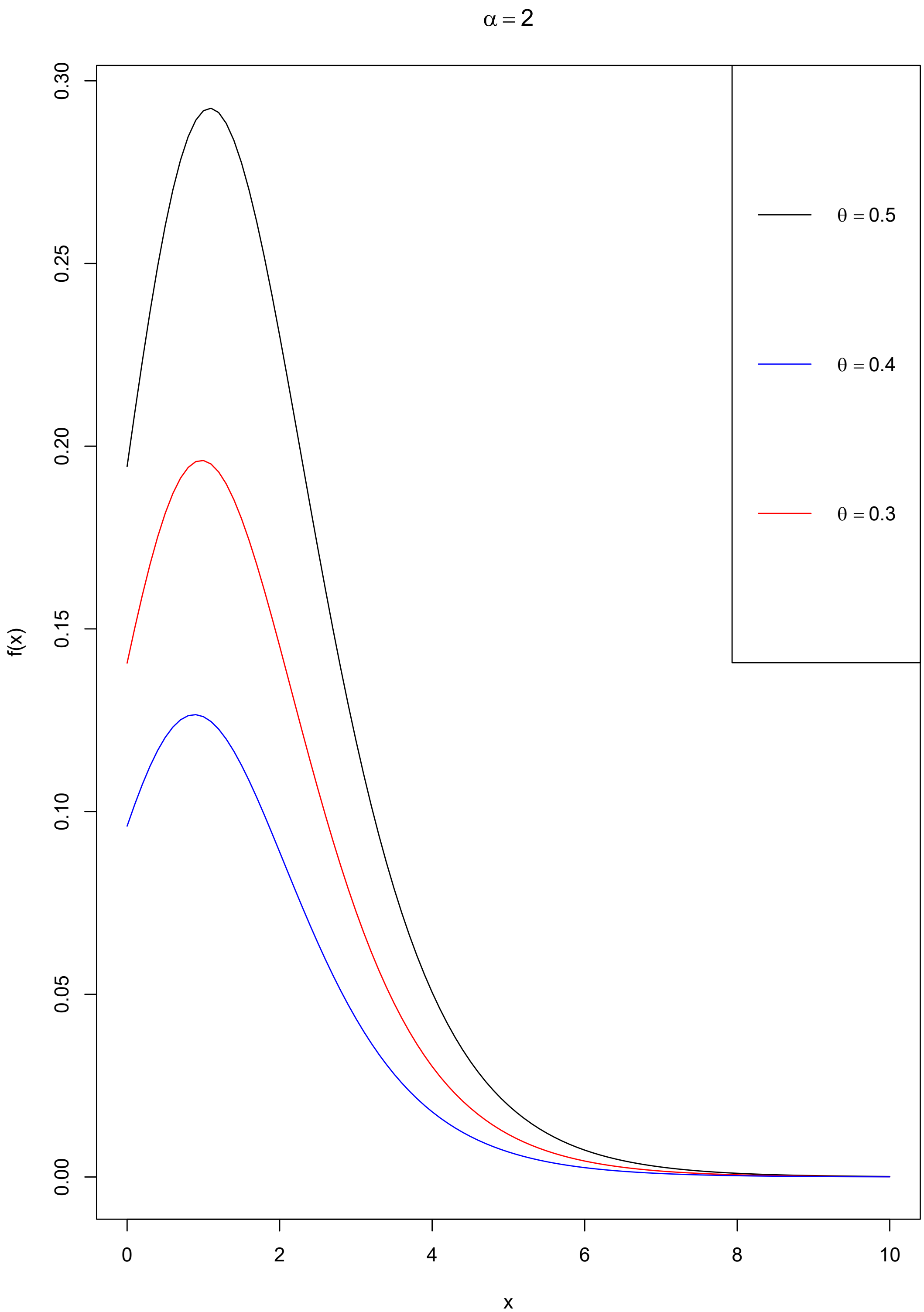
5.3.11 The logistic zero- truncated negative binomial distribution

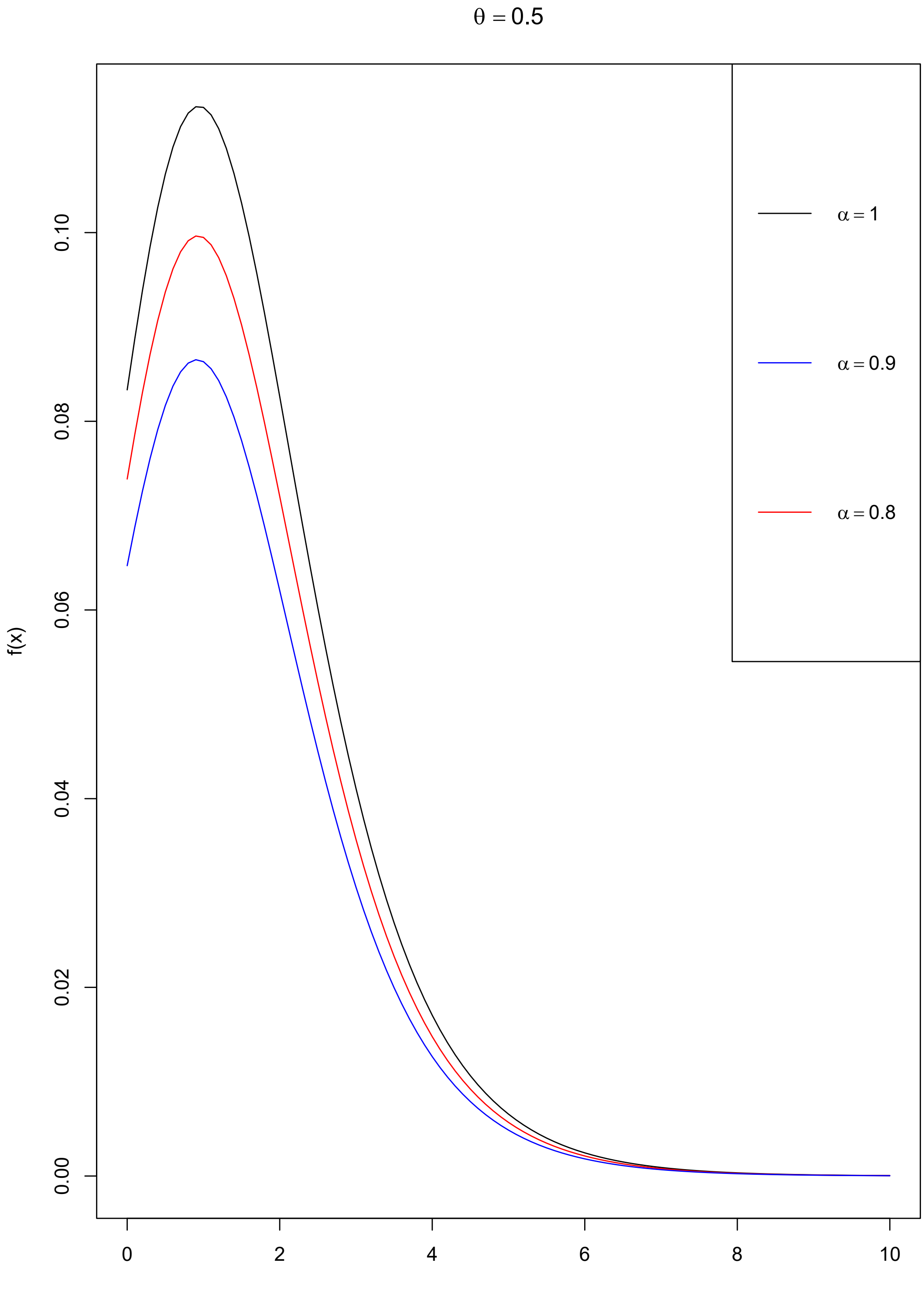
$$\begin{aligned} S(y) &= \sum_{n=1}^{\infty} \frac{ne^{-y}}{(1+e^{-y})^{n+1}} \binom{\alpha+n-1}{n} \frac{\theta^n}{(1-\theta)^{-\alpha}-1} \\ &= \frac{e^{-y}(1-\frac{\theta}{1+e^{-y}})^{-\alpha}-1}{(1+e^{-y})(1-\theta)^{-\alpha}-1} \end{aligned} \tag{5.63}$$

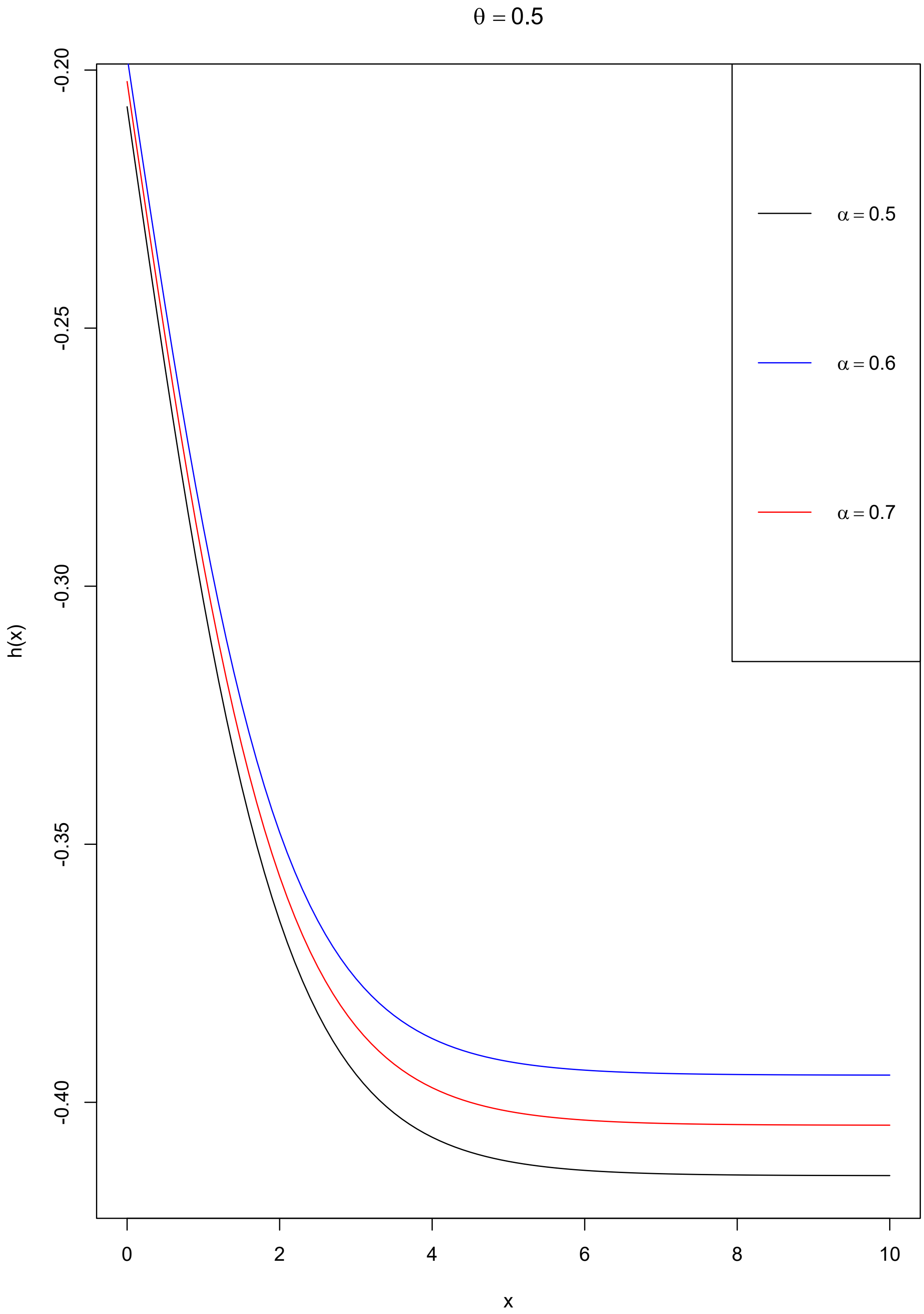
$$\begin{aligned} f(y) &= \frac{1}{(1-\theta)^{-\alpha}-1} \frac{d}{dS} \Big\{ \frac{(1-\frac{\theta}{1+e^{-y}})^{-\alpha}-1}{(1+e^{-y})} \Big\} \\ &= \frac{-\Big[-e^{-y}\Big\{(\frac{1+e^{-y}}{1+e^{-y}-\theta})^{\alpha}-1\Big\}\Big]}{(1+e^{-y})^2} \end{aligned} \tag{5.64}$$

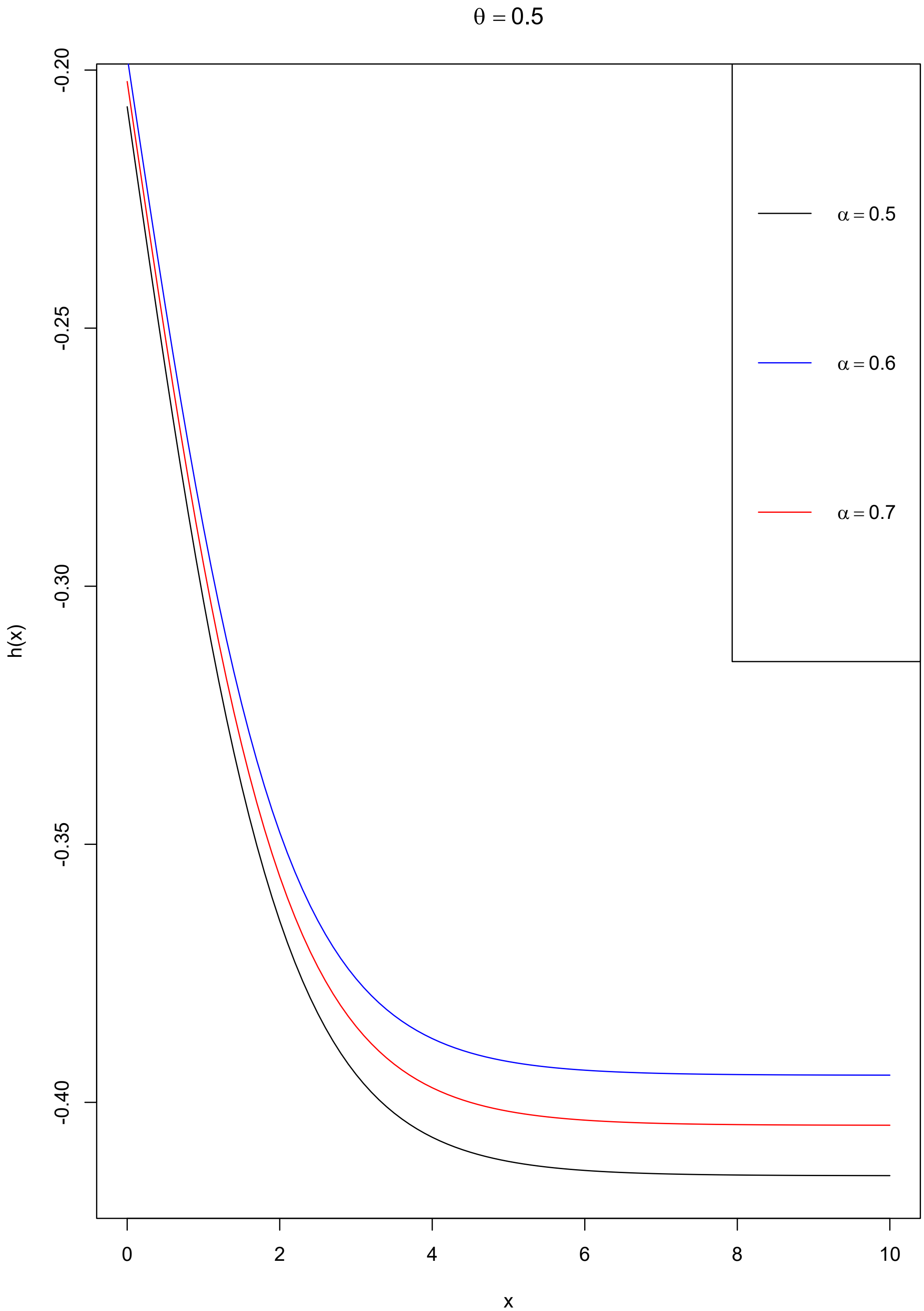
$$h(y) = \frac{-\{(1-\theta)^{-\alpha}-1\}}{1+e^{-y}} \tag{5.65}$$

5.3.12 Graphs of the pdf and Hazard of Logistic Negative-Binomial




θ = 0.5
f(x)
0.00
0.02
0.04
0.06
0.08
0.10
0
2
4
6
8
10
x
α = 1
α = 0.9
α = 0.8


θ = 0.5
h(x)
x
α = 0.5
α = 0.6
α = 0.7
-0.20
-0.25
-0.30
-0.35
-0.40
0
2
4
6
8
10

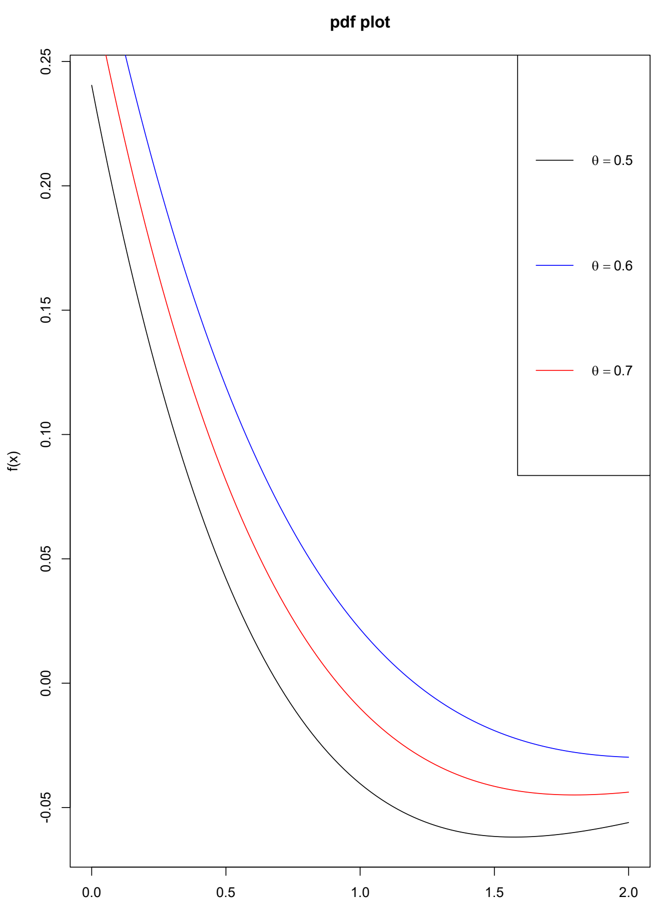

θ = 0.5
α = 0.5
α = 0.6
α = 0.7
h(x)
x
-0.20
-0.25
-0.30
-0.35
-0.40
0
2
4
6
8
10

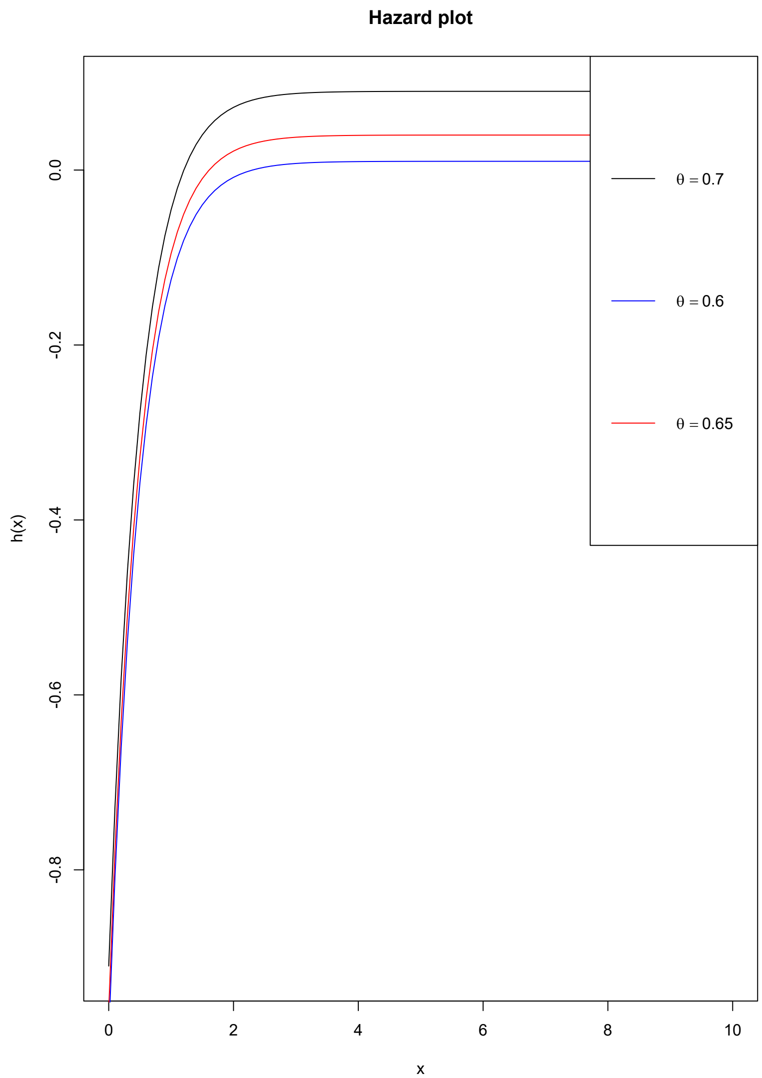
5.3.13 The logistic-logarithmic series distribution

$$S(y) = \sum_{n=1}^{\infty} \frac{ne^{-y}}{(1+e^{-y})^{n+1}} \frac{\theta^n}{-log(1-\theta)}$$
$$= \frac{\theta e^{-y}}{(1+e^{-y}-\theta)^2 log(1-\theta)}$$
$$= -\frac{\theta}{log(1-\theta)} \frac{d}{dy} \frac{e^{-y}}{(1+e^{-y}-\theta)^2}$$
$$= \frac{\theta e^{-y}(1-e^{-y}-\theta)}{log(1-\theta)(1+e^{-y}-\theta)} \quad \textbf{(5.66)}$$

$$h(y) = \frac{\theta e^{-y}(1-e^{-y}-\theta)}{log(1-\theta)(1+e^{-y}-\theta)} \times \frac{(1+e^{-y}-\theta)^2 log(1-\theta)}{\theta e^{-y}}$$
$$= (1-e^{-y}-\theta)(1+e^{-y}-\theta)$$
$$= 2\theta + e^{-2y} - \theta^2 - 1 \quad \textbf{(5.67)}$$

5.3.14 Graphs of the pdf and Hazard of Logistic Logaritmic





5.4 Summary

In this charpter we obtained results for distributions resulting from mixtures of order statistics for both minimum and maximum order statistics from the logistic distribution. The mixing distributions used were zero truncated poisson, binomial, negative binomial, geometric and the logarithmic distribution. Graphs of the resulting mixed distributions were simulated for various values of θ and α. A similar trend of increasing or decreasing has been observed for the pdf's and hazard functions.

6 CONTINUOUS MIXTURES OF LOGISTIC DISTRIBUTION WITH SCALE AND LOCATION PARAMETERS

6.1 Introduction

From literature, its only Nadarajah and Kotz (2006) who have studied mixtures of logistic distribution. They used 16 mixing distributions to obtain the pdfs of type II exponential mixtures which were expressed in terms of special functions.

In this charpter we shall construct the mixed distributions the logistic distribution with scale and location parameters. We shall also introduce a new distribution "The Logistic Inverse Gaussian" whose r^{th} moments and EM algorithm have been obtained.

6.2 Logistic Distribution with Scale and Location Parameters

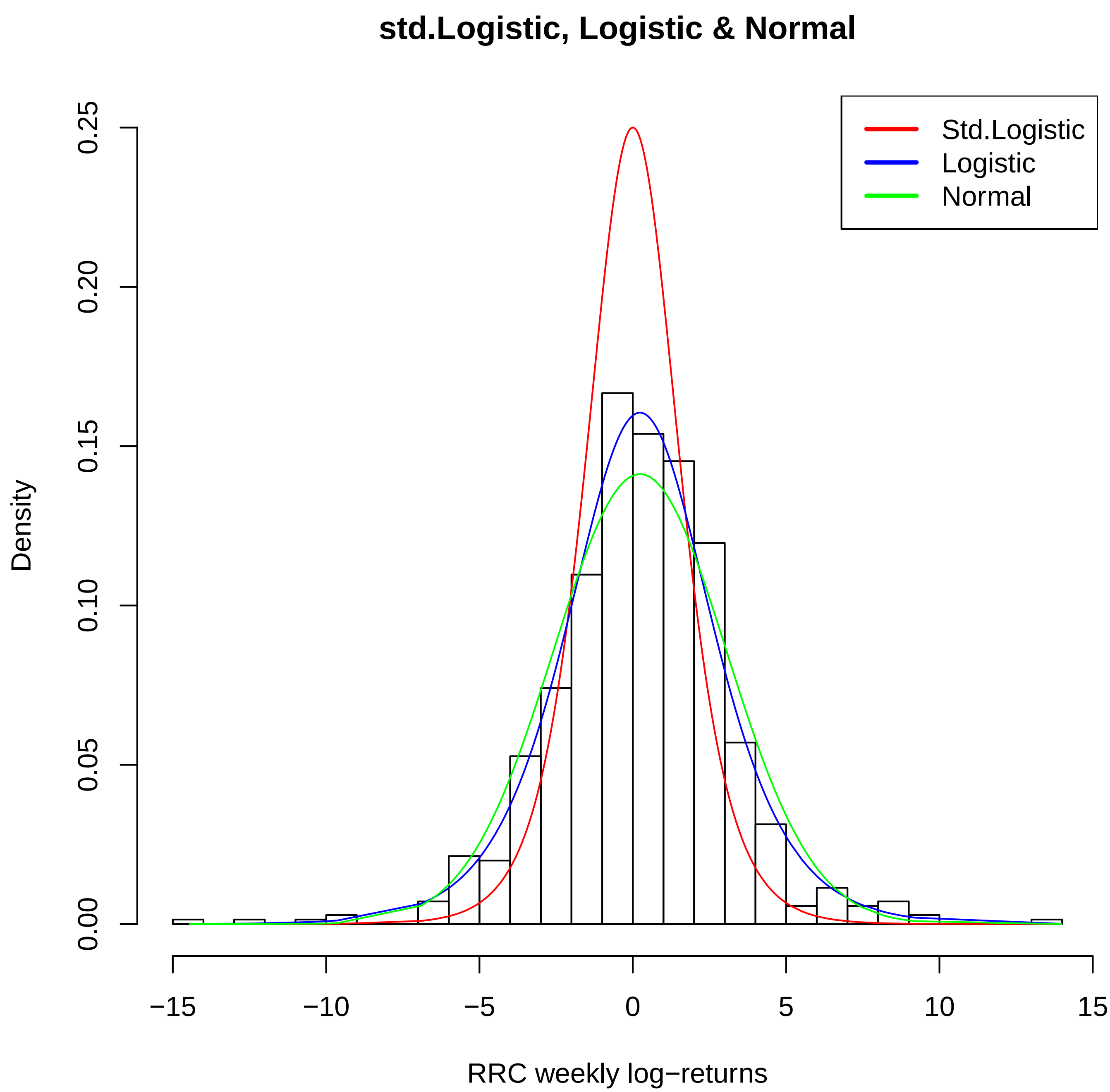
6.2.1 The Pdf and cdf

They considered the conditional pdf and cdf of the logistic distribution given by;

$$\begin{aligned} f(x|\lambda) &= \frac{e^{-\frac{x-\theta}{\lambda}}}{\lambda[1+e^{-\frac{x-\theta}{\lambda}}]^2}, \qquad -\infty < x < \infty, \lambda > 0, \theta > 0 \\ F(x|\lambda) &= \frac{1}{1+e^{-\frac{x-\theta}{\lambda}}}, \qquad -\infty < x < \infty, \lambda > 0, \theta > 0 \end{aligned} \tag{6.1}$$

Where θ and λ are scale and location parameters respectively.

Figure 6.1: Graphs of the standard logistic, logistic with scale and location parameter normal



The graph above shows a comparison between the normal, standard logistic and the logistic with scale and location parameters.

Hazard and survival functions

The conditional survival function;

$$\begin{aligned}S(x|\lambda) &= \int_x^\infty \frac{e^{-\frac{x-\theta}{\lambda}}}{\lambda[1+e^{-\frac{x-\theta}{\lambda}}]^2}dx\\ &= \frac{1}{\lambda}\int_x^\infty \frac{e^{-\frac{x-\theta}{\lambda}}}{[1+e^{-\frac{x-\theta}{\lambda}}]^2}dx\end{aligned}$$

let

$$\begin{aligned}z =&1+e^{-\frac{x-\theta}{\lambda}}dx\\ =&\frac{\lambda dz}{e^{-\frac{x-\theta}{\lambda}}}\\ =&\frac{1}{\lambda}\int_{1+e^{-\frac{x-\theta}{\lambda}}}^\infty \frac{e^{-\frac{x-\theta}{\lambda}}}{z^2}\frac{\lambda dz}{e^{-\frac{x-\theta}{\lambda}}}\\ =&\int_{1+e^{-\frac{x-\theta}{\lambda}}}^\infty \frac{1}{z^2}dz\\ =&\frac{1}{z}\Big|_{1+e^{-\frac{x-\theta}{\lambda}}}^\infty\\ S(x|\lambda) =&\frac{1}{1+e^{-\frac{x-\theta}{\lambda}}} \end{aligned} \qquad \textbf{(6.2)}$$

the conditional hazard function shall be given by;

$$\begin{aligned}h(x|\lambda) =&\frac{e^{-\frac{x-\theta}{\lambda}}}{\lambda[1+e^{-\frac{x-\theta}{\lambda}}]^2}\Big/\frac{1}{1+e^{-\frac{x-\theta}{\lambda}}}\\ =&\frac{e^{-\frac{x-\theta}{\lambda}}}{\lambda[1+e^{-\frac{x-\theta}{\lambda}}]}\end{aligned} \qquad \textbf{(6.3)}$$

6.2.2 Mathematical formulation of the problem

Let

$$f(x) = \int_0^\infty f(x|\lambda)g(\lambda)d\lambda$$

Where $f(x)$ is the mixed pdf $f(x|\lambda)$ is the conditional pdf $g(\lambda)$ is the mixing pdf.
By applying a linear transformation on the standard logistic;

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \qquad -\infty < x < \infty$$

let

$$x = \frac{y-\theta}{\lambda} \implies, y = x\lambda + \theta \implies \frac{dy}{dx} = \lambda$$

therefore

$$dx = \frac{dy}{\lambda}$$

$f(x)$ becomes $f(x|\lambda))$ as indicated in equation 6.1.
The problem is to find the mixed pdf $f(x)$, the survival function $S(x)$, hazard function $h(x)$ and the r^{th}moment $E(X^r)$ for various cases of mixing distributions $g(\lambda)$.
The mixtures shall be expressed in explicit form and in terms of modified Bessel functions of the third kind.

The r^{th} moment shall be obtained directly by definition

$$E(X^r) = \int_0^\infty x^r f(x)dx \tag{6.4}$$

The r^{th} moment shall be obtained indirectly using the conditional expectation approach stated below;

$$\begin{aligned}
E(X^r|\Lambda) &= \int_0^\infty x^r f(x|\lambda)dx \\
&= \int_0^\infty x^r \frac{e^{-\frac{x-\theta}{\lambda}}}{\lambda[1+e^{-\frac{x-\theta}{\lambda}}]^2} dx \\
&= \int_0^\infty x^r \lambda^{0-1} \left[1+e^{-\frac{x-\theta}{\lambda}}\right]^{-2} e^{-\frac{x-\theta}{\lambda}} dx \\
&= \int_0^\infty x^r \lambda^{0-1} \sum_{k=0}^\infty \binom{-2}{k} e^{-k(\frac{x-\theta}{\lambda})-\frac{x-\theta}{\lambda}} dx \\
&= \sum_{k=0}^\infty \binom{-2}{k} \int_0^\infty x^r \lambda^{0-1} e^{-k(\frac{x-\theta}{\lambda})-\frac{x-\theta}{\lambda}} dx \\
&= \sum_{k=0}^\infty \binom{-2}{k} \int_0^\infty x^r \Lambda^{0-1} e^{-\frac{1}{\lambda}[(k-1)(x+\theta)]} dx
\end{aligned}$$

let

$$\begin{aligned}
\lambda &= \sqrt{(k-1)(y+\theta)} x \\
d\lambda &= \sqrt{(k-1)(y+\theta)} dx \\
dx &= \frac{d\lambda}{\sqrt{(k-1)(y+\theta)}}
\end{aligned}$$

$$=\sum_{k=0}^{\infty}\binom{-2}{k}\int_0^{\infty}x^r\Big(\sqrt{(k-1)(y+\theta)x}\Big)^{0-1}e^{-\frac{1}{\sqrt{(k-1)(y+\theta)x}}[(k-1)(y+\theta)]}\frac{d\lambda}{\sqrt{(k-1)(y+\theta)}}$$

$$=r!\Big\{\sum_{k=0}^{\infty}\binom{-2}{k}\frac{E(\lambda^r)}{(k+1)^r}\Big\} \tag{6.5}$$

so that

$$EE\Big(X^r|\Lambda\Big)=E(X^r)$$

6.2.3 Distributions based on Bessel function of the third kind

The logistic mixing distribution

$$\begin{aligned}
g(\lambda)=&\int_0^{\infty}\frac{e^{-\frac{x-\theta}{\lambda}}}{\lambda[1+e^{-\frac{x-\theta}{\lambda}}]^2}\frac{e^{-\frac{x-\theta}{\lambda}}}{\lambda[1+e^{-\frac{x-\theta}{\lambda}}]^2}dx\\
=&\int_0^{\infty}\lambda^{0-2}e^{-\frac{x-\theta}{\lambda}-\frac{x-\theta}{\lambda}}[1+e^{-\frac{x-\theta}{\lambda}}]^{-2}[1+e^{-\frac{x-\theta}{\lambda}}]^{-2}d\lambda\\
=&\int_0^{\infty}\lambda^{0-2}\sum_{k=0}^{\infty}\binom{-2}{k}\sum_{l=0}^{\infty}\binom{-2}{l}e^{-2(\frac{x-\theta}{\lambda})-k(\frac{x-\theta}{\lambda})-l(\frac{x-\theta}{\lambda})}d\lambda\\
=&\sum_{k=0}^{\infty}\binom{-2}{k}\sum_{l=0}^{\infty}\binom{-2}{l}\int_0^{\infty}\lambda^{0-2}e^{-2(\frac{x-\theta}{\lambda})-k(\frac{x-\theta}{\lambda})-l(\frac{x-\theta}{\lambda})}d\lambda\\
=&\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{-2}{k}\binom{-2}{l}\int_0^{\infty}\lambda^{-(1)-1}e^{-\frac{1}{\lambda}[(x-\theta)(2+k+l)]}d\lambda\\
=&\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{-2}{k}\binom{-2}{l}\frac{\Gamma 1}{(x-\theta)(2+k+l)}\\
=&\frac{1}{(x-\theta)(2+k+l)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{-2}{k}\binom{-2}{l}
\end{aligned} \tag{6.6}$$

The Exponential mixing distribution

In this case the exponential distribution is used as a mixing distribution. The exponential distribution has two ways of representing its pdf;

$$f(x,\lambda)=\lambda e^{-\lambda x} \tag{6.7}$$

and

$$f(x,\lambda)=\frac{1}{\lambda}e^{-\frac{1}{\lambda}x} \tag{6.8}$$

Proposition 6.2.1. *The logistic exponential distribution is given as*

$$f(x) = -\frac{2}{\mu}\sum_{k=0}^{\infty}\binom{2}{k}k_0 2\sqrt{\frac{(k+1)(x-\theta)}{\mu}}$$
$$E(X^r) = r!\Big\{\sum_{k=0}^{\infty}\binom{-2}{k}\frac{E(\lambda^r)}{(n+1)^r}\Big\}$$

Proof. Using equations (2) and (9)

$$\begin{aligned}f(x) &= \int_0^\infty \frac{e^{-\frac{x-\theta}{\lambda}}}{\lambda[1+e^{-\frac{x-\theta}{\lambda}}]^2}\frac{1}{\mu}e^{-\frac{\lambda}{\mu}}d\lambda \\ &= \frac{1}{\mu}\int_0^\infty \lambda^{0-1}[1+e^{-\frac{x-\theta}{\lambda}}]^{-2}]e^{-\frac{x-\theta}{\lambda}-\frac{\lambda}{\mu}}d\lambda \\ &= \frac{1}{\mu}\int_0^\infty \lambda^{0-1}\binom{-2}{k}e^{-k(\frac{x-\theta}{\lambda})-\frac{x-\theta}{\lambda}-\frac{\lambda}{\mu}}d\lambda \\ &= \frac{1}{\mu}\sum_{k=0}^{\infty}\binom{-2}{k}\int_0^\infty \lambda^{0-1}e^{-\frac{\lambda}{\mu}-\frac{(k+1)(x-\theta)}{\lambda}}d\lambda \\ &= \frac{1}{\mu}\sum_{k=0}^{\infty}\binom{-2}{k}\int_0^\infty \lambda^{0-1}e^{-\frac{1}{\mu}(\lambda+(k+1)(x-\theta)\mu)\frac{1}{\lambda}}d\lambda\end{aligned}$$

Let the transformation

$$\lambda = \sqrt{\mu(k+1)(x-\theta)}z$$
$$d\lambda = \sqrt{\mu(k+1)(x-\theta)}dz$$

then

$$\begin{aligned}f(x) &= \frac{1}{\mu}\sum_{k=0}^{\infty}\binom{-2}{k}\int_0^\infty (\sqrt{\mu(k+1)(x-\theta)}z)^{0-1}e^{-\frac{1}{\mu}}(\sqrt{\mu(k+1)(x-\theta)} \\ &\quad z(k+1)(x-\theta))\frac{1}{\sqrt{\mu(k+1)(x-\theta)}z}\sqrt{\mu(k+1)(y-\theta)}dz \\ &= \frac{1}{\mu}\sum_{k=0}^{\infty}\binom{-2}{k}\int_0^\infty z)^{0-1}e^{-\frac{1}{\mu}}(\sqrt{\mu(k+1)(x-\theta)}[z+\frac{1}{z}]dz \\ &= \frac{2}{\mu}\sum_{k=0}^{\infty}\binom{-2}{k}\frac{1}{2}\int_0^\infty z^{0-1}e\{\frac{-2\sqrt{(k+1)(x-\theta)}}{\mu}[z+\frac{1}{z}]dz \\ &= -\frac{2}{\mu}\sum_{k=0}^{\infty}\binom{-2}{k}k_0\left(2\sqrt{\frac{(k+1)(x-\theta)}{\mu}}\right) \qquad (6.9)\end{aligned}$$

as required □

The Gamma I mixing distribution

The mixing distribution is given as ;

$$g(\lambda)=\frac{\beta^{\alpha}e^{-\beta\lambda}\lambda^{\alpha-1}}{\Gamma\alpha},\lambda>0,\alpha>0,\beta>0 \tag{6.10}$$

The mixed distribution shall be obtained as follows

$$\begin{aligned}
f(x)&=\int_0^{\infty}\frac{e^{-\frac{x-\theta}{\lambda}}}{\lambda[1+e^{-\frac{x-\theta}{\lambda}}]^2}\frac{\beta^{\alpha}e^{-\beta\lambda}\lambda^{\alpha-1}}{\Gamma\alpha}d\lambda\\
&=\frac{\beta^{\alpha}}{\Gamma\alpha}\int_0^{\infty}\frac{e^{-\frac{x-\theta}{\lambda}}}{\lambda[1+e^{-\frac{x-\theta}{\lambda}}]^2}e^{-\beta\lambda}\lambda^{\alpha-1}d\lambda\\
&=\frac{\beta^{\alpha}}{\Gamma\alpha}\int_0^{\infty}\lambda^{\alpha-2}[1+e^{-\frac{x-\theta}{\lambda}}]^{-2}e^{-\beta\lambda}d\lambda\\
&=\frac{\beta^{\alpha}}{\Gamma\alpha}\int_0^{\infty}\lambda^{\alpha-2}\sum_{k=0}^{\infty}\binom{-2}{k}e^{-\frac{k(x-\theta)}{\lambda}-\frac{x-\theta}{\lambda}-\beta\lambda}d\lambda\\
&=\frac{e^{\beta}\beta^{\alpha}}{\Gamma\alpha}\sum_{k=0}^{\infty}\binom{-2}{k}\int_0^{\infty}\lambda^{(\alpha-1)-1}e^{\frac{1}{\lambda}[(k+1)(\theta-x)-\lambda^2]}d\lambda
\end{aligned}$$

let

$$\lambda=\sqrt{(k+1)(\theta-x)}z$$

$$\begin{aligned}
d\lambda=&\sqrt{(k+1)(\theta-x)}dz\\
=&\frac{e^{\beta}\beta^{\alpha}}{\Gamma\alpha}\sum_{k=0}^{\infty}\binom{-2}{k}\int_0^{\infty}\left(\sqrt{(k+1)(\theta-x)}z\right)^{(\alpha-1)-1}*\\
&\qquad e^{-\frac{1}{\sqrt{(k+1)(\theta-x)}z}}\left\{(k+1)(\theta-x)+z^2(k+1)(\theta-x)\right\}dz\\
=&\frac{e^{\beta}\beta^{\alpha}}{\Gamma\alpha}\sum_{k=0}^{\infty}\binom{-2}{k}\int_0^{\infty}\left(\sqrt{(k+1)(\theta-x)}z\right)^{(\alpha-1)-1}e^{-\frac{(k+1)(\theta-x)}{\sqrt{(k+1)(\theta-x)}z}}\left\{1+z^2\right\}d\lambda\\
=&\frac{e^{\beta}\beta^{\alpha}}{\Gamma\alpha}\sum_{k=0}^{\infty}\binom{-2}{k}\int_0^{\infty}\left(\sqrt{(k+1)(\theta-x)}z\right)^{(\alpha-1)-1}e^{-\frac{\sqrt{(k+1)(\theta-x)}}{z}\left\{1+z^2\right\}}d\lambda\\
=&\frac{e^{\beta}\beta^{\alpha}}{\Gamma\alpha}\sum_{k=0}^{\infty}\binom{-2}{k}\int_0^{\infty}\left(\sqrt{(k+1)(\theta-x)}z\right)^{(\alpha-1)-1}e^{-\sqrt{(k+1)(\theta-x)}\left\{\frac{1}{z}+z\right\}}d\lambda\\
=&\frac{e^{\beta}\beta^{\alpha}}{\Gamma\alpha}\sum_{k=0}^{\infty}\binom{-2}{k}K_{\alpha-1}\left(\sqrt{(k+1)(\theta-x)}\right)
\end{aligned} \tag{6.11}$$

The Gamma II mixing distribution

The pdf of the mixing distribution is given by;

$$g(\lambda) = \frac{1}{\mu^\beta \Gamma\beta} e^{-\frac{\lambda}{\mu}} \lambda^{\beta-1}, \lambda > 0, \mu > 0 \tag{6.12}$$

The mixed distribution shall be obtained as follows;

$$f(x) = \int_0^\infty \frac{e^{-\frac{x-\theta}{\lambda}}}{\lambda[1+e^{-\frac{x-\theta}{\lambda}}]^2} \frac{1}{\mu^\beta \Gamma\beta} e^{-\frac{\lambda}{\mu}} \lambda^{\beta-1} d\lambda$$

$$f(x) = \frac{1}{\mu^\beta \Gamma\beta} \int_0^\infty \lambda^{\beta-2} e^{-\frac{x-\theta}{\lambda} - \frac{\lambda}{\mu}} [1+e^{-\frac{x-\theta}{\lambda}}]^{-2} d\lambda$$

$$f(x) = \frac{1}{\mu^\beta \Gamma\beta} \int_0^\infty \lambda^{(\beta-1)-1} e^{-k\frac{x-\theta}{\lambda} - \frac{x-\theta}{\lambda} - \frac{\lambda}{\mu}} \sum_{k=0}^\infty \binom{-2}{k} d\lambda$$

$$= \sum_{k=0}^\infty \binom{-2}{k} \frac{1}{\mu^\beta \Gamma\beta} \int_0^\infty \lambda^{(\beta-1)-1} e^{-k\frac{x-\theta}{\lambda} - \frac{x-\theta}{\lambda} - \frac{\lambda}{\mu}} d\lambda$$

$$= \sum_{k=0}^\infty \binom{-2}{k} \frac{1}{\mu^\beta \Gamma\beta} \int_0^\infty \lambda^{(\beta-1)-1} e^{\frac{1}{\lambda\mu}[\mu(k+1)(\theta-x)-\lambda^2]} d\lambda$$

let

$$\lambda = \sqrt{\mu(k+1)(\theta-x)}z$$

$$d\lambda = \sqrt{\mu(k+1)(\theta-x)}dz$$

$$= \sum_{k=0}^\infty \binom{-2}{k} \frac{1}{\mu^\beta \Gamma\beta} \int_0^\infty \left(\sqrt{\mu(k+1)(\theta-x)}z\right)^{(\beta-1)-1} \times$$

$$e^{\frac{1}{\mu\sqrt{\mu(k+1)(\theta-x)}z}[\mu(k+1)(\theta-x)-\mu(k+1)(\theta-x)z^2]} d\lambda$$

$$= \sum_{k=0}^\infty \binom{-2}{k} \frac{1}{\mu^\beta \Gamma\beta} \int_0^\infty \left(\sqrt{\mu(k+1)(\theta-x)}z\right)^{(\beta-1)-1} e^{-\frac{\mu(k+1)(\theta-x)}{\mu\sqrt{\mu(k+1)(\theta-x)}z}[1+z^2]} d\lambda$$

$$= \sum_{k=0}^\infty \binom{-2}{k} \frac{1}{\mu^\beta \Gamma\beta} \int_0^\infty \left(\sqrt{\mu(k+1)(\theta-x)}z\right)^{(\beta-1)-1} e^{\frac{\sqrt{\mu(k+1)(\theta-x)}}{\mu}[\frac{1}{z}+z]} d\lambda$$

$$= \sum_{k=0}^\infty \binom{-2}{k} \frac{1}{\mu^\beta \Gamma\beta} K_{\beta-1}\left(\sqrt{\frac{(k+1)(\theta-x)}{\mu}}\right) \tag{6.13}$$

The Inverse Gamma mixing distribution

The pdf of the mixing distribution is given as;

$$g(\lambda)=\frac{\beta^{\alpha}e^{\frac{-\beta}{\lambda}}\lambda^{-\alpha-1}}{\Gamma\alpha},\lambda>0,\alpha>0,\beta>0 \tag{6.14}$$

The mixed distribution shall be obtained as follows ;

$$\begin{aligned}
f(x)=&\int_0^\infty \frac{e^{-\frac{x-\theta}{\lambda}}}{\lambda[1+e^{-\frac{x-\theta}{\lambda}}]^2}\frac{\beta^{\alpha}e^{\frac{-\beta}{\lambda}}\lambda^{-\alpha-1}}{\Gamma\alpha}d\lambda\\
=&\frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty \lambda^{-\alpha-2}[1+e^{-\frac{x-\theta}{\lambda}}]^{-2}e^{-\frac{x-\theta}{\lambda}-\frac{\beta}{\lambda}}d\lambda\\
=&\frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty \lambda^{-\alpha-2}\sum_{k=0}^{\infty}\binom{-2}{k}e^{-k(\frac{x-\theta}{\lambda})-\frac{x-\theta}{\lambda}-\frac{\beta}{\lambda}}d\lambda\\
=&\frac{\beta^\alpha}{\Gamma\alpha}\sum_{k=0}^{\infty}\binom{-2}{k}\int_0^\infty \lambda^{-\alpha-2}e^{-\frac{1}{\lambda}(k+1)(\theta-x)-\beta}d\lambda\\
=&\frac{\beta^\alpha}{\Gamma\alpha}\sum_{k=0}^{\infty}\binom{-2}{k}\int_0^\infty \lambda^{-(\alpha-1)-1}e^{-\frac{1}{\lambda}(k+1)(\theta-x)-\beta}d\lambda\\
=&\frac{\beta^\alpha}{\Gamma\alpha}\sum_{k=0}^{\infty}\binom{-2}{k}\frac{-\Gamma(\alpha-1)}{\Big(-\frac{1}{\lambda}(k+1)(\theta-x)-\beta\Big)^{-\alpha-1}}\\
=&\frac{\beta^\alpha}{\Gamma\alpha}\sum_{k=0}^{\infty}\binom{-2}{k}\frac{-\alpha\Gamma(\alpha)}{\Big(-\frac{1}{\lambda}(k+1)(\theta-x)-\beta\Big)^{-\alpha-1}}\\
=&\frac{-\alpha\beta^\alpha}{\Big(-\frac{1}{\lambda}(k+1)(\theta-x)-\beta\Big)^{-\alpha-1}}\\
=&-\alpha\beta^\alpha\Big(-\frac{1}{\lambda}(k+1)(\theta-x)-\beta\Big)^{\alpha+1}\\
=&\alpha\beta^\alpha\Big(\frac{1}{\lambda}(k+1)(\theta-x)+\beta\Big)^{\alpha+1}
\end{aligned} \tag{6.15}$$

The Half Logistic mixing distribution

The pdf of the mixing distribution is given as;

$$g(\lambda)=\frac{2\mu e^{-\mu\lambda}}{(1+e^{-\mu\lambda})^2} \tag{6.16}$$

The mixed distribution shall be obtained as follows;

$$
\begin{aligned}
f(x) =& \int\limits_0^\infty \frac{e^{-\frac{x-\theta}{\lambda}}}{\lambda[1+e^{-\frac{x-\theta}{\lambda}}]^2} \frac{2\mu e^{-\mu\lambda}}{(1+e^{-\mu\lambda})^2} d\lambda \\
=& 2\mu \int_0^\infty \lambda^{0-1}[1+e^{-\frac{x-\theta}{\lambda}}]^{-2}(1+e^{-\mu\lambda})^{-2} e^{-\frac{x-\theta}{\lambda}-\lambda\mu} d\lambda \\
=& 2\mu \int_0^\infty \lambda^{0-1} \sum_{k=0}^{\infty} \binom{-2}{k} \sum_{l=0}^{\infty} \binom{-2}{l} e^{-k(\frac{x-\theta}{\lambda})-\frac{x-\theta}{\lambda}-l\lambda\mu-\lambda\mu} d\lambda \\
=& 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \sum_{l=0}^{\infty} \binom{-2}{l} \int\limits_0^\infty \lambda^{0-1} e^{-k(\frac{x-\theta}{\lambda})-\frac{x-\theta}{\lambda}-l\lambda\mu-\lambda\mu} d\lambda \\
=& 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \sum_{l=0}^{\infty} \binom{-2}{l} \int\limits_0^\infty \lambda^{0-1} e^{-\frac{1}{\lambda}(\theta-x)(k+1)-\lambda^2\mu(l-1)} d\lambda
\end{aligned}
$$

let

$$
\begin{aligned}
\lambda &= \sqrt{\mu(\theta-x)(k+1)}z \\
d\lambda &= \sqrt{\mu(\theta-x)(k+1)}dz
\end{aligned}
$$

$$
\begin{aligned}
=& 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \sum_{l=0}^{\infty} \binom{-2}{l} \int\limits_0^\infty \left(\sqrt{\mu(\theta-x)(k+1)}z\right)^{0-1} \times \\
& e^{-\frac{1}{\sqrt{\mu(\theta-x)(k+1)}z}[(\theta-x)(k+1)+\mu^2(\theta-x)(k+1)z^2(l-1)]} \sqrt{\mu(\theta-x)(k+1)}dz \\
=& 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \sum_{l=0}^{\infty} \binom{-2}{l} \int\limits_0^\infty \left(\sqrt{\mu(\theta-x)(k+1)}z\right)^{0-1} \times \\
& e^{-\frac{(\theta-x)(k+1)}{\sqrt{\mu(\theta-x)(k+1)}z}[1+\mu^2 z^2(l-1)]} \sqrt{\mu(\theta-x)(k+1)}dz \\
=& 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \sum_{l=0}^{\infty} \binom{-2}{l} \int\limits_0^\infty \left(\sqrt{\mu(\theta-x)(k+1)}z\right)^{0-1} \times \\
& e^{-\sqrt{\mu(\theta-x)(k+1)}[\frac{1}{z}+\mu^2 z(l-1)]} \sqrt{\mu(\theta-x)(k+1)}dz \\
=& 2\mu \sum_{k=0}^{\infty} \binom{-2}{k} \sum_{l=0}^{\infty} \binom{-2}{l} K_0\left(-\sqrt{\mu(\theta-x)(k+1)}\right)
\end{aligned}
\quad \textbf{(6.17)}
$$

6.3 Logistic Inverse Gaussian Distribution

6.3.1 Introduction

Let

$$X = \mu + ZU \quad \textbf{(6.18)}$$

Where μ is a constant, z is a positive random variable independent of U, a standard logistic random variable. Suppose $F(x)$,$G(z)$ and $H(u)$ are cdfs of X, Z and U respectively; and their corresponding pdfs being $f(x)$, $g(z)$ and $h(u)$. Then

$$\begin{aligned}
F(x) =& \mathrm{Prob}\{X \leq x\} \\
=& \mathrm{Prob}\{X \leq x, \;\; 0 < Z < \infty\} \\
=& \mathrm{Prob}\{\mu + ZU \leq x \;\; 0 < Z < \infty\} \\
=& \mathrm{Prob}\{U \leq \frac{x-\mu}{Z}, \;\; 0 < Z < \infty\} \\
=& \int_0^\infty \left(\int_{-\infty}^{\frac{x-\mu}{z}} h(u)du \right) g(z)dz \\
=& \int_0^\infty H(u)\Big| g(z)dz \\
=& \int_0^\infty H\left(\frac{x-\mu}{z}\right) g(z)dz \\
& and \\
f(x) =& \int_0^\infty \frac{1}{z} h\left(\frac{x-\mu}{z}\right) g(z)dz \\
=& \int_0^\infty f(x|z)g(z)dz, \;\; -\infty < x < \infty
\end{aligned}$$

where

$$\therefore f(x) = \int_0^\infty \frac{1}{z} \frac{e^{-\frac{x-\mu}{z}}}{[1+e^{-\frac{x-\mu}{z}}]^2} g(z)dz, \quad -\infty < x < \infty \tag{6.19}$$

is the conditional pdf and $g(z)$ is the mixing pdf Therefore

$$f(x|z) = \frac{1}{z} \frac{e^{-\frac{x-\mu}{z}}}{[1+e^{-\frac{x-\mu}{z}}]^2}, \quad -\infty < x < \infty, \;\; -\infty < \mu < \infty, \;\; z > 0 \tag{6.20}$$

Which is conditional pdf and $g(z)$ is the mixing distribution. Formula (6.18) is called a stochastic representation while formula (6.20) is hierarchical representation.

6.3.2 Generalized Inverse Gaussian (GIG) Distribution.

Generalized Inverse Gaussian (GIG) distribution is based on modified Bessel function of the third kind of order λ evaluated at ω is denoted by $K_\lambda(\omega)$ and defined as

$$K_\lambda(\omega) = \frac{1}{2} \int_0^\infty x^{\lambda-1} e^{-\frac{\omega}{2}(x+\frac{1}{x})} dx \quad -\infty < \lambda < \infty, \omega > 0. \tag{6.21}$$

By parametrization $\omega = \delta\gamma$ We have

$$K_\lambda(\delta\gamma) = \frac{1}{2} \int_0^\infty x^{\lambda-1} e^{-\frac{\delta\gamma}{2}(\frac{1}{x}+x)} dx$$

And by transformation,

$$x = \frac{\gamma}{\delta} z \implies dx = \frac{\gamma}{\delta} dz$$

We have

$$\begin{aligned} K_\lambda(\delta\gamma) =& \frac{1}{2}\int_0^\infty (\frac{\gamma}{\delta} z)^{\lambda-1} e^{-\frac{\delta\gamma}{2}[\frac{1}{\frac{\gamma}{\delta}z}+\frac{\gamma}{\delta}Z]} \frac{\gamma}{\delta} dz \\ =& \frac{1}{2}(\frac{\gamma}{\delta})^\lambda \int_0^\infty Z^{\lambda-1} e^{-\frac{1}{2}(\frac{\delta^2}{Z}+\gamma^2 z)} dz \end{aligned} \quad \textbf{(6.22)}$$

Therefore,

$$I = \int_0^\infty (\frac{\gamma}{\delta})^\lambda \frac{z^{\lambda-1} e^{-\frac{1}{2}(\frac{\delta^2}{z}+\gamma^2 Z)}}{2K_\lambda(\delta\gamma)} dz$$

Thus we have a pdf

$$g(z) = (\frac{\gamma}{\delta})^\lambda \frac{z^{\lambda-1} e^{-\frac{1}{2}(\frac{\delta^2}{z}+\gamma^2 z)}}{2K_\lambda(\delta\gamma)} \quad \text{for} \quad z>0, \quad -\infty<\lambda<\infty, \quad \delta>0, \quad \gamma>0 \quad \textbf{(6.23)}$$

This is a generalized inverse Gaussian (GIG) distribution. It has three parameters : λ, δ and γ

Thus we write it in short form

$$Z \sim GIG(\lambda, \delta, \gamma)$$

6.3.3 Moments of the generalized inverse Gaussian (GIG)distribution

The r^{th} moment of (GIG) is given by:

$$\begin{aligned} E(X^r) =& \int_0^\infty z^r g(z) dz \\ =& \int_0^\infty z^r (\frac{\gamma}{\delta})^\lambda \frac{z^{\lambda-1} e^{-\frac{1}{2}(\frac{\delta^2}{z}+\gamma^2 z)}}{2k_\lambda(\delta\gamma)} dz \\ \therefore E(X^r) =& \frac{(\frac{\gamma}{\delta})^\lambda}{2K_\lambda(\delta\gamma)} \int_0^\infty z^{\lambda+r-1} e^{-\frac{1}{2}(\frac{\delta^2}{z}+\gamma^2 z)} dz \\ =& \frac{(\frac{\gamma}{\delta})^\lambda}{2K_\lambda(\delta\gamma)} (\frac{\gamma}{\delta})^{-\lambda-r} 2K_{\lambda+r}(\delta\gamma) \int_0^\infty (\frac{\gamma}{\delta})^{\lambda+r} \frac{z^{\lambda+r-1} e^{-\frac{1}{2}(\frac{\delta^2}{z}+\gamma^2 z)}}{2K_{\lambda+r}(\delta\gamma)} dz \\ =& (\frac{\gamma}{\delta})^\lambda (\frac{\gamma}{\delta})^{-\lambda-r} \frac{2K_{\lambda+r}(\delta\gamma)}{2K_\lambda(\delta\gamma)} \cdot 1 \\ =& (\frac{\gamma}{\delta})^{-r} \frac{K_{\lambda+r(\delta\gamma)}}{K_\lambda(\delta\gamma)} \\ =& (\frac{\delta}{\gamma})^r \frac{K_{\lambda+r}(\delta\gamma)}{K_\lambda(\delta\gamma)} \end{aligned} \quad \textbf{(6.24)}$$

Where λ is a positive or a negative integer.

6.3.4 Logistic GIG distribution

Using GIG as a mixing distribution :

$$
\begin{aligned}
f(x) =& \int_0^\infty f(x/z)g(z)dz \\
=& \int_0^\infty \frac{1}{z}\frac{e^{-\frac{x-\mu}{z}}}{(1+e^{-\frac{x-\mu}{z}})^2}\cdot\frac{(\frac{\gamma}{\delta})^\lambda z^{\lambda-1}}{2K_\lambda(\delta\gamma)}e^{-\frac{1}{2}(\frac{\delta^2}{z}-\gamma^2 z)}dz \\
=& (\frac{\gamma}{\delta})^\lambda\frac{1}{2K_\lambda(\delta\gamma)}\int_0^\infty \frac{z^{\lambda-2}e^{-\frac{x-\mu}{z}-\frac{1}{2}(\frac{\delta^2}{z}+\gamma^2 z)}}{(1+e^{-\frac{x+\mu}{z}})^2}dz \\
\therefore f(x) =& (\frac{\gamma}{\delta})^\lambda\frac{1}{2K_\lambda(\delta\gamma)}\int_0^\infty \frac{Z^{(\lambda-1)-1}e^{-\frac{1}{2}[\gamma^2 z+\frac{2(x-\mu)}{z}]}}{(1+e^{-\frac{(x-\mu)}{z}})^2}dz \\
=& (\frac{\gamma}{\delta})^\lambda\frac{1}{2K_\lambda(\delta\gamma)}\int_0^\infty \sum_{j=0}^\infty \binom{-2}{j} e^{-j\frac{x-\mu}{z}} z^{(\lambda-1)-1}e^{-\frac{1}{2}[\gamma^2 z+\frac{\delta^2+2(x-\mu)}{z}]}dz \\
=& (\frac{\gamma}{\delta})^\lambda\frac{1}{2K_\lambda(\delta\gamma)}\sum_{j=0}^\infty \left\{\binom{-2}{j}\right\}\int_0^\infty z^{(\lambda-1)-1}e^{-\frac{1}{2}\gamma^2 z}e^{-\frac{1}{2}[\frac{\delta^2+2(j+1)(x-\mu)}{z}]}dz \\
\therefore f(x) =& (\frac{\gamma}{\delta})^\lambda\frac{1}{2K_\lambda(\gamma\delta)}\sum_{j=0}^\infty \left\{\binom{-2}{j}\right\}\int_0^\infty Z^{(\lambda-1)-1}e^{-\frac{1}{2}[\gamma^2 z+\delta^2+\frac{2(j+1)(x-\mu)}{z}]}dz \qquad \textbf{(6.25)}
\end{aligned}
$$

Let

$$
\begin{aligned}
\delta^2+2(j+1)(x-\mu) =& \delta^2[1+\frac{2(j+1)(x-\mu)}{\delta^2}] \\
=& \delta^2\phi(j,x)
\end{aligned}
$$

where $\phi(j,x) = 1+\frac{2(j+1)(x-\mu)}{\delta^2}$

$$
\begin{aligned}
\therefore f(x) =& (\frac{\delta}{\gamma})^\lambda\frac{1}{2K_\lambda(\delta\gamma)}\sum_{j=0}^\infty \binom{-2}{j}\int_0^\infty Z^{(\lambda-1)-1}e^{-\frac{1}{2}[\gamma^2 z+\frac{\delta^2\phi(j,x)}{z}]}dz \\
\therefore f(x) =& (\frac{\delta}{\gamma})^\lambda\frac{1}{2K_\lambda(\delta\gamma)}\sum_{j=0}^\infty \binom{-2}{j}\int_0^\infty Z^{(\lambda-1)-1}e^{-\frac{\gamma^2}{2}[z+\frac{\delta^2\phi(j,x)}{\delta^2 z}]}dz
\end{aligned}
$$

Let

$$
z = \frac{\delta\sqrt{\phi(j,x)}}{\gamma}t \implies dz = \frac{\delta\sqrt{\phi(j,x)}}{\gamma}dt
$$

$$\therefore f(x) = (\frac{\delta}{\gamma})^{\lambda} \frac{1}{2K_{\lambda}(\delta\gamma)} \sum_{j=0}^{\infty} \binom{-2}{j} \left(\frac{\delta\sqrt{\phi(j,x)}}{\gamma}\right)^{\lambda-1} \int_{0}^{\infty} t^{(\lambda-1)-1} e^{\gamma\delta\sqrt{\phi(j,x)}(t+\frac{1}{t})} dt$$

$$\therefore f(x) = (\frac{\delta}{\gamma})^{\lambda} \frac{1}{2K_{\lambda}(\delta\gamma)} \sum_{j=0}^{\infty} \binom{-2}{j} \left(\frac{\delta\sqrt{\phi(j,x)}}{\gamma}\right)^{\lambda-1} K_{\lambda-1}(\gamma\delta\sqrt{\phi(j,x)})$$

$$\therefore f(x) = (\frac{\delta}{\gamma}) \sum_{j=0}^{\infty} \left\{ \binom{-2}{j} \left(\frac{\delta\sqrt{\phi(j,x)}}{\gamma}\right)^{\lambda-1} \frac{K_{\lambda-1}(\gamma\delta\sqrt{\phi(j,x)})}{2K_{\lambda}(\delta\gamma)} \right\} \tag{6.26}$$

6.3.5 Logistic Inverse Gaussian (LIG) Distribution

Construction

The inverse Gaussisan is a special case of GIG with $\lambda = -\frac{1}{2}$. The pdf of logistic inverse Gaussian (LIG) is given by substituting $\lambda = -\frac{1}{2}$ in equation (6.26) i.e.

$$\begin{aligned} g_1(z) &= (\frac{\gamma}{\delta})^{-\frac{1}{z}} \frac{Z^{-\frac{1}{2}-1} e^{-\frac{1}{2}(\frac{\delta^2}{z}+\gamma^2 z)}}{2K_{-\frac{1}{2}}(\delta\gamma)} \\ &= (\frac{\gamma}{\delta})^{-\frac{1}{z}} \frac{Z^{-\frac{3}{2}} e^{-\frac{1}{2}(\frac{\delta^2}{z}+\gamma^2 z)}}{2\sqrt{\frac{\pi}{2\delta\gamma}} e^{-\delta\gamma}} \\ &= \frac{e^{\delta\gamma}}{[\frac{\gamma}{\delta}\frac{4\pi}{2\delta\gamma}]^{\frac{1}{2}}} z^{-\frac{3}{2}} e^{-\frac{1}{2}(\frac{\delta^2}{z}+\gamma^2 z)} \end{aligned} \tag{6.27}$$

The pdf of logistic inverse Gaussian (GIG) is given by substitution $\lambda = -\frac{1}{2}$ in (4.1) to get;

$$\begin{aligned} f_1(x) &= \frac{\delta}{\gamma} \sum_{j=0}^{\infty} \left\{ \binom{-2}{j} \left(\sqrt{\phi(j,x)}\right)^{-\frac{3}{2}} \frac{K_{-\frac{3}{2}}(\gamma\delta\sqrt{\phi(j,x)})}{2K_{-\frac{1}{2}}(\delta\gamma)} \right\} \\ &= \frac{\delta}{\gamma} \sum_{j=0}^{\infty} \left\{ \binom{-2}{j} \left(\sqrt{\phi(j,x)}\right)^{-\frac{3}{2}} \frac{(1+\frac{1}{\delta\gamma\sqrt{\phi(j,x)}})K_{-\frac{1}{2}}(\gamma\delta\sqrt{\phi(j,x)})}{K_{-\frac{1}{2}}(\delta\gamma)} \right\} \end{aligned}$$

$$\begin{aligned} f_1(x) &= \frac{\delta}{\gamma} \sum_{j=0}^{\infty} \left\{ \binom{-2}{j} \left(\sqrt{\phi(j,x)}\right)^{2} \left((1+\frac{1}{\delta\gamma\sqrt{\phi(j,x)}})\right) \frac{\frac{\sqrt{\pi}}{2\gamma\delta\sqrt{\phi}} e^{-\delta\gamma\sqrt{\phi}}}{\sqrt{\frac{\pi}{2\delta\gamma}} e^{-\delta\gamma}} \right\} \\ &= \frac{\delta}{\gamma} e^{\delta\gamma} \sum_{j=0}^{\infty} \left\{ \binom{-2}{j} \left(\sqrt{\phi(j,x)}\right)^{2} \left((1+\frac{1}{\delta\gamma\sqrt{\phi(j,x)}})\right) \frac{e^{\delta\gamma\sqrt{\phi(j,x)}}}{\sqrt{\phi(j,x)}} \right\} \end{aligned}$$

$$\textit{But} \binom{-2}{j} = (-1)\binom{2+j-1}{j} = (-1)\binom{j+1}{j} = (-1)^j(j+1)$$

$$\therefore f_(x) = \frac{\gamma}{\delta} e^{\gamma\delta} \sum_{j=0}^{\infty} (-1)^j (j+1) \left(\frac{1}{\sqrt{\phi(j,x)}}\right)^2 \left(1 + \frac{1}{\delta\gamma\sqrt{\phi(j,x)}}\right) e^{-\delta\gamma\sqrt{\phi(j,x)}} \quad \textbf{(6.28)}$$

The log-likelihood function of LIG Distribution

$$\ell = \log L = \log \prod_{i=1}^{n} f_1(x_i)$$

$$\therefore \ell = \sum_{i=1}^{n} \log f_1(x_i)$$

$$= \sum_{i=1}^{n} \left\{ \log\gamma - \log\delta + \gamma\delta + \log \sum_{j=0}^{\infty} (-1)^j (j+1) \left(\frac{1}{\sqrt{\phi(j,x)}}\right)^2 \left(1 + \frac{1}{\delta\gamma\sqrt{\phi(j,x)}}\right) e^{-\delta\gamma\sqrt{\phi(j,x)}} \right\}$$

Therefore,

$$\ell = \begin{cases} n\log\gamma - n\log\delta + n\delta\gamma + \\ \sum\limits_{i=1}^{n} [\log \sum_{j=0}^{\infty} (-1)^j (j+1) \left(\frac{1}{\sqrt{\phi(j,x)}}\right)^2 \left(1 + \frac{1}{\delta\gamma\sqrt{\phi(j,x)}}\right) e^{\delta\gamma\sqrt{\phi(j,x)}}] \end{cases} \Bigg\} \quad \textbf{(6.29)}$$

6.3.6 EM Algorithm for LIG Distribution

EM algorithm is a technique for obtaining maximum likelihood estimates for parameters when there's missing or latent data, by repeatedly performing expectation and maximization of the likelihood function until convergence of old and new parameter estimates is achieved. It was introduced by Dempater et. al (1977).

Assume that the true data are made of an observed part x and unobserved part z.

This then ensures the log-likelihood of the complete data x_i, z_i for $i = 1, 2, \ldots, n$ factorizing into two parts (Kostas, 2007).

$$L = \prod_{i=1}^{n} f(x_i|z_i) g(z_i)$$

$$= \prod_{i=1}^{n} f(x_i|z_i) \prod_{i=1}^{n} g(z_i) \quad \textbf{(6.30)}$$

The log-likelihood function is

$$\begin{aligned} l = \log L = & \log \prod_{i=1}^{n} f(x_i|z_i) + \log \prod_{i=1}^{n} g(z_i) \\ = & \sum_{i=1}^{n} \log f(x_i|z_i) + \sum_{i=1}^{n} \log g(z_i) \end{aligned} \tag{6.31}$$

Let

$$\ell_1 = \sum_{i=1}^{n} \log f(x_i|z_i)$$

and

$$\ell_2 = \sum_{i=1}^{n} \log g(z_i)$$

6.3.7 M-steps for the conditional pdf

$$\begin{aligned} f(x|z) = & \frac{1}{z} \frac{e^{-(\frac{x-\mu}{z})}}{(1+e^{-(\frac{x-\mu}{z})})^2} \\ \ell_1 = & \sum_{i=1}^{n} \log f(x_i|z_i) \\ = & \sum_{i=1}^{n} \log \frac{1}{z_i} \frac{e^{-(\frac{x_i-\mu}{z_i})}}{(1+e^{-\frac{(x_i-\mu)}{x_i}})^2} \\ = & \sum_{i=1}^{n} \left\{ -\log z_i - \frac{(x_i-\mu)}{z_i} - \log\left(1+e^{\frac{-(x_i-\mu)}{z_i}}\right)^2 \right\} \\ = & -\sum_{i=1}^{n} \log z_i - \sum_{i=1}^{n} \left(\frac{x_i-\mu}{z_i} - 2\sum_{i=1}^{n} \log\left(1+e^{-(\frac{x_i-\mu}{z_i})}\right)\right) \\ \frac{\partial \ell}{\partial \mu} = & \sum_{i=1}^{n} \frac{1}{z_i} 2 \sum_{i=1}^{n} \frac{1}{z_i} \frac{e^{\frac{-(x_i-\mu)}{z_i}}}{1+e^{-(\frac{x_i-\mu}{z_i})}} \\ \frac{\partial \ell_1}{\partial \mu} = & 0 \implies \sum_{i=1}^{n} \frac{1}{z_i} = 2\sum_{i=1}^{n} \frac{e^{-\frac{(x_i-\mu)}{z_i}}}{z_i(1+e^{-\frac{(x_i-\mu)}{z_i}})} \\ \therefore \sum_{i=1}^{n} \frac{1}{z_i} = & 2\sum_{i=1}^{n} \left[\frac{1}{z_i(1+e^{\frac{x_i-\mu}{z_i}})} \right] \end{aligned} \tag{6.32}$$

6.3.8 M-step for the mixing distribution

$$
\begin{aligned}
g(z) =& \frac{\delta e^{\delta\gamma}}{\Gamma(2\pi)} Z^{\frac{-3}{2}} e^{\frac{-1}{2}(\frac{\delta^2}{z}+\gamma^2 z)} \\
\therefore \ell_2 =& \sum_{i=1}^{n} \log g(z_i) \\
=& \sum \left\{ \log(2\pi) + \delta\gamma - \frac{3}{2}\log z_i - \frac{\delta^2}{2z_i} - \frac{\gamma^2}{2} z_i \right\} \\
=& n\log\delta - \frac{n}{2}\log(2\pi) + n\delta\gamma - \frac{3}{2}\sum_{i=1}^{n}\log z_i - \frac{\delta^2}{2}\sum_{i=1}^{n}\frac{1}{z_i} - \frac{\gamma^2}{2}\sum_{i=1}^{n} z_i \\
\frac{\partial \ell_2}{\partial\gamma} =& n\delta - \gamma\sum z_i \\
\frac{\partial \ell_2}{\partial\gamma} =& 0 \implies \hat{\delta} = \gamma\bar{z} \\
\frac{\partial \ell_2}{\partial\delta} =& \frac{n}{\delta} + n\gamma - \delta\sum_{i=1}^{n}\frac{1}{z_i} \\
=& \frac{n}{\delta} + n\frac{\hat{\delta}}{\bar{z}} - \delta\sum\frac{1}{z_i} \\
\therefore \frac{\partial \ell_2}{\partial\delta} =& 0 \implies n + n\frac{\delta^2}{\bar{z}} = \hat{\delta^2}\sum\frac{1}{z_i} \\
n =& \delta^2\left[\sum_{i=1}^{n}\frac{1}{z_i} - \frac{n}{\bar{z}}\right] \\
\hat{\delta} =& \sqrt{\frac{n}{\sum\frac{1}{z_i} - \frac{n}{\bar{z}}}}
\end{aligned} \tag{6.33}
$$

where $\bar{z} = \sum_{i=1}^{n} \frac{z_i}{n}$

6.3.9 E-STEP

Since there is no data for z and $\frac{1}{z}$ we shall estimate them by the posterior expectation $E(Z|x)$ and $E(\frac{1}{Z}|x)$ respectively.

A posterior pdf is defined as

$$f(Z|x) = \frac{f(x|z)g(z)}{\int_0^\infty f(x|z)g(z)dz} \tag{6.34}$$

so that

$$E(Z|x) = \frac{z\int_0^\infty f(x|z)g(z)dz}{\int_0^\infty f(x|z)g(z)dz} \tag{6.35}$$

Next

$$E(Z|x) = \frac{\int_0^\infty z f(x|z) g(z) dz}{\int_0^\infty f(x|z) g(z)} dz$$

$$= \frac{\int_0^\infty z \frac{1}{z} e^{-\frac{(x-\mu)}{z}} [1 + e^{-\frac{(x-\mu)}{z}}]^{-2} g(z) dz}{\int_0^\infty \frac{1}{z} e^{-\frac{(x-\mu)}{z}} [1 + e^{\frac{-(x-\mu)}{z}}]^{-2} g(z) dz}$$

$$\therefore E(Z|x) = \frac{\int_0^\infty e^{-\frac{(x-\mu)}{z}} [1 + e^{-\frac{(x-\mu)}{z}}]^{-2} g(z) dz}{\int_0^\infty z^{-1} e^{-\frac{(x-\mu)}{z}} [1 + e^{\frac{-(x-\mu)}{z}}]^{-2} g(z) dz} \quad \textbf{(6.36)}$$

But

$$e^{-\frac{x-\mu}{z}} [1 + e^{-\frac{(x-\mu)}{z}}]^{-2} g(z)$$

$$= e^{-\frac{x-\mu}{z}} \sum_{j=0}^\infty \binom{-2}{j} e^{-\frac{j(x-\mu)}{z}} \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}} z^{\frac{-3}{2}} e^{-\frac{1}{2}(\frac{\delta^2}{z} + \gamma^2 z)}$$

$$= \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}} \sum_{j=0}^\infty \binom{-2}{j} z^{-\frac{3}{2}} e^{-(j+1)\frac{(x-\mu)}{z} - \frac{1}{2}(\frac{\delta^2}{z} + \gamma^2 z)}$$

$$= \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}} \sum_{j=0}^\infty \binom{-2}{j} z^{-\frac{3}{2}} e^{-\frac{1}{2}[\frac{\delta^2 + 2(j+1)(x-\mu)}{z} + \gamma^2 z]}$$

$$= \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}} \sum_{j=0}^\infty \binom{-2}{j} z^{-\frac{3}{2}} e^{-\frac{1}{2}\left[\gamma^2 z + \delta^2 (1 + 2\frac{(j+1)(x-\mu)}{\delta^2}) \frac{1}{z}\right]}$$

$$= \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}} \sum_{j=0}^\infty \binom{-2}{j} z^{-\frac{3}{2}} e^{-\frac{1}{2}\left[\gamma^2 z + \frac{\delta^2 \phi(j,x)}{z}\right]} \quad \textbf{(6.37)}$$

where $\phi(j,x) = 1 + \frac{2(j+1)(x-\mu)}{\delta^2}$

$$\therefore E(Z|x) = \frac{\frac{1}{2} \int_0^\infty \sum_{j=0}^\infty \binom{-2}{j} z^{-\frac{3}{2}} e^{-\frac{1}{2}\left[\gamma^2 z + \delta^2 \frac{\phi(j,x)}{z}\right]} dz}{\frac{1}{2} \int_0^\infty z^{-1} \sum_{j=0}^\infty \binom{-2}{j} z^{-\frac{3}{2}} e^{-\frac{1}{2}\left[\gamma^2 z + \delta^2 \frac{\phi(j,x)}{z}\right]} dz}$$

$$= \frac{\sum_{j=0}^\infty \left\{ \binom{-2}{j} \frac{1}{2} \int_0^\infty z^{-\frac{1}{2}-1} e^{-\frac{1}{2}\left[\gamma^2 z + \delta^2 \frac{\phi(j,x)}{z}\right]} dz \right\}}{\sum_{j=0}^\infty \left\{ \binom{-2}{j} \frac{1}{2} \int_0^\infty z^{-\frac{3}{2}-1} e^{-\frac{1}{2}\left[\gamma^2 z + \delta^2 \frac{\phi(j,x)}{z}\right]} dz \right\}}$$

$$E(Z|x) = \frac{\sum_{j=0}^{\infty}\left\{\binom{-2}{j}\frac{1}{2}\int_0^{\infty} z^{-\frac{1}{2}-1}e^{-\frac{\gamma^2}{2}\left[z+\frac{\delta^2}{\gamma^2}\frac{\phi(j,x)}{z}\right]}dz\right\}}{\sum_{j=0}^{\infty}\left\{\binom{-2}{j}\frac{1}{2}\int_0^{\infty} z^{-\frac{3}{2}-1}e^{-\frac{\gamma^2}{2}\left[z+\frac{\delta^2}{\gamma^2}\frac{\phi(j,x)}{z}\right]}dz\right\}}$$

Let

$$z = \frac{\delta}{\gamma}\sqrt{\phi(j,x)}t \implies dz = \frac{\delta}{\gamma}\sqrt{\phi(j,x)}dt$$

$$\therefore E(Z|x) = \frac{\sum_{j=0}^{\infty}\left\{\binom{-2}{j}\left(\frac{\delta}{\gamma}\sqrt{\phi(j,x)}\right)^{-\frac{1}{2}}K_{-\frac{1}{2}}\left(\delta\gamma\sqrt{\phi(j,x)}\right)\right\}}{\sum_{j=0}^{\infty}\left\{\binom{-2}{j}\left(\frac{\delta}{\gamma}\sqrt{\phi(j,x)}\right)^{-\frac{3}{2}}K_{-\frac{3}{2}}\left(\delta\gamma\sqrt{\phi(j,x)}\right)\right\}}$$

$$= \frac{(\frac{\delta}{\gamma})^{-\frac{1}{2}}}{(\frac{\delta}{\gamma})^{\frac{-3}{2}}}\frac{\sum_{j=0}^{\infty}\left\{\binom{-2}{j}\left(\sqrt{\phi(j,x)}\right)^{-\frac{1}{2}}K_{-\frac{1}{2}}\left(\delta\gamma\sqrt{\phi(j,x)}\right)\right\}}{\sum_{j=0}^{\infty}\left\{\binom{-2}{j}\left(\sqrt{\phi(j,x)}\right)^{-\frac{3}{2}}K_{-\frac{3}{2}}\left(\delta\gamma\sqrt{\phi(j,x)}\right)\right\}}$$

$$= \frac{\sum_{j=0}^{\infty}\binom{-2}{j}\left(\sqrt{\phi(j,x)}\right)^{-\frac{1}{2}}\left[\frac{\pi}{2\delta\gamma\sqrt{\phi(j,x)}}\right]^{\frac{1}{2}}e^{-\delta\gamma\sqrt{\phi(j,x)}}}{(\frac{\delta}{\gamma})^{-1}\sum_{j=0}^{\infty}\binom{-2}{j}\left(\sqrt{\phi(j,x)}\right)^{-\frac{3}{2}}\left(1+\frac{1}{\delta\gamma\sqrt{\phi(j,x)}}\right)K_{-\frac{1}{2}}(\delta\gamma\sqrt{\phi((j,x)})}$$

$$\therefore E(Z|X) = \frac{\delta\sum_{j=0}^{\infty}\binom{-2}{j}\left(\sqrt{\phi(j,x)}\right)^{-\frac{1}{2}}\left[\frac{\pi}{2\delta\gamma\sqrt{\phi(j,x)}}\right]^{\frac{1}{2}}e^{-\delta\gamma\sqrt{\phi(j,x)}}}{\gamma\sum_{j=0}^{\infty}\binom{-2}{j}\left(\sqrt{\phi(j,x)}\right)^{-\frac{3}{2}}\left(1+\frac{1}{\delta\gamma\sqrt{\phi(j,x)}}\right)\left[\frac{\pi}{2\delta\gamma\sqrt{\phi(j,x)}}\right]^{\frac{1}{2}}e^{-\delta\gamma\sqrt{\phi(j,x)}}}$$

$$= \frac{\delta}{\gamma}\frac{\sum_{j=0}^{\infty}\binom{-2}{j}\left(\sqrt{\phi(j,x)}\right)^{-1}e^{-\delta\gamma\sqrt{\phi(j,x)}}}{\sum_{j=0}^{\infty}\binom{-2}{j}\left(\sqrt{\phi(j,x)}\right)^{-2}\left(1+\frac{1}{\delta\gamma\sqrt{\phi(j,x)}}\right)e^{-\delta\gamma\sqrt{\phi(j,x)}}} \tag{6.38}$$

Next,

$$E(\frac{1}{Z}|x) = \frac{\int_0^{\infty}\frac{1}{z}\cdot\frac{1}{z}e^{-\frac{(x-\mu)}{z}}\left[1+e^{\frac{-(x-\mu)}{z}}\right]^{-2}g(z)dz}{\int_0^{\infty}\frac{1}{z}\cdot e^{-\frac{(x-\mu)}{z}}\left[1+e^{\frac{-(x-\mu)}{z}}\right]^{-2}g(z)dz}$$

$$
\begin{aligned}
&=\frac{\frac{1}{2}\int\limits_0^\infty z^{-2}\sum\limits_{j=0}^{\infty}\binom{-2}{j}z^{-\frac{3}{2}}e^{-\frac{\gamma^2}{z}\left(z+\frac{\delta^2\phi(j,x)}{\gamma^2 z}\right)}dz}{\frac{1}{2}\int\limits_0^\infty z^{-1}\sum\limits_{j=0}^{\infty}\binom{-2}{j}z^{-\frac{3}{2}}e^{-\frac{\gamma^2}{z}\left(z+\frac{\delta^2\phi(j,x)}{\gamma^2 z}\right)}dz}\\
E(\frac{1}{Z}|x)&=\frac{\sum\limits_{j=0}^{\infty}\left\{\binom{-2}{j}\frac{1}{2}\int\limits_0^\infty z^{-\frac{5}{2}-1}e^{-\frac{\gamma^2}{z}\left(z+\frac{\delta^2\phi(j,x)}{\gamma^2 z}\right)}dz\right\}}{\sum\limits_{j=0}^{\infty}\left\{\binom{-2}{j}\frac{1}{2}\int\limits_0^\infty z^{-\frac{3}{2}-1}e^{-\frac{\gamma^2}{z}\left(z+\frac{\delta^2\phi(j,x)}{\gamma^2 z}\right)}dz\right\}}\\
&=\frac{\sum\limits_{j=0}^{\infty}\left\{\binom{-2}{j}\left(\frac{\delta}{\gamma}\sqrt{\phi(j,x)}\right)^{-\frac{5}{2}}K_{-\frac{5}{2}}\left(\delta\gamma\sqrt{\phi(j,x)}\right)\right\}}{\sum\limits_{j=0}^{\infty}\left\{\binom{-2}{j}\left(\frac{\delta}{\gamma}\sqrt{\phi(j,x)}\right)^{-\frac{3}{2}}K_{-\frac{3}{2}}\left(\delta\gamma\sqrt{\phi(j,x)}\right)\right\}}\\
&=\frac{(\frac{\delta}{\gamma})^{-\frac{5}{2}}}{(\frac{\delta}{\gamma})^{-\frac{3}{2}}}\frac{\sum\limits_{j=0}^{\infty}\binom{-2}{j}\left(\sqrt{\phi(j,x)}\right)^{-\frac{5}{2}}K_{-\frac{5}{2}}\left(\delta\gamma\sqrt{\phi(j,x)}\right)}{\sum\limits_{j=0}^{\infty}\binom{-2}{j}\left(\sqrt{\phi(j,x)}\right)^{-\frac{3}{2}}K_{-\frac{3}{2}}\left(\delta\gamma\sqrt{\phi(j,x)}\right)}\\
&=\left(\frac{\gamma}{\delta}\right)^{-1}\frac{\sum\limits_{j=0}^{\infty}\binom{-2}{j}\left(\sqrt{\phi(j,x)}\right)^{-\frac{5}{2}}K_{-\frac{5}{2}}\left(\delta\gamma\sqrt{\phi(j,x)}\right)}{\sum\limits_{j=0}^{\infty}\binom{-2}{j}\left(\sqrt{\phi(j,x)}\right)^{-\frac{3}{2}}K_{-\frac{3}{2}}\left(\delta\gamma\sqrt{\phi(j,x)}\right)}
\end{aligned}
\tag{6.39}
$$

6.4 Summary

The major contribution of this charpter is the introduction of the "Logistic Inverse Gaussian distribution" which has been obtained as a special case of the generalized inverse Gaussian (GIG).

The likelihood function and the posterior distribution based on LIG have been obtained. The EM algorithm has been calculated but not applied to data.

7 SUMMARY CONCLUSION AND RECOMMENDATIONS

This charpter presents the accomplishments of the thesis, summary of the results and the general recommendations for further areas of investigation following the study. A distributional framework of generalizations of the logistic distribution has been shown on the title page. The framework represents a summary of the entire work in a pictorial depiction.

The appendix shows the results of credit scoring applied in chapter two in logistic regression. The results were obtained using R codes.

7.1 Summary of Results

The objectives of the study stated in 1.4 have been achieved as follows.

The standard logistic distribution has been constructed explicitly using the method of mixtures, Burr differential equation, transformation and the difference of two independent random variables. The Transformation method required appropriate choice of transformations. Moments were obtained directly and by the mgf technique.

The cdf of the standard logistic was applied to data collected from a mobile phone money lending company (Mobipesa Ltd) in Nairobi- Kenya. Logistic regression was applied in determining the probability of default. Credit scores were then computed for all the 496 customers whose data was collected. The data was divided into train and test data. The probabilities of default obtained have helped to distinguish between good and bad customers by the lender. The results obtained are plausible because they indicated that the probability of default and the credit score are inversely related.

Various generalizations of the standard logistic distribution were obtained using the various methods of constructing the standard logistic. A clear pattern of construction was developed for constructing the generalized distributions. Generalized logistic distribution types GLI,$GLII$,$GLIII$ and $GLIV$. Their extended versions were also obtained.

The generalized distributions were also obtained using the beta I and beta II distributions through transformations. The beta generated approach has also been used to generalize the standard logistic distributions.

A new distribution " The Extended Standard logistic" has been introduced and constructed using five different methods.

Special cases were obtained for the generalized logistic distribution type IV and the extended $GLIV$.All the generalized logistic distributions $GL1$,$GLII$,$GLIII$ and $GLIV$ were obtained as special cases from the extended $GLIV$.The extended standard logistic and the logistic distribution were also obtained from the extended $GLIV$ as special cases.

The discrete mixture distributions of minimum and maximum order statistics from the standard logistic distribution have been constructed.

Construction of $\mathrm{pdf}'s$, survival functions and hazard functions of the discrete mixture distributions has also been done for minimum and maximum order statistics. The mixing distributions used are the zero truncated power series distributions namely; zero truncated Poisson, zero truncated binomial, zero truncated negative binomial, zero truncated (shifted) geometric and the logarithmic series distributions.

The logistic distribution with the shape and location parameter was considered. The mixed distributions obtained were based on modified Bessel functions of the third kind. The mixing distributions considered were the logistic, half logistic, exponential, The inverse Gaussian,inverse Gamma, I and Gamma II.

A new distribution the logistic inverse Gaussian has been introduced and studied in detail. Its properties like the r^{th} moments, log-likelihood and EM algorithm have been obtained.

7.2 Recommendations

For further research we recommend studies in the following areas that have not been covered;

i. A study of finite mixtures is not covered in the current study, however, its is captured in the distributional framework as part of the work. A study of finite mixtures of the logistic is therefore recommended.

ii. The study recommends the use of Tricomi confluent hypergeometric functions, however, confluent hypergeometric functions (Kummers and Tricomi) were listed among the tools to be used in the current study.

iii. A detailed study on the properties of the generalized logistic distributions. This study has mostly concentrated on the various methods in which they can be constructed. Applications to real time data also needs to be investigated . This study investigated the application of the standard logistic distribution to credit scoring.

iv. Properties of the new distribution" the extended standard logistic distribution" need to be investigated. The study has mostly concentrated on the properties of the logistic inverse Gaussian distribution,a new distribution proposed.Further work can be done on properties, estimations and fitting the generalized logistic distributions to data.

v. The study further recommends further use of data to investigate the discrete mixtures of minimum and maximum order statistics from a standard logistic distribution.

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APPENDIX

Appendix I- R-codes

```
# Load packages for modelling:
rm(list = ls())
library(lubridate)
library(tidyverse)
library(caret)
library(scorecard)
library(corrplot)
library(broom)
library(pROC)


#==================================
#   Stage 1: Loading Data
#==================================

# Import data and create default_flag by definition described in section 2.2.1:
library(readxl)
data <- read_excel("C:data.xlsx")
View(data)

# Split data:
set.seed(2122)
id <- createDataPartition(y = data$def, p = 0.7, list = FALSE)
train <- data[id, ]
test <- data[-id, ]

#===================================
#  Stage 2: Univariate Analysis
#===================================

# WOE binning:


bins <- woebin(train, y = "def",  positive = "def|1")

```


```
#===================================
#  Stage 2: Univariate Analysis
#===================================

# WOE binning:


bins <- woebin(train, y = "def",  positive = "def|1")
```


```
do.call("rbind", bins) %>%
  filter(!duplicated(variable)) %>%
  filter(total_iv >= 0.1) %>%
  pull(variable) -> var_IV_01

# WOE transformation:

train_woe <- woebin_ply(train, bins = bins) %>%
  as.data.frame()

def <- train_woe$def

# Correlations:

my_corr <- cor(train_woe)

my_corr %>%
  corrplot(., method = "number", tl.cex = 0.7)

#============================================
#  Stage 3: Logistic Stepwise Regression
#============================================
logit_stepwise <- glm(def ~., family = "binomial", data = train_woe) %>%
  step(trace = 0)

```

```r
#============================================
#  Stage 3: Logistic Stepwise Regression
#============================================
logit_stepwise <- glm(def ~., family = "binomial", data = train_woe) %>%
  step(trace = 0)


tidy(logit_stepwise) %>%
  mutate_if(is.numeric, function(x) {round(x, 3)}) %>%
  knitr::kable()

#=======================================
#     Stage 4: Score for test data
#=======================================

test_woe <- woebin_ply(test, bins = bins) %>%
  as.data.frame()

pd_test <- predict(logit_stepwise, test_woe, type = "response") %>% as.vector()


# Function for scoring based on PD predicted:

scaled_score <- function(pd_selected) {

  odds <- 72
  my_offset <- 500
  pdo <- 20
  b <- pdo / log(2)
  a <- my_offset - b*log(odds)

  scores <- a + b*log((1 - pd_selected) / pd_selected)
  return(round(scores, 0))

}
```

```r
#============================================
#  Stage 5: Evaluate Discriminatory Power
#============================================

df_scored_test <- test %>%
  mutate(SCORE = scores) %>%
  mutate(def = case_when(def == 1 ~ "Default", TRUE ~ "NonDefault"))

df_scored_test %>%
  group_by(def) %>%
  summarise_each(funs(min, max, median, mean, n()), SCORE) %>%
  mutate_if(is.numeric, function(x) {round(x, 0)}) %>%
  knitr::kable(caption = "Table 1: Scorecad Points by Group for Test Data (Stepwise Logistic)")

df_scored_test %>%
  group_by(def) %>%
  summarise(tb = mean(SCORE)) %>%
  ungroup() -> mean_score_test


theme_set(theme_minimal())

g1 <- df_scored_test %>%
  ggplot(aes(SCORE, color = def, fill = def)) +
  geom_density(alpha = 0.3) +
  geom_vline(aes(xintercept = mean_score_test$tb[1]), linetype = "dashed", color = "red") +
  geom_vline(aes(xintercept = mean_score_test$tb[2]), linetype = "dashed", color = "blue") +
  geom_text(aes(x = 390, y = 0.008, label = mean_score_test$tb[1] %>% round(0)), color = "red", size = 4) +
  geom_text(aes(x = 545, y = 0.008, label = mean_score_test$tb[2] %>% round(0)), color = "blue", size = 4) +
  theme(legend.title = element_blank()) +
  theme(legend.position = c(0.2, 0.8)) +
  theme(panel.grid = element_blank()) +
  theme(axis.text.y = element_blank()) +
  theme(plot.margin = unit(c(1.3, 1.3, 1.3, 1.3), "cm")) +
  labs(x = "Scorecard Point", y = NULL,
       title = "Figure 1: Scorecard Point Distribution by Group for Test Data, Stepwise Logistic Model",
```

```r

# Use this function:
my_auc_logit <- auc_for_test(pd_test)

# Function for presenting AUC/ROC curve:

my_ROC_curve <- function(auc_object) {

  sen_spec_df <- data_frame(TPR = auc_object$sensitivities,
                            FPR = 1 - auc_object$specificities)

  sen_spec_df %>%
    ggplot(aes(x = FPR, ymin = 0, ymax = TPR))+
    geom_polygon(aes(y = TPR), fill = "red", alpha = 0.3)+
    geom_path(aes(y = TPR), col = "firebrick", size = 1.2) +
    geom_abline(intercept = 0, slope = 1, color = "gray37", size = 1, linetype = "dashed") +
    scale_y_continuous(labels = scales::percent) +
    scale_x_continuous(labels = scales::percent) +
    theme_bw() +
    coord_equal() %>%
    return()
}


# ROC curve for Stepwise Logistic:

my_auc_logit %>%
  my_ROC_curve() +
  labs(x = "FPR (1 - Specificity)",
       y = "TPR (Sensitivity)",
       title = "Figure 2: Model Performance Based on Test Data",
       subtitle = paste0("AUC Value for Stepwise Logistic: ", my_auc_logit$auc %>% round(3))) -> g2

g2
```

Appendix II- Data sets

Full data set;

click here

Table of probability of default and credit scores for all the customers

click here