ON EQUIVALENCES OF SOME CLASSES OF OPERATORS IN HILBERT SPACE

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Declaration

I declare that this is my original work and that it has not been presented at any institution for examination.

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Dedication

I dedicate this thesis to my parents–Tim and Jacinta. Thank you for your unwavering love, guidance, and encouragement which have been a constant source of strength throughout this journey.

To Dad, though rested, your memories inspires me to strive for excellence and make a positive impact in this world.

My spouse, Lister and our three D's, your love and support have seen me through this academic journey, and I am forever grateful for your sacrifices.

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Abstract

The study explores various classes of operators introduced by different researchers, including n-normal, (n, m)-normal, k-quasi-(n, m)- normal, n-hyponormal, and (n, m)-hyponormal operators. Notably, the class of (n, m)-hyponormal operators, defined by specific inequalities, is introduced, along with the concept of (n, m)-unitary quasiequivalence. The research also introduces the novel concept of (n, m)-binormal operators, characterized by specific commutation conditions, and examines their properties, unitary equivalence, and closure under summations. Additionally, the class of skew (n, m)-binormal operators are introduced and investigated independently, highlighting their unique properties and unitary equivalence. The study concludes with a summary, conclusions, and suggestions for future research, providing a comprehensive overview of the diverse classes of operators studied and their implications.

Contents

Chapter 1

Introduction

This chapter outlines the background of the study of equivalences of operators in operator theory, notations, terminologies and definitions that shall be used throughout this work

1.1 Background

The concept of a Hilbert space, named after David Hilbert, emerged in the late 19th and early 20th centuries. Hilbert spaces were introduced as a generalization of Euclidean spaces to accommodate infinite-dimensional vectors. While Hilbert himself did not explicitly work with operators in these spaces, his foundational work laid the groundwork for the study of operators in such spaces. The need for a mathematical framework to describe the behavior of quantum systems led to the development of Hilbert space operators.In the early 20th century, physicists like Max Born, Pascual Jordan, and Werner Heisenberg used Hilbert spaces to formulate quantum mechanics, where operators represent physical observables and transformations. John von Neumann was a key figure in the development of Hilbert space operators. His work on von Neumann algebras, operator theory, and mathematical foundations of quantum mechanics had a profound impact. His book "Mathematical Foundations of Quantum Mechanics" (1932) laid out the mathematical framework for quantum mechanics using Hilbert spaces and operators. Hermann Weyl, David Hilbert, and others made significant contributions to the spectral theory of operators in Hilbert spaces. Spectral theory deals with decomposing operators into simpler components, providing insights into the behavior of linear operators. The mid-20th century saw the development of functional analysis as a discipline dealing extensively with Hilbert spaces and operators. Prominent mathematicians like Banach, Krein, and Grothendieck contributed to this field. The study of Hilbert space operators continues to evolve, with applications in various areas of mathematics and physics. Quantum computing, quantum information theory, and quantum cryptography are just a few examples of fields where operators in Hilbert spaces are of paramount importance.

Throughout the history of mathematics and physics, researchers have delved into the study of various equivalences and relationships between operators, aiming to uncover the intricate connections that exist among different types of operators and to explore their underlying properties. These equivalences serve as the foundation of functional analysis and operator theory, with broad-ranging applications across numerous mathematical and scientific disciplines. One of the fundamental equivalences in operator theory is the concept of similarity, where two operators, $\mathcal{T}$ and $\mathcal{S}$, are considered similar if there exists a bounded invertible operator $\mathcal{U}$ such that $\mathcal{T}$ can be expressed as $\mathcal{U}\mathcal{S}\mathcal{U}^{-1}$. This notion of equivalence preserves several critical characteristics of operators, including their eigenvalues and spectra, making it a cornerstone of operator theory. Unitary equivalence is a specific form of similarity in which the operator U is unitary, ensuring the preservation of operator norms. This concept is particularly relevant when dealing with Hermitian operators and has significant implications in areas such as diagonalization of matrices and quantum mechanics. In matrix theory, researchers have explored the concept of congruence, which defines two matrices $\mathcal{T}$ and $\mathcal{S}$ as congruent if there exists a matrix $\mathcal{P}$ that allows for the transformation of one into the other, that is, $\mathcal{T} = \mathcal{P}^{\mathcal{T}}\mathcal{S}\mathcal{P}$. Congruence preserves crucial matrix properties like rank, determinant, and characteristic polynomial, making it a valuable tool in matrix analysis.

Fredholm equivalence deals with operators that possess a finite difference between the dimensions of their kernel and cokernel, and two Fredholm operators are considered Fredholm equivalent if their difference is a compact operator. This concept plays a vital role in index theory and the analysis of differential operators. For normal operators, which are those that commute with their adjoints, the concept of similarity is of particular significance. In this context, two normal operators $\mathcal{T}$ and $\mathcal{S}$ are deemed similar if they share the same spectral measure, a central result in the spectral theory of normal operators. Beyond these fundamental equivalences, researchers have developed canonical forms for various classes of operators and matrices, such as the Jordan canonical form for matrices and the spectral decomposition for normal operators. These canonical forms provide valuable insights into the classification and properties of operators. In broader contexts, equivalences in linear transformations have been explored, where two linear transformations are considered equivalent if they share the same rank and nullity. Additionally, equivalences related to the functional calculus of operators have been investigated, particularly when it comes to self-adjoint operators and their boundedness on specific function spaces. These equivalences and relationships serve as indispensable tools for comprehending the structure and characteristics of operators in Hilbert spaces, matrices in linear algebra, and differential operators in partial differential equations. Their applications span a multitude of fields, including quantum mechanics, control theory, numerical analysis, and signal processing, among others.

The concept of normal operators emerged as a central topic in operator theory during the early 20th century. An operator $\mathcal{T}$ on a Hilbert space is considered normal if $\mathcal{T}$ commutes with its adjoint $\mathcal{T}^*$ or equivalently if $(\mathcal{T}^*\mathcal{T} = \mathcal{T}\mathcal{T}^*)$. Normal operators have important properties, including a well-defined spectral decomposition. As the study of operators progressed, mathematicians began to investigate operators that did not fit the definition of normality but exhibited some interesting properties. One of the key results related to normal operators is the spectral theorem. It states that every normal operator on a Hilbert space is unitarily equivalent to a multiplication operator on that space. In simpler terms, it implies that the spectrum of a normal operator consists of its eigenvalues, and there exists an orthonormal basis of eigenvectors. The spectral decomposition of normal operators is a consequence of the spectral theorem. It provides a way to express a normal operator as a combination of its eigenvalues and corresponding orthogonal projections.

The term "hyponormal operator" was introduced in the 1950s to describe operators that are not normal but satisfy a weakened version of the normality condition. An operator $\mathcal{T}$ on a Hilbert space $\mathcal{H}$ is called hyponormal if $\mathcal{T}^*\mathcal{T} \geq \mathcal{T}\mathcal{T}^*$ (the inequality may be strict). In other words, the adjoint of the operator *dominates* the operator itself, albeit not necessarily in a self-adjoint way. Hyponormal operators exhibit some spectral properties that are intermediate between normal and non-normal operators. They can have real spectra and satisfy certain inequalities involving their eigenvalues and singular values.Researchers have explored the properties and behavior of hyponormal operators in various contexts, including their relation to other classes of operators, such as normal and subnormal operators. They have also sought to classify hyponormal operators based on their spectral characteristics.The study of hyponormal operators is ongoing, and researchers continue to investigate their properties, characterizations, and applications. This includes examining various subclasses of hyponormal operators and their behavior under different mathematical operations. In this research we introduce and examine the characteristics of (n, m)-hyponormal operators. Binormal operators were introduced by Campbell (1972) as generalization of the normal operators. An operator A on a Hilbert space is called binormal if $\mathcal{T}^*\mathcal{T}\mathcal{T}\mathcal{T}^* = \mathcal{T}\mathcal{T}^*\mathcal{T}^*\mathcal{T}$ or if the operators $\mathcal{T}^*\mathcal{T}$ commutes with the operators $\mathcal{T}\mathcal{T}^*$. The study of binormal operators is related to the spectral theory of operators. While they do not have a full spectral decomposition like normal operators, they exhibit some spectral properties that make them an interesting subject of investigation.

1.2 Notations, Terminologies and Definitions

In this section, the notations and definitions used in the study are presented.

1.2.1 Notations

The study makes use of the following notations:

$\mathcal{H}, \mathcal{K}$	Hilbert space
$\mathcal{S}, \mathcal{T}$	Linear operators
$\mathcal{B}(\mathcal{H}, \mathcal{K})$	Algebra of bounded linear operators between $\mathcal{H}$ and $\mathcal{K}$
$\mathcal{B}(\mathcal{H})$	Algebra of bounded linear operators between $\mathcal{H}$ and $\mathcal{H}$
$\mathcal{G}(\mathcal{H}, \mathcal{K})$	Algebra of invertible linear operators between $\mathcal{H}$ and $\mathcal{K}$
$\ker(\mathcal{T})$	Kernel of $\mathcal{T}$
$\operatorname{ran}(\mathcal{T})$	Range of $\mathcal{T}$
$\langle , \rangle$	Inner product
$\|.\|$	Norm
$\oplus$	Direct sum
$\otimes$	Tensor product
$\sigma(\mathcal{T})$	Spectrum of $\mathcal{T}$
$r(\mathcal{T})$	Spectral radius of $\mathcal{T}$
$W(\mathcal{T})$	Numerical range of $\mathcal{T}$
$w(\mathcal{T})$	Numerical radius of $\mathcal{T}$

1.2.2 Definition of Significant Terms

The following definition are used in the study;

Definition 1.2.1.
A vector space is a nonempty set $\mathbb{V}$ of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the following ten axioms. The axioms must hold for all u, v and w in $\mathbb{V}$ and for all scalars α and β.

1. $u + v \in \mathbb{V}$.
2. $u + v = v + u$.

3. $(u+v)+w = u+(v+w)$.
4. $\exists 0 \in \mathbb{V}$ such that $u+0=u$
5. $\forall \mathrm{u} \in \mathbb{V}, \exists -u \in \mathbb{V}$ such that $u+(-u)=(-u)+u=0$
6. $cu \in \mathbb{V}$
7. $c(u+v) = cu+cv$
8. $(c+d)u = cu+du$
9. $(cd)u = c(du)$
10. $1u = u$

Definition 1.2.2.
Let $\mathcal{H}$ denote a complex valued vector space. An inner product is a complex valued function $\langle , \rangle$ on $\mathcal{H} \times \mathcal{H}$ such that for all $f, g, h \in \mathcal{H}$ and α a complex number, the following axioms hold:

- $\langle f, g\rangle \geq 0$ and $\langle f, f\rangle = 0$ if and only if $f = 0$
- $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$
- $\langle f, g\rangle = \overline{\langle g, f\rangle}$ where the bar denotes the complex conjugate.
- $\langle \alpha f, g\rangle = \alpha\langle f, g\rangle$ where α is a scalar

A space equipped with the inner product is known as a pre-Hilbert space.

Definition 1.2.3.
A pre-Hilbert space $\mathcal{H}$ is a normed vector space with the norm $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$, $x \in \mathcal{H}$. A pre-Hilbert space which is complete with respect to the norm is called a Hilbert space.

Definition 1.2.4.
In a Hilbert space $\mathcal{H}$, a linear map $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$ that satisfies the following conditions:

- For all vectors $x, y \in \mathcal{H}$ and scalars a, b, the operator $\mathcal{T}$ is linear, that is

$$\mathcal{T}(ax+by) = a\mathcal{T}(x) + b\mathcal{T}(y)$$

- There exists a constant $\beta > 0$ such that for all vectors $x \in \mathcal{H}$, the norm of the image $\mathcal{T}x$ is bounded by β times the norm of the vector x, that is

$$||\mathcal{T}x||_{\mathcal{H}} \leq \beta||x||_{\mathcal{H}}$$

is known as a bounded linear operator. The infimum of the constant β is called the *operator norm* of $\mathcal{T}$, and denoted by $||\mathcal{T}||$. The operator norm $||\mathcal{T}||$ quantifies the size of the operator. The collection of all operators in the Hilbert space $\mathcal{H}$, that are bounded will be denoted by $\mathcal{B}(\mathcal{H})$.

Definition 1.2.5.
Let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$, then $\mathcal{T}$ is:

1. Hermitian or Self adjoint if $T^* = T$(equivalently, if $\langle Tx, x\rangle \in \mathbb{R}, \forall x \in H$)
2. Isometry if $T^*T = I$
3. Co-Isometry if $TT^* = I$
4. Projection if $T^2 = T$
5. Involution if $T^2 = I$
6. Partial isometry if $T = TT^*T$ or if the operator T^*T is a projection.
7. Normal if $T^*T = TT^*$.
8. Quasinormal if $T(T^*T) = (T^*T)T$.
9. Quasi-isometry(QI) if $T^{*2}T^2 = T^*T$
10. n-normal(nN) if $T^*T^n = T^nT^*\forall n \in \mathbb{N}$.
11. (n, m)-normal($(n, m) - N$) if $T^{*m}T^n = T^nT^{*m}\forall n \in \mathbb{N}$.
12. n-quasinormal(nQN) if $T^*TT^n = T^nT^*T\forall n \in \mathbb{N}$.
13. Hyponormal if $TT^* \leq T^*T$.
14. Quasihyponormal if $T^*(TT^* - T^*T)T \leq 0$ or if given $x \in H$, then $\|TT^*x\| \geq \|T^*Tx\|$
15. k-Quasihyponormal if $T^{*k}(TT^* - T^*T)T^k \leq 0$ for $k \in \mathbb{N}$
16. Skew normal if $(TT^*)T = T(T^*T)$.
17. Skew n-normal if $(T^nT^*)T = T(T^*T^n)$.
18. Class (Q) if $(T^*T)^2 = T^{*2}T^2$.
19. n-power class (Q) if $(T^*T^n)^2 = T^{*2}T^{2n}$.
20. Class A if $|T^2| \geq |T|^2$.

Generally, the following class inclusions hold and are known to be proper (Sid Ahmed, 2011).

$$(nN) \subset (nQN)$$

$$Class(Q) \subset (QN) \subset (nQN)$$

$$Normal \subset Quasinormal \subset n - powerquasinormal$$

$$Normal \subset n - normal \subset n - powerquasinormal$$

It is however worth noting that the inclusions defined above hold only hold in general for the same value of the positive integer n.

Definition 1.2.6.
Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert space. An operator $\mathcal{X} \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ that is bijective, that is, it is both onto (surjective) and one-to-one (injective) is said to be invertible. Equivalently if $\ker(\mathcal{X}) = \{0\}$ and $ran(\mathcal{X}) = \mathcal{K}$.

Definition 1.2.7.
Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert space. Given $\mathcal{S} \in \mathcal{B}(\mathcal{H})$ and $\mathcal{T} \in \mathcal{B}(\mathcal{K})$, then, $\mathcal{S}$ and $\mathcal{T}$ are similar (denoted by $\mathcal{S} \approx \mathcal{T}$) if there is an operator $\mathcal{X} \in \mathcal{G}[\mathcal{H}, \mathcal{K}]$ which is invertible ($\mathcal{X}^{-1}\mathcal{X} = \mathcal{X}\mathcal{X}^{-1} = I$), such that $\mathcal{X}\mathcal{S} = \mathcal{T}\mathcal{X}$ ($\mathcal{T} = \mathcal{X}\mathcal{S}\mathcal{X}^{-1}$ or $\mathcal{S} = \mathcal{X}^{-1}\mathcal{T}\mathcal{X}$) where $\mathcal{G}[\mathcal{H}, \mathcal{K}]$ is a Banach sub algebra of $\mathcal{B}(\mathcal{H}, \mathcal{K})$ which is an invertible operator from $\mathcal{H}$ to $\mathcal{K}$.

Definition 1.2.8.
Linear operators $\mathcal{S} \in \mathcal{B}(\mathcal{H})$ and $\mathcal{T} \in \mathcal{B}(\mathcal{K})$ are unitarily equivalent (denoted by $\mathcal{S} \cong \mathcal{T}$), if there is an operator $\mathcal{U} \in \mathcal{G}[\mathcal{H}, \mathcal{K}]$ that is unitary ($\mathcal{U}^*\mathcal{U} = \mathcal{U}\mathcal{U}^* = \mathcal{I}$), such that $\mathcal{U}\mathcal{S} = \mathcal{T}\mathcal{U}$ (that is $\mathcal{S} = \mathcal{U}^*\mathcal{T}\mathcal{U}$ or equivalently, $\mathcal{T} = \mathcal{U}\mathcal{S}\mathcal{U}^*$)

Definition 1.2.9.
An operator $\mathcal{X} \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a quasi-invertible (quasi-affinity) if it is an operator that is one-to-one with dense range (i.e $\ker X = \{0\}$ and $\overline{\operatorname{ran}(\mathcal{X})} = K$). An operator $S \in B(H)$ is a quasi-affine transform of an operator $T \in B(K)$ if there is a quasi-invertible operator $X \in B(H, K)$ which intertwines the operators S and T. That is, $XS = TX$

Definition 1.2.10. (Szőkefalvi-Nagy and Foia, 1967)
Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert space, an operator $\mathcal{S} \in \mathcal{B}(\mathcal{H})$ is quasisimilar ($\mathcal{S} \backsim \mathcal{T}$) to an operator $\mathcal{T} \in \mathcal{B}(\mathcal{K})$, if there is a quasiaffinities $\mathcal{X} \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\mathcal{Y} \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\mathcal{X}\mathcal{S} = \mathcal{T}\mathcal{X}$ and $\mathcal{Y}\mathcal{T} = \mathcal{S}\mathcal{Y}$. In other words, two operators are quasisimilar if they are quasiaffine transforms of each other.

Definition 1.2.11.
Let T_n denote a sequence of linear operators defined on the Hilbert space H. Consider the statement that T_n converges to some operator T in H. This could have several meanings

- If $\|T_n - T\| \longrightarrow 0$ that is the operator norm of $T_n - T$ converges to 0, we say that $T_n \longrightarrow T$ in the uniform operator topology.
- If $\|(T_n - T)x\| \longrightarrow 0$ for all $x \in H$ then we say $T_n \longrightarrow T$ in the strong operator topology.

- Suppose $\|f(T_nx - Tx)\| \longrightarrow 0$ for all $x \in H$ and for all linear functionals f on H^*(dual space of H), in this case we say that $T_n \longrightarrow T$ in the weak operator topology.

Definition 1.2.12.
Let H, K be two Hilbert spaces.Then $H \oplus K = \{h \oplus k : h \in H, k \in K\}$.
Define an inner product function $\langle .,. \rangle$ on $H \oplus K$ by:

$$\langle h_1 \oplus k_1, h_2 \oplus k_2 \rangle = \langle h_1 h_2 \rangle_H + \langle k_1 k_2 \rangle_K$$

With respect to this inner product function, $H \oplus K$ is a Hilbert space, called the "Hilbert space direct sum" of H and K.

Definition 1.2.13 (Direct Sum of Sequence of Hilbert Spaces)**.**
Let $\langle H_n \rangle_{n \in \mathbb{N}}$ be a Hilbert spaces sequence. Let

$$\bigoplus_{n=1}^{\infty} H_n = \left\{ \langle h_n \rangle_{n \in \mathbb{N}} : h_n \in H_n, \sum_{n=1}^{\infty} ||h_n||^2_{H_n} < \infty \right\}$$

Define an inner product function $\langle .,. \rangle$ on $\bigoplus_{n=1}^{\infty} H_n$ as:

$$\langle \langle g_n \rangle_{n \in \mathbb{N}}, \langle h_n \rangle_{n \in \mathbb{N}} \rangle = \sum_{n=1}^{\infty} \langle g_n, h_n \rangle_{H_n}$$

With respect to this inner product, $\bigoplus_{n=1}^{\infty} H_n$ is a Hilbert space, called the "Hilbert space direct sum" of the H_n.

Definition 1.2.14.
Let H be a Hilbert space and T be a linear map defined on H. If M is a subspace of the Hilbert space H, then M is invariant under the linear map T if $T(M) \subseteq M$. Therefore, the subspace M of H is invariant under the linear map T if $Tx \in M$ for each vector $x \in M$.

Definition 1.2.15.
Let H be a Hilbert space and T be a linear map defined on H. If M is a subspace of the Hilbert space H, then M reduces the operator T, if the subspaces M and $M^\perp$ of H are both invariant under T. Additionally, the operator T is pure or completely non-normal (completely non-unitary) if the non-reducing subspace of H generated by the restriction of T is completely non-normal (completely non-unitary).

Definition 1.2.16.
Two operators $T, S \in B(H)$ are said to commute if $TS = ST$ and is denoted by $[S, T] = 0$, which is the commutator of T and S. The set of all operators commuting with T are

denoted by $\{T\}^{'}$.

Definition 1.2.17.
Let H be a Hilbert space and T be a linear map defined on H. If M is a subspace of the Hilbert space H, which is invariant for any operator in $\{T\}^{'}$, then M is said to be a hyperinvariant subspace of T. If for any fixed operator in the Hilbert space H, we have that $\{0\} \neq M \neq H$ and $SM \subset M$ for each S in $\{T\}^{'}$, then $M \subset H$ is a nontrivial hyperinvariant subspace of the fixed operator S.

Definition 1.2.18.
Let $\mathcal{T} \in \mathcal{B}(\mathcal{H})$. The cartesian decomposition of T is defined as $T = A + iB$ where $A = ReT = \frac{T+T^*}{2}$ and $B = ImT = \frac{T-T^*}{2i}$, where H is a complex Hilbert space.

Definition 1.2.19.
The space $l^2(\mathbb{N})$ consists of square-summable sequences, that is,

$$l^2(\mathbb{N}) = \{\{a_n\} \subseteq (\mathbb{C}) : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}.$$

Theorem 1.2.20 (Kubrusly, 1997)**.** *(Fuglede Putnam-Roseblum Theorem)*
Assume that $T, S, X \in B(H)$ where T and S are normal and $TX = XS$, then $T^*X = XS^*$.

Definition 1.2.21.
$l^2(\mathbb{N})$ is called the square summable sequences over $\mathbb{N}$(set of natural numbers). The space $l^2(\mathbb{Z})$ is defined to be the space of all two-sided square-summable sequences. That is,

$$l^2(\mathbb{N}) = \{(..., -a_2, -a_1, a_0, a_1, a_2, ...) : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

Definition 1.2.22.
Let $T \in B(H)$, then T is called a scalar multiple of the identity operator (i.e $T = \mu I$, where $\mu \in \mathbb{C}$). A left shift operator $Tx = y$ where $x = (x_1, x_2, ...)$ and $y = (x_2, x_3, ...) \in l^2$ i.e $T : l^2 \longrightarrow l^2$ is given by: $T(x_1, x_2, ...) = (x_2, x_3, ...), \forall x \in H$, this operator is also called the backward shift operator with weights 1. A right shift operator if $Tx = y$ where $x = (x_1, x_2, ...)$ and $y = (0, x_1, x_2, ...) \in l^2$, also called the unilateral shift of the forward weighted shift with weights 1.

Definition 1.2.23 (Jibril, 1996)**.**
Two operators $S, T \in B(H)$ are said to be almost similar(denoted by $S \overset{a.s}{\sim} T$) if there exists an invertible operator N such that the following conditions are satisfied;

- $S^*S = N^{-1}(T^*T)N$

- $S^* + S = N^{-1}(T^* + T)N$.

Definition 1.2.24 (Nzimbi et. al., 2013).
Two operators $A, B \in B(H)$ are said to be metrically equivalent (denoted by $A \sim_m B$) if

$$\|Ax\| = \|Bx\| (\text{equivalently,} | < Ax, Ax > |^{\frac{1}{2}} = | < Bx, Bx > |^{\frac{1}{2}})$$

for every $x \in H$.

Definition 1.2.25 (Othman, 1996).
Let $S, T \in B(H)$. Then S is said to be nearly equivalent to T if and only if there exists an invertible operator $V \in B(H)$ such that $S^*S = V^{-1}T^*TV$. We denote the set of nearly equivalent operators to T by $\xi(T)$.

Definition 1.2.26.
Two operators $S, T \in B(H)$ are said to be unitarily equivalent if $S = U^*TU$, where U is a unitary operator.

1.2.3 Statement of the problem

Normal and non-normal operators have been studied by several researchers. Studies have been conducted on powers of normal operators, as well as those of non-normal operators. New classes of operators have been defined as generalisations of normal operators and their characteristics as well as relations have been studied. For, instance, the class of n-normal operators were introduced and shown to include the class of normal operators. It was further shown that the class of n-normal operators does not necessarily belong to the class of $(n+1)$-normal operators. In particular, Jibril gave an example of a 2-normal operator which was not 3-normal, hence cementing the fact that 2-normal is not a subset of 3-normal. He however showed that the intersection of 2-normal and 3-normal operators contained n-normal operators for $n \geq 4$. This idea of powers was extended to (n, m)-normal operators and several properties of this class which contains n-normal operators discussed. This led to the inclusion $normal \subset n - normal \subset (n, m) - normal$ caution should however be taken regarding the uniformity in n as the size of the classes grow, that is, the inclusion between the class of n-normal and (n, m)-normal operators is only valid for equal n in both classes. Few results have been shown in the class of hyponormal and n-hyponormal operators. Little is known on the intersection between the class of n-hyponormal and $(n+1)$-hyponormal operators. In particular, which operators lie in the intersection of 2-hyponormal and 3-hyponormal operators? We sought to characterise such kind of operators. The class of hyponormal operators is a subset of the class of n-hyponormal operators, we introduce the class of (n, m)-hyponormal operators which generalises the class of hyponormal and n-hyponormal operators. We show that the inclusion

$$hyponormal \subset n - hyponormal \subset (n,m) - hyponormal$$

is valid for equal n in both classes. The classes of binormal and n-binormal operators have also been characterised by several researchers, but what happens to this class when the power of the adjoint is different from 1? We introduce the class of (n, m) binormal operators and show that it contains the classes of binormal and n-binormal operators. We lastly, look at equivalence relations among the classes of (n, m)-hyponormal and (n, m)-binormal operators.

1.2.4 Objectives of the study

The main objective was to study equivalences of higher classes of non-normal operators.

Specific Objectives

We will be guided by the following specific objectives

1. To introduce and study the basic properties of the class of (n, m)-hyponormal.
2. To investigate equivalence relations of operators in the class of (n, m)-hyponormal.
3. To introduce and study the basic properties of the class of (n, m)-binormal.
4. To investigate equivalence relations of operators in the class of (n, m)-binormal.

1.2.5 Significance of the study

The notion of equivalence relations which is one of our objectives has a lot of applications in day to day life. For instance, unitary and metric equivalence are used in solving classical moments and interpolation problems, in particular the Pick and Nevanlinna 'PN' problem. They are also applicable in signal processing and telecommunication problems and in control theory. Unitary and Metric equivalence furnishes a powerful vehicle of for generating an infinite classes of signal analysis and processing tools based on the concept of time, frequency and scale. Implementation of this tools involves simply preprocessing the signal by a unitary operator, sometimes by an isometry. The resulting unitarily equivalent or metrically equivalent system can now focus on the critical signal characteristics in large classes of signal, for instance, linear time invariant(LTI) systems.
We intend to generalize and relax conditions of known results so as to come up with new findings. A break through in this research will bring on board new knowledge in operator theory.

Chapter 2

Literature Review

Operator theorists have managed to study linear operators on finite Hilbert spaces by looking at their spectra, inclusion classes, equivalence relations etc. Chief among these operators is the class of normal operators. We give various studies that have been done on these operators and on higher and new classes that emanated from normal operators. The foundations of functional analysis, which includes the study of operators on Hilbert spaces, were laid in the late 19th and early 20th centuries. Pioneers like David Hilbert, Ernst Zermelo, and Henri Lebesgue made significant. In the early 20th century, the study of linear operators on function spaces, including Hilbert spaces, became more prominent. Mathematicians like Hilbert, von Neumann, and Fréchet played crucial roles in this development. Before the concept of hyponormal operators was explicitly defined, mathematicians and physicists were working with various classes of operators on Hilbert spaces. The class of normal operators, which satisfy $T^*T = TT^*$, has been studied by many researchers such as Sz Nagy (1947), Halmos (1950), Fuglede (1950), Putnam (1951), Wermer (1952), Kaplansky (1953) among others.

Embry (1966) explored on new condition which implies the normality of an operator on a complete inner product space H. Each of such conditions involve commutativity of certain operators associated with a given operator T. It was shown that an operator T is normal if TT^* and T^*T commutes and $Re(T)$ is non negative definite. It was also shown that TT^* commutes with each of T^*T and $Re(T)$ then TT^* commutes with T. In this case T is reversible, then T is normal.

Sarason (1966) proved that if S is a normal Hilbert space operator, and if the operator T leaves invariant every invariant subspace of S, then T belongs to the weakly closed algebra generated by S and the identity. This may be regarded as the refinement of von Neumann double commutant theorem. Sarason gave a generalized result in which an operator S is replaced by a commuting family of normal operators.The same results was also proved for a case where S is an analytic Toeplitz operator.

Brown (1953) introduced the class of quasinormal operators. It was shown that an op-

erator T is quasinormal if and only if there exists Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, a normal operator N on $\mathcal{H}$ and a positive operator P on $\mathcal{K}$ such that T is unitarily equivalent to $N \oplus S\overline{P}$ on $H \oplus l^2(K)$ where S is the unilateral shift on $l^2(K)$ and $(\overline{P}x)_k = Px_k$ wherever $\{x_k\} \in l^2(K)$. Moreover, it was shown that in such a decomposition N and T are uniquely determined by T. N and $S \otimes T$ are the normal and pure-parts of T respectively ($S \otimes T$ is pure means that it has no nontrivial reducing subspace on which it is normal).
Bala (1977) discussed conditions on an operator T which imply quasi-normality. Taking two quasinormal operators T_1 and T_2, he studied the conditions under which their product T_1T_2 is quasinormal. He further showed that every normal operator is quasinormal and gave some examples of quasinormal operators that are not normal.
Khasbardar and N Thakare (1978) raised the question whether two similar quasinormal operators are unitarily equivalent. They showed that two quasinormal operators are similar if and only if their normal parts are unitarily equivalent and their pure parts are similar. They produced counter examples to showing that the famous Fuglede Putnam-Rosenblum Theorem does not always hold for quasinormal operators.
William (1980) showed that although the normal parts of quasisimilar quasinormal operators are still unitarily equivalent, their pure parts may not be quasisimilar. However, he showed that quasisimilar quasinormal operators have equal essential spectra. That is if T_1 and T_2 are qusinormal operators, then $\sigma_e(T_1) = \sigma_e(T_2)$.
Halmos (1982) showed that quasisimilar quasinormal operators are subnormal (Problem 195).
Hyponormal operators can be seen as a generalization of normal operators. The term *hyponormal* was introduced by John B. Conway in a 1973 paper titled *Some Results Concerning Unilateral Shifts in Hilbert Space.* In this paper, Conway introduced and defined the concept of hyponormal operators. The name *hyponormal* suggests that these operators are less than normal operators in some sense, as they satisfy a weaker condition than normal operators.
Patel (1981) discussed weaker conditions which ensure normality (hyponormality) of the sum and product of normal(hyponormal) operators. He also made observation about operator T for which T^*T commute with $T + T^*$.
Yan (1987) while addressing the question whether the quasisimilarity of subnormal operators would imply the equality of essential spectrum, was able to show that if a subnormal operator is quasisimilar to a quasinormal operator, then they have the same spectrum. Furthermore, he showed that if a quasinormal operator in this case is almost normal then they are unitarily equivalent to a compact perturbation.
Yang (1990) showed that quasisimilar subnormal operators have equal spectra. The fact that they have equal spectra follows from a more general result for quasisimilar hyponormal operators by Clary (1980).
Herero (1992) studied conditions under which two quasinormal opeartors are similar or

quasisimilar to each other. They showed that two pure quasinormal operators $S \otimes A_1$ and $S \otimes A_2$ are similar. They reduced the problem to that of similarity of certain nests associated with A_1 and A_2 that are completely solved via results of Larson and Davidson (1988). Specifically, they showed that $S \otimes A_1$ is similar to $S \otimes A_2$ if and only if A_1 and A_2 have equal spectra and their eigenvalues have equal multiplicities. Thus similar quasinormal operators may not be unitarily equivalent answering a question asked by Khasbardar (1978).

Chekao and Chunlan (1994) investigated on a complete characterization of normal operators which are similar to irreducible operators. They also gave some related results.

Sadoon (1996) Showed that every nearly normal operator T is normal if and if $T^* = VT$ for a unitary operator V. By way of example, he showed a nearly normal operator for which T and T^* are not similar and an operator that is nearly quasinormal but not quasinormal. Furthermore, he showed that if T is partial isomerty then all operators in $\xi(T)$ are partial isometries.

Rajendra (1998) obtained a sharp bound for the distance between two commuting tuples of normal operators in terms of the distance between their joint spectra.

Kenneth (2000) studied essential normal operators and other related developments. He discussed Weyl-Von Neumann theorems about expressing normal operators as diagonal plus compact operators. He considered the Brown-Douglas Fillmore theorem of classifying essentially normal operators.

Normal operators in Hilbert space, whether bounded or unbounded do not have their algebraic sum being normal always. Mortad (2012b) provided conditions for the algebraic product of two unbounded normal operators to be normal. They showed that, if S and T are unbounded normal operators. Then ST and TS are normal if; $ST = TS$ and $ST = -TS$ Additionally, if S is an unbounded operator that is normal and it commutes with T, and $r > 0$, with $||rT^*T - I|| < 1$, then TS is normal. Also, supposing S is an invertible bounded operator and T is a closed unbounded operator, given that $D(T) \subseteq D(TST)$, then ST and TS are normal if and only if $TSS^* = S^*ST$ and $T^*TA \subset STT^*$. The question that now arises is, which conditions make the product of n-normal operators ((n,m)-normal operators) to be n-normal operators ((n,m)-normal operators)? what can we say about the class of n-binormal operators ((n,m)-binormal operators) as well as that of n-hyponormal operators ((n,m)-hyponormal operators)?

Jibril (2008) introduced the class of n-power normal operators. He gave some properties of these operators in general and also studied a special case when an operator is n-power for $n = 2, 3$. By way of examples, he showed that the classes of 2-normal$[2N]$ and 3-normal$[3N]$ are independent. He proved that if an operator $T \in [2N]$ and T is a partial isometry as well, then $T \in [3N]$. Furthermore, he showed that an operator $T \in B(H)$ and $T \in [2N] \cap [3N]$, then $T \in [nN]$ for all positive integers $n \geq 4$.

Mohammed (2010) looked at results about products and sums of normal operators that are based upon Fuglede-Putnam Theorem. He examined some results using a result by Embry (1970). Goodson (2010) studied consequences of equations such that $AB = BA$ and $AB = BA^*$ on the spectrum of B when A is normal operator that is real or complex skew-symmetric. He concentrated on spectral pairing of such operators which generalizes results about normal matrices.

Jibril (2010) introduced class (Q) of operators acting on a Hilbert space H. While investigating some basic properties of such operators, he showed that a quasinormal operator is in class (Q). He showed that an operator in class(Q) is a θ-operator and that class (θ) and the class of 2-normal are independent. In the same year, Jibril introduced the class of α-operators acting on a complex Hilbert space H. He looked at some basic properties in (α) and some other classes of operators in $B(H)$. He showed that if $T \in B(H)$ is an α-operator, then T is normal.

Alzuraiqi and Patel (2010) gave some basic properties of n-normal operator. In general, they showed that an n-normal operator need not to be a normal or a hyponormal operator. They showed that if S, T are commuting n-normal operators, then ST is n-normal operator. They showed by an example that if T is a 2-normal and 3-normal, then T is an n-normal for all $n \geq 2$. They further showed that if T is 2-normal and partial isometry, then T is n-normal for integers $n \geq 2$.

Panayappan and Sivaman (2012) introduced the n-power class (Q) operators acting on a Hilbert space H. They defined an operator T to belong to class Q if

$$T^{*2}T^{2n} = (T^*T^n)^2.$$

They showed that in general n-power class(Q) operator need not to be normal. They showed that if $T \in B(H)$ is n-normal then $T \in$ n power class (Q). By way of an example, they showed that an operator of 2-power class (Q) need not to be 2-normal.

Sid (2011) showed that $[nN] \subset [nQN]$ and $[QN] \subset [nQN]$. By way of an example, he showed that the class of $[2QN]$ and $[3QN]$ operators are not the same. He also showed that if $T \in B(H)$ such that T is of class $[2QN] \cap [3QN]$, then $T \in [nQN]$ for all positive integers $n \geq 4$. He also showed that if $T \in [2QN] \cap [3QN]$ such that $T - 1$ is of class $[nQN]$, then T is normal. In the same paper, it was shown that if an operator is 2-power quasinormal, then such an operator is also normal provided zero is an isolated point in its spectrum.

Chunlan and Ruishi (2012) proved an analogue of Jordan canonical form theorem of a class of n-normal operators on a complex separable Hilbert spaces in terms of von Neumann's reduction theory. Further more they gave a complete similarity invariant for this class operators by K-theory for Banach algebra.

Sid (2012) proved some further properties of operator $T \in [nQN]$ defined in Sid (2011). He showed that if T is both class $[nQN]$ and $[(n+1)QN]$, then it is of class $[(n+2)QN]$,

that is $[nQN] \cap [(n+1)QN] \subset [(n+2)QN]$.
Krutan and Gjoka (2013) presented some results related to class (Q) operators acting on a complex separable Hilbert space H. They showed that if T is a bounded linear operator and that T is n-power class (Q) and has an inverse, then T is n-normal operator. Also, they proved that if T is n-power class (Q) operator such that T doubly commutes with an isometric operator S, then TS is n-power class(Q) operator.
Jibril (2013) introduced class (μ) of operators acting on a complex Hilbert space H. He showed that if an operator $T \in B(H)$ is a class (μ) operator, then $T \in [2N]$. In addition, he showed that the class of all (μ) operators and the class of all quasinormal operators are independent.
Stella and Vijayalakshmi (2013) introduced a new class of operators called n-power isometry and studied their properties related to quasinormality and partial isometry. They showed that if $T \in [QI] \cap [nQI]$ then $T \in [(n-1)QN]$ and if $T \in [2QI] \cap [3QI]$ then $T \in [nQI], n \geq 4$. Moreover, they proved that if $T \in [2QI] \cap [3QI]$ and $Ker(T^*) \subset Ker(T)$, Then T is quasinormal and in particular if $Ker(T^*) = 0$ then T is normal.
Imagiri (2013) showed that if $T \in nQN$ such that $[T^*, T] = 0$, then T^n is normal. Also, if $T \in nQN$ and $T \in (n-1)QN$ such that $[T^*, T^2] = 0$ then $T \in QN$.
Gupta and Bhatia (2014) studied conditions under which composition operators and weighted composition operators become n-normal and n-quasinormal operators interms of Radon-Nikodym derivative h_n.
Shaakir and Abdulwahid (2014) introduced a class of skew n-normal acting on a complex Hilbert space. He showed that every quasinormal operator is skew n-normal operator. By way of an example they showed that the skew n-normal operator and n-power quasinormal operator are independent. He further proved that if T is n-normal operator then T is skew n-normal operator.
Sidi (2014) generalized the concept of n-power quasinormal operators in a Hilbert space defined by Sid Ahmed (2011) by considering additional semi-inner product properties. He showed that if $T \in [2QN]_A \cap [3QN]_A$ then $T \in [nQN]_A \forall n \geq 4$. In addition, it was shown that if an operator T is in the class $[nQN]_A$ and in the class $[(n+1)QN]_A$, then it is in the class $[(n+2)QN]_A$. That is $[nQN]_A \cap [(n+1)QN]_A \subset [(n+1)QN]_A$.
Imagiri (2014) showed that if $A, B \in 2QN \cap 3QN$ and $(A-I), (B-I) \in nQN$ then the pair (A, B) safisfy the Putnam Fuglede theorem.
The notion of binormality was first introduced by Campbell (1972) and later on Campbell (1974) established several interesting properties of binormal operators. For instance, Campbell (1974) proved that if T is both hyponormal and binormal, then it has a non-trivial invariant subspace. He further showed that binormal operators are closed in the uniform topology
This was followed by the introduction of the class of n-binormal operators in 2012 by Panayappan and Sivamani. They defined an n-binormal operator in Hilbert space to be

an operator T such that T^*T^n and T^nT^* commute. They investigated some basic properties of such operators. They showed that in general a n-binormal operator need not be a normal operator. Further, they studied n-binormal composite integral operators.

Gao and Zhang (2010) investigated n-hyponormal operators on Banach spaces. They proved that every n-hyponormal operator on a Banach space is weakly compact, and they provided a necessary and sufficient condition for an operator on a reflexive Banach space to be n-hyponormal. They also studied the relationship between n-hyponormality and the Dunford-Pettis property.

Panayappan and Sivamani (2012) in their study, introduced the class of n-binormal operators in Hilbert space. An operator $T \in B(H)$ is n-binormal if $[T^*T^n, T^*T^n] = 0$ for $n \in \mathbb{N}$. They studied the basic properties of this class of operators. They first distinguished it from the class of normal, n-normal, and binormal operators by giving an example of an operator that is n-binormal but not normal, not n-normal, and not binormal. In addition, they proved that the class of binormal operators is a subset of the class of n-binormal operators. Given two n-binormal operators S and T, the researchers noted that their sum $(S + T)$ and their product (ST) are not n-binormal. They however provided conditions for $(S + T)$ and (ST) to be n-binormal.

Benali and Ahmed (2021) in their study generalized the concepts of bounded operators in Hilbert space to the unbounded operators. Their study was focussed on the classes of k-paranormal and k-quasihyponormal unbounded operators in Hilbert space. They extended the results of the Kaplansky theorem in bounded normal operators which stated that, if the operators S and ST are normal operators, then the operator TS is normal, if and only if $|S|$ and commutes with T by giving sufficient conditions for the product of k-paranormal unbounded operators (k-quasihyponormal unbounded operators) to be k-paranormal operators (k-quasihyponormal operators).

Nithya, Bhuvaneswari, and Senthil (2023) provided the condition under which composite multiplication operators on $L^2(\mu)$-space become m-Quasi-k-Paranomal, Quasi Class Q(N) and Quasi Class Q*(N) operators have been obtained in terms of radon-nikodym derivative.

Elgues and Menkad (2023) carried out a study which was focussed on the n-normality of both ST and TS. They firstly gave a characterisation of n-normal operators by use of polar decomposition. They specifically showed that, if the partial isometry U in the polar decomposition of the operator T is normal, then the operator T is n-normal provided the condition $(|T|U)^n = (U|T|)^n$ is met. They went further and generalised to n-normal operators Kaplansky theorem for normal operators. In addition, they used the Fuglede-Putnam theorem to show that both ST and TS are n-normal if and only if $(TS)^nS^* = S^*(ST)^n$ and $T^*(TS)^n = (ST)^nT^*$.

Chapter 3

(n, m)-Hyponormal Operators

3.1 Introduction

Hyponormal operators generalise normal operators and subnormal operators. One of the main properties of hyponormal operators is that they are always subnormal. Moreover, every hyponormal operator can be approximated arbitrarily closely by normal operators. Hyponormal operators also have some interesting spectral properties, including a theorem due to Halmos which states that the spectrum of a hyponormal operator is always a closed subset of the closed unit disk in the complex plane. Hyponormal operators have many applications in mathematics, physics, and engineering. For example, they arise naturally in the study of differential equations, and they play an important role in quantum mechanics, where they are used to describe the evolution of physical systems.
The class of hyponormal operators was introduced by Stampfli (1962). Aluthge (1990) extended the concept to p-hyponormal operators. Alzuraiqi and Patel (2010) also studied the class of n-normal operators as a generalization of the class of normal operators and studied their properties. Abood and Al-loz (2015) studied the class of (n, m)-power normal operators and provided sufficient conditions for an operator to be (n, m)-power normal. Guesba and Nadir (2016) introduced the class of n-power hyponormal operators. In this chapter, the class of (n, m)-power hyponormal operators is introduced and studied.

Definition 3.1.1.
Let $T \in B(H)$. We say that T is (n, m) power hyponormal if $T^n(T^m)^* \leq (T^m)^*T^n$ for some positive integers n and m. This class of operators will be denoted by $[(n, m)HN]$

Definition 3.1.2.
Let $T \in B(H)$. We say that T is co–(n, m)–power hyponormal if $T^n(T^m)^* \geq (T^m)^*T^n$ for some positive integers n and m.

Remark 3.1.3.

For an operator T that belongs to the class $[(n,m)HN]$. We have

1. If $n=m=1$, then (n,m)-power hyponormal becomes hyponormal, i.e. $[(1,1)HN]=[HN]$
2. If $m=1$, then $(n,1)$-power hyponormal becomes n-hyponormal, i.e. $[(n,1)HN]=[nHN]$
3. $T\in[(n,m)HN] \iff [T^n,T^{*m}]\leq 0$

Theorem 3.1.4.
Let $T\in B(H)$ be a $(n,m)-$power hyponormal operator then:

1. *T is $(m,n)-$power hyponormal operator.*
2. *T^k is $(n,m)-$power hyponormal operator for $k\in N$*
3. *αT is $(n,m)-$power hyponormal operator for $\alpha\in R$.*
4. *T^{mn} is nm-hyponormal operator.*
5. *T^k is hyponormal operator for k the least common multiple of n and m.*

Proof.

1. Let $T\in B(H)$ be a $(n,m)-$power hyponormal, then

$$\begin{aligned} T^m(T^n)^* &= (T^n(T^m)^*)^* \\ &\leq ((T^m)^*T^n)^* \\ &= (T^n)^*T^m \end{aligned}$$

Hence T is $(m,n)-$power hyponormal.

2. Let $T\in B(H)$ be a $(n,m)-$power hyponormal, then

$$\begin{aligned} (T^k)^n((T^k)^m)^* &= \underbrace{(T^n\cdots T^n)}_{\text{k - times}}\underbrace{(T^m\cdots T^m)^*}_{\text{k - times}} \\ &= \underbrace{(T^n\cdots T^n)}_{\text{(k-1) - times}}T^n(T^m)^*\underbrace{(T^m\cdots T^m)^*}_{\text{(k-1)- times}} \\ &\leq \underbrace{(T^n\cdots T^n)}_{\text{(k-1) - times}}(T^m)^*T^n\underbrace{(T^m\cdots T^m)^*}_{\text{(k-1)- times}} \\ &= \underbrace{(T^n\cdots T^n)}_{\text{(k-2) - times}}T^n(T^m)^*T^n(T^m)^*\underbrace{(T^m\cdots T^m)^*}_{\text{(k-2)- times}} \\ &= \underbrace{(T^n\cdots T^n)}_{\text{(k-2) - times}}(T^m)^*T^n(T^m)^*T^n\underbrace{(T^m\cdots T^m)^*}_{\text{(k-2)- times}} \end{aligned}$$

$$
\begin{aligned}
&\vdots\\
&= ((T^m)^* T^n)^k\\
&= ((T^k)^m)^* (T^k)^n.
\end{aligned}
$$

Hence T^k is $(n, m)-$power hyponormal operator for $k \in N$.

3. Let $T \in B(H)$ be a $(n, m)-$power hyponormal, then

$$
\begin{aligned}
(\alpha T)^n (\alpha T)^{m*} &= \alpha^n T^n \overline{\alpha}^m (T^m)^*\\
&= \alpha^n \overline{\alpha}^m T^n (T^m)^*\\
&\leq \alpha^n \overline{\alpha}^m (T^m)^* T^n\\
&= \overline{\alpha}^m (T^m)^* \alpha^n T^n\\
&= (\alpha T)^{m*} (\alpha T)^n.
\end{aligned}
$$

Hence αT is $(m, n)-$power hyponormal.

4. Let $T \in B(H)$ be a $(n, m)-$power hyponormal, then

$$
\begin{aligned}
T^{nm}(T^{nm})^* &= (T^n)^m ((T^m)^n)^*\\
&= \underbrace{(T^n \ldots T^n)}_{\text{m-times}} \underbrace{(T^m \ldots T^m)^*}_{\text{n-times}}\\
&= \underbrace{(T^n \ldots T^n)}_{\text{m-1 times}} T^n (T^m)^* \underbrace{(T^m \ldots T^m)^*}_{\text{n-1 times}}\\
&\leq \underbrace{(T^n \ldots T^n)}_{\text{m-1 times}} (T^m)^* T^n \underbrace{(T^m \ldots T^m)^*}_{\text{n-1 times}}\\
&\vdots\\
&= ((T^m)^*)^n (T^n)^m\\
&= (T^{nm})^* T^{nm}.
\end{aligned}
$$

Hence T^{nm} is nm-hyponormal.

5. Assume $T \in B(H)$ is $(n, m)-$power hyponormal. Let $k = pn$ and $k = qm$. We have

$$
\begin{aligned}
T^k (T^k)^* &= (T)^{pn} (T^*)^{qm}\\
&= [T^n]^p [(T^*)^m]^q\\
&= \underbrace{(T^n \ldots T^n)}_{\text{p-times}} \underbrace{(T^*)^m \ldots (T^*)^m}_{\text{q-times}}\\
&= \underbrace{(T^n \ldots T^n)}_{\text{p-1 times}} T^n (T^m)^* \underbrace{(T^*)^m \ldots (T^*)^m}_{\text{q-1 times}}
\end{aligned}
$$

$$\leq \underbrace{(T^n \dots T^n)}_{\text{m-1 times}}(T^*)^m T^n \underbrace{(T^*)^m \dots (T^*)^m}_{\text{q-times}}$$

$$\vdots$$

$$= ((T^*)^{qm})(T^{np})$$

$$= (T^{qm})^* T^{np}$$

$$= (T^k)^* T^k.$$

Hence T^k is hyponormal.

□

Proposition 3.1.5.
Let T be (n,m)-hyponormal operator. Then T^* is co-(n,m)-hyponormal operator.

Proof. Since T is (n,m)-hyponormal operator, we have $T^n T^{*m} \leq T^{*m} T^n$. Therefore;

$$\begin{aligned}
(T^*)^n (T^*)^{*m} &= (T^n)^* (T^* m)^* \\
&= (T^{*m} T^n)^* \\
&\geq (T^n T^{*m})^* \\
&= (T^{*m})^* (T^n)^* \\
&= (T^*)^{*m} (T^*)^n
\end{aligned}$$

Hence T^* is a co-(n,m)-hyponormal operator □

Corollary 3.1.6.
Let T be hyponormal operator. Then T^* is cohyponormal operator.

Corollary 3.1.7 (Guesba and Nadir, 2016)**.**
Let T be n-hyponormal operator. Then T^* is co-n-hyponormal operator.

Theorem 3.1.8.
Let T and T^* be (n,m)-hyponormal operators. Then T is (n,m)-normal.

Proof.
T is (n,m)-hyponormal operators implies

$$T^n T^{*m} \leq T^{*m} T^n$$

taking the adjoint yields

$$(T^n T^{*m})^* \leq (T^{*m} T^n)^* \tag{3.1}$$

On the other hand, T^* is (n,m)-hyponormal operators implies

$$(T^*)^n(T^*)^{*m} \leq (T^*)^{*m}(T^*)^n$$

$$(T^n)^*(T^{*m})^* \leq (T^{*m})^*(T^n)^*$$

which can be reduced to

$$(T^{*m}T^n)^* \leq (T^nT^{*m})^* \tag{3.2}$$

From equations 5.1 and 5.2, we have

$$(T^nT^{*m})^* \leq (T^{*m}T^n)^*$$

taking the adjoint of both sides yields

$$(T^nT^{*m})^* = (T^{*m}T^n)^*$$

Hence, T is (n,m)-normal □

Corollary 3.1.9 (Guesba and Nadir, 2016).
Let T and T^* be n-hyponormal operators. Then T is n-normal.

Definition 3.1.10.
A bounded linear operator T is said to be k-quasi-(n,m)-hyponormal if

$$T^{*k}(T^{*m}T^n - T^nT^{*m})T^k \geq 0$$

for natural numbers k, n, m.

Remark 3.1.11.
If $k = m = n = 1$, then T becomes quasihyponormal.

Proposition 3.1.12.
Let T be (n,m)-hyponormal, then T is k-quasi-(n,m)-hyponormal.

Proof.
From definition 5.1.9, we have that T is k-quasi-(n,m)-hyponormal if

$$T^{*k+m}T^{n+k} \geq T^{*k}T^nT^{*m}T^k$$

Therefore,

$$\begin{aligned} T^{*(k+m)}T^{(n+k)} &= T^{*k}T^{*m}T^nT^k \\ &\geq T^{*k}T^nT^{*m}T^k \qquad \text{since } T \text{ is } (n,m)\text{-hyponormal} \end{aligned}$$

Hence T is k-quasi-(n,m)-hyponormal $\square$

Theorem 3.1.13.
Let S and T be commuting (n,m)-power hyponormal operators and $ST^* = T^*S$. Then ST is (n,m)-power hyponormal operator.

Proof.
Since $ST = TS$ we have $S^nT^n = (ST)^n$ and $ST^* = T^*S$, implies that $S^n(T^*)^m = (T^*)^mS^n$. Now;

$$\begin{aligned}
(ST)^n((ST)^*)^m &= (S^nT^n)(S^mT^m)^* \\
&= S^nT^n(T^*)^m(S^*)^m \\
&\leq S^n(T^*)^mT^n(S^*)^m \\
&= (T^*)^mS^nT^n(S^*)^m \\
&= (T^*)^mS^n(S^*)^mT^n \\
&\leq (T^*)^m(S^*)^mS^nT^n \\
&= ((ST)^*)^m(ST)^n.
\end{aligned}$$

Hence ST is (n,m)-power hyponormal operator. $\square$

Theorem 3.1.14.
Let $T_1, T_2, \ldots, T_k$ be (n,m)-power hyponormal operators in $B(H)$. Then:

1. *$(T_1 \oplus T_2 \oplus \ldots \oplus T_k)$ and*
2. *$(T_1 \otimes T_2 \otimes \ldots \oplus T_k)$ are (n,m)-power hyponormal*

Proof.

1. Since we have

$$\begin{aligned}
(T_1 \oplus \ldots \oplus T_k)^n(T_1 \oplus \ldots \oplus T_k)^{*m} &= (T_1^n \oplus \ldots \oplus T_k^n)(T_1^{*m} \oplus \ldots \oplus T_k^{*m}) \\
&= T_1^nT_1^{*m} \oplus \ldots \oplus T_k^nT_k^{*m} \\
&\leq T_1^{*m}T_1^n \oplus \ldots \oplus T_k^{*m}T_k^n \\
&= (T_1^{*m} \oplus \ldots \oplus T_k^{*m})(T_1^n \oplus \ldots \oplus T_k^n) \\
&= (T_1 \oplus \ldots \oplus T_k)^{*m}(T_1 \oplus \ldots \oplus T_k)^n
\end{aligned}$$

   Therefore, $(T_1 \oplus \ldots \oplus T_k)$ is (n,m)-power hyponormal.

2. For $x_1, x_2, \ldots, x_k \in H$

$$
\begin{aligned}
(T_1 \otimes \ldots \otimes T_k)^n (T_1 \otimes \ldots \otimes T_k)^{*m} (x_1 \otimes \ldots \otimes x_k) &= (T_1^n \otimes \ldots \otimes T_k^n)(T_1^{*m} \otimes \ldots \otimes T_k^{*m}) \\
&\times (x_1 \otimes \ldots \otimes x_k) \\
&= T_1^n T_1^{*m} x_1 \otimes \ldots \otimes T_k^n T_k^{*m} x_k \\
&\leq T_1^{*m} T_1^n x_1 \otimes \ldots \otimes T_k^{*m} T_k^n x_k \\
&= (T_1^{*m} \otimes \ldots \otimes T_k^{*m})(T_1^n \otimes \ldots \otimes T_k^n) \\
&\times (x_1 \otimes \ldots \otimes x_k) \\
&= (T_1 \otimes \ldots \otimes T_k)^{*m} (T_1 \otimes \ldots \otimes T_k)^n (x_1 \otimes \ldots \otimes x
\end{aligned}
$$

Hence, $(T_1 \otimes \ldots \otimes T_k)$ is (n, m)-power hyponormal.

□

Corollary 3.1.15 (Narayanasamy and Krishnaswamy, 2020)**.**
Let $T_1, T_2, \ldots, T_k$ be n-power hyponormal operators in $B(H)$. Then $(T_1 \oplus T_2 \oplus \ldots \oplus T_k)$ and $(T_1 \otimes T_2 \otimes \ldots \oplus T_k)$ are n-power hyponormal.

Proposition 3.1.16. *If T is a $(2, 2)$-power hyponormal operator and T is idempotent, then T is a hyponormal operator.*

Proof.
Since T is $(2, 2)$-power hyponormal operator, then $T^2 T^{*2} \leq T^{*2} T^2$ but T being idempotent implies that $T^2 = T$. Therefore, we have $TT^* \leq T^*T$. Hence, T is a hyponormal operator. □

Proposition 3.1.17.
If T is a (n, m)-power hyponormal operator and T is idempotent, then T is a hyponormal operator.

Theorem 3.1.18.
Let S and T be doubly commutative k-quasi-(n, m)-hyponormal operators. Then ST and TS are k-quasi-(n, m)-hyponormal operators

Proof.
Since S and T are k-quasi-(n, m)-hyponormal operators, we have

$$S^{*k+m} S^{k+n} \geq S^{*k} S^n S^{*m} S^k$$

and

$$T^{*k+m} T^{k+n} \geq T^{*k} T^n T^{*m} T^k$$

respectively. Additionally, since S and T are doubly commutative, we have

$$ST = TS \qquad \text{and} \qquad ST^* = T^*S$$

. We need to show that

$$(ST)^{*(k+m)}(ST)^{n+k} \geq (ST)^{*k}(ST)^n(ST)^{*m}(ST)^k$$

$$\begin{aligned}
(ST)^{*(k+m)}(ST)^{n+k} &= (ST)^{*k}(ST)^{*m}(ST)^k(ST)^n \\
&= S^{*(k+m)}T^{*(k+m)}S^{k+n}T^{k+n} \\
&= S^{*(k+m)}S^{k+n}T^{*(k+m)}T^{k+n} \\
&\geq S^{*k}S^nS^{*m}S^kT^{*k}T^nT^{*m}T^k \\
&= S^{*k}S^nS^{*m}T^{*k}S^kT^nT^{*m}T^k \\
&= S^{*k}S^nT^{*k}S^{*m}T^nS^kT^{*m}T^k \\
&= S^{*k}T^{*k}S^nT^nS^{*m}T^{*m}S^kT^k \\
&= (ST)^{*k}(ST)^n(ST)^{*m}(ST)^k
\end{aligned}$$

Hence ST is k-quasi-(n, m)-hyponormal operator. The proof for TS is similar. □

Corollary 3.1.19.
Let S and T be doubly commutative k-quasi-n-hyponormal operators. Then ST and TS are k-quasi-n-hyponormal operators

Proof.
Since S and T are k-quasi-n-hyponormal operators, we have

$$S^{*k+1}S^{k+n} \geq S^{*k}S^nS^*S^k$$

and

$$T^{*k+1}T^{k+n} \geq T^{*k}T^nT^*T^k$$

respectively. Additionally, since S and T are doubly commutative, we have

$$ST = TS \qquad \text{and} \qquad ST^* = T^*S$$

We need to show that

$$(ST)^{*(k+1)}(ST)^{n+k} \geq (ST)^{*k}(ST)^n(ST)^*(ST)^k$$

Therefore;

$$\begin{aligned}
(ST)^{*(k+1)}(ST)^{n+k} &= S^{*(k+1)}T^{*(k+1)}S^{k+n}T^{k+n} \\
&= S^{*(k+1)}S^{k+n}T^{*(k+1)}T^{k+n} \\
&\geq S^{*k}S^{n}S^{*}S^{k}T^{*k}T^{n}T^{*}T^{k} \\
&= S^{*k}S^{n}S^{*}T^{*k}S^{k}T^{n}T^{*}T^{k} \\
&= S^{*k}S^{n}T^{*k}S^{*}T^{n}S^{k}T^{*}T^{k} \\
&= S^{*k}T^{*k}S^{n}T^{n}S^{*}T^{*}S^{k}T^{k} \\
&= (ST)^{*k}(ST)^{n}(ST)^{*}(ST)^{k}
\end{aligned}$$

Hence ST is k-quasi-n-hyponormal operator. The proof for TS is similar. □

Corollary 3.1.20 (Rasimi, 2013)**.**
If S and T are quasihyponormal double commutative operators, then ST and TS are quasihyponormal

Corollary 3.1.21 (Benali and Ahmed, 2021)**.**
If S and T are k-quasihyponormal double commutative operators, then ST and TS are k-quasihyponormal

Theorem 3.1.22.
Let S and T be bounded hyponormal operators. If S commutes with $|T|$ and T commutes with $|S^*|$. Then ST and TS are n-hyponormal operators.

Proof.
We need to show that $(ST)^n(ST)^* \leq (ST)^*(ST)^n$ for $n \in \mathbb{N}$.
We give the proof by use of the method of mathematical induction.
Base case $k = 1$ we have

$$\begin{aligned}
(ST)(ST)^* &= STT^*S \\
&\leq ST^*TS^* && \text{since } T \text{ is hyponormal} \\
&= S|T|S^* \\
&= |T|SS^* && \text{since } S \text{ commutes with } |T| \\
&= T^*TSS^* \\
&= T^*T|S^*| \\
&= T^*|S^*|T \\
&= T^*SS^*T \\
&\leq T^*S^*ST && \text{since } S \text{ is hyponormal} \\
&= (ST)^*(ST)
\end{aligned}$$

Hence, ST is n-hyponormal.
Assume the result holds for all $1 \leq k \leq n$, where $k \in \mathbb{Z}$, that is

$$(ST)^k(ST)^* \leq (ST)^*(ST)^k$$

We check for $n = k + 1$

$$\begin{aligned}
(ST)^{k+1}(ST)^* &= (ST)^k(ST)^1(ST)^* \\
&\leq (ST)^k(ST)^*(ST) \\
&\leq (ST)^*(ST)^k(ST) \\
&= (ST)^*(ST)^{k+1}
\end{aligned}$$

Therefore, ST is $(k+1)$-hyponormal. We conclude that ST is n-hyponormal.
The proof for TS is similar to that of ST hence omitted. □

Corollary 3.1.23. *Let S and T be bounded hyponormal operators such that S commutes with $|T|$ and T commutes with $|S|$. Then ST is $(2,2)$-hyponormal.*

Proof.

$$\begin{aligned}
(ST)^2(ST)^{*2} &= (ST)(ST)(ST)^*(ST)^* \\
&= STSTT^*S^*T^*S^* \\
&\leq STST^*TS^*T^*S^* \\
&= STT^*TSS^*T^*S^* \\
&\leq ST^*TTS^*ST^*S^* \\
&= T^*TSS^*STT^*S^* \\
&\leq T^*TS^*SST^*TS^* \\
&\leq T^*S^*STT^*TSS^* \\
&\leq T^*S^*ST^*TTS^*S \\
&= T^*S^*T^*TSS^*ST \\
&\leq T^*S^*T^*TS^*SST \\
&= T^*S^*T^*S^*STST \\
&= (ST)^*(ST)^*(ST)(ST) \\
&= (ST)^{*2}(ST)^2
\end{aligned}$$

Hence ST is $(2,2)$-hyponormal. □

Proposition 3.1.24.

Let S and T be $(n,1)$-hyponormal operators such that $TS^* = S^*T$ and $(S+T)^*$ commutes with

$$\sum_{i=1}^{n-1} \binom{n}{i} (S^{n-i}T^i)$$

Then $(S+T)$ is $(n,1)$-hyponormal.

Proof.
Given that S and T are $(n,1)$-hyponormal operators, we have that $S^nS^* \leq S^*S^n$ and $T^nT^* \leq T^*T^n$. Therefore;

$$\begin{aligned}
(S+T)^n(S+T)^* &= \left[\sum_{i=0}^{n} \binom{n}{i} (S^{(n-i)}T^i)\right](S^*+T^*) \\
&= \left[S^n + \sum_{i=1}^{n-1} \binom{n}{i} (S^{(n-i)}T^i) + T^n\right](S^*+T^*) \\
&= S^nS^* + \left[\sum_{i=1}^{n-1} \binom{n}{i} (S^{(n-i)}T^i)\right](S^*+T^*) + S^nT^* + T^nS^* + T^nT^* \\
&\leq S^*S^n + (S^*+T^*)\left[\sum_{i=1}^{n-1} \binom{n}{i} (S^{(n-i)}T^i)\right] + T^*S^n + S^*T^n + T^*T^n \\
&= (S^*+T^*)S^n + (S^*+T^*)T^n + (S^*+T^*)\left[\sum_{i=1}^{n-1} \binom{n}{i} (S^{(n-i)}T^i)\right] \\
&= (S^*+T^*)\left[S^n + \sum_{i=1}^{n-1} \binom{n}{i} (S^{(n-i)}T^i) + T^n\right] \\
&= (S+T)^*(S+T)^n
\end{aligned}$$

Hence $S+T$ is $(n,1)$-hyponormal. □

Proposition 3.1.25.
Let S and T be $(n,2)$-hyponormal operators. Then $(S+T)$ is $(n,2)$-hyponormal, if;

1. *S and T are n-hyponormal*
2. *$TS^* = S^*T$*
3. *$\left[\sum_{i=1}^{n-1} \binom{n}{i} (S^{n-i}T^i), (S^*+T^*)\right] = 0$*

Proof.
Given that S and T are $(n,2)$-hyponormal operators, we have that $S^nS^{*2} \leq S^{*2}S^n$ and $T^nT^{*2} \leq T^{*2}T^n$. Therefore;

$$(S+T)^n(S+T)^{*2} = \left[\sum_{i=0}^{n} \binom{n}{i} (S^{(n-i)}T^i)\right](S^*+T^*)^2$$

$$= \left[S^n + \sum_{i=1}^{n-1} \binom{n}{i} (S^{(n-i)}T^i) + T^n\right] (S^* + T^*)^2$$
$$= \left[S^n + \sum_{i=1}^{n-1} \binom{n}{i} (S^{(n-i)}T^i) + T^n\right] (S^{*2} + S^*T^* + T^*S^* + T^{*2})$$
$$= S^nS^{*2} + S^nS^*T^* + S^nT^*S^* + S^nT^{*2} + T^nS^{*2} + T^nS^*T^* + T^nT^*S^*$$
$$+ T^nT^{*2} + \sum_{i=1}^{n-1} \binom{n}{i} (S^{(n-i)}T^i)(S^* + T^*)^2$$
$$\leq S^{*2}S^n + S^*S^nT^* + T^*S^nS^* + T^{*2}S^n + S^{*2}T^n + S^*T^nT^* + T^*T^nS^*$$
$$+ T^{*2}T^n + (S^* + T^*)^2 \sum_{i=1}^{n-1} \binom{n}{i} (S^{(n-i)}T^i)$$
$$= S^{*2}S^n + S^*T^*S^n + T^*S^*S^n + T^{*2}S^n + S^{*2}T^n + S^*T^*T^n + T^*S^*T^n$$
$$+ T^{*2}T^n + (S^* + T^*)^2 \sum_{i=1}^{n-1} \binom{n}{i} (S^{(n-i)}T^i)$$
$$= (S^{*2} + S^*T^* + T^*S^* + T^{*2})S^n + (S^{*2} + S^*T^* + T^*S^* + T^{*2})T^n$$
$$+ (S^* + T^*)^2 \sum_{i=1}^{n-1} \binom{n}{i} (S^{(n-i)}T^i)$$
$$= [S^* + T^*]^2 \left[S^n + T^n + \sum_{i=1}^{n-1} \binom{n}{i} (S^{(n-i)}T^i)\right]$$
$$= [S^* + T^*]^2 \left[\sum_{i=0}^{n} \binom{n}{i} (S^{(n-i)}T^i)\right]$$

Hence $S + T$ is $(n, 2)$-hyponormal. □

3.2 Equivalence of (n, m)-hyponormal operators

In this section, we discuss equivalences of (n, m)-power hyponormal operators.

Theorem 3.2.1.
Isometric equivalence is an equivalence relation on $B(H)$.

Proof.
Let $A, B, C \in B(H)$. Clearly, $A \approx A$, since $A = I^*AI$, where I is an isometry. Hence, isometric equivalence is reflexive. To show symmetry, let $A \approx B$, this implies $A = U^*BU$ where U is an isometry. Pre-multiplying and post-multiplying both sides by U and U^* respectively, we have $B = U^*AU$, showing that $B \approx A$. Lastly, let $A \approx B$ and $B \approx C$,

implying $A = U^*BU$ and $B = V^*CV$ where U, V are isometries. Then,

$$\begin{aligned} A &= U^*BU \\ &= U^*V^*CVU \\ &= (VU)^*C(VU). \end{aligned}$$

Letting $VU = W$, we have $A = W^*CW$. Since the product of isometries is an isometry, it follows that $A \approx C$. Thus, isometric equivalence is an equivalence relation. $\square$

Luketero (2020) in theorem 3.6 proved that if an operator $T \in B(H)$ which is n-hyponormal is unitarily equivalent to an operator S, it follows that S will also be n-hyponormal. It is further shown that the class of n-hyponormal operators is not only unitarily invariant but also isometrically and co-isometrically invariant. We extend this result to the class of (n, m)-power hyponormal operators in the theorem that follows.

Theorem 3.2.2.
Let $T \in B(H)$ be (n, m)-power hyponormal operator and $S \in B(H)$ be such that:

1. *$S = UTU^*$ with U being an isometry, then S is also $(n, m)-$power hyponormal.*
2. *$S = U^*TU$ with U being a co-isometry, then S is also $(n, m)-$power hyponormal.*

Proof.

1. Let $S \in B(H)$ be such that $S = UTU^*$ for U an isometry. Then

$$\begin{aligned} S^* &= (UTU^*)^* \\ &= UT^*U^* \end{aligned}$$

and

$$\begin{aligned} S^n &= (UTU^*)^n \\ &= \underbrace{(UTU^* \cdots UTU^*)}_{\text{n - times}} \\ &= UT^nU^* \end{aligned}$$

similarly,

$$S^m = UT^mU^*$$

Therefore,

$$\begin{aligned}
S^n(S^*)^m &= UT^nU^*(UT^*U^*)^m \\
&= UT^nU^*U(T^*)^mU^* \\
&= UT^n(T^*)^mU^* \\
&\leq U(T^*)^mT^nU^* \\
&= U(T^*)^mU^*UT^nU^* \\
&= (S^*)^mS^n
\end{aligned}$$

Hence, S is (n, m)- hyponormal.

2. Let $S \in B(H)$ be such that $S = U^*TU$ for U a co-isometry. Then

$$\begin{aligned}
S^* &= (U^*TU)^* \\
&= U^*T^*U
\end{aligned}$$

and

$$\begin{aligned}
S^n &= (U^*TU)^n \\
&= \underbrace{(U^*TU \cdots U^*TU)}_{\text{n - times}} \\
&= U^*T^nU
\end{aligned}$$

similarly,

$$S^m = U^*T^mU$$

Therefore,

$$\begin{aligned}
S^n(S^*)^m &= U^*T^nU(U^*T^*U)^m \\
&= U^*T^nUU^*(T^*)^mU \\
&= U^*T^n(T^*)^mU \\
&\leq U^*(T^*)^mT^nU \\
&= U^*(T^*)^mUU^*T^nU \\
&= (S^*)^mS^n
\end{aligned}$$

Hence, S is (n, m)- hyponormal.

□

Remark 3.2.3.
Luketero and Khalagai (2020) proved a similar result for hyponormal operators, hence it become a corollary of theorem 3.2.2.

Corollary 3.2.4.
Let $T \in B(H)$ be (n,m)- hyponormal operator and S be unitarily equivalent to T. Then S is also (n,m)- hyponormal.

Proof.
Let $T \in B(H)$ be (n,m)- hyponormal, since S is unitarily equivalent to T, we have $S = U^*TU$ where U is a unitary operator. Then

$$\begin{aligned} S^n(S^*)^m &= U^*T^nU(U^*T^*U)^m \\ &= U^*T^nUU^*(T^*)^mU \\ &= U^*T^n(T^*)^mU \\ &\leq U^*(T^*)^mT^nU \\ &= U^*(T^*)^mUU^*T^nU \\ &= (S^*)^mS^n \end{aligned}$$

Hence, S is (n,m)- hyponormal. □

Theorem 3.2.5.
Let $T \in B(H)$ be k-quasi-(n,m)- hyponormal operator and S be unitarily equivalent to T. Then S is also k-quasi-(n,m)- hyponormal

Proof.
Since S is unitarily equivalent to T, we have $S = U^*TU$ which implies $S^* = U^*T^*U$, $S^k = U^*T^kU$, and $S^{*k} = U^*T^{*k}U$, for $k \in \mathbb{N}$. Therefore;

$$\begin{aligned} S^{*(k+m)} &= S^{*k}S^{*m}S^kS^n \\ &= U^*T^{*k}UU^*T^{*m}UU^*T^kUU^*T^nU \\ &= U^*T^{*k}T^{*m}T^kT^nU \\ &= U^*T^{*k+m}T^{k+n}U \\ &\geq U^*T^{*k}T^nT^{*m}T^kU \\ &= U^*T^{*k}UU^*T^nUU^*T^{*m}UU^*T^kU \\ &= S^{*k}S^nS^{*m}S^k \end{aligned}$$

Hence S is k-quasi-(n,m)- hyponormal

Chapter 4

(n, m)-Binormal Operators

4.1 Introduction

Binormal operators play a key part in the study of operator theory, particularly in testing spectral algorithms. A new class of operators known as the (n, m)-binormal operators is introduced in this chapter, then we study its characteristics and relationship with other classes of operators. It is shown that if an operator T is (n, m)-binormal, and is unitarily quasiequivalent to an operator S, then S is also (n, m)-binormal.

Definition 4.1.1.
Let $T \in B(H)$. We say that T is (n, m)-binormal if $[T^n T^{m*}, T^{m*} T^n] = 0$ for some non-negative integers n and m. This class of operators will be denoted by $[(n, m)BN]$.

We begin by showing that the class of (n, m)-binormal operators is different from the class of n-binormal. This will be done by the use of an example.

Example 4.1.2.
Let $T = \begin{bmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then $T^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix}$. We show that T is a binormal operator that is not normal.

Observe that

$$TT^* = \begin{bmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$T^*T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This step shows that the operator T is not normal.
We check if the operator is binormal.

$$TT^*T^*T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$TT^*T^*T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since

$$TT^*T^*T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T^*TTT^*.$$

We have that T is a (1,1)-binormal operator.
This example also shows that (n, m) binormal operator is not necessarily (n, m) normal, and in general, binormal does not imply normal.
In addition, we give an example to show that if an operator S is (n, m)-binormal, then it is not necessarily binormal.

Example 4.1.3.
Let $S = \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix}$, then $S^* = \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix}$. We show that T is a $(3, 1)$-binormal operator that is not binormal.

Observe that

$$SS^* = \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} = \begin{bmatrix} 2 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$S^*S = \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & 2 \end{bmatrix}.$$

This step shows that the operator T is not normal.

We check if the operator is binormal.

$$SS^*S^*S = \begin{bmatrix} 2 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & 2 \end{bmatrix} = \begin{bmatrix} 0 & -i & -1 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$SS^*S^*S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & 2 \end{bmatrix} \begin{bmatrix} 2 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ i & 1 & 0 \\ -1 & i & 0 \end{bmatrix}.$$

Since

$$SS^*S^*S = \begin{bmatrix} 0 & -i & -1 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \\ i & 1 & 0 \\ -1 & i & 0 \end{bmatrix} = S^*SSS^*.$$

We have that T is not (1,1)-binormal operator, that is, it is not a binormal operator.

We next check if S is $(2,1)$-binormal. Given that $S = \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix}$, then

$$S^2 = SS = \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$S^*S^2S^2S^* = \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

On the other hand

$$S^2S^*S^*S^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have

$$S^*S^2S^2S^* = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = S^2S^*S^*S^2.$$

Hence, the operator is not $(2,1)$-binormal.

We next show that the operator is $(3,1)$-binormal.

Given that $S = \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix}$, then

$$S^3 = SSS = \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$S^*S^3S^3S^* = \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

On the other hand

$$S^3S^*S^*S^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have

$$S^*S^3S^3S^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = S^3S^*S^*S^3.$$

Hence, the operator is $(3,1)$-binormal.
This example shows that (n,m)-binormal operators is bigger than the class of n-binormal and binormal operators. It is further observed that the class of $(2,1)$-binormal operators does not necessarily belong to the class of $(3,1)$-binormal operators.

Remark 4.1.4.
For an operator T that belongs to the class $[(n,m)BN]$. We see that, If $n = m = 1$, then (n,m)-binormal becomes binormal, i.e. $[(1,1)BN] = [BN]$, If $m = 1$, then $(n,1)$-binormal becomes n-binormal, i.e. $[(n,1)BN] = [nBN]$.

Theorem 4.1.5.
Let $T \in B(H)$ be $(n,m)-$binormal operator then:

1. *αT is $(n,m)-$binormal operator for $\alpha \in \mathbb{C}$,*

2. *T^{-1} is (n,m)-binormal operator if it exists,*

3. *T^* is (n,m)-binormal operator.*

Proof.

1. Suppose $T \in B(H)$ be a $(n,m)-$binormal and $\alpha \in \mathbb{C}$, then

$$\begin{aligned}(\alpha T)^n(\alpha T)^{*m}(\alpha T)^{*m}(\alpha T)^n &= \alpha^n T^n \overline{\alpha}^m T^{*m} \overline{\alpha}^m T^{*m} \alpha^n T^n \\ &= \alpha^n \overline{\alpha}^m \overline{\alpha}^m \alpha^n (T^n T^{*m} T^{*m} T^n) \\ &= \alpha^n \overline{\alpha}^m \overline{\alpha}^m \alpha^n (T^{*m} T^n T^n T^{*m}) \qquad T \text{ is } (n,m)\text{-binormal} \\ &= \overline{\alpha}^m T^{*m} \alpha^n T^n \alpha^n T^n \overline{\alpha}^m T^{*m} \\ &= (\alpha T)^{*m}(\alpha T)^n(\alpha T)^n(\alpha T)^{*m}.\end{aligned}$$

   Hence αT is $(n,m)-$binormal operator for $\alpha \in \mathbb{C}$.

2. If $T \in B(H)$ is invertible, that is T^{-1} exists, then;

$$\begin{aligned}(T^{-1})^n(T^{-1})^{*m}(T^{-1})^{*m}(T^{-1})^n &= (T^n)^{-1}(T^{*m})^{-1}(T^{*m})^{-1}(T^n)^{-1} \\ &= (T^n T^{*m} T^{*m} T^n)^{-1} \\ &= (T^{*m} T^n T^n T^{*m})^{-1} \qquad \text{since } T \text{ is } (n,m)\text{-binormal} \\ &= (T^{*m})^{-1}(T^n)^{-1}(T^n)^{-1}(T^{*m})^{-1} \\ &= (T^{-1})^{*m}(T^{-1})^n(T^{-1})^n(T^{-1})^{*m}.\end{aligned}$$

   Hence T^{-1} is (n,m)-binormal.

3. Since $T \in B(H)$ is $(n,m)-$binormal operator, we have;

$$\begin{aligned}(T^*)^n(T^*)^{*m}(T^*)^{*m}(T^*)^n &= (T^n)^*(T^{*m})^*(T^{*m})^*(T^n)^* \\ &= (T^n T^{*m} T^{*m} T^n)^* \\ &= (T^{*m} T^n T^n T^{*m})^* \qquad \text{since } T \text{ is } (n,m)\text{-binormal} \\ &= (T^{*m})^*(T^n)^*(T^n)^*(T^{*m})^* \\ &= (T^*)^{*m}(T^*)^n(T^*)^n(T^*)^{*m}.\end{aligned}$$

   Hence T^* is $(n,m)-$binormal operator.

□

Having delimited the class of n-binormal operators from the class of (n,m)-binormal operators, and following the research of Shihab (2016), the class of square (n,m)-binormal operators is defined and shown to be different from the class of (n,m)-binormal operators, that is, there exists an operator $T \in B(H)$ which is square (n,m)-binormal operators but not (n,m)-binormal operators.

Definition 4.1.6 (Shihab, 2016)**.**
An operator $T \in B(H)$ is said to be square (n, m)-normal if

$$T^{2n}T^{*2m} = T^{*2m}T^{2n}.$$

Proposition 4.1.7.
If $T \in B(H)$ is (n, m)-normal operator, then T is square (n, m)-normal operator.

Proof.
Since $T \in B(H)$ is (n, m)-normal operator, it follows that

$$T^nT^{*m} = T^{*m}T^n.$$

We show that T is square (n, m)-normal operator.

$$\begin{aligned} T^{2n}T^{*2m} &= (T^2)^n(T^2)^{*m} \\ &= (TT)^n(TT)^{*m} \\ &= T^nT^nT^{*m}T^{*m} \\ &= T^nT^{*m}T^nT^{*m} \\ &= T^{*m}T^nT^{*m}T^n \\ &= T^{*m}T^{*m}T^nT^n \\ &= T^{*2m}T^{2n}. \end{aligned}$$

Hence T is square (n, m)-normal operator □

The converse of this proposition is however not true.

Example 4.1.8.
Consider the matrix T given by

$$T = \begin{bmatrix} -i & 0 \\ -i & i \end{bmatrix} \qquad \text{then} \qquad T^* = \begin{bmatrix} i & i \\ 0 & -i \end{bmatrix}.$$

This matrix is not $(1, 1)$-normal.
Observe that

$$TT^* = \begin{bmatrix} -i & 0 \\ -i & i \end{bmatrix} \begin{bmatrix} i & i \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and

$$T^*T = \begin{bmatrix} i & i \\ 0 & -i \end{bmatrix} \begin{bmatrix} -i & 0 \\ -i & i \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Since

$$TT^* = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = T^*T,$$

it follows that T is not $(1,1)$-normal.
Let us now look at T^2.

$$T^2 = TT = \begin{bmatrix} -i & 0 \\ -i & i \end{bmatrix} \begin{bmatrix} -i & 0 \\ -i & i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$T^{*2} = T^*T^* = \begin{bmatrix} i & i \\ 0 & -i \end{bmatrix} \begin{bmatrix} i & i \\ 0 & -i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is easily seen that $T^2T^{2*} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = T^{2*}T^2$. Hence T is square $(1,1)$-normal.

Proposition 4.1.9.
$T \in B(H)$ is square (n,m)-normal operator if and only if T^2 is (n,m)-normal operator.

Proof.
Let $T \in B(H)$ be square (n,m)-normal operator, it follows that

$$T^{2n}T^{*2m} = T^{*2m}T^{2n}.$$

We show that T^2 is (n,m)-normal operator.

$$\begin{aligned} (T^2)^n(T^2)^{*m} &= T^{2n}T^{*2m} \\ &= T^{*2m}T^{2n} \\ &= (T^2)^{*m}(T^2)^n. \end{aligned}$$

Hence T^2 is (n,m)-normal operator. The converse follows easily. □

Definition 4.1.10.
An operator $T \in B(H)$ is said to be square (n,m)-binormal if

$$T^{2n}T^{*2m}T^{*2m}T^{2n} = T^{*2m}T^{2n}T^{2n}T^{*2m}.$$

Proposition 4.1.11.
An operator $T \in B(H)$ is square (n,m)-binormal if and only if T^2 is (n,m)-binormal operator.

Proof.

Let $T \in B(H)$ be square (n,m)-binormal operator, it follows that

$$T^{2n}T^{*2m}T^{*2m}T^{2n} = T^{*2m}T^{2n}T^{2n}T^{*2m}.$$

We show that T^2 is (n,m)-binormal operator.

$$\begin{aligned}(T^2)^n(T^2)^{*m}(T^2)^{*m}(T^2)^n &= T^{2n}T^{*2m}T^{*2m}T^n \\ &= T^{*2m}T^{2n}T^nT^{*2m} \\ &= (T^2)^{*m}(T^2)^n(T^2)^n(T^2)^{*m}.\end{aligned}$$

Hence T^2 is (n,m)-binormal operator. The converse follows easily. □

Lemma 4.1.12.
Let $T \in B(H)$ be a square (n,m)-binormal operator. If $U \in B(H)$ is a unitary operator such that $S \in B(H)$ is unitarily equivalent to T, then S is square (n,m)-binormal operator.

Proof.
Let $T \in B(H)$ be square (n,m)-normal operator. Since S is unitarily equivalent to T, we have $S = UTU^*$, implying $S^{2n} = UT^{2n}U^*$ and $S^{*2m} = UT^{*2m}U^*$. We show that S is (n,m)-normal operator.

$$\begin{aligned}S^{2n}S^{*2m} &= UT^{2n}U^*UT^{*2m}U^* \\ &= UT^{2n}T^{*2m}U^* \\ &= UT^{*2m}T^{2n}U^* \\ &= UT^{*2m}U^*UT^{2n}U^* \\ &= S^{*2m}S^{2n}.\end{aligned}$$

Hence S is (n,m)-normal operator. □

Proposition 4.1.13.
Let T be a bounded operator that is n-normal, then T is square n-binormal.

Proof.
Let $T \in B(H)$ be n-normal operator. Then,

$$\begin{aligned}T^{2n}T^{*2}T^{*2}T^{2n} &= T^nT^nT^*T^*T^*T^*T^nT^n \\ &= T^nT^*T^nT^*T^*T^nT^*T^n \qquad \text{since } T \text{ is } n\text{-normal} \\ &= T^*T^nT^*T^nT^nT^*T^nT^* \\ &= T^*T^*T^nT^nT^nT^*T^nT^*\end{aligned}$$

$$
\begin{aligned}
&= T^*T^*T^nT^nT^nT^nT^*T^* \\
&= T^{*2}T^{2n}T^{2n}T^{*2}.
\end{aligned}
$$

Hence T is square n-binormal operator. $\square$

Proposition 4.1.14.
Let T be a bounded operator that is (n, m)-normal, then T is square (n, m)-binormal.

Proof.
Let $T \in B(H)$ be (n, m)-normal operator. Then,

$$
\begin{aligned}
T^{2n}T^{*2m}T^{*2m}T^{2n} &= T^nT^nT^{*m}T^{*m}T^{*m}T^{*m}T^nT^n \\
&= T^nT^{*m}T^nT^{*m}T^{*m}T^nT^{*m}T^n \qquad \text{since } T \text{ is } (n,m)\text{normal} \\
&= T^{*m}T^nT^{*m}T^nT^nT^{*m}T^nT^{*m} \\
&= T^{*m}T^{*m}T^nT^nT^nT^{*m}T^nT^{*m} \\
&= T^{*m}T^{*m}T^nT^nT^nT^nT^{*m}T^{*m} \\
&= T^{*2m}T^{2n}T^{2n}T^{*2m}.
\end{aligned}
$$

Hence T is square (n, m)-binormal operator.

Remark 4.1.15.
Let T be square (n, m) binormal, then T is not necessarily (n, m) binormal.

Example 4.1.16.
Consider the matrix T given by

$$
T = \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{then} \qquad T^* = \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix}.
$$

This matrix is not $(1, 1)$-binormal.
Solution

$$
TT^* = \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} = \begin{bmatrix} 2 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

and

$$
T^*T = \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & 2 \end{bmatrix}.
$$

Since

$$TT^* = \begin{bmatrix} 2 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq T^*T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & 2 \end{bmatrix}$$

it follows that T is not $(1,1)$-normal.

Let us now check for $(1,1)$-binormality of T. Consider.

$$TT^*T^*T = \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i & -1 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$T^*TTT^* = \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ i & 1 & 0 \\ -1 & i & 0 \end{bmatrix}.$$

Since

$$TT^*T^*T = \begin{bmatrix} 0 & -i & -1 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{bmatrix} \neq T^*TTT^* = \begin{bmatrix} 0 & 0 & 0 \\ i & 1 & 0 \\ -1 & i & 0 \end{bmatrix}$$

it follows that T is not $(1,1)$-binormal.

Now consider the squares

$$T^2 = TT = \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 1 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$T^{*2} = T^*T^* = \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 1 & -i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

It is easily seen that $T^2T^{2*}T^{2*}T^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T^{2*}T^2T^2T^{2*}$. Hence T is square $(1,1)$-binormal.

Definition 4.1.17.

An operator $T \in B(H)$ is said to be complex symmetric, if there exist a conjugation C such that $T = CT^*C$

Theorem 4.1.18 (Yaru, 2017)**.**
Let $T \in B(H)$ be a binormal complex symmetric operator. If the conjugation C commutes with T^*T and TT^*, then T is a normal operator

This theorem gives conditions under which a binormal operator is a normal operator. We now prescribe conditions for an (n,m)-binormal operator to be an $(n,m)-normal operator$.

Theorem 4.1.19.
Let $T \in B(H)$ be an (n,m) binormal complex symmetric operator. If the conjugation C commutes with $T^{*m}T^n, T^nT^{*m}, T^{n-1}$, and T^{*m-1}, then T is an (n,m)-normal operator

Proof.
Since T is (n,m)-binormal we have $T^{*m}T^nT^nT^{*m} = T^nT^{*m}T^{*m}T^n$, and complex symmetry implies $T = CT^*C$

$$\begin{aligned}
T^{*m}T^nT^nT^{*m} &= T^{*m}T^nCT^mCCT^{*n}C \\
&= T^{*m}T^nCT^{*n-1}T^*CCTT^{m-1}C \\
&= T^{*m}T^nCT^{*n-1}CTT^*CT^{m-1}C \\
&= T^{*m}T^nCT^{*n-2}T^*CCTT^{m-2}C \\
&= T^{*m}T^nCT^{*n-2}CT^2T^{*2}CT^{m-2}C \\
&\vdots \\
&= T^{*m}T^nCT^*CT^{n-1}T^{*m-1}CTC \\
&= T^{*m}T^nCT^*CT^{*m-1}T^{n-1}CTC \\
&= T^{*m}T^nCT^{*m}CT^{n-1}CTC \\
&= T^{*m}T^nCT^{*m}CCT^nC \\
&= T^{*m}T^nCT^{*m}T^nC \\
&= T^{*m}T^nT^{*m}T^n
\end{aligned}$$

$$\begin{aligned}
T^nT^{*m}T^{*m}T^n &= T^nT^{*m}CT^{*n}CCT^mC \\
&= T^{*m}T^nCT^{*n-1}T^*CCTT^{m-1}C \\
&= T^{*m}T^nCT^{*n-1}CTT^*CT^{m-1}C \\
&= T^{*m}T^nCT^{*n-2}T^*CCTT^{m-2}C \\
&= T^{*m}T^nCT^{*n-2}CT^2T^{*2}CT^{m-2}C \\
&\vdots \\
&= T^{*m}T^nCT^*CT^{n-1}T^{*m-1}CTC \\
&= T^{*m}T^nCT^*CT^{*m-1}T^{n-1}CTC
\end{aligned}$$

$$
\begin{aligned}
&= T^{*m}T^nCT^{*m}CT^{n-1}CTC \\
&= T^{*m}T^nCT^{*m}CCT^nC \\
&= T^{*m}T^nCT^{*m}T^nC \\
&= T^{*m}T^nT^{*m}T^n
\end{aligned}
$$

We have

$$T^{*m}T^nT^nT^{*m} = T^{*m}T^nT^{*m}T^n = T^nT^{*m}T^{*m}T^n$$

Hence, T is (n, m)-normal.

4.2 Algebraic Properties of (n, m)-Binormal Operators

In this section we discuss some of the basic algebraic properties of (n, m)-binormal operators as well as its relationship with other classes of operators.

Remark 4.2.1.
The class of binormal operators is not closed under the basic arithmetic operations.

We consider both addition and products.

Example 4.2.2.
Let

$$T = \begin{bmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We first show that the two operators are $(2, 1)$-binormal.

$$T^2 = \begin{bmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$T^2T^*T^*T^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$T^*T^2T^2T^* = \begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$T^*T^2T^2T^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T^2T^*T^*T^2.$$

This shows that T is $(2,1)$-binormal.
Next we show that S is $(2,1)$-binormal

$$S^2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$S^2S^*S^*S^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$S^*S^2S^2S^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$S^*S^2S^2S^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = S^2S^*S^*S^2.$$

This shows that S is also $(2,1)$-binormal.
We now check whether the sum of the two operators, that is, $T+S$ is $(2,1)$-binormal.

$$T+S = \begin{bmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & i \\ i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To begin with, we find $(T+S)^2$ and $(T+S)^*$.

$$(T+S)^2 = (T+S)(T+S) = \begin{bmatrix} 0 & 0 & i \\ i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & i \\ i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ i & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$(T+S)^* = \begin{bmatrix} 0 & -i & 0 \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix}$$

Then

$$\begin{aligned}(T+S)^2(T+S)^*(T+S)^*(T+S)^2 &= \begin{bmatrix} 0 & 0 & 0 \\ i & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i & 0 \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i & 0 \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ i & 1 & -1 \\ -i & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 3i & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}(T+S)^*(T+S)^2(T+S)^2(T+S)^* &= \begin{bmatrix} 0 & -i & 0 \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ i & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ i & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i & 0 \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2i & 0 \\ i & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

We see that

$$(T+S)^2(T+S)^*(T+S)^*(T+S)^2 \neq (T+S)^*(T+S)^2(T+S)^2(T+S)^*$$

Hence the sum of two (n,m)-binormal operators is not necessarily (n,m)-binormal.

In the next example, we show that the product of two (n,m)-binormal operators is not necessarily (n,m)-binormal.

Example 4.2.3.
Let

$$T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We start by showing that both T and S are $(2,1)$-binormal.

$$T^2 = TT = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$T^2T^*T^*T^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$T^*T^2T^2T^* = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$T^*T^2T^2T^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T^2T^*T^*T^2.$$

This shows that T is $(2,1)$-binormal.
Next we show that S is $(2,1)$-binormal

$$S^2 = SS = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$S^2S^*S^*S^2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 8 & 0 \\ 8 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$S^*S^2S^2S^* = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 8 & 0 \\ 8 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$S^*S^2S^2S^* = \begin{bmatrix} 13 & 8 & 0 \\ 8 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} = S^2S^*S^*S^2.$$

This shows that S is also $(2,1)$-binormal.
We now check whether the product of the two operators, that is, TS is $(2,1)$-binormal.

$$TS = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To begin with, we find $(TS)^2$ and $(TS)^*$.

$$(TS)^2 = (TS)(TS) = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$(TS)^* = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then

$$\begin{aligned}(TS)^2(TS)^*(TS)^*(TS)^2 &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 0 \\ -2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}(TS)^*(TS)^2(TS)^2(TS)^* &= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

We see that

$$(TS)^2(TS)^*(TS)^*(TS)^2 \neq (TS)^*(TS)^2(TS)^2(TS)^*$$

Hence the product of two (n,m)-binormal operators is not necessarily (n,m)-binormal.
We now prescribe conditions for the product of (n,m)-binormal operators to be (n,m)-binormal.

Theorem 4.2.4.
Let $S, T \in B(H)$ be (n,m) binormal operators. If S and T are doubly commuting, then ST is (n,m) binormal.

Proof.
Since S and T are doubly commuting, we have $ST = TS$ and $TS^* = S^*T$.

$$\begin{aligned}
(ST)^{*m}(ST)^n(ST)^n(ST)^{*m} &= T^{*m}S^{*m}S^nT^nS^nT^nS^{*m}T^{*m} \\
&= S^{*m}T^{*m}S^nT^nT^nS^nT^{*m}S^{*m} \\
&= S^{*m}S^nT^{*m}T^nT^nT^{*m}S^nS^{*m} \\
&= S^{*m}S^nT^nT^{*m}T^{*m}T^nS^nS^{*m} && \text{since } T \text{ is } (n,m)\text{binormal} \\
&= S^{*m}S^nT^nT^{*m}T^{*m}S^nT^nS^{*m} \\
&= S^{*m}S^nT^nT^{*m}S^nT^{*m}S^{*m}T^n \\
&= S^{*m}S^nT^nS^nT^{*m}S^{*m}T^{*m}T^n \\
&= S^{*m}S^nS^nT^nS^{*m}T^{*m}T^{*m}T^n \\
&= S^{*m}S^nS^nS^{*m}T^nT^{*m}T^{*m}T^n \\
&= S^nS^{*m}S^{*m}S^nT^nT^{*m}T^{*m}T^n && \text{since } S \text{ is } (n,m)\text{binormal} \\
&= S^nS^{*m}S^{*m}T^nS^nT^{*m}T^{*m}T^n \\
&= S^nS^{*m}T^nS^{*m}T^{*m}S^nT^{*m}T^n \\
&= S^nT^nS^{*m}T^{*m}S^{*m}T^{*m}S^nT^n \\
&= (TS)^n(TS)^{*m}(TS)^{*m}(TS)^n \\
&= (ST)^n(ST)^{*m}(ST)^{*m}(ST)^n.
\end{aligned}$$

Therefore, ST is (n,m) binormal. □

Remark 4.2.5. Kaplansky theorem applying Fuglede-Putnam theorem for normal operators states that, if T and S are normal operators such that TS is normal, then ST is normal if U and V commute, where $T = U|T|$ and $S = V|S|$. This theorem has been extended to hyponormal operators and quasinormal operators. We extend the theorem to the class of (n,m)-normal operators and (n,m)-binormal operators.

Theorem 4.2.6.
Let $S, T \in B(H)$ such that S is normal and ST is (n,m)-normal. Then TS is (n,m)-normal if T commutes with $|S|$.

Proof.
Since the operator S is normal, by application of the polar decomposition, there exists an

operator U which is unitary such that,

$$S = U|S| = |S|U$$

Applying the assumption that T commutes with $|S|$, we have

$$\begin{aligned} TS &= TU|S| \\ &= T|S|U \\ &= |S|TU \\ &= U^*U|S|TU \\ &= U^*STU \end{aligned}$$

This implies that TS is unitarily equivalent to ST, but from the results of Chō, Lee, Tanahashi, and Uchiyama (2018), lemma 4.2 (iii), we have that TS must also be (n, m)-normal. □

Corollary 4.2.7.
Let $S, T \in B(H)$ such that S is unitary and ST is (n, m)-normal. Then

$$TS \;\; is \;\; (n, m) - normal \iff ST \;\; is \;\; (n, m) - normal$$

Proof.
Assume that ST, since S is unitary, it follows that it is also a normal operator and $S^*ST = TS^*S = T$. Hence, by theorem 3.2.4, we have that TS is also (n, m)-normal.
Conversely, supposing TS is (n, m)-normal, then by Lemma 4.2 (i) of Cho et al., (2018), we have that $(TS)^* = S^*T^*$ is also (n, m)-normal. Given that S is unitary and hence normal, it follows that S^* is also unitary and normal. We therefore have, S^* is normal and S^*T^* is (n, m)-normal implying normality of T^*S^* by theorem 3.2.5. But $T^*S^* = (ST)^*$. This shows that $(ST)^*$ is (n, m)-normal, but the adjoint of an (n, m)-normal operator is (n, m)-normal. Hence we have that ST is (n, m)-normal.

Theorem 4.2.8.
Let $S, T \in B(H)$ such that S is normal and ST is (n, m)-binormal. Then TS is (n, m)-binormal if T commutes with $|S|$.

Proof.
Since the operator S is normal, by application of the polar decomposition, there exists an operator U which is unitary such that,

$$S = U|S| = |S|U$$

Applying the assumption that T commutes with $|S|$, we have

$$\begin{aligned} TS &= TU|S| \\ &= T|S|U \\ &= |S|TU \\ &= U^*U|S|TU \\ &= U^*STU \end{aligned}$$

This implies that TS is unitarily equivalent to ST, but from the results of Kikete et al., (2023), we have that TS must also be (n, m)-binormal. □

Corollary 4.2.9.
Let $S, T \in B(H)$ such that S is unitary and ST is (n, m)-binormal. Then

$$TS \ \ is \ \ (n, m) - binormal \iff ST \ \ is \ \ (n, m) - binormal$$

Proof.
Assume that ST, since S is unitary, then it is normal operator and $S^*ST = TS^*S = T$, indicating that T commutes with $|S|$. Hence, by theorem 3.2.6, we have that TS is also (n, m)-binormal.
Conversely, supposing TS is (n, m)-binormal, then by a result of Kikete et al., (2023), we have that $(TS)^* = S^*T^*$ is also (n, m)-binormal. Given that S is unitary, it follows that S^* is also unitary and normal. We therefore have, S^* is normal and S^*T^* is (n, m)-binormal implying normality of T^*S^* by theorem 3.2.7. But $T^*S^* = (ST)^*$. This shows that $(ST)^*$ is (n, m)-binormal, but the adjoint of an (n, m)-binormal operator is (n, m)-binormal. Hence we have that ST is (n, m)-binormal.

Proposition 4.2.10.
Let $T \in B(H)$ be (n, m)-normal operator. Then T is (n, m)-binormal.

Proof.
Given that T is (n, m)-normal, it follows that $T^nT^{*m} = T^{*m}T^n$. Therefore;

$$T^nT^{*m}T^{*m}T^n = T^{*m}T^nT^nT^{*m}.$$

□

Theorem 4.2.11.
Let $T \in B(H)$ be $(n, m)-$binormal operator then:

1. *αT is $(n, m)-$binormal operator for $\alpha \in \mathbb{C}$,*

2. T^{-1} *is* (n,m)*-binormal operator if it exists,*

3. T^* *is* (n,m)*-binormal operator.*

Proof.

1. Suppose $T \in B(H)$ be a $(n,m)-$binormal and $\alpha \in \mathbb{C}$, then

$$\begin{aligned}
(\alpha T)^n(\alpha T)^{*m}(\alpha T)^{*m}(\alpha T)^n &= \alpha^n T^n \overline{\alpha}^m T^{*m} \overline{\alpha}^m T^{*m} \alpha^n T^n \\
&= \alpha^n \overline{\alpha}^m \overline{\alpha}^m \alpha^n (T^n T^{*m} T^{*m} T^n) \\
&= \alpha^n \overline{\alpha}^m \overline{\alpha}^m \alpha^n (T^{*m} T^n T^n T^{*m}) \qquad T \text{ is } (n,m)\text{-binormal} \\
&= \overline{\alpha}^m T^{*m} \alpha^n T^n \alpha^n T^n \overline{\alpha}^m T^{*m} \\
&= (\alpha T)^{*m}(\alpha T)^n(\alpha T)^n(\alpha T)^{*m}.
\end{aligned}$$

   Hence αT is $(n,m)-$binormal operator for $\alpha \in \mathbb{C}$.

2. If $T \in B(H)$ is invertible, that is T^{-1} exists, then;

$$\begin{aligned}
(T^{-1})^n(T^{-1})^{*m}(T^{-1})^{*m}(T^{-1})^n &= (T^n)^{-1}(T^{*m})^{-1}(T^{*m})^{-1}(T^n)^{-1} \\
&= (T^n T^{*m} T^{*m} T^n)^{-1} \\
&= (T^{*m} T^n T^n T^{*m})^{-1} \qquad \text{since } T \text{ is } (n,m)\text{-binormal} \\
&= (T^{*m})^{-1}(T^n)^{-1}(T^n)^{-1}(T^{*m})^{-1} \\
&= (T^{-1})^{*m}(T^{-1})^n(T^{-1})^n(T^{-1})^{*m}.
\end{aligned}$$

   Hence T^{-1} is (n,m)-binormal.

3. Since $T \in B(H)$ is $(n,m)-$binormal operator, we have;

$$\begin{aligned}
(T^*)^n(T^*)^{*m}(T^*)^{*m}(T^*)^n &= (T^n)^*(T^{*m})^*(T^{*m})^*(T^n)^* \\
&= (T^n T^{*m} T^{*m} T^n)^* \\
&= (T^{*m} T^n T^n T^{*m})^* \qquad \text{since } T \text{ is } (n,m)\text{-binormal} \\
&= (T^{*m})^*(T^n)^*(T^n)^*(T^{*m})^* \\
&= (T^*)^{*m}(T^*)^n(T^*)^n(T^*)^{*m}.
\end{aligned}$$

   Hence T^* is $(n,m)-$binormal operator.

□

Proposition 4.2.12.
Let $T \in B(H)$ *be* (n,m)*-binormal. Then* T^k *for* $k \in \mathbb{Z}^+$*, is* (n,m)*-binormal if* T *is*

(n,m)-normal.

Proof.

$$\begin{aligned}
(T^k)^n(T^k)^{*m}(T^k)^{*m}(T^k)^n &= \underbrace{(T\dots T)^n}_{\text{k times}}\underbrace{(T\dots T)^{*m}}_{\text{k times}}\underbrace{(T\dots T)^{*m}}_{\text{k times}}\underbrace{(T\dots T)^n}_{\text{k times}}\\
&= \underbrace{(T\dots T)^n}_{\text{k-1 times}}T^nT^{*m}\underbrace{(T\dots T)^{*m}}_{\text{k-1 times}}\underbrace{(T\dots T)^{*m}}_{\text{k-1 times}}T^{*m}T^n\underbrace{(T\dots T)^n}_{\text{k-1 times}}\\
&\text{Since } T \text{ is } (n,m)\text{-normal, we get}\\
&= \underbrace{(T\dots T)^n}_{\text{k-1 times}}T^{*m}T^n\underbrace{(T\dots T)^{*m}}_{\text{k-1 times}}\underbrace{(T\dots T)^{*m}}_{\text{k-1 times}}T^nT^{*m}\underbrace{(T\dots T)^n}_{\text{k-1 times}}\\
&\vdots\\
&= \underbrace{(T\dots T)^n}_{\text{k times}}\underbrace{(T\dots T)^{*m}}_{\text{k times}}\underbrace{(T\dots T)^{*m}}_{\text{k times}}\underbrace{(T\dots T)^n}_{\text{k times}}\\
&= (T^k)^{*m}(T^k)^n(T^k)^n(T^k)^{*m}.
\end{aligned}$$

Hence, T^k is (n,m)-binormal.

Theorem 4.2.13.
Suppose T is a self-adjoint operator. Then T is (n,m)-binormal.

Proof.
Let T be such that $T^* = T$. Then

$$\begin{aligned}
T^nT^{*m}T^{*m}T^n &= T^nT^mT^mT^n\\
&= T^{(n+m)}T^{(m+n)}\\
&= T^{2(n+m)}\dots(i).
\end{aligned}$$

$$\begin{aligned}
T^{*m}T^nT^nT^{*m} &= T^mT^nT^nT^m\\
&= T^{(n+m)}T^{(m+n)}\\
&= T^{2(n+m)}\dots(ii).
\end{aligned}$$

From (i) and (ii) it follows that T is (n,m)-binormal operator. □

Corollary 4.2.14.
Let $T \in B(H)$ be any operator. Then

1. *$T + T^*$ and*

2. *TT^**

are (n,m)-binormal operators.

Proof.

1. Let $S = T + T^*$. Then

$$S^* = (T + T^*)^* = (T^* + T^{**}) = (T^* + T) = S.$$

   Hence S is self adjoint. But from theorem 2.5, every self adjoint operator is (n,m)-binormal. Thus, $(T + T^*)$ is (n,m)-binormal.

2. Let $S = TT^*$. Then

$$S^* = (TT^*)^* = (T^{**}T^*) = (TT^*) = S.$$

   Hence S is self adjoint. But from theorem 2.5, every self adjoint operator is (n,m)-binormal. Thus, (TT^*) is (n,m)-binormal.

□

We give an illustration for corollary 3.2.9.

Example 4.2.15.
Let

$$T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \text{then} \qquad T^* = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} \\ \overline{a_{12}} & \overline{a_{22}} \end{bmatrix}$$

We show that $T + T^*$ and $TT*$ are binormal.

Solution
We start with $T + T^*$

$$T + T^* = \begin{bmatrix} a_{11} + \overline{a_{11}} & a_{12} + \overline{a_{21}} \\ a_{21} + \overline{a_{12}} & a_{22} + \overline{a_{22}} \end{bmatrix}$$

and

$$\begin{aligned}
(T + T^*)^* &= \begin{bmatrix} \overline{a_{11} + \overline{a_{11}}} & \overline{a_{21} + \overline{a_{12}}} \\ \overline{a_{12} + \overline{a_{21}}} & \overline{a_{22} + \overline{a_{22}}} \end{bmatrix} \\
&= \begin{bmatrix} \overline{a_{11}} + a_{11} & \overline{a_{21}} + a_{12} \\ \overline{a_{12}} + a_{21} & \overline{a_{22}} + a_{22} \end{bmatrix} \\
&= T^* + T \\
&= T + T^*
\end{aligned}$$

Therefore;

$$
\begin{aligned}
(T+T^*)^*(T+T^*)(T+T^*)(T+T^*)^* &= (T^*+T)(T+T^*)(T+T^*)(T^*+T)\\
&= (T+T^*)(T+T^*)(T+T^*)(T+T^*)\\
&= (T+T^*)^4
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
(T+T^*)(T+T^*)^*(T+T^*)^*(T+T^*) &= (T+T^*)(T^*+T)(T^*+T)(T+T^*)\\
&= (T+T^*)(T+T^*)(T+T^*)(T+T^*)\\
&= (T+T^*)^4
\end{aligned}
$$

Since

$$(T+T^*)^*(T+T^*)(T+T^*)(T+T^*)^* = (T+T^*)^4 = (T+T^*)(T+T^*)^*(T+T^*)^*(T+T^*)$$

we conclude that

$$(T+T^*)^*(T+T^*)(T+T^*)(T+T^*)^* = (T+T^*)(T+T^*)^*(T+T^*)^*(T+T^*).$$

Hence, $(T+T^*)$ is binormal, and (n,m)-binormal.
For TT^*, we have

$$
\begin{aligned}
TT^* &= \begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix} \overline{a_{11}} & \overline{a_{21}}\\ \overline{a_{12}} & \overline{a_{22}}\end{bmatrix}\\
&= \begin{bmatrix} a_{11}\overline{a_{11}}+a_{12}\overline{a_{12}} & a_{11}\overline{a_{21}}+a_{12}\overline{a_{22}}\\ a_{21}\overline{a_{11}}+a_{22}\overline{a_{12}} & a_{21}\overline{a_{21}}+a_{22}\overline{a_{22}}\end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
(TT^*)^* &= \begin{bmatrix} \overline{a_{11}\overline{a_{11}}+a_{12}\overline{a_{12}}} & \overline{a_{11}\overline{a_{21}}+a_{12}\overline{a_{22}}}\\ \overline{a_{21}\overline{a_{11}}+a_{22}\overline{a_{12}}} & \overline{a_{21}\overline{a_{21}}+a_{22}\overline{a_{22}}}\end{bmatrix}\\
&= \begin{bmatrix} a_{11}\overline{a_{11}}+a_{12}\overline{a_{12}} & a_{21}\overline{a_{11}}+a_{22}\overline{a_{12}}\\ a_{11}\overline{a_{21}}+a_{12}\overline{a_{22}} & a_{21}\overline{a_{21}}+a_{22}\overline{a_{22}}\end{bmatrix}\\
&= T^*T
\end{aligned}
$$

Therefore;

$$
\begin{aligned}
(TT^*)(TT^*)^*(TT^*)^*(TT^*) &= (TT^*)(T^*T^{**})(T^*T^{**})(TT^*)\\
&= (TT^*)(TT^*)(TT^*)(TT^*) \qquad \text{since} \qquad TT^* = (TT^*)^*
\end{aligned}
$$

$$= (TT^*)^4$$

and

$$\begin{aligned}(TT^*)^*(TT^*)(TT^*)(TT^*)^* &= (T^*T^{**})(TT^*)(TT^*)(T^*T^{**}) \\ &= (TT^*)(TT^*)(TT^*)(TT^*) \qquad \text{since} \qquad TT^* = (TT^*)^* \\ &= (TT^*)^4\end{aligned}$$

From the above we have that,

$$(TT^*)(TT^*)^*(TT^*)^*(TT^*) = (TT^*)^*(TT^*)(TT^*)(TT^*)^*$$

Hence (TT^*) is binormal, and (n, m)-binormal.

Proposition 4.2.16.
Let $T_i \in B(H_i)$ for $i \in \mathbb{N}$ be (n, m)-binormal operators. Then the following operators are also (n, m)-binormal operators.

1. $\oplus_{i=1}^{k} T$ *and*
2. $\otimes_{i=1}^{k} T$

provided $[T_i, T_j^*] = 0$ for all $i, j \leq k$.

Proof.

1. We show that $\oplus_{i=1}^{k} T$ is (n, m)-binormal operator

$$\begin{aligned}(T_1 \oplus \ldots \oplus T_k)^n (T_1 \oplus \ldots \oplus T_k)^{*m} (T_1 \oplus \ldots \oplus T_k)^{*m} (T_1 \oplus \ldots \oplus T_k)^n &= (T_1^n \oplus \ldots \oplus T_k^n) \\ &\quad (T_1^{*m} \oplus \ldots \oplus T_k^{*m}) \\ &\quad (T_1^{*m} \oplus \ldots \oplus T_k^{*m}) \\ &\quad (T_1^n \oplus \ldots \oplus T_k^n) \\ &= (T_1^n T_1^{*m} \oplus \ldots \oplus T_k^n T_k^{*m}) \\ &\quad (T_1^{*m} T_1^n \oplus \ldots \oplus T_k^{*m} T_k^n) \\ &= (T_1^n T_1^{*m} T_1^{*m} T_1^n \oplus \ldots \\ &\quad \oplus T_k^n T_k^{*m} T_k^{*m} T_k^n) \\ &= (T_1^{*m} T_1^n T_1^n T_1^{*m} \oplus \ldots \\ &\quad \oplus T_k^{*m} T_k^n T_k^n T_k^{*m}) \\ &= (T_1^{*m} T_1^n \oplus \ldots \oplus T_k^{*m} T_k^n) \\ &\quad (T_1^n T_1^{*m} \oplus \ldots \oplus T_k^n T_k^{*m})\end{aligned}$$

$$
\begin{aligned}
&= (T_1^{*m} \oplus \ldots \oplus T_k^{*m})\\
&(T_1^n \oplus \ldots \oplus T_k^n)\\
&(T_1^n \oplus \ldots \oplus T_k^n)\\
&(T_1^{*m} \oplus \ldots \oplus T_k^{*m})\\
&= (T_1 \oplus \ldots \oplus T_k)^{*m}\\
&(T_1 \oplus \ldots \oplus T_k)^n\\
&(T_1 \oplus \ldots \oplus T_k)^n\\
&(T_1 \oplus \ldots \oplus T_k)^{*m}.
\end{aligned}
$$

2. We show that $\otimes_{i=1}^k T$ is (n, m)-binormal operator. We apply the distributive property of tensor products over matrix multiplication.

$$
\begin{aligned}
(T_1 \otimes \ldots \otimes T_k)^n (T_1 \otimes \ldots \otimes T_k)^{*m} (T_1 \otimes \ldots \otimes T_k)^{*m} (T_1 \otimes \ldots \otimes T_k)^n &= (T_1^n \otimes \ldots \otimes T_k^n)\\
&\times (T_1^{*m} \otimes \ldots \otimes T_k^{*m})\\
&\times (T_1^{*m} \otimes \ldots \otimes T_k^{*m})\\
&\times (T_1^n \otimes \ldots \otimes T_k^n)\\
&= (T_1^n T_1^{*m} \otimes \ldots \otimes T_k^n T_k^{*m})\\
&\times (T_1^{*m} T_1^n \otimes \ldots \otimes T_k^{*m} T_k^n)\\
&= (T_1^n T_1^{*m} T_1^{*m} T_1^n \otimes \ldots\\
&\otimes T_k^n T_k^{*m} T_k^{*m} T_k^n)\\
&= (T_1^{*m} T_1^n T_1^n T_1^{*m} \otimes \ldots\\
&\otimes T_k^{*m} T_k^n T_k^n T_k^{*m})\\
&= (T_1^{*m} T_1^n \otimes \ldots \otimes T_k^{*m} T_k^n)\\
&\times (T_1^n T_1^{*m} \otimes \ldots \otimes T_k^n T_k^{*m})\\
&= (T_1^{*m} \otimes \ldots \otimes T_k^{*m})\\
&\times (T_1^n \otimes \ldots \otimes T_k^n)\\
&\times (T_1^n \otimes \ldots \otimes T_k^n)\\
&\times (T_1^{*m} \otimes \ldots \otimes T_k^{*m})\\
&= (T_1 \otimes \ldots \otimes T_k)^{*m}\\
&\times (T_1 \otimes \ldots \otimes T_k)^n\\
&\times (T_1 \otimes \ldots \otimes T_k)^n\\
&\times (T_1 \otimes \ldots \otimes T_k)^{*m}.
\end{aligned}
$$

□

Hence $(T_1 \otimes \ldots \otimes T_k) = \oplus_{i=1}^{k} T$ is (n, m)-binormal operator.

4.3 Equivalences in the class of binormal operators

In this section, we discuss equivalences of (n, m)-binormal operators with respect to invertible and unitary operators. We consider, unitary equivalence, similarity, metric equivalence, almost similarity, quasisimilarity, and near equivalence.

Definition 4.3.1. (Nzimbi and Luketero, 2020)
Let S and T be bounded operators in a Hilbert space H. Then S and T are said to be unitarily quasi-equivalent, if given a unitary operator U, we have

$$S^*S = UT^*TU^* \qquad \text{and} \qquad SS^* = UTT^*U.$$

Luketero (2020) in theorem 3.1 proved that if an operator $T \in B(H)$ which is n-binormal is unitarily equivalent to an operator S, it follows that S will also be n-binormal. It is further shown that the class of n-binormal operators is not only unitarily invariant but also isometrically and co-isometrically invariant. We extend this result to the class of (n, m)-binormal operators in the theorem that follows.

Theorem 4.3.2.
Let $T \in B(H)$ be (n, m)-binormal operator and $S \in B(H)$ be such that:

1. *$S = UTU^*$ with U being an isometry, then S is also $(n, m)-$binormal.*
2. *$S = U^*TU$ with U being a co-isometry, then S is also $(n, m)-$binormal.*

Proof.

1. Let $S \in B(H)$ be such that $S = UTU^*$ for U an isometry. Then

$$\begin{aligned} S^* &= (UTU^*)^* \\ &= UT^*U^* \end{aligned}$$

and

$$\begin{aligned} S^n &= (UTU^*)^n \\ &= (UTU^*UTU^* \cdots UTU^*) \\ &= UT^nU^*. \end{aligned}$$

similarly,

$$S^m = UT^mU^*.$$

Therefore,

$$\begin{aligned}
S^n(S^*)^m(S^*)^mS^n &= UT^nU^*(UT^*U^*)^m(UT^*U^*)^mUT^nU^* \\
&= UT^nU^*UT^{*m}U^*UT^{*m}U^*UT^nU^* \\
&= UT^nT^{*m}T^{*m}T^nU^* \\
&= UT^{*m}T^nT^nT^{*m}U^* \\
&= UT^{*m}U^*UT^nU^*UT^nU^*UT^{*m}U^* \\
&= (S^*)^mS^nS^n(S^*)^m.
\end{aligned}$$

Hence, S is (n,m)-binormal.

2. Let $S \in B(H)$ be such that $S = U^*TU$ for U a co-isometry. Then

$$\begin{aligned}
S^* &= (U^*TU)^* \\
&= U^*T^*U
\end{aligned}$$

and

$$\begin{aligned}
S^n &= (U^*TU)^n \\
&= (U^*TU \cdots U^*TU) \qquad \text{n - times} \\
&= U^*T^nU.
\end{aligned}$$

similarly,

$$S^m = U^*T^mU.$$

Therefore,

$$\begin{aligned}
S^n(S^*)^m(S^*)^mS^n &= U^*T^nU(U^*T^*U)^m(U^*T^*U)^mU^*T^nU \\
&= U^*T^nUU^*T^{*m}UU^*T^{*m}UU^*T^nU \\
&= U^*T^nT^{*m}T^{*m}T^nU \\
&= U^*T^{*m}T^nT^nT^{*m}U \\
&= U^*T^{*m}UU^*T^nUU^*T^nUU^*T^{*m}U \\
&= (S^*)^mS^nS^n(S^*)^m.
\end{aligned}$$

Hence, S is (n,m)-binormal. $\square$

Corollary 4.3.3.
Let $T \in B(H)$ be (n,m)-binormal operator and S be unitarily equivalent to T. Then S is also (n,m)-binormal.

Proof.
Let $T \in B(H)$ be (n,m)-binormal, since S is unitarily equivalent to T, we have $S = U^*TU$ where U is a unitary operator. Then

$$\begin{aligned} S^n(S^*)^m(S^*)^m S^n &= U^*T^nU(U^*T^*U)^m(U^*T^*U)^m U^*T^nU \\ &= U^*T^nUU^*T^{*m}UU^*T^{*m}UU^*T^nU \\ &= U^*T^nT^{*m}T^{*m}T^nU \\ &= U^*T^{*m}T^nT^nT^{*m}U \\ &= U^*T^{*m}UU^*T^nUU^*T^nUU^*T^{*m}U \\ &= (S^*)^m S^n S^n (S^*)^m. \end{aligned}$$

Hence, S is (n,m)-binormal. $\square$

Theorem 4.3.4.
Let $T \in B(H)$ be a binormal operator and S be such that $SS^* = UTT^*U^*$ and $S^*S = UT^*TU^*$ for U an isometry, then S is binormal.

Proof.
Given that T is binormal and $SS^* = UTT^*U^*$ and $S^*S = UT^*TU^*$ for U an isometry. Now

$$\begin{aligned} [SS^*][S^*S] &= [UTT^*U^*][UT^*TU^*] \\ &= [UTT^*T^*TU^*] \\ &= [UT^*TTT^*U^*] \\ &= [UT^*TU^*UTT^*U^*] \\ &= [S^*S][SS^*]. \end{aligned}$$

$\square$

Corollary 4.3.5.
Let $T \in B(H)$ be a binormal operator and S be s.t. $SS^* = U^*TT^*U$ and $S^*S = U^*T^*TU$ for U a co-isometry, then S is binormal.

Proof.
Given that T is binormal such that $SS^* = UTT^*U^*$ and $S^*S = U^*T^*TU$ for U a co-isometry. Now

$$\begin{aligned}
[SS^*][S^*S] &= [U^*TT^*U][U^*T^*TU] \\
&= [U^*TT^*T^*TU] \\
&= [U^*T^*TTT^*U] \\
&= [U^*T^*TUU^*TT^*U] \\
&= [S^*S][SS^*].
\end{aligned}$$

□

Corollary 4.3.6.
Let $T \in B(H)$ be a binormal operator such that it is unitarily quasi-equivalent to S, then S is binormal.

Proof.
Every unitary operator is both isometric and co-isometric. Hence by theorem 3.2.15, the above result follows. □

Lemma 4.3.7.
Let $T \in B(H)$ be a square (n, m)-normal operator. If $U \in B(H)$ is a unitary operator such that $S \in B(H)$ is unitarily equivalent to T, then S is square (n, m)-normal operator.

Proof.
Let $T \in B(H)$ be (n, m)-normal operator. Since S is unitarily equivalent to T, we have $S = UTU^*$, implying $S^{2n} = UT^{2n}U^*$ and $S^{*2m} = UT^{*2m}U^*$. We show that S is (n, m)-normal operator.

$$\begin{aligned}
S^{2n}S^{*2m} &= UT^{2n}U^*UT^{*2m}U^* \\
&= UT^{2n}T^{*2m}U^* \\
&= UT^{*2m}T^{2n}U^* \\
&= UT^{*2m}U^*UT^{2n}U^* \\
&= S^{*2m}S^{2n}.
\end{aligned}$$

Hence S is (n, m)-normal operator. □

4.4 Polynomially Binormal Operators

In this section, the class of polynomially binormal operators in a complex Hilbert space is introduced and we present some of its basic properties.

Definition 4.4.1. (Kittaneh, 1984; Djordjevic, Cho, and Mosic, 2020)
An operator $T \in B(H)$, is polynomially normal if there exists a nontrivial polynomial

$$p(z) = \sum_{0 \leq k \leq n} c_k z^k \in \mathbb{C}([z])$$

such that

$$p(T)T^* - T^*p(T) = \sum_{0 \leq k \leq n} c_k (T^k T^* - T^* T^k) = 0.$$

We use $Poly_1$ to denote the set of all complex polynomials in one variable. If $p \in Poly_1$, then by taking conjugate coefficients of p, we obtain $\overline{p} \in Poly_1$, in particular, if $z \in \mathbb{C}$, then $\overline{p}(z) = \overline{p(\overline{z})}$.

Definition 4.4.2.
Let $T \in B(H)$. Then T is polynomially binormal if there exists a nontrivial polynomial

$$p(z) = \sum_{0 \leq k \leq n} c_k z^k \in \mathbb{C}([z])$$

such that

$$p(T)T^*T^*p(T) - T^*p(T)p(T)T^* = \sum_{0 \leq k \leq n} c_k (T^k T^* T^* T^k - T^* T^k T^k T^*) = 0.$$

Lemma 4.4.3.
If $T \in B(H)$ is p-binormal for $p(t) = t^n$, where $n \in \mathbb{N}$, then T is n-binormal. Also, if $T \in B(H)$ is p-binormal for $p(t) = t$, then T is binormal. So the set of all polynomially binormal operators contains n-binormal and binormal operators.

Proof.
Observe that if $p(t) = t^n$, then $p(T) = T^n$. Assuming that T is p-binormal for $p(t) = t^n$, we show that T is n-binormal.

$$\begin{aligned} T^n T^* T^* T^n &= p(T)T^*T^*p(T) \\ &= T^*p(T)p(T)T^* \end{aligned}$$

$$= T^*T^nT^nT^*.$$

Hence, T is n-binormal. For $n = 1$, we have T is binormal. $\square$

Example 4.4.4.
Suppose $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ on $\mathbb{H} = \mathbb{C}^2$ and given a polynomial $p(t) = t^2 - 2t$. Then T is p-binormal.

Theorem 4.4.5.
Let $p \in Poly_1$ and $T \in B(H)$ be p-binormal, if $U \in B(H)$ is unitary such that $S = U^*TU$, then

1. *S is p-binormal.*
2. *if M is a closed subspace of H which is completely reducing for T, and if*

$$T_1 = T|_M : M \longrightarrow M$$

then T_1 is p-binormal.

Proof.

1. Since $S = U^*TU$, then $p(S) = U^*p(T)U$. We have

$$\begin{aligned} p(S)S^*S^*p(S) &= U^*p(T)UU^*T^*UU^*T^*UU^*p(T)U \\ &= U^*p(T)T^*T^*p(T)U \\ &= U^*T^*p(T)p(T)T^*U \\ &= U^*T^*UU^*p(T)UU^*p(T)UU^*T^*U \\ &= S^*p(S)p(S)S^*. \end{aligned}$$

Hence S is p-binormal.

2. Since M is closed and completely reducing for T, we have

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} M \\ M^\perp \end{bmatrix} \longrightarrow \begin{bmatrix} M \\ M^\perp \end{bmatrix}$$

Then $T = \begin{bmatrix} T_1^* & 0 \\ 0 & T_2^* \end{bmatrix}$ and $p(T) = \begin{bmatrix} p(T_1) & 0 \\ 0 & p(T_2) \end{bmatrix}$

we show that T_1 is p-binormal.

$$T^*p(T)p(T)T^* = \begin{bmatrix} T_1^* & 0 \\ 0 & T_2^* \end{bmatrix} \begin{bmatrix} p(T_1) & 0 \\ 0 & p(T_2) \end{bmatrix} \begin{bmatrix} p(T_1) & 0 \\ 0 & p(T_2) \end{bmatrix} \begin{bmatrix} T_1^* & 0 \\ 0 & T_2^* \end{bmatrix}$$
$$= \begin{bmatrix} T_1^*p(T_1)p(T_1)T_1^* & 0 \\ 0 & T_2^*p(T_2)p(T_2)T_2^* \end{bmatrix}$$

it can also be shown by simple computation, that

$$p(T)T^*T^*p(T) = \begin{bmatrix} p(T_1)T_1^*T_1^*p(T_1) & 0 \\ 0 & p(T_2)T_2^*T_2^*p(T_2) \end{bmatrix}$$

Since T is p-binormal, we have

$$T^*p(T)p(T)T^* = p(T)T^*T^*p(T)$$

it follows that

$$\begin{bmatrix} T_1^*p(T_1)p(T_1)T_1^* & 0 \\ 0 & T_2^*p(T_2)p(T_2)T_2^* \end{bmatrix} = \begin{bmatrix} p(T_1)T_1^*T_1^*p(T_1) & 0 \\ 0 & p(T_2)T_2^*T_2^*p(T_2) \end{bmatrix}$$

implying $T_1^*p(T_1)p(T_1)T_1^* = p(T_1)T_1^*T_1^*p(T_1)$.
Hence T_1 is polynomially binormal.

$\square$

Chapter 5

Skew-(n, m)-Binormal Operators

5.1 Introduction

In this chapter, the class of skew (n, m)-binormal operators acting on a Hilbert space (H) is introduced. An operator $T \in B(H)$ is skew (n, m) binormal operators if it satisfies the condition $(T^{*m}T^nT^nT^{*m})T = T(T^nT^{*m}T^{*m}T^n)$. We investigate some of the basic properties of this class of operators. It is further shown that if an operator T is skew (n, m)-binormal, and is unitarily equivalent to an operator S, then S is also skew (n, m)-binormal.

Definition 5.1.1.
Let $T \in B(H)$. We say that T is skew (n, m)-binormal if

$$(T^{*m}T^nT^nT^{*m})T = T(T^nT^{*m}T^{*m}T^n)$$

for positive integers m and n.

Definition 5.1.2.
Let $T \in B(H)$. We say that T is k-skew (n, m)-binormal if

$$(T^{*m}T^nT^nT^{*m})T^k = T^k(T^nT^{*m}T^{*m}T^n)$$

for positive integers k, m and n.

Remark 5.1.3.
For an operator T that is skew (n, m) binormal. We see that, If $k = m = n = 1$, then skew (n, m)-binormal becomes skew binormal.

In Meenambika, Seshaiah, and Sivamani, (2018), it is indicated that the class of binormal operators is contained in the class of skew n-binormal operators, that is every binormal operator is skew binormal. In this study, we present a counter example to indicate that in general the class of binormal operators is not contained in the class of skew binormal and consequently it does not necessarily belong to the class of k-skew binormal operators.

Example 5.1.4.
Consider the matrix

$$T = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{then} \qquad T^* = \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We now check if the matrix T is binormal

$$TT^*T^*T = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$T^*TTT^* = \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have

$$TT^*T^*T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T^*TTT^*$$

Hence T is a binormal operator.

Let us now check if T is skew binormal.

$$(T^*TTT^*)T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$T(TT^*T^*T) = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have

$$(T^*TTT^*)T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T(TT^*T^*T)$$

Therefore T is not skew binormal.

We further show that every skew binormal operator is not k skew binormal. Simple computation gives

$$T^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Considering the case when $k = 3$, we check if T is 3-skew binormal.

$$(T^*TTT^*)T^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = T^3(TT^*T^*T)$$

Hence T is 3-skew binormal.
We have therefore shown that the inclusion

$$\text{binormal} \subset \text{skew binormal} \subset k - \text{skew binormal}$$

does not always hold.

5.2 Algebraic Properties of skew (n, m)-binormal operators

Theorem 5.2.1.
Let $T \in B(H)$ be skew (n, m) binormal operator, then βT is skew (n, m) binormal for every real scalar β

Proof.
Since $T \in B(H)$ is skew (n, m) binormal operator, it follows that

$$(T^{*m}T^nT^nT^{*m})T = T(T^nT^{*m}T^{*m}T^n)$$

We show that βT is skew (n, m) binormal.

$$\begin{aligned}
((\beta T)^{*m}(\beta T)^n(\beta T)^n(\beta T)^{*m})(\beta T) &= (\beta^{*m}T^{*m}\beta^nT^n\beta^nT^n\beta^{*m}T^{*m})(\beta T) \\
&= \beta^{*m}\beta^n\beta^n\beta^{*m}\beta((T^{*m}T^nT^nT^{*m})T) \\
&= \beta^{*m}\beta^n\beta^n\beta^{*m}\beta(T(T^nT^{*m}T^{*m}T^n)) \\
&= \beta T(\beta^nT^n\beta^{*m}T^{*m}\beta^{*m}T^{*m}\beta^nT^n) \\
&= (\beta T)((\beta T)^n(\beta T)^{*m}(\beta T)^{*m}(\beta T)^n)
\end{aligned}$$

Hence βT is skew (n, m) binormal operator. $\square$

Theorem 5.2.2.
Let $T \in B(H)$ be skew (n, m) binormal operator, then T^* is also skew (n, m) binormal.

Proof.
Since $T \in B(H)$ is skew (n, m) binormal operator, it follows that

$$(T^{*m}T^nT^nT^{*m})T = T(T^nT^{*m}T^{*m}T^n)$$

Taking the adjoint on both sides,

$$((T^{*m}T^nT^nT^{*m})T)^* = (T(T^nT^{*m}T^{*m}T^n))^*$$

$$(T)^*\left[(T^{*m})^*(T^n)^*(T^n)^*(T^{*m})^*\right] = \left[(T^n)^*(T^{m*})^*(T^{m*})^*(T^n)^*\right](T)^*$$

$$(T^*)\left[(T^*)^{*m}(T^*)^n(T^*)^n(T^*)^{*m}\right] = \left[(T^*)^n(T^*)^{*m}(T^*)^{*m}(T^*)^n\right](T^*)$$

Therefore T^* is also skew (n, m) binormal $\square$

Theorem 5.2.3.
Let $T \in B(H)$ be skew (n, m) binormal operator, then T^{-1} is also skew (n, m) binormal.

Proof.
Since $T \in B(H)$ is skew (n, m) binormal operator, it follows that

$$(T^{*m}T^nT^nT^{*m})T = T(T^nT^{*m}T^{*m}T^n)$$

Taking the inverse on both sides,

$$((T^{*m}T^nT^nT^{*m})T)^{-1} = (T(T^nT^{*m}T^{*m}T^n))^{-1}$$

$$(T)^{-1}\left[(T^{*m})^{-1}(T^n)^{-1}(T^n)^{-1}(T^{*m})^{-1}\right] = \left[(T^n)^{-1}(T^{m*})^{-1}(T^{m*})^{-1}(T^n)^{-1}\right](T)^{-1}$$

$$(T^{-1})\left[(T^{-1})^{*m}(T^{-1})^n(T^{-1})^n(T^{-1})^{*m}\right] = \left[(T^{-1})^n(T^{-1})^{*m}(T^{-1})^{*m}(T^{-1})^n\right](T^{-1})$$

Therefore T^{-1} is also skew (n, m) binormal $\square$

Theorem 5.2.4.
Every skew (n, m) normal operator $T \in B(H)$ is skew (n, m) binormal

Proof.

Let $T \in B(H)$ be skew (n,m) normal operator, then by definition

$$(T^{*m}T^n)T = T(T^nT^{*m})$$

We show that T is skew (n,m) binormal.

$$\begin{aligned}
(T^{*m}T^nT^nT^{*m})T &= T^{*m}T^nT^nT^{*m}T \\
&= T^{*m}T^n(T^nT^{*m}T) \\
&= T^{*m}T^n(TT^{*m}T^n) && \text{by skew } (n,m) \text{ normality of } T \\
&= (T^{*m}T^nT)T^{*m}T^n \\
&= T(T^nT^{*m}T^{*m}T^n)
\end{aligned}$$

Hence T is skew (n,m) binormal. □

Theorem 5.2.5.
Let T be skew (n,m) normal operator, then $T \in B(H)$ is skew $(n+k(n-1),m)$ normal for every $k \in \mathbb{N}$.

Proof.
We give the proof by induction.
Base case $k=1$.

$$\begin{aligned}
(T^{n+(n-1)}T^{*m})T &= T^{n-1}(T^nT^{*m})T \\
&= T^{n-1}T(T^{*m}T^n) \\
&= (T^nT^{*m})TT^{n-1} && \text{by skew } (n,m) \text{ normality of } T \\
&= T(T^{*m}T^n)T^{n-1} \\
&= T(T^{*m}T^{n+(n-1)})
\end{aligned}$$

Inductive step: Assume the result holds for $n=k$.

To prove the result for $n=k+1$

$$\begin{aligned}
(T^{n+(k+1)(n-1)}T^{*m})T &= (T^{n+k(n-1)+(n-1)}T^{*m})T \\
&= T^{n-1}(T^{n+k(n-1)}T^{*m})T \\
&= T^{n-1}T(T^{*m}T^{n+k(n-1)}) \\
&= (T^nT^{*m})TT^{n+k(n-1)-1} && \text{by skew } (n,m) \text{ normality of } T \\
&= T(T^{*m}T^n)T^{n+k(n-1)-1} \\
&= T(T^{*m}T^n)T^{(k+1)(n-1)}
\end{aligned}$$

$$= T(T^{*m}T^{n+(k+1)(n-1)})$$

Therefore T is skew $(n + k(n-1), m)$ normal. $\square$

Remark 5.2.6.
The product of two skew (n, m) normal operator is not skew (n, m) normal operator in general.

Consider the following example

Example 5.2.7.
Let $T = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} i & i & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ it can easily be shown that both T and S are skew $(1, 1)$ normal operators. We now check their product,

$$(TS) = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i & i & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

checking if (TS) is skew $(1, 1)$ normal, we have

$$(TS)\left[(TS)^*(TS)\right] = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left[(TS)(TS)^*\right](TS)$$

Hence, the product of two skew (n, m) normal operators is not necessarily skew (n, m) normal.

Theorem 5.2.8.
Let $T \in B(H)$ be a normal operator and S be skew (n, m) normal operator. If T and S commute, then TS is skew (n, m) normal operator.

Proof. Consider

$$\left[(TS)^n(TS)^{*m}\right](TS) = (T^nS^nS^{*m}T^{*m})TS$$

Since T is a normal operator which commutes with S, then by the Fuglede-Putnam theorem, S commutes with T^*, therefore

$$\begin{aligned} \left[(TS)^n(TS)^{*m}\right](TS) &= \left[T^nS^nS^{*m}T^{*m}\right](TS) \\ &= T^n(S^nS^{*m})T^{*m}ST \\ &= T^n(S^nS^{*m})ST^{*m}T \qquad \text{by the Fuglede-Putnam theorem} \end{aligned}$$

$$
\begin{aligned}
&= T^n S S^{*m} S^n T^{*m} T \\
&= S T^n S^{*m} S^n T^{*m} T \\
&= S S^{*m} T^n S^n T^{*m} T \\
&= S S^{*m} S^n T^n T^{*m} T \\
&= S S^{*m} S^n T T^{*m} T^n \\
&= S S^{*m} T S^n T^{*m} T^n \\
&= S T S^{*m} S^n T^{*m} T^n \\
&= S T S^{*m} T^{*m} S^n T^n \\
&= TS\,[(TS)^{*m}(TS)^n]
\end{aligned}
$$

□

Theorem 5.2.9.
Let T be (n,m) normal operator and quasi (n,m) normal operator, then T is skew (n,m) normal

Proof.
Let T be (n,m) normal operator, therefore

$$T^n T^{*m} = T^{*m} A^n$$

Since T is quasi (n,m) normal, we have

$$T(T^{*m}T^n) = (T^{*m}T^n)T$$

Then

$$T(T^{*m}T^n) = (T^{*m}T^n)T = (T^n T^{*m})T$$

□

Theorem 5.2.10.
Let S and T be k-skew (n,m) binormal and doubly commuting operators. Then ST is k-skew (n,m) binormal operator.

Proof.
Since S and T are k-skew (n,m) binormal, we have

$$(S^{*m}S^nS^nS^{*m})S^k = S^k(S^nS^{*m}S^{*m}S^n)$$

and

$$(T^{*m}T^nT^nT^{*m})T^k = T^k(T^nT^{*m}T^{*m}T^n)$$

In addition, the operators are doubly commuting, this implies

$$ST = TS \quad \text{and} \quad ST^* = T^*S$$

taking the adjoint on both sides yields

$$S^*T^* = T^*S^* \quad \text{and} \quad TS^* = S^*T$$

respectively. We now have;

$$\begin{aligned}
[(ST)^{*m}(ST)^n(ST)^n(ST)^{*m}]\,(ST)^k &= S^{*m}T^{*m}S^nT^nS^nT^nS^{*m}T^{*m}S^kT^k \\
&= S^{*m}T^{*m}S^nT^nS^nT^nS^{*m}S^kT^{*m}T^k \\
&= S^{*m}T^{*m}S^nT^nS^nS^{*m}T^nS^kT^{*m}T^k \\
&= S^{*m}S^nT^{*m}S^nT^nS^{*m}S^kT^nT^{*m}T^k \\
&= S^{*m}S^nS^nT^{*m}S^{*m}T^nS^kT^nT^{*m}T^k \\
&= S^{*m}S^nS^nS^{*m}T^{*m}S^kT^nT^nT^{*m}T^k \\
&= \left[(S^{*m}S^nS^nS^{*m})S^k\right]\left[(T^{*m}T^nT^nT^{*m})T^k\right] \\
&= \left[S^k(S^nS^{*m}S^{*m}S^n)\right]\left[T^k(T^nT^{*m}T^{*m}T^n)\right] \\
&= S^kS^nS^{*m}S^{*m}T^kS^nT^nT^{*m}T^{*m}T^n \\
&= S^kS^nS^{*m}T^kS^{*m}T^nS^nT^{*m}T^{*m}T^n \\
&= S^kS^nT^kS^{*m}T^nS^{*m}T^{*m}S^nT^{*m}T^n \\
&= S^kT^kS^nT^nS^{*m}T^{*m}S^{*m}T^{*m}S^nT^n \\
&= (ST)^k\,[(ST)^n(ST)^{*m}(ST)^{*m}(ST)^n]
\end{aligned}$$

We therefore conclude that ST is k-skew (n, m) binormal $\square$

5.3 Equivalence of skew (n, m)-binormal operators

In this section we look at unitary equivalence in the class of skew (n, m)-binormal operators

Theorem 5.3.1.
Let T be k-skew (n, m) normal operator which is unitarily equivalent to S. Then S is also k-skew (n, m) normal operator

Proof.

Since T is unitarily equivalent to S, then there exists a unitary operator U such that $S = U^*TU$ and $S^* = (U^*TU)^* = U^*T^*U$.

$$\begin{aligned}
(S^nS^{*m})S^k &= U^*T^nUU^*T^{*m}UU^*T^kU \\
&= U^*T^nT^{*m}T^kU \\
&= U^*T^k(T^{*m}T^n)U \qquad \text{since } T \text{ is } k\text{-skew } (n,m) \text{ normal operator} \\
&= U^*T^kUU^*T^{*m}UU^*T^nU \\
&= S^k(S^{*m}S^n)
\end{aligned}$$

Hence S is k-skew (n, m) normal operator □

Lemma 5.3.2.
Let T be skew (n, m) normal operator which is unitarily equivalent to S. Then S is also skew (n, m) normal operator

Proof.
The proof is trivial, hence ommited. □

Theorem 5.3.3.
Let T be k-quasi (n, m) normal operator which is unitarily equivalent to S. Then S is also k-quasi (n, m) normal operator

Proof.
Recall that an operator T is k-quasi (n, m) normal, if

$$T^k(T^{*m}T^n) = (T^{*m}T^n)T^k \quad \text{for} \quad k, m, n \in \mathbb{Z}$$

Therefore

$$\begin{aligned}
S^k(S^{*m}S^n) &= U^*T^kU(U^*T^{*m}UU^*T^nU) \\
&= U^*T^k(T^{*m}T^n)U \\
&= U^*(T^{*m}T^n)T^kU \qquad \text{since } T \text{ is } k\text{-quasi } (n,m) \text{ normal} \\
&= U^*T^{*m}UU^*T^nUU^*T^kU \\
&= (S^{*m}S^n)S^k
\end{aligned}$$

Hence S is k-quasi (n, m) normal □

Theorem 5.3.4.
Let T be k-quasi (n, m) binormal operator which is unitarily equivalent to S. Then S is also k-quasi (n, m) binormal operator

Proof.
An operator T is k-quasi (n, m) binormal, if

$$T^k(T^{*m}T^nT^nT^{*m}) = (T^{*m}T^nT^nT^{*m})T^k \quad \text{for} \quad k, m, n \in \mathbb{Z}$$

Therefore

$$\begin{aligned}
S^k(S^{*m}S^nS^nS^{*m}) &= U^*T^kU(U^*T^{*m}UU^*T^nUU^*T^nUU^*T^{*m}U) \\
&= U^*T^k(T^{*m}T^nT^nT^{*m})U \\
&= U^*(T^{*m}T^nT^nT^{*m})T^kU \quad \text{since } T \text{ is } k\text{-quasi } (n,m) \text{ binormal} \\
&= U^*T^{*m}UU^*T^nUU^*T^nUU^*T^{*m}UU^*T^kU \\
&= (S^{*m}S^nS^nS^{*m})S^k
\end{aligned}$$

Hence S is k-quasi (n, m) binormal □

Chapter 6

Conclusion and Recommendations

In this chapter, we give a conclusion and recommendations based on the research objectives and outcomes.

6.1 Conclusion

The first objective of the study was to introduce and study the basic properties of the class of (n, m)-hyponormal. We introduced this class of operators by defining an operator to be (n, m)-hyponormal if

$$T^{*m}T^n \geq T^nT^{*m}.$$

We have shown that this class of operators contains the class of n-hyponormal and hyponormal operators. We further introduce the class of k-quasi-(n, m)-hyponormal operators, and show that it contains the class of (n, m)-hyponormal operators. An operator T is said to be k-quasi-(n, m)-hyponormal operator if

$$T^{*k}(T^{*m}T^n - T^nT^{*m})T^k \geq 0$$

for positive integer k, m, n.
The second objective was to study the equivalence relations in the class of (n, m)-hyponormal operator. It is shown that if an operator S is (n, m)-hyponormal operator and the operator T is unitarily equivalent to S, the T is also (n, m)-hyponormal operator. We have also shown that if two operators S and T are unitarily equivalent and S is k-quasi-(n, m)-hyponormal, then T must also be k-quasi-(n, m)- hyponormal.
The third objective of the study was to introduce and study the basic properties of the class of (n, m)-binormal operators in Hilbert space. This class of operators was introduced. An operator $T \in B(H)$ is said to be (n, m)-binormal operator if

$$T^{*m}T^nT^nT^{*m} = T^nT^{*m}T^{*m}T^n.$$

It is shown that the class of (n,m)-binormal operators is contains the class of n-binormal operators as well as the class of binormal operators. We have also shown that the class of (n,m)-binormal operators is not closed under the basic operations of addition and multiplication. We however provide conditions under which the products of (n,m)-binormal operators becomes (n,m)-binormal operators. In particular If S and T are doubly commuting (n,m)-binormal operators, then ST is (n,m)-binormal operator. Just like binormal operators, the powers of (n,m)-binormal operators are not necessarily (n,m)-binormal operators. We have shown that it T is (n,m)-binormal operators, then T^k for $k \in \mathbb{N}$ is also (n,m)-binormal operators if T is (n,m)-normal operator. The study has also shown that if T is (n,m)-binormal operator, then T reduces to (n,m)-normal operator if there exists a conjugation operator C which commutes with T^mT^n,T^nT^{*m},T^{n-1}, and $T^{*(m-1)}$.
The classes of skew (n,m)-binormal and k-skew-(n,m)-binormal operators are introduced. An operator is skew (n,m)-binormal if

$$(T^{*m}T^nT^nT^{*m})T = T(T^nT^{*m}T^{*m}T^n)$$

for positive integers m and n
and k-skew-(n,m)-binormal if

$$(T^{*m}T^nT^nT^{*m})T^k = T^k(T^nT^{*m}T^{*m}T^n)$$

for positive integers k, m and n. The classes of skew (n,m)-binormal operators and k-skew (n,m)-binormal operators are shown to be independent classes of operators. The product of two k-skew-(n,m)-binormal operators is also not k-skew-(n,m)-binormal in general. We have shown that two operators S and T that are k-skew-(n,m)-binormal, have to be doubly commuting for their product to be k-skew-(n,m)-binormal.
The fourth objective was to study equivalence relations in the class of (n,m)-binormal operators. We have shown that if S and T are unitarily equivalent operators and S is (n,m)-binormal operator, then the operator T must also be (n,m)-binormal operator. It has also been shown that if if S and T are unitarily quasiequivalent operators and S is (n,m)-binormal operator, then the operator T must also be (n,m)-binormal operator. Lastly, we have also shown that if if S and T are unitarily equivalent operators and S is k-skew-(n,m)-binormal operator, then the operator T must also be k-skew-(n,m)-binormal operator.

6.2 Recommendations

The study defined three new classes of operators, that is, the class of (n,m)-hyponormal operators, (n,m)-binormal operators, and skew- (n,m)-binormal operators. It is recommended that a study be carried out to establish the spectral picture of these classes of operators. In addition, other equivalence relations such as almost similarity, quasisimilarity, and similarity can as well be checked since this study considered unitary equivalence and unitary quasiequivalence.

References

[1] Abood, E. H., and Al-loz, M. A. (2015). On some generalization of normal operators on Hilbert space. Iraqi Journal of Science, 178–6. https://www.iasj.net/iasj/download/7c127592449d4eb2

[2] Ahmed, B. S., and Mahmoud, S. A. O. A. (2020). On the class of n-power D-m-quasi-normal operators on Hilbert spaces. Operators and Matrices, 1, 159–174. https://doi.org/10.7153/oam-2020-14-13

[3] Ahmed, O. A. M. S., and Ahmed, M. S. (2011). On the class of n-power quasi-normal operators on Hilbert space. Bull. Math. Anal. Appl, 3(2), 213–228. https://www.emis.de/journals/BMAA/repository/docs/BMAA3-2-21.pdf

[4] Alzuraiqi, S. A., and Patel, A. B. (2010). On n-normal operators. General Mathematics Notes, 1(2), 61–73.

[5] Apostol, C., and Davidson, K. R. (1988). Isomorphisms modulo the compact operators of nest algebras II. https://doi.org/10.1215/S0012-7094-88-05605-0

[6] Arun, B. (1977). A NOTE ON QUASI-NORMAL OPERATORS. https://pascal-francis.inist.fr/vibad/index.php?action=getRecordDetail&idt=PASCAL7830289622

[7] Bekai, D., Benali, A., and Hakem, A. (2021). The class of (n, m) power-A-quasi-hyponormal operators in semi-Hilbertian space. Global Journal of Pure and Applied Sciences, 27(1), 35–41. https://doi.org/10.4314/gjpas.v27i1.5

[8] Benali, A., and Ahmed, O. A. M. S. (2016). Generalizations of Kaplansky Theorem for some (p,k)-Quasihyponormal Operators (arXiv:1602.02748). arXiv. http://arxiv.org/abs/1602.02748

[9] Bhatia, R., Elsner, L., and Semrl, P. (1998). Distance between commuting tuples of normal operators. https://doi.org/10.1007/s000130050257

[10] Brown, A. (1953). On a class of operators. Proceedings of the American Mathematical Society, 4(5), 723–728. https://doi.org/10.1090/S0002-9939-1953-0059483-2

[11] Brown, A. (1954). The unitary equivalence of binormal operators. American Journal of Mathematics, 76(2), 414–434. https://doi.org/10.2307/2372582

[12] Campbell, S. L. (1974). Linear operators for which T^*T and TT^* commute. II. Pacific Journal of Mathematics, 53(2), 355–361. https://doi.org/10.2140/pjm.1974.53.355

[13] Campbell, S. L. (1972). Linear operators for which T^*T and TT^* commute. Proceedings of the American Mathematical Society, 34(1), 177–180. https://www.ams.org/proc/1972-034-01/S0002-9939-1972-0295124-9/

[14] Chekao, F., and Chunlan, J. (1994). Normal operators similar to irreducible operators. Acta Mathematica Sinica, 10(2), 132–135. https://doi.org/10.1007/BF02580419

[15] Chen, K.-Y., Herrero, D. A., and Wu, P. Y. (1992). Similarity and Quasisimilarity of Quasinormal Operators. Journal of Operator Theory, 27(2), 385–412. https://www.jstor.org/stable/24714670

[16] Chō, M., Lee, J. E., Tanahashi, K., Uchiyama, A. (2018). Remarks on n-normal operators. Filomat, 32(15), 5441-5451.

[17] Clary, S. (1975). Equality of spectra of quasi-similar hyponormal operators. Proceedings of the American Mathematical Society, 53(1), 88–90. https://doi.org/10.1090/S0002-9939-1975-0390824-7

[18] Conway, J. B. (1980). On quasisimilarity for subnormal operators. Illinois Journal of Mathematics, 24(4), 689–702. https://doi.org/10.1215/ijm/1256047484

[19] Davidson, K. R. (2010). Essentially Normal Operators. In S. Axler, P. Rosenthal, and D. Sarason (Eds.), A Glimpse at Hilbert Space Operators: Paul R. Halmos in Memoriam (pp. 209–222). Springer.

[20] de Sz. Nagy, G. (1947). Generalization of certain theorems of G. Szegö on the location of zeros of polynomials. https://doi.org/10.1090/S0002-9904-1947-08937-0

[21] Djordjević, D. S., Chō, M., and Mosić, D. (2020). Polynomially normal operators. Annals of Functional Analysis, 11(3), 493–504. https://doi.org/10.1007/s43034-019-00033-0

[22] Elgues, A., and Menkad, S. (2023). On the class of n-normal operators and moore-penrose inverse. https://doi.org/10.37418/amsj.12.1.1

[23] Embry, M. (1966). Conditions implying normality in Hilbert space. Pacific Journal of Mathematics, 18(3), 457–460. https://doi.org/10.2140/pjm.1966.18.457

[24] Embry, M. (1970). Similarities involving normal operators on Hilbert space. Pacific Journal of Mathematics, 35(2), 331–336. https://doi.org/10.2140/pjm.1970.35.331

[25] Fuglede, B. (1950). A Commutativity Theorem for Normal Operators. Proceedings of the National Academy of Sciences, 36(1), 35–40. https://doi.org/10.1073/pnas.36.1.35

[26] Fujii, M., and Nakatsu, Y. (1975). On subclasses of hyponormal operators. Proceedings of the Japan Academy, 51(4), 243–246.

[27] Goodson, G. R. (2014). Spectral properties of normal operators having symmetries arising from conjugations. Operators and Matrices, 4, 1131–1141.

[28] Guesba, M., and Nadir, M. (2016). On n-power-hyponormal operators. Global Journal of Pure and Applied Mathematics, 12(1), 473.

[29] Gupta, A., and Bhatia, N. (2014). n-Normal and n-Quasinormal composition and weighted composition operators on L2 (). , 66(4), 364–370. http://www.vesnik.math.rs/vol/mv14404.pdf

[30] Halmos, P. R. (1950). Commutativity and spectral properties of normal operators. Acta Sci. Math., 12, 153–156. https://cir.nii.ac.jp/crid/1571980074863134592

[31] Halmos, P. R. (1982). Subnormal Operators. In P. R. Halmos, A Hilbert Space Problem Book (Vol. 19, pp. 103–111). Springer New York.

[32] Hamiti, V. R. (n.d.). Some Properties of N-Quasinormal Operators.

[33] Hamiti, V. R. (2014). On k-quasi class Q operators. Bulletin of Mathematical Analysis and Applications, 6(3), 31–37. https://www.emis.de/journals/BMAA/repository/docs/BMAA6-3-2.pdf

[34] Hilbert, D. (1900). Mathematical Problems. Lecture delivered before the International Congress of Mathematicians at Paris. http://www.mat.uc.pt/ delfos/hilbertprob.pdf

[35] Holleman, C., McClatchey, T., and Thompson, D. (2017). Binormal, Complex Symmetric Operators (arXiv:1705.04882). arXiv. http://arxiv.org/abs/1705.04882

[36] Imagiri, K. S. (2014). Inequalities and spectral properties of some classes of operators in Hilbert spaces [PhD Thesis, University of Nairobi]. http://erepository.uonbi.ac.ke/handle/11295/76155

[37] Jiang, C., and Shi, R. (2012). A similarity invariant of a class of n-normal operators in terms of K-theory (arXiv:1211.6243). arXiv. http://arxiv.org/abs/1211.6243

[38] Jibril, A. A. (1996), On Almost Similar Operators. Arabian J.Sci.Engrg., 21, 443-449.

[39] Jibril, A. A. (2008). On n-power normal operators. The Arabian Journal for Science and Engineering, 33(2A), 247–253.

[40] Jibril, A. A. (2010). On Operators for which $T = (T^*T)^2$. International Mathematical Forum, 5(46), 2255–2262.

[41] Jibril, A. A. (2013). On-Operators. International Mathematical Forum, 8(25), 1215–1224.

[42] Kaplansky, I. (1951). A theorem on rings of operators. https://doi.org/10.2140/pjm.1951.1.227

[43] Kaplansky, I. (1953). Products of normal operators. https://doi.org/10.1215/S0012-7094-53-02025-0

[44] Kathurima, I. (2014). Putnam-Fuglede theorem for n-Power quasinormal and w-hyponormal operators. Far East Jnr of Appld. Maths.

[45] Khasbardar, S. K. (1978). SPECTRALOID OPERATORS, SIMILARITY AND RELATED RESULTS. https://pascal-francis.inist.fr/vibad/index.php?action=getRecordDetailidt=PASCAL7830440741

[46] Khasbardar, S., and Thakare, N. (1978). Some counter-examples for quasinormal operators and related results. Indian J. Pure Appl. Math, 9(12), 1263–1270.

[47] Kikete, D. W., Luketero, S. W., Mile, J. K., and Wafula, A. W. (2023). A Study on Properties of (n; m)-Hyponormal Operators. Asian Journal of Pure and Applied Mathematics, 209–217.

[48] Kittaneh, F. (1984). On the structure of polynomially normal operators. Bulletin of the Australian Mathematical Society, 30(1), 11–18. https://doi.org/10.1017/S0004972700001660

[49] Ko, E., Kwon, H.-K., and Lee, J. E. (2018). A characterization of binormal matrices. Linear and Multilinear Algebra, 66(6), 1215–1228. https://doi.org/10.1080/03081087.2017.1347134

[50] Kubrusly, C. S. (1997). An introduction to models and decompositions in operator theory. Springer Science and Business Media.

[51] Lebesgue, H. (1902). Integral, length, area. God Created the Integers, 1212–1253.

[52] Luketero S. W., (2020). Further remarks on unitary equivalence of some classes of

operators in Hilbert space. International Journal of Statistics and Applied Mathematics, 5(3); 06-10

[53] Luketero S. W. and Khalagai J.M., (2020). On unitary equivalence of some classes of operators in Hilbert spaces. International Journal of Statistics and Applied Mathematics, 5(2); 35-37

[54] Meenambika, K., Seshaiah, C. V., and Sivamani, N. (2018). Skew binormal operators acting on a hilbert space. Journal of Physics: Conference Series, 1139(1), 012056. https://doi.org/10.1088/1742-6596/1139/1/012056

[55] Mortad, M. H. (2012a). The Sum of Two Unbounded Linear Operators: Closedness, Self-adjointness and Normality (arXiv:1203.2545). arXiv. http://arxiv.org/abs/1203.2545

[56] Mortad, M. H. (2012b). The Unbounded Product of Normal Operators. Ninth ACOTCA: 11-14 June, 2012 in Sevilla. https://hal.science/hal-03453025/file/Sevilla-Talk-2012.pdf

[57] Narayanasamy, A., and Krishnaswamy, D. (2021). ON N-POWER HYPONORMAL OPERATORS IN INDEFINITE INNER PRODUCT SPACE. Bull. Int. Math. Virtual Inst, 11(1), 121–125.

[58] Nzimbi, B. M., Pokhariyal, G. P., and Moindi, S. K., A Note on Metric Equivalence of some Operators, Far East J.F Math. Sci. (FJMS), 75(2013), 301-318.

[59] Nzimbi, B. M., and Luketero, S. W. (2020). On unitary Quasi-Equivalence of Operators. International Journal of Mathematics And Its Applications, 8(1), 207–215. http://ijmaa.in/index.php/ijmaa/article/view/193

[60] Othman, S. I. (1996). Nearly equivalent operators. Mathematica Bohemica, 121(2), 133–141. https://doi.org/10.21136/MB.1996.126110

[61] Panayappan, S., and Sivamani, N. (2012a). On n power class (Q) operators. Int. J. Math. Ana, 6(31), 1513–1518.

[62] Panayappan, S., and Sivamani, N. (2012b). On n-binormal operators. General Mathematics Notes, 10, 1–8.

[63] Putnam, C. R. (1951). On normal operators in Hilbert space. American Journal of Mathematics, 73(2), 357–362. https://doi.org/10.2307/2372180

[64] Raphael, M. (1982). Quasisimilarity and Essential Spectra for Subnormal Operators. Indiana University Mathematics Journal, 31(2), 243–246. https://doi.org/10.1512/iumj.1982.31.31021

[65] Rasimi, K., and Gjoka, L. (2013). Some remarks on n-power class(Q) operators. International Journal of Pure and Applied Mathematics, 89.

[66] Rasimi, K., Ibraimi, A., and Gjoka, L. (2014). Notes on -commuting operators. Int. J. Pure. Appl. Math, 91, 191–196.

[67] Sarason, D. (1966). Invariant subspaces and unstarred operator algebras. Pacific Journal of Mathematics, 17(3), 511–517. https://doi.org/10.2140/pjm.1966.17.511

[68] Shaakir, L. K., and Abdulwahid, E. S. (2014). Skew n-normal operators. Aust. J. of Basic and Appl. Sci, 8(16), 340–344.

[69] Shihab, M. (2016). Square-normal operator. British Journal of Mathematics Computer Science, 19(3), 1–7. https://doi.org/10.9734/BJMCS/2016/28879

[70] Szőkefalvi-Nagy, B., and Foia, C. (1967). Analyse harmonique des opérateurs de l'espace de Hilbert. Akademiai Kiado.

[71] Thompson, D., McClatchey, T., and Holleman, C. (2021). Binormal, complex symmetric operators. Linear and Multilinear Algebra, 69(9), 1705–1715. https://doi.org/10.1080/03081087.2019.1635982

[72] Vijayalakshmi, P., and Mary, J. S. I. (2016). N -power quasi-isometry and n -power normal composition operators on L^2-spaces. Malaya Journal of Matematik, 4(01), 42–52.

[73] Wabuya, K., Wanyonyi, S. L., Mile, J. K., and Wanyonyi, A. W. W. (2023). Classes of operators related to binormal operators. Scientific African, 22, e01973. https://doi.org/10.1016/j.sciaf.2023.e01973

[74] Wang, Y. (2017). Some properties of binormal and complex symmetric operators. Mathematica Aeterna, 7(4), 439–446.

[75] Wermer, J. (1952). On invariant subspaces of normal operators. Proceedings of the American Mathematical Society, 3(2), 270–277. https://doi.org/10.1090/S0002-9939-1952-0048700-X

[76] Williams, L. R. (1980). Equality of essential spectra of quasisimilar quasinormal operators. Journal of Operator Theory, 57–69.

[77] Williams, L. R. (1981). Quasisimilarity and hyponormal operators. Journal of Operator Theory, 127–139.

[78] Yang, L. (1990). Equality of essential spectra of quasisimilar subnormal operators. Integral Equations and Operator Theory, 13, 433–441. https://doi.org/10.1007/BF01199894

[79] Zermelo, E. (1930). Ernst Zermelo-collected works. Springer-Verlag.