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**On the classification of semisimple Lie algebras**

By

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A dissertation submitted to the School of mathematics in Partial fulfillment for a degree of Master of Science in Pure mathematics.

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## DECLARATION

I the undersigned declare that this project is my original work and has not been presented for a degree in any other university.

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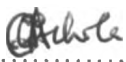
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## **DEDICATION**

To my wife MUKARUGINA Dancille for her creative initiative

## **MOTIVATION**

In august (15<sup>th</sup>-28<sup>th</sup>) , 2009 University of Dales Salam hosted great mathematicians from East Africa and other parts of the world.

Besides Dr Andrew Maffei and Dr Elnora(university of Roma) , Prof Ramadas (ICTP) who were the main speakers at the East Africa workshop on Algebra(Finite reflection group) this study is a part of the lecture given at the conference by the above speakers.

From this I got the topic of this work.

## NOTATIONS AND MEANINGS

$\subseteq$  : Subset of

$Gl(n, F)$  : The set of all linear groups of invertible matrices with entries in  $F$

$ad$  : Adjoint representation

$gl(V)$  : The set of linear transformation of a vector space  $V$  to itself

$A_n = Sl(n+1)$  : Lie algebra of the special linear group in  $(n+1)$  variables

$C_n = Sp(2n)$  : The Lie algebra of the symplectic group in  $2n$  variables

$B_n = O(2n+1)$  : The Lie algebra of the special orthogonal group in  $(2n+1)$  variables,

$D_n = O(2n)$  : The Lie algebra of the special orthogonal group in  $2n$  variables,

$Rad(L)$  : Radical of Lie algebra  $L$ .

$\oplus$  : Direct sum

$\cup$  : Union

$\Rightarrow$  : Imply

$\in$  : belong to

$\neq$  : Not equal (different from)

$\approx$  : Isomorphic to

## **ACKNOWLEDGEMENT**

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Finally I am indebted to my family for their unfailing support throughout my school days.

Above all thanks to God for bringing me this far.

## ABSTRACT

Lie algebra over a field  $F$  ( $F = \mathbb{C}$ ) is a vector space  $L$  over  $F$  equipped with a skew symmetric bilinear operation called the Lie bracket, which satisfied the Jacobi identity.

Lie algebras, have Jordan decomposition into semisimple and nilpotent parts, with representation theory of nilpotent Lie algebras being intractable in general. The finite dimensional representation and classification of semisimple Lie algebras are completely understood, after work of Elie Cartan.

A classification of semisimple Lie algebras  $L$  is analyzed by choosing a Cartan subalgebra which is essentially a generic maximal subalgebra  $H$  of  $L$  on which the Lie bracket is zero (abelian). The representation of  $L$  is decomposed into weight spaces which are eigen spaces for the action of  $H$  (root space decomposition). From which the analysis of representation is easily understood by the possible weights which can occur (root system).

The classification of semisimple Lie algebras by Dynkin gives the four classical Lie algebras and five exceptional simple Lie algebras over the finite algebraically field of characteristic zero.

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## CHAPTER I: GENERAL INTRODUCTION

### I.1. Literature review

But, in the further development of a branch of Mathematics, the human mind, encouraged by the success of its solution, becomes conscious of its independence. It evolves from its self alone, often without, by means of logical combination, generalization, specialization, by separating and collecting ideas in fortune new ways, new and fruitful problems, and appears then itself as the real questioner.

“David Hilbert, Mathematical problem”

The theory of Lie algebras has been developed significantly over the last two centuries, due largely to the efforts of Lie, Killing, Engel, Cartan, Weyl, Iwasawa and Harish Chandra.

Our work is based on the classification of complex semisimple Lie algebras.

In 1866 Sophus Lie began to read more and more mathematics and the library of University of Christiania (Norway) showed clearly that his interests were steadily twining in the direction of Mathematics. It was during the year 1867 that Lie had first brilliant new mathematical idea. It came to him in the middle of the night, and filled with excitement, he rushed to see his friend, woke him up and shouted “I have found it, it is quite simple”. Earlier, Lie had problems of career choice; he could not decide what subject to pursue. This was not the end of Lie’s problems of course, but now he knew the career he wanted and it would be fair to say that from that moment on Lie became a Mathematician. The type of mathematics that Lie would study become more clearly defined during 1868 when he avidly read papers on geometry by Plucker and Poncelet.

In 1869, Crelle’s Journal accepted Lie’s paper based on the inspiration which had struck him in 1867.

Between 1873 and 1881 Lie and Slow prepared an edition of Abel’s complete works which was not published when Abel died. Lie had started examining partial differential equations, hoping that he could find a theory which was analogous to the Galois theory of equations. He wrote “the theory of differential equations is the most important discipline in modern Mathematics”. He examined his contact transformations considering how they affected a process due to Jacobi of generating further solutions of differential equations from a given one. This led to combining the transformations in a way that Lie called an infinitesimal group, but which is not a group with the usually definition, rather what is today called Lie algebra.

In 1884, Killing, introduced Lie algebras in Program Schrift published by the Lyceum Hosianium in Brawnsberg. His aim was to systematically study all space forms, that is, geometries with specific properties relating to infinitesimal notions. In his program schrift he translated this geometrical aim into the problem of classifying all finite dimensional real Lie algebras. At this stage Killing was not aware of Lie's work and therefore his definition of Lie algebra was made quite independently of Lie.[12]

Between 1888 and 1890, the classification theorems were presented by Killing in his paper Die Zusammensetzung Der Stetigen.

In 1885, Killing with Engel discussed the semisimple Lie algebras, which theory they knew about. Killing in a letter of 12 April 1886 conjectured that the only simple algebras were those related to the special linear group and orthogonal group. In the same letter he conjectured other theorems about Lie algebras.

Killing on 27 April 1887, come up with the definition of a semisimple Lie algebra (his definition that such an algebra had no abelian ideals is equivalent to the definition that such an algebra has no soluble ideal).

On 23<sup>rd</sup> May, Killing discovered that his conjecture about simple algebras was not correct, for he had discovered the group  $G$ , and by 18 October he had discovered the complete list of simple Lie algebras.

Elie Cartan (1869-1951) revised and completed the work of Killing on the classification of Semisimple Lie algebras over  $C$  and extended it to give a classification of their representations. He also classified the semisimple Lie algebras over the real numbers, and he used this to classify symmetric spaces.

Weyl (1885-1955) proved that the finite dimensional representations of Semisimple Lie algebras and Lie groups are semisimple (completely reducible).

Noether (1882-1935), Hass (1898-1979), Brauer (1901-1977) and Albert (1905-1972) found a classification of Semisimple algebras over number fields, which gives a classification of the classical algebraic group over the same field.

Chevalley (1909-1984), proved the existence of the simple Lie' algebras and of their representations without using the classification. He was one of the initiators of the systematic study of algebraic groups over arbitrary fields.

Iwasawa (1917-1998) found the Iwasawa decomposition which is fundamental for the structure of real semisimple Lie groups.

Harish –Chandra (1923-1983) independently of Chevalley, also showed the existence of the simple lie algebras and of their representations without using the classification.[24]

In this work, we study the classification of semisimple Lie algebras, by classifying all possible root systems. The amazing results due to Killing with some repair work by Elie Cartan, is that with only five exceptions, the root system of the classical algebras exhaust all possibilities. We will be limited to the classification of Complex semisimple Lie algebras.

## **1.2. Objectives of the study**

This work aims to summarize the major concepts related to the study of semisimple Lie algebras and their classifications.

Specifically, we will:

- analyze the main properties of semisimple Lie algebras.
- apply these properties to the classification of semisimple Lie algebras
- Give applications of these properties.

## **1.3. Methodology**

The approach of classification of semisimple Lie algebra  $L$  is as follows:

- Find the maximal abelian subalgebra  $H$  consisting of elements that are diagonalizable in every representation.
- Restrict the adjoint representation of  $L$  to  $H$  and show that  $L$  is the direct sum of weight spaces with respect to  $H$  (Root space decomposition).
- Prove the general results about the set of weights (Root systems).
- Show that the isomorphism type of  $L$  is completely determined by its roots.
- Classify such root systems.
- Show that an isomorphism of the root systems of two complex semisimple Lie algebras lifts to an isomorphism of these algebras themselves.

The reference material comes from the books, journals and material from the World Wide Web. We use the traditional mathematical definition theorem-proof sequence.

## **1.4. Significance**

This work is a part of continuing efforts by Algebraists and Physicists aimed at a fuller understanding of the relationship between group theory and Lie algebras.

Hence, the significance of semi simplicity comes firstly from the decomposition which states that every finite dimensional Lie algebras is the semi direct product of solvable ideal(its radical) and a semisimple algebras. Semisimple Lie algebras have a very elegant classification; over an



algebraically closed field are completely classified by their root system, which are in turn classified by Dynkin diagrams.

Further, the representation theory of semisimple Lie algebras is much cleaner than for general Lie algebras. For example, the Jordan decomposition in semisimple Lie algebra coincides with the Jordan decomposition in its representation, this is not the case for Lie algebras in general.

### 1.5. Definitions of key terms

The following are definitions of some key concepts of algebra used in this study.

#### Definition 1: (Group)

A group consists of a set  $G$  and a binary operation “.” defined on  $G$  for which the following conditions are satisfied:

- i)  $(a.b).c = a.(b.c)$  for all  $a, b, c \in G$  (associativity)
- ii) There is  $a \in G$  such that  $1.a = a.1 = a$  for all  $a \in G$  (identity)
- iii) Given  $a \in G$ , there is  $b \in G$  such that  $a.b = b.a = 1$  (inverse)

#### Definition 2: (Ring)

A set  $\mathbb{R}$  of elements for which addition (+) and multiplication (.) are defined is called a ring if the following axioms are satisfied.

The set  $\mathbb{R}$  is an abelian group under addition.

- i) For any two elements  $a$  and  $b$  of  $\mathbb{R}$  the product  $a.b$  is defined and is an element of  $\mathbb{R}$  (closure).
- ii) For any elements  $a, b$  and  $c$  of  $\mathbb{R}$ ,  $a.(b.c) = (a.b).c$  (associative law).
- iii) For any three elements  $a, b$  and  $c$  of  $\mathbb{R}$ ,  $a.(b+c) = a.b + a.c$  and  $(b+c).a = b.a + c.a$  (Distributive law)

#### Definition 3: (Field)

This a ring  $\mathbb{R}$ , such that the elements of  $\mathbb{R}$  different from 0, forms an abelian group under multiplication.

#### Definition 4: (Vector space)

A set  $V$  of elements is called a vector space over a field  $F$  if it satisfies the following properties:

- i) The set  $V$  is an abelian group under addition

- ii) For any vector  $v$  in  $V$  and for any  $c$  in  $F$ , is defined  $cv$  in  $V$  (Field elements are called scalars and elements of  $V$  are called vectors)
- iii) If  $v$  is a vector,  $c$  and  $d$  are scalars then  $(c+d)v = cv + dv$  (distributive law)
- iv) If  $u$  and  $v$  are vectors in  $V$  and  $c$  is a scalars then  $c(u+v) = cu + cv$  (distributive law).
- v) If  $v$  is a vector in  $V$ ,  $c$  and  $d$  are scalars, then  $(cd)v = c(dv)$
- vi)  $1v = v$  for any  $v$  in  $V$ .

**Definition 5 :( Linear Transformation)**

Let  $V$  and  $W$  be vector spaces over a field  $F$ . A map  $T : V \rightarrow W$  is said to be linear if satisfies

$$T(\lambda u + \beta v) = \lambda T(u) + \beta T(v) \text{ for all } u, v \text{ in } V \text{ and } \lambda, \beta \text{ in } F$$

**Definition 6: (Algebra)**

An algebra  $A$  over a field  $F$  is a vector space  $A$  over  $F$  with additional operation called multiplication of vectors which associates to each pair of vectors  $u, v \in A$  a vector  $uv \in A$  such that is associative, is distributive with respect over the addition and  $\alpha(uv) = (\alpha u)v = u(\alpha v)$  for each scalar  $\alpha$  in  $F$ .

If there is a vector  $1 \in A$  such that  $1v = v1 = v \forall v \in A$  then  $A$  is called a linear algebra with unity over  $F$ . If  $uv = vu, \forall u, v \in A$  then it is said to be commutative.

**Definition7: (Ideal)**

A subset  $I \subset R[x_1, x_2, \dots, x_n]$  is an ideal if it satisfies:

- i)  $0 \in I$
- ii) If  $f, g \in I$ , then  $f + g \in I$
- iii) If  $f \in I$  and  $h \in R[x_1, x_2, \dots, x_n]$  then  $hf$  and  $fh \in I$

**Definition 8: (Bilinear form)**

A bilinear form on a vector space  $V$  is a function  $B : V \times V \rightarrow F$  that assigns to each pair of ordered vectors  $v, w \in V$  a scalar  $B(v, w) \in F$  such that :

- i)  $B(\alpha v + u, w) = \alpha B(v, w) + B(u, w)$
- ii)  $B(v, u + \alpha w) = B(v, u) + \alpha B(v, w) \forall u, v, w \in V$  and  $\alpha \in F$ .

A bilinear form is non degenerate if  $\forall v \neq 0, \exists w \in V$  such that  $B(v, w) \neq 0$ . Otherwise it is called degenerate. A bilinear form  $B$  is symmetric if  $B(u, v) = B(v, u) \forall u, v \in V$ .

**Definition 9: (Matrix)**

An  $n \times m$  Matrix over  $F$  is a rectangular array with  $nm$  elements of  $F$  of  $n$  rows and  $m$  columns

given as:  $M_{n \times m} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$ .

A square matrix is one for which  $n = m$ .

The trace of a square matrix  $M$  written  $\text{Tr}(M)$  is defined to be the sum of the diagonal elements of  $M$ . The set of  $n \times n$  invertible matrices is denoted by  $GL(n, F)$ .

$GL(n, F)$  is a group.

**Definition 10: (Finite matrix group)**

A finite group  $G \subset GL(n, F)$  is called finite matrix group.

**Definition 11: (Representation of a group)**

A representation of a group  $G$  is homomorphism from  $G$  into  $GL(n, F)$  for some  $n, n$  is the degree of the representation.

**1.6. Frame work of the study**

The content of the work is organized as follows:

General introduction given in Chapter I, describes the origin and the development of Lie algebras.

Chapter II presents the general concepts and basic notions of Lie algebras.

In Chapter III, the standard results on Semisimple Lie algebras are proved. The Killing form is introduced and used to describe the Cartan subalgebra in the next chapter.

Chapter IV of this work gives the classification of Semisimple Lie algebras.

The last chapter contains conclusion, comments and current research.

## CHAPTER II: BASIC CONCEPTS OF LIE ALGEBRAS

In this chapter we present the basic notions of Lie algebras.

### II.1. Preliminaries [4, 13]

#### II.1.1. Definition (Lie algebra)

A finite dimensional Lie algebra is a finite dimensional vector space  $L$  over a field  $F$  together with a map  $[.,.]: L \times L \rightarrow L$  such that:

i)  $[.,.]$  is bilinear :  $[x + v, y] = [x, y] + [v, y]$

$$\forall x, y, v \in L,$$

$$\text{and } [\alpha x, y] = \alpha[x, y] \forall x, y \in L \text{ and } \alpha \in F,$$

$$[x, y + w] = [x, y] + [x, w] \forall x, y, w \in L \text{ and}$$

$$[x, \beta y] = \beta[x, y] \forall x, y \in L \text{ and } \beta \in F$$

ii)  $[.,.]$  is skew symmetric:  $[x, x] = 0$  for all  $x \in L$ .

iii) the Jacobi identity holds:  $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$  for all  $x, y, z$  in  $L$

Note that the first and the second property in definition (II.1.1), together imply

$$[x, y] = -[y, x] \text{ for all } x, y \in L :$$

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x] \quad (2.1)$$

#### Examples

a) Any vector space  $L$  with the  $[x, y] = 0$  for all  $x, y \in L$

is an abelian Lie algebra.

b) Let  $V$  be a vector space and  $L$  the ring of all linear transformations. Then define  $[x, y] = xy - yx$  for all  $x, y \in L$ . Obviously,  $[.,.]$  is bilinear, and  $[x, x] = 0$ .

For the Jacobi identity we have:

$$\begin{aligned} [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= [x, yz - zy] + [y, zx - xz] + [z, xy - yx] = \\ &xyz - xzy + yzx - yxz + zxy - zyx - yzx + zyx - zxy + xzy - xyz + yxz = 0 \end{aligned}$$

So indeed this gives us a Lie algebra which we will call the general linear Lie algebra  $gl(V)$ .

### II.1.2. Definition (Lie algebra homomorphism)

Let  $L_1$  and  $L_2$  be Lie algebras; a linear map  $\varphi : L_1 \rightarrow L_2$  is called a Lie algebra homomorphism

if  $\varphi([x, y]) = [\varphi(x), \varphi(y)]$  for all  $x, y \in L_1$ . If a Lie algebra homomorphism  $\varphi$  is a bijection, then  $\varphi$  is called a Lie algebra isomorphism.

A Lie algebra isomorphism  $\varphi : L \rightarrow L$  is called a Lie algebra automorphism. If  $L$  is a Lie algebra then we define for  $\varphi : L \rightarrow gl(L)$  then  $[\varphi(x), \varphi(y)] = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$

See example (II.1.2).

### II.1.3. Definition (adjoint representation "ad")

Let  $L$  be Lie algebra, for  $x \in L$ , we define a linear map  $ad_x : L \rightarrow L$  by  $ad_x(y) = [x, y]$  (2.2)

Thus  $ad$  (i.e the map  $x \rightarrow ad_x$ ) in fact is a linear map from  $L$  into the space of linear operators from  $L$  to  $L$ . The  $ad$  function is useful in the sense that it makes things more readable instead of writing  $[x, [x, [x, [x, [y]]]]$  we will now simply write  $(ad_x)^5(y)$ .

We have the following:

### II.1.4. Lemma

If  $L$  is a Lie algebra, then  $ad$  is a Lie algebra homomorphism from  $L$  to  $gl(L)$

#### Proof

Let  $L$  be a Lie algebra, and let  $x, y \in L$ , then we have, for every  $z \in L$  (by Jacobi identity)

$$\begin{aligned} ad_{[x,y]}(z) &= [[x, y], z] = -[z, [x, y]] = [x, [y, z]] + [y, [z, x]] = [x, [y, z]] - [y, [x, z]] \\ &= ad_x ad_y(z) - ad_y ad_x(z) = [ad_x, ad_y](z) \end{aligned} \quad (2.3)$$

So indeed  $ad_{[x,y]} = [ad_x, ad_y]$ . So  $ad$  is a homomorphism.

### II.1.5. Definition (derivation)

A derivation  $d$  is a Lie algebra homomorphism satisfying  $d([x, y]) = [d(x), (y)] + [x, d(y)]$

### II.1.6. Lemma

If  $L$  is a Lie algebra and  $x \in L$ , then  $ad_x$  is a derivation, i.e  $ad_x([y, z]) = [y, ad_x(z)] + [ad_x(y), z]$

## Proof

Let  $x \in L$ , and observe the action of  $ad_x$  on  $[y, z]$  for  $y, z \in L$

$$\begin{aligned} ad_x([y, z]) &= [x, [y, z]] = [y, [x, z]] - [z, [x, y]] = [y, ad_x(z)] - [z, ad_x(y)] \\ &= [y, ad_x(z)] + [ad_x(y), z]. \end{aligned} \tag{2.4}$$

### II.1.7. Definition (Subalgebra)

A subspace  $K$  of  $L$  is called a subalgebra if  $[x, y] \in K$  whenever  $x, y \in K$ . In particular  $K$  is a Lie algebra in its own right.

### II.1.8. Definition (center, normalizer)

The center and normalizer of Lie algebra  $L$  are denoted and defined respectively:

$$\begin{aligned} Z(L) &= \{Z \in L \mid [x, Z] = 0 \text{ for all } x \in L\} \\ N_L(K) &= \{x \in L \mid [x, K] \subset K\} \end{aligned}$$

Where  $K$  is a subalgebra of  $L$ .

## II.2. Representation

Let  $L$  be a Lie algebra over the field  $F$ , and  $V$  a vector space over  $F$ .

### II.2.1. Definition (representation)

A representation of  $L$  in  $V$  is a map:  $\varphi: L \rightarrow \text{End}(V)$ : such that

- c)  $\varphi$  is linear, and
- d)  $\varphi([x, y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$  for  $x, y \in L$ .

If  $V$  is a finite dimensional, the above is equivalent to saying that  $\varphi$  is a homomorphism of  $L$  into  $gl(V)$ . A well known representation is the adjoint representation of Lie algebra  $\varphi: L \rightarrow gl(L)$

$$x \rightarrow ad_x$$

## II.3. Ideals

We note that we defined  $gl(V)$  as the set of linear transformation of a vector space  $V$  to itself, viewed as a Lie algebra.

### II.3.1. Definition (linear Lie algebra)

A Lie algebra  $L$  is called a linear Lie algebra if it is isomorphic to a subalgebra of  $gl(V)$  for some vector space  $V$ .

The notion of ideals in rings extends to ideals in Lie algebras.

### II.3.2. Definition (ideal)

A subspace  $I$  of a Lie algebra  $L$  is called an ideal if  $[x, y] \in I$  for  $x \in L$  and  $y \in I$ .

#### Example:

Let  $L$  be Lie algebra. Obviously,  $\{0\}$  and  $L$  are ideals. A more interesting example is the center of  $L$ :  $Z(L) = \{z \in L / [x, z] = 0 \text{ for all } x \in L\}$  (2.5)

Indeed if we let  $y \in L$  and  $z \in Z(L)$ , then  $[y, z] = 0 \in Z(L)$ .

### II.3.3. Lemma

If  $I$  and  $J$  are both ideals of  $L$ , then  $I + J = \{x + y | x \in I, y \in J\}$  is an ideal and so  $[I, J] = \{\sum [x_i, y_i] | x_i \in I, y_i \in J\}$

#### Proof

Let  $I$  and  $J$  be ideals of Lie algebra  $L$ . It is straight forward that  $I + J$  is an ideal of  $L$ , so we focus on  $[I, J]$ . Let  $y \in L$  and  $z \in [I, J]$  so  $z = [a_1, b_1] + \dots + [a_i, b_i]$ , with  $a_i \in I$  and  $b_i \in J$ .

$$\text{Then by bilinearity, } [y, z] = [y, [a_1, b_1]] + \dots + [y, [a_i, b_i]] \quad (2.6)$$

By Jacobi identity, we have for every term of this equation

$$[y, [a_i, b_i]] = [a_i, [y, b_i]] + [[y, a_i], b_i] \quad (2.7)$$

And both of these terms are of the form  $[a, b]$  with  $a \in I$  and  $b \in J$ . So indeed  $[y, z] \in [I, J]$ , as desired.

The ideal  $I + J$  above is called the sum of  $I$  and  $J$ .

### II.3.4. Definitions

- **Abelian Lie algebra**

A Lie algebra  $L$  is called abelian if  $[L, L] = 0$  (i.e.  $[x, y] = 0 \forall x, y \in L$ ). In this case we say that Lie algebra  $L$  is called abelian if and only if  $Z(L) = L$ .

- **Simple Lie algebra**

A Lie algebra  $L$  is said to be simple if  $[L, L] \neq 0$  and  $L$  has no ideals except  $\{0\}$  and  $L$  its self (i.e. If it is non abelian and has no proper ideals).

### Remark

If we consider  $ad$  as a map from  $L$  into the space of linear operators on  $L$ , we see that its kernel is equal to the center of  $L$ :  $Ker(ad) = \{x \in L \mid ad_x(y) = 0 \text{ for all } y \in L\} = Z(L)$  (2.8)

So, if  $L$  is a simple Lie algebra,  $Ker(ad) = \{0\}$ , hence  $ad$  is an isomorphism of  $L$  to  $gl(L)$ , so any simple Lie algebra is a linear Lie algebra.

### Example

Let  $V$  be a vector space over  $F$  of dimension  $n$ . Recall that the trace of a matrix  $M$  is the sum of its diagonal elements, commonly denoted by  $Tr(M)$ , and independent of the choice of basis.

Then we let  $sl(V)$  (or  $sl_n(F)$  if  $V = F^n$ ) denote the set of endomorphism of  $V$  having trace zero.

Since  $Tr(x+y) = Tr(x) + Tr(y)$  and  $Tr(xy) = Tr(yx)$ , we know that  $sl(V)$  is a subalgebra of  $gl(V)$ . It is called the special linear algebra and the dimension of  $sl(V)$  is  $n^2 - 1$ .

### 11.3.5. Classical Lie algebras and exceptional simple Lie algebras

Over a field of characteristic 0 there exist four families of classical Lie algebras and five exceptional simple Lie algebras. The accompanying proofs [13].

The four families of classical Lie algebras are:

- i) The Lie algebra of the special linear Lie algebra in  $(n+1)$  variables:  $A_n (n \geq 1)$ : also denoted  $sl(n+1)$ , and most commonly represented of  $(n+1) \times (n+1)$  matrices with trace 0. This Lie algebra has dimension  $(n+1)^2 - 1$ .
- ii) The Lie algebra of the symplectic Lie algebra in  $2n$  variables:  $C_n (n \geq 3)$ : also denoted by  $Sp(2n)$ .

If we let  $V$  be a vector space of dimension  $2n$ , and denotes its elements as row vectors, we define the non degenerate bilinear form  $g$  on  $V$  by defining the matrix  $G$ :

$$G = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \text{ then its dimension is } n(2n+1) \quad (2.9)$$

- iii) The Lie algebra of the special orthogonal Lie in  $2n+1$  variables is:  $B_n (n \geq 2)$ , also denoted  $O(2n+1)$ . Its dimension is  $n(2n+1)$ . (2.10)

- iv) The Lie algebra of the special orthogonal Lie algebra in  $2n$  variables:  $D_n (n \geq 4)$ , also denoted  $O(2n)$ . The dimension of  $O(2n)$  is  $n(2n-1)$ . (2.11)



## Remark:

We note that we could also define  $B_n$  and  $C_n$  for  $n \geq 1$  and  $D_n$  for  $n \geq 3$  but to avoid repetitions (because  $A_1 = B_1 = C_1, B_2 = C_2$  and  $A_3 = D_3$ ) we usually use the numbering above.

The exceptional Lie algebras are denoted by  $G_2$  (of dimension 14),  $F_4$  (of dimension 52),  $E_6$  (of dimension 78),  $E_7$  (of dimension 133) and  $E_8$  (of dimension 248).

## II.4. Solvability and nilpotency [6]

### II.4.1. Definition (solvability)

Let  $L$  be a Lie algebra. We define a sequence of ideals of  $L$  by:

$$L^{(0)} = L, L^{(1)} = [L, L], L^{(2)} = [L^{(1)}, L^{(1)}] = [[L, L], [L, L]], L^{(3)} = [L^{(2)}, L^{(2)}], \quad (2.12)$$

$L$  is called solvable if  $L^{(n)} = 0$  for some  $n$ . So, from the definition we immediately see that Abelian Lie algebras are always solvable, and Simple Lie algebras are never solvable.

### II.4.2. Definition (nilpotency)

Let  $L$  be a Lie algebra. We define a sequence of ideals of  $L$  by:

$$L^0 = L, L^1 = [L, L], L^2 = [L, L^1] = [L, [L, L]], L^3 = [L, L^2], \quad (2.13)$$

$L$  is called nilpotent if  $L^n = 0$  for some  $n$ .

## Example

We let  $L$  be a Lie algebra over the field  $F$  generated by  $a, b$  and  $c$ , such that

$$[a, b] = [a, c] = a \text{ and } [b, c] = 0. \text{ This Lie algebra satisfies the Jacobi identity since} \\ [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 - [b, a] + [c, a] = -(-a) + (-a) = 0 \quad (2.14)$$

Furthermore,  $[L, L] = aF$ , so  $[[L, L], [L, L]] = 0$  and  $L$  is solvable. However,  $L$  is not nilpotent because for example  $[b, [b, [\dots, [b, a]]]]$  is non zero for any arbitrary number  $b$ 's in front.

From the definitions above, we note that

- Every nilpotent Lie algebra (ideal) is solvable; since  $[L, L] \subseteq L$ .
- A solvable Lie algebra (ideal), however is not necessary nilpotent.

### II.4.3. Definition (radical and nilradical) [10]

The radical of  $L$ , denoted by  $Rad(L)$ , is the largest solvable ideal of  $L$ .

Similarly, we define the nilradical of  $L$ , denoted by  $NilRad(L)$ , as the largest nilpotent ideal of  $L$ .

### Example

Consider  $L =$  The set of  $2 \times 2$  upper triangular matrices.

$$L = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

$L^{(0)} = L, L^{(1)} = [L, L] = [x, y] \forall x, y \in L$ . Then for  $x = \begin{pmatrix} u & v \\ 0 & w \end{pmatrix}$  and  $y = \begin{pmatrix} k & l \\ 0 & m \end{pmatrix}$

We have:

$$\begin{aligned} L^{(1)} = [x, y] &= xy - yx = \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \begin{pmatrix} k & l \\ 0 & m \end{pmatrix} - \begin{pmatrix} k & l \\ 0 & m \end{pmatrix} \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} = \\ &= \begin{pmatrix} uk & ul + vm \\ 0 & mw \end{pmatrix} - \begin{pmatrix} ku & kv + lw \\ 0 & mw \end{pmatrix} = \begin{pmatrix} 0 & (ul + vm) - (kv + lw) \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$L^{(2)} = [L^{(1)}, L^{(1)}] = [s, t] \forall u, v \in L^1$ . Hence for  $s = \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}$  and  $t = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$

We have

$$L^{(2)} = [s, t] = st - ts = \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So  $L^{(2)} = 0$  and  $L$  is solvable. The  $Rad(L) = L$ .

It is straight forward to see that a Lie algebra has unique radical.

Suppose  $I$  and  $J$  are solvable ideals of Lie algebra  $L$ , then:

- $J/(I \cap J)$  is solvable (because  $J$  is solvable)
- $(I + J)/I$  is solvable (because  $\frac{(I + J)}{I} \approx \frac{J}{(I \cap J)}$ ), so  $I + J$  is solvable

#### II.4.4 Definition (semisimple Lie algebra)

A Lie algebra  $L$  is said to be semisimple if  $Rad(L) = 0$  (i.e if it does not contain any non zero abelian ideal).

#### II.4.5. Theorem [6]

Let  $L$  be a Lie algebra. Then  $\frac{L}{Rad(L)}$  is semi simple.

## Proof

Let  $\varphi$  be the natural map of  $L$  onto  $\frac{L}{\text{Rad}(L)}$ . If  $I$  is a solvable ideal of  $\frac{L}{\text{Rad}(L)}$ , then  $\varphi^{-1}(I)$  must be a solvable ideal of  $L$ , and we have  $\text{Rad}(L) \subseteq \varphi^{-1}(I)$ . Moreover, by maximality of  $\text{Rad}(L)$  we have  $\varphi^{-1}(I) \subseteq \text{Rad}(L)$  hence  $\text{Rad}(L) = \varphi^{-1}(I)$ . This shows that  $I = 0$ ,

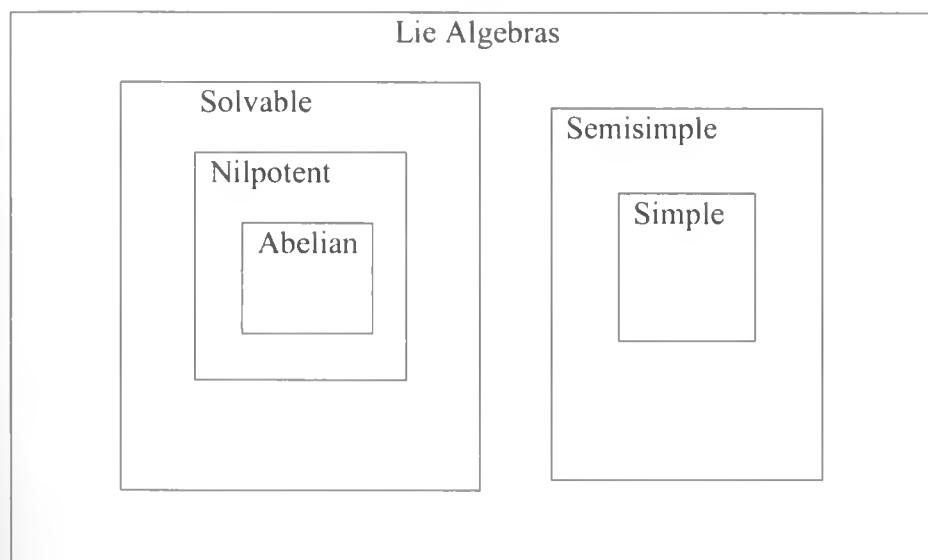
Hence  $L$  is semisimple.

### 11.4.6.Theorem [13]

$L$  is semisimple if and only if it is isomorphic to a direct sum of simple Lie algebras.

We end this chapter by giving the following figure:

**Fig 1:** Relationship of solvability, nilpotency, abelian, semisimple and simple Lie algebras.



From the above we deduce the following:

The sum of two solvable ideals is a solvable ideal.

A Lie algebra  $L$  is simple if  $L$  is non abelian and has no proper nonzero ideal and is semisimple if it has no nonzero solvable ideals. Every simple Lie algebra is semisimple and every semisimple Lie algebra has 0center.

We have the sequence:

$$0 \rightarrow T \rightarrow L \rightarrow L/T \rightarrow 0 \text{ With } T \text{ solvable and } L/T \text{ semisimple.}$$

In particular, there is no nonzero Lie algebra that is both solvable and semisimple.

## CHAPTER III: SEMISIMPLE LIE ALGEBRAS

In this chapter we prove the standard results on semisimple Lie algebras and introduce the Killing form used to describe the Cartan subalgebra.

### III.1. Definitions

#### III.1.1. Definition (L-module)

Let  $L$  be a Lie algebra, a vector space  $V$ , endowed with an operation

$$L \times V \rightarrow V$$

$(x, v) \rightarrow x.v$  or just  $xv$  is called an  $L$ -module if the following conditions are satisfied:

- i)  $(ax + by).v = a(x.v) + b(y.v)$
  - ii)  $x.(av + bw) = a(x.v) + b(x.w)$
  - iii)  $[xy].v = x.y.v - y.x.v \quad (x, y \in L; v, w \in V; a, b \in F)$
- (3.1)

#### III.1.2. Definition (L-module homomorphism)

Let  $V$  and  $W$  be  $L$ -modules,

A linear map  $\varphi: V \rightarrow W$  such that  $\varphi(x.v) = x.\varphi(v)$ ,  $\forall x \in L, v \in V$  is called  $L$ -module homomorphism.

#### III.1.3. Definition (completely reducible L-module)

An  $L$ -module is said to be irreducible if it has only precisely two  $L$ -modules (itself and 0)

An  $L$ -module is completely reducible if  $L$  is a direct sum of irreducible  $L$ -submodules. (i.e. each  $L$ -submodule  $W$  of  $V$  has a complement  $W'$  (an  $L$ -submodule such that  $V = W \oplus W'$ )).

### III.2. Properties semisimple Lie algebras

#### III.2.1. Complete reducibility [13]

A consequence of semisimplicity is a theorem due to Weyl:

[Every finite dimensional representation is completely reducible].

While in other contexts complete reducibility is equivalent to being semisimple, for Lie algebras the two notions are different.

Lie algebras whose finite dimensional representations are all completely reducible, are called reductive Lie algebras. The linear Lie algebra  $gl(n)$  is not semisimple, but its finite dimensional

representations are completely reducible. Here semisimple means that the Lie algebras is semisimple (a sum of semisimple Lie algebras), not that its representations are semisimple (sum of simple representations).

### III.2.2. Centerless

Since the center of a Lie algebra  $L$  is an abelian ideal, if  $L$  is semisimple, then its center is zero. We note that every ideal, quotient and product of a semisimple Lie algebra is again semisimple.

### III.2.3. Linear

The adjoint representation  $ad: L \rightarrow End(L)$  is injective, and so a semisimple Lie algebra is also a linear Lie algebra under the adjoint representation.

### III.2.4. Jordan decomposition [23]

Any endomorphism  $x$  of a finite dimensional vector space over an algebraically closed field can be decomposed uniquely into a diagonalizable (or semisimple) and nilpotent parts.

$x = s + n$ , where  $s$  is the semisimple and  $n$  nilpotent parts and  $s$  and  $n$  commute with each other. Moreover each of  $s$  and  $n$  is a polynomial in  $x$ . This is a consequence of Jordan decomposition.

If  $x \in L$ , then the image of  $x$  under the adjoint map decomposes as:  $ad(x) = ad(s) + ad(n)$ . The elements  $s$  and  $n$  are uniquely determined by  $x$ .

For any representation  $\rho$  of  $L$ ,  $\rho(x) = \rho(s) + \rho(n)$  is the Jordan decomposition of  $\rho(x)$  in the endomorphism ring of the representation space.

### Generalization

With these properties, semisimple Lie algebras admit certain generalizations:

Firstly, many statements that are true for semisimple Lie algebras are true more generally for reductive Lie algebras. Abstractly, a reductive Lie algebra is one whose adjoint representation is completely reducible, while concretely, a reductive Lie algebra is a direct sum of a semi simple Lie algebra and an abelian Lie algebra.

### Examples

The following are examples of semisimple Lie algebras:

$A_n : Sl(n)$ , the special linear group

$$Sl(n, c) = \{x \in gl(n, c) / Tr X = 0\} \quad n \geq 2$$

$C_n : Sp(2n)$ , the symplectic Lie algebra

$$Sp(2n) = \{x \in gl(n, c) / X^T J_{n,n} + J_{n,n} X = 0\} \quad n \geq 1$$

$$\text{Where } J_{n,n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

(3.2)

$S(O_n)$ : the special orthogonal Lie algebra

$$S(O_n) = \{x \in gl(n, c) / X + X^T = 0\} \quad n \geq 3$$

This family is divided into two families:

$$B_n : SO(2n+1) \text{ and } D_n = SO(2n)$$

(3.3)

These families together with the five exceptional Lie algebras  $G_2, F_4, E_6, E_7$  and  $E_8$  are in fact the only semisimple Lie algebras over the complex field.

### III.3. Engel's theorem

Engel's theorem connects nilpotency of Lie algebras with ordinary nilpotency of operators on a vector space.

#### III.3.1. Theorem (first form)

Let  $V$  be a vector space, let  $L$  be a sub Lie algebra of the general linear Lie algebra  $gl(V)$ , consisting of nilpotent as linear operators. Then  $L$  is nilpotent Lie algebra.

#### III.3.2. Theorem (second form)

If  $L$  is a Lie algebra such that all operators  $adx$ , with  $x$  in  $L$ , are nilpotent, then  $L$  is nilpotent.

The proof follows from the following proposition:

"Let the Lie algebra  $L$  act on the non zero vector space  $V$  by nilpotent operators, then the null space  $N = \{v \in V / Xv = 0 \text{ for } \forall x \text{ in } L\}$  is not zero."

Proofs [10,13].

### III.4. Killing form [23]

The Killing form is a symmetric bilinear form that plays a basic role in the theory of Lie algebras.

#### III.4.1. Definition

Consider a Lie algebra  $L$  over a field  $F$ . Every endomorphism  $(adx)(adx)$  of  $L$  with the help of the Lie bracket, as  $adx(y) = [x, y]$ . Suppose  $L$  is of finite dimension, the trace of the composition

of two such endomorphism defines a symmetric bilinear form  $B(x, y) = \text{Tr}(adxady)$ , with the values in  $F$ , is the Killing form on  $L$  (where  $\text{Tr}$  means trace).

### III.4.2. Properties [23, 13]

1. The Killing form  $B$  is bilinear and symmetric
2. The Killing form is an invariant form, in the sense that it has the associativity property:  $B([x, y], z) = B(x, [y, z])$  where  $[,]$  is the Lie bracket.
3. If  $L$  is a simple Lie algebra, then any invariant symmetric bilinear form on  $L$  is a scalar multiple of the Killing form.
4. The Killing form is also invariant under automorphism  $s$  of the Lie algebra  $L$ , that is  $B(s(x), s(y)) = B(x, y)$  for  $s$  in  $\text{Aut}(L)$ .
5. The Cartan criterion states that Lie algebra is semisimple if and only if the Killing form is non degenerate.
6. The Killing form of nilpotent Lie algebra is identically zero.
7. If  $I$  and  $J$  are two ideals on Lie algebra  $L$  with zero intersection, then  $I$  and  $J$  are orthogonal subspaces with respect to the Killing form.
8. If a given Lie algebra  $L$  is a direct sum of its ideals  $I_1, I_2, \dots, I_n$ , then the Killing form of  $L$  is the direct sum of the Killing form of the individual summands.

#### Example (Computation of the Killing form)

Let us compute the Killing form of  $Sl(2, F)$  where  $(x, h, y)$  are the standard basis with

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$adh(x) = hx - xh = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2x$$

$$adh(y) = hy - yh = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} = -2y$$

$$adh(h) = hh - hh = 0$$

$$adh = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$



$$adx(x) = xx - xx = 0$$

$$adx(h) = xh - hx = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} = -2x$$

$$adx(y) = xy - yx = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h$$

$$adx = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$ady(x) = yx - xy = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -h$$

$$ady(h) = yh - hy = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} = 2y$$

$$ady(y) = yy - yy = 0$$

$$ady = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

Let  $B(a, b) = \text{Tr}(ada \cdot adb)$ , where  $a = \alpha_1 x + \alpha_2 h + \alpha_3 y$  and  $b = \beta_1 x + \beta_2 h + \beta_3 y$

$$B(a, b) = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} \text{Tr}(adx \cdot adx) & \text{Tr}(adx \cdot adh) & \text{Tr}(adx \cdot ady) \\ \text{Tr}(adh \cdot adx) & \text{Tr}(adh \cdot adh) & \text{Tr}(adh \cdot ady) \\ \text{Tr}(ady \cdot adx) & \text{Tr}(ady \cdot adh) & \text{Tr}(ady \cdot ady) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \quad (3.4)$$

$$B(a, b) = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

### III.5. Cartan's criterions [10]

#### III.5.1. Theorem (First criterion)

This describes the solvability in terms of the Killing form.

A Lie algebra  $L$  is solvable if and only if its Killing form  $B$  vanishes identically on the derived Lie algebra  $L$ .

### III.5.2. Proposition

Let  $L$  be a sub Lie algebra of  $gl(V)$  with the property that  $Tr(xy) = 0$  for all  $x, y$  in  $L$ , then the derived Lie algebra  $L'$  is nilpotent.

Proof [10]

### III.5.3. Theorem (Second criterion) [13]

This describes the basic connection between semi simplicity and the Killing form.

“A Lie algebra  $L$  is semi simple if and only if its dimension is non zero and its Killing form  $B$  is non degenerate.”

Recall that  $B$  is non degenerate: if for some  $x_0$  in  $L$ , the value  $K(x_0, y) = 0$  for all  $y$  in  $L$ , then  $x_0 = 0$

### III.5.4. Corollary [10]

A Lie algebra  $L$  is semisimple if and only if it is direct sum of simple Lie algebras.

### III.6. Representation of $Sl(2, C) = A_1$

Our purpose is to describe all representation of  $Sl(2)$  in order to have something concrete to look at and also we will use the results in studying the structure and the representation of semisimple Lie algebras.

#### III.6.1. Weights and maximal vectors

Let  $L$  denotes  $Sl(2, F)$  whose standard basis consists of  $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ,  $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Then:

$$[hx] = hx - xh = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2x$$

$$[hy] = hy - yh = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} = -2y$$

$$[xy] = xy - yx = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h \quad (3.5)$$

Let  $V$  denotes an arbitrary  $L$ -module. Since  $h$  is semi simple,  $h$  acts diagonally on  $V$ . This yields a decomposition of  $V$  as a direct sum of eigenspaces  $V_\lambda = \{v \in V / h.v = \lambda v\}, \lambda \in K$ .

Whenever  $V_\lambda \neq 0$ , we call  $\lambda$  a weight of  $h$  in  $V$  and we call  $V_\lambda$  a weight space.

### III.6.2. Lemmas

#### Lemma 1.

If  $v \in V_\lambda$ , then  $x.v \in V_{\lambda+2}$  and  $y.v \in V_{\lambda-2}$ .

#### Proof

We are given  $V_\lambda = \{v \in V / hv = \lambda v\}$ . We use the commutation relations, i.e. we note that

$[hx]$  acts as  $hx - xh$  thus we have:

$$h.(x.v) = [h, x].v + x.h.v = 2x.v + \lambda.x.v = (\lambda + 2)x.v \text{ and also}$$

$$h.(y.v) = [h, y].v + y.h.v = -2y.v + y.\lambda v = (\lambda - 2)y.v$$

$$\text{Hence } x.v \in V_{\lambda+2} \text{ and } y.v \in V_{\lambda-2}. \quad (3.6)$$

#### Remark

Since  $\dim V$  is finite, and the sum  $V = \bigcup_{\lambda \in F} V_\lambda$  is direct, there must exist  $V_\lambda \neq 0$ , such that  $V_{\lambda+2} = 0$

For such  $\lambda$ , any non zero vector in  $V_\lambda$  will be called a maximal vector of weight space.

Assume now that  $V$  is an irreducible  $L$ -module, choose a maximal vector,

$$\text{say } v_0 \in V_\lambda, \text{ set } v_{-1} = 0, v_i = \left(\frac{1}{i!}\right) y^i . v_0 (i \geq 0).$$

#### Lemma 2

$$i) h.v_i = (\lambda - 2i)v_i$$

$$ii) y.v_i = (i+1)v_{i+1} \quad (3.7)$$

$$iii) x.v_i = (\lambda - i + 1)v_{i-1} \quad (i \geq 0)$$

#### Proof

$$i) \text{ From Lemma 1, we have } h.(y.v_i) = [h, y].v_i + y.hv_i = -2iy.v_i + y.\lambda v = (\lambda - 2i)y.v_i$$

$$\text{The result } h.v_i = (\lambda - 2i)v_i \text{ follows.} \quad (3.8)$$

ii) By definition

$$\begin{aligned}v_i &= \left(\frac{1}{i!}\right) y^i \cdot v_0 \quad (i \geq 0) \\v_{i+1} &= \left(\frac{1}{(i+1)!}\right) y^{i+1} v_0 = \frac{1}{i!(i+1)} y^i \cdot y \cdot v_0 = \left(\frac{y^i}{i!} v_0\right) \frac{1}{(i+1)} y \\v_{i+1} &= \frac{v_i}{(1+i)} y \Rightarrow y \cdot v_i = (i+1)v_{i+1}\end{aligned}\tag{3.9}$$

iii) Use induction on  $i$ , the case  $i = 0$  being clear (since  $V_{-1} = 0$ , by convention)

Hence  $ix \cdot v_i = x \cdot y \cdot v_{i-1}$  (by definition)

$$\begin{aligned}ix \cdot v_i &= [x, y] \cdot v_{i-1} + y \cdot x \cdot v_{i-1} = h v_{i-1} + y \cdot x \cdot v_{i-1} = ((\lambda - 2(i-1))v_{i-1} + (\lambda - i + 2)y \cdot v_{i-2}) \\&= (\lambda - 2i + 2)v_{i-1} + (i-1)(\lambda - i + 2)v_{i-1} \quad (\text{by (ii)})\end{aligned}$$

We get :  $ix \cdot v_i = i(\lambda - i + 1)v_{i-1}$

Then divide both sides by  $i$  and the result follows. (3.10)

## CHAPTER IV: CLASSIFICATION OF SEMISIMPLE LIE ALGEBRAS

In this chapter we develop the structure theory of general semisimple Lie algebras over the complex field and bring the complete classification of Semisimple Lie algebras.

### IV.1. Maximal toral subalgebra

#### IV.1.1. Definitions

##### Semisimple elements

Let  $L$  be a finite dimensional semisimple Lie algebra over  $\mathbb{C}$ . An element  $x \in L$  is called semisimple, if its Jordan decomposition is  $x = x + 0$ , that is the nilpotent part is equal to zero. This means that  $x$  acts diagonalisably on every  $L$ -module.

##### Maximal toral subalgebra

Let  $L$  be a finite dimensional semisimple Lie algebra over  $\mathbb{C}$ . A toral subalgebra  $T$  is a subalgebra consisting of semisimple elements.

A toral subalgebra  $T \subseteq L$  is called maximal toral subalgebra if  $L$  has no toral subalgebra properly containing  $T$ .

Every finite dimensional semisimple Lie algebra over  $\mathbb{C}$  has a maximal toral subalgebra.

#### IV.1.2. Lemma (Maximal toral Subalgebra is abelian) [13]

Let  $L$  be a finite dimensional semisimple Lie algebra over  $\mathbb{C}$ . Every maximal toral subalgebra  $T$  of  $L$  is abelian.

##### Cartan subalgebra

Let  $L$  be a finite dimensional semisimple Lie algebra over  $\mathbb{C}$ . A maximal abelian toral subalgebra is called Cartan subalgebra by the above lemma every such  $L$  has Cartan subalgebra since every maximal toral subalgebra is abelian.

##### Example

For  $Sl(3, F)$ , ( $F = \mathbb{C}$ ) where  $(h_1, h_2, x_1, x_2, x_3, y_1, y_2, y_3)$  is the standard basis with

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad x_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We note that the span of  $\{h_1, x_1, y_1\}$  is a subalgebra of  $Sl(3, F)$  which is isomorphic to  $Sl(2, F)$ , (it is seen by ignoring the third row and the third column in each matrix).

The two dimensional subspace  $H$  of  $Sl(3, F)$  spanned by  $h_1$  and  $h_2$  is a cartan subalgebra of  $Sl(3, F)$ .

$$\text{Hence } H = \alpha h_1 + \beta h_2 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha + \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

### IV.1.3. Theorem (Cartan subalgebras are self centralizing) [13]

Let  $H$  be cartan subalgebra of a finite dimensional semisimple Lie algebra  $L$  over  $\mathbb{C}$ .

Then  $C_L(H) = H$  where  $C_L(H)$  is the centralizer of  $H$  in  $L$ .

## IV.2. Root space decomposition [1]

### IV.2.1. Definition

Let  $H$  be a Cartan subalgebra and  $\Phi \subseteq H^*$  be the set of non zero weights. Note that the zero map ( $h \rightarrow 0$ ) is a weight and  $L_0 = H$  by the theorem above, the space  $L$  is the direct sum of the weight spaces for  $H$ . Where  $H^*$  is the dual space to the Cartan subalgebra  $H$  (the weight space).

$L = H \oplus \bigcup_{\alpha \in \Phi} L_\alpha$ , this decomposition is called the root space decomposition of  $L$  with respect to

$H$ , where  $L_\alpha = \{x \in L \mid [x, h] = (h\alpha).x \text{ for all } h \in H\}$  and  $\alpha$  ranges over  $H^*$ .

The set  $\Phi$  is called the set of roots of  $L$  with respect to  $H$  and  $L_\alpha$  for  $\alpha \in \Phi \cup \{0\}$  are called the root spaces. Since  $L$  is finite dimensional  $\Phi$  is finite.

### IV.2.2. Properties of $\Phi$ .

Suppose that  $\alpha, \beta \in \Phi \cap \{0\}$  then:

i)  $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$

ii) If  $\alpha + \beta \neq 0, B(L_\alpha, L_\beta) = \{0\}$

iii) The restriction of  $B$  to  $L_0$  is non degenerate.

## Proof

Let  $x \in L_\alpha$  and  $y \in L_\beta$  then

$$\begin{aligned} [[x, y], h] &= [[x, h], y] + [x, [h, y]] = (h\alpha)[x, y] + (h\beta)[x, y] \\ &= (h(\alpha + \beta))[x, y], \text{ thus } [x, y] \in L_{\alpha+\beta} \end{aligned}$$

This proves (i).

For (ii) we conclude from  $\alpha + \beta \neq 0$ , that there is some  $h \in H$  with  $h(\alpha + \beta) \neq 0$ .

Then  $(h\alpha)B(x, y) = B([x, h], y) = B(x, [h, y]) = -(h\beta)B(x, y)$ , and thus  $h(\alpha + \beta) \cdot B(x, y) = 0$

Therefore,  $B(x, y) = 0$ .

For (iii), suppose that  $z \in L_0$  and  $B(z, x_0) = 0$  for all  $x_0 \in L_0$ . Since every  $x \in L$  can be written as

$$x = x_0 + \sum_{\alpha \in \Phi} x_\alpha \text{ with } x_\alpha \in L_\alpha \text{ we immediately get } B(z, x) = 0 \text{ for all } x \in L.$$

From (ii) contracting the non degeneracy of  $B$  on  $L$ . (4.1)

## IV.3. Root system [1]

Let  $E$  be a finite dimensional vector space over  $\mathbb{R}$  with a positive definite symmetric bilinear form  $(-, -): E \times E \rightarrow \mathbb{R}$  (positive definite means that  $(x | x) > 0$  if and only if  $x \neq 0$ )

### IV.3.1. Definitions

#### Reflection

For  $v \in E$ , the map  $S_v: E \rightarrow E$

$$x \rightarrow x - 2 \frac{\langle x, v \rangle}{\langle v, v \rangle} v \text{ is called a reflection along } v. \text{ It is linear, interchanges}$$

$v$  and  $-v$  and fixes the hyper plane orthogonal to  $v$ . As an abbreviation, we use

$$\langle x | v \rangle = 2 \frac{\langle x, v \rangle}{\langle v, v \rangle} \text{ for } x, v \in E, \text{ note that } \langle - | - \rangle \text{ is only linear in the first component.}$$

We have  $S_v(x) = x - \langle x | v \rangle v$ . The group of reflections generated by  $S_v$  is called the Weyl group and is denoted by  $W$ .

## Root system

A subset  $\Delta \subseteq E$  is called a root system, if

1.  $\Delta$  is finite,  $\text{span}(\Delta) = E$  and  $0 \notin \Delta$
2. If  $\alpha \in \Delta$ , then so is  $-\alpha$
3. If  $\alpha \in \Delta$ , then the only scalar multiples of  $\alpha$  in  $\Delta$  are  $\alpha$  and  $-\alpha$
4. If  $\alpha \in \Delta$ , then  $S_\alpha$  permutes the elements of  $\Delta$ .
5. If  $\alpha, \beta \in \Delta$ , then  $\langle \alpha, \beta \rangle \in \mathbb{Z}$ .
6. Any  $\Delta$  satisfying these five above properties is said to be a reduced root system whereas any  $\Delta$  for property 3 is not satisfied is called unreduced (non reduced) root system. In this work we deal with reduced root system.

### IV.3.2. Properties of root system [6]

Let  $\Delta$  be a root system in the vector space  $E$  with inner product  $\langle \cdot, \cdot \rangle$ .

1. If  $\alpha \in \Delta$ , then  $-\alpha \in \Delta$
2. If  $\alpha$  in  $\Delta$  is reduced, then the only members of  $\Delta \cup \{0\}$  proportional to  $\alpha$  are  $\pm\alpha, \pm 2\alpha$  and  $0, \pm 2\alpha$  can not occur if  $\Delta$  is reduced.
3. If  $\alpha \in \Delta$  and  $\beta \in \Delta \cup \{0\}$ , then  $2 \frac{\langle \beta, \alpha \rangle}{|\alpha|^2} \in \{0, \pm 1, \pm 2, \pm 3, \pm 4\}$  and  $\pm 4$  only occurs in a non reduced system for  $\beta = \pm 2\alpha$ .
4. If  $\alpha$  and  $\beta$  are non proportional members of  $\Delta$  such that  $|\alpha| \leq |\beta|$ , Then  $2 \frac{\langle \beta, \alpha \rangle}{|\beta|^2} \in \{0, 1\}$ .
5. If  $\alpha, \beta \in \Delta$  with  $\langle \alpha, \beta \rangle > 0$ , then  $\alpha - \beta$  is a root or zero.

If  $\langle \alpha, \beta \rangle < 0$ , then  $\alpha + \beta$  is a root or zero.

6. If  $\alpha, \beta \in \Delta$  and neither  $\alpha + \beta$ , nor  $\alpha - \beta$  in  $\Delta \cup \{0\}$ , then  $\langle \alpha, \beta \rangle = 0$ .

We say that a root  $\alpha \in \Delta \subset E$  is simple if  $\alpha > 0$  and  $\alpha$  does not decompose as  $\alpha = \beta_1 + \beta_2$  with both  $\beta_1$  and  $\beta_2$  positive roots. The set of positive roots is denoted by  $\Pi$ .



### IV.3.3. Bases for root system [7]

Let  $\Delta$  be a root system in a vector space  $E$ .

A subset  $B \subseteq \Delta$  is called a base of  $\Delta$ , if

1.  $B$  is a vector space basis of  $E$ , and
2. Every  $\alpha \in \Delta$  can be written as  $\alpha = \sum_{\beta \in B} k_{\beta} \beta$  with  $k_{\beta} \in \mathbb{Z}$  such that the non zero coefficients  $k_{\beta}$  are either positive all or all negative

#### Remark

For a fixed base  $B$ , we call  $\alpha$  positive if all its non zero coefficients with respect to  $B$  are positive and negative otherwise.

We denote the subset of  $\Delta$  of positive roots by  $\Delta^+$  and the subset of negative roots by  $\Delta^-$ . Some coefficients can be equal to zero, only the non-zero ones need to have the same sign.

For any base  $B$ , the set  $-B$  is also a base.

#### Example of base, root and positive root

Consider  $Sl(3, F)$  and evaluate the commutation relations, we have:

$$\begin{aligned}
 [h_1, x_1] &= 2x_1, & [h_1, y_1] &= -2y_1, & [h_2, x_2] &= 2x_2, & [x_2, y_2] &= -2y_2, & [x_2, y_2] &= h_2, & [x_1, y_1] &= h_1, & [h_1, h_2] &= 0 \\
 [h_2, x_1] &= -x_1, & [h_2, y_1] &= y_1, & [h_1, x_2] &= -x_2, & [h_1, y_2] &= y_2, & [h_2, y_2] &= -2y_2, & [h_1, x_3] &= x_3, & [h_1, y_3] &= -y_3 \\
 [h_2, x_3] &= x_3, & [h_2, y_3] &= -y_3, & [x_1, y_1] &= h_1, & [x_2, y_2] &= h_2, & [x_3, y_3] &= h_1 + h_2, & [x_1, x_2] &= x_3, & [y_1, y_2] &= -y_3 \\
 [x_1, y_2] &= 0, & [x_2, y_1] &= 0, & [x_1, y_3] &= 0, & [y_1, y_3] &= 0, & [x_2, x_3] &= 0, & [h_2, y_3] &= 0, & [x_2, y_3] &= y_1, & [x_3, y_2] &= x_1 \\
 [x_1, y_3] &= -y_2, & [x_3, y_1] &= -x_2
 \end{aligned} \tag{4.2}$$

By (IV.2.1):

$$\begin{aligned}
 [h_1, z] &= \alpha_1 z \\
 [h_2, z] &= \alpha_2 z
 \end{aligned} \tag{4.3}$$

Where  $z$  is a root vector corresponding the root  $\alpha$ , ( $z$  is simultaneous eigen vector for  $adh_1$  and  $adh_2$ ).

The commutations relations above tell us what roots for  $Sl(3, F)$  are.

They are six roots:

|          |   |       |       |       |       |       |
|----------|---|-------|-------|-------|-------|-------|
| $\alpha$ | (2,-1), (-1,2), (1,1), (-2,1), (1,-2) (-1,-1) |       |       |       |       |       |
| $Z$      | $x_1$   | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |

The first root  $\alpha = (2, -1)$  comes from  $[h_1, x_1] = 2x_1$  and  $[h_1, x_2] = -x_2$  where  $\alpha_1 = 2, \alpha_2 = -1$ .

The second  $\alpha = (-1, 2)$  comes from  $[h_2, x_1] = -x_1$  and  $[h_2, x_2] = 2x_2$  where  $\alpha_1 = -1, \alpha_2 = 2$ .

Others are obtained by the same process. The roots above plus  $h_1$  and  $h_2$  form a basis for  $Sl(3, F)$ . These six roots form a root system conventionally called  $A_2$ .

The roots  $\alpha_1 = (2, -1)$  and  $\alpha_2 = (-1, 2)$  are called the positive root since they have the property that all of the roots can be expressed as a linear combination of  $\alpha_1$  and  $\alpha_2$  with integer coefficients.

This is verified by the following computation:

$$\alpha_1 = (2, -1), \alpha_2 = (-1, 2), \alpha_1 + \alpha_2 = (1, 1), -\alpha_1 = (-2, 1), -\alpha_2 = (1, -2) \text{ and } -\alpha_1 - \alpha_2 = (-1, -1)$$

#### IV.3.4. Isomorphism of root systems

Let  $\Delta_1 \subseteq E_1$  and  $\Delta_2 \subseteq E_2$  be two root systems. An isomorphism between the two root systems  $\Delta_1$  and  $\Delta_2$  is bijective linear map:  $\Psi : \Delta_1 \rightarrow \Delta_2$  such that :

i)  $\Delta_1 \Psi = \Delta_2$

ii) For any  $\alpha, \beta \in \Delta_1$ , we have  $\langle \alpha, \beta \rangle = \langle \alpha\Psi, \beta\Psi \rangle$

#### IV.3.5. Cartan Matrix

##### IV.3.5.1. Proposition

With  $l = \dim E$ , there are  $l$  simple roots  $\{\alpha_1, \alpha_2, \dots, \alpha_l\} = \Pi$  which are linearly independent.

If  $\beta$  is a root and is decomposed by  $\beta = a_1\alpha_1 + a_2\alpha_2 + \dots + a_l\alpha_l$  then all  $a_i \neq 0$  have the same sign and all  $a_i$  are integers.

##### IV.5.2. Definition (Cartan matrix)

Let  $\Delta$  be a reduced root system in an  $l$ -dimension vector space  $E$  and let  $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$  denotes the simple roots in a fixed ordering.

The  $l$  by  $l$  matrix  $A = (A_{ij})$  given by  $A_{ij} = 2 \frac{\langle \alpha_i, \alpha_j \rangle}{|\alpha_i|^2}$  is called the Cartan matrix of  $\Delta$  and  $\Pi$ .

This matrix depends on the ordering of the simple roots but distinct orderings lead

to Cartan matrices which are conjugate by a permutation matrix.

##### IV.3.5.3. Properties of Cartan matrices [7]

The Cartan matrix  $A = (A_{ij})$  of  $\Delta$  and  $\Pi$  has the following properties:

1.  $A_{ij}$  is in  $\mathbb{Z}$  for all  $i, j$
2.  $A_{ii} = 2$  for all  $i$
3.  $A_{ij} \leq 0$  for  $i \neq j$
4.  $A_{ij} = 0$  if and only if  $A_{ji} = 0$

5. There exists a diagonal matrix  $D$  with positive diagonal entries such that  $DAD^{-1}$  is symmetric positive definite.

An arbitrary square matrix  $A$  satisfying the above properties is called Cartan matrix.

Two Cartan matrices are isomorphic if they are conjugate by a permutation matrix.

#### IV.3.5.4. Theorem

Let  $\Pi$  and  $\Pi'$  be two sets of simple roots in  $\Delta$ . There exists one and only one element  $s \in W$  such that  $s(\Pi) = \Pi'$

#### IV.3.5.4. Corollary

Let  $\Delta$  be an abstract root system and let  $\Delta^+$  and  $\Delta'^+$  be two positive systems, with corresponding simple systems  $\Pi$  and  $\Pi'$ . The Cartan matrices of  $\Pi$  and  $\Pi'$  are isomorphic.

#### Proof

By the above theorem, we obtain an  $s \in W(\Delta)$  such that  $\Pi' = s(\Pi)$ . We fix an enumeration of  $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$  and choose an enumeration of  $\Pi' = \{\beta_1, \beta_2, \dots, \beta_l\}$  such that  $\beta_j = s(\alpha_j)$  for all  $j \in \{1, 2, \dots, l\}$ .

So we have:

$$2 \frac{\langle \beta_i, \beta_j \rangle}{|\beta_i|^2} = 2 \frac{\langle s\alpha_i, s\alpha_j \rangle}{|s\alpha_i|^2} = 2 \frac{\langle \alpha_i, \alpha_j \rangle}{|\alpha_i|^2} . \quad (4.4)$$

Since  $s$  is orthogonal and the resulting Cartan matrices are equal after a permutation of indices, which means that they are isomorphic.

### IV.4. Complete classification of semisimple Lie algebras

#### IV.4.1. Coxeter graphs and Dynkin diagrams

In this section, we will classify all possible root systems; we will use the axioms in definition (IV.3.1)

#### Lemma (finiteness lemma)

Let  $\Delta$  be a root system in a finite dimensional real vector space  $E$  equipped with a positive symmetric bilinear form:

$(-|-) : E \times E \rightarrow \mathbb{R}$ . Let  $\alpha, \beta \in \Delta$  with  $\beta \notin \{\alpha, -\alpha\}$ , then  $\langle \alpha, \beta \rangle \cdot \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$ .

**Proof**

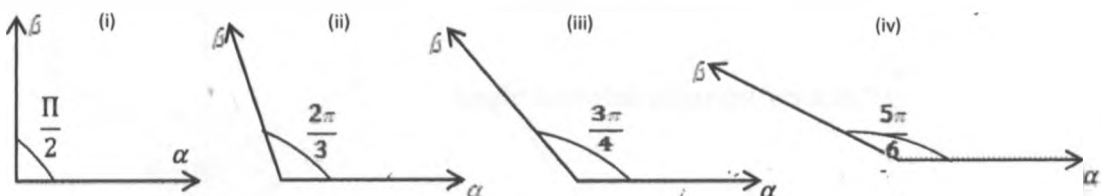
By (4) of (IV.3.1) the product is an integer. We have  $(x|y)^2 = (x|x)(y|y)\cos^2\theta$ . If  $\theta$  is the angle between two non zero vectors  $x, y \in E$ . Thus  $\langle x, y \rangle \cdot \langle y, x \rangle = 4\cos^2\theta$  and this must be an integer. If  $\cos^2\theta = 1$ , then  $\theta$  is an integer multiple of  $\pi$  and so  $\alpha$  and  $\beta$  are linearly dependent which is impossible because of our assumption and (2).

We immediately conclude that there are only very few possibilities for  $\langle \alpha|\beta \rangle \cdot \langle \beta|\alpha \rangle$ , the angle  $\theta$  and the ratio  $\frac{(\beta|\beta)}{(\alpha|\alpha)}$ , we assume that  $(\beta|\beta) \geq (\alpha|\alpha)$ . (4.5)

**Table1.** The following table gives the different cases of  $\langle \alpha|\beta \rangle \cdot \langle \beta|\alpha \rangle$ , which can occur

| Cases | $\langle \alpha \beta \rangle$ | $\langle \beta \alpha \rangle$ | $\theta$          | $\frac{(\beta \beta)}{(\alpha \alpha)}$ |
|-------|--------------------------------|--------------------------------|-------------------|---|
| (i)   | 0                              | 0                              | $\frac{\pi}{2}$ , | -                                       |
|       | 1                              | 1                              | $\frac{\pi}{3}$   | 1                                       |
| (ii)  | -1                             | -1                             | $\frac{2\pi}{3}$  | 1                                       |
|       | 1                              | 2                              | $\frac{\pi}{4}$   | 2                                       |
| (iii) | -1                             | -2                             | $\frac{3\pi}{4}$  | 2                                       |
|       | 1                              | 3                              | $\frac{\pi}{6}$   | 3                                       |
| (iv)  | -1                             | -3                             | $\frac{5\pi}{6}$  | 3                                       |

**Fig 2.** The relative positions corresponding to  $\alpha$  and  $\beta$  are illustrated below



**Proposition**

Any root system of rank two is equivalent to one of four shown in the figure 2 below

**Fig 3:**

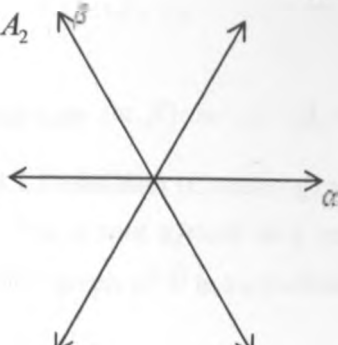
(i)  $A_1 \oplus A_2$



$\alpha \perp \beta$

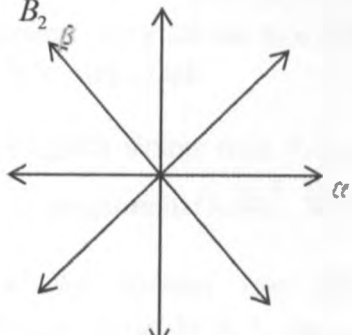
Any ratio  $|\alpha|:|\beta|$  permissible

(ii)  $A_2$



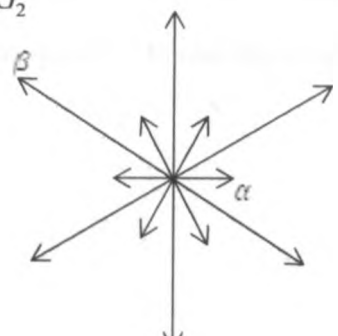
All vectors of the same norm,  
Angle between adjacent vectors =  $\frac{\pi}{3}$ .

(iii)  $B_2$



Ratio  $|\beta|:|\alpha| = \sqrt{2}$   
Angle between adjacent vectors =  $\frac{\pi}{4}$

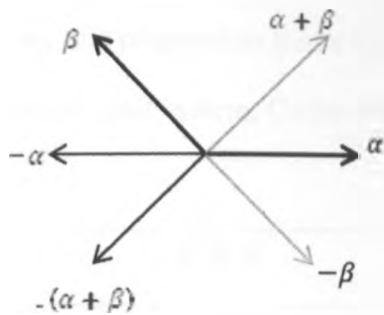
(iv)  $G_2$



Ratio  $|\beta|:|\alpha| = \sqrt{3}$   
Angle between adjacent vectors =  $\frac{\pi}{6}$

From the above figure we have figure 3, which illustrates the base of the root system.

**Fig 3.** In the following diagram we have colored a base of the root system in black and in red.



Hence both  $(\alpha, \beta)$  and  $(\alpha + \beta, -\beta)$  are bases.

#### IV.4.2. Definition (Coxeter graph)

Let  $\Delta$  be a root system in a real vector space  $E$  and let  $B = (b_1, b_2, \dots, b_n)$  be a base of  $\Delta$ . The Coxeter graph of  $B$  is an undirected graph with  $n$  vertices, one for every element  $b_i$  and with

$\langle b_i, b_j \rangle \langle b_j, b_i \rangle$  edges between vertexes  $b_i$  and  $b_j$ , for all  $1 \leq i < j \leq n$ .

#### IV.4.3. Dynkin diagrams

The last step in reducing the problem of classification to the essential minimum are Dynkin diagrams. We associate to a reduced root system  $\Delta$  with simple roots  $\Pi$  and Cartan matrix  $A$  the following graph:

- Each simple root  $\alpha_i$  is represented by a vertex, and we attach to that vertex a weight proportional to  $|\alpha_i|^2$ . We will omit to write the weights if they are the same.
- We connect two given vertices corresponding to two distinct simple roots  $\alpha_i$  and  $\alpha_j$  by  $A_{ij}A_{ji}$  edges. The resulting graph is called the Dynkin diagram of  $\Pi$ . [7]

#### Definition (irreducible root system)

A root system  $\Delta$  is called irreducible, if it cannot be written as the disjoint union


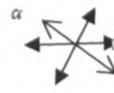
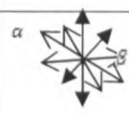



$\Delta_1 \cup \Delta_2$  such that  $(\alpha | \beta) = 0$  whenever  $\alpha \in \Delta_1$  and  $\beta \in \Delta_2$ .

A root system is irreducible if and only if its Dynkin diagram is connected.

If two root systems are isomorphic then they have the same Dynkin diagram. In particular, the Dynkin diagram does not depend on the choice of base. So Dynkin diagrams are the same as isomorphism types of root systems.

It follows from that proposition that a Dynkin diagram is connected if and only if  $\Delta$  is irreducible.

**Table 2:** Example root system, Cartan matrix and Dynkin diagram

|                | $A_1 \oplus A_1$  | $A_2$   | $G_2$   |
|----------------|---|---|---|
| Root system    |  |  |  |
| Cartan Matrix  | $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$                                    | $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$                                  | $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$                                    |
| Dynkin diagram |  |  |  |

#### IV.4.4. Classification

Now we give an outline of the classification of Cartan matrices. We will work simultaneously with Cartan matrices and their associated Dynkin diagrams.

First we observe two operations on the Dynkin diagrams and their counterparts on Cartan matrices.

1. Remove the  $i^{\text{th}}$  vertex and all attached edges from an abstract Dynkin diagram. The counterpart operation on an abstract cartan matrix is removing the  $i^{\text{th}}$  row and column from the matrix.
2. If the  $i^{\text{th}}$  and  $j^{\text{th}}$  vertices are connected by a single edge, their weights are equal. Collapse the two vertices to a single one removing the connected edge, retaining all other edges.

The counterpart operation collapses the  $i^{\text{th}}$  and  $j^{\text{th}}$  row and column replacing the 2by2 matrix from the  $i^{\text{th}}$  and  $j^{\text{th}}$  indices:

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \text{ by 1by1 matrix}$$

Using the properties of Cartan matrices plus operation (1) we get the following:

##### IV.4.4.1. Proposition

Let  $A$  be a Cartan matrix. If  $i \neq j$ , then

1.  $A_{ij}A_{ji} < 4$
  2.  $A_{ij} \in \{0, -1, -2, -3\}$
- (4.6)

An important step which uses the above proposition in the classification is the following.

#### IV.4.4.2. Proposition

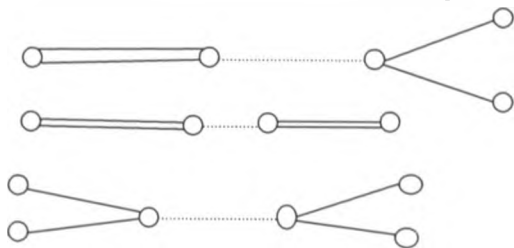
The Dynkin diagram associated to the  $l$  by  $l$  Cartan matrix  $A$  has the following properties:

1. There are at most  $l$  pairs of vertices  $i < j$  with at least one edge connecting them.
2. There no loops
3. There at most three edges attached to one vertex.

Using these tools we obtain the following classification of irreducible Dynkin diagrams in five steps.

Note that reducible Dynkin diagrams are not connected and therefore they are obtained by putting irreducible ones side by side.

**Step 1:** None of the following configuration occurs



Otherwise we use operation in (IV.4.2.2) to collapse all the single line part in the center to a single vertex leading to a violation of (IV.4.2.2).

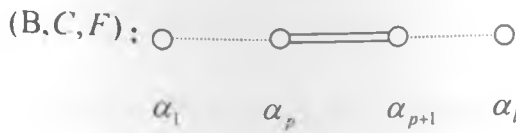
**Step 2:** We do a raw classification of the maximal number of lines connecting two vertices.

- There is a triple line. By (IV.4.2.2) the only possibility is :



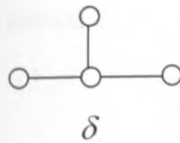


- There is a double line, but no triple line. The graph in the middle of the figure in step (1) shows that only one pair of vertices connected by two edges exists.



- There are only single lines.

In this situation we have  $\delta$  :

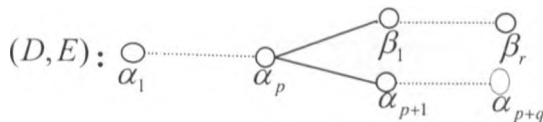


We call  $\delta$  triple point. If there is no triple point, then the absence of loops implies that the diagram is



If there is a triple point there is only one because of the third diagram in the figure in step 1.

So the other possibility is:



**Step 3:** Now we address the problem of possible weights going through the three point of the previous step in the reverse order:

- If the  $i^{\text{th}}$  and  $j^{\text{th}}$  vertices are connected by a single line then  $A_{ij} = A_{ji} = -1$  which implies that the weights  $w_i$  and  $w_j$  of these vertices are equal. Thus in the case (A) and (D,E) all weights are equal, and we may take them to be 1. In this situation we omit writing the weights in the diagram.

- In the case  $(B, C, F)$  we have  $A_{p,p+1} = -2$  and  $A_{p+1,p} = -1$  the defining property (5)

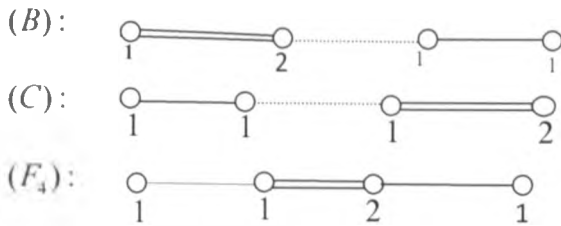
of Cartan matrices leads to  $|\alpha_{p+1}|^2 = 2|\alpha_p|^2$ . Take  $\alpha_k = 1$  for  $k \leq p$  we get

$\alpha_k = 2$  for  $k \geq p+1$ .

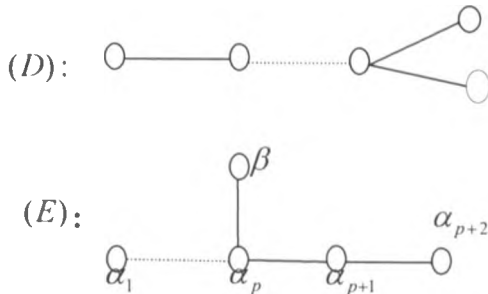
- In the case  $(G_2)$  we get  $|\alpha_1|^2 = 1$  and  $|\alpha_2|^2 = 3$

**Step 4:** The remaining steps deal with special situations. In this step we cover the case (B, C and F).

In this case only these diagrams are possible.



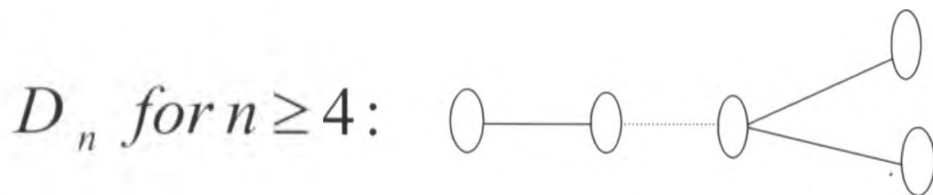
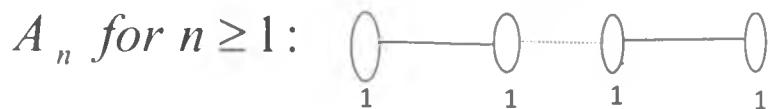
**Step 5:** In the case  $(D, E)$  the only possibilities are

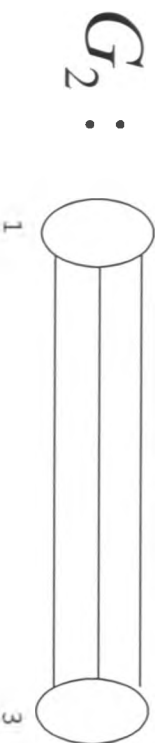
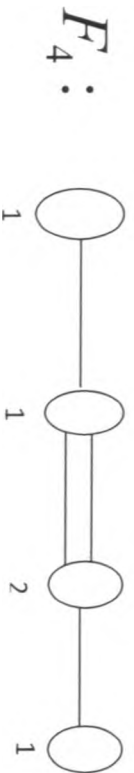
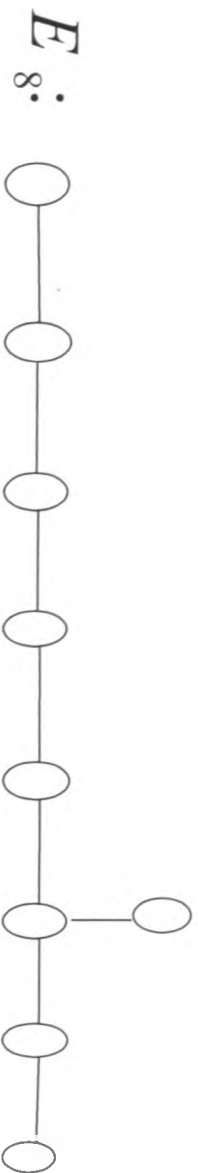
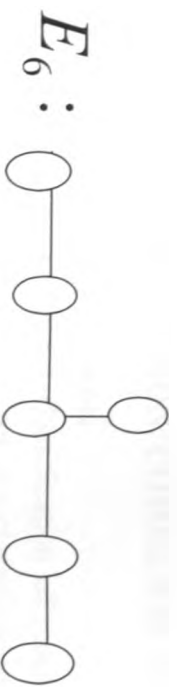


Where  $p \in \{3, 4, 5\}$

These steps lead to the following connected Dynkin diagrams:

Where  $n$  refers to the number of the vertices of the Dynkin diagram. The restrictions of  $n$  in the first four items are made to avoid identical diagrams.



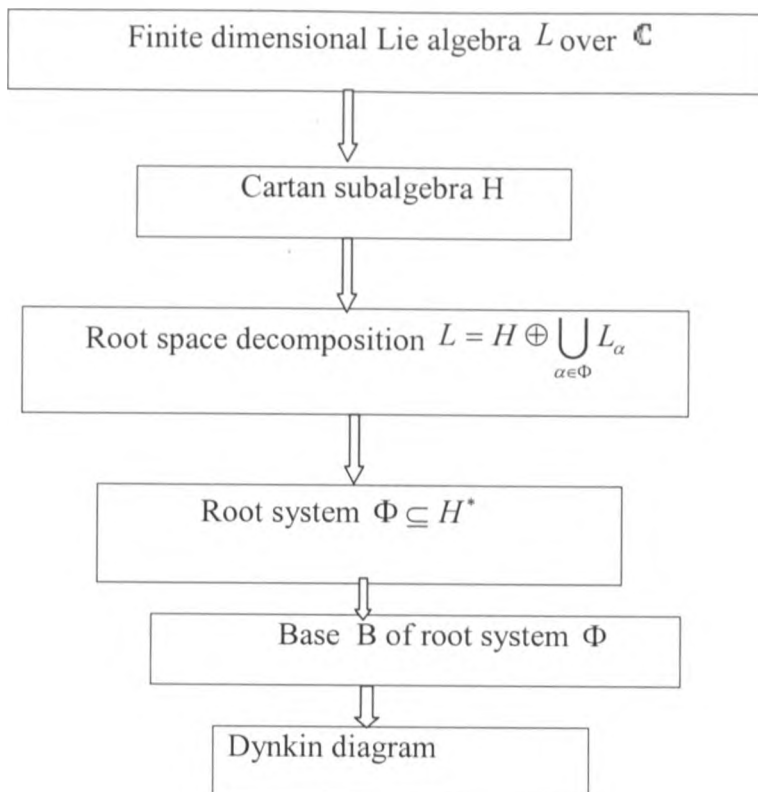


## CHAPTER V: CONCLUSION, COMMENTS AND CURRENT RESEARCH

### V.1. Conclusion and comments

In this work we have got a classification of reduced abstract Dynkin diagrams (respectively Cartan matrices), now which gives us a classification of reduced abstract root systems.

To achieve this we have followed the following diagram:



From this we know all resulting diagrams that can possibly occur.

During the classification of semisimple Lie algebras the notions of indices, roots, edges and vertices are important.

For any base  $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$  for a root system  $\Phi$ , the Dynkin diagram for  $\Phi$  is a graph having vertices  $v_1, v_2, \dots, v_r$ . Between two vertices, we place no edge, one edge, two edges, or three edges. If the roots  $\alpha_i$  and  $\alpha_j$  are orthogonal, then we put no edge between  $v_i$  and  $v_j$ . In the case  $\alpha_i$  and  $\alpha_j$  are not orthogonal, we put one edge between  $v_i$  and  $v_j$ ; if  $\alpha_i$  and  $\alpha_j$  have the same length, two edges if the longer of  $\alpha_i$  and  $\alpha_j$  is  $\sqrt{2}$  longer than the shorter; and three edges if the longer of  $\alpha_i$  and  $\alpha_j$  is  $\sqrt{3}$  longer than the shorter.

root systems  $G_2, F_4, E_6, E_7$  and  $E_8$  which define five exceptional simple complex Lie algebras of dim 14, 52, 78, 133 and 248.

Isomorphism following from Dynkin diagrams gives:

$$D_2 \approx A_1 \oplus A_2 \Rightarrow SO(4) = SI(2) \oplus SI(2) = SO(3) \oplus SO(3)$$

$$A_3 \approx C_2 \Rightarrow SI(3) = SO(6)$$

$$B_2 \approx C_2 \Rightarrow SO(5) = Sp(4).$$

I wish to recommend other researchers including myself to the following topics related to this work:

- ✓ Classification of semisimple Lie algebras over a non algebraically closed field, where the classification deals with a real Lie algebra with a given complexification, this can be done by Satake diagrams which are Dynkin diagrams with additional data.
- ✓ Classification of semisimple Lie algebras by using the extend and affine Dynkin diagrams.

## V.2. Current researches related to the present study

Work activities associated with Lie algebras especially on the classification of semisimple Lie algebras have been carried out and findings (results) of the studies on the Lie algebras are still important.

Dynkin established fundamental results about the structure of simple subalgebras of arbitrary semisimple Lie algebras. Modern representation theory makes significant use of Dynkin's results due to a fundamental relationship between representations and subalgebras of a given Lie algebra.

The work of Dimitrov, N.Snyder, E. Dan Cohen and Ivan Penkovmc (2000) has been based on the structure theory for the classical simple finitely Lie algebras:  $SI(\infty), So(\infty)$  and  $Sp(\infty)$  has contributed to obtain a complete structure theory of these Lie algebras including a generalization of Dynkin's results. Now there are significant simplification in the infinite case which make the results elementary and attractive.

Also their work have been also based on : Cartan, Borel and Parabolic subalgebras of semisimple algebras; The Lie algebras  $SI(\infty), So(\infty)$  and  $Sp(\infty)$  and their general cartan subalgebras.

A.M. Cohen, F.G.M.T Cypres, J.Draisma(2001) in Special elements in Lie algebras they have deduced: from the classification of semisimple Lie algebras, the classification of finite dimensional modular simple Lie algebras is complete for algebraically closed field of characteristic greater than 3 and the complete classification of the simple modular Lie algebras has been announced by Premet and Strade. It provides a nice way of distinguishing the classical Lie algebras from the others by means of extremal elements. An element  $x$  of a Lie algebra  $L$

defined over a field  $K$  is called extremal if the image of  $L$  under the square of left multiplication by  $x$  (in formula  $ad^2x(L) \subseteq Kx$ ). By the two collections of simple Lie algebras, the classical algebras and the Cartan type algebra are distinguished by the adjoint action of an extremal element  $x$  of  $L$ .

In (2003) Cohen and Iwanys worked on the classification of Lie algebra  $L$  generated by non sandwich extremal elements (if  $ad^2x = 0$ ) they got the connection with the classical Lie algebras. But the problem: to find nice presentation for the simple classical Lie algebras involving their extremal elements remained unsolved.

D.A. Roozmond (2005) Lie algebras generated by extremal elements (Master's thesis), he studied from the classification of semisimple Lie algebras Lie algebras generated by extremal elements by considering the general case for a Lie algebra generated by  $n$  extremal elements.

He found for a particular case that a Lie algebra generated by four elements is isomorphic to  $D_4$  two families of classical Lie algebras can be generated by extremal elements. For him the problems: To generate every simple Lie algebra by extremal elements and to analyze Lie algebras generated by six extremal elements, or focus on Lie algebras over fields of non zero characteristic were not solved and are still open.

Karim Baur (2006) Richardson elements for classical elements, Journal of algebras 297: 168-185 from the classification of semisimple Lie algebras they studied Parabolic subalgebras of semisimple Lie algebras and describe a normal form for Richardson elements in the classical case.

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