UNIVERSITY OF NAIROBI

FACULTY OF SCIENCE AND TECHNOLOGY

MASTERS PROJECT IN PURE MATHEMATICS

FUNDAMENTAL PROPERTIES OF N-POWER OPERATORS AND THEIR CHARACTERIZATION

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A project submitted in partial fulfilment of the requirement for the award of degree of Master of Science in Pure Mathematics

DECLARATION

The project is my original work and has never been submitted anywhere for purposes of any award of a degree in any university or other institutions of learning.

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Dedication

I dedicate this project to my family and in particular my wife Anne and My mother Irene without forgetting my children Errol, Elbethel and Erwin.

Abstract

The project will study class of n-power operators and their properties. The study will focus n-power normal operators, n-power hyponormal operators,n-power posinormal operators,n-power quasi-normal operators and n-power quasi-isometry operators. We shall look at their basic properties as well as their spectral and numerical properties.

Contents

1 Chapter One:Introduction

1.1 Literature Review

Operator theory including its subtopics such as spectral theory came into focus after 1900.Around this time, Fredholm's report on the theory of integral equations was published. He gave a complete analysis of integral equations which he referred to as Fredholm's equations that extended results from linear algebra to a class of operators. He also defined the determinant to a class of operators and was the first to use the term resolvent operator.

1902, Lebesque introduced an important category of spaces known as L^P. Around the same time, Hilbert founded spectral theory as a result of a series of articles by Fredholm. Now, the word "spectrum" was adopted by Hilbert in 1897 from an article by Wilhelm Wirtinger. Hilbert used the notion of integral equations and found results underL^2spaces,square-integrable functions and discussed also some results for the scenario where integral operator was symmetric. In 1906, Hilbert discovered continuous spectrum,a work away from integral equations.

Around 1913, Frigyes Riez,introduced the concept of algebra of operators where he studied of bounded operators on the Hilbert space L^2.In his work,he introduced other concepts like Riez representation theorem, orthogonal projectors and spectral integrals.

In 1916, Riez found the theory of completely continuous operators now referred to as compact operators. He further extended Fredholm's work on the spectral theorem of compact operators. Further developments came in between 1929-32 when spectral theorem of self-adjoint and normal operators were discovered by Marshall Stone and John Von Neumann.
Neumann also introduced concepts that are widely used in operator theory like closure of an operator, adjoint operators, unbounded operators and extension of operators. In 1932, Stefan Banach published a first text on operator theory which included the closed-graph theorem, Weak convergence and the fixed point theorem.

Israil Gel'fand in 1941 extended the spectral theorem to elements of a normed algebra and introduced the spectral radius formula as well as C- algebra and the character of an algebra.
Since Gel'fand's time, operator theory has been an enormous branch of Mathnematics. Many authors have defined new classes of operators and new interesting results have been captured. Patel and Ramanujan (1981) introduced and studied normal operators. An operator T is called normal if $TT = TT$

Adnan Jibril(2008) extended the notion of normal operators to n-power normal operators and showed that an operator T B(H) is n-power normal operator if and only if T^n is normal. An operator T is n-power normal if T commutes with T^n, that is $T^nT = TT^n$.

A. brown in 1953 introduced the concept of quasi-normal operator. An operator T is quasi-normal if T commutes with TT. Sid Ahmed(2011) studied the concept of n-power quasi-normal operators as an extention quasi-normal operators. An operator TB(H) is n-power quasi-normal if T^n commutes with TT.

Panayappan and Sivaman(2012) coined the concept of n-binormal operators.P.R Halmos and Stampfli introduced hyponormal operators in 1962. T is called hyponormal if $T^{T \geq TT}$

Guesba et al. extended the concept of hyponormal operators to n-power hyponormal operators in 2016. An operator T is called n-power-hyponormal if $T^{T^n \geq T^nT}$
.
In this project, we shall study the fundamental properties of n-power operators. For each class of n-power operator, we shall study their subclasses and spectral and numerical range properties where applicable.

1.2 Notation and Definitions

Notations
Hilbert spaces will be denoted by H and $B(H)$ will denote the algebra of bounded operators in the Hilbert space H. T and S will denote operators and I will denote the identity operator on the Hilbert space.

The spectrum, the point spectrum and the residual spectrum of an operator T will be denoted by $\sigma(T), \sigma_a(T)$ and $\sigma_p(T)$ respectively.

The residual spectrum of T is denoted by $\sigma_r(T)$, Ker(T) is used as the kernel of T and λ as the eigenvalue of T.

The range of T will be denoted $R(T)$ while the nullity of T will be denoted by $N(T)$

Definitions
Definition
Let X be a vector space over a field of complex numbers.A norm on X is a mapping $\|.\| : X \to \mathbb{C}$ such that it satisfies the following axioms

1) $\|ax\| = |c|\|x\| c \in \mathbb{C}, x \in X$

2) $\|x\| \geq 0$

3)$\|x\| = 0$ if and only if $x = 0$ for all $x \in X$

4) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

Definition

Let T $inB(H)$,then there exists an operator T^* known as the adjoint of T such that $< x, Ty >>=< T^*x, y >$ for all $x, y \in H$

An operator T$\in$ B(H) is said to be

Self-adjoint if $T = T^*$

Normal if $T^*T = TT^*$

Isometry if $T^*T = I$

Quasinormal if $T(T^*T) = (T^*T)T$
Hyponormal if $T^*T \geq TT^*$

Subnormal if it has a normal extension

Binormal if T^*T and TT^* commute

Unitary if $T^*T = TT^* = I$

Partial isometry if $T = TT^*T$

Seminormal if either T or T is hyponormal

Involution if $T^2 = I$

Idempotent if $T^2 = T$

Co-isometry if $TT^* = I$

Nilpotent if $T^n = 0$ for some n

Quasi-nilpotent if $\sigma(T) = \{0\}$

Definition

An operator T is called a normaloid if and only if the spectral radius is equal to its operator norm. i.e $r(T) = \|T\|$

Definition

An operator T is called posinormal if it has a positive interrupter or in other words if there exists a positive operator P such that the self commutator $[T^*, T]$ of $[T^*, T] = T^*(I - P)T$

Definition
An operator L_k is called an integral operator with the kernel k is defined by

$L_k f(x) = \int k(x,y)f(y)dy$

Definition
An operator T is called a quasi-isometry if $T^{*2}T^2 = T^*T$

Definition
An operator P is called an interrupter for T if $TT^* = T^*PT$

Definition
The commutator of two operators K and L which is denoted by [K, L] can defined as [K, L] = KL - LK

Definition
Two bounded linear operators K and L in the Hilbert space H are said to be unitarily equivalent if there exists a unitary operator A ∈ G(K, L) such that $AK = LA$

Definition
The set of all $\lambda \in$ C such that $\lambda I - T$ is not invertible is called the spectrum of T and $\lambda \in C$ is an eigenvalue of the operator T if there exists $x \in H$
Definition
For $\lambda \in \sigma(T)$ such that $\lambda I - T$ is not bounded from below is known as the point spectrum of T.

Definition
For $\lambda \in \sigma(T)$ such that $\lambda I - T$ is one-to- one but unbounded from below is called the approximate spectrum of T.

Definition
For $\lambda \in \sigma(T)$ such that $\lambda I - T$ has no dense range is known as the residual spectrum of T.

Furthermore, An operator T ∈ B(H) is called:

n-power normal operator if $T^nT^* = T^*T^n$.

n- power hyponormal if $T^*T^n \geq T^nT^*$.

n-power Quasinormal if $T^nT^*T = T^*TT^n$.

n-power posinormal if $R(T^n) \subset R(T^)$.

n-power quasi-isometry if $T^{n-1}T^{*2}T^2 = T^*TT^{n-1}$ for some integer n.

Definition
T∈B(H) is called an isoloid if every isolated point of the σ(T) is a member of the point spectrum of T.
Definition
$\lambda \in$ C is an eigenvalue of T if $ker(T\lambda I) \neq 0$

Definition
The Numerical range $W(T)$ of an operator T is a subset of the complex number C given by: $W(T) = \{< Tx, x >, x \in H, \|x\| = 1$ with the property that $W(\alpha I + \beta T) = \alpha + \beta W(T)$for all $\alpha, \beta \in C$

Definition
The numerical radius denoted by $r(T)$ of an operator T on the Hilbert space H is given by $r(T) = sup\{|\lambda| : \lambda \in W(T)\}$ i.e sup $\{| < Tx, x > |, \|x\| = 1\}$

Definition
A set X is called convex if for every two points $x, y \in X$ we have $tx+(1-t)y \in X$ for all $t \in [0, 1]$.

Defition
The convex hull of an operator T is the smallest convex set containing T.

2 Chapter Two

2.1 n-Power normal operators

Before we go to n-power normal operators, we start with a discussion of normal operators

Normal operators

Definition

An operator T on a Hilbert space H is said to be normal if it commutes with its adjoint.i.e $TT^* = T^*T$

These operators include self-adjoint operators, non-negative operators such as orthogonal projections skew-adjoint, Unitary operators and normal matrices

The following inclusion hold.

Normal $\subset$ Quasinormal $\subset$ Subnormal $\subset$ Hyponormal $\subset$ Paranormal $\subset$ Normaloid.

Basic properties of normal operators

Theorem

Let T=$[\begin{array}{cc} a & b \\ c & d \end{array}]$ where a, b, c, d $\in C$ be a normal operator. Then T is said to be a normal operator if $|c| = |b|$ and also if $a = d$ or $c = b$

Proof

Suppose T = $[\begin{array}{cc} a & b \\ c & d \end{array}]$. Then $T^* = [\begin{array}{cc} a & c \\ b & d \end{array}]$.

Therefore

$$TT^* = [\begin{array}{cc} a & b \\ c & d \end{array}][\begin{array}{cc} a & c \\ b & d \end{array}]$$

$$=[\begin{array}{cc} a^2 + b^2 & ac + bd \\ ca + db & c^2 + d^2 \end{array}]$$

$$T^*T = [\begin{array}{cc} a & c \\ b & d \end{array}][\begin{array}{cc} a & b \\ c & d \end{array}]$$

$$=[\begin{array}{cc} a^2 + c^2 & ab + cd \\ ba + dc & b^2 + d^2 \end{array}]$$

Thus, $TT^* = T^*T$ implies that,

$$a^2 + c^2 = a^2 + b^2$$

$\Rightarrow c^2 = b^2$ and hence $|c| = |b|$

Also note that

ac + bd = ab + cd

$\Rightarrow$ ac- ab = cd-bd

$\Rightarrow$a(c-b)= d(c-b)

$\Rightarrow$a(c-b)-d(c-b)=0

$\Rightarrow$ a-d = 0 or c-b =0

i.e a=d or c=b

This completes the proof

Theorem

Let $T \in B(H)$ be a normal operator and if $S \in B(H)$ unitarily equivalent to T, then S is a normal operator

Proof

$S^*S = (U^*T^*U)(U^*TU)$

$= U^*T^*TU$

$=U^*TT^*U$

$=SU^*T^*U$

$=SU^*US^*$

$=SS^*$

This completes the prove.

Remark

The sum of two normal operators is not in general normal as shown in the example below.

Example.

Let A and B be matrices defined as

A = $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. Then A and B are normal operators .However,

A + B = $\begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}$ is not normal.

Proposition

Let T $\in B(H)$ and S $\in B(H)$ be two normal operators. Then, if T commutes with S^* then T + S is normal

Proof

We show that T + S is a normal operator

Now, $(T+S)(T+S)^* = (T+S)(T^*+S^*)$

$= TT^* + TS^* + ST^* + SS^*$

$=T^*T + S^*T + T^*S + S^*S$

Since T and S are normal operators, we have that

$=(T^* + S^*) + (T + S)$

$=(T + S)^*(T + S)$
This implies that T+S is a normal operator

Remark
The product of two commuting normal operators T and S is again normal if T commutes with S^* and S commutes with T^*.
Proof
Let us consider
$(TS)(TS)^* = (TS)(T^*S^*)$

$=TSS^*T^*$

$=TS^*ST^*$

S^*TT^*S

$=S^*T^*TS$

$=(TS)^*(TS)$
This implies that TS is a normal operator
Theorem
Let T $\in B(H)$. Then T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$
Proof
Now, for all $x \in H$ we have that
$(T^*Tx, x) - (TT^*x, x) = (Tx, Tx) - (T^*x, T^*x) = \|Tx\|^2 - \|T^*x\|^2$.
Also, if $T_1, T_2 \in B(H)$ then we have that $(T_1x, x) = (T_2x, x)$ if and only if $T_1 = T_2$.
Hence $\|Tx\| = \|T^*x\|$.
Corollary
Let $T \in B(H)$. If T is a normal operator, then $\sigma(T) \subset \{\overline{(Tx, x)|\|x\| = 1}\}$
Proof
Now, suppose that $\lambda \in \sigma(T)$, then there is a sequence (x_n) of unit vectors such that

$\|Tx_n - \lambda x_n\| \to 0$

Therefore, this implies that

$(Tx_n - \lambda x_n, x_n) \to 0$

$(Tx_n, x_n) \to \lambda$

Theorem
If T is a normal operator, then T and T has the same kernel and range i.e

$Ker(T) = ker(T^{)}$ and $R(T) = R(T^{)}$.

Theorem

The range of a normal operator T is dense if and only if T is injective.This implies that the kernel of T is the orthogonal complement of its range

Theorem(Putnam-Fuglede Theorem)

The Putnam-Fuglede theorem states that if A and B are normal operators and if X is an operator such that AX = XB. Then $A^*X = XB^*$

n-power normal operator

Definition

An operator T on a complex Hilbert space H is said to be an n-power normal operator if $T^nT^* = T^*T^n$ for any positive integer n. T $\in B(H)$ is called an n-power normal operator if T^n commutes with T^*

Remark

1- power normal operators are normal operator

2-power normal operators

Definition

An operator T is said to be 2-power normal if $T^2T^* = T^*T^2$. We denote the class of 2-power normal operators by [2N].

Basic properties of 2-power normal operators

An operator T = $\left(\begin{array}{cc} a & b \\ c & d \end{array} \right)$ where a, b,c,d $\in$ C is said to be 2-power normal if and only if (a + d) = 0 and $|b| = |c|$ or b(d a) = c(d a).

Proof

$T^2 = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right)\left(\begin{array}{cc} a & b \\ c & d \end{array} \right)$

$=\left[\begin{array}{cc} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{array} \right]$

$T^* = \left[\begin{array}{cc} a & c \\ b & d \end{array} \right]$

$T^2T^* = \left[\begin{array}{cc} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{array} \right]\left[\begin{array}{cc} a & c \\ b & d \end{array} \right]$

$=\left[\begin{array}{cc} a^3+abc+ab^2+bd^2 & a^2c+bc^2+abd+bd^2 \\ a^2c+acd+b^2c+bd^2 & ac^2+dc^2+b^2c+d^3 \end{array} \right]$

$T^*T^2 = \left[\begin{array}{cc} a & c \\ b & d \end{array} \right]\left[\begin{array}{cc} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{array} \right]$

$\left[\begin{array}{cc} a^3+abc+ac^2+dc^2 & a^2b+abd+bc^2+cd^2 \\ ba^2+b^2c+acd+d^2c & ab^2+b^2d+bcd+d^3 \end{array} \right]$

Then $T^2T^* = T^*T^2$ implies that

$$a^3+abc+ab^2+b^2d = a^3+abc+ac^2+dc^2$$

$\Rightarrow ab^2 + b^2d = ac^2 + dc^2$

$=b^2(a+d) = c^2(a+d)$
$\Rightarrow b^2(a+d) - c^2(a+d) = 0$

$=(b^2 - c^2)(a+d) = 0$

$b^2 - c^2 = 0$ or a+d = 0

$\Rightarrow |b| = |c|$ or a+d = 0

Remark
The sum of two commuting 2-power normal operators need not be 2-power normal.
Example
Let $S = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. S and T are two commuting 2-power normal. But $S+T = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$, $(S+T)^2 = \begin{pmatrix} 4 & 2 \\ 0 & 0 \end{pmatrix}$ is not normal.
Thus S +T is not 2-power normal.
Lemma
Let S, T $\in$ B(H) be 2-power normal operators such that ST = TS = 0 then T + S is n-power normal
Proof
Now, since ST + TS = 0 , $S^2T^2 = T^2S^2$. Therefore,
$(S+T)^2 = S^2 + T^2$ is normal. This implies that (S + T) is an 2-power normal operator.

Theorem
Let S, T $\in$ B(H) be 2-power normal operators and ST + TS = 0 then TS is n-power normal
Proof
since ST + TS = 0, we have $(ST)^2 = -S^2T^2 = -T^2S^2$. This shows that ST is a 2-power normal operator.
Remark
2-power normal operators may not necessarily have a translation-invariant as shown in the following Example
Let T=$\begin{pmatrix} 0 & T_1 \\ 0 & 0 \end{pmatrix}$ where $T_1 : H_1 \rightarrow H$. Then the operator T is a 2-power normal operator but $[(T-\lambda)^{*2}, (T-\lambda)^2] = \begin{pmatrix} -4|\lambda|^2T_1T^1 & 0 \\ 0 & 4|\lambda|^2T_1^*T_1 \end{pmatrix}$ not necessarily equal to 0 unless $\lambda = 0$. This implies that $(T-\lambda)^2$ is not normal and therefore $(T-\lambda)$ is not necessarily a 2-power normal operator

Remark
A 2-power normal operator is 2k-power normal for any integer k.

3-power normal operators
An operator T is said to be 3-normal if $T^3T^* = T^*T^3$
Proposition
Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d $\in C$. Then T is 3-normal if and only if $(a^2 + bc + ad + d^2) = 0$ and ($|a| = |c|$ or $\overline{C}(d - a) = b(\overline{d} - \overline{a})$)
Example
Consider $T = \begin{pmatrix} 2 & 1 \\ 0 & -1 + \sqrt{3i} \end{pmatrix}$. T is 3-power normal since $T^3 = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$ is normal. Hence T is 3-power normal
Remark
A 3-power normal operator is 3k-normal for any integer K
Remark
Example
Let $T = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}$. Then $T^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ is a normal operator. But $T^3 = \begin{pmatrix} 8 & 4 \\ 0 & -8 \end{pmatrix}$ is not normal. So T is 2-power normal but is not 3-normal
Example
Suppose $T = \begin{pmatrix} 2 & 2 \\ -2 & 0 \end{pmatrix}$. Then $T^3 = \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$ is a normal operator. $T^2 = \begin{pmatrix} 0 & 4 \\ -4 & -4 \end{pmatrix}$ is not a normal operator. This implies that T is a 3-normal operator and not a 2-power normal operator.

General properties of n-power normal operators

Proposition
An operator T $\in$ B(H) is n-normal if and only if T^n is normal for some integer n.
Proof
If T is n-power normal, then this implies that $T^nT^* = T^*T^n$.
This shows that $T^n(T^*)^n = T^*T^n(T^*)^{n-1}$=$T^*(T^nT^*)(T^*)^{n-1}$=$T^*T^nT^*)(T^*)^{n-2}$

=$(T^*)^nT^n$.
Hence T is n-power normal.
Conversely, let T^n be normal, then $T^nT = TT^n$. This implies that by Fuglede theorem that $(T^n)^*T = T(T^n)^*$ that is $T^*T^n = T^nT^*$. Hence T is n-normal.
Corollary
n-power normal operators are closed under scalar multiplication and unitary equivalence.
Proof

Suppose T is an n-power normal operator and α is a scalar, then $(\alpha T)^n(\alpha T)^* = \alpha^n\overline{\alpha}(T^nT^*) = \overline{\alpha}\alpha^n(T^*T^n)$ and $(\overline{\alpha}T^*)(\alpha^nT^n = (\alpha T)^*(\alpha T)^n$. This implies that αT is n-normal

Theorem

Suppose T is k-power normal and (k+1)-power normal. If T or T^* is injective, then T is a normal operator.

Proof

T is (k+1)-normal implies that $T^{k+1}T^* = T^*T^{k+1}$. Also, T is k-normal implies $T^kT^* = T^*T^k$. Therefore, $T^k(TT^* - T^*T)$= 0. Now, since T is injective, this implies that $TT^* - T^*T$= 0. Hence T is a normal operator. Also, T^* is injective and since T^* is k-power normal and (k+1)-power normal, this implies T^* is normal and therefore, T is a normal operator.

Corollary

Let T $\in B(H)$ be n-power normal. Then T^m is n-power normal for any positive integer m.

Remarks

A bounded normal operator is n-normal for any integer n.

2. All non zero nilpotent operators are n-normal for any n.

Proposition

Suppose T is a bounded linear operator which is n-normal. Then

i) T^* is n-normal

ii) If T^{-1} exists then (T^{-1}) is n-power normal

iii)If S$\in$ B(H) is unitarily equivalent to T, then S is n-normal

iv) Suppose M is a closed subspace of H such that it reduces T, then S = T/M is an n-power normal operator

Proof

i) Let T be n-power normal, then T^n is normal. This implies that $(T^*)n = (T^n)^*$ is a normal operator. Hence T^* is n-normal.

ii) Let T be n-power normal, then T^n is normal. Now, $(T^n)^{-1} = (T^{-1})^n$, T^{-1} is n-power normal.

iii)Suppose T is n-power normal and S is unitarily equivalent to T. This implies that there exists a unitary operator U such that $S = UTU^*$. T^n normal implies S^n is normal. Hence S is n-power normal.

iv) T n-normal implies T^n is normal. Therefore, T^n/M is a normal operator. Since M is a closed subspace of H and reduces T implies M is invariant under T and $T^n/M = (T/M)^n$. This shows that $(T/M)^n$ is normal and hence T/M is an n-power normal operator.

Theorem

If S, T are n-power normal operators which commute, then ST is an n-power

normal operator
Proof
Since S, T are commuting n-power normal operators,
S^n, T^n are commuting normal operators. So S^nT^n is a normal operator.
Since $S^nT^n = (ST)^n, (ST)^n$ is normal. ST is n-normal.
Two commuting n-normal operation their sum not necessarily n-normal.
Corollary
If T is an n-power normal operator, then T^k is an n-normal operator for any integer k.
Example.

Corollary
If T is a 2-power normal operator and also partial isometry, then T is n-power normal for all integers n∈ B(H)

Theorem
Let $T \in B(H)$ be an n-power normal operator. If T is quasinilpotent,then T is nilpotent and hence T is a subscalar.
Proof
Now, since T is quasinilpotent, $\sigma(T) = \{0\}$. By spectral mapping theroem,we have that $\sigma(T^n) = \sigma(T)^0 = \{0\}$. This implies that T^n is quasinilpotent and normal. Therefore, $T^n = 0$that is T is nilpotent and T is algebraic operator and hence T is a subscalar.

2.2 n-power hyponormal operators

We now go to n-power hyponormal operators. Before we look at this new class of operators, we gather some basic facts about hyponormal operators which forms the foundation of n-power hyponormal operators.

Hyponormal operators

Definition
An operator T $\in$ B(H) is called hyponormal if $T^*T \geq TT^*$ or the self commutator of T is a positive operator i.e $[T^*, T] = T^*T - TT^* \geq 0$. Hyponormal operators include Normal, quasi-normal, subnormal and integral operators.
Examples of hyponormal operators
Let T = $\left[\begin{array}{cc} 2 & 1 \\ 1 & 0 \end{array} \right]$ is a hyponormal operator since it is a normal operator
Remark

An operator T $\in B(H)$ is called cohyponormal if T^* is hyponormal i.e $T^*T - TT^* \leq 0$
Theorem
Let T $\in B(H)$. Then T is hyponormal if and only if $\|Tx\| \geq \|T^*x\|$ for every $x \in H$.
Proof
Suppose $T \in B(H)$. Then $T^*T \geq TT^*$ if and only if $< T^*Tx, x > \geq < TT^*x, x >$. This means that $\|Tx\|^2 \geq \|T^*x\|^2$ for every $x \in H$ and hence $\|Tx\| \geq \|T^*x\|$.

Theorem
Let T be a bounded operator on H. If T is hyponormal, then $|T^{k+1}| = |T|^{k+1}$
Proof
If T is hyponormal, then

$$\|T^2x\| = (Tx, Tx) = (T^*Tx, x)$$

$$\leq \|T^*Tx\|\|x|$$

$$\leq |TTx||x|$$

$$=\|T^2x\|\|x\|$$
If $\|x\| = 1$, implies that $\|Tx\|^2 \leq \|T^2x\|$ and hence $sup\{\|Tx\|^2\} \leq sup\{\|T^2x\|\}_{\|x\|=1}$
$\Rightarrow \|T\|^2 \leq \|T^2\|$
Now if we suppose that the above result holds for n= 1,2,3...., k, then we have that,
$\|T^kx\|^2 = (T^kx, T^kx) = (T^*T^kx, T^{k-1}x)$

$$\leq \|T^*T^kx\|\|T^{k-1}x\|$$

$\leq \|T^{k+1}x\|\|T^{k-1}x\|$(T is hyponormal)

$\leq \|T^{k+1}\|\|x\|\|T^{k-1}\|\|x\|$

$=\|T^{k+1}\|\|T^{k-1}\|\|x\|^2$

If $\|x\| = 1$ then we have
$\|T^k x\|^2 \leq \|T^{k+1}\|\|T^{k-1}\|$ and hence we have, $sup\|T^k x\|^2 \leq \|T^{k+1}\|\|T^{k-1}\|$ for $\|x\| = 1$.
By inductive hypothesis we have that,

$\|T^k\|^2 = [\|T\|]^2 \leq \|T^{k+1}\|\|T\|^{k-1}$.

Or $\|T\|^{k+1} \leq \|T^{k+1}\|$

But $\|T^{k+1}\| \leq \|T\|^{k+1}$
hence we have that $\|T^{k+1}\| = \|T\|^{k+1}$

Theorem
Let T $\in$ B(H) be a hyponormal operator and $\lambda_1, \lambda_2 \in \sigma_a(T)$ that $\lambda_1 \neq \lambda_2$. Suppose that x_n and y_n are sequences of unit vectors of the hilbert space H such that $\|T - \lambda_1 I\| \to 0$ and $\|T - \lambda_2 I\| \to 0$ then $< x_n, y_n >\to 0$
Lemma
If T is a hyponormal on H, such that $Tx = \lambda x$ then $T^* = \overline{\lambda} x$
Lemma
Let T be hyponormal on H, then $\|T\| = r(T)$
Proof
Let $x \in H$, $\|x\| = 1$, then $\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) \leq \|T^*Tx\| \leq \|T^2 x\|$.
But $\|T\|^2 \leq \|T^2\| \leq \|T\|^2$ which shows that $\|T\|^2 = \|T^2\|$
Therefore,
$\|T^n x\|^2 = (T^n x, T^n x) = (T^*T^n x, T^{n-1}x)$

$\leq \|T^*T^n x\|.\|T^{n-1}x\| \leq \|T^{n-1}x\|.\|T^{n-1}x\|$

Hence, $\|T^n\|^2 \leq \|T^{n-1}\|.\|T^{n-1}\|$ and combining the two equality from above, yields $\|T^n\| = \|T\|^2$. Since $r(T) = lim_{n\to\infty}\|T^n\|^{\frac{1}{n}} = lim_{n\to\infty}\|T\|$.

n-power hyponormal operators

Definition
An operator T in a complex Hilbert space H is said to be n-power hyponormal if $T^*T^n \geq T^nT^*$ for any positive integer n
For n = 1,
then $T^*T^1 \geq T^1T^*$.
which is equivalent to $T^*T \geq TT^*$. Hence 1-power hyponormal operators are hypo-normal operators.
For n=2,implies $T^*T^2 \geq T^2T^*$ or equivalently, $T^*T^2 - T^2T^* \geq 0$, then T is called 2-power hyponormal operator

Proposition
Suppose S, T $\in B(H)$ are 2-hyponormal operators, such that $TS^* = S^*T$ and ST = TS = 0, then T+S and ST are 2-power hyponormal.
Proof
$(T+S)^*(T+S)^2 = (T+S)^*(T^2 + TS + ST + S^2)$
Since $ST = TS = 0$ implies that
$(T+S)^*(T+S)^2 = (T+S)^*(T^2 + S^2)$

$=T^*T^2 + T^*S^2 + S^*T^2 + S^*S^2$
Using the fact that $TS^* = S^*T$ we have that

$\geq T^*T^2 + T^2S^* + T^2S^*S^*S^2$
$= T^*(T^2 + S^2) + S^*(T^2 + S^2)$

$(T+S)^2(T+S)^*$
This shows that T+S is 2-power hyponormal
Also
$(ST)^2(ST)^* = S^2T^2T^*S^*$

$\leq S^2T^*T^2S^*$

$= T^*S^2S^*T^2$

$\leq T^*S^*S^2T^2$
$=(ST)^*(ST)^2$
Hence, ST is a 2-power hyponormal operator
For n=3,implies $T^*T^3 \geq T^3T^*$ or equivalently, $T^*T^3 - T^3T^* \geq 0$, then T is called 3-power hyponormal operator
Denoted as [3HN]
Proposition
Suppose T is [3HN] and $T^2 = -T^{*2}$. Then T is 3-normal operator.
Proof
$T^3T^* = TT^2T^* = -TT^{*3}$
and
$T^*T^3 = T^*T^2T = -T^{*3}T$
Hence T is 3-power hyponormal
Proposition
Let T be [3HN] which is idempotent. Then T is [2HN]
Proof
Since T is [3HN], implies $T^*T^3 \geq T^3T^*$
Since T is idempotent imply
$T^*T^2 \geq T^2T^*$
Hence [2HN].

General properties of n-power hyponormal operators

N-power hyponormal operators contain normal as well as n-power normal operators.
Proposition
If S, T $\in$ B(H) are unitarily equivalent and T is n-power hyponormal, then S is also n-power hyponormal
Proof
Let T be an n-power-hyponormal operator and S be unitary equivalent of T. Then there exists unitary operator U such that S= UTU^* so S^n= UT^nU. We have S^nS^* = (UT^nU^*)$(UTU^*)^*$

$= UT^nU^*UTU^*$

$=UT^nT^*U^*$

$\leq UT^*T^nU^*$

$=S^*S^n$

Hence, $S^nS^* \leq S^*S^n$, which implies that S is a n-power hyponormal operator.

Proposition
If T is an n-power hyponormal operator, then T^* is also an n-power hyponormal operator
Proof
If T is an n-power hyponormal operator, then we have that
$T^*T^n \geq T^nT^*$
Now, this implies that $(T^*T^n)^* \geq (T^nT^*)^*$
$\Rightarrow (T^n)^*(T^*)^* \geq (T^*)^*(T^n)^*$

$\Rightarrow (T^*)^nT \geq T(T^*)^n$

This implies that T^* is n-power hyponormal operator.
Corollary
If T and T^* are n-power hyponormal operators, then T is a n-normal
Theorem
If S and T are commuting n-power hyponormal operators such that ST^*= T^*S, then ST is an n-power hyponormal operator

Proof
Since ST = TS, so $S^nT^n = (ST)^n$ and $ST^* = T^*S, soS^nT^* = T^*S^2$
Therefore
$ST^* = T^*S \Rightarrow TS^* = S^*T$

Now,
$(ST)^n(ST)^* = S^nT^nT^*S^*$

$\leq S^nT^*T^nS^*$

$=T^*S^nS^*T^n$

$\leq T^*S^*S^nT^n$
Hence
$(ST)^n(ST)^* \leq (ST)^*(ST)^n$ This shows that ST is an n-power hyponormal operator

Proposition
Let T $\in B(H)$be an n-power hyponormal operator. Then T^* is co-n-power hyponormal
Proof
Since T is n-power hyponormal operators, implies that
$T^*T^n \geq T^nT^* \Rightarrow (T^*T^n)^* \geq (T^nT^*)^*$
$\Rightarrow (T^n)^*(T^*)^* \geq (T^*)^*(T^n)^*$

$\Rightarrow (T^*)^nT \geq T(T^*)^n$
This further implies that $T(T^*)^n \geq (T^*)^nT$ and hence T^* is a co-n-power hyponormal operator.

Theorem
Let T be an n-power hyponormal operator such that it is idempotent. Then T is n-1 -power hyponormal operator.
Proof
Since T is n-power hyponormal operator, then
$T^*T^n \geq T^nT^*$. Now, since T is also idempotent implies that
$T^*T^{n-1} \geq T^{n-1}T^*$
Hence T is n-1 power hyponormal.

Theorem
Suppose $T_1, T_2, ...T_m$ are n-power hyponormal operators. Then,$(T_1 \oplus T_2 \oplus ...... \oplus T_m)$ is an n-power hyponormal operator.
Proof
$(T_1, T_2, ...T_m)$ is n-power hyponormal means that
$(T_1 \oplus T_2 \oplus .... \oplus T_m)^n(T_1 \oplus T_2 \oplus ...... \oplus T_m)^*$

$=(T_1^n \oplus T_2^n \oplus ..... \oplus T_m^n)(T_1^* \oplus T_2^* \oplus ..... \oplus T_m^*)$

$=T_1^nT_1^* \oplus .... \oplus T_m^nT_m^*$

$\leq T_1^*T_1^n \oplus .... \oplus T_m^*T_m^n$(since T is n-power hyponormal)

$=(T_1^* \oplus T_2^* \oplus ..... \oplus T_m^*)(T_1^n \oplus T_2^n \oplus ..... \oplus T_m^n)$

$=(T_1 \oplus T_2 \oplus ...... \oplus T_m)^*(T_1 \oplus T_2 \oplus .... \oplus T_m)^n$
This implies that $(T_1, T_2, ...T_m)$ is an n-power hyponormal operator.

Theorem
Suppose $T_1, T_2, ...T_m$ are n-power hyponormal operators. Then $(T_1 \otimes T_2 \otimes ...... \otimes T_m)$ is an n-power hyponormal operator.
Proof
Let $x_1, x_2, ..., x_m \in H$. Since implies that $(T_1 \otimes T_2 \otimes ...... \otimes T_m)^n (T_1 \otimes T_2 \otimes ...... \otimes T_m)^* (x_1 \otimes .... \otimes x_m)$

$=(T_1^n(\mathrm{T}_1 \otimes T_2 \otimes ...... \otimes T_m)$ is a hyponormal operators $\mathrm{T}_2^n \otimes ...... \otimes T_m^n)(T_1^* \otimes T_2^* \otimes ...... \otimes T_m^*)((x_1 \otimes .... \otimes x_m)$

$=T_1^n T_1^* x_1 \otimes .... \otimes T_m^n T_m^* x_m$

$\leq T_1^* T_1^n x_1 \otimes ..... \otimes T_m^* T_m^n x_m$

$= (T_1^* \otimes T_2^* \otimes ...... \otimes T_m^*)(T_1^n \otimes T_2^n \otimes ...... \otimes T_m^n)(x_1 \otimes .... \otimes x_m)$

$= (T_1 \otimes T_2 \otimes ...... \otimes T_m)^*(T_1 \otimes T_2 \otimes ...... \otimes T_m)^n (x_1 \otimes .... \otimes x_m)$
This implies that
$(T_1 \otimes T_2 \otimes ...... \otimes T_m)^n (T_1 \otimes T_2 \otimes ...... \otimes T_m)^* \leq (T_1 \otimes T_2 \otimes ...... \otimes T_m)^* (T_1 \otimes T_2 \otimes ...... \otimes T_m)^n$
Therefore $(T_1 \otimes T_2 \otimes ...... \otimes T_m)$ is a hyponormal operator

2.3 n-power posinormal operators

3.1 Posinormal operators
Definition
An operator T$\in$ B(H) is posinormal if there exists a positive operator P$\in$B(H) such that $TT^* = T^*PT$. In this case P is called an interrupter for T.
If T^* is posinormal then T is coposinormal
Remark
T $\in B(H)$ is a posinormal and coposinormal operator if and only $FR(T) = R(T^*)$.
Posinormal operators are normal operators where P is the identity operator.

Remark
The integral powers of a posinormal operator need not be posinormal, Kubrusly et al (2016).

Examples of posinormal matrices
Let T = $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and N = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then T and N are posinormal matrices.
Basic properties of Posinormal operators.
Theorem
The product of two posinormal operators need not be posinormal.
Example
Let T = $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and N = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then TN = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not posinormal.
Theorem
Let T $\in B(H)$ be a posinormal operator. If P is an interrupter and S is an isometry, then STS^* is posinormal

Theorem
Let T be a posinormal operator with a closed range, then T^n is posinormal for all $n \geq 1$
Proof
If T is a posinormal operator with a closed range,then it implies that T^2 has also a closed range. Since $R(T)$ and $R(T^2)$ are closed, the same applies to $R(T^*)$ and that of $R(T^{*2})$. This result yield $kerT = kerT^2$ and this shows that $(kerT)^\perp = (kerT^2)^\perp$ such that $\overline{R(T^*)} = \overline{R(T^{*2})}$. These two ranges are closed which implies that $R(T^*) = R(T^{*2})$ and thus $R(T^*) = R(T^{*2})$ for all $n \geq 1$. Since T is a posinormal operator, we have that $R(T) \subseteq R(T^*)$ and thus $R(T^n) \subseteq R(T) \subset R(T^*) = R(T^{*n})$. Hence T^n is a posinormal operator for all $n \geq 1$.
Theorem
Let T be a bounded operator on the Hilbert space H. Then the following are equivalent statements.

a) T is a posinormal operator

b)$R(T) \subseteq R(T^*)$

c)$TT^* \leq \lambda^2 T^*T$ equivalently $\|T^*x\| \leq \lambda\|Tx\|, x \in H, \lambda \geq 0$

d)There exists C $\in B(H)$ such that $T = T^*C$
Moreso, if (1), (2), (3) and (4) holds, then thereis a unique operator S such that

(i) $\|S\|^2 = inf\{\lambda, TT^* \leq \lambda^2 T^*T\}$

(ii) $N(T) = N(S)$

(iii) $R(S) \subseteq \overline{R(T)}$

Theorem
Let T $\neq 0$ be a posinormal operator and S is unitarily equivalent to T. Then S is also posinormal.

Theorem
Let T $\in B(H)$ be a posinormal operator with interrupter P,then $\|P\| \geq 1$
$\|T\|^2 = \|TT^*\| = \|T^*PT\| \leq \|T^*\|\|p\|\|T\| = \|P\|\|T\|^2$. Now, on cancellation we get $1 \leq \|P\|$. This completes the prove.
Theorem
Every invertible positive operator is posinormal
Proof
Suppose T is an invertible operator. This implies that

$T^* = T^*(T^{-1}T) = (T^*T^{-1})T$ and so T^* is also posinormal.

Let T $\in B(H)$ be a posinormal operator on the Hilbert space H with the interrupter P. If $\lambda \neq 0$, then we have that λT is also a posinormal operator.
Proof
$(\lambda T)(\lambda T)^* = |\lambda|^2 TPT$

$=(\lambda T)^* P(\lambda T)$
This implies that λT is a posinormal operator

Theorem

Definition
An operator T∈B(H) is called an n-power posinormal if $R(T^n) \subseteq R(T^*)$.

T is called n-power coposinormal if T^*if T is an n-power posinormal.

1.The class of 1-power-posinormal operators is a class of posinormal operators which was introduced by Rhaly

2-power posinormal operators
An operator T is said to be 2-power posinormal if $R(T^2) \subseteq R(T^*)$ and T is 2-power coposinormal if T^* is 2-power posinormal.
Remarks
Suppose n is a positive integer, then the following holds for a 2-power posinormal operator.
a) If T is a 2-power posinormal operator, then T is n-power posinormal for any integer m$\geq$ 2.
b) If T is a 2-power posinormal operator, then $N(T) \subseteq N(T^{*2})$
c)If T is 2-power posinormal, then T is 3-power posinormal
Proposition
If T is a 2-power posinormal and T^* is an isometry, then T is unitary
Proof
Let T be posinormal and P be an interrupter for T. We have that,
$TT^* = T^*PT$
Since T^* is an isometry,we have
$TT^* = I$. This gives
$I = T^*PT$
Multiplying the later identity from the left by T and from the right by T^*, we obtain
$I = TT^* = P$ and hence we have
$I = TT^* = T^*T$

General properties of n-power posinormal operators
Theorem
Let T $\in B(H)$be an n-power-posinormal operator. Then T is an n-power-normal operator
Proof
Suppose that T is an n-power normal operator. This implies that T^n is normal.This further implies that T^n is posinormal and hence T is an n-power-posinormal operator.

Theorem
Let T be an n-power posinormal operator such that $R(T) = R(T^n)$, then T is a posinormal operator.
Proof

T being an n-power posinormal operator implies that $R(T^n) \subseteq R(T^*)$. Since $R(T) = R(T^n)$ further implies that $R(T) = R(T^*)$ and hence T is posinormal.

Theorem

Let T be a k+n-power posinormal operator and T^{*n} be an isometry. Then T is a k-power posinormal operator.

Proof

Suppose P is an n+k-power-interrupter which is positive for the operator T.This implies that $T^{n+k}T^{*n+k} = T^*PT$

Since T^{*n} is an isometry implies that $T^nT^{*n} = I$ we have that $T^{n+k}T^{*n+k} = T^kT^{*k}$

This shows that $T^kT^{*k} = T^*PT$ and hence T is a k-power-posinormal operator

2.If N(T) = 0, implies T^* is surjective and consequently, T is an n-power-posinormal operators

3.(i) If T is n-power posinormal operator, then T is m-power posinormal for m$\geq$ n

(ii) If T is n-power-posinormal, then N(T) $\subseteq N(T^{*n})$

(iii) If T is n-power posinormal, then T is n+1-power posinormal.

(iv) If T is posinormal, then T^k is n-power-posinormal for any integer k

If T is a posinormal operator, then T^2 is not posinormal

In general if T is n-power-posinormal, then T^n may not be posinormal for any integer n.

Theorem

Let A, B $\in B(H)$ on the Hilbert space H. Then the following statements are equivalent

1) $R(A) \subseteq R(B)$

2) $AA^* \leq \lambda^2 BB^*$

3)There exists C $\in B(H)$ such that A = BC

If (1), (2) and (3) holds, then there exists a unique operator T such that the following also hold

(a) $\|T\| = inf\{\lambda, AA^* \leq \lambda^2 BB^*\}$

(b)$N(A) = N(T)$

(c) $R(T) \subseteq \overline{R(B^*)}$

Theorem
Let T∈B(H), then the following statements are equivalent

i) T has a positive n-interrupter

ii) $T^nT^* \leq \lambda^2 T^*T$

iii) T is an n-power posinormal operator that is $R(T^n) \subseteq R(T^*)$

iv) There exists C∈ B(H) such that $T^n = T^*C$.
If (i), (ii), (iii), and (iv) hold, then there exists a unique operator S such that
a) $\|S\|^2 = inf\{\lambda, T^nT^{n*} \leq \lambda^2 T^*T\}$
b)$N(T^n) = N(T)$
c)$R(S) \subseteq \overline{R(T^n)}$
Proof
(i) ⇒ (ii)
Suppose that $T^nT^{*n} = T^*PT$ where P is the positive interrupter for the operator T.
From we have, $< T^nT^{*n}x, x > = < \sqrt{P}Tx, \sqrt{P}Tx >$
$\|\sqrt{P}Tx\| \leq \|\sqrt{P}\|^2 \|Tx\|^2$

$= \|\sqrt{P}\| < T^*Tx, x >$
This implies that (ii) holds with $\lambda > \|\sqrt{P}\|$.
Also applying the above theorem and by taking $A = T^n$ and $B = T^*$ we have that $(ii) \Rightarrow (iii) \Rightarrow (iv)$. Also, if (iv) holds, then this implies that (i) holds if we take $P = C^*C$.
For $(a) \Rightarrow (b) \Rightarrow (c)$ we take $A = T^n$ and $B = T^*$ as in (a) , (b) and (c) the theorem above.

2.4 n-power Quasi-normal operators

Definition
Quasi-normal operators
A bounded linear operator T on a complex Hilbert space is said to be Quasi-normal if T and TT^* commute i.e $TT^*T = T^*TT$. This class contains all normal operators as well as isometries.
Example of a Quasi-normal operator is a unilateral or foward or right shift operator.
Theorem
Any invertible Quasi-normal operator is a normal operator.
Proof
Suppose that T is a quasi-normal operator. Then this implies that $T(T^*T) = (T^*T)T$

$$\Rightarrow T(T^*T)T^{-1} = (T^*T)TT^{-1}$$

Since T is a normal operator, we have that
$TT^*(TT^{-1}) = (T^*T)TT^{-1}$

$$\Rightarrow TT^* = T^*T$$

Hence T is a normal operator

Theorem
If T and S are two quasi-normal operators such that $TS = ST = T^*S = S^*T = 0$, then we have that T + S is quasi-normal
Proof
$(T+S)[(T+S)^*(T+S)] = (T+S)[(T^*+S^*)(T+S)]$

$$= (T+S)[(T^*T + T^*S + S^*T + S^*S)]$$

$$= (T+S)(T^*T + S^*S)(\text{Since } T^*S = S^*T = 0)$$

$$= T(T^*T) + T(S^*S) + S(T^*T) + S(S^*S)$$

$$= T(T^*T) + S(S^*S) \text{ Since T and S are quasinormal operators}$$

$= (T^*T)T + (S^*S)S$

$$(T+S)^*(T+S)(T+S)$$

This implies that T + S is a quasi-normal operator

We look at n-power Quasi-normal operators.
Deinition
An operator T is said to be n-power Quasi-normal if $T^nT^*T = T^*TT^n$ for some integer n
This class of n-power quasi-normal operators contains n-normal and quasi-normal operators

Remark
(i) 1- power quasi-normal operator is quasi-normal

2-power Quasi-normal operators

Definition
An operator T$\in$ B(H) is called 2-power Quasi-normal operator if $T^2T^*T = T^*T^3$
Theorem
Let T be a self adjoint operator. Then T is a 2-power quasi normal operator if and only if it is binormal
Proof
Since T is self adjoint implies that $T^* = T$
Now, let T be a 2-power quasi normal operator, then

$$T^2T^*T = T^*TT^2$$

$$\Rightarrow TTT^*T = T^*TTT$$

$$\Rightarrow TTT^*T = T^*TTT^*$$

$$\Rightarrow TT^*T^*T - T^*TTT^* = 0$$

$\Rightarrow [TT^*, T^*T] = 0$. This implies that T is binormal.
Conversely, let T be binormal, by definition we have that,
$TT^*T^*T = T^*TTT^*$
Since T is a self adjoint operator, we have that, $T^* = T$

$$\Rightarrow TTT^*T = T^*TTT$$

$\Rightarrow T^2T^*T = T^*TT^2$
Hence, T is a 2 power quasi normal operator

Theorem
Let T $\in B(H)$ be a 2-power quasi-normal and also 3-power quasi-normal operator such that $[T^*T, TT^* = 0]$, then T^2 is quasi-normal.
Proof
$(T^{*2}T^2)T^2 = T^*(T^*T)T^3$

$$=T^*T^2T^*T^2$$

$$=(T^*T)(TT^*)T^2$$

$$=(TT^*)(T^*T)T^2$$

$$=TT^*T^2T^*T$$

$=T(T^*T)(TT^*)T$

$T^*(T^{*2}T^2)$
This completes the prove

Theorem

Let T $\in B(H)$ be n-power quasi-normal operator which is partial isometry,then T is an n+1-power quasi-normal

Proof

Since T is a partial isometry operator, it implies that

$$TT^*T = T$$

If we multiply the above equation on the left by T^*T^{n+1} and using the fact that T is an n-power quasi-normal operator, we have

$T^*T^{n+2} = T^*T^{n+2}T^*T$

$=T^nT^*TTT^*T$

$= T^{n+1}T^*T$

Hence T is n+1-power quasinormaloperator.

Remarks

(i)All quasi-normal operators are n-power quasi-normal for any integer n.
(ii) n-power normal operators are n-power quasi-normal operators.
(iii) T$\in$ B(H) is n-power quasinormal if and only if $[T^n$,T^*T$]$ = $[T^n, T^*]$T = 0.
(iv) T$\in$ B(H) is n-power quasi-normal if and only if $T^n[T]^2 = [T]^2T^n$

Theorem

Let T be n power quasi normal which is a self adjoint operator. Then T^* is also n power quasi normal operator

Proof

Given that T is a n power quasi normal operator implies that $T^nT^*T = T^*TT^n$...................(i)
Also, since T is self-adjoint implies $T^* = T$.......................(ii)
Now, replacing T^* by T in (i), we get,
$(T^*)^n(T^*)^*T^* = (T^*)^nTT^* = T^nT^*T$...................(iii)
and also, $(T^*)^*T^*(T^*)^n = TT^*(T^*)^n = T^*TT^n$.............(iv)
Now, from (i), (iii) and (iv) we have that T^* is also n-power quasi normal operator

Lemma

Self adjoint operators are n power quasi normal operators

Proof

Let T be a self adjoint operator. This implies that $T = T^*$
Therefore,
$T^nT^*T = T^nTT = T^{n+2}$....................(i)
$T^*TT^n = TTT^n = T^{2+n}$
Hence, $T^nT^*T = T^*TT^n$..................(ii)
Therefore T is n power quasi normal operator

Theorem

Let T be a self adjoint operator. Then T^{-1} is also n power quasi normal operator

Proof

If T is a self adjoint operator implies $T = T^*$

We also have $(T^{-1})^* = (T^*)^{-1} = T^{-1}$ [Since $T = T^*$]

$(T^{-1})^* = T^{-1}$ which shows that T^{-1} is a self adjoint operator

From the above thereom that every self adjoint operator is n power quasi normal operator and hence T^{-1} is a self adjoint operator and therefore T^{-1} is n power quasi normal.

Theorem

Let T $\in B(H)$ be an n-power quasi-normal operator, then T is a 2n-power quasi-normal operator

Proof

(i)Since T is an n-power quasi-normal operator, it implies that

$T^nT^*T = T^*TT^n$ If we Multiply the left by T^n, we obtain

$T^{2n}T^*T = T^*TT^{2n}$

Thus T is a 2n-power quasi-normal operator

Theorem

Let T $\in B(H)$ be an n-power normal operator which has a dense range in H . Also, suppose that T is invertible, then T^{-1} is an n-power quasi-normal operator

Proof

Since T is of class [nQN], we have for y$\in$ R(T) : y $= Tx$, $x \in$ H, and $\|(T^nT^* - T^*T^n)y\| = \|(T^nT^* - T^*T^n)Tx\| = \|(T^nT^*T - T^*T^{n+1})x\|$ =0

Thus, T is n-power normal on R(T) and hence T is of class [nN]. In case T is invertible, then it is an invertible operator of class [nN] and so

$T^nT^* = T^*T^n$

This in turn shows that

$T^{-n}(T^{*-1}T^{-1} = [(TT^*)T^n]^{-n} = [T^{n+1}T^*]^{-1} = [T^{*-1}T^{-1}]T^{-1}$

Which proves the result.

Theorem

Let T ,S $\in B(H)$ be n-power quasi-normal operators such that ST= TS $= T^*S$= $ST^* = 0$, then TS is an n-power quasi-normal

Proof

$(TS)^n(TS)^*TS = T^nS^nT^*S^*TS$

$= T^nT^*TS^nS^*S$

$=T^*T^{n+1}S^*S^{n+1}$

$=(TS)^*(TS)^{n+1}$
This shows that TS is n-power quasi-normal

Theorem
Let S , T $\in B(H)$ be n-power quasi-normal operators such that ST = TS = $T^*S = ST^*= 0$, then S + T is an n-power quasi-normal
Proof
$(T+S)^n(T+S)^*(T+S)= (T^n+S^n)(T^*T+S^*S)$
$= T^nT^*T+S^nS^*S$
$=T^*T^{n+1}+S^*S^{n+1}$
$=(T+S)^*(T+S)^{n+1}$
Which proves that T+S is n-power quasi-normal.
Proposition
If T$\in$B(H) is a class of [2QN]$\cap$[3QN], then T a n-power quasi-normal for any integer n.
Proof
We proof by using mathematical induction. Suppose n=5 and since T$\in$[2QN]
$T^2T^*T = T^*T^3$
Multiplying to the left by T^3 we have
$T^5T^*T = T^3T^*T^3$
We have $T^5T^*T = T^3T^*T^3$
$= T^*T^4T^2$
$=T^*T^6$
Assuming that the result holds for n$\geq$5 i.e
$T^nT^*T = T^*TT^n$
Then
$T^{n+1}T^*T = TT^*T^{n+1}$
$TT^*T^3T^{n-2}$
$T^3T^*TT^{n-2}$
$T^*T^4T^{*(n-2)}$
$=T^*T^{n+2}$
Which implies that T is of class [(n+1)QN]

Theorem
Let T $\in B(H)$ be an n-power quasi- normal, then $N(T^n)\subset N(T^{*n})$
Proof
If we let $T^nx = 0$ then we have that $T^{*n}(T^*T)T^{n-1}x = 0$
By hypothesis, $T^*TT^{*n}T^{n-1}x = 0$
This implies that $TT^{*n}T^{n-1}x = 0$,
Thus we have $T^{*n}T^{n-2}x = 0$
Since T is an n-power quasi-normal, $T^*TT^{*n}T^{n-2}x = 0$
Hence, $T^{*n}T^{n-2}x = 0$
If we repeat this process we can find that $T^{*n} = 0$

Theorem
Let T and T^* be n-power quasi-normal, then T^n is normal

Proof

By the above theorem and hypothesis, we have $N(T^{*n}) = N(T^n)$

Now that T is an n-power quasi-normal, it implies that $[T^nT^* - T^*T^n]T^n = 0$ that is $[T^nT^* - T^*T^n] = 0$ on R(T).Since $N(T^*) \subset N(T^n)$ gives $[T^nT^* - T^*T^n] = 0$ on $N(T^*)$ and the hence the result.

Proposition

If T∈B(H) is n-power quasi-normal operator such that $N(T^*) \subset N(T)$, then T is n-power normal operator

Theorem

Let T be an n- power quasi normal and λ be any real scalar, then λT is also a n-power quasi normal operator

Proof

Now, since T is a n power quasi normal operator implies that $T^nT^*T = T^*TT^n$.....................(i)

Also, λ as any real scalar shows that $(\lambda T)^* = \overline{\lambda}T^* = \lambda T^*$. We have that

$[(\lambda T)^*]^n = (\lambda T^*)^n = \lambda^n T^{*n}$

Using (i) above, we have,

$(\lambda T)^n(\lambda T)^*(\lambda T) = \lambda^n T^n \lambda T^* \lambda T = \lambda^{n+2}T^nT^*T$........................(ii)

$(\lambda T)^*(\lambda T)(\lambda T)^n = \lambda T^* \lambda T \lambda^n T^n = \lambda^{n+2}T^*TT^n$..............................(iii)

From (i) , (ii) and (iii), we have that λT is n power quasi normal operator.

2.5 n-power quasi-isometry

5.1 Quasi - isometry
Definition
An operator T is called a quasi-isometry if $T^{*2}T^2 = T^*T$
The class of quasi-isometry is an extention of isometries. Thus, every isometry is a quasi-isometry.
Theorem
If T is a quasi-isometry and if $\|T\| = 1$, then T is hyponormal.
Proof
We proof by hypothesis,
$\|Tx - T^*T^2x|^2 = \|Tx\|^2 + \|T^*T^2x\|^2 - 2Re < Tx, T^*T^2x >$
This implies that,
$=\|Tx\|^2 + \|T^*T^2x\|^2 - 2\|Tx\|^2$

$\leq \|Tx\|^2 + \|Tx\|^2 - 2\|Tx\|^2 = 0$
This shows that $T = T^*T^2$ (2.1)
This further implies that $T^* = T^{*2}T$. From this equation we have that $N(T) \subset N(T^*)$ or $N(U) \subset N(U^*)$. It is obvious that $U^*U \geq UU^*$ and since $P^2 \leq I$, we have that $U^*P^2U = U^*U \geq UU^* \geq UP^2U^*$. This results to the equation $PU^*(T^*T)UP \geq P(TT^*)P$ (2.2)
Now that $P^2(TT^*) = TT^*$ (2.1), this implies that P commutes with TT^*. Combining this with (2.2) we have that
$T^*T = T^{*2}T^2 \leq P(TT^*)P = P^2(TT^*) = TT^*$
Hence T is a hyponormal operator

Corollary
Let T be a quasi-isometry. Then T is quasi-isometry if and only if it is a partial isometry.
Corollary
Let T is a quasi-isometry and quasinilpotent, then T= 0
Proof
As r(T) = 0, $|T^n| \leq 1$ for some positive integer n. Since T^n is also a quasi-isometry, $|T^n| = 1$. Hence T^n is hyponormal and the desired assertion follows from the relation $\|T^n\| = r(T^n)$.
Example
Let T = $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then T is a quasi-isometry matrix

We now introduce the new class operators called the n-power quasi-isometries which is an extension of quasi-isometries.
Definition
An operator T∈B(H) is called an n-power quasi-isometry if $T^{n-1}T^{*2}T^2 = T^*TT^{n-1}$ for some integer n.
Suppose n= 1,then 1-power quasi-isometry is just a quasi-isometry operator.
The corresponding classes of n-power quasi-isometries are independent of each other as illustrated in the example below

Example 1

Let T = $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is a quasi-isometry operator but not 2-power quasi-isometry

Example 2

Let T = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, by calculation, T is a 2-power quasi-isometry but not a quasi-isometry operator

According to P.Vijayalakshmi et al. The following assertions hold:

Theorem

Let T $\in B(H)$ be an n-power quasi-isometry. If S is unitarily equivalent to T, then S is also an n-power quasi-isometry operator.

Proof

If S is unitarily equivalent to T, then $S = UTU^*$,where U is a unitary operator. Since T is an n-power quasi-isometry operator, implies that $T^{n-1}T^{*2}T^2 = T^*TT^{n-1}$.

This implies that,

$$S^{n-1}S^{*2}S^2 = (UT^{n-1}U^*)(UT^{*2}U^*)(UT^2U^*)$$

$$=U(T^{n-1}T^{*2}T^2)U^*$$

$$=U(T^*T^n)U^*$$

$$=S^*S^n$$

Theorem

Let T , S $\in B(H)$ be n-power quasi-isometries.If T doubly commute with S, then TS is an n-power quasi-isometry operator.

Proof

$(TS)^{n-1}(TS)^{*2}(TS)^2 = T^{n-1}S^{n-1}S^{*2}T^{*2}T^2S^2$

$$=S^{n-1}S^{*2}S^2T^{n-1}T^{*2}T^2$$

$$=S^*SS^{n-1}T^*TT^{n-1}$$

$$=(TS)^*(TS)(TS)^{n-1}$$

Hence TS is an n-power quasi-isometry operator.

Theorem

Let T,S $\in B(H)$ be two n-power quasi-isometry operators. If $ST = TS = T^*S = ST^* = 0$ then T+S is an n-power quasi-isometry operator

Proof

$(T+S)^{n-1}(T+S)^{*2}(T+S)^2 = (T^{n-1}+S^{n-1})(T^{*2}+S^{*2})(T^2+S^2)$ Since $TS = ST = T^*S = ST^* = 0$ we have that

$= T^{n-1}T^{*2}T^2 + S^{n-1}S^{*2}S^2$

$=(T + S)^*(T + S)(T + S)^{n-1}$
Hence T + S is an n-power quasi-isometry operator

Theorem
Let $T \in [QI] \cap [2QI]$ then T is an n-power-quasi-isometry operator, for all n¿2.
Proof
Since $T \in [QI] \cap [2QI]$ we have
$T^*2T^2 = T^*T$ (2.1)

$TT^*2T^2 = T^*TT$ (2.2)
Combining (2.1) and (2.2), we have
$TT^*2T^2 = T^*2T^2T$ (2.3)
By equations (2.1) and (2.3),we have that $(T^*T)T^{n-1} = (T^*2T^2)T^{n-1} = (T^*2T^2T)T^{n-2} = (TT^*2T^2)T^{n-2}$
Again applying (2.3) in $(T^*T)T^{n-1} = T(T^*2T^2T)T^{n-3}$, we get $(T^*T)T^{n-1} = T(TT^*2T^2)T^{n-3}$

Theorem
Let $T \in [2QI] \cap [3QI]$ then T is an n-power-quasi-isometry operator, for $n \geq 4$
Proof
Since T $\in [2QI] \cap [3QI]$ implies that

$T^2T^{*2}T^2 = T^*TT^2$ (3.1)

We also have that $TT^{*2}T^2 = T^*TT$.(3.2)

Combining equations (3.1) and (3.2), we have

$T(TT^{*2}T^2) = (TT^{*2}T^2)T$ (3.3)

Applying (3.2) and (3.3) above, we obtain the equation,

$T^*TT^{n-1} = (T^*TT)T^{n-2} = (TT^{*2}T^2)T^{n-2} = (TT^{*2}T^2)TT^{n-3} = T(TT^{*2}T^2)T^{n-3} = T^2T^{*2}T^2T^{n-3}$
Repeating the above process several times we get $T^*TT^{n-1} = T^{n-1}T^{*2}T^2$.
Hence T is an n-power quasi-isometry operator. Theorem
Let T $\in B(H)$ be an n-power quasi-isometry operator.If $T \in [QI][nQI]$, then T is n-1 power quasi-isometry.
Proof
$T \in [QI] \cap [nQI]$ implies that

$T^{*2}T^2 = T^*T$ (3.6)

$$T^{n-1}T^{*2}T^2 = T^*TT^{n-1} \tag{3.7}$$

By (3.7), we have that $T^*TT^{n-1} = T^{n-1}T^{*2}T^2$

Also by (3.6) we obtain $T^*TT^{n-1} = T^{n-1}T^*T$. This implies that T is an n-1 power quasi-isometry.

Series of inclusions of n-power operators
(i) n-nilpotent operators $\subset$ n-normal operators
(ii) n-normal operators $\subset$n-hyponormal operators
(iii) n-normal operators $\subset$ n-posinormal operators
(iv) n-normal operators $\subset$ n-quasi normal operators

3 Chapter Three: Spectral properties of n-power operators

Spectral properties of normal operators
In this chapter, we discuss basic results on the spectral properties of n-power operators. Spectral theory of operators goes beyond the eigenvalues and eigenvectors of matrices to a wider perspective of arrangements of operators in a number of mathematical spaces. Franklin(1968) presented a foundation of studying spectral properties from algebraic to proving spectral theorem for normal operators.Ghaemi(2000), introduced the relationship between spectral decomposition of operators, the functional calculii of operators and the structure of Banach space. Curie(2005), presented a thesis that focused on the spectral structure of second order self-adjoint differential operators on a graph.
Before we discuss the spectral properties of n-power operators, we first look at various results on normal operators.
Spectral properties of Normal operators
Theorem
λ is an eigenvalue of an operator T if and only if its conjugate $\overline{\lambda}$ is an eigenvalue of T^*
Proof
By the normality of the operator T, for each $x \in H$ we have that$\|(T - \lambda I)x\| = \|(T - \lambda I)^*x\| = \|(T^* - \overline{\lambda}I)x\|$
Hence this proves the theorem.
Theorem
Let T $\in B(H)$ be a normal operator. Then if $\lambda_1 \neq \lambda_2$ are complex numbers, then ker$(T - \lambda_1 I) \perp ker(T - \lambda_2 I)$
Proof
Let $x, y \in H$ and $\lambda_1 \neq \lambda_2 \in C$ such that $Tx = \lambda_1 x$ and $Ty = \lambda_2 y$. Then,
$\lambda_1(x, y) = (Tx, y) = (x, T^*y) = (x, \overline{\lambda_2} y) = \lambda_2(x, y)$
Now, since $\lambda_1 \neq \lambda_2$, we have that $(x, y) = 0$
Theorem
Let T be a normal operator. Then $r(T) = w(T)$
Theorem
Eigenvectors of a normal operator T that corresponds to different eigenvalues are orthogonal and stabilize the orthogonal complement to its eigenspaces.

Theorem
3.The residual spectrum of a normal operator T is always empty.
Proof
Let $T \in B(H)$ be a normal operator.Also let $x \in H$. Then we have that $\|T^*x\|^2 =< T^*x, T^*x >=< x, TT^*x >=< x, T^*Tx >=< Tx, Tx >= \|Tx\|^2$
This implies that $ker(T^*) = ker(T)$. Now, since $\lambda, \lambda - T$ is normal, we therefore have that $ker(\overline{\lambda} - T^*) = ker(\lambda - T)$. This shows that $\overline{\lambda} \in \sigma_p(T^*)$ if and only if $\lambda \in \sigma_p(T)$ completing the proof.
Theorem

4.The operator norm of a normal operator T is equal its spectral radius
Proof
If T is a self- adjoint operator, then $< Tx, Tx >=< x, T^2x >$, therefore, $\|T^2\| = \|T\|^2$ and $\|T^{2n}\| = \|T\|^{2n}$. Therefore $r(T) = lim_{n\to\infty}\|T\|$. For the case where T is normal, we have that $\|Tx\|^2 =< Tx, Tx >=< x, T^*Tx >$, Therefore,$\|T^*T\| = \|T^*\|$. Hence, $|(T^*T)^n\| = \|T^n\|^2$ and then $\|T^*T\| = \rho(T^*T) = \rho(T)^2 = \|T\|$

Theorem
Every point in the spectrum of a normal operator is an approximate eigenvalue
Proof
Let T be a normal operator, then $T - \lambda$ is also a normal operator for every complex number λ. Therefore, $\|(T - \lambda)x\| = \|(T - \lambda)^*x\| = \|(T^* - \overline{\lambda})x\|$ for all vectors x. This shows that λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*. This implies that $\sigma_p(T) = \sigma_{comp}(T)$. Hence the residual spectrum is empty and therefore the rest of the spectrum is $\sigma_a(T)$

3.1 Spectral properties n-power normal operators

spectral properties of 2-normal operators
Theorem
Let T be a 2-normal operator satisfying the condition $\sigma(T) \cap (-\sigma(T) \subset 0$, then $\sigma(T) = \sigma_a(T)$
Proof
It is already known that $\sigma(T) = \sigma_a(T) \cup \sigma_r(T)$, Owing to this, we now only show that $\sigma_r(T) \subset \sigma_a(T)$. Now, let $\lambda \in \sigma_r(T)$. This implies that there exists a non-zero vector $x \in H$ such that $T^*x = \overline{\lambda}x$. Furthermore,$T^{*2}x = \overline{\lambda}^2x$ and therefore we have that $T^2x = \lambda^2x$.
1) If $\lambda \neq 0$, implies that $(T + \lambda)(T - \lambda)x = 0$. Since $-\lambda \notin \sigma(T)$ shows that $(T - \lambda)x = 0$ and therefore $\lambda \in \sigma_p(T)$.
2) If $\lambda = 0$, then $T^2x = 0$, therefore we have that $0 \in \sigma_p(T)$ and hence $\sigma(T) = \sigma_a(T)$
Theorem
Let $T \in B(H)$ be a 2-normal operator satisfying the condition $\sigma(T) \cap (-\sigma(T) \subset 0$,
1) If $\lambda \neq w$ are eigenvalues of T such that $x, y \in H$ are corresponding eigenvectors respectively then $< x, y >= 0$
2) If $\lambda \neq w$ are eigenvalues of T $\sigma_a(T)$ and x_n, y_n are the sequence of unit vectors in H such that $(T - \lambda)x_n \to 0$ and $(T - w)y_n \to 0 asn\infty$, then $lim_{n\to\infty} < x_n, y_n >= 0$
Proof
1. follows from 2, therefore we shall only proof part 2 2. Now, from the previous theorems, $(T^2 - \lambda^2)x_n \to 0$ as $n \to \infty$ and $(T^2 - w^2)y_n \to 0$ and T^2 is normal, it therefore holds that $(T^{*2} - \overline{w}^2)y_n \to 0$. Hence

$lim_{n\to\infty}\lambda^2 < x_n, y_n >= lim_{n\to\infty} < \lambda^2 x_n, y_n >= lim_{n\to} < T^2 x_n, y_n >= lim_{n\to} < x_n, T^2 y_n >= lim_{n\to} w^2 < x_n, y_n >$.
If $\lambda^2 = w^2$ then $(\lambda + w)(\lambda - w) = 0$. Since $\lambda \neq w$, we have that $\lambda = -w$. By $\sigma(T) \cap (-\sigma(T) \subset 0$,implies that $\lambda = w = 0$ which is not possible for distinct values. Therefore, $lim_{n\to} < x_n, y_n >= 0$ which completes the prove.
Remark
If T is 2-normal operator, then the ker(T) is not its reducing subspace

We now show the spectral properties of n-normal operators
n-normal operators
Theorem
The following statements are equivalent
1) $T - \lambda$ is n-normal for all $\lambda \geq 0$
2) T is normal.
3) $T - \lambda$ is an n-normal operator for all $\lambda \in C$
Proof
We show that $(1) \Rightarrow (2)$. T and $T - \lambda$ are n-normal, implies that
$(T - \lambda)^*(T - \lambda)^n - (T - \lambda)^n(T - \lambda)^*$
$= \Sigma_{j=1}^{n-1}(-1)^j\binom{n}{j}t^j(T^*T^{n-j} - T^{n-j}T^*)$
$=(-1)^{n-1}nt^{n-1}(T^*T - TT^*) + \Sigma_{j=1}^{n-2}(-1)^j\binom{n}{J}t^j(T^*T^{n-j} - T^{n-j}T^*) = 0$
Therefore, we have that
$(-1)^{n-1}n(T^*T - TT^*) + \Sigma_{j=1}^{n-2}(-1)^j\binom{n}{J}\frac{t^j}{t^{n-1}}(T^*T^{n-j} - T^{n-j}T^*) = 0$
Taking $t \to \infty$ it holds that $T^*T - TT^* = 0$ and hence T is normal

3.2 Spectral properties of n-power hyponormal operators

spectral properties of hyponormal operators
Theorem
The operator norm $\|T\|$ of a hyponormal operator is equal to it's spectral radius
Proof
Let $x \in H$ and for a positive integer n, by assumption, we have that
$\|T^*T^n x\| \leq \|T^{n+1}x\|$
This implies that
$\|T^*T^n\| \leq \|T^{n+1}\|$
Therefore
$\|T^n\|^2 = \|T^{*n}T^n\| \leq \|T^{*(n-1)}T^*T^n\| \leq \|T^{*(n-1)}\|\|T^*T^n\| \leq \|T^{*(n-1)}\|\|T^{n+1}\| = \|T^{n-1}\|\|T^{n+1}\|$
Now, if we let $\|T^k\| = \|T\|^k p << n$ and we get
$\|T\|^{n+1} \leq \|T^{n+1}\|$ and hence $\|T\|^{n+1} = \|T^{n+1}\|$

3.3 Spectral properties of n-power posinormal operators

we begin a discussion of basic spectral properties of posinormal operators
Theorem

Let T be a posinormal operator. Then $kerT = kerT^2$
Proof
It is always obvious that $kerT \subseteq kerT^2$. Now let $x \in kerT^2$. Since T is a posinormal operator, it implies that $Ran(T) \subseteq Ran(T^*)$ and this further implies that there exists $y \in H$ such that $Tx = T^*y$. We have that $0 = T(Tx) = T(T^*y)$ so that $0 =< TT^*y, y >=< T^*y, T^*y >= \|T^*y\|^2$. This implies that $T^*y = 0$ and $Tx = T^*y = 0$. Thus $x \in kerT$ and hence $kerT^2 \subseteq kerT$. This completes the prove

3.4 Spectral properties of n-power quasi-isometry

We begin with basic spectral properties of quasi-isometry operators
Theorem
Let T $\in B(H)$ be a quasi-isometry operator. If $\lambda \in \sigma_p(T)$ then $\overline{\lambda} \in \sigma_p(T^*)$
Proof
Suppose that $\lambda \in \sigma_p(T)$. Let $\lambda = 0$. If $0 \in \mathbb{C} \setminus \sigma_p(T^*)$ then from $T^{*2}T^2 = T^*T, T^*T^2 = T$ or $T^*T = T^*$. This implies that T is an isometry and this is a contradiction because $0 \in \sigma_p(T)$. Now we consider a case where $\lambda \neq 0$. For a vector $x \neq 0$ we have that $Tx = \lambda x$. Since, $T^{*2}T^2 = T^*T$ we obtain $\lambda T^*x = \lambda^2 T^{*2}x$. If $|\lambda| = 1$ we have $(T^* - \overline{\lambda}I)T^*x = 0$. We establish that $\overline{\lambda} \in \sigma_p(T^*)$.We now need to show that $T^*x \neq 0$ and if $T^*x = 0$ implies that $0 =< x, T^*x >=< Tx, x >= \lambda < x, x >$ and hence $\lambda = 0$ because $x \neq 0$. This contradicts the fact that $|\lambda| = 1$

Theorem
Let T be a quasi-isometry operator. If $\lambda \in \sigma_a(T)$, then $\overline{\lambda} \in \sigma_a(T^*)$
Proof
$\lambda \in \sigma_a(T)$. If $\lambda = 0$, then we can show that $0 \in \sigma_a(T^*)$. Now we suppose that $\lambda \neq 0$ and we choose a sequence (x_n) of unit vectors such that $(T - \lambda I)x_n \to 0$.

$$-\lambda^2 T^{*2}x_n + \lambda T^*x_n = T^{*2}(T^2x_n - \lambda^2 x_n) - T^*(Tx_n - \lambda x_n) \to 0 \text{ as } n \to \infty$$

$(\lambda T^* - I)T^*x_n \to 0$. Since $\lambda = lim < Tx_n, x_n >= lim < x_n, T^*x_n >$
If $\lambda \neq 0$ means that $(T^*x_n) \nrightarrow 0$. We can choose a subsequence (T^*x_{nk}) of (T^*x_n)so that $\|T^*x_{nk}\| \geq M$ for some positive integer M.
The set $y_k = \frac{T^*x_{nk}}{\|T^*x_{nk}\|}$. Then (y_k) is a sequence of unit vectors such that $(\lambda T^* - I)y_k \to 0$ or $(T^* - \overline{\lambda}I)y_k \to 0$as $|\lambda| = 1$.

Theorem
Let $\lambda_1 \neq \lambda_2 \neq 0$ be two distinct eigenvalues of a quasi-isometry operator T. Then the corresponding eigenspaces of λ_1 and $\lambda_2$are orthogonal.
Proof
Let λ_1 and λ_2 be two nonzero eigenvalues of T. If $Tx = \lambda_1 x$ and $Ty = \lambda_2 y$. Then we have that $0 =< T^2x, T^2y > - < Tx, Ty >= \lambda_1 \overline{\lambda_2(\lambda_1\overline{\lambda_2} - 1))} < x, y >$

Since $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ implies that $\lambda_1 \overline{\lambda_2 \neq 0}$ and $|\lambda| = 1$.
Also, if $\lambda_1 \neq \lambda_2$, this will mean that $\lambda_1 = \frac{1}{\lambda_2}$ or $\lambda_1 \overline{\lambda_2} = 1$. Thus we have that $< x, y >= 0$

Spectral properties of n-power quasi-isometry

Theorem

Let T $\in [2QI] \cap [3QI]$ such that $ker(T^*) \subset ker(T)$ then T is a quasi-normal operator and $ker(T^*) = 0$

Proof

By hypothesis, we have can have $TT^{*2}T^2 = T^*TT$ and also $T^2T^{*2}T^2 = T^*TT^2$

$$T(TT^{*2}T^2) = (T^*T)T^2$$

$\Rightarrow T(T^*TT) = (T^*T)T^2$.This further implies that $(TT^* - T^*T)T^2 = 0$ equivalently $T^{*2}(TT^* - T^*T) = 0$. But since $ker(T^*) \subset ker(T)$ we have $TT^*(TT^* - T^*T) = 0$ and $ker \mid T^* \mid^2 = ker(T^*) \Rightarrow T^*(TT^* - T^*T) = 0$ thus $(TT^* - T^*T)T = 0$

Hence T is a quasinormal operator

Theorem

Let T $\in [QI] \cap [2QI]$ then $r(T) = 1$

Proof

Since $T \in [QI] \cap [2QI]$ by the previous theorem implies that T is a quasinormal operator and hence $r(T) = \|T\| = 1$.

4 Chapter Four: Numerical Ranges of N-power operators

Definition
Let T $\in B(H)$. The set $W(T) = \{< Tx, x >| \ x \in H, \|x\| = 1\}$ is known as the numerical range of T.
Remark
For any $x \in H$, such that $\|x\| = 1$ we have that $|< Tx, x >| \leq \|Tx\|\|x\| = \|Tx\|$

$$\leq \|T\|\|x\| = \|T\|.$$

Thus $|< Tx, x >| \leq \|T\|$ for all $x \in H$
Hence, this implies that $W(T)$ is a bounded set
General properties of numerical ranges of operators
Theorem
Let T $\in B(H)$. Then $\lambda \in W(T)$ if and only if $\overline{\lambda} \in W(T^*)$
Proof
Suppose that $\lambda \in W(T)$. This implies that there exists $x \in H, \|x\| = 1$ such that $\lambda \in < Tx, x >$. Thus $\overline{\lambda} \in \overline{< Tx, x >} = \overline{< x, T^*x >}$

$$= < T^*x, x >$$

Hence $\overline{\lambda} \in W(T^*)$
Conversely, suppose that $\overline{\lambda} \in W(T^*)$
Then by definition there exists $x \in H, \|x\| = 1$ such that $\overline{\lambda} \in < T^*x, x >$. Now, $\lambda = \overline{\overline{\lambda}} = \overline{< T^*x, x >}$

$$= \overline{< x, Tx >}$$

$$=< Tx, x >$$

Hence $\lambda \in W(T)$
Theorem
Let T $\in B(H)$. Then $\sigma(T) \subset W(T)$
Proof
Let $\lambda \in \sigma(T)$. By definition, there exists $x \in H, \|x\| = 1$ such that $(T - \lambda I)x = 0$. This implies that $Tx = \lambda x$.
Now, $\lambda = \lambda\|x\| = \lambda < x, x >$

$$=< \lambda x, x >$$

Since $Tx = \lambda x$ we have

$$=< Tx, x >$$

Hence $\lambda \in W(T)$
Theorem
Let $\lambda \in W(T)$ such that $| \ \lambda \ | = \|T\|$, then $\lambda \in \sigma_p(T)$
Proof
Suppose that $\lambda \in W(T)$. By definition there exists $x \in H, \|x\| = 1$ such that

$\lambda =< Tx, x >$

$| \lambda |=|< Tx, x >|< \|Tx\| \|x\| = \|Tx\| \leq \|T\|$.
Therefore, we have that $|< Tx, x >|= \|Tx\| \|x\| \rightarrow Tx = \lambda x$ for some $\lambda \in \mathbb{C}$
However, $\lambda \in< Tx, x >=< \lambda x, x >= \lambda$
Hence $Tx = \lambda x$

Numerical properties of normal operators

Theorem
The convex hull of the spectrum of a normal operator is equal to its closed numerical range
Proof
Let T be a normal operator and λ be a complex number.By the normality of $T - \lambda I$ implies that the following statements are equivalent

(i) $\lambda \in \overline{W(T)}$

(ii) $0 \notin \overline{W(T - \lambda I)}$

(iii) The spectrum of $T - \lambda I$ lies on the one side of the origin

(iv) 0 is not in the convex hull of $\sigma(T - \lambda I)$

(v) λ is not in the convex hull of $\sigma(T)$

Lemma
If $W(T) = \|T\|$ this implies that $r(T) = \|T\|$
Proof
Now, without loss of generality,when we multiply by a suitable constant we have that
$\|T\| = 1$. It is always obvious that $W(T) = \|T\|$ and there exists a sequence x_n of unit vectors such that $| < Tx_n, x_n > | \rightarrow 1$.
Hyponormal operators
Let T $\in B(H)$ be a hyponormal opeartor. Then the closure of the numerical range of T coincides with the convex hull of its spectrum
Proof
In general, the inclusion $\sigma(T) \subset \overline{W(T)}$ holds.Now suppose that $\lambda \in W(T)$ which is not in the convex hull of the spectrum. By an affine change of variables, we can assume that $\sigma(T)$ is contained in a disk centered at zero of radius r and $\lambda > r$. Since the spectral radius of the operator T equals its norm, we therefore find that $\|T\| \leq r$ and on the other hand $< Tx, x >= \lambda$ for a unit vector x. This implies that $|\lambda| \leq | < Tx, x > | \leq r$, a contradiction.

Conclusion
n-power operators is a class of operators that has been extensively been researched on with a keen focus on their sum, product and unitary equivalence. These classes of operators is growing so fast and many reslts are being discovered.
My project has focused on five classes of n-power operators that is n-power normal, n-power hypormal, n-power quasi-normal, n-power posinormal and n-power quasi-isometry operators. We have seen that n-power operators is an extension of lower classes of operators such as normal, hyponormal, quasi-normal, posinormal and quasi-isometries respectively.

We have also seen that these classes of operators are unitarily invariant and if they doubly commute, then their sum and product are also n-power operators However, we have also seen that little research has been done about their spectral and numerical properties. My keen interest in future is study and discover results on their spectral and numerical ranges.

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