INVARIANT AND HYPER-INVARIANT SUBSPACES OF SOME CLASSES OF OPERATORS IN HILBERT SPACES

Irene Mumbua Muteti
Reg No: I80/56999/2020
Department of Mathematics
University of Nairobi

A Thesis Submitted in Fulfilment of the Requirements for the Award of the Degree of Doctor of Philosophy in Pure Mathematics of the University of Nairobi

DECEMBER 2023

Declaration

This thesis is my original work and has not been presented for a degree award in any other university.

Irene Mumbua Muteti (PhD)
I80/56999/2020

Signature_ ______________ Date: 08/12/2023

This thesis has been submitted for examination with our approval as university supervisors.

Prof. Bernard M. Nzimbi

Signature_______________________ Date 08/12/2023
Department of Mathematics

Prof. Jairus Khalagai

Signature_______________________ Date 08/12/2023
Department of Mathematics

Dr. Stanley Imagiri

Signature_______________________ Date 08/12/2023
Department of Mathematics

Dedication

To my daughters Magdaline and Claire may you be inspired by this work.

Acknowledgment

I thank the Almighty God for taking me this far. My sincere gratitude is due to my supervisors namely Prof. Bernard M. Nzimbi, Prof. J.M. Khalagai, and Dr. S.K. Imagiri for their support throughout this journey through their guidance in this research. Despite the challenges experienced your expertise and patience constantly gave me encouragement and kept me focused.I acknowledge the work of the Chairman of Mathematics Department, University of Nairobi, Prof. Stephen Luketero whose instruction, guidance and assistance helped me a lot in this research. I am grateful to my family for being there for me whenever I needed them and for always believing in me. Your sacrifice, patience and financial support could never go unmentioned. I am greatly thankful to my father William Muteti and Mother Magdalena Syombua for having faith in me and for their encouragement and support throughout my studies till this far.

My sincere gratitude goes to the University of Nairobi for giving me the opportunity to undertake my graduate studies.

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NOTATIONS

$B(H)$	-Banach algebra of bounded linear operators on a Hilbert space H.
$\mathbb{D}$	- Open unit disk in the complex plane.
$\partial\mathbb{D}$	- Unit circle in the complex plane.
$r(T)$	- The spectral radius of T
$c.n.n$	- Completely non normal.
$c.n.u$	- Completely non unitary.
$n.h.s$	- Non trivial hyperinvariant subspace.
$M^{\perp}$	- Orthogonal complement of a subspace M.
$M \oplus N$	- Direct sum of M and N.
$W^{*}(T)$	- Unital weakly closed von Neumann algebra generated by T.
$\sigma(T)$	-Spectrum of T.
$\rho(T)$	-resolvent set of T.
$W(T)$	- Numerical range of T.
$Ker(T)$	- Kernel of T.
$Ran(T)$	- Range of T.
$Lat(T)$	- Invariant subspaces of T.
$Red(T)$	- Reducing subspaces of T.
$Hyperlat(T)$	- Hyperinvariant subspaces of T.

Abstract

Direct sum decomposition of bounded linear operators gives a way of studying complicated operators since it decomposes these operators into parts whose structures are user friendly. It is known that some operators are not decomposable, however, every reducible operator is decomposable. Invariant subspaces of an operator, their classification play an explicitly central role in studying complicated operators, for it is known that, every operator with a nontrivial invariant subspace is reducible. Reducing subspaces are special invariant subspaces which are useful in the direct sum decomposition. The motivations behind the study of invariant subspaces come from the interest in the structure of an operator and from approximation theory to a wide variety of problems in physics(quantum theory), computer science (data mining) and chemistry(lattice theory of crystal analysis).

In this thesis, we have investigated the existence of invariant and hyperinvariant subspace for some operators such as quasinormal,nilpotent, quasinilpotent, hyponormal, paranormal among others. We have shown that quasinormal as well as nilpotent operators have nontrival invariant subspace. Quasinilpotent operators have nontrivial invariant subspaces if the operator and its adjoint satisfy Single Value Extension Property. Conditions in which hyponormal and higher classes have nontrivial invariant subspaces are given. We have studied the structure of lattices of some of the operators in relation to certain equivalence classes. We have shown that self adjoint operator T and its adjoint have equal lattices. It has also been shown a normal operator and its adjoint have isomorphic invariant subspace lattices but fails for quasinormal operators. It has been shown that isomorphism of hyperlattices of operators does not imply quasisimilarity nor similarity of operators.

CHAPTER 1

INTRODUCTION

1.1 Background

The invariant subspace problem asks whether every operator on a complex separable Hilbert space has a non-trivial invariant subspace. This problem has its origin approximately in 1950 when, (according to Aronszajn and Smith [4]), J. von Neumann (unpublished work), proved that every compact operator on a separable infinite dimensional complex Hilbert space has a non-trivial invariant subspace. The invariant subspaces of an operator play a central role in operator theory. They are a direct analogue of the eigen-vectors of a linear operator. Reducing subspaces are useful in the direct sum decomposition of an operator. Note that, every reducing subspace M of $T \in B(H)$ is also invariant subspace of T, that is, M is $T-$ reducing implies M is $T-$ invariant. The motivation behind the study of invariant subspaces comes from the interest of the structure of operators and from approximation theory. The knowledge of the invariant subspace lattice of an operator T gives information about the vectors which can be approximated by linear combinations of $T^n x$ for some $x \in H$ and every positive integer n. The knowledge of the hyperinvariant subspace of $T\ B(H)$ gives information about the commutant of T,(the set of all operators S such that $TS = ST$).

1.2 Definitions and Terminologies

In what follows, capital letters H, H_1, H_2, K, K_1, K_2 denote Hilbert spaces or subspaces of Hilbert space, and $T, T_1, T_2, S_1, S_2, A, B$ etc denote bounded linear operators where by an operator we mean a bounded linear transformation from H into H. $B(H_1, H_2)$ denotes the set of bounded linear operators from H_1 to H_2. For an operator $T \in B(H)$, T^* denotes the adjoint of T, while $Ker(T), Ran(T), \bar{M}, M^{\perp}$ stand for the kernel of T, range of T, closure of M and orthogonal complement of a closed subspace M of H, respectively and $\sigma(T)$ denotes the spectrum of T, $\|T\|$ denotes the norm of T, $r(T)$ the spectral radius of T, while $W(T)$, denotes the numerical range of T. By 0 and I we denote the zero and identity operators on H, respectively.

Definition 1.2.1. *An operator $T \in B(H)$ is said to be:*

normal *if $T^*T = TT^*$,*

self-adjoint *or* ***Hermitian*** *if $T = T^*$,*

skew-adjoint *if $T^* = -T$,*

an involution *if $T^2 = I$,*

a projection *if $T^* = T$ and $T^2 = T$,*

unitary *if $T^*T = TT^* = I$,*

symmetric *if $T = T^* = T^{-1}$, that is, T is self-adjoint unitary,*

isometric *if $T^*T = I$,*

co-isometric *if $TT^* = I$*

complex symmetric *if there exists a conjugation S such that $STS = T^*$. where by a conjugation we mean an isometry or an involution.*

a partial isometry *if $T = TT^*T$, that is, T^*T is a projection,*

quasi-normal *if $T(T^*T) = (T^*T)T$, that is, if T commutes with T^*T,*

nilpotent *if $T^n = 0$ for some positive integer n,*

quasi-nilpotent *if $\sigma(T) = \{0\}$,*

binormal *if $(T^*T)(TT^*) = (TT^*)(T^*T)$,*

complex symmetric *operator if there exists a conjugation S such that $STS = T^*$. By a conjugation we mean an isometry or antilinear involution.*

A-self adjoint *if $T^* = A^{-1}TA$, where A is a self-adjoint invertible operator,*

compact *if for each bounded sequence $\{x_n\}$ in the domain H, the sequence $\{Tx_n\}$*

contains a subsequence converging to some limit in the range,

a scalar *if it is a scalar multiple of the identity operator, that is, if* $T = \alpha I$ *where* $\alpha \in \mathbb{C}$

hyponormal *if* $T^*T \geq TT^*$, *that is,* $T^*T - TT^* \geq 0$,

cohyponormal *if its adjoint is hyponormal, that is,* T *is cohyponormal if* $TT^* \geq T^*T$,

p-hyponormal *if* $(T^*T)^p \geq (TT^*)^p$ *where* $0 < p \leq 1$,

quasihyponormal *if* $||T^*Tx|| \leq ||TTx||$ *for all* x *in* H, *equivalently if* $T^*(T^*T - TT^*)T \geq 0$,

paranormal *if* $\|Tx\|^2 \leq \|T^2x\|$, *for all unit vectors* $x \in H$, *equivalently if* $\|Tx\| \leq \|T\| \|x\|$, *for every* $x \in H$,

****-paranormal*** *if* $\|T^*x\|^2 \leq \|T^2x\|$, *for all unit vectors* $x \in H$,

n-*-paranormal *if* $\|T^*x\|^n \leq \|T^nx\|$, *for all unit vectors* $x \in H$,

totally *-paranormal *if* $\|(T-\lambda I)^*x\|^2 \leq \|(T-\lambda I)^2x\| \|x\|$, *for all unit vectors* $x \in H$ *and* $\lambda \in \mathbb{C}$,

k-quasihyponormal *if* $T^{*k}(T^*T - TT^*)T^k \geq 0$, *for* $k \geq 1$ *some integer and every* $x \in H$,

p-quasihyponormal *if* $T^*((T^*T)^p - (TT^*)^p)T \geq 0$,

(p,k)-quasihyponormal *if* $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$, *where* $0 < p < 1$ *and* k *is a positive integer,*

log-hyponormal *if* $logT^*T \geq logTT*$,

dominant *if for any* $\lambda \in C$ *there corresponds a number* M_λ *such that* $\|(T-\lambda I)\|^* x \leq M_\lambda \|(T-\lambda I)x\|$ *for all* $x \in H$,

algebraic *if there exists a nonzero polynomial* p *satisfying* $p(T) = 0$.

Definition 1.2.2. *An operator* T *is* ***seminormal*** *if it is either hyponormal or cohyponormal, equivalently it is either* T *or* T^* *hyponormal.*

Remark 1.2.3. *Clearly, if an operator $T \in B(H)$ is both hyponormal and cohyponormal, then T must be normal.*
Note also that, every hyponormal operator is seminormal but the converse is not true in general.

Definition 1.2.4. *A partial order is a binary relation denoted by $\preceq$ over a set X which is reflexive, anti-symmetric and transitive.*

A set with a partial order is called a partially ordered set (*poset*).
For x, y elements of a partially ordered set X, if $x \leq y$ or $y \leq x$, then x and y are comparable. Otherwise they are incomparable.
A partial order under which every pair of elements is comparable is called a total order or linear order or linear ordering.
A totally ordered set is called a chain. A subset of a poset in which no two distinct elements are comparable is called an antichain.
An ordered set X is called directed if for every $x, y \in X$, there is $z \in X$ such that $x \leq z$ and $y \leq z$.
An element $x \in X$ is the greatest element of the ordered set X if $y \leq x$ for every $y \in X$.
An element $x \in X$ is the least element of the ordered set X if $x \leq y$ for every $y \in X$.
An element $x \in X$ is maximal in the ordered set X if there is no y in X such that $x < y$. Note that the greatest element if it exists is the maximal but not conversely.
$x \in X$ is minimal if there is no $y \in X$ such that $x > y$.
$x \in X$ is an upper bound of $A \subset X$ if $y \leq x$ for every $y \in A$.
$x \in X$ is a lower bound of A if $x \leq y$ for every $y \in A$.

Definition 1.2.5. *If every two elements $x, y \in X$ possess both a least upper bound and a greatest lower bound, denoted by $sup\,(x, y)$ and $inf\,(x, y)$ known as supremum and infimum of (x, y) respectively. This ordered set $(X, \leq)$ is called a lattice.*

Definition 1.2.6. *Let $T \in B(H)$. A subspace M of a Hilbert space H is* ***invariant*** *under operator T if $T(M) \subseteq M$, that is $x \in M$ implies $Tx \in M$ for every $x \in M$. $Lat(T)$ will denote the lattice of all invariant subspaces of T, that is, $Lat(T) = \{M \subseteq H : T(M) \subseteq M\}$.*
A subspace M of H is said to be a ***reducing subspace*** *of $T \in B(H)$ if it is invariant under both T and T^*, equivalently, if both M and $M^{\perp}$ are invariant under T. $Red(T)$ will denote the lattice of all reducing subspaces of T, that is, $Lat(T) = \{M \subseteq H : M \in Lat(T) \cap Lat(T^*)\}$.*

Remark 1.2.7. *Note that, the set of all invariant subspaces for $T \in B(H)$, that is, $Lat(T)$ is a lattice but $Red(T)$ need not be a lattice.*

Definition 1.2.8. *If $T \in B(H)$, we denote by $\{T\}'$ the commutant of T, that is, $\{T\}' = \{S \in B(H) : ST = TS\}$.*

Definition 1.2.9. *A subspace $M \subset H$ is said to be nontrivial hyperinvariant subspace (n.h.s) for a fixed operator $T \in B(H)$ if $0 \neq M \neq H$ and $SM \subseteq M$ for each $S \in \{T\}'$, that is, it is invariant under every operator commuting with T. The lattice of all hyperinvariant subspaces of T will be denoted by $HyperLat(T)$.*

Definition 1.2.10. *The bicommutant or double commutant of $T \in B(H)$ is defined by $\{T\}'' = \{A \in B(H) : AS = SA$, for all $S \in \{T\}'\}$.*

Definition 1.2.11. *An operator $T \in B(H)$ is hyper-reducing if M reduces every operator in the commutant of T and the collection of all subspaces hyper-reducing for $T \in B(H)$ is denoted by $HyperLat(T)$.*

Definition 1.2.12. *An operator $T \in B(H)$ is a generalized scalar operator if there exists a continuous algebra homomorphism $\Phi : H \to K$ satisfying $\Phi(1) = I$, the identity operator on K and $\Phi(z) = T$ where z denotes the identity function on*

H, that is,the function Φ is an operator valued distribution known as the spectral distribution of T.

Definition 1.2.13. *An operator $T \in B(H)$ is subscalar if it is similar to a restriction of a generalized scalar operator to one of its closed invariant subspaces.*

An arbitrary operator $T \in B(H)$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry satisfying $KerU = Ker|T| = KerT$ and $KerU^* = KerT^*$. Associated with T is the operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ called the Aluthge tranform of T denoted by $\tilde{T}$.

Definition 1.2.14. *An operator $T = U|T|$ in $B(H)$ is w-hyponormal if $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$ where $|\tilde{T}^*| = (\tilde{T}^*\tilde{T})^{\frac{1}{2}}$*

Definition 1.2.15. *An operator $T \in B(H)$ is a **left shift** on $H = \ell^2(\mathbb{N})$ if $Tx = y$ for all $x = (x_1, x_2, \ldots)$ and $y = (x_2, x_3, \ldots)$ while it is a **right shift** operator if $Tx = y$ where $x = (x_1, x_2, \ldots)$ and $y = (0, x_1, x_2, \ldots)$.*

Definition 1.2.16. *An operator $T \in B(H)$ is completely non-unitary (c.n.u) if the restriction of it to any non-zero reducing subspace is not unitary.*

Remark 1.2.17. *: If M is an invariant subspace for $T \in B(H)$, then relative to the decomposition $H = M \oplus M^\perp$, T can be written as $T = \begin{bmatrix} Z & X \\ 0 & Y \end{bmatrix}$ for operators $X : M^\perp \to M$ and $Y : M^\perp \to M^\perp$, $Z = T|_M : M \to M$ is a part of T. The operator $X = 0$ if and only if M reduces T. In such a case, the operator T is decomposed into the orthogonal direct sum of the operators $Z = T|_M$ and $Y = T|_{M^\perp}$: $T = Z \oplus Y$.*
A direct summand of an operator T is the restriction of it to a reducing subspace.
An operator is reducible if it has nontrivial reducing subspace (equivalently, if it has a proper nonzero direct summand), otherwise it is irreducible.

In 1985, Conway and Gillespie [15] posed an interesting problem on to what extend the operator T is determined by the lattice theoretic structure of its invariant subspace lattice $Lat(T)$. Since nothing is known about $Lat(T)$ for a general operator T, one is led to consider special classes of operators, where $Lat(T)$ is well described. Conway and Gillespie settled the problem for self-adjoint operators, they characterized the isomorphism of the lattices of reducing subspaces for any two normal operators.

Definition 1.2.18. *Two lattices L_1 and L_2 are said to be spatially or unitarily isomorphic if there exists a unitary operator $U : L_1 \to L_2$ such that $L_2 = U(L_1)$.*

Definition 1.2.19. *The set $\rho(T)$ of all complex numbers λ for which $\lambda I - T$ is invertible is called the resolvent set of T, that is,*

$\rho(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\}$ *and* $Ran(\lambda I - T) = H\}$.

The complement of the resolvent set $\rho(T)$ denoted by $\sigma(T)$, is called the spectrum of T. In other words, $\sigma(T) = \mathbb{C}\backslash\rho(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) \neq \{0\}$ and $Ran(\lambda I - T) \neq H\}$, which is the set of all λ such that $\lambda I - T$ fails to be invertible, that is, fails to have a bounded inverse on $Ran(\lambda I - T) = H$.

Definition 1.2.20. *A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of an operator $T \in B(H)$ if there exists a non-zero vector $x \in H$ such that $Tx = \lambda x$.*

Definition 1.2.21. *The set of all those $\lambda \in \mathbb{C}$ such that $\lambda I - T$ has no inverse, denoted by $\sigma_p(T)$ is called the point spectrum of T. Equivalently, $\sigma_p(T) = \{\lambda \in \mathbb{C} : ker(\lambda I - T) \neq \{0\}\}$ which is the set of all eigenvalues of T.*

Remark 1.2.22. *Note that in finite dimensional settings, $\sigma(T) = \sigma_p(T)$.*

Definition 1.2.23. *The set of all those $\lambda \in \mathbb{C}$ for which $\lambda I - T$ has a densely defined but unbounded inverse on its image, denoted by $\sigma_c(T)$ is called the continuous spectrum of T. Equivalently, $\sigma_c(T) = \{\lambda \in \mathbb{C} : ker(\lambda I - T) = \{0\}$ and $\overline{Ran(\lambda I - T)} = H$ and $Ran(\lambda I - T) \neq H\}$.*

If $\lambda I - T$ has an inverse that is not densely defined, then λ belongs to the residual spectrum of T denoted by $\sigma_r(T)$, that is, $\sigma_r(T) = \{\lambda \in \mathbb{C} : ker(\lambda I - T) = \{0\}$ and $\overline{Ran(\lambda I - T)} \neq H\}$.

Remark 1.2.24. *The parts $\sigma_p(T), \sigma_c(T), \sigma_r(T)$ are pairwise disjoint and $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$. Thus, the collection $\sigma_p(T), \sigma_c(T), \sigma_r(T)$ forms a partition of $\sigma(T)$.*

Definition 1.2.25. *An operator $T \in B(H)$ has finite ascent if $Ker(T^n) = Ker(T^{n+1})$.*

Definition 1.2.26. *An operator $T \in B(H, K)$ is invertible if it is injective (one-to-one) and surjective .*

Definition 1.2.27. *Two operators A and B are said to commute if $AB = BA$, denoted by $[A, B] = 0$.*

Definition 1.2.28. *The set of all bounded linear operators $T \in H$ such that T^*T and $T^* + T$ commute is denoted by θ.*

Definition 1.2.29. *Two operators $A \in B(H)$ and $B \in B(K)$ are similar denoted by $A \approx B$ if there exists an invertible operator $N \in B(H, K)$ such that $NA = BN$ or equivalently $A = N^{-1}BN$.*

Definition 1.2.30. *Jibril [30] Two operators A and B are said to be almost similar (a.s) denoted by $A \overset{a.s}{\sim} B$ if there exists an invertible operator N such that the following two conditions hold:*
$A^*A = N^{-1}(B^*B)N$ and
$A^* + A = N^{-1}(B^* + B)N$.

Definition 1.2.31. *An operator $N \in B(H, K)$ is quasi-invertible or a quasi-affinity if it is an injective operator with dense range (i.e, $Ker(N) = \{0\}$, and $\overline{RanN} =$*

K;) equivalently $Ker(N) = \{0\}$ and $Ker(N^*) = \{0\}$, thus $N \in B(H,K)$ is quasi-invertible if and only if $N^* \in B(H,K)$ is quasi-invertible.

Definition 1.2.32. *An operator $A \in B(H)$ is a quasi-affine transform of $B \in B(K)$ if there exists a quasi-invertible operator $N \in B(H,K)$ such that $NA = BN$,that is N intertwines A and B. Thus, A is a quasi-affine transform of B if there exists a quasi-invertible operator intertwining A and B.*

Definition 1.2.33. *Nagy and Foias [47] Two operators $A \in B(H)$ and $B \in B(K)$ are quasi-similar denoted by $A \cong B$, if they are quasi-affine transforms of each other, equivalently, if there exists quasi-invertible operators $N \in B(H,K)$ and $M \in B(K,H)$ such that $AN = NB$ and $MB = AM$.*

Definition 1.2.34. *Two operators $A \in B(H)$ and $B \in B(K)$ are unitarily equivalent, denoted by $A \equiv B$ if there exists a unitary operator $U \in B(H,K)$ such that $UA = BU$, that is, $A = U^*BU$.*

Remark 1.2.35. *Two operators are considered the "same" if they are unitarily equivalent since they have the same properties of invertibility, normality and spectral picture (norm, spectrum and spectral radius).*

Definition 1.2.36. *Nzimbi [51] Two operators $A \in B(H)$ and $B \in B(K)$ are said to be metrically equivalent, denoted by, $A \overset{m}{\sim} B$ if $\|Ax\| = \|Bx\|$, that is, $|< Ax, Ax >|^{\frac{1}{2}} = |< Bx, Bx >|^{\frac{1}{2}}$ for all $x \in H$, or $A^*A = B^*B$.*

Definition 1.2.37. *Two operators $S, T \in B(H)$ are said to be nearly-equivalent written $S \overset{n.e}{\sim} T$ if there exists a unitary operator U such that $T^*T = US^*SU^*$*

Remark 1.2.38. *Note that near-equivalence of operators is weaker than metric equivalence of operators.*

Note also that, $S, T \in B(H)$ are nearly-equivalent if S^*S and T^*T are unitarily equivalent.

Definition 1.2.39. *An operator $T \in B(H)$ is said to be decomposable if for every open cover $\{U, V\}$ in $\mathbb{C}$ there are closed $T-$invariant subspaces X and Y such that $H = X + Y, \sigma(T|_X) \subset \overline{U}$ and $\sigma(T|_Y) \subset \overline{V}$.*

Bishop's property (β) [7]: An operator $T \in B(H)$ has Bishop's property (β) provided that for every open subset U of $\mathbb{C}$ and for every sequence of analytic functions $f_n : U \to H$ for which $(T - \lambda I)f_n(\lambda)$ converges uniformly to zero on each compact subset of U, thus $f_n(\lambda) \to 0$ as $n \to \infty$.

The surjectivity spectrum of an operator $T \in B(H)$ is $\sigma_{su}(T) = \{\lambda \in \mathbb{C} : (\lambda I - T)H \neq H\}$.

$E_T(F)$ denotes the algebraic spectral subspace for a subset F in $\mathbb{C}$ defined as $E_T(F) = span\{M \in Lat(T) : \sigma_{su}(T|_M) \subseteq F\}$, which is the largest $T-$ invariant subspace of H for which the surjectivity spectrum of T is a subset of F.

Clearly $E_T(F) = \cap_{\lambda \notin F, n \in \mathbb{N}} (\lambda I - T)^n H$, where this inclusion becomes equality is for instance, T is a normal operator on a Hilbert space H.

The analytic spectrum of $T \in B(H)$ is defined as $H_T(F) = \{x \in H : \sigma_T(x) \subseteq ofF\}$, where $\sigma_T(x)$ is the local spectrum of T at x defined by $\sigma_T(x) = C \backslash \rho_T(x)$, where $\rho_T(x)$ is the local resolvent set of T at the point $x \in H$, defined as the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \to H$ which satisfies $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in U$.

Clearly, $H_T F$ is a hyperinvariant subspace of T.

Property I: A bounded linear operator $T \in B(H)$ is said to have property (I) provided that T has Bishop's property (β) and there exists an integer $p > 0$ such that for a closed subset $F \subset \mathbb{C}$, $H_T(F) = E_T(F) = \cap_{\lambda \in \mathbb{C} \backslash F} (T - \lambda I)^p H$ for all closed sets $F \subseteq \mathbb{C}$, where $H_T(F)$ denotes the analytic spectral subspace and $E_T(F)$ denotes the algebraic spectral subspace of T.

Decomposition property δ: An operator $T \in B(H)$ is said to have the decomposition property δ if given an arbitrary open covering $\{U_1, U_2\}$ of $\mathbb{C}$, every

$x \in H$ has a decomposition $x = u_1 + u_2$ where $u_1, u_2 \in H$ satisfy $u_k = (T - \lambda I)f_k(\lambda)$ for all $\lambda \in \mathbb{C}\backslash\overline{U_k}$ and some analytic function $f_k : \mathbb{C}\backslash\overline{U_k} \to H$ for $k = 1, 2$.

Single Value Extension Property (SVEP) [21]: An operator $T \in B(H)$ has the SVEP at $\lambda_0 \in \mathbb{C}$ if for every open neighbourhood U of λ_0, the only analytic function $f : U \to H$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ is the zero function for all $\lambda \in U$, that is, $f \equiv 0, f$ is identically zero. The operator T is said to have SVEP if it has SVEP for every $\lambda \in \mathbb{C}$

Remark 1.2.40. *Note that if T has Bishop's property, then it has the single value extension property but the converse is not true. SVEP does not imply Bishop's property.*

It has been observed by Albrecht, Eschmeier and Neumann [1] that an operator $T \in B(H)$ is decomposable if and only if it has properties (β) and (δ). Albrecht et all [1] further showed that properties (β) and (δ) are completely dual to each other in the sense that an operator $T \in B(H)$ satisfies (β) if and only if the adjoint operator T^* on the dual space H^* satisfies (δ) and the corresponding statement remains valid if both properties are interchanged.

Invariant subspaces are important in studying the spectral properties and canonical forms of operators. The fact that every operator on a finite dimensional complex vector space is unitarily equivalent to an upper triangular matrix is as a result of the existence of nontrivial invariant subspaces for operators on finite-dimensional spaces.

It has been shown by Brown [8] that every subnormal operator has a nontrivial invariant subspace. Brown et all [10] proved that very contraction whose spectrum contains a unit circle has a nontrivial subspace. Hyponormal operators whose spectrum has a nonempty interior has a nontrivial invariant subspace was proved by Brown [9].

The following are some basic results on invariant and hyperinvaraint subspaces.

Proposition 1.2.41. *Suppose $T \in B(H)$ and let M be a closed subspace of H. Then the following statements are equivalent.*

(i) $T(M) \subseteq M$.

(ii) $T(M^{\perp}) \subseteq M^{\perp}$.

(iii) $[T, P_M] = 0$, where P_M denotes the orthogonal projection of H onto M.

Proof. (ii) $\Rightarrow$ (i) $T(M^{\perp}) \subseteq M^{\perp}$ implies that $T(M) \subseteq M$, since $M = (M^{\perp})^{\perp}$.

(i) $\Rightarrow$ (iii) Suppose $T \in B(H)$ such that $T(M) \subseteq M$ and $x \in H$. Then $T(P_M x - P_M(Tx)) = T(P_M x - P_M(T(P_M x + P_{M^{\perp}} x))) = T(P_M x - P_M(T(P_M x))) = 0$. This shows that $[T, P_M] = 0$ for all $T \in B(H)$. Thus $[T, P_M] = 0$.

(iii) $\Rightarrow$ (ii): Suppose that $[T, P_M] = 0$ and let $m \in M$ and $y \in M^{\perp}$. Then we have $< Ty, m >=< Ty, P_M m >=< P_M Ty, m >=< TP_M y, m >= 0$, and hence $T(M^{\perp}) \subseteq M^{\perp}$. □

The following result shows that the kernel and range of an operator $T \in B(H)$ are invariant subspaces.

Theorem 1.2.42. *If $T \in B(H)$,then the following subspaces are invariant under T :*

(i) $Ker(T) \in Lat(T)$.

(ii) $Ran(T) \in Lat(T)$.

Proof. (i) If $x \in Ker(T)$, then $Tx = 0$, and hence $Tx \in Ker(T)$. Thus $Ker(T)$ is invariant under T.

(ii) Note that, since the operator T is bounded on a Hilbert space H, it is bounded below and hence its range is a closed subspace of H. Thus $T(Ran(T))$ is contained in $Ran(T)$. Let $x \in Ran(T)$, then $Tx \in Ran(T)$. Thus range T is invariant under T. □

Theorem 1.2.43. *Let $T \in B(H)$ be a nonzero operator. If the operator equation $STS = TS$ has a nontrivial solution, then T has a nontrivial invariant subspace.*

Proof. Let $T \in B(H)$ and S be a nontrivial solution in $B(H)$ to the equation $STS = TS$. If $TS = 0$, then $Ker(T)$ and $\overline{Ran(S)}$ are nontrivial invariant subspaces for both T and S. Suppose $TS \neq 0$, since $(I-S)TS = 0$ and $S \neq I$, we have $\{0\} \neq Ran(TS) \subseteq Ker(I-S) \neq H$ so that $Ker(I-S)$ is nontrivial. $Ker(I-S)$ is a subspace of H since $I-S$ lies in $B(H)$. $Ker(I-S)$ is $T-$invariant. If $x \in Ker(I-S)$, then $Sx = x$ so that $Tx = TSx = STSx = STx$, hence $Tx \in Ker(I-S)$. In both cases, T has a nontrivial invariant subspace. $\square$

Theorem 1.2.44. *Let $T \in B(H)$ and M be a subspace of H. Then*
(a) M is $T-$invariant if and only if $M^{\perp}$ is $T^{*}-$ invariant.
(b) M is invariant for every operator that commutes with T if and only if $M^{\perp}$ is invariant for every operator that commutes with T^{*}.

Proof. (a) Let $y \in M^{\perp}$. If $Tx \in M$ whenever $x \in M$, then $< x, T^{*}y >=< Tx, y >= 0$ for every $x \in M$, and therefore $T^{*}y \perp M$ which implies that $T^{*}y$ lies in $M^{\perp}$. Thus $T(M) \subseteq M$ implies $T^{*}(M^{\perp}) \subseteq M^{\perp}$. Conversely, since this holds for every operator in $B[H]$, it follows that $T^{*}(M^{\perp}) \subseteq M^{\perp}$ implies $T(M) \subseteq M$. Hence M is $T-$invariant if and only if $M^{\perp}$ is $T^{*}-$invariant.
(b)Let $\{T\}'$ be the commutant of T. Then $L \in \{T\}'$ if and only if $L^{*} \in \{T^{*}\}'$. Suppose M is invariant for every operator that commutes with T, that is, M is $L-$invariant whenever $L \in \{T\}'$. By part (a), $M^{\perp}$is $L^{*}-$invariant whenever $L^{*} \in \{T^{*}\}'$. Thus $M^{\perp}$ is invariant for every operator that commutes with T^{*}. Dually, if $M^{\perp}$ is invariant for every operator that commutes with T^{*}, then $(M^{\perp)^{\perp}} = \overline{M} = M$ is invariant for every operator that commutes with $(T^{*})^{*} = T$. $\square$

Remark 1.2.45. *Theorem 1.2.43 says that an operator on a Hilbert space has a nontrivial invariant if and only if its adjoint has.*

An operator $T \in B(H)$ has a nontrivial invariant(reducing) subspace if the dimension of H is greater than 1.

Kubrusly [34] On a finite dimensional Hilbert space with dimension greater than one every nonzero operator has an eigenvector and hence a nontrivial invariant subspace

1.3 Series of inclusions of classes of operators

It can be shown that the following inclusions of classes of operators hold:

Normal $\subseteq$ Quasinormal $\subseteq$ Subnormal $\subseteq$ Hyponormal $\subseteq$ Semi-normal

Normal $\subseteq$ Quasinormal $\subseteq$ Hyponormal $\subseteq$ Normaloid

Normal $\subseteq$ Quasinormal $\subseteq$ Hyponormal $\subseteq$ M-hyponormal

Normal $\subseteq$ Quasinormal $\subseteq$ Subnormal $\subseteq$ Hyponormal $\subseteq$ Paranormal

Normal $\subseteq$ Quasinormal $\subseteq$ Subnormal $\subseteq$ Hyponormal $\subseteq$ *-Paranormal

Normal $\subseteq$ Quasinormal $\subseteq$ Subnormal $\subseteq$ Hyponormal $\subseteq$ totally *-Paranormal

1.4 Statement of the problem

In this thesis, we study and investigate the invariant and hyperinvariant subspaces for hyponormal, p-hyponormal, quasinilpotent operators and other classes of operators. We also intend to investigate the structure of invariant, hyperinvariant and reducing subspace lattices for some operators in some equivalence relations.

1.4.1 Objectives of the study

The general objective of this study is to establish operators which have nontrivial invariant and hyperinvariant subspaces.

The specific objectives of this study are:

(i) to investigate the existence of non trivial invariant subspaces for hyponormal, p-hyponormal,paranormal, *-paranormal and quasinilpotent operators.

(ii)to investigate the existence of non trivial hyper-invariant subspaces for the same classes of operators.

(iii) to investigate the structure of invariant, hyperinvariant and reducing subspace lattices of some operators in some equivalence relations.

1.4.2 Significance of the study

The significance of existence of nontrivial invariant subspaces extends beyond its abstract mathematical formulation. Existence of nontrivial invariant subspaces have profound implication for a wide range of mathematical area such as operator theory, functional analysis and linear algebra. In addition, applications in signal processing, quantum mechanics and Control theory depend greatly on understanding the existence and properties of invariant subspaces.

The study of invariant subspaces has profound connections to spectral theory. The spectral properties of an operator are closely related to the existence and structure of invariant subspaces. For instance, operators with rich spectral behavior tend to have more nontrivial invariant subspaces. Spectral theory provides powerful tools for understanding the structure of operators and characterizing their invariant subspaces.

Quantum mechanics is a fundamental theory that describes the behavior of particles at the atomic and subatomic level. Quantities such as position, momentum, energy are represented by linear operators known as observables acting on vector space referred to Hilbert space. The concept of invariant subspaces applies when studying the properties of observables and time-revolution operators. The presence of invariant subspaces corresponds to conserved quantities in quantum systems. The conserved quantities are associated with physical quantities that do not change overtime such as total angular momentum of total energy. The existence of invariant subspaces under the action of the angular momentum operator corresponds to the conservation of angular momentum. This helps in understanding the behavior of particles with rotational symmetry such as atoms and molecules. The concept of spin which is a fundamental property of particles arises from study of invariant subspaces. Spin is an intrinsic form of angular momentum possessed by particles. The existence of invariant subspaces associated with the spin operator leads to the quantization of spin values and the prediction of spin related phenomena such as the Stern-Gerlach experiment. In control theory, invariant subspaces are vital in analyzing the controllability and observability of dynamical systems. Controllability deals with the ability to steer a system from one state to another using control inputs. The existence of invariant subspaces indicates that certain states can not be reached from the given initial conditions as they a confined to these invariant subspaces. The absence of invariant subspaces guarantees that the system is fully controllable enabling one to maneuver the system to any desired state through appropriate control inputs. Observability in control theory deals with the ability to estimate the internal states of a system based on the available measurement outputs. The existence of invariant subspaces for the adjoint of the system operator implies that certain states cannot be uniquely determined from the available measurements while absence of nontrivial invariant subspaces quarantees that the system is fully observable hence one can estimate all the states using available measurement outputs. Lattices are applicable in computer science as they are algorithmic tools to solving problems in crytography and cryptanalysis.

CHAPTER 2

LITERATURE REVIEW

Direct sum decomposition, $T = T_1 \oplus T_2$, of an operator T acting on an Hilbert space H, follows from the well known fact that, properties satisfied by the direct summands, that is, T_1 and T_2, are always the same properties satisfied by the direct sum, that is, T. Thus, studying the behaviour of T gets relaxed to studying the behaviour of the parts T_1 and T_2, since these parts are known to have a simpler structure than 'their mother operator', T. Unfortunately, for any T to be guaranteed of such a direct sum decomposition, there must exists atleast one non-trivial reducing subspace in H, for T. From the invariant problem for linear operators, (that is, the question of coming up with a non-trivial invariant subspace for every operator on H), one expects to encounter some operators on H, (especially, when the dimension of H is neither 1 nor finite), which cant be expressed as direct sum decompositions. Diagonal or equivalently, diagonalizable operators are easy to study since they have simple structures. Linear operators which are not diagonalizable, might atleast be expressed as direct sum decompositions of probably diagonalizable operators. Unfortunately, linear operators acting on Hilbert spaces are neither diagonalizable, nor reducible in general. However, every normal operator is either diagonalizable or similar to a known diagonalizable operator. On the other edge, every reducible operator can be expressed as a direct sum decomposition of a normal and a completely non-normal operator. Generally, n-Power normal, n-power quasinormal and w-hyponormal operators are not only non-normal, but also irreducible.

The question on whether every operator has a non trivial invariant and hyperinvariant subspaces has its origin approximately in 1935 when according to Aronszajn and Smith [4], J. von Neumann(unpublished) in 1950 proved that every compact operator

T on a Hilbert space has a non trivial invariant and hyperinveraint subspaces, that is, closed subspaces for all operators S such that $ST = TS$. As a consequence of the spectral theorem, all normal operators have nontrivial invariant and hyperinvariant subspaces unless they are equal to the scalar operator since the scalar operator commutes with all operators.

Aronszajn and Smith proved in [4] that every compact operator on a Hilbert space has a nontrivial invaraint and hyperinvariant subspace. Later in 1966, Bernstein and Robinson [6] proved that a polynomially compact operator $p(T)$ on a Hilbert space has an invariant subspace. Halmos [27] in 1966, gave a shorter proof of the same result by a similar method avoiding the nonstandard analysis.

In 1973, Lomonosov [39] proved that every operator T commuting with a nonzero compact operator has nontrivial invariant and hyperinvariant subspaces unless it is a scalar multiple of the identity operator. More generally, if T commutes with an operator S and S commutes with a nonzero compact operator, then T has a nontrivial invariant subspace. In 1968, Arveson and Feldman [5] proved that given a Hilbert space H and $T \in B(H)$, satisfying $\|TP_n - P_nTP_n\| \to 0$ for some sequence (P_n) of orthogonal projection operators which converge strongly to the identity operator (such operators are called quasitriangular) and assuming that the norm closed algebra generated by T and I contains a non-zero compact operator, then T has a non trivial invariant subspace.

Hoover [29] in 1972 studied hyperinvariant subspaces and proved the result that if S and T are quasisimilar operators acting on the Hilbert spaces H and K, respectively, and if S has a hyperinvariant subspace, then so does T. If in addition,if S is normal, then the lattice of hyperinvariant subspaces for T contains a sub-lattice which is lattice isomorphic to the lattice of spectral projection for S.

In 1973, Pearcy and Salinas [53] proved that if $T \in B(H)$ is a quasitriangular operator and $R(T)$ (the norm closure of the rational functions of T) contains a non-zero compact operator, then there exists a non-trivial subspace invariant under all operators in $R(T)$.

In 1950, Halmos asked whether every subnormal operator has a nontrivial invariant subspace. The problem was solved by Scott Brown [8] in 1978 where he proved that every subnormal operator on a Hilbert space H has a nontrivial invariant subspace, equivalently, if M is an infinite dimensional invariant subspace for a normal operator N, then M contains a proper subspace other that $\{0\}$ which is $N-$invariant. Brown [9] extended this result to hyponormal operators with thick spectra in 1987 . Brown [9] showed that if T is a hyponormal operator on H such that $\sigma(T)$ has a non-empty interior, then T has a non-trivial invariant subspace, that is, every hyponormal operator whose spectrum has nonempty interior has a nontrivial invariant subspace. It is not known whether every hyponormal operator has nontrivial invariant subspace. This study will focus on the class of hyponormal operator among other classes.

Results for hyperinvariance have been studied by C. S. Kubrusly [34] in 1997 and has shown that similarity preserves nontrivial invariant subspaces while quasisimilarity preserves hyperinvariant subspaces.In 1984, Putinar [55] proved that hyponormal operators satisfy the Bishop's property. In 1984, C. J. Read [57] proved that there exists quasi-nilpotent operators (and hence decomposable) on Banach spaces without nontrivial closed invariant subspaces. Herrero [28] in 1977, proved that the structure of the hyperlattice of an operator is not preserved under quasisimilarity. In 1988, Brown et al [10] proved that every contraction operator on a Hilbert space with spectrum containing the unit circle has a nontrivial invariant subspace. Prunaru [54] in 1997,proved that, a bounded operator T on a Hilbert space H such that $p(T)^*p(T)-p(T)p(T)^* \geq 0$ for every polynomial p has a nontrivial invariant subspace. In 2004, Ambrozie and Muller [2] proved that every polynomially bounded operator T on a Hilbert space such that the spectrum of T contains the unit circle has a nontrivial invariant closed subspace. Gamal [24] in 2004, proved that the lattices of invariant subspaces remain isomorphic under quasiaffine transforms. In 2005, Liu [38] proved that the converse proposition of the famous Lomonosov Theorem is true, and obtained some new necessary and sufficient condition for the invariant closed subspace problem. In 2007, Foias et al [23] showed that a the class of subnormal operators has a nontrivial hyperinavriant subspace. Nzimbi [49] in 2018 showed that the hyperlattice of a unitary operator is equal to its reducing subspaces. Rashid [56] in 2019 showed that if $T \in B(H)$ is w-hyponormal and has decomposition property δ, then T has a non trivial invariant closed linear subspace. In the same paper it has also been shown that such an operator with rich spectrum has a nontrivial invariant subspace.

The invariant subspace problem can as well be solved by investigating the structure of the spectrum of an operator in question since it is shown by Kaplansky, [31] in 1953 that the spectral space of any class of operators includes properly, the spectral space of all operators from its sub classes. Dunford, [20] in 1954 showed that the spectrum of a self-adjoint operator lies along the real line, that of a unitary lies on the unit circle, that of a projection consists of the points 0 and 1, and that of a normal operator can be any compact set in the complex plane. Thus knowledge of the location of spectrum of an operator can show whether the operator has nontrivial invariant subspace or not. The spectrum of higher classes of operators is not trivial but once some conditions are imposed it can shed light about the spectrum. Derming and Juk [16] showed that if an operator is similar to its adjoint such that the zero is not in the numerical range of the intertwining operator, then the spectrum of this operator is contained in the real line.
The set of all bounded linear operators T on H such that T^*T and T^*+T commute is denoted by θ. The operators in class θ have properties similar to those of hyponormal operators. It was conjectured that every operator in class θ is subnormal. Campbell

[13] showed that there is a nonhyponormal operator in class θ which means that this class of operators are different from the class of hyponormal and that of subnormal operators. However, if an operator in the class θ is hyponormal, then it is subnormal [11]. In this thesis we investigate a subclass of class θ which consists of operators T such that the spectrum of $T, \sigma(T)$ does not intersect with the real line, that is, $\sigma(T) \cap \mathbb{R} \neq \phi$. The complete structure of these operators is developed in [Campbell and Gellar [12]]. Stampfli [60] proved that an operator T whose spectrum is a $k - spectral$ set, that is, $\|f(T)\| \leq k \|f\|_{\sigma(T)}$ for every rational function f with pole off $\sigma(T)$ has a nontrivial invariant subspace. Brown extended the result of subnormal to hyponormal operators with thick spectra in 1987. He showed that if T is a hyponormal operator on H such that $\sigma(T)$ has a non-empty interior, then T has a non-trivial invariant subspace. In 1990, Eschmeier and Prunaru [22] showed that if an operator on a Banach space satisfies Bishop's property and has an open set V whose intersection with the spectrum is dominating for V has a nontrivial invariant subspace. Kim [33], proved that every operator in the class θ whose outer boundary of its spectrum is the outer boundary of a Caratheodory domain has a nontrivial invariant subspace. He went ahead and gave a condition on spectra of operators in the class θ which give information about invariant subspaces.

2.1 Knowledge Gap

From the extant literature, it has been shown that normal and subnormal operators have nontrivial invariant subspaces. We extend this study to the class of quasinormal operators, nilpotent operators, quasinilpotent operators, hyponormal operators, p-hyponormal operators, paranormal operators on whether these operators have nontrivial invariant and hyperinvariant subspaces. Invariant and hyperinvariant subspace lattices of some of these operators is investigated. The structure of these subspace lattices for these operators in some equivalence relations is also examined.

CHAPTER 3

INVARIANT AND HYPERINVARIANT SUBSPACES OF SOME OPERATORS

In this chapter we investigate operators which have nontrivial invariant and hyperinvariant subspaces. Recall that a subspace M of a Hilbert space H is an invariant subspace for an operator $T \in B(H)$ if $T(M) \subseteq M$. If $T \in B(H)$, we denote by $\{T\}'$ the commutant of T, that is, $\{T\}' = \{S \in B(H) : ST = TS\}$. A subspace $M \subset H$ is said to be nontrivial hyperinvariant subspace (n.h.s) for a fixed operator $T \in B(H)$ if $0 \neq M \neq H$ and $SM \subseteq M$ for each $S \in \{T\}'$, that is, it is invariant under every operator commuting with T. It has been shown by Neumann, [48] that normal operators have nontrivial invariant subspaces. This result was extended to the class of subnormal operators by Brown [8]. We will extend this study to quasinormal, hyponormal, quasinilpotent and paranormal operators among others classes of operators.

3.1 On invariant subspaces of quasinormal operators

Recall that an operator T is quasinormal if T and T^*T commute, that is, if $(T^*T - TT^*)\,T = 0$. Note that every normal operator is quasinormal and so is every isometry.

We show that, every quasinormal operator $T \in B(H)$ has a nontrivial invariant subspace. First we need the following results:

Proposition 3.1.1. *Kubrusly, [34] Let T and L be nonzero operators on a Hilbert space H. If $LT = 0$, then $Ker(L)$ and $\overline{Ran(T)}$ are nontrivial invariant subspaces for both T and L.*

This leads to the following corollary.

Corollary 3.1.2. *Every nilpotent operator has a nontrivial invariant subspace.*

Proof. Recall that, an operator is nilpotent if $T^n = 0$. Thus $T^n = T(T^{n-1})$ which can be written as a product of two operators and thus by Proposition 3.1.1, $Ker(T)$ and $\overline{Ran(T^{n-1})}$ are nontrivial invariant subspaces. $\square$

The following is an example of a nilpotent operator. We show that $Ker(T)$ and $\overline{Ran(T^2)}$ are nontrivial invariant subspaces of T.

Example 3.1.3. *Let* $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. *Then* $T^3 = 0$. *Thus* T *is a nilpotent operator.* $T^3 = T^2 T$, *where* $T^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $Ker(T) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle = \overline{Ran(T^2)}$ *is a nontrivial invariant subspaces of* T *since* $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in span \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Remark 3.1.4. *Note that, if an operator* $T \in B(H)$ *and its adjoint* T^* *satisfy the Single Value Extension Property (SVEP), then* T *is decomposable and has nontrivial invariant subspaces.*

We give an alternative proof to show that nilpotent operators have nontrivial invariant subspaces.

We first need the following result:

Theorem 3.1.5. *Nzimbi and Luketero [50] Let* H *and* K *be Hilbert spaces. If* $A \in B(H)$ *has the SVEP at* $\lambda_0 \in \mathbb{C}$ *and* $B \in B(K)$ *is a quasiaffine transform of* A, *then* B *has the SVEP at* λ_0.

Remark 3.1.6. *Theorem 3.1.5 shows that SVEP is stable under quasiaffine transformation.*

This leads to the following result:

Theorem 3.1.7. *Let H and K be Hilbert spaces. Suppose $A \in B(H)$ and $B \in B(H)$ are quasisimilar. Then A has SVEP at λ_0 if and only if B has SVEP at λ_0.*

Proof. Suppose that $X \in B(H, K)$ is a quasiaffinity such that $AX = XB$, let $f : U \to K$ be an analytic function defined on an open neighbourhood U of λ_0 such that $(\lambda I - B) f(\lambda) = 0$ for all $\lambda \in U$. Then $X(\lambda I - B) f(\lambda) = (\lambda I - A) Xf(\lambda) = 0$. Using the SVEP of A at λ_0, we conclude $f(\lambda) = 0$ for all $\lambda \in U$. Thus B has the SVEP. Conversely let $Y \in B(K, H)$ be a quasiaffinity such that $YA = BY$, let $g : U \to H$ be an analytic function defined on an open neighbourhood U of μ_0 such that $(\mu I - A) f(\mu) = 0$ for all $\mu \in U$. Then $Y(\mu I - A) g(\mu) = (\mu I - A) Yg(\mu) = 0$. Using the SVEP of B at μ_0, we conclude $g(\mu) = 0$ for all $\mu \in U$. Thus A has the SVEP. □

Remark 3.1.8. *From Theorem 3.1.7 we notice that, SVEP is stable under quasimilarity.*

Theorem 3.1.9. *Apostol, [3] Let $T \in B(H)$. If T is a nilpotent operator, then T and T^* are quasisimilar.*

We deduce the following result.

Theorem 3.1.10. *Let $T \in B(H)$. If T is a nilpotent operator, the T^* has SVEP.*

Proof. Since T is nilpotent implies that T has SVEP. Also T and T^* are quasisimilar by Theorem 3.1.9, we conclude T^* has SVEP by Theorem 3.1.7. □

Theorem 3.1.11. *Let $T \in B(H)$ be a nilpotent operator, then T has nontrivial invariant subspace.*

Proof. The proof follows from the fact that an operator T is decomposable if it satisfies the SVEP as well as its adjoint T^*. By Theorem 3.1.10, we notice that a nilpotent operator T and its adjoint T^* satisfy the SVEP and thus is decomposable and hence has nontrivial invariant subspace by Remark 3.1.4. $\square$

Example 3.1.12. *Let* $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. *Then* $T^3 = 0$. *Thus* T *is a nilpotent operator.* $T^* = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ *where the quasiaffinities are* $X = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *and* $Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, *that is,* $XT^* = TX$ *and* $T^*Y = YT$, *hence* T *and* T^* *are quasimilar. Note that* $M = span \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ *is a nontrivial invariant subspaces of* T, *and* T^* *since* $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in span \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ *and* $T^* = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in span \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

We extend this result to the class of quasinilpotent operators. Note that every nilpotent operator is quasinilpotent.

Corollary 3.1.13. *Let* $T \in B(H)$ *be quasinilpotent, If* T *and* T^* *have SVEP, Then* T *has nontrivial invariant subspaces.*

Proof. The proof follows from the fact that an operator T is decomposable if it has the SVEP together its adjoint T^*. Thus T has nontrivial invariant subspace by Remark 3.1.4. $\square$

The following result by Neumann, [48] shows that quasinormal operators have nontrivial invariant subspaces. We need the following result.

Theorem 3.1.14. *Neumann, [48] Normal operators have nontrivial invariant subspaces.*

We show that quasinormal operators have nontrivial invariant subspaces.

Corollary 3.1.15. *If $T \in B(H)$ is quasinormal, then it has a nontrivial invariant subspace.*

Proof. If T is quasinormal, then $(T^*T - TT^*)T = 0$. If $T \neq 0$ is quasinormal, then either $T^*T - TT^* = 0$ or $T^*T - TT^* \neq 0$. If $T^*T - TT^* = 0$ then T is a normal operator and hence by Theorem 3.1.14 T has a nontrivial invariant subspace. If $T \neq 0$ and $T^*T - TT^* \neq 0$ but $(T^*T - TT^*)T = 0$ then using Proposition 3.1.1, then $Ker(T^*T - TT^*)$ and $\overline{Ran(T)}$ are both nontrivial $T-$ invariant subspaces. $\square$

Remark 3.1.16. *Every quasinormal operator in particular, every isometry has a nontrivial invariant subspace.*

Example 3.1.17. *Let* $T = \begin{bmatrix} 0 & 1 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, *then* $T^* = \begin{bmatrix} 0 & -i & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. *Thus* $T(T^*T) = \begin{bmatrix} 0 & 1 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (T^*T)T$. *Hence T is quasinormal and* $M = span \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ *is a nontrivial invariant subspace of T since and* $TM = \begin{bmatrix} 0 & 1 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in span \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = M$

Example 3.1.18. *Define $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $T(x) = T(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$ for $x = (x_1, x_2, x_3, \dots) \in \ell^2(\mathbb{N})$. Then $T^*(x) = T^*(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$ and $T^*T(x) = T^*(0, x_1, x_2, x_3, \dots) = (x_1, x_2, x_3, \dots) = x$ Thus $T^*T = I$ hence T is an isometry. Using Remark 3.1.16 we conclude T has nontrivial invariant subspaces.*

3.2 Invariant Subspaces of Hyponormal and other classes of Operators in Hilbert Space

The class of hyponormal operators contains that class of subnormal operators. Thus if $T \in B(H)$ is subnormal then T is hyponormal but hyponormal operators are not necessarily subnormal. Thus subnormality is stronger than hyponormality. The following result by Brown [8], shows that every subnormal operator has a nontrivial invariant subspace.

Proposition 3.2.1. *Brown, [8] Every subnormal operator has a non-trivial invariant subspace.*

In this section we extend our study to the class of hyponormal operators and other classes of operators. We first show that hyponormal operators satisfy SVEP. We first need the following result.

Proposition 3.2.2. *Let $T \in B(H)$. Then the following hold:*
(i) $Tx = \lambda x$ implies $T^*x = \overline{\lambda}x$.
(ii) if $Tx = \lambda x$ and $Ty = \mu y$ for $\lambda \neq \mu$ then $\langle x, y\rangle = 0$.

Proof. The first property holds since $T - \lambda I$ is also hyponormal. For $\lambda \neq \mu$ we have $\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle = \langle x, \overline{\mu}y\rangle = \mu\langle x, y\rangle$. Since $\lambda \neq \mu$ then $\langle x, y\rangle = 0$. □

The following result shows that hyponormal operators satisfy the single value extension property (SVEP).

Theorem 3.2.3. *Let $T \in B(H)$ be a hyponormal operator, Then T satisfies single value extension property.*

Proof. Let f be a vector valued analytic function such that $(T - \lambda I)f(\lambda) = 0$ for every $\lambda \in \mathbb{C}$.
Then $Tf(\lambda) = \lambda f(\lambda)$ for every $\lambda \in \mathbb{C}$.
Fix $\lambda \in \mathbb{C}$, then for every $\mu \neq \lambda \in \mathbb{C}$ we have by property (ii) of Proposition 3.2.2 $\langle f(\lambda), f(\mu)\rangle = 0$.
By Pythagorean Theorem if follows that
$\|f(\lambda) - f(\mu)\|^2 = \|f(\lambda)\|^2 + \|f(\mu)\|^2$. Letting $\mu \longrightarrow \lambda$ gives $f(\lambda) = 0$. Therefore f is identically zero since λ is an arbitrary element in $\mathbb{C}$. □

Remark 3.2.4. *The unilateral forward shift operator on $\ell^2(\mathbb{N})$ is hyponormal hence satitisfies SVEP. Its adjoint the unilateral backward shift operator is cohyponormal and does not satisfy SVEP since it is not decomposible by Lange [35]. We have seen that for an operator to be decomposable, the it should satisfy SVEP together with its adjoint. This shows that generally hyponormal operators do not have nontrivial invariant subspace since there exists adjoints of hyponormal operators which do not satisfy SVEP.*

Thus we will investigate conditions in which hyponormal, p-hyponormal,log-hyponormal, p-quasihyponormal and other classes of operators have nontrivial invariant subspaces. Since normal operators have nontrivial invariant subspaces, we investigate conditions in which these operators are normal. We first need the following result: Recall that, an operator $Y \in B[H, K]$ intertwines $T \in B[H]$ to $L \in B[K]$ if $YT = LY$. In this case we say that T is intertwined to L.

Remark 3.2.5. *T is a quasiaffine transform of L if there exists quasi invertible operator intertwining T to L. T is said to be densely intertwined to L if there exists an operator with dense range intertwining T to L. In the next result we show a sufficient* *condition for transferring nontrivial invariant subspaces from L to T whenever T is densely intertwined to L.*

Lemma 3.2.6. *[34] Let $T \in B[H], L \in B[K]$ and $Y \in B[H,K]$ be such that $YT = LY$. Suppose $M \subset K$ is a nontrivial invariant subspace for L. If $\overline{Ran(Y)} = K$ and $Ran(Y) \cap M \neq \{0\}$, then the inverse image of M under Y, that is, $Y^{-1}(M)$, is a nontrivial invariant subspace for T.*

Corollary 3.2.7. *If the intertwining operator is surjective, that is $YT = LY$ and $Ran(Y) = K$ then $Y^{-1}(M)$ is a nontrivial invariant subspace for T whenever M is a nontrivial invariant subspace for L.*

Theorem 3.2.8. *Similarity of operators preserves non-trivial invariant subspaces.*

Proof. Suppose T and S are similar. Then there exists an invertible operator $Y \in B(H)$ such that $T = Y^{-1}SY$. If M is a non-trivial invariant subspace for T, then $SYM = YTM \subset YM$. Since M is non-trivial and Y is invertible, then YM is a non-trivial invariant subspace for S. Thus M is $T-$ invariant if and only if M is $S-$ invariant. □

This leads to the following result.

Corollary 3.2.9. *If two operators are similar and if one of them has a nontrivial invariant subspace, then so has the other.*

The following results hold since similarity preserves nontrivial invariant subspaces.

Theorem 3.2.10. *Stampfli, [59] Let T be a hyponormal operator. If T is similar to a normal operator, then T is normal.*

Using Theorem 3.2.10 we deduce the following result.

Corollary 3.2.11. *Let T be a hyponormal operator. If T is similar to a normal operator, then T has a nontrivial invariant subspace.*

Proof. Theorem 3.2.10 shows that if a hyponormal operator T is similar to a normal operator then T is normal. The rest of the proof follows from Theorem 3.1.14 which shows that every normal operator has a nontrivial invariant subspace. □

Theorem 3.2.10 was extended by Duggal,[17] to the class of p-hyponormal operators as follows.

Lemma 3.2.12. *Duggal, [[17], Proposition 4] Let T_1 be a p-hyponormal operator and T_2 be normal. If there exists an operator Y with dense range such that, $T_1Y = YT_2$, then T_1 is normal .*

Using Lemma 3.2.12 we deduce the following result.

Corollary 3.2.13. *Let T_1 be a p-hyponormal operator and T_2 be normal. If there exists an operator Y with dense range such that, $T_1Y = YT_2$, then T_1 is normal and hence T_1 has a non-trivial invariant subspace.*

Proof. Lemma 3.2.12 shows that if there exists an operator with dense range intertwining a p-hyponormal operator to a normal operator, then the p-hyponormal operator is normal hence has nontrivial invariant subspace since normal operators have nontrivial invariant subspaces by Theorem 3.1.14. □

Masuo, [41] extended Lemma 3.2.12 to the class of log-hyponormal operators and gave the following result:

Lemma 3.2.14. *Masuo,[[41], Theorem 1.5] Let T_1 be a log-hyponormal operator and T_2 be normal. If there exists an operator Y with dense range such that, $T_1Y = YT_2$, then T_1 is normal and hence T_1 has a non-trivial invariant subspace.*

Using Lemma 3.2.14 we deduce the following result.

Corollary 3.2.15. *Let T_1 be a log-hyponormal operator and T_2 be normal. If there exists an operator Y with dense range such that, $T_1Y = YT_2$, then T_1 has a non-trivial invariant subspace.*

Proof. The proof follows from Corollary 3.2.13. □

Duggal, [18] extended Lemma 3.2.14 to the class of p-quasihyponormal operators as follows:

Lemma 3.2.16. *Duggal, [[18], Lemma 2.4] Let T_1 be an invertible p-quasihyponormal operator and T_2 be normal. If there exists an operator Y with dense range such that, $T_1 Y = Y T_2$, then T_1 is normal.*

Lemma 3.2.16 leads to the following result.

Corollary 3.2.17. *Let T_1 be an invertible p-quasihyponormal operator and T_2 be normal. If there exists an operator Y with dense range such that, $T_1 Y = Y T_2$, then T_1 has a non-trivial invariant subspace.*

Proof. The proof follows from Corollary 3.2.13. □

Duggal, [17] extended Lemma 3.2.16 to the class of w-hyponormal operators as follows:

Lemma 3.2.18. *Duggal, [[17], Lemma 2.5] Let T_1 be a w-hyponormal operator and T_2 be normal. If there exists an operator Y with dense range such that, $T_1 Y = Y T_2$, then T_1 is normal and hence T_1 has a non-trivial invariant subspace.*

Corollary 3.2.19. *Let T_1 be a w-hyponormal operator and T_2 be normal. If there exists an operator Y with dense range such that, $T_1 Y = Y T_2$, then T_1 has a non-trivial invariant subspace.*

Proof. The proof follows from Corollary 3.2.13. □

Proposition 3.2.20. *Sheth, [58] If $T^* \in B(H)$ is a hyponormal operator and $S^{-1}TS = T^*$ for an operator S, where $0 \notin \overline{W(S)}$, then T is self-adjoint.*

We deduce the following result from Proposition 3.2.20.

Corollary 3.2.21. *If $T^* \in B(H)$ is a hyponormal operator and $S^{-1}TS = T^*$ for an operator S, where $0 \notin \overline{W(S)}$, then T has nontrivial invariant subspace.*

Proof. From Proposition 3.2.20 we conclude that T is normal, since every self-adjoint operator is normal hence has nontrivial invariant subspaces by Theorem 3.1.14. □

Remark 3.2.22. *Corollary 3.2.21 shows that a hyponormal operator which is similar to its adjoint is normal and hence has a nontrivial invariant subspace.*

Proposition 3.2.20 was extended to the class of p-hyponormal operators by Hyoun Kim, [32] as follows.

Proposition 3.2.23. *Kim, [32] If T or T^* is p-hyponormal, and S is an operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint.*

We deduce the following result from Proposition 3.2.23.

Corollary 3.2.24. *If T or T^* is p-hyponormal, and S is an operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T has a nontrivial invariant subspace.*

Proof. Proof follows from Corollary 3.2.21 and the fact that every self-adjoint operator is normal. □

Note that for self-adjoint operators their spectrum lies on the real line. The location of spectrum from higher classes is not trivial. The following result sheds more light on the spectrum of a hyponormal or p-hyponormal after imposing some extra condition.

Lemma 3.2.25. *Derming, [16] If T is a hyponormal or a p-hyponormal operator and $T = ST^*S^{-1}$ for any operator S, where $0 \notin \overline{W(S)}$, $\sigma(T) \subset \mathbb{R}$.*

This result was generalized for any operator $T \in B(H)$ by Xia, [61] as follows.

Lemma 3.2.26. *Derming, [16] If $T \in B(H)$ be any operator such that $T = ST^*S^{-1}$ for any operator S, where $0 \notin \overline{W(S)}$, $\sigma(T) \subset \mathbb{R}$.*

This leads to the following corollary.

Corollary 3.2.27. *If $T \in B(H)$ be any operator such that $T = ST^*S^{-1}$ for any operator S, where $0 \notin \overline{W(S)}$, $\sigma(T) \subset \mathbb{R}$ and hence T has nontrivial invariant subspace.*

Remark 3.2.28. *Note that Lemma 3.2.26 holds for all operators whose spectrum lies on the real line. Corollary 3.2.27, in other words states that if an operator is similar to its adjoint, then it has nontrivial invariant subspace.*

The following results hold since unitarily equivalence implies similarity and similarity preserves nontrivial invariant subspace.

Lemma 3.2.29. *Herrero, [[28], Corollary 7] Let $T, S \in B(H)$ be p-hyponormal operator. If $TX = XS$, where $X \in B(H)$ is injective with dense range, then T is normal unitarily equivalent to S..*

This result was extended to the class of $k-$quasihyponormal operators by Garcia, [25] as follows.

Theorem 3.2.30. *Garcia, [25] If $T \in B(H)$ is $k-$quasihyponormal operator and $S \in B(H)$ is normal operator for which $TY = YS$ where $Y \in B(H)$ is injective with dense range, then T is a normal operator unitarily equivalent to S. .*

Theorem 3.2.30 leads to the following result.

Corollary 3.2.31. *If $T \in B(H)$ is $k-$quasihyponormal operator and $S \in B(H)$ is normal operator for which $TY = YS$ where $Y \in B(H)$ is injective with dense range, then T is a normal operator unitarily equivalent to S and hence T has a non-trivial invariant subspace.*

An operators is decomposable if and only if it has decomposition property δ and Bishop's property β. This leads to the following result.

Theorem 3.2.32. *Albrecht, [1] Let $T \in B(H)$ be a bounded linear operator on a Hilbert space H of dimension greater than 1. If $T \in B(H)$ has property (β) and decomposition property (δ), then T has a non-trivial invariant closed linear subspace.*

Laursen, [[37],Proposition 1.5.11] proved that Theorem 3.2.32 holds for subscalar operators and came up with the following result.

Corollary 3.2.33. *Every subscalar operator with property (δ) on a Hilbert space of dimension greater than 1 has a non-trivial invariant closed linear subspace.*

Remark 3.2.34. *Corollary 3.2.33 applies to all k-quasihyponormal operators, isometries, M-hyponormal, w-hyponormal, log-hyponormal or hyponormal operators with property (δ) since these operators are subscalar and satisfy SVEP.*

Putinar, [55] proved every hyponormal operator satisfies property β in the form of a corollary.

Corollary 3.2.35. *Putinar, [55] Every hyponormal operator has property (β).*

Note that Bishop's property implies SVEP but the converse is not true. This leads to the following result.

Corollary 3.2.36. *Every hyponormal operator with property (δ) on a Hilbert space of dimension greater that 1 has a nontrivial invariant closed linear subspace.*

Proof. Corollary 3.2.35 shows that T has property (β). Suppose that $T \in B(H)$ has both property (β) and decomposition property (δ) on a Hilbert space H of dimension greater than 1. Then T is decomposable. We show that if $\sigma(T)$ contains at least two points, then T has a non-trivial invariant closed linear subspace. Since T is decomposable, it follows that T has a nontrivial invariant closed linear subspace. It remains to consider the case of operator $T \in B(H)$ such that H is at least two-dimensional and $\sigma(T)$ is a singleton. It follows that $T = \lambda I + N$ for some

$\lambda \in \mathbb{C}$ and some nilpotent operator $N \in B(H)$. Let $p \in \mathbb{Z}$ be the smallest integer for which $N^p = 0$, and choose an $x \in H$ for which $N^{p-1}x \neq 0$. The linear subspace generated by $N^{p-1}x$ is a one-dimensional $T-$invariant linear subspace of H. □

This result can be extended to the class of paranormal, *-paranormal and totally *-paranormal operators as follows.
Rashid in 2019 [56] showed that a paranormal operator satisfies Bishop's property. This leads to the following result:

Corollary 3.2.37. *Muteti, [43] Every paranormal operator with property (δ) on a Hilbert space of dimension greater that 1 has a nontrivial invariant closed linear subspace.*

Proof. Proof follows from Corollary 3.2.36 since paranormal operator satisfy (β) as shown by Rashid [56] in 2019. □

Proposition 3.2.38. *Duggal, [19] *-paranormal operators satisfy property* (β).

This leads to the following results.

Corollary 3.2.39. *Every *-paranormal operator with property (δ) on a Hilbert space of dimension greater that 1 has a nontrivial invariant closed linear subspace.*

Proof. Proof follows from Corollary 3.2.36 since *-paranormal operators satisfy property (β) from Proposition 3.2.38. □

Recall that an operator $T \in B(H)$ has finite ascent if $Ker(T^n) = Ker(T^{n+1})$.

Proposition 3.2.40. *Laursen, [36] Any operator T with finite ascent has SVEP.*

Lemma 3.2.41. *If T is totally *-paranormal, then $ker(T) = ker(T^2)$.*

Remark 3.2.42. *Lemma 3.2.42 shows that an operator T which is totally *-paranormal has finite ascent and hence satisfies SVEP which leads to the following result.*

Proposition 3.2.43. *Every totally *-paranormal operator has SVEP.*

Proof. The proof follows from Lemma 3.2.42 since $ker(T) = ker(T^2)$ implies that the operator T has finite ascent. The rest of the proof follows from Proposition 3.2.40. $\square$

This leads to the following result.

Corollary 3.2.44. *Every totally *- paranormal operator with property (δ) on a Hilbert space of dimension greater that 1 has a nontrivial invariant closed linear subspace.*

Proof. Proposition 3.2.43 shows that a totally *- paranormal operator T satisfies SVEP and the decomposition property ensure its adjoint operator T^* has SVEP hence the operator T is decomposable. The rest of the proof follows from the fact decomposable operators have nontrivial invariant subspaces. $\square$

Remark 3.2.45. *An operator is decomposable if it has both the Bishop's property (β) (or Dunford's property(C)) and decomposition property (δ). A decomposable operator has nontrivial invariant subspaces and vice versa.*

The following result gives a condition in which a hyponormal operator is normal. Recall that, an operator $T \in B(H)$ is a complex symmetric operator if there exists a conjugation S such that $STS = T^*$ where by a conjugation we mean an isometry or an involution.

Theorem 3.2.46. *Muteti, [46] An operator $T \in B(H)$ which is both hyponormal and complex symmetric is normal.*

Proof. Let T be a hyponormal operator ,$\|Tx\| \geq \|T^*x\|$ for all $x \in H$. Since T is complex symmetric, there is a conjugation S so that $T = ST^*S$, that is $ST = T^*S$.

Thus

$$\|Tx\| = \|ST^*Sx\| = \|T^*Sx\| \leq \|TSx\| = \|ST^*x\| = \|T^*x\|.$$

Thus $\|Tx\| = \|T^*x\|$. Hence T is normal. $\square$

The following result shows that a hyponormal operator has invariant subspaces if it is complex symmetric.

Theorem 3.2.47. *Let $T \in B(H)$ be a hyponormal operator which is complex symmetric. Then T has an invariant subspace.*

Proof. From Theorem 3.2.46, we observe that T is normal. Using Theorem 3.1.14 we get the required result. $\square$

Theorem 3.2.48. *If T^2 is normal, then T is both binormal and complex symmetric.*

Proof. It T^2 is normal, then T is binormal by[[14], Theorem 1] and complex symmetric by [[26], Corollary 3]. $\square$

In [11], Campbell remarks that a hyponormal operator is normal if and only if its square is normal. We relax that hypothesis from normality to paranormality.

Lemma 3.2.49 ([11], Theorem 4)**.** *A binormal operator is hyponormal if and only if it is paranormal.*

Lemma 3.2.50. *Muteti, [46] A binormal, complex symmetric operator T is normal if and only if it is paranormal.*

Proof. If T is paranormal and binormal, it is hyponormal by Lemma 3.2.48. $\square$

If T is hyponormal and complex symmetric, it is normal by Theorem 3.2.46.

Theorem 3.2.51. *Muteti, [46] A paranormal operator T is normal if and only if T^2 is normal.*

Proof. Suppose T^2 is normal, by Theorem 3.2.48, T is binormal and complex symmetric. Since T is paranormal, it is hyponormal by Lemma 3.2.49. Then T is normal by Lemma 3.2.50. □

Proposition 3.2.52. *Muteti, [46] A paranormal operator T has a nontrivial invariant subspace if its square is normal.*

Proof. By Theorem 3.2.51 T is normal and the rest of the proof follows from Theorem 3.1.14. □

3.3 On Invariant Subspaces of Class θ operators

Let K be a compact subset of $\mathbb{C}$. The polynomially convex hull ηK of K is defined as $\eta K = \{\lambda \in \mathbf{C} : |p(\lambda)| \leq \|p\|_K\}$ for all polynomial p.

The outer boundary of K means the boundary of the polynomially convex hull ηK of K. In what follows,$C(K)$ denotes the Banach algebra consisting of all continuous functions on K with the supremum norm, $P(K)$ the uniform closure of all polynomials in $C(K)$ and $R(K)$ the uniform closure of all rational functions with poles off K in $C(K)$. A caratheodory domain is an open connected subset of $\mathbb{C}$ whose boundary coincide with its outer boundary.

Corollary 3.3.1. *Every operator T in the class θ has Bishop's property (β).*

Using the Corollary 3.3.1 and Brown's result [[9], Theorem 3] we have:

Theorem 3.3.2. *Let T be in class θ. If the spectrum of T satisfies $\sigma_r(T) \neq \sigma_c(T)$, Then T has a nontrivial invariant subspace.*

Proof. For a compact set K in $\mathbb{C}$, if the set $K \cap G$ is not dominating in G for any nonempty open set G, then $R(K) = C(K)$ [[9], Theorem 3]. Hence there exists an open set V such that $V \cap \sigma(T)$ is dominating for V. By Corollary 3.3.1, T has Bishop's property (β). Therefore T has a nontrivial invariant subspace. $\square$

Proposition 3.3.3. *Let T be an operator in class θ.*

(a) If the outer boundary of the spectrum of $T, \sigma(T)$, is a convex Jordan curve, that is, it is the boundary of a convex set and a Jordan curve, then the polynomially convex hull $\eta\sigma(T)$ of the spectrum of T is a $k-$spectral set for T.

(b) If the spectrum of T does not meet the real line, that is, $\sigma(T) \bigcap \mathbb{R} = \phi$, then it is $k-$spectral set for T.

Theorem 3.3.4. *Let T be in class θ satisfying $\sigma(T) \bigcap \mathbb{R} = \Phi$. If $P(\sigma(T)) = C(\sigma(T))$, then T is reductive.*

Proof. Since the spectrum of T is a $k-$spectral set by Proposition 2.2, we can find a bounded homomorphism $\rho : P(\sigma(T)) \to B(H)$ defined by $\rho(f) = f(T)$ such that $\|\rho(f)\| = \|f(T)\| \leq k \|f\|_{\sigma(T)}$. Since $P(\sigma(T)) = C(\sigma(T))$, the homomorphism is defined on $C(\sigma(T))$, that is,

$$\rho : C(T) \to B(H)$$

for all $f \in C(\sigma(T))$. Since ρ is unital, that is, $\rho(1) = 1, \rho$ is positive, that is, for a positive function f in $\sigma_c(T), \rho(f)$ is a positive operator [Paulsen [52], Proposition 2.11] and so it is completely positive [Paulsen [52], Theorem 3.11] which means that ρ is completely bounded. Hence ρ is similar to a *-homomorphism from [Paulsen [52], Corollaries 9.2 and 9.12], that is, there exists a *-homomorphism Φ and an invertible operator S such that $\rho(f) = S\Phi(f)S^{-1}$ for all $f \in C(\sigma(T))$.

For $z \in C(\sigma(T)), \Phi(z)\Phi(z)^* = \Phi(z)\Phi(\bar{z}) = \Phi(z\bar{z}) = \Phi(\bar{z})\Phi(z) = \Phi(z)^*\Phi(z)$.

So $\Phi(z)$ is a normal operator. This means that $T = \rho(z)$ is similar to a normal operator. Since every operator in class θ which is similar to a normal operator is normal [Mecheri [42], Proposition 4.1], it follows that T is normal.

Let T be an invariant subspace for T, since $\bar{z}$ is in $C(\sigma(T))$, there exists a sequence

$\{p_n\}$ of polynomials such that $p_n(z) \to \bar{z}$ uniformly on $\sigma(T)$.

Thus $\|p_n(T) - T^*\| = \|\rho(p_n) - \rho(\bar{z})\| = \|\rho(p_n - \bar{z})\| \leq k \|p_n - \bar{z}\|_{\sigma(T)} \to 0$. Thus M is invariant for T^*, since M is an invariant subspace of $p_n(T)$ for each n. Hence T is reductive. □

It is known that if T is in the operator class θ and $\sigma(T)$ is a subset of a vertical line, then T is normal [Campbell et all [12], Proposition 4] We extend this result for the case the spectrum of T does not meet the real line, that is, $\sigma(T) \bigcap \mathbf{R} = \phi$ as follows:

Corollary 3.3.5. *Let T be an operator in the class θ satisfying $\sigma(T) \bigcap \boldsymbol{R} = \phi$. If $R(\sigma(T)) = C(\sigma(T))$, then T is a normal operator.*

Theorem 3.3.6. *Let T be an operator in the class θ satisfying $\sigma(T) \bigcap \boldsymbol{R} = \phi$. If $R(\sigma(T)) = C(\sigma(T))$, then T has an invariant subspace.*

Proof. Muteti [45] By Corollary 3.3.5 , T is normal and combining with Theorem 3.1.14, then T has nontrivial invariant subspaces since normal operators have nontrivial invariant subspaces. □

3.4 Hyperinvariant Subspaces of Some Operators

In this section we investigate operators which have nontrivial hyperinvariant subspaces. The knowledge of hyperinvariant subspaces of an operator T gives information on the operators which commute with T, that is, the set of all the operators S such that $TS = ST$. The commutant of an operator T is useful since it contains all quasiaffine transforms of an operator and its very nature reveals information about operators quasisimilar, similar or unitarily equivalent to T.

Theorem 3.4.1. *Kubrusly, [34] Let $T \in B(H)$. Then*

(a) $Ker(T)$ and $\overline{Ran(T)}$ are hyperinvariant subspaces for T.

(b)If dimension of H is greater that 1 and T has no nontrivial invariant subspace, then $Ker(T) = \{0\}$ and $\overline{Ran(T)} = H$.

Remark 3.4.2. *A linear transformation between linear spaces, say $T : X \to Y$ is injective if and only if $Ker(T) = \{0\}$. Thus by Theorem 3.4.1 part(b) means that if an operator has a nontrivial subspace, then it is quasiinvertible.*
Theorem 3.4.1 part (a) and Proposition 3.1.1 ensure that, if the product of two nonzero operators are null, then both have nontrivial hyperinvariant subspaces.

Proposition 3.4.3. *Let H be a Hilbert space of dimension greater than 1. Then we have the following assertions.*
(i) a nonzero nilpotent operator on H has a nontrivial hyperinvariant subspace.
(ii) every nonscalar algebraic operator on H has a nontrivial hyperinvariant subspace.

Proof. (i)$Ker(T)$ is a hyperinvariant subspace for T using Theorem 3.4.1 part (a). If T is nonzero and nilpotent, then $T^n = 0$ and $T^{n+1} = TT^n = 0$ for some positive integer n so that $Ker(T)$ is nontrivial.
(ii) If T is an algebraic operator, then there exists a minimal polynomial p such that $p(T) = 0$. If T is nonscalar, then the degree of p is greater than 1. Thus $p(T) = (\lambda I - T)q(T) = 0$ for some scalar λ and for some polynomial q. Since $(\lambda I - T) \neq 0$ because T is nonscalar and $q(T) \neq 0$ because p is minimal. By Theorem 3.4.1(a), $Ker(\lambda I - T)$ is a nontrivial invariant subspace for every operator that commutes with $\lambda I - T$, and hence for every operator that commute with T. Hence, $Ker(\lambda I - T)$ is a nontrivial hyperinvariant subspace for T. $\square$

Remark 3.4.4. *Proposition 3.4.3 (ii) ensures that every operator in the commutant of a nonscalar algebraic operator has a nontrivial hyperinvariant subspace, that is, if an operator has no nontrivial hyperinvariant subspace, then there is no nonscalar algebraic operator in its commutant.*

Corollary 3.4.5. *Every non-scalar normal operator has a non-trivial hyperinvariant subspace.*

Lemma 3.4.6. *Kubrusly, [34] Let $T \in B[H], L \in B[K], X \in B[H, K]$ and $Y \in B[K, H]$ be such that $XT = LX$ and $YL = TY$. Suppose M is a nontrivial hyperinvariant subspace of L. If $\overline{R(X)} = K$ and $N(Y) \cap M = \{0\}$, then $Y(M) \neq \{0\}$ and for each nonzero x in $Y(M), \overline{T_x}$ is a nontrivial hyperinvariant subspace for T.*

Remark 3.4.7. *Kubrusly, [34] If $XT = LX$ and $YL = TY$ with $\overline{R(X)} = K$ and $N(Y) = \{0\}$, that is T and $T^*$is densely intertwined to L and L^*, then there exists a $x \in Y(M)$ such that $\overline{T_x}$ is a nontrivial hyperinvariant subspace for T whenever M is a nontrivial hyperinvariant subspace for L.*

This leads to the following corollary.

Corollary 3.4.8. *If two operators are quasisimilar and if one of them has a nontrivial hyperinvariant subspace, then so has the other.*

Remark 3.4.9. *Corollary 3.4.8 ensures that an operator quasisimilar to a nonscalar normal operator has a nontrivial invariant subspace.*

The following result shows that similarity implies quasisimilarity. This is brought by the fact that invertible operators are quasiinvertible. Note that the converse is not true.

Proposition 3.4.10. *If $T, S \in B(H)$ are similar operators, then they are quasisimilar.*

Proof. Suppose $S, T \in B(H)$ are similar, then there exists an invertible(quasi-invertible) operator $Y \in B(H, K)$ such that $YT = YS$. Thus $Y^{-1}S = TY^{-1}$, where $Y^{-1} \in B(K, H)$. Hence T and S are quasisimilar. □

From Proposition 3.4.10 we deduce the following corollaries.

Corollary 3.4.11. *Let T be a hyponormal operator. If T is similar to a normal operator, then T has a nontrivial hyper-invariant subspace.*

Proof. The proof follows from Proposition 3.4.10. □

Corollary 3.4.12. *Let T_1 be a p-hyponormal operator and T_2 be normal. If there exists a bijective operator Y such that, $T_1Y = YT_2$, then T_1 has a non-trivial hyper-invariant subspace.*

Proof. The proof follows from Proposition 3.4.10 since bijective operators are invertible. □

Corollary 3.4.13. *Let T_1 be a log-hyponormal operator and T_2 be normal. If there exists a bijective operator Y such that, $T_1Y = YT_2$, then T_1 has a non-trivial hyper-invariant subspace.*

Proof. The proof follows from Proposition 3.4.10 since bijective operators are invertible. □

Corollary 3.4.14. *Let T_1 be an invertible p-quasihyponormal operator and T_2 be normal. If there exists a bijective operator Y such that, $T_1Y = YT_2$, then T_1 has a non-trivial hyper-invariant subspace.*

Proof. The proof follows from Proposition 3.4.10 since bijective operators are invertible. □

Corollary 3.4.15. *Let T_1 be a w-hyponormal operator and T_2 be normal. If there exists aIf there exists a bijective operator Y such that, $T_1Y = YT_2$, then T_1 has a non-trivial hyper-invariant subspace.*

Proof. The proof follows from Proposition 3.4.10 since bijective operators are invertible. □

CHAPTER 4

OPERATOR EQUIVALENCES AND SUBSPACE LATTICES

Recall that $Lat(T), Hyperlat(T)$, or $Red(T)$ refers to the subspace lattice of all invariant, hyperinvariant and reducing subspaces of T, respectively. In this chapter we investigate the invariant and hyperinvariant lattices of some operators. The structure of invariant and hyperinvariant subspace lattices for operators in some equivalence relations is also examined.

4.1 Lattices of some operators

Proposition 4.1.1. *Muteti,[44] Let $T, S \in B(H)$ and M be a nontrivial invariant subspace for both T and S. Then M is $TS-$invariant.*

Proof. If M is invariant for both T and S, then we have $T(M) \subseteq M$ and $S(M) \subseteq M$. Thus we have $TSM = T(SM) \subseteq T(M) \subseteq M$. Therefore M is $TS-$ invariant. □

Proposition 4.1.2. *Muteti, [44] Let $T, S \in B(H)$ and M be a nontrivial invariant subspace for both T and S. Then M is $ST-$invariant.*

Proof. If M is invariant for both T and S, then we have $T(M) \subseteq M$ and $S(M) \subseteq M$. Thus we have $STM = S(TM) \subseteq S(M) \subseteq M$. Therefore M is $ST-$ invariant. □

Question 4.1.1. *Muteti, [44] If M is $TS-$invariant, is it true that M is $T-$invariant or $S-$invariant?*

We answer this question with the following example.

Example 4.1.3. *Let* $TS = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. *We observe that* $Lat(TS) = \left\{ \{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \boldsymbol{R}^2 \right\}$.

However TS *can be written, not uniquely, as the product of* $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *and* $S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. *We notice that,* $M = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ *is invariant for* TS *but it is not invariant for* T *and* S.

This leads to the following remark:

Remark 4.1.4. *Muteti, [44] Let* M *and* $T, S \in B(H)$. *If* M *is* $TS-$*invariant, then* M *is not necessarily* $T-$ *or* $S-$ *invariant.*

However if T commutes with S, such that, M is $TS-$invariant and either M is $T-$invariant or $S-$invariant then M is invariant under both T and S. This leads to the following result.

Proposition 4.1.5. *Let* $T, S \in B(H)$ *such that* T *and* S *commute and* $M \subseteq H$ *is invariant under* T. *If* TS *is* M-*invariant then* M *is* S-*invariant.*

Proof. Since M is TS-invariant and and S and T commute the M is ST-invariant, that is, $(TS)M = (ST)M \subseteq M$.
Since M is T-invariant, then $(TS)M = (ST)M = S(TM) = SM \subseteq M$. Thus M is S-invariant. □

The following results show that taking powers of an operator T preserves invariance and reducing subspaces.

Theorem 4.1.6. *Let* $T \in B(H)$ *and* $M \subseteq H$. *The following statements are true for any integer* $n > 1$.
(i) If $M \in Lat(T)$ *then* $M \in Lat(T^n)$.
(ii)If $M \in Red(T)$ *then* $M \in Red(T^n)$.

Proof. The proof of (i) and (ii) follows by mathematical induction on $n \in \mathbb{N}$. In the proof of (ii), we use the fact that $M \in Red(T)$ implies that $TM \subseteq M$ and $T^*M \subseteq M$. $\square$

Example 4.1.7. *Let* $T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. *Clearly,* $M = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ *is T-invariant. It is easy to check that* $T^n = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ *for all $n \geq 2$ and* $M = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ *is also T^n-invariant.*

Also $T^* = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. $N = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$ *is T^*-invariant.* $(T^*)^n = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ *for all $n \geq 2$ and* $N = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$ *is $(T^*)^n$-invariant.*

Remark 4.1.8. *Note that $Lat(T)$ need not be isomorphic to $Lat(T^n)$ as shown by the following example.*

Example 4.1.9. *Let* $T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. *Then* $T^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *for all $n \geq 2$ and* $Lat(T) = \{\{0\}, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2\}$ *and* $Lat(T^n) = \{\{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2\}$.

The following are the Hasse diagrams for $Lat(T)$ and $Lat(T^n)$, respectively.

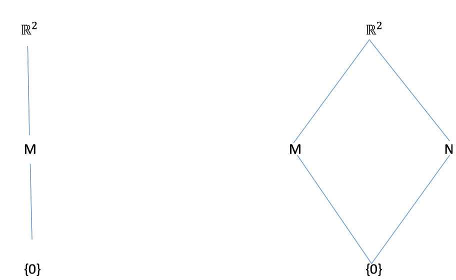

Figure 4.1: $\boldsymbol{Lat(T)}$ and $\boldsymbol{Lat(T^n)}$ respectively

Theorem 4.1.10. *Let $T \in B(H)$ and $M \subseteq H$. If $M \in Hyperlat(T)$ then $M \in Hyperlat(T^n)$ for any integer $n > 1$.*

Proof. We need to prove that $M \in Lat(S)$ where $S \in \{T\}'$ implies that $M \in Lat(S^n)$, where $S^n \in \{T^n\}'$.

By Theorem 4.1.6 (i), if $M \in Lat(S)$ then $M \in Lat(S^n)$, where $S^n \in \{T^n\}'$. By mathematical induction on $n \in \mathbb{N}$, if $S \in \{T\}'$ then $S^n \in \{T\}'$, $T^n \in \{S\}'$ and $S^n \in \{T^n\}'$. Thus if $M \in Hyperlat(T)$ then $M \in Hyperlat(T^n)$. $\square$

Example 4.1.11. *Let* $T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. *Let* $S \in \{T\}'$, *then* $S = \begin{bmatrix} a+d & a \\ 0 & d \end{bmatrix}$, $a, d \in \mathbb{R}$. *Thus* $M = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ *is* S*-invariant.* $S^n = \begin{bmatrix} (a+d)^n & (a+d)^n - d^n \\ 0 & d^n \end{bmatrix}$ *for all* $n \geq 2$ *and* $M = \left\langle \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle$ *is also* S^n*-invariant.*

We investigate the lattices of self-adjoint operators. Recall that an operator is self-adjoint if $T = T^*$.

Corollary 4.1.12. *Let* T *be a self-adjoint operator, then* $Lat(T) = Lat(T^*)$.

Proof. The proof follows from the definition. $\square$

Example 4.1.13. *Let* $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *Then* $T = T^*$ *and* $Lat(T) = \left\{ \{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\}$ $Lat(T^*)$

Let $M = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ *and* $N = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$. *The Hasse diagrams of* $Lat(T)$ *and* $Lat(T^*)$ *are as follows.*

We show that the hyperlaticces of a self adjoint operator are the same. We first need the following result.

Corollary 4.1.14. *Longstaff [40] Let* $T, S \in B(H)$. *If* $Lat(T) = Lat(S)$, *then* $Hyperlat(T) = Hyperlat(S)$.

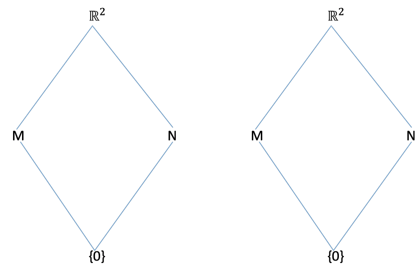

Figure 4.2: $Lat(T)$ and $Lat(T^*)$, respectively

Corollary 4.1.14 can be strengthened as follows.

Corollary 4.1.15. *Let $T \in B(H)$ be self-adjoint. Then $Hyperlat(T) = Hyperlat(T^*)$.*

Proof. We have $Lat(T) = Lat(T^*)$ from Corollary 4.1.1. The rest of the proof follows from Corollary 4.1.14. □

Example 4.1.16. *The commutant of operators T and T^* in Example 4.1.1 is the matrix $\{T\}' = \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} = \{T^*\}'$. Note that $S = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{R} \in \{T\}'$ and hence $Hyperlat(T) = \left\{ \{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\} = Hyperlat(T^*)$. The Hasse diagrams for $Hyperlat(T)$ and $Hyperlat(T^*)$ are similar to Figure 4.2.*

Corollary 4.1.17. *Let $T \in B(H)$ be self-adjoint. If $M \subseteq H$ is such that $TM = M$, then M reduces T.*

Proof. If $TM = M$, then $T^*M = TM = M$. Thus M invariant under both T and T^*, hence M reduces T. □

Corollary 4.1.18. *Let $T \in B(H)$ be self-adjoint. If $M \subseteq H$ is such that $TM = M$, then $Red(T) = Lat(T)$.*

Proof. This follows from Corollary 4.1.4 and the fact that $Red(T) \subseteq Lat(T)$, for any operator. □

Example 4.1.19. *Let* $T = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. *The* $Lat(T) = \left\{ \{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\}$. *The adjoint of operator* T *is* $T^* = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ *and* $Lat(T^*) = \left\{ \{0\}, \left\langle \begin{pmatrix} o \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\}$. *Thus* $Red(T) = \{\{0\}, \mathbb{R}^2\}$ *which shows* $Red(T) \subseteq Lat(T)$.

The following result shows that for self-adjoint operators the lattice of T is equal to its reducing subspaces.

Corollary 4.1.20. *Let* $T \in B(H)$ *be self-adjoint. Then* $Red(T) = Lat(T)$.

Proof. $Red(T) = \{M \subseteq H : TM \subseteq M, T^*M \subseteq M\}$
$= M \subseteq H : M \in Lat(T) \bigcap Lat(T^*)$
$= Lat(T)$. □

Recall that $T \in B(H)$ is reductive if every invariant subspace of T reduces T, that is, $Lat(T) \subseteq Red(T)$.

Theorem 4.1.21. *An operator* $T \in B(H)$ *is reductive if and only if* $Lat(T) = Red(T)$.

Proof. Suppose T is reductive, then $Lat(T) \subseteq Red(T)$. Since $Red(T) \subseteq Lat(T)$ thus $Red(T) = Lat(T)$.
Conversely, suppose $Lat(T) = Red(T)$. Then $Lat(T) \subseteq Red(T)$ and $Red(T) \subseteq Lat(T)$. By the first conclusion, $Lat(T) \subseteq Red(T)$, that is, every invariant subspace of T is $T-$ reducing hence T is reductive. □

Remark 4.1.22. *Corollary 4.1.22 shows that for a self-adjoint operator,* $Red(T) = Lat(T)$, *hence self-adjoint operators are reductive by Theorem 4.1.21.*

We show that for any self adjoint operator the $HyperRed(T) = Hyperlat(T)$. We first need the following results. Recall, that T is hyper-reducing if M reduces every operator in the commutant of T and the collection of all subspaces hyper-reducing for $T \in B(H)$.

Theorem 4.1.23. *Let $T \in B(H)$. Then $HyperRed(T) = Lat(\{T\}') \bigcap Lat(\{T^*\}')$.*

Proof. $HyperRed(T) = \{M \subseteq H : M \in Red(\{T\}')\}$

$\{M \subseteq H : SM \subseteq M, S^*M \subseteq M, S \in \{T\}'\}$

$= \{M \subseteq H : M \in Lat(S) \bigcap Lat(S^*), S \in \{T\}'\}$

$= \{M \subseteq H : M \in Lat(\{T\}') \bigcap Lat(\{T^*\})'\}$

$= Lat(\{T\}') \bigcap Lat(\{T^*\}')$. □

Theorem 4.1.24. *Let $T \in B(H)$. Then $HyperRed(T) = Hyperlat(T) \bigcap Hyperlat(T^*)$.*

Proof. The proof follows from Theorem 4.1.23 and the fact that $Lat(\{T\}') = Hyperlat(T)$ and $Lat(\{T^*\}') = Hyperlat(T^*)$, for any operator $T \in B(H)$. □

Corollary 4.1.25. *Let $T \in B(H)$ be self-adjoint. Then $HyperRed(T) = Hyperlat(T)$.*

Proof. The proof follows from Theorem 4.1.24 and self-adjointness of operator T. □

Remark 4.1.26. *Note that self-adjoint operators are normal but the converse is not true.*

Example 4.1.27. *Let* $T = \begin{bmatrix} 2i & 0 \\ 0 & 2i \end{bmatrix}$ *then* $T^* = \begin{bmatrix} -2i & 0 \\ 0 & -2i \end{bmatrix}$.

$T^*T = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = TT^*$. *Hence T is normal but not self-adjoint since $T \neq T^*$.*

We have seen that for self-adjoint operators $Lat(T) = Lat(T^*)$ and $Hyperlat(T) = Hyperlat(T^*)$. We extend this study to the class of normal operators.

Corollary 4.1.28. *Let $T \in B(H)$ be a normal operator. Then $Lat(T) = Lat(T^*)$.*

Proof. Suppose M is T^*T-invariant. Since T and T^* commute then M is TT^*-invariant, that is, $(TT^*)M = (T^*T)M \subseteq M$. Thus $(TT^*)M \subseteq M$ hence M is TT^*-invariant.

Let M be T-invariant, then $(TT^*)M = (T^*T)M = T^*(TM) = T^*M \subseteq M$. Thus M is T^*-invariant. Conversely, let M be T^*-invariant, then $(T^*T)M = (TT^*)M = T(T^*M) = TM \subseteq M$. Thus M is T-invariant. Hence $Lat(T) = Lat(T^*)$. $\square$

Example 4.1.29. *From Example 4.17,* $T = \begin{bmatrix} 2i & 0 \\ 0 & 2i \end{bmatrix}$ *and* $T^* = \begin{bmatrix} -2i & 0 \\ 0 & -2i \end{bmatrix}$.

The lattices are $Lat(T) = \left\{ \{0\}, \left\langle \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle, \left\langle \begin{matrix} 0 \\ 1 \end{matrix} \right\rangle, \mathbb{C}^2 \right\} = Lat(T^*)$.

Corollary 4.1.30. *Let* $T \in B(H)$ *be normal operator. Then* $Hyperlat(T) = Hyperlat(T^*)$.

Proof. The proof follows from Corollary 4.1.28 which shows that $Lat(T) = Lat(T^*)$. The rest of the proof follows from Corollary 4.1.14. $\square$

Remark 4.1.31. *Self-adjoint operator are normal. Corollary 4.1.30 can be used to show that,* $Hyperlat(T) = Hyperlat(T^*)$ *for any self-adjoint operator.*

Since normal operators are quasinormal, we investigate if $Lat(T) \equiv Lat(T^*)$ for quasinormal operators. Consider the following example.

Example 4.1.32. *Let* $T = \begin{bmatrix} 0 & 1 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, *then* $T^* = \begin{bmatrix} 0 & -i & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. *Clearly,* $T(T^*T) = \begin{bmatrix} 0 & 1 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (T^*T)T$. *Hence* T *is quasinormal* $Lat(T) = \left\{\{0\}, \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{C}^3\right\} = Lat(T^*)$. *Hence* $Lat(T) = Lat(T^*)$.

The Hasse diagram of $Lat(T)$ and $Lat(T^*)$ is as follows where $H = \mathbb{C}^3$ and $M = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$.

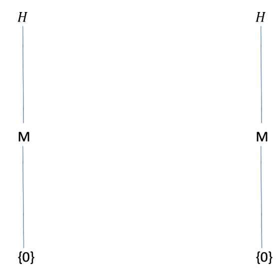

Figure 4.3: $\boldsymbol{Lat(T)}$ and $\boldsymbol{Lat(T^*)}$, respectively

Question 4.1.2. *If T and T^* are quasinormal, is T a normal operator?*

Consider the operator T in Example 4.1.32. Notice that,T commutes with T^*T and T^* commutes with TT^* since $T^*(TT^*) = \begin{bmatrix} 0 & -i & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (TT^*)T^*$. We also notice that T is normal since $T^*T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = TT^*$. Thus if T and T^* are quasinormal, then T is normal. This leads to the following result.

Corollary 4.1.33. *Let T and T^* be quasinormal operators, then $Lat(T) \equiv Lat(T^*)$.*

Remark 4.1.34. *Note that the adjoint of a quasinormal operator is not always quasinormal as shown below.*

Example 4.1.35. *Define $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $T(x) = T(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$ for $x = (x_1, x_2, x_3, \ldots) \in \ell^2(\mathbb{N})$. Then $T^*(x) = T^*(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$ and $T^*T(x) = T^*(0, x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots) = x$ Thus $T^*T = I$. T commutes with T^*T since $T^*T = I$ but T^* does not commute with the projection TT^*.*

We look at the lattice of the forward unilateral shift operator together with its adjoint. The matrix representation of the operators T^* is

$$T^* = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \cdots \end{bmatrix} \tag{4.1}$$

and the matrix representation T is

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix} \tag{4.2}$$

$Lat(T^*) = \left\{ \{0\}, span \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \ell^2(\mathbb{N}) \right\}$ and $Lat(T) = \{\{0\}, \ell^2(\mathbb{N})\}$. Let $H = \ell^2(\mathbb{N})$ and $M = span \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$.

The Hasse diagrams of $Lat(T^*)$ and $Lat(T)$ are as follows.

Thus for quasinormal operators, $Lat(T) \not\equiv Lat(T^*)$, in general. This result applies for hyponormal,p-hyponormal, paranormal operators among others.

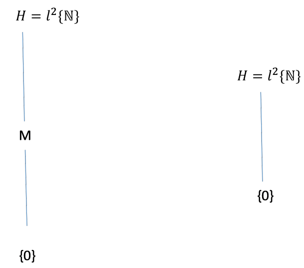

Figure 4.4: $\boldsymbol{Lat(T^*)}$ and $\boldsymbol{Lat(T)}$, respectively

4.2 Lattices of unitarily equivalent operators

Recall that two operators $T, S \in B(H)$ are said to be unitarily equivalent if there exists a unitary operator $U \in B(H)$ such that $UT = SU$ or equivalently $T = U^*SU$.

Theorem 4.2.1. *Let $T \in B(H)$ be a normal operator. Then $Lat(T) = Lat(T^*)$.*

Theorem 4.2.2. *Let $T \in B(H)$ be normal and suppose $S \in B(H)$ is unitarily equivalent to T, then S is normal.*

Proof. Suppose S is unitarily equivalent to T, there exists a unitary operator $U \in B(H)$ such that $S = U^*TU$. Taking adjoints on both sides, we have $S^* = U^*T^*U$. Thus

$$\begin{aligned} S^*S &= (U^*T^*U)(U^*TU) \\ &= U^*T^*TU \\ &= U^*TT^*U \\ &= SU^*T^*U \\ &= SU^*US^* \\ &= SS^* \end{aligned}$$

Hence S is normal. $\square$

Remark 4.2.3. *Theorem 4.2.2 shows that unitary equivalence preserves normality.*

This leads to the following result.

Theorem 4.2.4. *Two normal operators that are similar are unitarily equivalent.*

Theorem 4.2.4 can be strengthened as follows.

Proposition 4.2.5. *Let $T, S \in B(H)$ be normal operators, then S is unitarily equivalent to T if and only if S is similar to T.*

Corollary 4.2.6. *Let $T, S \in B(H)$ be normal operators that are unitarily equivalent, then $Lat(T) \equiv Lat(S)$.*

Proof. Proof follows from Proposition 4.2.5 and the fact that similar normal operators have isomorphic lattices. □

Corollary 4.2.7. *Let $T, S \in B(H)$ be unitarily equivalent projections, then $Lat(T) = Lat(S)$.*

Example 4.2.8. *Let $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. A simple computation shows that $P^2 = P$ and $Q^2 = Q$ and $P^* = P, Q^* = Q$ hence P and Q are projections unitarily equivalent where $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Thus $Lat(P) = \left\{ \{0\}, \left\langle \begin{matrix} 1 \\ 0 \end{matrix} \right\rangle, \left\langle \begin{matrix} 0 \\ 1 \end{matrix} \right\rangle, \mathbb{R}^2 \right\} = Lat(Q)$. Thus $Lat(P)$ is equal $Lat(Q)$.*

4.3 Lattices of similar operators

In this section we investigate lattices of operators under the similarity equivalence relation on $B(H)$. We will also give conditions when operators which are similar have isomorphic lattices. Recall that, two operators $A \in B(H)$ and $B \in B(K)$ are similar denoted by $A \approx B$ if there exists an invertible operator $N \in B(H, K)$ such that $NA = BN$ or equivalently $A = N^{-1}BN$ and two operators $A \in B(H)$ and

$B \in B(K)$ are quasi-similar denoted by $A \cong B$, if they are quasi-affine transforms of each other, equivalently, if there exists quasi-invertible operators $N \in B(H,K)$ and $M \in B(K,H)$ such that $AN = NB$ and $MB = AM$. We show that if two operators are similar, then their lattices are not isomorphic in general.

Example 4.3.1. *Consider the following operators T and S in $\mathbb{R}^2$.* $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $S = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. *A simple computation shows that T and S are similar under the invertible operator* $N = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. $Lat(T) = \left\{ \{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\}$ *and* $Lat(S) = \{\{0\}, \mathbb{R}^2\}$. *Thus $Lat(T)$ is not isomorphic to $Lat(S)$.*
Let $M = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$. *The Hasse diagrams for $Lat(T)$ and $Lat(S)$ are as shown below.*

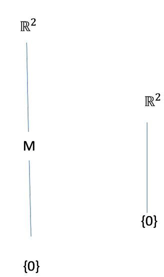

Figure 4.5: *Lat(T)* and *Lat(S)*

Remark 4.3.2. *Let $T, S \in B(H)$. If T and S are similar, then $Lat(T)$ is not necessarily isomorphic to $Lat(S)$.*

Question 4.3.1. *When does similarity preserve the structure of the lattices of operators?*

We answer Question 4.3.1 using the following example.

Example 4.3.3. *Consider* $T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ *and* $S = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. *A simple computation shows that* T *and* S *are similar under the invertible operator* $N = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}$.

$Lat(T) = \left\{ \{0\}, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\}$ *and* $Lat(S) = \left\{ \{0\}, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\}$. *Thus* $Lat(T) = Lat(S)$.

The commutant of T *and* S *are* $\{T\}' = \begin{bmatrix} a & 0 \\ a-d & d \end{bmatrix} : a, d \in \mathbb{R}$ *and* $\{S\}' = \begin{bmatrix} a & 0 \\ d-a & d \end{bmatrix} : a, d \in \mathbb{R}$ *respectively.* $Lat\{T\}' = \left\{ \{0\}, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\}$ *and* $Lat\{S\}' = \left\{ \{0\}, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\}$. *Thus* $Lat\{T\}' = Lat\{S\}'$. *Note that the operators* T *and* S *satisfy the property that* $T^2 = T$ *and* $S^2 = S$.

Corollary 4.3.4. *Let* $T, S \in B(H)$ *be similar normal operators, then* $Lat(T) = Lat(S)$.

Corollary 4.2.1 can be strengthened as follows.

Corollary 4.3.5. *Let* $P, Q \in B(H)$ *be similar projections, then* $Lat(P) = Lat(Q)$.

We show that the hyperlattices of similar normal operators are isomorphic. We first need the following result.

Corollary 4.3.6. *Let* $T, S \in B(H)$ *be similar normal operators. Then* $Hyperlat(T) = Hyperlat(S)$.

Proof. By Corollary 4.3.4 $Lat(T) = Lat(S)$. The rest of the proof follows from Corollary 4.1.14. □

Example 4.3.7. *Let* $T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ *and* $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. *A simple computation shows that* T *and* S *are similar under the invertible operator* $N = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. *Also* T *and* S *are normal operators since* $T^*T = TT^*$ *and* $S^*S = SS^*$. $Lat(T) = \left\{ \{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\} = Lat(S)$. *The commutant of* T *is* $\{T\}' = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in \mathbb{R} = \{S\}'$. $Hyperlat(T) = \left\{ \{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\} = Hyperlat(S)$.

Corollary 4.2.4 can be relaxed as follows.

Corollary 4.3.8. *Let* $P, Q \in B(H)$ *be similar projections, then* $Hyperlat(P) = Hyperlat(Q)$.

Proof. Proof follows from Corollary 4.3.6. □

The following example shows the hyperlattices of two operators which are isomorphic but the operators are neither similar nor quasisimilar.

Example 4.3.9. *Let* $T = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$ *and* $S = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}$.
$Lat(T) = \left\{ \{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\}$ *and* $Lat(S) = \{\{0\}, \mathbb{R}^2\}$. *Thus* $Lat(T) \neq Lat(S)$.

Note that $\{T\}' = \{X : X = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \alpha, \beta \in \mathbb{R}\}$ and $\{S\} = \{Y : Y = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \alpha \in \mathbb{R}\}$.

Thus $Hyperlat(T) = \left\{ \{0\}, \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\} = Hyperlat(S)$.
Thus their hyperinvariant subspaces are isomorphic.

Remark 4.3.10. *Let $T, S \in B(H)$. If $HyperLat(T) \equiv HyperLat(S)$, then $lat(T)$ is not necessarily isomorphic to $Lat(S)$.*
Note that T and S in Example 4.3.9 are anticommutative operators which are not quasisimilar. Thus isomorphism of hyperlattice does not in general imply quasi-similarity nor similarity of operators.

Theorem 4.3.11. *Let $T, S \in B(H)$ such that T is similar to S, then S^* and T^* are similar.*

Remark 4.3.12. *Note that similarity does not preserve the lattice of operators and so is their adjoints.*

Example 4.3.13. *Using Example 4.2.1, we have* $T^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ *and* $S^* = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$. $Lat(T) = \left\{ \{0\}, \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \mathbb{R}^2 \right\}$ *and* $Lat(S) = \{\{0\}, \mathbb{R}^2\}$. *Thus* $Lat(T^*)$ *is not isomorphic to* $Lat(S^*)$.
Let $M = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$. *The Hasse diagrams for* $Lat(T^*)$ *and* $Lat(S^*)$ *are as shown below.*

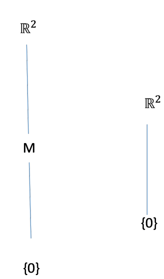

Figure 4.6: $Lat(T^*)$ and $Lat(S^*)$

Remark 4.3.14. *Note that $Lat(T) \equiv Lat(T^*)$ and $Lat(S) \equiv Lat(S^*)$.*

Corollary 4.3.15. *Let $T, S \in B(H)$ be similar normal operators, then $Lat(T^*) \equiv Lat(S^*)$.*

Proof. The proof follows from corollary 4.3.14 and Theorem 4.3.11. $\square$

Corollary 4.3.15 can be strengthened as follows.

Corollary 4.3.16. *Let $P, Q \in B(H)$ be similar projections. Then $Lat(P) \equiv Lat(Q)$.*

4.4 Lattices of almost similar operators

Recall that two operators T and S are said to be almost similar if there exists an invertible operator $N \in B(H)$ such that $T^*T = N^{-1}(S^*S)N$ and $T^* + T = N^{-1}(S^* + S)N$. The following result shows that similarity implies almost similarity.

Proposition 4.4.1. *[30] If $T, S \in B(H)$ such that T and S are unitarily equivalent, then $T \overset{a.s}{\sim} S$.*

We seek to find out when almost similarity implies similarity. We first need the following result.

Proposition 4.4.2. *[30] If $T, S \in B(H)$ such that $T \overset{a.s}{\sim} S$ and S is hermitian, then T is hermitian.*

Theorem 4.4.3. *If $T, S \in B(H)$ such that $T \overset{a.s}{\sim} S$ and if T is hermitian, then T is similar to S.*

Proof. Suppose $T \overset{a.s}{\sim} S$. There exists an invertible operator N such that $T^*T = N^{-1}(S^*S)N$ and $T^* + T = N^{-1}(S^* + S)N$. Since T is hermitian, by Proposition 4.4.2 S is hermitian. Thus $T^2 = N^{-1}(S^2)N$ and $2T = N^{-1}(2S)N$. This implies T and S are similar. $\square$

Remark 4.4.4. *We note that both operators T and S are hermitian and hence normal operators. Theorem 4.4.3 can be extended to similarity as follows.*

Corollary 4.4.5. *Let $T, S \in B(H)$ be normal operators such that $T \overset{a.s}{\sim} S$ then T is similar to S.*

Proof. The proof follows from Theorem 4.4.3. □

Corollary 4.4.6. *Let $T, S \in B(H)$ be normal operators such that $T \overset{a.s}{\sim} S$ then $Lat(T) = Lat(S)$.*

Proof. Proof follows from Corollary 4.4.5 which shows that almost similar operators are similar and and since similar normal operators have equal lattices, then the result follow Corollary 4.3.4. □

Theorem 4.4.7. *[30] If $T \in B(H)$ is normal, then $T \overset{a.s}{\sim} T^*$.*

This leads to the following result. We investigate when projections which are almost similar have isomorphic lattices. We first need the following result.

Theorem 4.4.8. *Let P and Q be orthogonal projections on a Hilbert space H. Then the following statements are equivalent.*
(i) P and Q are almost similar.
(ii) P and Q are similar.
(iii) P and Q are unitarily equivalent.

Proof. (i) $\Rightarrow$ (ii) Since P and Q are almost similar, there exist an invertible operator N such that $P^*P = N^{-1}(Q^*Q)N$ and $P^* + P = N^{-1}(Q^* + Q)N$. Since P is Q are orthogonal projections, by their idempotent and self-adjoint properties, we have $P^2 = N^{-1}(Q^2)N$ and $2P = N^{-1}(Q)N$. This implies P and Q are similar.
(ii) $\Rightarrow$ (iii) follows from the fact that self adjoint operators are normal and similar normal operators are unitarily equivalent.

(iii) $\Rightarrow$(i) Suppose P and Q are unitarily equivalent. Then there exists a unitary operator $U \in B(H)$ such that $P = U^*QP$. Taking adjoints on both sides we obtain $P^* = UQ^*P^*$.

Thus $P^*P = U^*Q^*UU^*QU = U^*Q^*QU = U^{-1}Q^*QU$ and $P^* + P = U^*Q^*U + U^*QU = U^*(Q^* + Q)U = U^{-1}(Q^* + Q)U$. Hence $P \overset{a.s}{\sim} Q$. □

Corollary 4.4.9. *Let P and Q be almost similar projections, then $Lat(P) = Lat(Q)$.*

Proof. The proof follows from Theorem 4.4.8. □

Corollary 4.4.9 can be strengthened as follows.

Corollary 4.4.10. *Let P and Q be almost similar projections, then $HyperLat(P) = HyperLat(Q)$.*

4.5 Lattices of metrically equivalent operators

In this section we give conditions under which two metrically equivalent operators have isomorphic lattices. Recall that operators $T, S \in B(H)$ are metrically equivalent if $T^*T = S^*S$.

Theorem 4.5.1. *[49] Let $S, T \in B(H)$. If S and T are unitarily equivalent, then they are metrically equivalent.*

The converse of Theorem 4.5.1 is not generally true as illustrated by the following example.

Example 4.5.2. *Let* $T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, *and* $S = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$ *T and S are metrically equivalent but not unitarily equivalent. Note that, $Lat(T) = \{\{0\}, \mathbb{C}^2\} = Lat(S)$.*

The following result shows when metric equivalence implies unitary equivalence.

Theorem 4.5.3. *Let T and S be metrically equivalent projections, then T and S are unitarily equivalent.*

Proof. Since T is metrically equivalent to S, there exists a unitary operator such that $T = USU^*$ by Nzimbi e tal [51]. Since T and S are projections, we have that $T = T^2 = T^*T = S^*S = USS^*U^* = US^2U^* = USU^*$. This shows that T and S are unitarily equivalent. $\square$

The following result shows when positive operators are metrically equivalent.

Proposition 4.5.4. *Let $T, S \in B(H)$ be positive operators, then T and S are metrically equivalent if and only if $T = S$.*

Note that, orthogonal projections are positive. Proposition 4.5.1 can be relaxed as follows.

Corollary 4.5.5. *Two orthogonal projections Q and P in $B(H)$ are metrically equivalent if and only if $P = Q$.*

Proof. Let P and Q be metrically equivalent. Then $P^*P = Q^*Q$. Since orthogonal projections are self-adjoint, then we have $P^2 = Q^2$ hence $P = Q$.
Conversely, let $P = Q$. By the idempotent property, $P^2 = Q^2$ and by the self-adjoint property, $P^*P = Q^*Q$ hence P and Q are metrically equivalent. $\square$

This leads to the following result.

Corollary 4.5.6. *Let $P, Q \in B(H)$ be metrically equivalent orthogonal projections, then $Lat(P) = Lat(Q)$.*

Corollary 4.5.6 can be strengthened as follows.

Corollary 4.5.7. *Let $P, Q \in B(H)$ be metrically equivalent projections, then $Hyperlat(P) = Hyperlat(Q)$.*

Example 4.5.8. *Let $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. A simple computation shows that $P^2 = P, Q^2 = Q$ and $P^* = P, Q^* = Q$ hence P and Q are projections* *which are metrically equivalent where* $Lat(P) = \left\{\{0\}, \left\langle\begin{matrix}1\\0\end{matrix}\right\rangle, \left\langle\begin{matrix}0\\1\end{matrix}\right\rangle, \mathbb{R}^2\right\} = Lat(Q)$.

Thus $Lat(P)$ *is equal* $Lat(Q)$.

Note that $\{P\}' = \{Q\}' = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : a, d \in \mathbb{R}\}$ *and so* $HyperLat(P) = \left\{\{0\}, \left\langle\begin{matrix}1\\0\end{matrix}\right\rangle, \left\langle\begin{matrix}0\\1\end{matrix}\right\rangle, \mathbb{R}^2\right\} = HyperLat(Q)$.

CHAPTER 5

CONCLUSION AND RECOMMENDATION

5.1 Conclusion

In this thesis we have made key contributions about operators with nontrivial invariant and hyperinvariant subspaces. We have extended results on some classes of operators to higher classes of operators.

In chapter three we have shown that nilpotent operators have nontrivial invariant subspaces using Proposition 3.1.1 which shows that if the product of two operators, say, $AB = 0$, then $Ker(A)$ and $\overline{Ran(B)}$ are nontrivial invariant subspaces. Note that an operator T is nilpotent if $T^n = 0$ and so T^n can be written as a product of two operators $T^n = T(T^n)$. An example on the same is given as shown by Example 3.1.1. An alternative proof is also given using SVEP. It is known that if an operator T and its adjoint (T^*) satisfy the SVEP, then T is decomposable and decomposable operators have nontrivial invariant subspaces. A nilpotent operator T satisfies SVEP and so is its adjoint since T is quasisimilar to its adjoint and quasisimilarity preserves SVEP. Thus a nilpotent operator is decomposable as shown in Theorem 3.1.11 and hence has notrivial invariant subspaces. This result is extended to the class of quasinilpotent operator as shown by Corollary 3.1.13. The class of quasinormal operators have been studied and it has been shown that quasinormal operators have nontrivial invariant subspaces as shown in Corollary 3.1.15. An example of the unilateral forward shift operator which is an isometry is shown to have nontrivial invariant subspaces in Example 3.1.16. This is as a result of the fact that every isometry is quasinormal and quasinormal operators have nontrivial invariant subspace.

This study has been extended to the class of hyponormal operators where it is shown that, hyponormal operators do not in general have nontrivial invariant subspaces. Conditions in which hyponormal and higher classes of operators have nontrivial invariant subspaces are given. Since similarity preserves nontrivial invariant subspaces, Corolllary 3.2.11 shows that a hyponormal operator which is similar to a normal operator has nontrivial invariant subspace. This result has been extended to the class of p-hyponormal,log-hyponormal, p-quasihyponormal, w-hyponormal operators where its is shown that if there exist an operator with dense range intertwining these operators to a normal operator, then they have nontrivial invariant subspace as shown by Corollaries 3.2.13, 3.2,15, 3.2.17 and 3.2.19, respectively. It is also shown that every operator whose spectrum lies on the real line has nontrivial invariant subspace as shown by Corollary 3.2.27. Since hyponormal operators satisfy the Bishop's property, it is shown that hyponormal operators satisfying decomposition property (δ) have nontrivial invariant subspaces in Corollary 3.2.36. This result has been extended to paranormal, *-paranormal and totally *- paranormal operators as shown in corollaries 3.2.37, 3.2.39 and 3.2.44. Theorem 3.2.47 shows that a hyponormal operator which is complex symmetric has nontrivial invariant subspace. Since similar operators are as well quasisimilar and quasimilarity preserves nontrivial hyperinvariant subspaces, Corollary 3.4.11 shows hyponormal operators have nontrivial hyperinvariant subspaces. In chapter four, lattices of some operator have been investigated. It has been shown that, if M is an invariant subspace for operators T and S then it is invariant for the product of these operators, that is, M is $TS-$ and ST-invariant as shown by Propositions 4.1.1 and 4.1.2 but the converse is not true in general. This has been illustrated using Example 4.1.3. However, if M is $TS-$ or $ST-$ invariant such that the operators commute and M is either $T-$ or $S-$ invariant then M is invariant under both T and S as shown by Proposition 4.1.5. It is also shown in Theorem 4.1.6 that, if M is

$T-$invariant then M is T^n- invariant however $Lat(T) \not\equiv Lat(T^n)$. This is illustrated in Example 4.1.9.

The lattices of self-adjoint operators has been studied. Corollaries 4.1.12 and 4.1.15 show that for a self-adjoint operator T, $Lat(T) = Lat(T^*)$ and $Hyperlat(T) = Hyperlat(T^*)$, repectively. It is also shown by Corollary 4.1.18 that, reducing subspaces of a self-adjoint operator T is equal to the invariant subspaces of T, that is, $Red(T) = Lat(T)$. An operator $T \in B(H)$ is reductive if and only if $Lat(T) = Red(T)$. Since this property holds for self-adjoint operators, then self adjoint operators are reductive. It has also been shown that in Corollary 4.1.25 that, hyperreducing subspaces and hyperinvariant subspaces are the same. The study has been extended to the class of normal operator where it is shown that for a normal operator T, invariant subspaces of T are the same as that its adjoint(T^*) and so is their hyperinvariant subspaces from Corollaries 4.1.28 and 4.1.30, respectively. The study has been extended to the class of quasinormal operators. It has been shown that the lattice of a quasinormal operator are isomorphic if both the operator and its adjoint are quasinormal. This results from the fact that if an operator and its adjoint are quasinormal then the operator is normal and normal operators have isomorphic lattices. An example to show that the adjoint of a quasinormal operator is not always quasinormal is given in Example 4.1.35. The lattices of this operator and its adjoint is studied and it is shown that their lattices are not isomorphic.

The lattices of operators and equivalence relation have been investigated. It is shown in Example 4.3.1 that similarity equivalence does not preserve the structure of lattices. However similar normal operators or projections have isomorphic lattices hyperinvariant lattices. The result hold for operators which are unitarily equivalent, almost similar operators as well as for metrically equivalent operators. An example of operators with isomorphic hyperinvariant lattices is given where the lattices are not isomorphic. However the operators do not satisfy any of the stated equivalence relations as shown in Example 4.3.9.

5.2 Recommendation

In our research, we were able to show that quasinormal operators, nilpotent operators have nontrivial invariant subspaces. It is also shown the a quasinilpotent operator has nontrivial invariant subspaces if it satisfies SVEP together with its adjoint. It is not known if all quasinilpotent operators satisfy SVEP. This is an area one can consider to study in the future. We have seen that hyponormal operators do not have nontrivial invariant subspaces in general. This result holds for p-hyponormal, w-hyponormal, paranormaln operators among others. We have shown that if an operator is similar to a normal operator then it has nontrivial invariant subspaces. Also operators whose spectrum lies in the real line have nontrivial invariant subspaces. Operators satisfying Bishop's property and decomposition property have nontrivial invariant subspaces. Thus other conditions in which these operators have nontrivial invariant subspaces can as well be considered as an area of study.

It is shown that similarity equivalence does not preserve the structure of lattices. However similar normal operators or projections have isomorphic lattices hyperinvariant lattices. The result hold for operators which are unitarily equivalent, almost similar operators as well as for metrically equivalent operators. This study can be extended to other equivalence relations such as unitary quasi-equivalent of operators. Conditions under which these equivalence relations have isomorphic hyperlattices can as well be investigated.

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