UNIVERSITY OF NAIROBI

COLLEGE OF BIOLOGICAL AND PHYSICAL SCIENCES

SCHOOL OF MATHEMATICS

Project Report

STUDY OF SOME EXACT SOLUTIONS OF NAVIER-STOKES EQUATIONS FOR STEADY FLOW OF VISCOUS FLUID

A project report submitted to the school of Mathematics in partial fulfillment for a degree of Masters of science in Applied Mathematics

Mpele James

I the undersigned declare that this project report is my original work and to the best of my knowledge has not been presented for the award of a degree in any other university

Mpele James

REG: I56/70848/2007

.... 10-08-2009

Signature Date

Declaration By Supervisor

This project report has been submitted for examination with my approval as supervisor

Dr. C.B. Singh

School of Mathematics.

UNIVERSITY OF NAIROBI P.O. Box 30197 00100, NAIROBI SCHOOL OF MATHEMATICS

.... 10/08/09

Signature Date

STUDY OF SOME EXACT SOLUTIONS OF NAVIER-STOKES EQUATIONS FOR STEADY FLOW OF VISCOUS FLUID

Mpele James

August 2009

Contents

Acknowledgements

In the first place,I thank God the Almighty who provided the breath of life to enable me reach this stage of completing my project.
Iam very grateful to the Government of the United Republic of Tanzania and Dar-es-Salaam University College of Education (DUCE) in particular for allowing me to undertake this master's degree in Applied Mathematics.
I highly appreciate the financial support from my main sponsor,The African Network of Scientific and Technological Institutions(**ANSTI**)using the funds provided by the Dutch Government.I also recognise the partial scholarship award I received from,The African Mathematics Millennium Science Initiative(**AMMSI**).Without these financial support,I would not have managed to complete my studies.
I wish to thank my lecturer and supervisor Dr.C.B.Singh who tirelessly worked very close with me,giving valuable comments and suggestions leading to successful completion of this project work.My sincere appreciations also goes to the other lecturers viz Prof.Genesh Porkhariary,Prof.Wandela Ogana, Dr.Hudson Jamen Were, Dr.John Muriuki, Dr.Charles Nyandwi and Dr.Adu Wasike in addition to all other staff in the school of Mathematics.My special gratitude also goes to my family members and friends for having been kind,caring,loving and patiently until they see me through my studies.Moreover I would like to thank my IT-advisor,Mr.Benjamin Kikwai for his generosity of assisting whenever I had an obstacle.Finally I wish to thank my classmates Jane Ireri,Sammy Njogu,Mary Nyawira and Fatemah Chagpar who have been supportive in one way or another throughout my stay at the school of Mathematics.

To all, may the Almighty God reward you abundantly.

Mpele James **UoN-Kenya**
August 2009

Dedication

I dedicate this work to my parents and all family members.

Abstract

The Navier-Stokes equations are the systein of non-linear partial differential equations governing the motion of a Newtonian fluid. This study is based on three kind of flows of obtaining the Exact solutions of Navier-Stokes equations which are:Flows for which the non-linear terms of Navier-Stokes equations vanish identically such as parallel flow,Flow with similarity properties such that the flow equations reduces to set of ODEs-Ordinary differential equations(Stagnation point flow was used as an example of such flow)and Flows for which the vorticity function is so chosen that the governing equations reduces to a linear equation.In each case the governing equations were formulated and the method of obtaining the exact solutions was analysed.

Background of the study

The non-linear nature of the Navier-Stokes equation makes it difficult to achieve exact solution.Researchers have been employing different methods to study them.Most of the exact solutions of the Navier-Stokes equation for fluid flow with constant viscosity and constant thermal conductivity are obtained by a variety of method and adress specific fluid-dynamic problem resulting in minimal cross referencing.

Much of the early history of the exact solutions has been recorded by the Truesdell[1954];also many books such as Batcherlor[1967],contain brief accounts of the most important exact solutions.More extensive accounts are to be found in treatises on the the flow of a viscous fluid,such as Dryden,Murnaghan and Bateman[1932],Schlichting[1979],Whithman[1963] and Langerstrom[1996].

Satya [1966], studied some exact solutions of the Navier-Stokes equation of viscous liquid motion in spherical polar coordinates $\left(r, \theta, \phi\right)$ with axial symmetry,the line OZ(i.e $\theta = 0$)being the axis of symmetry in the annulus of a convergent tunnel bounded by two porous coaxial cones with variable suction and injection and the results were found to be in agreement with those discussed in Schlichting's book(Schlichting [1960]) and the solution discussed by Agarwal(Agarwal [1957]).The investigation of the axially symemetric flow of a viscous liquid through a convergent tunnel bounded by a porous wall $\theta = \alpha$ and $\theta = \beta$ $(0 < \beta < \alpha < \frac{\pi}{2})$between the sections $r = a$ and $r = b$,where $0 < b < a$ and a is finite since $r \neq 0$ and $\eta = \cos\theta \neq \mp 1$ .The conclusion was that the flow of viscous liquid with axial symmetry along a plane boundary($\theta = \frac{\pi}{2}$) which ejects liquid with velocity kr^2,and in which the velocity along the axis of symmetry is zero.The velocity and pressure distributions were

$$v_r = -k_1 r^2 \cos\theta \sin^2\theta$$

$$v_\theta = k_1 r^2 \sin^3\theta$$

$$p = C - 4k_1 \rho \nu r \cos\theta$$

where C is an arbitrary constant $0 \leq r \leq a$ and $0 \leq \theta \leq \frac{\pi}{2}$.

Gupta & Goyal [1970], studied Plane Coutte flow between two parallel plates with uniform suction at the stationary plate,on pressure,longitudinal and transverse velocities of the plane coutte flow has been studied without taking the transverse velocity as independent of x and by introducing the non-dimensional quantities in such a way that the results of the plane coutte flow without suction can directly be obtained by taking λ equal to zero.
Sinha & Choudhary[1965];attempted to find the exact solution of the Navier-Stokes equations for the steady laminar flow of a viscous incompressible fluid between two coaxial porous cylinder rotating with constant angular velocities.A solution was obtained under the assumption of uniform conditions along the axis of the cylinders.The cylinder being porous, a hyperbolic radial velocity distribution has been superimposed over the circumferential velocity produced due to rotation.There is a Bernouli type pressure variation in the radial direction.When the inner cylinder is at rest,the shearing stress at it and the torque transmitted to it decreases as $\sigma(=\frac{v_0 y_1}{\nu})$ increases.When $\sigma = 0$,the results transform to the known results for coutte flow between uniformly rotating coaxial cylinders.
Chandna & Ukpong[1992];Studied the exact solutions of the Navier-Stokes equations of flows for chosen vorticity functions.Solutions were obtained for the equations of the motion of the steady incompressible viscous planar generalized Beltrami flows when the vorticity distribution was given by
$\nabla^2\psi = \psi + f(x,y)$ for some choosen forms of $f(x,y,z)$.
Singh[2007]; Solved the exact solutions of Navier-Stokes equations on Hydromagnetic Steady flow of viscous Incompressible fluid between Two parallel Infinite plates under the influence of Inclined Magnetic field.The analytical expression for fluid velocity were obtained graphically for different values of Hartmann numbers and at different inclinations.
Singh[2008];Studied the exact solution of Navier-Stokes equation for Coutte Flow Between Two Parallel Infinite plates in the presence of Transverse magnetic field.The study was for steady laminar flow of viscous incompressible fluid between two parallel infinite plates when the upper plate is moving with constant velocity and lower plate is held stationary under the influence of transverse magnetic field.The resulting expressions was solved by the application of Laplace transform and analytical expression was obtained.Further analysis showed the velocity profile decreases as the Hertmman number increases.He suggested that a similar approach can be used

to solve some of the meteorological problems which involve differential equation and are difficult to solve directly by applying boundary conditions.

Chapter 1

Preliminaries

1.1 Introduction

The equation of viscous incompressible fluid flow,known as the Navier-Stokes(N.-S.)equations named after the Frenchman(*Claude Louis Marie Henri Navier*)and Englishman(*George Gabriel Stokes*)who proposed them in the early to mid 19*th* century,can be expressed as:

$$\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\vec{q} \tag{1.1}$$

$$\nabla.\vec{q} = 0 \tag{1.2}$$

where ρ is the density of fluid (taken to be known constant) ; $\vec{q} \equiv (u_1, u_2, u_3)$ is the velocity vector which will often be written as $(u, v, w)^T$; p is the air pressure ; μ is viscosity, and $\vec{F}$=X$\hat{i}$+Y$\hat{j}$+Z$\hat{k}$ is the body force. $\frac{D}{Dt}\equiv \frac{\partial}{\partial t}$+u$\frac{\partial}{\partial x}$+v$\frac{\partial}{\partial y}$+w$\frac{\partial}{\partial z}$ which is the material derivative or substantial derivative expressing the Langrarian,or total acceleration of the fluid particle ; ∇^2 is the Laplacian,i.e.$\nabla^2 \equiv \hat{i}\frac{\partial^2}{\partial x^2} + \hat{j}\frac{\partial^2}{\partial y^2} + \hat{k}\frac{\partial^2}{\partial z^2}$ and $\nabla.$ is the divergence operator. The first equation is a three-component vector equation which is just **Newton's** second law of motion applied to a fluid particle,the left hand side of equation(1.1) is the sum of forces acting on a fluid element. Substituting into(1.1) and re-arranging the similar terms we obtain a **system** of Navier-Stokes equations as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\nabla^2 u,$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\nabla^2 v,$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\nabla^2 w$$

The *Navier-Stokes equations*Agrawal[1957] are the system of non-linear partial differential equations governing the motion of a *Newtonian fluid*, which may be liquid or gas. In essence,they represent the balance between the rate of change of momentum of an element of fluid and the forces on it, as does Newton's second law of motion for a particle, where the stress is linearly related to the rate of strain of the fluid.These equations have been widely accepted as an excellent model of the macroscopic motions of most real fluids,including air and water,and are used by countless engineers,physicists,chemists,mathematicians,meteorologists, oceanographers ,geologists and biologists. Navier-Stokes isn't really an equation;but a group of equations that's used to solve fluid dynamics problems.The idea is to describe precisely the forces in a moving fluid.Suppose,for example,one want to design a wing for a new air plane.To figure out how air will flow around the wing,how much lift it will generate,and how much drag it will produce,one need to be able to precisely model the behavior of air flowing around it.That's a very typical example of what the Navier-Stokes equations are used for.

The equations themselves are in some sense sort of straight forward.In all practical interactions;there a collection of fundamental conserved properties:mass,momentum,angular momentum and energy.One can express the velocity and pressure in terms of how the fluid motion preserves those conserved properties.The problem with Navier-Stokes equations is that they are group of extremely difficult partial differential equations.They don't actually tell us what the values of the variables are;they talk about the relationships between rates of change.So far we (Mathematicians) haven't been able to actually *solve*the Navier-Stokes equations in a way that gives us a useful closed-form solution.

There, however, notable and and useful models of fluids whose motions are not governed by the Navier-Stokes equations. For example there *non-Newtonian fluids* which are governed by a non-linear stress tensor, and *visco-elastic* fluids in which the stress depends on the strain as well as on the rate of strain of the fluid and retains a 'memory' of previous deformation;Lolyd[1981]. An *exact solution* may seem to be more or less than a solution,because either a given set of fields $\vec{q}$, p, for given ρ , μ and body force $\vec{F}$ satisfies the governing equations or it does not. By exact solution we mean a solution which has a simple explicit form,usually an expression in finite terms of elementary or other well known special functions.Sometimes an exact solution is taken to be one which can be reduced to a solution of ordinary differential equations.Rarely we go even further,and take an exact solution to be the solution of a partial differential equation, provided that the equation has fewer independent variables than the Navier-Stokes equations themselves. This is in contrast with to an approximate solution which is taken to be a field,simple or complicated,which approximates a solution either in numerical sense or asymptotic limit,for example vanishingly small viscosity thus the logical distinctions between solutions are blurred,but in practice the distinctions made are usually clear and useful.The exact solutions are,essentially,a subset of the solutions of the Navier-Stokes equations which happen to have relatively simple mathematical expressions and which are,mostly simple physically.The essence of this account, then,is the explicitness and relatively simplicity of the expression of the solutions.Many exact solutions of the Navier-Stokes equations are unstable and therefore unobservable in practice.In this work we are not going to include the stability flow of the Navier-Stokes equations. In the early decades of the development of the mathematical theory of the motion of a viscous fluid,exact solutions were the only solutions available.Researchers solved what problems they could,rather than solving the practical problems in hand.Inevitably the solvable problems were the simple ones,usually idealised with a strong symmetry.From the mid-nineteenth century,and early twentieth century,asymptotic method were developed,and thereafter numerical method.Nevertheless,the exact solution remain a valuable and irreplaceable resource. They immediately convey more physical insight than a numerical table.

Exact solutions are important for the following reasons:

a)The solutions represent fundamental fluid dynamic flows.Also,owing to the uniform validity of exact solutions,the basic phenomena described by the Navier-Stokes equations can be more closely studied.

b)The exact solution serves as standards for checking the accuracies of the many approximate methods,whether they are numerical,asymptotic,or empirical.Current advances in computer technology make the complete numerical integration of the Navier-Stokes equations more feasible.However,the accuracy of the results can only be ascertained by a comparison with an exact solution;C.Y.Wang[1991].

1.2 Systems of Coordinates for Navier-Stokes equations.

The Navier-Stokes equations for an incompressible fluid occurs in *three* most common co-ordinates systems,notably,the Cartesian co-ordinates,the Cylindrical polar co-ordinates and Spherical polar co-ordinates.

(i)*Cartesian co-ordinate*: In Cartesian co-ordinates(x, y, z) so that $\vec{q}$=u$\hat{i}$+v$\hat{j}$+w$\hat{k}$.The Navier-Stokes and Continuity equations (1.1) and (1.2)then become

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+X+\nu\nabla^2 u, \tag{1.3}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+Y+\nu\nabla^2 v, \tag{1.4}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+Z+\nu\nabla^2 w, \tag{1.5}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{1.6}$$

where the body force per unit mass $\vec{F}=(X,Y,Z)$,∇^2 represent the three-dimensional Laplacian operator.The vorticity $\omega=\xi\hat{i}+\eta\hat{j}+\zeta\hat{k}=(\xi,\eta,\zeta)$ where

$$\xi=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z},\eta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x},\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}. \tag{1.7}$$

For two-dimensional flow ,independent of the co-ordinate z,say,the continuity equation(1.6) is satisfied by introducing the stream function ψ such that

$$u=\frac{\partial\psi}{\partial y},v=-\frac{\partial\psi}{\partial x}, \tag{1.8}$$

and the only non-zero component of the vorticity is

$$\zeta=-\nabla^2\psi, \tag{1.9}$$

which satisfies, for a conservative body force $\vec{F}$,

$$\frac{D\zeta}{Dt}=\nu\nabla^2\zeta, \tag{1.10}$$

so that

$$\frac{\partial}{\partial t}(\nabla^2\psi) - \frac{\partial(\psi, \nabla^2\psi)}{\partial(x,y)} = \nu\nabla^4\psi. \tag{1.11}$$

(ii) *Cylindrical polar co-ordinate*: We define cylindrical polar co-ordinates (r, θ, z) such that

$$x = r\cos\theta, y = r\sin\theta, r \geq 0, 0 \leq \theta < 2\pi, \tag{1.12}$$

with corresponding velocity components $\mathbf{v} = (v_r, v_\theta, v_z) = v_r\mathbf{r} + v_\theta\theta + v_z\mathbf{z}$, vorticity and body force components $\omega = (\omega_r, \omega_\theta, \omega_z)$, $\mathbf{F} = (F_r, F_\theta, F_z)$ respectively. The component of Navier-Stokes equations are, then,

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + F_r + \nu(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}), \tag{1.13}$$

$$\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial \theta} + F_\theta + \nu(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}), \tag{1.14}$$

$$\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + F_z + \nu\nabla^2 v_z, \tag{1.15}$$

with continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0. \tag{1.16}$$

The components of vorticity are given by

$$\omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}, \omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_r}{\partial r}, \omega_z = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}. \tag{1.17}$$

For a rotationally symmetric flow, independent of θ, we introduce a different stream function ψ, such that with

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}, \tag{1.18}$$

the continuity equation is satisfied identically.

(iii) *Spherical polar coordinates*: We define spherical polar co-ordinates (r, θ, ϕ) such that

$$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta, r \geq 0, 0 \leq \theta \leq 2\pi, 0 \leq \phi < 2\pi. \tag{1.19}$$

with corresponding velocity component $\mathbf{v}=(v_r, v_\theta, v_\phi)= v_r\mathbf{r}+v_\theta\theta+v_\phi\phi$,vorticity and body-force components $\omega=(\omega_r,\omega_\theta,\omega_\phi)$,$\mathbf{F}=(F_r,F_\theta,F_\phi)$ respectively.The components of Navier-Stokes equations are,then,

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+F_r+\nu(\nabla^2 v_r-\frac{2v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{2v_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}), \tag{1.20}$$

$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}=-\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}+F_\theta+\nu(\nabla^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta}{r^2\sin^2\theta}-\frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}), \tag{1.21}$$

$$\begin{aligned}&\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi v_r}{r}+\frac{v_\theta v_\phi\cot\theta}{r}=\\&-\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}+F_\phi+\nu(\nabla^2 v_\phi-\frac{v_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}+\frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}),\end{aligned} \tag{1.22}$$

with continuity equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}=0. \tag{1.23}$$

The component of vorticity are given by

$$\begin{aligned}\omega_r&=\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial \theta}(v_\phi\sin\theta)-\frac{\partial v_\theta}{\partial \phi}\right\},\\ \omega_\theta&=\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{1}{r}\frac{\partial}{\partial r}(r v_\phi),\\ \omega_\phi&=\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)-\frac{1}{r}\frac{v_r}{\partial \theta}.\end{aligned} \tag{1.24}$$

The Stokes stream function,for a rotationally symmetric flow independent of ϕ, is now defined such that

$$v_r=\frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta}, v_\theta=\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}. \tag{1.25}$$

1.3 Derivation of Navier-Stokes Equations.

Let there be a viscous fluid occupying certain region.In this region let **v** be the volume enclosed by the surface S that moves with the fluid and so contains the same fluid particles at all times. Within S, let dv be the volume element surrounding the fluid particle p of density ρ.The mass ρdv of this element remains constant throughout.Then if $\vec{q}$ is the velocity of the fluid particle,the momentum **M** of volume **V** is given by

$$\mathbf{M} = \iiint_V \rho\vec{q}dv \tag{1.26}$$

where the integral has been carried out over the entire volume V .Let p be the normal pressure force which has an outward unit normal $\hat{n}$.The surface force due to p is

$$\therefore - \iint_A p\hat{n}ds = - \iiint_V \nabla p dv \left[By \quad Gauss Divergence \ Theorem\right] \tag{1.27}$$

The force acting on the volume **V** due to *frictional force* is:

$$\iiint_V \mu\nabla^2\vec{q}dv \tag{1.28}$$

Again let $\vec{F}$ be the *external force* per uit mass acting on the fluid,so that the total force acting on the fluid within the surface S at any time **t** is:

$$\iiint_V \vec{F}\rho dv. \tag{1.29}$$

Thus the total force acting on the volume **V**[of the *Euler's equations*for the perfect fluid] will be:

$$\iiint_V \vec{F}\rho dv - \iiint_V \nabla p dv + \iiint_V \mu\nabla^2\vec{q}dv \tag{1.30}$$

By *Newton's second law*,since the rate of change of Linear momentum is equal to the total force acting on the mass of the fluid,we get,

$$\frac{DM}{Dt} = \iiint_V (\rho\vec{F} - \nabla p + \mu\nabla^2\vec{q})dv. \tag{1.31}$$

Using equation(1.26) and the product rule of differentiation,equation (1.31) above can be written as:

$$\iiint_V \frac{D\vec{q}}{Dt}.\rho dv + \iiint_V \vec{q}\frac{D}{Dt}(\rho dv) = \iiint_V (\rho\vec{F} - \nabla p + \mu\nabla^2\vec{q})dv. \tag{1.32}$$

$(\frac{D}{Dt})$ being *zero* since ρdv is constant.Now since volume V can be taken as arbitrary volume of the fluid in the region considered.Thus equation (1.32) can be written as:

$$\rho\frac{D\vec{q}}{Dt} = \rho\vec{F} - \nabla p + \mu\nabla^2\vec{q}$$
$$Or \qquad \frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla p + \nu\nabla^2\vec{q}. \tag{1.33}$$

where $\nu = \frac{\mu}{\rho}$ is taken to be the $\left[\textit{Kinematics of viscosity}\right]$. Equation (1.33) is Navier-Stokes equation in vector form. Now if we let,

$$\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}\, and$$
$$\vec{F} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

then, Navier-Stokes equation takes the form:

$$\frac{D}{Dt}\left[u\hat{i} + v\hat{j} + w\hat{k}\right] = \left(X\hat{i} + Y\hat{j} + Z\hat{k}\right) - \frac{1}{\rho}\left[\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y} + \hat{k}\frac{\partial p}{\partial z}\right] + \nu\left[\hat{i}\nabla^2 u + \hat{j}\nabla^2 v + \hat{k}\nabla^2 w\right]. \tag{1.34}$$

Equating the coefficient of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on both sides of equation (1.34) we obtain:

$$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$
$$\frac{Dv}{Dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 V \tag{1.35}$$
$$\frac{Dw}{Dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w$$

Equation(1.35) is the Navier-Stokes equation in Cartesian form.Using the definition of *Material derivative*; $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$,equation (1.35) may be written as:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \tag{1.36}$$
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w$$

Equation (1.36) is the Navier-Stokes equations in a simplified Cartesian form.

1.4 Significance of the Terms Related to Navier-Stokes Equations.

- **Body force terms F(X,Y,Z):** The body force due to gravity is important in flow problems in which free liquid surface exists or when the fluid is non-homogeneous,i.e.its density changes from one point to another so that there exists a density gradient. If a fluid is rotating about an axis,body force due to centripetal action must be considered.In case of homogeneous fluid flow within closed boundaries,there is an equilibrium between the weight of a fluid and the buoyant force acting on it.In such a case.body force due to gravity does not influence the fluid motion and hence can be neglected from Navier-Stokes equation.

- **Viscous terms** $\left[(\nu\nabla^2 u), \text{etc.}\right]$:The no-slip condition between the fluid and the solid boundary requires that the fluid velocity must be equal to that of boundary(i.e zero for a stationary boundary).In other words,both normal and tangential velocity components must be zero.In frictionless flow,however,only the normal component of the velocity is required to be zero.Two independent boundary conditions must,therefore,be fulfilled,which require a partial differential equation(PDE)of second order.For that,it is not permissible to ignore the viscous terms in the PDE,even for very small values of ν,if true behaviors of the viscous fluid is to be determined in the vicinity of the boundary.

- **Pressure terms** $\left[(\frac{\partial p}{\partial x}), \text{etc.}\right]$:The pressure gradient terms is incorporated in Navier-Stokes equations to show the pressure distribution across a fluid flow,subjected to different boundary terms.

- **Inertia terms** $\left[(\frac{\partial u}{\partial x}), \text{etc.}\right]$:For high Reynolds number flow,inertia terms dominate over the viscous terms and hence,the viscous terms can be neglected to lead a fair approximation. However,in very low Reynolds number flow (known as **creep flow**),the velocity components are very small and higher order inertia terms can be neglected,which converts the Navier-Stokes equation into a linear PDE,which is much easier to solve.

1.5 Limiting Cases of the Navier-Stokes Equations.

- **Potential Flow Case.**

  In potential flow,viscous forces tend to zero.Akshoy,[2005].

  For incompressible flow $u=\frac{\partial\phi}{\partial x}$, $v=\frac{\partial\phi}{\partial y}$, $w=\frac{\partial\phi}{\partial z}$,where $\phi(x,y,z)$ is the velocity potential function.

  Using the equation of continuity for incompressible flow, $\nabla.\vec{q}$=0,

  and since $\vec{q}=u\hat{i}+v\hat{j}+w\hat{k}$, then the Navier-Stokes equation reduces to **Euler's** equation of motion as shown below;

$$\nabla.\vec{q}=(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}).(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z})=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0. \qquad (1.37)$$

$$\Longrightarrow \nabla^2\phi=0.$$

  Viscous terms of Navier-Stokes equations are$(\nu\nabla^2\vec{q}))=0$.

  As in case of incompressible flow,viscous terms in Navier-Stokes equations are zero and become viscous term independent, i.e.

$$\frac{D\vec{q}}{Dt}=\vec{F}-\frac{1}{\rho}\nabla p \qquad (1.38)$$

  Equation (1.38) is *Euler's equation of motion.*

- **Creep Flow case.**

  Creep flow case occurs at very low Reynold numbers.At these Reynold numbers(i.e.at very small velocity,small linear dimensions of the body or of the flow passage and large viscosity of fluid),the inertia forces are much smaller than the viscous forces.We know that for steady flow,inertia force=$\rho L^2V^2$and viscous force=$\mu A(\frac{\partial u}{\partial x})$.As for creep flow,$\vec{q}$ is very small,hence,we neglect higher order terms of the inertia force,like $u(\frac{\partial u}{\partial x})$, $v\frac{\partial v}{\partial y}$,etc.Equation(1.1)above becomes,

$$(\hat{i}\frac{\partial u}{\partial t}+\hat{j}\frac{\partial v}{\partial t}+\hat{k}\frac{\partial w}{\partial t})=\vec{F}-\frac{1}{\rho}\nabla p+\nu\nabla^2\vec{q} \qquad (1.39)$$

$$\Longrightarrow \frac{\partial\vec{q}}{\partial t}=\vec{F}-\frac{1}{\rho}\nabla p+\nu\nabla^2\vec{q}. \qquad (1.40)$$

Creep flow analysis is of potential importance for laminar flow in pipes and open channels,for seepage flow of water and oil underground,for motion of very small bodies such as spheres in a highly viscous fluid and in the theory of lubrication.

Chapter 2

Essential Concepts Related to the study of Fluids Dynamics in relation to Viscous Fluids.

Fluid dynamics is the science of flow of fluids.

2.1 Difference between solid and fluid.

The following definitions are important in understanding the difference between solid and fluid.
Rigidy or shear modulus(N)$=\frac{\text{Shear stress}}{\text{Shearing strain}}=\frac{\tau}{\alpha}$

Bulk modulus(K)$=\frac{\text{Unit change in pressure}}{\text{Volumetric Strain}}=\frac{-dp}{\frac{dv}{v}}=\rho\frac{dp}{d\rho}$
wheredp is the change in pressure, $d\rho$ is the change in density and dv is the change in volume of fluid from the original volume V.
Compressibility$(\beta)=\frac{1}{K}$.

- **Perfect Solid(N$\neq$ 0and K$\neq$ 0)**

Stressed on perfect solid,the external forces acting on it are balanced by the internal forces arising from the elastic strain(or static deformation)produced.The external forces may be tensile,comperehensive or tangential(shear)in nature.If the force per unit area is less than the yield stress of the material,the deformations disappear when the force is removed.This is true within the elastic limit of the material.Perfect solid hence obeys Hook's law within the elastic limit.

- **Perfect Fluid($N = 0$ and $K = 0$)**

Fluid is a substance,which deforms continuously under the application of a tangential(shear)force,no matter how small the force might be.In other words,if shear force is applied to a fluid,it deforms continously regardless the magnitude of the force.The following points are worthwhile for fluid.
◇ Fluid has no tensile strength or very little of it.
◇ Fluid can resist the compressive force only when it is kept in a container(as compressive forces≡normal pressure forces).
◇Fluid is frictionless,as it can not transmit tangential(shear)forces.
Fluid can be divided into two categories,namely,*liquid* and *gas.*

- **Perfect Liquid(K=∞, i.e.Incompressible)**

◇Liquid possesses a definite volume,which varies slightly with temperature and pressure.
◇All known liquids vaporise at narrow pressures above zero,depending on temperature.
◇Liquid being composed of relatively closed-packed molecules with strong cohesive forces,tends to form a free surface in a gravitational field if unconfined from above.

- **Ideal Fluids.**

Ideal fluids have no viscosity and no surface tension and these are incompressible.However in nature,no such ideal fluid are found.

- **Real Fluids.**

These are fluids having viscosity,surface tension and compressibility in addition to the density.All kinds of fluids found in nature are considered as real fluids.

2.2 Difference between Incompressible and Compressible Fluids.

Velocity of sound at isentropic condition can be expressed as:

$$\mathbf{a} = \sqrt{(\frac{dp}{d\rho})_s} \tag{2.1}$$

For incompressible fluid,$d\rho = 0$.Therefore,velocity of sound(a)is infinite in incompressible fluid,i.e.pressure pulses emmited any where in the fluid are thus felt simultaneously at all other points.

- **Characteristics of Incompressible Fluids.**

⋆Fluid velocity(v)is small compared with the velocity of sound(a).

⋆Fractional variation in density is insignificant.

⋆Fractional variation in temperature and pressure may be very large.

- **Characteristics of compressible Fluid.**

⋆Fluid velocities are appreciable as compared with the velocity of sound.

⋆Fractional variation of density,temperature and pressure are of significant magnitude.

2.3 Viscosity.

Viscosity is the property of fluid by virtue of which it offers resistance to flow.The shear stress at a point in a moving fluid is directly proportional to the rate of shear strain. The above figure

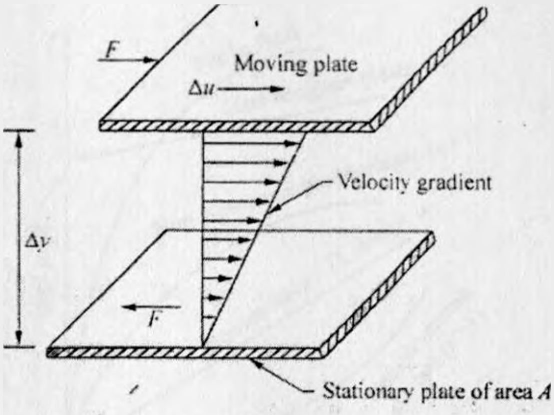


Fig. Real flow through a parallel plate.

represents two plates at a distanceΔy.The intermediate portion of two plates is filled up by some fluid.Now,the upper plate is moving at a velocity$\Delta u$by means of a force **F**.The lower plate is kept stationary.Since the particles of all the real fluids stick to the solid surfaces,the fluid particles sticking to the lower plate will have zero velocity and the particle sticking to the upper plate will have a uniform velocityΔu.This variation of velocities in y-direction leads to a velocity gradient.

If force **F** acts over an area of contact **A**,then the shear stress defined as $\tau = \frac{F}{A}$. Newton postulated that shear stress is proportional to the rate of shear strain, i.e.$\tau \propto \frac{\Delta u}{\Delta y}$. In the limiting case,as $\Delta y \rightarrow 0, \frac{\Delta u}{\Delta y} \rightarrow \frac{du}{dy}$.Hence,

$$\tau \propto \frac{du}{dy}$$

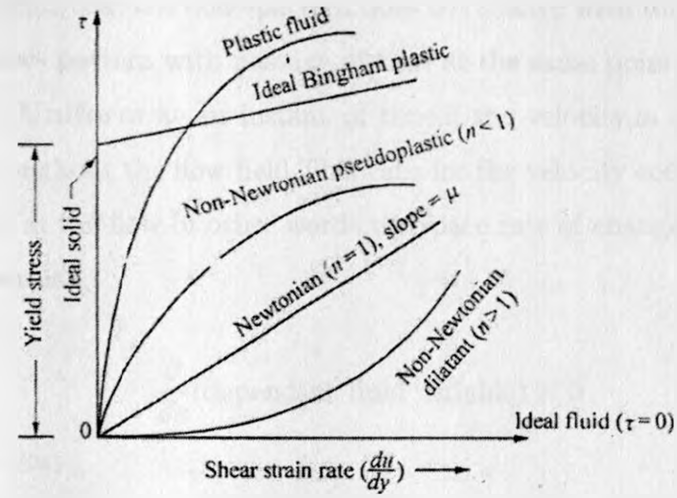
$$\tau = \mu \frac{du}{dy} \tag{2.2}$$

where,μ=coefficient of absolute or dynamic viscosity.The law above is called **Newton's law of viscosity**and fluids obeying this law are known as **Newtonian fluids**.

The unit of μ in SIsystem is Ns/m^2.In CGS system of units,μ is expressed in poise.$1poise = 0.1Ns/m^2$

The coefficient of *kinematic viscosity*(v)is defined as the ratio of coefficient of dynamic viscosity to density of fluid.i.e

$$v = \frac{\mu}{\rho} \tag{2.3}$$



The unit of **v**in SI system is m^2/s.In CGS system of units,v is expressed in stoke.$1stoke = 10^-4m^2/s$.

2.4 Steadness and Uniformity of Flow.

A flow is said to be **Steady**,if all the dependent fluid variables(such as velocity,pressure,density,ten any point in the flow do not change with time.In other words,the time rate of the dependent variables at a position is zero.Mathematically we write,

$$\frac{\partial}{\partial t}(dependent fluidvariables) \equiv 0 \tag{2.4}$$

It follows that

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0 = \frac{\partial p}{\partial t} = \frac{\partial \rho}{\partial t}, etc. for steady flow.$$

A flow is regarded as **Unsteady**,if the dependent fluid variables alter with the passage of time at a position in a flow.mathematically,we write

$$\frac{\partial}{\partial t}(\text{dependent fluid variables}) \neq 0 \qquad \text{for unsteady flow.} \tag{2.5}$$

flow implies that the flow -pattern does not change with time whereas, unsteadiness ...ging flow- pattern with passage of time at the same point in space.

...d to be **Uniform** at an instant of time,if the velocity,in magnitude,direction and ...ical throughout the flow field.This calls for the velocity components to be the same ...ositions in the flow.In other words,the space rate of change in velocity components must varnish,

$$\frac{\partial}{\partial s}(\text{dependent fluid variable}) \equiv 0 \qquad (2.6)$$

follows that

$$= 0 = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial y}, \qquad \text{etc.for} \qquad \text{uniform flow.}$$

...y components at different locations are different at the same instant of time,the flow ... **non-uniform**.

...fers to *no change with time* and uniformity refers to *no change in space*.Therefore, ...e steady or unsteady quite independent of its being uniform or non-uniform.All the ...ations are possible,which are given in the table below with examples.

Type	Examples
...y uniform flow	Flow at constant rate through a duct of uniform cross section.
...non-uniform flow	Flow at constant rate through a tapering pipe.
...dy uniform flow	Flow at varying rates through a long straight pipe of uniform cross section
... non-uniform flow	Flow at varying rates through a tapering pipe.

...cceleration in Fluid Flow.

Velocity component in fluid flow are in general functions of *space* and *time*

$$u = u(x, y, z, t), v = v(x, y, z, t), w = w(x, y, z, t)$$

By differential calculus,an infestimal(small) change in u is given by

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z + \frac{\partial u}{\partial t}\delta t \tag{2.7}$$

and the acceleration component in x,y and z-directions are respectively given by

$$\begin{aligned} a_x &= \frac{\delta u}{\delta t} = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \\ a_y &= \frac{\delta v}{\delta t} = \frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \\ a_z &= \frac{\delta w}{\delta t} = \frac{Dw}{Dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \end{aligned} \tag{2.8}$$

From equation(2.16) see that the operator for total differential with respect to time is ,$\frac{D}{Dt}$ in a convective field is related to the partial differential equations as

$$\frac{D}{Dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t} \tag{2.9}$$

Using vector calculus, we know that:

$$Gradient : \nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$Velocity : \vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

Hence equation (2.17) can be written as

$$\frac{D}{Dt} = (\vec{V}.\nabla) + \frac{\partial}{\partial t} \tag{2.10}$$

Equation(2.18) represents a definition of **material** or **substantial derivative** operator in vector notation,thus it is valid in any coordinate system.

$\frac{D}{Dt}$→ **Substantial(or total)derivative**, which is physically the instantaneous change following a moving fluid element.
$(\vec{V}.\nabla)$→ **Convective derivative**,which is physically the time- rate change due to the movement of fluid element from one location to another in the flow field,where the follow properties are spatially different.

$\frac{\partial}{\partial t}\rightarrow$ **Local(or temporal)derivative**,which is physically the time-rate change at a fixed oint.

lence,the total acceleration is given as:

$$\frac{D\vec{V}}{Dt}=(\vec{V}.\nabla)\vec{V}+\frac{\partial\vec{V}}{\partial t}=\frac{Du}{Dt}\hat{i}+\frac{Dv}{Dt}\hat{j}+\frac{Dw}{Dt}\hat{k}=a_x\hat{i}+a_y\hat{j}+a_z\hat{k} \tag{2.11}$$

r in words

Total acceleration=Convective acceleration+Local acceleration.

Chapter 3

Solutions of Flows For Chosen Vorticity Functions.

These are flows for which the vorticity function is chosen so that the governing equation iterms of the stream function reduces to a linear equation(Generalized Beltrami Flow-Wang[1991])
Let $\vec{q}$ be the velocity and $\zeta \equiv \nabla \times \vec{q}$ be the vorticity.The vorticity equation is obtained by taking the curl on the steady Navier-Stokes equations such that:

$$\nabla \times (\zeta \times \vec{q}) = -\nu \nabla \times \nabla \times \zeta. \tag{3.1}$$

For our case,the generalized Beltrami flows the left-hand side of equation(3.1) is identically zero,such that $\nabla \times (\nabla \times \zeta) = 0$
Wang[1991] employed this approach of choosing the vorticity by taking $\nabla^2\psi = K\psi$, $\nabla^2\psi = f(\psi)$, $\nabla^2\psi = y + (K^2 - 4\pi^2)\psi$, $\nabla^2\psi = A\psi + Cy$ and $\nabla^2\psi = K(\psi - Ry)$ respectively.For this case We study the generalized Beltrami flows when the vorticity function $\omega = -\nabla^2\psi$ is given by: $\nabla^2\psi = \psi + Ay^2 + Bxy + Cx + Dy$, $\nabla^2\psi = \psi + Ay^2 + Cx + D$, $\nabla^2\psi = \psi + Cx + Dy$, where A,B,C,D are constants.

Governing Equations and Their Solutions:

In two dimensions-Steady plane incompressible viscous fluid flow,in the absence of external forces,the Navier-Stokes equation will be governed by the system:

$$\begin{aligned} \bar{u}_{\bar{x}} + \bar{v}_{\bar{y}} &= 0 \\ \bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} + \frac{1}{\rho}\bar{p}_{\bar{x}} &= \mu\bar{\nabla}^2\bar{u} \\ \bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}} + \frac{1}{\rho}\bar{p}_{\bar{y}} &= \mu\bar{\nabla}^2\bar{v}. \end{aligned} \tag{3.2}$$

where $\bar{u}\left(\bar{x},\bar{y}\right)$, $\bar{v}\left(\bar{x},\bar{y}\right)$ are the velocity components.$\bar{p}\left(\bar{x},\bar{y}\right)$ is the pressure function,ρ is the constant density,μ is the constant viscosity and $\bar{\nabla}^2 = \frac{\partial^2}{\partial\bar{x}^2} + \frac{\partial^2}{\partial\bar{y}^2}$ is the **Laplacian** operator.The vorticity function for this flow is given by

$$\bar{\omega} = \bar{v}_{\bar{x}} - \bar{u}_{\bar{y}} \tag{3.3}$$

Letting U,L to be the characteristic velocity and length respectively,we introduce the non-dimensional variables,given by

$$x = \frac{\bar{x}}{L}, y = \frac{\bar{y}}{L}, u = \frac{\bar{u}}{U}, v = \frac{\bar{v}}{U}, \omega = \frac{L\bar{\omega}}{U}, p = \frac{\bar{p}}{\rho U^2} \tag{3.4}$$

in system(3.2) and equation(3.3).Applying the integrability condition$p_{xy} = p_{yx}$ to the linear momentum equations we find that u, v, w should satisfy the system:

$$\begin{aligned} u_x + v_y &= 0 \\ u\omega_x + v\omega_y &= \frac{1}{R}\nabla^2\omega \\ v_x - u_y &= \omega. \end{aligned} \tag{3.5}$$

where $R = \frac{\rho UL}{\mu}$is the Reynolds number. Introducing the stream function such $\psi(x,y)$ such that

$$u = \psi_y, v = -\psi_x \tag{3.6}$$

in system (3.5),we find that $\psi(x,y)$ must satisfy

$$\nabla^4\psi + R\frac{\partial(\psi, \nabla^2\psi)}{\partial(x,y)} = 0 \tag{3.7}$$

Our concentration is the study of the flow for which the vorticity distribution takes the forms:

$$(a)\omega = -\nabla^2\psi = -(\psi + Ay^2 + Bxy + Cx + Dy) \tag{3.8}$$

$$(b)\omega = -\nabla^2\psi = -(\psi + Ay^2 + Cx + Dy) \tag{3.9}$$

$$(c)\omega = -\nabla^2\psi = -(\psi + Cx + Dy) \tag{3.10}$$

where A,B,C,D are real constants.

Form(a):

Substituting equation(3.8) in the compatibility equation(3.7) we obtain,

$$R(2Ay + Bx + D)\psi_x - R(By + C)\psi_y + \psi + Ay^2 + Bxy + Cx + Dy + 2A = 0 \tag{3.11}$$

Using the *canonical coordinates* given by

$$\xi = Ay^2 + Bxy + Cx + Dy, \eta = y, \tag{3.12}$$

where$(By + C) \neq 0$,equation(3.11) may be written as

$$-R(B\eta + C)\psi_\eta + \psi + \xi + 2A = 0 \tag{3.13}$$

Equation(3.13) is solved and we obtain

$$\psi = f(\xi)(By + D)^{\frac{1}{RB}} - (Ay^2 + Bxy + Cx + Dy + 2A) \tag{3.14}$$

where f is an arbitrary function of ξ.Introducing equations(3.14) into (3.8),we obtain

$$\left\{R^2\left[C^2(C^2+D^2)+2BCD\xi+B^2\xi^2\right]f''(\xi)+2R\left[C(RAC+D)-B\xi\right]f'(\xi)+\left[1-RB-R^2C^2\right]f(\xi)\right\}$$
$$+2RC\left\{2R\left[C(AD+BC)+AB\xi\right]f''(\xi)+2A\left[RB+1\right]f'(\xi)-RBf(\xi)\right\}\eta$$
$$+R\left\{2R\left[C^2(2A^2+3B^2)+ABCD+AB^2\xi\right]+f''(\xi)+2AB\left[RB+1\right]f'(\xi)-RB^2f(\xi)\right\}\eta$$
$$+4R^2BC\left\{\left[A^2+B^2\right]f''(\xi)\right\}\eta^3+R^2B^2\left\{\left[A^2+B^2\right]f''(\xi)\eta^4\right\}=0 \tag{3.15}$$

Since ξ,η,are independent variables and $\left\{1, \eta, \eta^2, \eta^3, \eta^4\right\}$ is a linearly independent set,therefore the coefficients of the various powers of η are zero.Taking the coefficients of $\eta^4, \eta^3, \eta^2, \eta$ and 1 equal to zero,we obtain

$$\therefore f(\xi) = c_1\xi + c_2 \tag{3.16}$$

$$2A(RB+1)c_1 - RBc_2 - RBc_1\xi = 0 \tag{3.17}$$

wherec_1,c_2 are arbitrary constants.Since $\{1, \xi\}$ is a linearly independent set,it follows therefore from equation(3.17) that $2A(RB+1)c_1 - RBc_2 = 0, RBc_2 = 0$ giving $c_1 = c_2 = 0$.Using $c_1 = c_2 = 0$ in equation (3.16),we obtain $f(\xi) = 0$.

From equation(3.11), the stream function is given by

$$\psi(x,y) = -(Ay^2 + Bxy + Cx + Dy + 2A) \tag{3.18}$$

The exact integral of this flow is

$$\left.\begin{aligned} u = -(2Ay + Bx + D), v = By + C, \text{and} \\ p = p_0 - \frac{1}{2}\left[B^2(x^2+y^2) + 2(BD - 2AC)x + 2BCy\right] \end{aligned}\right\} \tag{3.19}$$

where p_0 is an arbitrary constant.

Equation (3.18) represents an impigment of two constant-vorticity oblique flows with stagnation point

$$(x,y) = \left(\tfrac{2AC-BD}{B^2}, -\tfrac{C}{B}\right) \tag{3.20}$$

for non-zero values of A, B, C and E,Wang(1991)[5].

Form(b):

Substituting equation(3.9) in (3.7),we obtain

$$R(2Ay + D)\psi_x - RC\psi_y + \psi + Ay^2 + Cx + Dy + 2A = 0 \tag{3.21}$$

Choosing the canonical coordinates

$$\xi = Ay^2 + Cx + Dy, \eta = y \tag{3.22}$$

where $C \neq 0$ and equation(3.17) above will take the form

$$-RC\psi_\eta + \psi + \xi + 2A = 0 \tag{3.23}$$

Solving the above equation we obtain

$$\psi = g(\xi)exp\left(\tfrac{1}{RC}y\right) - (Ay^2 + Cx + Dy + 2A) \tag{3.24}$$

where g is an arbitrary function ofξ.Substituting equation(3.23) into equation (3.8) we obtain

$$\left[R^2C^4g''(\xi)+2R^2AC^2g'(\xi)+(1-R^2C^2g(\xi))\right]+2RCg'(\xi)(2A\eta+D)+R^2C^2g''(\xi)(2A\eta+D)^2=0 \tag{3.25}$$

Sinceξ,η are independent variables and $\{1,(2A\eta+D)^2\}$ is a linearly independent set,it follows that

$$g''(\xi)=0, \quad g'(\xi)=0, \quad (1-R^2C^2)g(\xi)=0 \tag{3.26}$$

From $(1-R^2C^2)g(\xi)=0$,we get the three possibilities:$g(\xi)=0$,$R^2C^2\neq 1$;$R^2C^2=1$,$g(\xi)\neq 0$;$g(\xi)=0$,$R^2C^2=1$. The stream function in equation(4.24)is given by

$$\psi(x,y)=\begin{cases} -(Ay^2+Cx+Dy+2A) & g=0, R^2C^2\neq 1 \\ Kexp\left(\frac{1}{RC}y\right)-(Ay^2+Cx+Dy+2A); & R^2C^2=1, g\neq 0 \\ -(Ay^2+Cx+Dy+2A) & g=0, R^2C^2=1 \end{cases} \tag{3.27}$$

where $g\neq 0$ implies $g=K$(non-zero constant).

when the stream function is given by

$$\psi(x,y)=-(Ay^2+Cx+Dy+2A); R^2C^2=1 \quad \text{or} \quad R^2C^2=1 \tag{3.28}$$

the exact integral for the flow is

$$u=-(2Ay+D), v=C, \quad \text{and} \quad p=p_0+2ACx \tag{3.29}$$

where p_0 is an arbitrary constant.

The solution of equation(3.29) may be realized on a plate situated along the line $y=\frac{-D}{2A}$ with uniform suction or blowing.$C>0$ and $C<0$,respectively , for blowing and suction at the plate.The exact integral for the flow is givev by the stream function

$$\psi(x,y)=Kexp\left(\frac{1}{RC}y\right)-(Ay^2+Cx+Dy+2A); R^2C^2=1 \tag{3.30}$$

is

$$u=\frac{K}{RC}exp\left(\frac{1}{RC}y\right)-(2Ay+D), v=C, \quad \text{and} \quad p=p_0+2ACx \tag{3.31}$$

where p_0 is an arbitrary constant.

If K=RCD in equation(3.30) and (3.31),the velocity profile in(3.31) can be realized on a plate located along the line $y = 0$ with uniform sunction.The velocity profile attains the form

$$u = Dexp\left(\frac{1}{RC}y\right) - (2Ay + D), v = C \tag{3.32}$$

only asymptotically,and so may be regarded as the asymptotic suction profile[20].$C > 0$ and $C < 0$ for blowing and suction at the plate,respectively.

Form (c):

Substituting equation(3.9)into equation equation(3.7) gives:

$$RD\psi_x - RC\psi_y + \psi + Cx + Dy = 0 \tag{3.33}$$

The canonical coordinates

$$\xi = Cx + Dy, \eta = y; C \neq 0 \tag{3.34}$$

are employed in equation(3.33) to obtain

$$-RC\psi_\eta + \psi + \xi = 0 \tag{3.35}$$

The solution of this equation is

$$\psi = h(\xi)exp\left(\frac{1}{RC}y\right) - (Dx + Ey) \tag{3.36}$$

where$h$ is an arbitrary function of ξ.We substitute equation(3.35) into(3.10) to obtain

$$R^2C^2(C^2 + D^2)h''(\xi) + 2RCDh'(\xi) + (1 - R^2C^2)h(\xi) = 0 \tag{3.37}$$

The general solution of (3.36) is

$$h(\xi) = \begin{cases} A_1exp(\lambda_1\xi) + A_2exp(\lambda_2\xi) & ; R^2(C^2 + D^2) - 1 > 0 \\ (B_1 + B_2\xi)exp\left(\frac{-RD}{C}\xi\right) & ; R^2(C^2 + D^2) - 1 = 0 \\ C_1Cos(m\xi + C_2)exp\left[-\frac{D}{RC(C^2+D^2)}\xi\right] & ; R^2(C^2 + D^2) - 1 < 0 \end{cases} \tag{3.38}$$

where

$$\lambda_{1,2} = \frac{-D \mp C\sqrt{R^2(C^2 + D^2) - 1}}{RC(C^2 + D^2)}, \qquad m = \frac{\sqrt{1 - R^2(C^2 + D^2)}}{R(C^2 + D^2)} \tag{3.39}$$

and A_1,B_1,B_2,C_1,C_2 are arbitrary constants.

We now examine the **three** possibilities separately

(i) When $R^2(C^2+D^2)-1>0$ The stream function from (3.35) and (3.37), is

$$\psi(x,y) = A_1 exp\left[\lambda_1 Cx + \left(\lambda_1 D + \frac{1}{RC}\right)y\right] + A_2 exp\left[\lambda_2 Cx + \left(\lambda_2 D + \frac{1}{RC}\right)y\right] - (Cx+Dy) \quad (3.40)$$

The exact integral of this flow is

$$u = \left(\lambda_1 D + \frac{1}{RC}\right) A_1 exp\left[\lambda_1 Cx + \left(\lambda_1 D + \frac{1}{RC}\right)y\right] + \left(\lambda_2 D + \frac{1}{RC}\right) A_2 exp\left[\lambda_2 Cx + \left(\lambda_2 D + \frac{1}{RC}\right)y\right] - D,$$

$$v = -D\left\{\lambda_1 A_1 exp\left[\lambda_1 Cx + \left(\lambda_1 D + \frac{1}{RC}\right)y\right] + \lambda_2 A_2 exp\left[\lambda_2 Cx + \left(\lambda_2 D + \frac{1}{RC}y\right)y\right] - 1\right\},$$

$$p = p_0 + 2\left[1 - \frac{1}{R^2(C^2+D^2))}\right] A_1 A_2 exp\left[\frac{2(Dy-Ex)}{R(C^2+D^2)}\right] \quad (3.41)$$

where $p_0$is an arbitrary constant and λ_1,λ_2 are given by equation(3.38)

This flow represents an impingement of an oblique uniform stream with an oblique rotational,divergent flow,with stagnation point

$$(x,y) = -\frac{RC}{2\sqrt{R^2(C^2+D^2)-1}} C\ln\left(\frac{-A_1}{A_2}\right) - D\sqrt{R^2(C^2+D^2)-1}\ln\left\{\frac{-4A_1A_2\left[R^2(C^2+D^2)-1\right]}{R^2(C^2+D^2)}\right\} \quad (3.42)$$

$$+D\ln\left(-\frac{A_1}{A_2}\right) + C\sqrt{R^2(C^2+D^2)-1}\ln\left\{\frac{-4A_1A_2\left[R^2(C^2+D^2)-1\right]}{R^2(C^2+D^2)}\right\}$$

where, A_1, A_2 are non-zero real constants and either $A_1>0$,$A_2<0$,$A_2>0$.For fixed values of R,C and D ,the stagnation point shifts upward when the absolute value of A_2 is larger than that of A_1. If $A_1$and $A_2$are of the same sign,the above phenomenon does not take place,and we have a flow without a stagnation point.

When **(ii)**$R^2(C^2+D^2)-1=0$

Using equation(3.37) in (3.35),the stream function is

$$\psi(x,y) = \left[B_1 + B_2(Cx+Dy)\right] exp\left[R(Cy-Dx)\right] - (Cx+Dy) \quad (3.43)$$

This flow has the exact integral

$$u = \left\{ BB_2 + RC\left[B_1 + B_2(Cx + Dy)\right] \right\} exp\left[R(Cy - Dx)\right] - E,$$

$$v = \left\{ - DB_2 + RD\left[B_1 + B_2(Cx + Dy)\right] \right\} exp\left[R(Cy - Dx)\right] + D, \tag{3.44}$$

$$p = p_0 - \frac{1}{2R^2} B_2^2 exp\left[2R(Cy - Dx)\right]$$

wherep_0 is an arbitrary constant.

If $B_2$is a positive real constant,this flow represent an impingement of an oblique uniform stream with an an oblique rotational,divergent flow,with stagnation point

$$(x, y) = -\frac{1}{C^2 + D^2}\left(\frac{CB_1}{B_2} - \frac{D}{R}\ln B_2, \frac{DB_1}{B_2} + \frac{C}{R}\ln B_2\right) \tag{3.45}$$

For fixed values of R,and C,the stagnation point shifts upward if B_1 and D are of opposite sign and the absolute value of $B_1$is larger than B_2.

If B_2 is a negative real constant,equation(3.41) represents an oblique uniform stream which abuts on an oblique rotational,convergent flow.

(iii)when $R^2(C^2 + D^2) - 1 < 0$

From (3.28) and (3.37), the stream function is given by:

$$\psi(x, y) = C_1 \cos\left[m(Cx + Dy) + C_2\right] exp\left[\frac{Cy - Dx}{R(C^2+D^2)}\right] - (Cx + Dy) \tag{3.46}$$

The exact integral for this flow is

$$u = \frac{C_1}{R(C^2 + D^2} C\cos\left[m(Cx + Dy) + C_2\right] - mRD(C^2 + D^2)\sin\left[(Cx + Dy) + C_2\right] exp\left[\frac{Cy - Dx}{R(C^2+D^2)}\right] - D,$$

$$v = \frac{C_1}{R(C^2 + D^2)}\left\{ D\cos\left[m(Cx + Dy) + C_2\right] + mRC(C^2 + D^2)\sin\left[m(Cx + Dy) + C_2\right] \right\} exp\left[\frac{Cy - Dx}{R(C^2+D^2)}\right] + ($$

$$p_0 + \frac{1}{2}\left[1 - \frac{1}{R^2(C^2+D^2)}\right] C_1^2 \cos 2\left[mix(Cx + Dy) + C_2\right] exp\left[\frac{2(Cy - Dx)}{R(C^2+D^2)}\right]$$

(3.47)

where P_0 is an arbitrary constant,and m is given by equation(3.38)

If $C_1 > 0$,the stagnation point for this flow are

$$(x, y) = \left(\frac{RC\left[(2n + 1)\frac{\pi}{2} - C_2\right]}{\sqrt{1 - R^2(C^2+D^2)}} + RD\ln\left[\frac{C_1\sqrt{1 - R^2(C^2+D^2}}{R(C^2+D^2)}\right], \frac{RD\left[(2n + 1)\frac{\pi}{2} - C_2\right]}{\sqrt{1 - R^2(C^2+D^2)}} - RC\ln\left[\frac{C_1\sqrt{1 - R^2(C^2+D^2)}}{R(C^2+D^2)}\right]\right)$$

(3.48)

F

ıpter 4

ıct solutions of Navier-Stokes ıations for parallel Flow.

.sic difficulty in solving Navier-Stokes equations arises due to the presence of nonlin-
ıdratic)inertia terms[i.e.$u\frac{\partial u}{\partial x}$,$v\frac{\partial v}{\partial y}$,etc] on the L.H.S of equation(1.36) .However, there some
vial solutions of the Navier-Stokes equations in which the non-linear inertia terms are
ally *zero*,Akshoy[2005].One such flow is **parallel flow**,in which only one velocity term is
.vial and all the fluid particles move in one direction only.

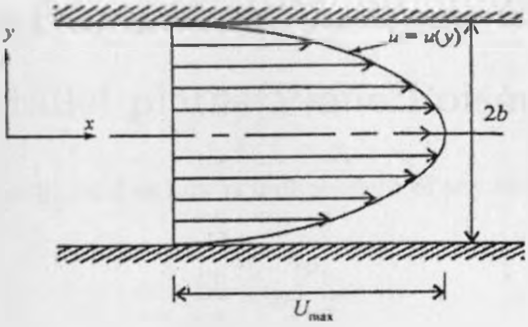
ɔles of parallel flows includes:Steady Two-Dimensional Laminar Flow Through Two Straight
əl Plates(Plane Poiseuille Flow),Coutte Flow between Two Plates,Flow between Two con-
: Rotating Cylinders,Generalized Beltrami flows,Flows in long Cylinders driven by a Pres-
ɾadient,Ekman Flow and Planar Generalized Beltrami Flow.

:t us choose x to be the direction along which all fluid particles travel,i.e. $u \neq 0, v = 0, w =$
ng this into continuity equation,we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\Longrightarrow \frac{\partial u}{\partial x} = 0$$

$$\Longrightarrow u = u(y, z, t)$$



Hence the Navier-Stokes equations for incompressible flow becomes

$$\frac{\partial u}{\partial t} + u\cancel{\frac{\partial u}{\partial x}} + \cancel{v}\cancel{\frac{\partial u}{\partial y}} + \cancel{w}\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2})$$

$$\cancel{\frac{\partial v}{\partial t}} + u\cancel{\frac{\partial v}{\partial x}} + \cancel{v}\frac{\partial v}{\partial y} + \cancel{w}\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}} + \cancel{\frac{\partial^2 v}{\partial z^2}})$$

$$\cancel{\frac{\partial w}{\partial t}} + u\cancel{\frac{\partial w}{\partial x}} + \cancel{v}\frac{\partial w}{\partial y} + \cancel{w}\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu(\cancel{\frac{\partial^2 w}{\partial x^2}} + \cancel{\frac{\partial^2 w}{\partial y^2}} + \cancel{\frac{\partial^2 w}{\partial z^2}})$$

Hence,we obtain;

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$$

$$\Longrightarrow p = p(x) \qquad \text{alone,}$$

And,

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \tag{4.1}$$

For Steady two-dimensional flow trough parallel plates,equation(4.1)can be further simplified and the analytical solution can be obtained;Singh & Okwoyo[2008]and Singh[2007] applied equation(4.1) to obtain the exact results in their findings.

4.1 Steady Two-Dimensional Laminar Flow Trough Two Straight Parallel plates(Plane Poiseuille Flow)

Since the flow is Steady, $\left[\frac{\partial u}{\partial t}=0\right]$, and as flow is independent of any variation in z-direction, $\left[\frac{\partial^2 u}{\partial z^2}=0\right]$,Thus equation(4.1),becomes

$$\frac{\partial p}{\partial x}=\mu\frac{\partial^2 u}{\partial y^2} \tag{4.2}$$

As $p=p(x)$ and $u=u(y)$,equation (4.2) can be written as an ordinary differential equation,

$$\therefore \frac{dp}{dx}=\mu\frac{d^2 u}{dy^2} \tag{4.3}$$

which is subjected to boundary conditions: $u=0$,at $y=\pm b$.

$$\Longrightarrow \frac{du}{dy}=\frac{1}{\mu}\frac{dp}{dx}y+C_1$$

$$\Longrightarrow u=\frac{1}{2\mu}\frac{dp}{dx}y^2+C_1y+C_2$$

Applying boundary conditions:$C_1=0$ and $C_2=\frac{1}{\mu}\frac{dp}{dx}\frac{b^2}{2}$

The solution is

$$u=-\frac{1}{2\mu}\frac{dp}{dx}(b^2-y^2) \tag{4.4}$$

Equation (4.4) is the relation describing **Plane Poiseuille flow**,which means that the velocity profile is *parabolic*.

At, $y=0$, $u=u_{max}$,then from equation (4.4) above we obtain:

$$U_{max}=\frac{-b^2}{2\mu}\frac{dp}{dx}$$

But the average velocity is given by: $U_{av}=\frac{\text{Flow rate(Q)}}{\text{Flow area}}=\frac{\int_{-b}^{b}udy}{(2b)\times 1}$, keeping *unit* thickness,we then have,

$$\Longrightarrow U_{av}=\frac{2}{2b}\int_0^b udy$$

$$\Longrightarrow U_{av}=\frac{1}{b}\int_0^b -\frac{1}{2\mu}\frac{dp}{dx}(b^2-y^2)dy \qquad \text{By using equation(4.4)}$$

$$\Longrightarrow U_{av}=-\frac{1}{2\mu}\frac{dp}{dx}\frac{1}{b}\left\{\left[b^2y\right]_0^b-\left[\frac{y^3}{3}\right]_0^b\right\}$$

$$\Longrightarrow U_{av} = -\frac{1}{2\mu}\frac{dp}{dx}\frac{2}{3}b^2$$

$$\therefore \frac{U_{av}}{U_{max}} = \frac{2}{3}$$

$$\text{Or} \qquad U_{max} = \frac{3}{2}U_{av} \tag{4.5}$$

The shearing stress at the wall for parallel flow(as described in the diagram below),

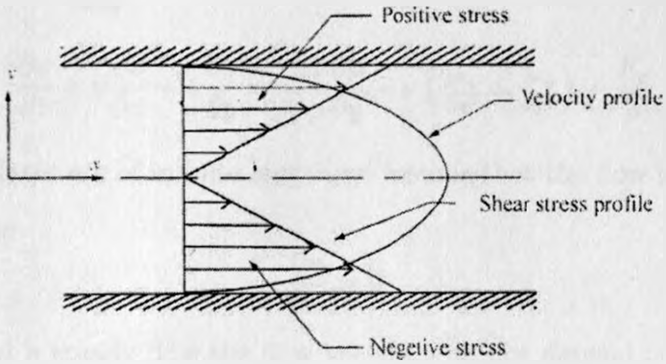


Fig. Shear stress and velocity profile for parallel flow in a straight channel.

$$|\tau_{yx}|_b = \mu\left(\frac{\partial u}{\partial y}\right)_b = b\frac{dp}{dx} = -2\mu\frac{U_{max}}{b}$$

Local friction coefficient,C_f is defined as

$$C_f = \frac{|(\tau_{yx})_b|}{\frac{1}{2}\rho U_{av}^2} = \frac{3\mu U_{av}/b}{\frac{1}{2}\rho U_{av}^2} = \frac{12}{\rho U_{av}(2b)/\mu}$$

$$\Longrightarrow C_f = \frac{12}{Re} \tag{4.6}$$

4.2 Generalised Coutte Flow

One plate is held stationary and other plate is moving at a velocity U as it is shown in the figure below.As both the plates are infinitely long in z-direction, thus z-dependence is not there. Therefore the governing equations are:Singh[2008]

The equation of continuity for the incompressible horizontal flow is;

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{4.7}$$

where$u$,v and w are components of velocity of the fluid in the x,y,and z directions,for this case $w = 0$

The equation of motion in two direction are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{F_x}{\rho} \tag{4.8}$$

and

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{F_y}{\rho} \tag{4.9}$$

Since the plates are of infinite length,we assume that the flow is only along the x-axis and depend on y. Thus

$$\frac{\partial u}{\partial x} = 0 \tag{4.10}$$

Since we have assumed a steady flow,the flow variables do not depend on time.Thus equations (4.8) and(4.9) can now be written as:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} + \frac{F_x}{\rho} \tag{4.11}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{F_y}{\rho} \tag{4.12}$$

Since,$\frac{\partial u}{\partial x} = 0$(By equation 4.10),substituting equation (4.10)into (4.7),we therefore obtain:

$$\frac{\partial v}{\partial y} = 0 \tag{4.13}$$

which implies that velocity component$v$ is independent of y.

As there is no fluid flow along y-axis we have:

$$V = 0 \tag{4.14}$$

In the absence of body forces, equations(4.11) and (4.12) reduces to:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{4.15}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{4.16}$$

From (4.16) it is clear that p is independent of y because $\left[\frac{\partial p}{\partial y} = 0\right]$.Keeping this in mind and differentiating equation (4.15) with respect to$x$ we obtain:

$$0 = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} \tag{4.17}$$

$$\Longrightarrow \frac{\partial^2 p}{\partial x^2} = 0 \tag{4.18}$$

$$\Longrightarrow \frac{\partial p}{\partial x} = -p \tag{4.19}$$

where p in the above equation is assumed to be a constant. Using using equation (4.19) we obtain from (4.15)

$$0 = \frac{p}{\rho} + \nu\frac{d^2u}{dy^2} \tag{4.20}$$

Or

$$\frac{d^2u}{dy^2} = -\frac{p}{\rho} \div \nu$$

$$\frac{d^2u}{dy^2} = -\frac{p}{\rho} \div \frac{\mu}{\rho}$$

$$\frac{d^2u}{dy^2} = -\frac{p}{\rho} \times \frac{\rho}{\mu}$$

$$\Longrightarrow \frac{d^2u}{dy^2} = -\frac{p}{\mu} \tag{4.21}$$

Integrating equation(4.21) we obtain

$$\frac{du}{dy} = -\frac{p}{\mu}y + A \tag{4.22}$$

where A is a constant of integration,Again integrating equation (4.22) we get:

$$u = \frac{-p}{2\mu}y^2 + Ay + B \tag{4.23}$$

where B is another constant of integration. A and B may be calculated from given boundary conditions.The boundary conditions are $u = 0$ at $y = o$and $u = U$at $y = h$(refer to the diagram).Incoperating equation(4.23)with the given boundary conditions we obtain

$$B = 0 \tag{4.24}$$

$$A = \frac{U}{h} + \frac{p}{2\mu}h \tag{4.25}$$

Substituting equations (4.24) and(4.25) into(4.22) we obtain

$$u = \frac{-p}{2\mu}y^2 + \left(\frac{U}{h} + \frac{p}{2\mu}h\right) y \tag{4.26}$$

Equation(4.26) can further be simplified and we obtain;

$$\frac{u}{U} = \frac{y}{h} + P\frac{y}{h}\left(1 - \frac{y}{h}\right) \tag{4.27}$$

where $P = -\frac{h^2}{2\mu U}\left(\frac{dp}{dx}\right)$(a non-dimensional pressure gradient). Equation(4.27) describe **generalised Coutte flow**.Akshoy[2006]. For generalised Coutte flow,($\frac{dp}{dx} \neq 0$)and the discharge per unit width of plate is given by

$$q = \int_0^h udy = \frac{Uh}{2} - \frac{Uh}{6}P$$

$$\Longrightarrow q = \frac{Uh}{2}\left(1 - \frac{P}{3}\right) \tag{4.28}$$

And shear stress is given as;

$$\tau = \mu\frac{du}{dy} = \mu\left(\frac{U}{h} + \frac{PU}{h} - \frac{2PUy}{h^2}\right) \tag{4.29}$$

For*simpleCoutte flow*,$\frac{dp}{dx} = 0$(i.e.$P = 0$),equation(4.27)becomes

$$\frac{u}{U} = \frac{y}{h} \tag{4.30}$$

Equation(4.30) describes **simple Coutte flow**. The location of maximum or minimum velocity in the channel is found out by by setting the derivative($\frac{du}{dy}$)equal to *zero*.From equation (4.27),one can write

$$u = \frac{y}{h}U + PU\frac{y}{h}\left(1 - \frac{y}{h}\right)$$

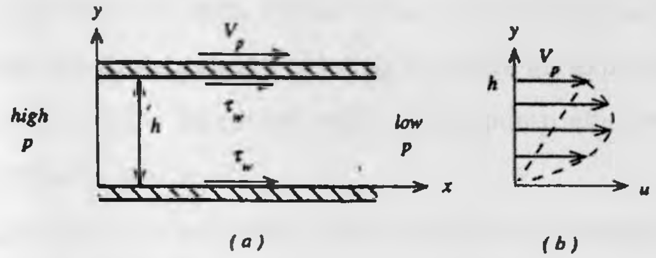
$$\Longrightarrow \frac{du}{dy} = \frac{U}{h} + \frac{PU}{h}\left(1 - 2\frac{y}{h}\right) \tag{4.31}$$

Setting $\frac{du}{dy} = 0$,for maximum or minimum velocity,yields:

$$\frac{y}{h} = \frac{1}{2} + \frac{1}{2P} \tag{4.32}$$

For P=0,simple Coutte flow exists.For $P > 0, \frac{dp}{dx} < 0$;pressure gradient will *assist* the viscously induced motion to overcome the shear force at the lower surface.For$P < 0, \frac{dp}{dx} > 0$.Pressure gradient will resist the motion induced by the motion of the upper plate.In this case,a region of *reverse flow* may occur near the lower surface. The values of maximum and minimum velocities can be obtained by substituting the value of y from equation(4.32) into equation(4.27) as

$$\begin{aligned} U_{max} &= \frac{U(1+P^2)}{4P} \text{for } P \geq 1 \\ U_{min} &= \frac{U(1-P)^2}{4P} \text{for} P \leq 1 \end{aligned} \tag{4.33}$$



4.3 Combined plane Coutte and Poiseuille Flow:

Because the form of the Navier-Stokes equation for both coutte flow and Poiseuille flow is *linear*,i.e it contains $u\{y\}$to the first power,we can add together these two solutions of Coutte flow yet has an impressed pressure gradient(negative pressure gradient).

$$u = V_p\left(\frac{y}{h}\right) + \frac{1}{2\mu}\left(\frac{-dp}{dx}\right)y(h-y) \tag{4.34}$$

Figure (a)above shows the linear combination of Coutte and Poiseuille plane flows between parallel plates with relative motion and figure (b) is an impressed pressure gradient results in a velocity profile I.e the sum of the two flows. The volumetric flow rate Q is simply the sum of the rates for the respective flows,

$$\frac{Q}{W} = \frac{V_p h}{2} + \frac{h^3}{12\mu}\left(\frac{-dp}{dx}\right) \tag{4.35}$$

An expression on the shear stressτ_w on the upper(moving)wall is given by

$$\tau_w = -\mu\left(\frac{V_p}{h}\right)\mathbf{i}_x + \frac{h}{2}\left(\frac{-dp}{dx}\right)\mathbf{i}_x \tag{4.36}$$

Example:A friction pump consists of a solid cylinder of diameter D and length$w$ that rotates clockwise at an angular speedΩ inside a hollow coaxial cylinder of inside diameter$D+2h$,as shown in the figure below.The fluid flow into the pump is pulled clockwise through a complete circle by the friction with the moving inner cylinder surface.The inflow and outflow passages to the pump are separated by a septum that prevents leakage from the higher outflow pressure p_{out} to the lower inflow pressure p_{in},the pressure difference $\Delta p \equiv p_{out} - p_{in}$ being maintained by clockwise flow through the pump.

Derive expressions for(a)the volumetric flow rate Q through the pump and (b)the clockwise torque$T$ that must be applied to the rotor for the steady operation as functions of the pressure,Δp.(c)If the power p_{out} produced by the pumped fluid is $Q\Delta p$,derive an expression for the value of Δp that maximizes P_{out} and find the numerical value of the pump efficiency$\eta \equiv \frac{P_{out}}{P_{in}}$ for this value ofΔp,where the input power is $P_{in} = \Omega T$.

Solution: (a)The pump flow is a combined plane Coutte and Poiseuille flow in which the wall velocity V_p and pressure gradient $\frac{dp}{dx}$ are:

$$V_p = \frac{\Omega D}{2}; \frac{dp}{dx} = \frac{P_{out} - P_{in}}{\pi D} = \frac{\Delta P}{\pi D}$$

Inserting these values into equation(4.35) for the volumetric flow rate Q,

$$Q = W\left[\frac{\Omega Dh}{4} + \frac{h^3}{12\mu}\left(\frac{-\Delta p}{\pi D}\right)\right] = \frac{W\Omega D^3}{4h}\left(1 - \frac{h^2\Delta p}{3\pi\mu\Omega D^2}\right)$$

(b)τ_w is the magnitude of the shear stress acting on the wall of the rotating cylinder in opposition to its motion,then the torque$T$ that must be applied to the cylinder in the clockwise direction is:

$$T = \tau_w\frac{D}{2}(\pi DW) = \frac{\pi D^2 W}{2}\left[\frac{\mu\Omega D}{2h} + \frac{h\Delta p}{2\pi D}\right] = \frac{\pi\mu W\Omega D^3}{4h}\left(1 + \frac{h^2\Omega p}{3\pi\mu\Omega D^2}\right)$$

where we have the equation(4.36)to evaluate τ_w.

(c) The power P_{out} is:

$$P_{out} = Q\Delta p = \frac{W\Omega Dh}{4}\left(1 - \frac{h^2\Delta p}{3\pi\mu\Omega D^2}\right)\Delta p$$

To maximize P_{out},differentiate with respect to Δp and set equal to zero.

$$1 - \frac{2h^2\Delta p}{3\pi\mu\Omega D^2} = 0; \Delta p = \frac{3\pi\mu\Omega D^2}{2h^2}$$

which result in a maximum P_{out} of:

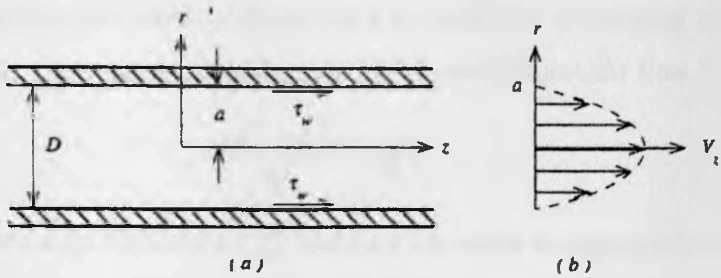
$$max P_{out} = \frac{W\Omega Dh}{4}\left(\frac{1}{2}\right)\left(\frac{3\pi\mu\Omega W\Omega D^2}{2h^2}\right) = \frac{3\pi\mu\Omega^2 D^3}{16h}$$

The corresponding value ofP_{in} is:

$$P_{in} = \Omega T = \frac{\pi\mu W\Omega^2 D^3}{4h}\left(\frac{5}{2}\right) = \frac{5\pi\mu W\Omega^2 D^3}{8h}$$

The ratio of these powers is the pump efficiency η_p :

$$\eta = \frac{P_{out}}{P_{in}} = \frac{3\pi}{16} \times \frac{8}{5\pi} = 30\text{percent}$$



4.4 Circular Poisseuille Flow

Laminar flow through a circular tube is of interest because a circular cross-section is the most common form tube or pipe.In,fact,it was the research on blood flow in animal capillaries and veins that first revealed the relationship between volumetric flow rate and pressure change in pipes of circular cross-section.In mammals,both,blood flow in capillaries and veins and air flow in lung alveoli are examples of circular Poiseuille flow.Other common examples are the flow through a sodastraw or through a hypodermic needle.

Because of the circular symmetry of the container,Circular Poiseuille flow is an axially-symmetric flow best described by using cylindrical coordinates,with circular coordinates,Navier-Stokes equations simplifies to the form:

$$\frac{d^2V_z}{dr^2} + \frac{1}{r}\left(\frac{dv_z}{dr}\right) = \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{1}{\mu}\frac{dp}{dz} \tag{4.37}$$

here we have used the total derivative because V_z is the function of radial distance$r$alone and p is the function of axial distance z alone.Multiplying the previous equation by r and integrating once,we obtain

$$r\frac{dV_z}{dr} = \frac{r^2}{2\mu}\frac{dp}{dz} + c_1 \tag{4.38}$$

The shear stress at the tube centerline $r = 0$must be zero because of axial symmetry and thus $\frac{dV_z}{dr} = 0$at$r = 0$ thereby requiring that $c_1 = 0$ .Dividing equation 4.38 by r and integrating,we get

$$V_z = \frac{r^2}{4\mu}\frac{dp}{dz} + c_2 \tag{4.39}$$

By choosing $c_2 = -a^2(\frac{dp}{dz})(4\mu)$, the velocity V_z at the tube wall $r = a$ becomes zero,as it should be at a stationary wall,and the velocity distribution for circular Poiseuille flow is:

$$V_z = \frac{a^2 - r^2}{4\mu}\left(-\frac{dp}{dz}\right) \tag{4.40}$$

where we include the minus sign multiplying $\frac{dp}{dz}$ because the latter is negative for positive V_z, i.e .for flow in the z-direction(this is called parabolic velocity distribution).

The volumetric flow Qcan be found by integrating the axial velocity V_z across the tube cross-section:

$$Q = \int_0^a V_z(2\pi r)dr = \frac{\pi}{2\mu}\left(-\frac{dp}{dz}\right)\int_0^a r(a^2 - r^2)dr = \frac{\pi}{2\mu}\left(\frac{-dp}{dz}\right)|\frac{r^2a^2}{2} - \frac{r^4}{4}|_0^a$$
$$= \frac{\pi a^4}{8\mu}\left(-\frac{dp}{dz}\right) = \frac{\pi D^4}{128\mu}\left(-\frac{dp}{dz}\right) \tag{4.41}$$

The volumetric flow rate trough a circular tube is very sensitive to the tube diameter$D$,varying as the fourth power of the diameter.For a fixed pressure difference across a coronary artery,for example,a reduction of the flow area$\frac{\pi D^2}{4}$ by a factor of two would decrease the blood volume flow rate by a factor of four.

The average flow velocity$\bar{v}$ is obtained by dividing the volumetric flow rate$Q$ by the tube area$\frac{\pi D^2}{4}$:

$$\bar{v} = \frac{4Q}{\pi D^2} = \frac{D^2}{32\mu}\left(-\frac{dp}{dz}\right) \tag{4.42}$$

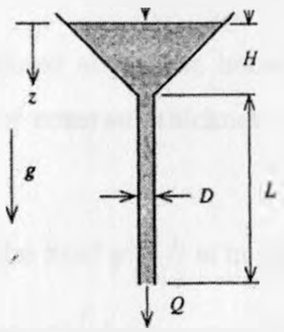
An expression for the wall shear stress τ_w may be derived by applying the linear momentum theorem to a cylindrical sample of the fluid inside the circular tube of length L and diameter$D$.The pressure difference$\left(\frac{-dp}{dz}\right)L$ acting on a cylinder produces a force$(\frac{-dp}{dz})L\left(\frac{\pi D^2}{4}\right)$ in the z direction that must be balanced by the shear stress force$\tau_w(\pi DL)$ acting in the opposite direction:

$$\tau_w(\pi DL) = \frac{\pi D^2 L}{4}\left(\frac{-dp}{dz}\right)$$

$$\tau_w = \frac{D}{4}\left(-\frac{dp}{dz}\right) = 8\mu\left(\frac{\bar{v}}{D}\right) \tag{4.43}$$

Example:

The kinematic of viscosity of a mixture of waste oils is to be measured by use of a laboratory funnel.The oil is poured into the funnel at a steady rate Q ,as shown in the figure below,maintaining the level of oil in the funnel at a distance $H = 3cm$ above the entrance to the funnel tube,which has the length $L = 30cm$ and diameter $D = 3mm$. The time required for $100cm^3$ to pass through the funnel is measured to be $152s$. Calculate the kinematic viscosity ν of the oil mixture.



Solution:

Solving equation (4.41) for $\nu = \frac{\mu}{\rho}$:

$$\nu = \frac{\mu}{\rho} = \frac{\pi D^4}{128 Q \rho}\left(-\frac{dp}{dz}\right).$$

Assuming that the flow in the top part of the funnel is inviscid, the value of p at the entrance to the funnel tube is the same as that at the liquid surface, i.e, P_a. At the funnel exist, $P = P_a - \rho g(H + L)$. Thus the pressure gradient is:

$$\frac{dp}{dz} = \frac{(P_a - \rho g(H+L)) - P_a}{L} = -\rho g\left(1 + \frac{H}{L}\right)$$

Substituting in the equation for ν:

$$\nu = \frac{\pi D^4 g}{128 Q}\left(1 + \frac{H}{L}\right)$$

$$= \frac{\pi(3.0)E(-3)m)^4(9.807m/s^2)}{128(1.0E(-3)m^3 \div 152s)}(1 + \frac{3}{30}) = 2.682E(-4)m^2/s$$

(If the funnel length L were doubled in this experiment, the flow rate Q would decrease only slightly, in proportion to $1 + \frac{1}{L}$).

4.5 Flow Down an Inclined Plane

Underlying Assumptions

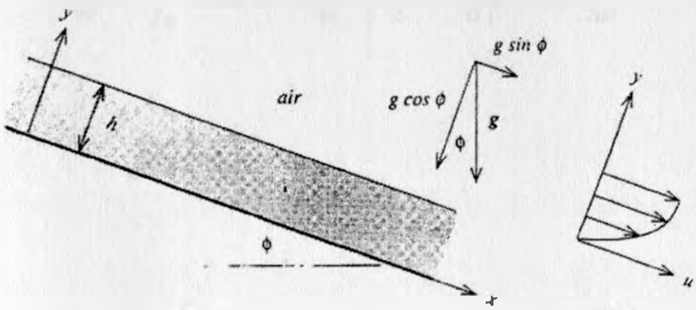
- Assuming a viscous fluid is flowing in only one solid surface.

- Assuming that the fluid motion is caused by a component of gravity force parallel to the solid surface.

- Assuming a plane surface inclined above the horizontal surface by an angle ϕ and covered with the liquid layer of constant thickness **h** that flows parallel to the plate in the down hill direction.

- Assume the upper surface of the fluid $y = h$ is in contact with air.

- Assume the air pressure is constant $\Big($thus a negligible shear stress on a liquid surface$\Big)$.



In the description of this motion,we select the x- component of the Navier-Stokes equation(because the flow is along the x-axis only).

$$\frac{D\vec{q}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g} + \nu\nabla^2\vec{q} \quad \text{if} \quad \nabla.\vec{q} = 0; \qquad \mu = \text{constant.} \tag{4.44}$$

noting that $\frac{D\vec{q}}{Dt} = 0$, $\frac{\partial p}{\partial x} = 0$ because the air pressure is constant and the x component of $\mathbf{g}$ is $g\sin\phi$:

$$\Longrightarrow 0 = 0 + g\sin\phi + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{d^2u}{dy^2} = -\frac{g\sin\phi}{\nu}$$

Integrating twice on y:

$$u = -\frac{(g\sin\phi)y^2}{2\nu} + c_1 y + c_2$$

and applying the boundary conditions that $u = 0$ at$y = 0$ and $\tau_{xy} = \mu\frac{du}{dy} = 0$ at $y = h$ enables us to find the velocity distribution $u\{y\}$:

$$u = \frac{g\sin\phi}{\nu}\left(hy - \frac{y^2}{2}\right) \tag{4.45}$$

This velocity profile in figure (*b*) ,is *parabolic* with a maximum value u_{max} at upper surface ($y = h$):

$$u_{max} = \frac{gh^2\sin\phi}{2\nu} \tag{4.46}$$

The volumetric flow rate per unit distance normal to the plane of the flow, $\frac{Q}{W}$ is found by integrating equation() above on y:

$$\frac{Q}{W} = \int_0^h u dy = \frac{g\sin\phi}{\nu}\left|\frac{hy^2}{2} - \frac{y^3}{6}\right|_0^h = \frac{gh^3\sin\phi}{3\nu} \tag{4.47}$$

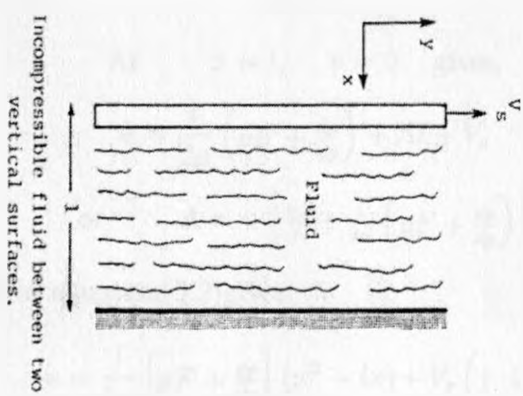
Chapter 5

Summary and Conclusion

- In this chapter we present applications of Navier-Stokes equations for some problems.

5.1 Some Application Problems

Application 1: Starting with the Navier-Stokes equation,obtain the velocity profile which describes the incompressible flow between two parallel vertical plates.The right plate is at rest,while the left is moving upward at a constant velocity,v_s.Assume the flow is laminar.



Solution:The Navier-Stokes equation in the y-direction is,

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-\rho F \tag{5.1}$$

where,

$$\frac{\partial v}{\partial t}=0 \quad \text{for steady flow.}$$

$$u=0=w$$

$$\frac{\partial v}{\partial y}=0=\frac{\partial v}{\partial z}$$

$$\frac{\partial p}{\partial y}=\frac{dp}{dy}=\text{constant.}$$

Substituting in equation (5.1)

$$0=-\frac{dp}{dy}+\mu\frac{d^2v}{d^2x}-\rho F$$

$$\text{or} \quad \frac{d^2v}{d^2x}=\frac{1}{\mu}\left[\rho F+\frac{dp}{dy}\right]$$

Integrating,

$$v=\frac{x^2}{2\mu}\left[\rho F+\frac{dp}{dy}\right]+Ax+B \tag{5.2}$$

Applying the boundary conditions,

$$\text{At} \quad x=0, \quad v=v_s \quad \text{gives,}$$

$$v_s=B$$

$$\text{At} \quad x=l, \quad v=0 \quad \text{gives,}$$

$$0=\frac{l}{2\mu}\left(\rho F+\frac{dp}{dy}\right)+Al+V_s$$

$$\text{or} \quad A=-\left[\frac{v_s}{l}+\frac{l}{2\mu}\left(\rho F+\frac{dp}{dy}\right)\right]$$

Substituting for A and B in equation(5.2)gives:

$$v=\underbrace{\frac{1}{2\mu}\left[\rho F+\frac{dp}{dy}\right](x^2-lx)}+\underbrace{V_s\left(1-\frac{x}{l}\right)} \tag{5.3}$$

Note: The first expression in equation (5.3) is the equation of a *symmetric parabola* and the the second expression in above equation is the equation of a *straight line* with negative slope.

Application2: Derive the differential equations of motion for a fluid of constant viscosity and density which is flowing over an impulsively accelerated,infinitely long horizontal flat plate.Assume that the flow is laminar.

Solution: Writing the Navier-Stokes equation for the x-direction,

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho B_x \tag{5.4}$$

Assuming the following:

1.No body force in the x-direction; $B_x = 0$

2.One dimensional flow ; $v = w = 0$

3.No change of flow variables in the z-direction;

$$\frac{\partial}{\partial z}=\frac{\partial^2}{\partial z^2}=0$$

Then,

$$\frac{\partial}{\partial x}=\frac{\partial^2}{\partial x^2}=0 \quad \text{for the long plate in the } x\text{-direction.}$$

Substituting these conditions into equation(5.4) yields:

$$\rho\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial y^2}$$

Similarly for the y-direction,the $y$component of the Navier-Stokes equation becomes,

$$\frac{\partial p}{\partial y}=\rho B_y=-\rho F$$

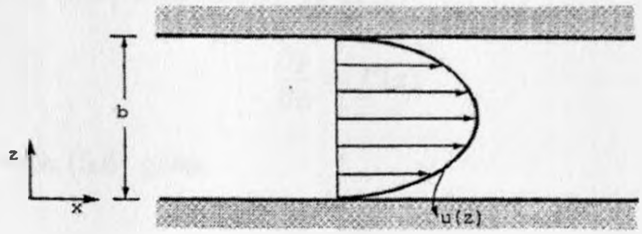
$$\text{or} \quad \frac{\partial p}{\partial y}=-\rho F$$

and the z-direction gives,

$$0=\rho B_z$$

Thus,there is no body force component in the z-direction.

Application3: Consider a fluid with constant viscosity and density which is flowing between two fixed parallel plates.The velocity profile is given by $u = c(bz - z^2)$, where b is the distance



Laminar Flow Between Two Parallel Plates.

between the plates and c is a constant.If the pressure at the point $(0,0,0)$is p_0, find the pressure distribution at any point in the flow region.Assume that $v = w = 0$.

Solution: Writing the Navier-Stokes equation for steady flow in the x-direction,

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho F_x \tag{5.5}$$

Where,

$$\rho F_x = 0 \quad \text{(no body force in the x-direction)}$$

$$v = w = 0 \quad \text{(given)}$$

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y} \quad \text{(given)}$$

$$\text{and} \qquad \frac{\partial^2 u}{\partial z^2} = -2c$$

Substituting in equation(5.5) we obtain:

$$\frac{\partial p}{\partial x} = -2\mu c \tag{5.6}$$

The y and z components of the Navier-Stokes equation are:

$$\frac{\partial p}{\partial y} = 0 \tag{5.7}$$

Thus p is not a function of y.

$$\frac{\partial p}{\partial z} = -\rho F \tag{5.8}$$

Integrating equation(5.8) with respect to z,

$$p = -\rho F z + f(x) \tag{5.9}$$

To find $f(x)$,differentiate (5.9) with respect to x,

$$\frac{\partial p}{\partial x} = f'(x) \tag{5.10}$$

Comparing to (5.10) with (5.6) gives:

$$f'(x) = -2\mu c$$

$$\text{or} \qquad f(x) = -2\mu c x + c' \tag{5.11}$$

Substituting equation (5.11) into equation (5.9) yields:

$$p(x,z) = c' - 2\mu c x - \rho F z$$

Substituting $p = p_0$ at $(0,0,0)$ gives

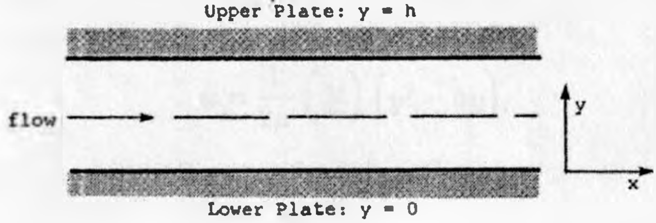
$$c' = p_0$$

and the final result is:

$$p(x,z) = p_0 - 2\mu c x - \rho F z$$

Application 4: Using the Navier-Stokes equation,determine the velocity profile for the incompressible steady flow of water between two parallel plates at rest.Assume one dimensional laminar flow with constant viscosity.

<u>Solution:</u>For one dimensional flow,



Flow between two parallel plates of infinite width.

$$v = w = 0$$

$$\text{Thus,} \quad \frac{\partial u}{\partial x} = 0 \ \ \text{(from the continuity equation)}$$

$$\text{Also,} \quad \frac{\partial u}{\partial t} = 0 \ \ \text{for steady flow(no body forces in the x-direction)}$$

Substituting the above in the x-direction of the Navier-Stokes equation and simplifying,

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \tag{5.12}$$

Similarly for the y and z directions:

$$\frac{\partial p}{\partial y} = -\rho F$$

$$\text{and} \quad \frac{\partial p}{\partial z} = 0$$

For small'h' the variation of p in the y-direction will be negligible and hence,

$$\frac{\partial p}{\partial y} = 0$$

$$\text{Thus} \quad \frac{\partial p}{\partial x} = \frac{dp}{dx} = \text{constant.(because } u \neq f(x))$$

Substituting in (5.12) and integrating twice,

$$u = \frac{1}{2\mu}\left(\frac{dp}{dx}\right) y^2 + c_1 y + c_2$$

$$\text{At } \ y = 0, u = 0, \ \ \text{gives}, c_2 = 0$$

$$\text{At } \ y = h, u = 0 \ \ \text{gives},$$

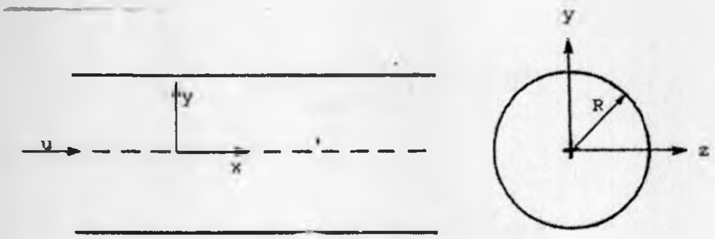
$$c_1 = -\frac{1}{2\mu}\left(\frac{dp}{dx}\right)\frac{h^2}{h} = -\frac{1}{2\mu}\left(\frac{dp}{dx}\right) h$$

Substituting,

$$u = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)\left(y^2 - hy\right)$$

Application5:

An incompressible viscous fluid undergoes steady.laminar flow through a circular pipe of radius R.Derive the equation relating the velocity at any point to the maximum velocity.Assume one dimensional flow.



Laminar Flow in a Circular Pipe.

Solution: Writing the Navier-Stokes equation in cylindrical coordinates for steady, laminar, constant property, with fully developed flow within a circular pipe,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{\mu}\frac{\partial p}{\partial x} \tag{5.13}$$

Since the the flow is fully-developed,

$$\frac{\partial u}{\partial x} = 0$$

Thus, u is the function of r alone. In addition, since p is not a function of r and the left-hand side of equation (5.13) can not depend on x (since u depends upon r, but not x),

$$\frac{\partial p}{\partial x} = \frac{dp}{dx} = \text{constant.}$$

Thus, equation (5.13) becomes,

$$\frac{1}{r}\frac{d}{dr}(r\frac{du}{dr}) = \frac{1}{\mu}\frac{dp}{dx}$$

Integrating twice,

$$u = \frac{1}{4\mu}\frac{dp}{dx}(r^2) + c_1 r + c_2$$

$$\text{At } r = \pm R \qquad u = 0, \quad \text{which gives}$$

$$0 = \frac{1}{4\mu}\left(\frac{dp}{dx}\right) R^2 + c_1 R + c_2$$

$$0 = \frac{1}{4\mu}\left(\frac{dp}{dx}\right) R^2 - c_1 R + c_2$$

Solving,

$$c_1 = 0$$

$$c_2 = -\frac{1}{4\mu}\left(\frac{dp}{dx}\right) R^2$$

Substituting,

$$u = \frac{1}{4\mu}\left(\frac{dp}{dx}\right)(r^2 - R^2) \tag{5.14}$$

Now,the maximum velocity occurs at $r = 0$,

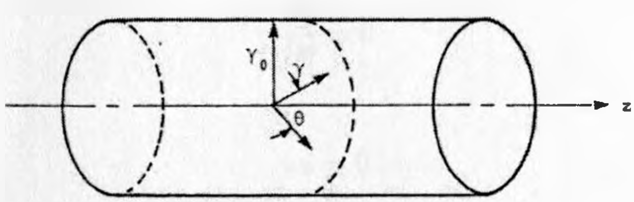
$$u_{max} = U = -\frac{R^2}{4\mu}\left(\frac{dp}{dx}\right)$$

Substituting in equation (5.14) and simplifying,

$$u = U\left[1 - (\tfrac{r}{R})^2\right]$$

Application6:

In idealized laminar flow,as defined for the circular pipe shown in the figure below,the temperature is constant.Velocities in the z-direction are not zero,the viscosity and density are functions of temperature only,the potential Ω is zero and the flow is at steady state.Also,the pressure drop per unit length of conduit is taken to be a constant.Determine the equation of motions of motions for idealized laminar flow.



Coordinate System for Flow in Circular Pipe.

<u>solution:</u> The continuity equation in cylindrical coordinates is given by,

$$\left(\frac{\partial\rho}{\partial t}\right)_{r,\theta,z} + \frac{1}{r}\left\{\left(\frac{\partial r\rho v_x}{\partial r}\right)_{\theta,z,t} + \left(\frac{\partial\rho v_\theta}{\partial\theta}\right)_{r,z,t} + \left(\frac{\partial r\rho v_z}{\partial z}\right)_{r,\theta,t}\right\} = 0.$$

where r is the radial direction,θ the azimuth,and z the axial direction.The designations v_r,v_θ and$v_z$denote the components of the velocity in the radial,azimuthal,and axial direction respectively.For idealized flow,the above equation becomes,

$$\frac{\partial\rho}{\partial t} + v_r\frac{\partial\rho}{\partial r} + v_\theta\frac{\partial\rho}{\partial\theta} + v_z\frac{\partial\rho}{\partial z} + \frac{\rho}{r}\left[\frac{\partial rv_r}{\partial r} + \frac{\partial v_\theta}{\partial\theta} + \frac{\partial rv_z}{\partial z}\right] = 0 \tag{5.15}$$

since ρ is a constant,

$$\frac{\partial r v_r}{\partial r} + \frac{\partial v_\theta}{\partial \theta} + \frac{\partial r v_z}{\partial z} = 0 \tag{5.16}$$

There are no radial or azimuthal components of velocity.Therefore:

$$\frac{\partial r v_z}{\partial z} = 0 \tag{5.17}$$

Since the potential is zero,the following equation,

$$\rho\frac{\partial v_r}{\partial t} = \rho\phi_r - \frac{\partial p}{\partial r} + \frac{1}{3}\frac{\partial}{\partial r}\left[\frac{\mu}{r}\left(\frac{\partial r v_r}{\partial r} + \frac{\partial v_\theta}{\partial \theta} + \frac{\partial r v_z}{\partial z}\right)\right] + \left[\frac{1}{r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial v_r}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\mu\frac{\partial v_r}{\partial \theta}\right) + r\frac{\partial}{\partial z}\left(\mu\frac{\partial v_r}{\partial z}\right)\right] - \frac{1}{r^2}\left[\mu v_r + 2\mu\frac{\partial v_\theta}{\partial \theta}\right]$$

where μ = absolute viscosity,becomes

$$\frac{\partial p}{\partial r} = 0 \tag{5.18}$$

when equations (5.16) and (5.17) are used together with the fact that $v_r = 0$.

Similarly, the following equation

$$\rho\frac{\partial v_\theta}{\partial t} = \rho\phi_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{3r}\frac{\partial}{\partial \theta}\left[\frac{\mu}{r}\left(\frac{\partial r v_r}{\partial r} + \frac{\partial v_\theta}{\partial \theta} + \frac{\partial r v_z}{\partial z}\right)\right] + \left[\frac{1}{r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\mu\frac{\partial v_\theta}{\partial \theta}\right) + r\frac{\partial}{\partial z}\left(\mu\frac{\partial v_z}{\partial z}\right)\right] - \frac{1}{r^2}\left[\mu v_\theta - 2\mu\frac{\partial v_r}{\partial \theta}\right]$$

becomes

$$\frac{\partial p}{\partial \theta} = 0 \tag{5.19}$$

since

$$v_\theta = 0 \tag{5.20}$$

Since steady state-state conditions are assumed,the following equations,

$$\rho\frac{\partial_{v_z}}{\partial t} = \rho\phi_z - \frac{\partial p}{\partial z} + \frac{1}{3}\frac{\partial}{\partial z}\left[\frac{\mu}{r}\left(\frac{\partial r v_r}{\partial r} + \frac{\partial v_\theta}{\partial \theta} + \frac{\partial r v_z}{\partial z}\right)\right] + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\mu r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\mu\frac{\partial v_z}{\partial \theta}\right) + r\frac{\partial}{\partial z}\left(\mu\frac{\partial v_z}{\partial z}\right)\right]$$

and

$$\frac{\partial v_z}{\partial t} = \frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}$$

gives:

$$-\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial_{v_z}}{\partial r}\right) = 0 \tag{5.21}$$

The fact that v_r and v_θ are zero and the symmetry of the flow condition gives:

$$\frac{\partial v_z}{\partial_\theta} = 0 \tag{5.22}$$

Thus for idealized laminar flow in a pipe the pressure gradient may be expressed interms of the viscous force in the following way:

$$\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(\mu r \frac{\partial v_z}{\partial r}\right) \tag{5.23}$$

Equation (5.23) may be solved for the idealized flow by integrating once with respect to r so that

$$\frac{r^2}{2}\frac{\partial p}{\partial z} = \mu r \frac{\partial v_z}{\partial r} + A(z) \tag{5.24}$$

where $A(z)$ is an arbitrary function of z and corresponds to a constant of integration in ordinary integration.Since the flow is symmetrical about the axis,

$$\frac{\partial v_z}{\partial r} = 0 \tag{5.25}$$

$$\text{at} \quad r = 0 \qquad \text{so that} \quad A(z) = 0$$

In the idealized flow T is taken as a constant and

$$\left(\frac{\partial \mu}{\partial p}\right)_T = 0$$

and therefore μ is also a constant.Thus,a second radial integration may be made to give

$$\frac{r^2}{4}\frac{dp}{dz} = \mu v_z + B(z) \tag{5.26}$$

The velocity v_z is zero when

$$r = r_0$$

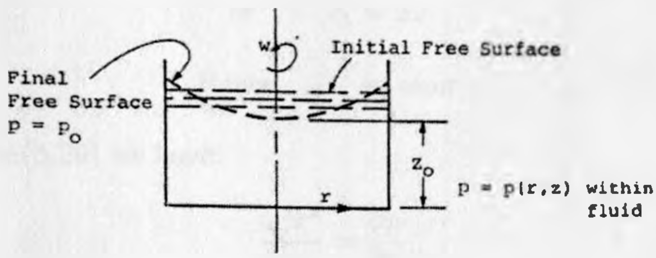
and so,

$$B(z) = \frac{r_0^2}{4}\frac{dp}{dz}$$

Thus finally we obtain,

$$v_z = \frac{dp}{dz}\frac{(r^2 - r_0^2)}{4\mu} \tag{5.27}$$

Application7: Consider an incompressible liquid in a cylindrical vessel which has been undergoing constant angular motion for a time interval which is of such a duration that the liquid has assumed a fixed orientation in the vessel.Show that the steady-state,the free surface forms a



paraboloidal surface given by $z - z_0 = \left(\frac{\omega^2}{2g}\right) r^2$.Assume the viscosity of fluid is constant.

Solution:Writing the Navier-Stokes equation in cylindrical coordinates for laminar flow,

r-component : $\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r$

θ−component:$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (rv_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] +$

z − component:$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial (rv_z)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$

For steady state,

$$v_z = v_r = 0$$

$$\text{and} \qquad v_\theta = f(r) \quad \text{only.}$$

Therefore the above equation now becomes:

$$\rho\frac{v_\theta^2}{r} = \frac{\partial p}{\partial r} \tag{5.28}$$

$$0 = \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) \tag{5.29}$$

$$0 = -\frac{\partial p}{\partial z} - \rho g$$

$$\text{or} \qquad \frac{\partial p}{\partial z} = -\rho g \tag{5.30}$$

Integrating equation(5.30) we obtain:

$$v_\theta = \frac{1}{2}c_1 r + \frac{c_2}{r}$$

where $c_2 = 0$ for a finite value of v_θ at $r = 0$ At $R = 0$,the vessel radius,

$$v_\theta = \omega R = \frac{c_1 R}{2}$$

$$\text{or} \qquad c_1 = 2\omega$$

$$\text{Hence,} \qquad v_\theta = \omega r$$

Substituting in equation(5.29) we have:

$$\frac{\omega^2 r^2}{r} = \frac{\partial p}{\partial r}$$

$$\text{or} \qquad \frac{\partial p}{\partial r} = \rho\omega^2 r$$

Integrating,

$$p = \frac{1}{2}\rho\omega^2 r^2 + f(z) \tag{5.31}$$

To find the unknown function f(z),we differentiate the above expression with respect to z,

$$\frac{\partial p}{\partial z} = f'(z)$$

Comparing with equation(5.30),

$$f'(z) = -\rho g$$

$$\text{Thus,} \qquad f(z) = -\rho g z + c$$

Substituting into equation(5.31),

$$p = -\rho g z + \frac{1}{2}\rho\omega^2 r^2 + c$$

At $r = 0$, $p = p_0$ and $z = z_0$ gives,

$$c = p_0 + \rho g z_0$$

Thus,

$$p = \rho g(z_0 - z) + \frac{1}{2}\rho\omega^2 r^2 + p_0$$

$$\text{or} \qquad p - p_0 = -\rho g(z - z_0) + \frac{1}{2}\rho\omega^2 r^2$$

It is known that$p = p_0 = p_{atm}$(for open vessels)at all radii on the free surface,and hence the equation of free surface is,

$$0 = -\rho g(z - z_0) + \frac{1}{2}\rho\omega^2 r^2$$

$$\text{or} \qquad z - z_0 = \left(\frac{\omega^2}{2g}\right) r^2.$$

Bibliography

[1] Agrawal,H.L.(1957):A new exact solution of the equation with axial symmetry.*Q.J.Mech.Appl.Math.*10:42-44

[2] Akshoy,R.P and Pijush Roy(2005):Engineering Thermodynamics and Fluid Mechanics,*Prentice-Hall of India,New Delthi*-110001

[3] C.B.Singh & J.Okwoyo(2008):Coutte Flow Between Two parallel Infinite plate in the presence of Transverse Magnetic field.*J.Kenya Meteorol.soc*;2(2)93-97

[4] C.B.Singh(2007):Hydromagnetic Steady Flow of Viscous Incompressible Fluid between two parallel Infinite plates under the influence of Inclined magnetic Field;*Kenya Journal of Sciences series,A Vol.*12 No.1.

[5] C.Y.Wang(1991):Exact solutions of the steady-state Navier-Stokes equations;*Ann.Rev.Fluid Mech.*.23:159-77

[6] C.Y.Wang(1990(b):Exact solution of the Navier-Stokes equations-The generalized Beltrami flows,review and extension.*Acta Mech.*81;69-74

[7] C.Y.Wang(1989)(a):Exact solutions of unsteady Navier-Stokes equations.*Appl.Mech.Rev.*42:S269-82

[8] C.Y.Wang(1985):Stagnation flow on the surface of a quiescent fluid-an exact solution of the Navier-Stokes equations.Q.Appl.Math.43:215-23.

[9] Dorrepaal,J.M(1986):An exact solution of the Navier-Stokes equations which describes non-orthogonal stagnation point flow in two dimensions,*J.Fluid Mech.*163:141-47

G.S.Beavers & D.D.Joseph(1990),*J.Fluid Mech.*30

Govindaraju;K.V(1981):Confluent flows of a viscous fluid(a special case):*Quarat.of applied mathematics,Appl.*

Hui,W.H(1987):Exact solutions of the unsteady two-dimensional Navier-Stokes equations,*J.Appl.Math.phys.*(ZAMP),38:689-702.

Irmay.S.Zuzovsky.M.Exact solutions of the Navier-Stokes equations in two-ways flow.*First J.Technol.*8:307-15,1970.

4] James;A.Fay(2007):Introduction to Fluid Mechanics-*Prentice-Hall of India Private Limited New Delhi*-110001.

5] J.L.Bansal(1977):Viscous fluid Dyanamics;New-Delhi:Oxford&IHB publishing Co.1977

16] J.Armengol *et al*(2008);Bernoulli correction to viscous losses:Radial flow between two parallel discs;*Ann.J.Phy.*76,730

17] Kambe T.A(1986):Class of exact solutions of the Navier-Stokes equations.*Fluid Dyn.Res.1*,21-31

[18] Kirk T.Mc Donald(2008).Radial viscous Flow between two parallel Annular plates.*Princeton*;NJ08544

[19] Lolyd,S.P.(1981):The infinistemal group of the Navier-Stokes equations,*Acta Mech*;38,pp.85-98.

[20] Marris,A.W(1981):Remarks on plane universal Navier-Stokes motions.*Arch.Rat.Mech.Anal*;77;pp95-102.

[21] Moses,H.E,(1972):Exact vorticity equations of the incompressible Navier-Stokes equations.*J.De Mech.*,11,442,44.

[22] N.Rudraiah,R.Veerabhadraiah,B.C.Chandrasekhara and S.I.Nagaraj(1979),PGSAM *series*vol.2,Bangalore University,chap.4,p.83

[23] R.Berker(1979):A new solution of the Navier-Stokes equations for the motion of fluid contained between two parallel plates rotating about the same axis.*Arch.Mech,*31:265-280.

[24] R.Berker(1982):An exact solution of the Navier-Stokes equation the vortex with curvillinear axis.*Int.J.Eng.Sc.*20:217-30'.

[25] R.E,Biosvert,Ames,W.F & Srivastava(1983),U.N.Group properties and new solutions of Navier-Stokes and new solutions of Navier-Stokes equations;*J.Eng.Math,vol.17,pp.*203-221,1993.

[26] Stallybrass,M.P:A class of exact solutions of the Navier-Stokes equations.Plane unsteady flow,*Int.J.of Engng.SC.vol.*21,No2,pp.179-186.

[27] S.Ganesh(2006):Magnetohydrodynamic flow of viscous fluid between Two parallel porous plates;*Journal of Applied Sciences*6(11):2420-2425.

[28] Taylor,G.I(1923) :On the decay of vortices in a viscous fluid.*phil.Mag,Series*6,48-671-674