COMPARING GROUP SCREENING DESIGNS

BY

ALEX K. MWANGI

This project is submitted for degree of Master of Science in Mathematical Statistics in the Department of Mathematics

UNIVERSITY OF NAIROBI

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DECLARATION

This project is my original work and has not been presented for a degree in any other university.

Signature: .....

Date: 5/10/99

ALEX K. MWANGI

This project has been submitted for examination with my approval as the university supervisor.

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DEDICATION

This project is dedicated to all members of my family particularly my parents Mr & Mrs J. M. Kanyi, my brothers and sister whose support has made me what I am today.

CONTENTS

ABSTRACT

The problem of detecting defective members in a large population, consisting of defective and non-defective members has been tackled in various ways. One procedure used in this kind of investigation is the group testing procedure. In this procedure, defective members of a population are weeded out with as few tests (runs) as possible.

Designs which classify all members in the population as defective or non-defective have been called screening designs. Work in this area was pioneered by Dorfman[1] in 1943 and Sterret[14] in (1957).Watson[16] in 1961 and Patel[9] in 1962 have approached the problem from the point of view of designs of experiment and called these designs "group screening designs".Patel and Manene[10] in 1987 worked along the line of Sterret and called their designs "Stepwise group screening designs".In this project various group screening procedures are compared.We restrict ourselves to Dorfman, Sterret and Hwang's procedures and extensions of these procedures.

Chapter I reviews the basic concepts of group screening and work done in this area.

In chapter II, various group screening procedures are defined. The work done in this area and other related areas by several authors in the past is described.

In chapter III, the procedures are compared using the expected number of runs as a basis of comparison.

CHAPTER ONE

INTRODUCTION

1.1 Introduction

Suppose that we have a population of 'f' members of which 'j' are defective and the rest (f-j) are non-defective. Each member can be tested to find out whether it is defective or not.

In a two-stage group screening design, the factors (members) are divided into groups in the first stage. These are the first order group-factors. The group-factors are tested for their defects and classified as defective or non-defective. In the second stage, factors within group-factors found to be defective are tested individually.

In a three-stage group screening design, the factors are divided into groups in the first stage. These are known as the first order group-factors. The group-factors are tested and identified as defective or non-defective. In the second stage of the experiment, any first order group-factor found to be defective is divided further into smaller group-factors called second order group-factors which are then tested and classified as defective or non-defective. Finally, in the third stage all the factors belonging to the second order group-factors found to be defective in the second stage are tested individually and identified as defective or non-defective. The three-stage group screening can be extended to m-stage group screening design.

In a one-type step-wise group screening design, factors are divided into group-factors known as first order group-factors.The group-factors are then tested for their defects and classified as either defective or non-defective. In the first of the type one search steps, we start with any defective first order group-factor and test the factors within it one by one till we find a defective factor. This defective factor is kept separate. In the second of type one search steps, we test the

remaining factors in a pool. If the pooled test is negative, then the test procedure is terminated. Otherwise the first and second of the type one search steps are repeated successively till the analysis terminates with a test on a non-defective group-factor or a group-factor of size one.The type one search steps are performed for all the first order group-factors found to be defective in the initial step.

In a two type step-wise group screening design, the factors are divided into group factors, known as the first order group-factors, which are then tested for their effects and classified as defective or non-defective in the initial step. Each of the first order group-factors classified as defective in the initial step is further divided into smaller group-factors called second order group-factors. In the first of type one search steps, we start with any defective first order group-factor and test the second order group-factors within it one by one till we find a defective second order group factor. The defective second order group factor is kept separate.In the second of the type one search steps we test the remaining second order group-factors in a pool. If the pool test is negative, we terminate the test procedure. Otherwise in the third of the type one search steps , we continue testing the remaining second order group-factors one by one till we find another defective second order group-factor. This is also kept separate. The second and the third of type one search steps are repeated successively till all the defective second order group factors are isolated.The type one search steps are performed for all the first order group-factors found to be defective in the initial step.

Finally, in the first of the type two search steps, we start with any second order group-factor found to be defective in the type one search steps. We test the factors within it one by one till we find a defective factor. The defective factor is kept separate. In the second of the type two search steps

we test the remaining factors in a pool. If the pooled test is negative, the test procedure is terminated. Otherwise in the third of type two search steps, we continue testing the factors individually till another defective factor is found. This is again isolated. The second and third of the type two search steps are repeated successively till the test procedure terminates with a test on a non-defective group or a group-factor of size one. The type two search steps are performed for all the second order group-factors found to be defective at the end of type one search steps. The two type step-wise group screening design can easily be extended to an r-type (r >2) step-wise group screening design.

In the method proposed by Hwang [2] in (1972), the knowledge of an upper bound *m* of *j* (number of defective members) is assumed. When the probability distribution of *j* (not necessarily binomial) is known, then if any chosen number *m*, is used as the "upper bound", we can compute the probability that $j < m$; with at least this probability, all the defectives will be identified in not more than a specified number of tests. Let *m* be the given upper bound and f the population size. The corresponding problem of using group testing to detect all members in the population is referred to as the (m,f) problem. If $f \leq 2m-2$, test the members individually. In the first step, if $f \geq 2m-1$, compute $l = f-m=1$. Also compute a non-negative integer $\propto$ satisfying $2^{\alpha+1} > l\!/ \; m \geq 2^{\alpha}$ .In the second step, take a group of size 2^{α} to test. If the group is non-defective, we dispose of these 2^{α} members as non-defective and go back to the first step for the remaining problem $(m,f-2^{\alpha})$. If the group is non-defective, find one defective member in α tests by binary search and dispose of all non-defective groups encountered during that stage. After $1+\alpha$ tests we go back to the first step for the remaining problem $(m-,f')$ where $f' \leq f-1$.

1.2 Literature review

The idea of group testing was first proposed by Dorfman [1] in 1943, as an economical method of testing blood samples of army inductees in order to detect the presence of infection. He proposed that to trace the presence of infection in blood, a sample of each member is taken and pooled together, then the pooled blood sample is tested. If the test is negative, the pooled blood samples free from infection and all the inductees in the sample could be passed with no further tests. Otherwise the blood sample of each of the individuals making up the pool is tested individually to determine which of the inductees are infected. If the prevalence rate of infection were low, the expected total number of tests and thus the expected total cost of blood-testing would be reduced.

Dorfman s method was improved by Sterrett [14] in (1957), who proposed that individual testing of the members of a defective group should cease once a defective member us found. Then the remaining members should be tested simultaneously in a pool. If the result was negative, then stop testing that sample. Otherwise testing members individually continued till another defective item was found. The remaining items were again tested in a pool. The procedure was continued until all the defective members in the defective pooled sample were isolated. This procedure reduced the number of runs obtained by dorfman's by eight-per cent for a prevalence rate of five-per cent.

Watson [16] in (1961) applied Dorfman's method in group screening problems where a large number of variables are screened by group testing to identify the important ones. He studied two-stage group screening designs with and without errors in observation using equal sized groups. Assuming continuous variations in group-sizes, he obtained the optimum group-sizes by minimising the total expected number of runs with respect to the group-sizes using ordinary calculus techniques.

Patel [9] in (1962) and Li [4] in (1962) extended the two-stage group screening procedure with equal prior probabilities to a multi-stage group screening procedure when responses are observed with negligible error. The work was restricted to the case when all the factors were defective with equal a-priori probability. By assuming continuous variation, Patel obtained the optimum group sizes that minimise the expected number of runs with respect to the group-sizes. He also discussed the choice of the number of stages to be used. He compared the procedures at different stages with respect to the minimum expected number of runs. He also compared two-stage procedure with higher stage procedures. Li [4] also considered the group screening problems and proposed a method which is essentially a multi-cycle version of Dorfman' s method. The members of a particular cycle are divided into groups for group testing. Those groups which are defective (contain important variables in his language) are then pooled together to become the members for the next cycle. However, he used the maximum number of tests as a criterion of efficiency.

Hwang [2] in (1972) proposed a method which assumes the knowledge of an upper bound of the defective items. This method is twin to a merging algorithm suggested by Hwang and Lin for merging two disjoint ordered sets by making paired comparisons. The method is designed to reduce the maximum number of tests. When the number of defectives is known (hence an upper bound is known), the proposed method compares favourably with Li' s method. When the probability distribution of the defective items (not necessarily binomial) is known, then if any chosen number is used as the "upper bound" in the proposed method, we can compute the probability that the number of defectives is less than or equal to the upper bound; with at least this probability, all the defectives will be identified in not more than a specified number of tests.

Sobel and Groll [13] listed many industrial applications of group testing. They set up recursion equations to determine the optimal size of the group to be tested next-optimal under the restraint that if one group is found defective, the group to be tested must be one of its subgroups. While closed-form solutions were not obtained, numerical solutions for values of the population from one to twelve for the probability of any member being non-defective (q) and values of the population from thirteen to a hundred for q =0.90, 0.95 and 0.99. Hunter and Mezaki[17] in (1964) used group screening method to select the best catalyst from a list of possible catalysts for the oxidation of methane. This was done by arranging possible catalysts for the reaction in logical groups and testing each group in a single run, the less active catalyst can be isolated and the total number of runs reduced.

Kleijnen [3] in (1975) compared group screening designs with other types of factor screening designs. He investigated the assumptions made by Watson [16] and derived some new results on two-stage group screening by allowing the possibility of two-factor interactions.

Mauro and Smith [5] in (1982) discussed the performance of two-stage group screening designs when the assumptions that the direction of possible effects are known or are correctly assumed a-priori is relaxed. The case of zero error variance is considered. They assumed that for all defective factors, the magnitude of the effect is the same but the direction of the effects could be different. To gauge the effect of cancellation, they defined the relative testing cost as another measure of screening efficiency.

Patel and Ottieno [11] in (1984) considered two-stage group screening designs with equal prior probabilities of factors to be defective and with no errors in observations. They used the method of finite differences to obtain the optimum

group-sizes and compared their results with Watson's results obtained by assuming continuous variations in the group-sizes. In another paper, Patel and Ottieno [12] discussed two-stage group screening designs with unequal prior probabilities of factors to be defective and with no errors in observations. They obtained the optimum group-sizes by assuming continuous variations. They have also shown that two-stage group screening design with unequal probabilities of factors to be defective has fewer runs than the corresponding designs in which all the factors are assumed to have the same a-priori probability of being defective.

Patel and Odhiambo [8] in (1986) studied the multi-stage group screening designs with unequal prior probabilities of to be defective and with no errors in observations. They described a procedure for grouping the factors in the absence of concrete priori information, so that the relative testing cost is minimal. They also showed that under quite general conditions, the designs with unequal prior probabilities to be defective require fewer runs than the equivalent designs in which the group-factors contain the same number of factors.

Patel and Manene [10] in (1987) studied one-type step-wise group screening designs with equal prior probabilities of factors to be defective and with no errors in observation. Their approach is similar to Sterret's approach except that in the initial step, the group-factors are tested in a factorial experiment. They compared the one-type step-wise group screening procedures with the m-stage group screening procedures (m=2,3,4) and found that the one-type step-wise group screening has fewer runs than the m-stage group screening for $0.035 \leq p \leq 0.40$.

Odhiambo and Manene [7] in (1987) have considered the performance of one-type stepwise group screening in terms of the expected number of runs and the expected number of incorrect decisions. They presented a method for obtaining optimal one-type step-wise designs for the cases in which the direction of each

defective factor is assumed to be known a-priori and the observations are subject to error.

Odhiambo [6] in (1986) discussed the performance of multistage group screening designs when the assumption that the direction of possible effects are known or are correctly assumed a-priori is relaxed. The case of zero error variance is considered. He assumed that for all effective factors, the magnitude of the effect is the same but the direction of the effects could be different. He obtained expressions for the relative testing cost and that for the percentage measure of efficiency of detecting active factors for an m-stage design ($m \geq 2$). He also defined a linear cost function for the m-stage group screening design. He showed that, whereas it is possible to reduce the relative testing cost by increasing the number of stages, the efficiency for detecting active factors decreases as the number of stages increases, for given p_1 and p_2 due to cancellation of effects. He also showed that two-stage group screening definitely performed better than three-stage group screening design for $p>0.1$ and three-stage performed better than four-stage group screening design when $p>0.03$.

1.3 Objective and significance of the study

The main objective of group screening experiments is to cut down the number of runs or observations needed, thus reducing the cost and time used for the experiment. We say that a design is more efficient than another if the expected number of runs in the former design is less or equal to that in the latter design for all p ($0<p<1$) with strict inequality holding true for at least one value of p (the a priori probability of a factor to be defective).

This study aims at comparing different group screening procedures using the expected number of runs as the basis of comparison. We restrict ourselves to Dorfman's procedure which was extended by Li [4] in (1962) and Patel [9] in (1962), Sterret's procedure which Patel and Manene [10] in (1987) extended and called it step-wise group screening procedure and Hwang's procedure.

Once the procedures are compared, we are in a position to tell which one has the fewest expected number of runs for given upper limit of number of defectives or upper limit of probability of a defective.

CHAPTER TWO

DESCRIPTION OF PROCEDURES

2.1 DORFMAN GROUP TESTING PROCEDURE

1. Introduction

The concept of group testing was first introduced by Dorfman as an economical method of testing blood samples of army inductees in order to detect the presence of infection. He applied this method to weeding out all syphilitic cases among those called for induction into the Armed Forces. Instead of testing each blood sample individually, Dorfman proposed to pool k samples in a single analysis. Presence of syphilitic antigen in the pool led to a decision to make k individual tests; absence of such antigen in the pool led to immediate clearance of all k inductees without further testing. Dorfman was mainly concerned with savings expected to result from application of his procedure. This depended on the group size, k, and the prevalence rate of the disease. If the latter was known, it was possible to choose a group size that maximised the expected savings in testing.

On the assumption that testing is error-free, for a prevalence rate of 5 % the optimum size of the group is 5 and the per cent expected savings is 57 %; for a prevalence rate of 1 % the optimum size is 11 and the expected saving increases to 80 %. When the prevalence rate is 30 %, however, the optimum size decreases to 3 and the expected savings is barely 1%.

Assumptions

1. The prevalence rate, p, is sufficiently small.
2. It is easier, or at least as economical to obtain an observation on a group as on an individual of the group tested separately.
3. The inspection is error-free.

2.Testing Procedure

Notation

Let

p be the prevalence rate, that is the probability that a random selection will yield an infected individual.

Then,

the probability of selecting at random an individual free from infection is 1-p.

Further

$(1-p)^k$ is the probability of obtaining by random selection a group of k individuals all of whom are free from infection.

and

$p = 1-(1-p)^k$ is the probability of obtaining by random selection a group of k individuals with at least one infected member.

Now

$\frac{f}{k}$ is the number of groups of size k in a population of size f.

Thus

$p'\frac{f}{k}$ is the expected number of infected groups of k in a population of f with a prevalent rate of p.

The expected number of tests required by the grouping procedure would be

$$E[R] = \frac{f}{k} + k\left(\frac{f}{k}\right)p'$$

$$= \frac{f}{k} + f\,p' \qquad (2.1.1)$$

that is the number of groups plus the number of individual in the groups which require retesting. The ratio of the number of tests required by the group technique to the number required by the individual technique is a measure of its expected relative cost. This is given by

$$C = \frac{E(R)}{f} = \frac{1}{k} + p'$$

$$= \frac{k+1}{k} - (1-p)^{k} \qquad (2.1.2)$$

3. Optimum Group Sizes

The extent of the saving attainable by use of the group method depends on the group size and the prevalence rate. Dorfman showed that for a prevalence rate of 0.01, only 20% of the individual tests would be required when group tests with groups of 11 are used. The attainable savings decreased as the prevalence rate increased. For a prevalence rate of 0.15, 72% of the tests required by individual testing are required when groups of size 3 are used.

The group testing technique works well where the following two conditions are satisfied.

1. The prevalence rate must be sufficiently small to make worth while economics possible;
2. When it is easier or more economical to obtain an observation on a group than on the individuals of the group separately.

2.1.1 PATEL'S GROUP TESTING PROCEDURE

1. Introduction

In 1962 Patel [6]modified Dorfman's procedure by extending the two-stage group screening procedure to multi-stage group screening procedure with no errors in observation. His work was restricted to the case when all the factors were defective with equal a-priori probabilities; f was finite and large, and none of the factors interacted. This guaranteed that the effects were additive and there was no chance of cancellation of effects. A factor is defined to be defective if it produces a non-zero change in the mean response.

2. Two-stage Procedure

Let there be 'f' factors under investigation. The 'f' factors are initially divided into g_1 group-factors, each containing k_1 factors. These are referred to as first order group-factors. Further let each of the g_1 first order group-factors be divided into $g_2 = k_1$ second order group-factors, each containing $k_2 = 1$ factor.

Then $f = g_1 g_2$. The experiment has two stages;

In stage one we test the first order group-factors for their effects.
while in stage two we test the second order group-factors belonging to the first order group-factors found to be defective for their effects.

In the first stage, there are $R_{(1)} = g_1 + 1$ runs. Let n_1 be the number of first order group-factors found to be defective in the first stage. Then the probability distribution of n_1 is

$$f(n_1) = \binom{g_1}{n_1} p_1^{n_1} (1-p_1)^{g_1-n_1} \qquad (2.1.3)$$

where $p_1 = (1-q^{k_1})$ is the probability that a first order group-factor is defective.

The number of runs $R_{(2)}$ in the second stage is $n_1 g_2$.

$$\begin{aligned} E(R_{(2)}) &= E(n_1 g_2) \\ &= g_1 g_2 p_1 \\ &= f p_1 \end{aligned} \qquad (2.1.4)$$

Therefore, on the average, the total number of runs R_2 in a two-stage group screening experiment is

$$\begin{aligned} E(R_2) &= R_{(1)} + E(R_{(2)}) \\ &= 1 + g_1 + f p_1 \end{aligned} \qquad (2.1.5)$$

3.Three-stage Group Screening Procedure

Let the f factors initially be divided into g_1 group-factors, each consisting of k_1 factors. These are referred to as the first order group-factors. Further let each of the g_1 first order group-factors be divided into g_2 second order group-factors, each consisting of k_2 factors, which for uniformity may also be referred to as $k_2 = g_3$ third order group-factors of k_3 factor each. Then $f = g_1 g_2 g_3$. The experiment has three stages:-

In stage one we test first order group-factors for their effects. In stage two we test the second order group-factors belonging to the first order group-factors found to be defective for their effects. In the third stage, we test the factors which belong to the second order group-factors found to be defective for their effects.

In the first stage, there are $R_{(1)} = g_1 + 1$ runs. Let n_1 be the number of first order group-factors found to be defective. Then the probability distribution of n_1 is

$$f_1(n_1) = \binom{g_1}{n_1} p_1^{n1}(1-p_1)^{g_1-n1} \tag{2.1.6}$$

where $p_1 = (1-q^{k_1})$ is the probability that a first order group-factor is defective. The number of runs $R_{(2)}$ in the second stage is $n_1 g_2$ with its mean value given by

$$E(R_{(2)}) = E(n_1 g_2) = g_1 g_2 p_1 \tag{2.1.7}$$

Next, let n_2 be the number of second order group-factors that are found to be defective in the second stage. Then, given n_1, the distribution of n_2 is

$$f(n_2|n_1) = \binom{n_1 g_2}{n_2} p_{2|1}^{n1}(1-p_{2|1})^{n_1 g_2 - n2} \tag{2.1.8}$$

where $p_{2|1}$ is the probability that a second order group-factor is defective, given that it is within a first group-factor which is defective. The probability $p_{2|1}$ is given by

$$p_{2|1} \quad p_2 / p_1 \tag{2.1.9}$$

where $p_2 = 1 - q^{k2}$.

The mean value of the number of runs $R_{(3)}$ in the third stage is given by

$$E(R_{(3)}) = \underset{n1\,n2}{E\,E}\ (n_2 \mid n_1)\, g_3$$

$$= \underset{n1}{E}\ (n_1) g_2\, g_3\, p_2 \mid 1$$

$$= g_1 g_2\, g_3\, p_1 p_2 \mid 1$$

$$= f p_2 \qquad (2.1.10)$$

Hence, on the average, the total number of runs, R_3 in a three stage group screening experiment is

$$E(R_3) = R_{(1)} + E(R_{(2)}) + E(R_{(3)})$$

$$= g_1 + 1\ g_1 g_2 p_1 + f p_2 \qquad (2.1.11)$$

4.Four - Stage group-screening procedure

The experiment plan for the three-stage group screening may be extended to four-stage group screening. Let each second order group-factor be further divided into g_3 third order group-factors, each containing k_3 factors or $k_3 = g_4$ fourth order group-factors of $k_4 = 1$ factor each Then, $k_2 = k_3 g_3$. The expected number of runs in this stage may be found as follows;

Let $p_{3|2}$ be the probability that a third order group-factor is defective, given that it is within a second order group-factor which is defective. Also let $p_{3|1}$ be the probability that a third group-factor is defective given that it is within a first order which is defective.

Then

$$P_{3|2} \cdot p_{2|1} = p_{3|1} \tag{2.1.12}$$

i.e.

$$p_{3|2} \cdot p_2 / p_1 = p_3 / p_1$$

where

$p_3 = 1 - q^{k3}$, so that

$$p_{3|2} = p_3 / p_2 \tag{2.1.13}$$

Let n_3 be the number of third order group-factors which are defective in the third stage. Then given n_2, the distribution of n_3 is

$$f(n_3 | n_2) = \binom{n_2}{n_3} p_{3|2}^{n3} (1 - p_{3|2})^{n2g3 - n3} \tag{2.1.14}$$

The mean value of the number of runs $R_{(4)}$ in the fourth stage is given by

$$E(R_{(4)}) = \underset{n_2}{E}\,\underset{n_1}{E}\,(n_3 \mid n_2)\, g_4$$

$$= \underset{n_1}{E}\,(n_2)\, g_3\, g_4\, p_{3|2}$$

$$= \underset{n_1}{E}\,\underset{n_2}{E}\,(n_2 \mid n_4) g_3\, g_4\, p_{3|2}$$

$$= g_1 g_2 g_3 g_4 p_1 p_{2|1} p_{3|2}$$

$$= f\, p_3 \qquad (2.1.15)$$

Hence, on the average, the total number of runs R_4 in a four stage group screening experiment is

$$E(R_4) = g_1 + 1 + g_1 g_2 p_1 \qquad + g_1 g_2 g_3 p_2 + f\, p_3 \qquad (2.1.16)$$

5. Extension to (n+1) Stages

In the general case there are n+1 stages of experimentation. The f factors are divided into g_1 first order group-factors of k_1 factors each; each of the g_1 first order group-factors is divided into g_2 second order group-factors of k_2 factors each..... and each of the g_{n-1} (n−1)st order group-factors are divided into g_n of k_n factors each. Letting the k_n factors be called g_{n+1}(n+1)st order group-factors of $k_{n+1} = 1$ factor each, it follows that;

$$f = g_1 g_2 \ldots g_{n+1} \qquad (2.1.17)$$

The first stage consists of testing the g_1 first order group-factors in g_1+1 runs. The (r+1)st stage is an experiment with the (r+1)st order group-factors which belong to the n_r defective r-th order group-factors tested in $n_r g_{r+1}$ runs, (r = 1,2,...,n).

The expected number of runs for all stages is

$$E(R_{n+1}) = 1 + g_1 + \sum_{i=2}^{n+1} \prod_{j=1}^{i} g_j p_{j-1}$$

$$= 1 + f/k_1 + f \sum_{i=2}^{n+1} p_{i-1}/k_i \qquad (2.1.18)$$

where $p_{i-1} = 1-q^{k_{i-1}}$

Assuming continuous variation, the values of $k_1, k_2, \ldots k_n$ that minimise $E(R_{n+1})$ are given by the equations

$$\frac{\partial E(R_{n+1})}{\partial k_1} = -\frac{f}{k_1^2} - \frac{fq^{k_1}}{k_2}\log q = 0$$

$$\frac{\partial E(R_{n+1})}{\partial k_2} = -\frac{f p_1}{k_2^2} - \frac{fq^{k_2}}{k_3}\log q = 0$$

$$\vdots$$

$$\frac{\partial E(R_{n+1})}{\partial k_{n-1}} = -\frac{f p_{n-2}}{k_{n-1}^2} - \frac{fq^{k_{n-1}}}{k_n}\log q = 0$$

$$\frac{\partial E(P_{n-1})}{\partial k_n} = -\frac{f\, p_{n-1}}{k_n^2} - f\, q^{k_n} \log q = 0 \qquad (2.1.19)$$

Equations (2.1.14) may be approximated by using

$$p_i = 1 - q^{k_i} = 1 - (1-p)^{k_i} \sim k_i p$$

and

$$q^{k_1} \log q = (1-p)^{k_1} \log(1-p) \sim -p.$$

for small p.

Equations (2.1.14) becomes

$$\frac{-f}{k_1^2} + \frac{fp}{k_2} = 0 \ ,$$

$$\frac{-f\, k_1 p}{k_2^2} + \frac{fp}{k_2} = 0$$

.

.

.

$$\frac{-f\, k_1 p}{k_n^2} + f p_1 = 0 \qquad (2.1.20)$$

which readily give

$$k_1 \sim \frac{1}{p^{n/n+1}}, k_2 \sim \frac{1}{p^{n-1/n+1}}, \ldots, k_n \sim \frac{1}{p^{1/n+1}} \qquad (2.1.21)$$

From (2.1.16) and $k_{r-1} = k_r g_r$, $k_0 = f$, it follows that

$$g_1 \sim fp^{n/n+1} \quad , \quad g_2 \sim \frac{1}{p^{1/n+1}}, \ldots, \; g_n \sim \frac{1}{p^{1/n+1}} \qquad (2.1.22)$$

The values of k_r and g_r ($r = 1, 2, 3, \ldots, n$) in (2.1.16) and (2.1.17) give approximately the values that minimise $E(R_{n+1})$. Substituting in (2.1.13) gives

$$\text{Min } E(R_{n+1}) \sim (n+1) f p^{n/n+1} + 1 \qquad (2.1.23).$$

6.Choice of number of stages

The minimised value of the expected number of runs in group-screening experiments with different number of stages can now be compared with the help of formula (2.1.23). For instance

$$\min E(R_2) < E(R_1) = f+1$$

if

$$2fp^{1/2} + 1 < f+1$$

which implies that

$$p < 0.25 \qquad (2.1.24)$$

Also

$$\min E(R_3) < \min E(R_2)$$

If

$$3fp^{2/3} + 1 < 2fp^{1/2} + 1$$

which implies that

$$p < (2/3)^6 \sim 0.088 \qquad (2.1.25)$$

Further,

$$\min E(R_4) < \min E(R_3)$$

If

$$4fp^{3/4} + 1 < 3fp^{2/3} + 1$$

which implies that

$$p < (3/4)^{12} \sim 0.032 \qquad (2.1.26)$$

In general

$$\min E(R_n) < \min E(R_{n-1})$$

If

$$p < (n-1/n)^{n/n-1} = (1-1/n)^{n(n-1)}$$

$$\sim e^{-(n-1)} \quad \text{for } n \text{ large} \qquad (2.1.27)$$

These results indicate that a one-stage procedure is best for $p < 0.25$, a two-stage is best for $0.088 < p < 0.25$, and a three-stage procedure is best for $0.032 < p < 0.088$.

A one stage procedure may be compared with higher stage procedures. For a three stage procedure

$$\min E(R_3) < f+1$$

If

$$3fp^{2/3}+1 < f+1$$

which implies

$$p < (1/3)^{3/2} \sim 0.19 \qquad (2.1.28)$$

For any n,

$$\min E(R_n) < f+1$$

If

$$p < (1/n)^{n/n-1}$$

$$\sim 1/n \quad \text{for } n \text{ large} \qquad (2.1.29)$$

A two-stage procedure may be compared with higher stage procedures. For a four-stage procedure,

$$\min E(R_4) < \min E(R_2)$$

If

$$4fp^{3/4}+1 < 2fp^{1/2}+1$$

which implies that

$$p < 1/16 \sim 0.0625 \qquad (2.1.30)$$

For an n stage procedure

$$\text{Min } E(R_n) < \min E(R_2)$$

If

$$nfp^{n-1/n}+1 < 2fp^{1/2}+1$$

which implies that

$$p < (2/n)^{2n/n-2}$$

$$\sim 4/n^2 \quad \text{for } n \text{ large} \qquad (2.1.31)$$

2.1.2 LI'S GROUP TESTING PROCEDURE

1. Introduction

Li described a statistical method for group testing designs in which a relatively small number of critical variables or interactions must be quickly selected from a large group. These critical variables or interactions are assumed to have effects too large to be masked by the experimental error, or the combined effects of the important variables.

The group of independent variables is divided into subgroups of suitable sizes, each of which is treated as a single combined variable. One cycle of tests eliminates the subgroups which contain only unimportant variables. The remaining subgroups are then redivided and tested to eliminate all except those containing the critical variables . This process grouping and group-testing may be repeated as often as desired . The optimum number of regroupings or test cycles for up to 1000,000 independent variables, and the best subgroup sizes for the test cycles, have been calculated.

For certain applications, this method can reduce the number of tests required to as mall fraction of that required with conventional or non-sequential (or one-cycle) procedures. The efficiency of this screening method partly results from the collection and use of information after each cycle, to set up to the best plans for succeeding test cycles. The conventional or one-cycle Method lacks this 'information feedback' feature, requires the experimenter to make a test on each variable or interaction and does not permit him to make corrections or change strategies as the testing proceeds.

Examples of such application

1. Identifications of the variables responsible for an epidemic of failures in a product which can be assembled by many different methods and processing

steps, from many different combinations of components and materials.

2. Improvement of an extremely complex product, such as an electronic system, a missile, or a jet plane, involving thousands of dimensions i.e. lengths, widths thickness, depths, diameters angles, curvatures tolerances, and their ratios , by a means of, say 100 tests in which 5,000 of these dimensions or ratios are screened for the few which are highly critical.

3.Determination by means of say, 200 tests on samples as to which of 10,000 drugs or procedures is most likely to provide a cure for specific disease.

Notations

f = the number of independent variables in the group included in the experiment;

j = the number of defective variables;

c = the number of cycles of tests in the experiment;

g_i = the number of subgroups in the i-th cycle of tests;

k_i = the subgroup size, or the number of variables combined into each subgroup, in the i-th cycle of tests;

r_i = the number of tests in the i-th cycle of tests, generally as g_i 's;

R_c = the total number of tests in a c-cycle experiment.

Assumptions

1. Out of a very large number f of variables, only a small number p are defective.Each of the f variables has only two levels of conditions of testing.
2. The j defective variables have much greater effects than all of the defective variables combined.

3. The tests are fairly reproducible, i.e. the error of experiment is small.
4. There no interactions among variables, i.e. the change in response caused by a variable going from one 'level' to another does not depend on the 'levels' of the other variables.

2. One-cycle Experiment

In this type of experiment, the group of f variables is studied in a single test cycle, one test being used for each variable, i.e.

$$c = 1 \tag{2.1.32}$$

$$k_1 = 1 \tag{2.1.33}$$

and

$$R_1 = j_1 = g_1 = f \tag{2.1.34}$$

This design is inefficient because like the classical one-factor approach, only one variable is varied at a time.

3. Two-cycle Experiment

In this type of experiment the group of f variables is first divided into and tested in g_1 subgroups, of size k_1 each. The j defective variables will show up in j or less of the g_1 subgroups. The second cycle of tests, therefore, has to deal with only jk_1 or less variables, i.e.

$$c = 2 \tag{2.1.35}$$

$$f = g_1 k_1 \tag{2.1.36}$$

$$jk_1 \geq g_2 \tag{2.1.37}$$

$$k_1 > k_2 = 1 \tag{2.1.38}$$

$$R_2 = \sum_{i=1}^{2} r_i = \sum_{i=1}^{2} g_i = g_1 + g_2 \leq \frac{f}{k_1} + jk_1 \tag{2.1.39}$$

To minimise R_2 for the likely case where the j defective variables show up in exactly j subgroups (since $f >> k_1 >> j$), set

$$\frac{dR_2}{dk_1} = \frac{-f}{k_1^2} = 0 \tag{2.1.40}$$

which implies

$$k_1 = \left(\frac{f}{j}\right)^{1/2} \tag{2.1.41}$$

and

$$\frac{d^2R_2}{dk_1^2} = \frac{2f}{k_1^2} > = 0 \tag{2.1.42}$$

demonstrating that R_2 is a true minimum at

$$k_1 = \left(\frac{f}{j}\right)^{1/2}$$

Therefore,

$$g_1 = g_2 = (fj)^{1/2} \tag{2.1.43}$$

and

$$R_2 = g_1 = g_2 = (fj)^{1/2} + (fj)^{1/2} = 2(fj)^{1/2} \quad (2.1.44)$$

4. c-cycle Experiment

In this general experiment

$$c=c \quad (2.1.45)$$

$$f = g_1 k_1 \quad (2.1.46)$$

$$jk_1 \geq g_2 k_2 \quad (2.1.47)$$

$$jk_2 \geq g_3 k_3 \quad (2.1.48)$$

.

.

.

$$jk_{c-2} \geq g_{c-1} k_{c-1} \quad (2.1.49)$$

$$jk_{c-1} \geq g_c \quad (2.1.50)$$

$$k_1 > k_2 > k_3 \ldots > k_i > \ldots > k_{c-1} > k_c = 1 \quad (2.1.51)$$

$$R_c = \sum j_i = \sum g_i \leq \frac{f}{k_1} + \frac{jk_1}{k_2} + \frac{jk_2}{k_3} + \ldots + \frac{jk_{c-2}}{k_{c-1}} + jk_{c-1} \quad (2.1.52)$$

When the j defective variables show up in exactly j subgroups at every cycle, R_c has a true minimum which occurs at the following values of k_i ;

$$k_1 = \left(\frac{f}{j}\right)^{(c-1)/c} \quad (2.1.53)$$

$$k_2 = \left(\frac{f}{j}\right)^{(c-2)/c}$$

.

$$k_{c-1} = (\frac{f}{j})^{1/c} \qquad (2.1.54)$$

and

$$R_c = \sum_{i=1}^{c} j_i = \sum_{i=1}^{c} g_i \le (fj^{c-1})^{1/c} + (fj^{c-1})^{1/c} + \ldots + (fj^{c-1})^{c-1}$$
$$= c\,(fj^{c-1})^{1/c} \qquad (2.1.55)$$

5. Best Number of Test Cycles

The best c for given f/j or j/f occurs when $R_{c-1} > R_c < R_{c+1}$ or

$$b+1 = \frac{R_{c+1}}{R_c} = \frac{c+1}{c}(j/f)^{1/c(c+1)} > 1 \text{ for best c} \qquad (2.1.56)$$

where

$$b = \frac{R_{c+1} - R_c}{R_c} = \frac{R_{c+1}}{R_c}\,(\frac{c+1}{c})(\frac{j}{f})^{1/c(c+1)} - 1$$

That is

$$\frac{f}{j} < (\frac{c+1}{c})^{c(c+1)} \qquad \text{for best c} \qquad (2.1.57)$$

The values of f/j or j/f at which a change in the best c occurs, can this be calculated.

2.2 STERRET'S GROUP TESTING PROCEDURE AND ITS EXTENTIONS

1.Introduction

Instead of analysing each sample of a defective group, Sterret[11]in (1957) proposed to continue making individual tests only until a defective was found. If the proportion of defective items in the population was small it was regarded as likely that a new subgroup, consisting of the remaining untested units, will prove on testing, to be free of defective items. If this does happen, the work was finished. Otherwise, individual testing was resumed until another defective item was found where upon the remaining items were tested as a group, and so on until a decision was reached in regard to each item. The reversion to a group test was repeated as many times as needed.

Sterret calculated his procedure, when applied continuously (without a stopping rule), increased Dorfman's efficiencies on the average by 'about 6% ' if optimum group sizes (which were different from those in Dorfman's procedure were used in each case).

2.Notation

The probability that a pool containing k samples has exactly j defective members is given by $p_k(j)$; the expected value of the number of analysis required to isolate the j defectives is $E_k(j)$.

Given a universe of f elements with p per cent defective, E (f, k, p) is the total expected value of the number of analyses required to investigate the universe by pooling k samples at a time.

3.procedure

Using the definition of expectation of a random variable

$$E(f, k, p) = \frac{f}{k} \sum_{j=0}^{k} \{P_k(j)\, E_k(j)\} \qquad (2.2.1)$$

Before E (f,k,p) can be evaluated it must be shown that

$$E_k(j) = \frac{j}{j+1}\, k + j + 1 + \frac{j}{j+1} - 2j\, \frac{1}{k} \qquad (2.2.2)$$

When there are no defective elements in a pool one laboratory analysis will suffice, i.e.

$E_k(0) = 1$ from equation 2.2.2

Now

$$E_k(n) = 1 + \frac{n}{k}\{1 + E_{k-1}(n-1)\} +$$

$$\sum_{i=1}^{k-n} [(\prod_{j=1}^{i} \frac{k-(j+n-1)}{k-(j-1)}) \frac{n}{k-i} \{(i+1) + E_{k-|i+1|}(n-1)\}] \qquad (2.2.3)$$

The first term on the right-hand side of equation (2.2.3) represents the initial group test. The factor n/k in the next term is the probability that the first sample tested is defective; the factor $\{1 + E_{k-1}(n-1)\}$ is the sum of the number of tests required to find a defective on the first trial and the average number of tests needed to find (n- 1) defectives in the remaining pool of k –1 members. The probability that the first i samples are not defective is

$$\prod_{j=1}^{i} [k-(j+n-1)]/[k-(j-1)]$$

while the probability that the (i+1)st element tested is defective is n (k-i). The number of tests required to find the first defective is (i+1) and $E_{k-[i+1]}(n-1)$ is the expected number of tests required among k- [i+1] members.

Equation (2.2.3) reduces to the form given by equation (2.2.2) when values of $E_{k-[i+1]}(n-1)$ obtained from equation (2.2.2) are properly substituted. The poof, then, of the formula for $E_K(j)$ follows by induction.

4. Approximation to E(f,k,p)

The probability connected with all but the first few terms of E(f,k,p) are insignificant for small p. Therefore an approximation to E (f, k, p) is defined as

$$E'(f, k,p) = \left(\frac{f}{k}\right) \sum_{j=0}^{m} \{p_k(j)\, E_k(j)\}$$

Where m is the smallest integer such that

$$\sum_{j=0}^{m} p_k(j) > 0.99$$

The number of terms required to calculate E' (f,k,p) is m+1. This is also the minimum number of subdivisions into which an element must be divided by a laboratory technician to be at least 99% per cent confident that he will know the history of the group before exhausting any element.

2.2.1 STEP-WISE GROUP SCREENING METHOD

1. Introduction

In 1987 Patel and Manene [10] studied one-type stepwise group screening designs with equal prior probability of factors to be defective and with no errors in observation. Their approach was along Sterret's line except that in the initial step, the group-factors were tested in a factorial experiment.

In a stepwise group screening, the first step is to divide the factors into groups referred to as 'group-factors'. These group-factors are then tested for their effects. Those found to be nondefective are set aside. In the second step, we start with any defective group-factor and test the factors within it one by one until we find a defective factor. The factors which are found to be nondefective are set aside, keeping the defective factor separate. The remaining factors are then grouped into a group-factor. This process is done for all group-factors found to be defective in the first step. The test procedure carried out in the first and in the second steps is repeated in subsequent steps successively until the analysis terminates with a test on a nondefective group-factor or with a group-factor of size one.

2. Expected Number of Runs

In the first step, we partition 'f' factors into 'g' group-factors, each consisting of 'k' factors ($f = kg$). If p^* is the probability that a group-factor.

In the first step is defective, then

$$p^* = \sum_{i=1}^{k} \binom{k}{i} p^i q^{k-i}$$

$$= 1 - q^k \qquad (2.2.4)$$

In the first step, all the 'g' group-factors are tested for their effects. Thus the number of tests required in the step is given by

$$R_1 = g + 1 \qquad (2.2.5)$$

Where the one extra run is the control run. The density function of n, the number of defective group-factors in the initial step is,

$$f(n) = p(\underline{n} = n)$$

$$= \binom{g}{n} p^{*n} (1-p)^{g-n} \quad , \quad n = 0, 1, \ldots, g$$

$$= \text{Otherwise} \qquad (2.2.6)$$

Thus

$$E(n) = g\,p^*$$

$$= \frac{f}{k}(1 - q^k) \qquad (2.2.7)$$

In the subsequent steps, the analysis of the n group-factors found to be defective in the first step is continued. Let $p_k(j)$ denote the probability that a group-factor of size k contains exactly j defective factors if it is known to contain at least one defective factor. Then,

$$P_k(j) = (1-q)^{-1}\binom{k}{j} p^j (1-p)^{k-j} \qquad j = 1, 2, \ldots, k \qquad (2.2.8)$$

Let $E_k(R_j)$, $(j = 1, 2, \ldots, k)$, be the expected number of runs required to classify as defective or nondefctive all the factors within a group-factor of size k which is known to be defective if it contains exactly i defective factors.

Then

$$E_k(R_j) = \frac{jk}{j+1} + j + \frac{j}{j+1} - 2j \qquad (2.2.9)$$

Let $R_s^{(1)}$ denote the number of runs required to analyse a defective group-factor, i.e. classify as defective or nondefective all the factors within a group-factor of size k that is known to be defective. Then

$$E(R_s^{(1)}) = \sum_{j=1}^{k} E_k(R_j)\, p_k(j)$$

$$= (1-q^k)^{-1} \sum_{j=1}^{k} \left\{ \frac{j(k+1)}{j+1} + j - \frac{2j}{k} \right\} \binom{k}{j} p^j q^{k-j}$$

$$= (1-q^k)^{-1} \left[(k+1) + kp - 2p - \frac{1}{p}\{1-q^{k+1}\} \right] \qquad (2.2.10)$$

Let R_s denote the number of tests required to analyse n group-factors found to be defective in the first step. Then

$$R_s = n\, E(R_s^{(1)}) \qquad (2.2.11)$$

Let R denote the total number of runs required to screen out the defective factors from the 'f' factors under investigation in a step-wise group screening experiment. Then

$$E(R) = R_1 + E[R_s]$$

$$= R_1 + E[n\, E(R_s^0)]$$

$$= 1 + fp + \frac{2fp}{k} + f - \frac{f}{kp}\{1 - q^{k-1}\} \qquad (2.2.12)$$

3.Optimum Size of the Group-factor in the First Step

Assuming 'p' to be small, let

$$y_k = E(R) = 1 + fp + \frac{2fp}{k} + f - \frac{f}{kp}\{1 - q^{k+1}\} \qquad (2.2.13)$$

Then

$$\Delta y_k = y_{k+1} - y_k$$

$$= \frac{\{1 - 2p(1-p)\}}{k(k+1)p} - fq^{k+1}\left[\frac{kp+1}{(k+1)kp}\right] \qquad (2.2.14)$$

The forward difference Δy_k changes sign just before a minimum of y_k, thus, the minimum is just after the formal solution of the equation

$$\Delta y_k = 0 \qquad (2.2.15)$$

which implies

$$\frac{1}{(1-p)^{k+1}} = \frac{kp+1}{1-2p+2p^2} \qquad (2.2.16)$$

Expanding the L.H.S. and the R.H.S. in the above equation up to order p^2 we get

$$1 + (k+1)p + (k+1)(k+2)\frac{p^2}{2} = (kp+1)(1+2p) + 2p^2$$

i.e.

$$pk^2 - pk - 2 - 2p = 0 \qquad (2.2.17)$$

which implies

$$k = \left(\frac{2}{p}\right)^{1/2} + \frac{1}{2} + \frac{9}{8}\left(\frac{p}{2}\right)^{1/2} \quad \text{up to order } p \qquad (2.2.18)$$

which is the size of the group-factor in the first step which minimises the expected total number of runs in a step-wise group screening design.

Rewriting E (R) in the form

$$E(R) = 1 + fp + \frac{2fq}{k} + f - \frac{f}{kp} + \frac{f}{kp}\exp\left[-(k+1)(p+p^2/2)\right] \qquad (2.2.19)$$

And substituting the value of k, we obtain.

$$\text{Min } E(R) = 1 + f(2p)^{1/2} + \frac{5}{12}fp \quad \text{up to order } p. \qquad (2.2.20).$$

2.2.2 TWO-TYPE STEPWISE GROUP SCREENING DESIGN

1.Introduction

In a two-type step-wise group screening design, the factors are partitioned into first order group-factors. The first order group-factors are then tested for their effects and classified as either defective or nondefective. Each first order group-factor found to be defective in the initial step is further divided Into smaller group-factors. Type one search steps are then used to classify all the second order group-factors as either defective or nondefective. Finally factors within the second order within the second order group-factors found to be defective in type one search steps are classified as defective or nondefective using type two search steps.

He made the following assumptions; suppose that there are 'f' factors under investigation then:

(i) All factors have independently the same prior probability, p, of being defective (q=1-p).

(ii) Defective factors have the same positive effect.

(iii) None of the factors interact.

(iv) The directions of possible effects are known.

2. The Procedure

When screening with two of search steps, we first divide the 'f' factors into g_1 first order group-factors each of size k_1 ($f = k_1 g_1$). In the initial step the first order group-factors are tested for their effects. Those that are found to be nondefective are set aside, keeping the defective ones separate. Each defective first order group-factor is divided into g_2 second order group-factors each containing

k_2 factors ($k_1 = k_2 g_2$). In the first of the type one search steps we start with any defective first order group-factor. We test the second order group-factors within it one by one till we find a defective second order group-factor. The defective second order group-factor is kept separate. In the second of the type one search steps, we test the remaining second order group-factors in a pooled group. If the pooled group test is negative, the test procedure is terminated. Otherwise in the third of the type one search steps, we continue testing the remaining second order group-factors ono by one till another defective second order group-factor is found. This is also kept separate. The second and the third of the type one search steps are repeated successively till the analysis terminates with a test on a nondefective pooled group-factor or a test with a single second order group-factor. This test procedure is performed for all the n_1 first order group-factors found to be defective in the initial step.

Suppose n_2 second order group-factors are found to be defective at the end of the type one search steps. In the type two search, steps the defective factors are isolated in a similar procedure as was used to isolate defective second order group-factors in the type one search steps.

3. The expected number of runs

Let 'p' be the a-priori probability that a factor is defective. Suppose p_1^* is the probability that a first order group-factor is defective and p_2^* is the probability that a second order group-factor is defective.

Then, (2.2.21)

$$P_1^* = 1 - q^{k_1}$$

And (2.2.22)

$$P_2^* = 1 - q^{k_2}$$

In the initial step we test all the g_1 first order group-factors for their effects in an experiment. The number of runs required is

$$R_1 = g_1 + 1 \qquad (2.2.23)$$

Let n_1 be the number of first order group-factors found to defective in the initial step. Then the probability function of n_1 is

$$f(n_1) = \text{prob}(\underline{n}_1 = n_1)$$

$$= \binom{g_1}{n_1} p_1^{*}(1-p^{*}_1)^{g_1-n_1} \qquad n_1 = 0, 1, \ldots, g_1 \qquad (2.2.24)$$

Thus

$$E(n_1) = g_1 p_1^{*}$$

$$= \frac{f}{k_1}(1-q^{k1}) \qquad (2.2.25)$$

Denote by $p^{*}_{2|1}$ the probability that a second order group-factor is defective given that it is within a defective first order group-factor. Then

$$p^{*}_{2|1} = \frac{p^{*}_2}{p^{*}_1} \qquad (2.2.26)$$

If n_2 is the number of second order group-factor found to be defective at the end of type one search steps, then the probability function of n_2 for given n_1 is

$$f(n_2 \mid n_1) = \text{prob}(n_2 = n_2 \mid n_1 = n_1) \qquad n_2 = 0, 1, \ldots, n_1 g_2 \qquad (2.2.27)$$

Thus

$$E(n_2 \mid n_1) = n_1 g_2 p^{*}_{2|1} \qquad (2.2.28)$$

Therefore

$$E(n_2) = \underset{n_1 n_2}{E} E(n_2 | n_1)$$

$$= g_2 p_{2|1} * E(n_1)$$

$$= g_2 g_1 p_{2|1} * p_1 *$$

$$= \frac{f}{k_2}(1-q^{k_2}) \qquad (2.2.29)$$

Let $p_{g_2}(j_1)$ be the probability that a defective first order group-factor contains exactly j_1 defective second order group-factors and $p_{k_2}(j_2)$ the probability that a defective second order group-factor contains exactly j_2 defective factors. Then,

$$p_{g_2}(j_1) = (1-q^{k1})^{-1} \binom{g2}{j1} p^{*j_1}_2 (1-p^{*}_2)^{g_2-j_1} \qquad j_1 = 0, 1, \ldots g_2$$

(2.2.30)

and

$$p_{k_2}(j_2) = (1-q^{k2})^{-1} \binom{k_2}{j_2} p^{j2}(1-q)^{k_2-j_2} \qquad j_2 = 0, 1, \ldots, k_2$$

(2.2.31)

Denote by $E_{g_2}(R_{j_1})$ the expected number of runs require to classify as defective or nondefective all the second order group-factors within a defective first order group-factor if it contains exactly j_1 defective second order group-factors. Further let $E_{k_2}(R_{j_2})$ denote the expected number of runs required to classify as

defective or nondefective all the factors within a defective second order group-factor if it contains exactly j_2 defective factors. Then,

$$E_{g_2}(R_{j_1}) = \frac{j_1 g_2}{j_1+1} + j_1 + \frac{j_1}{j_1+1} - \frac{2j_1}{g_2} \qquad (2.2.32)$$

and

$$E_{k_2}(R_{j_2}) = \frac{j_2 k_2}{j_2+1} + j_2 + \frac{j_2}{j_2+1} - \frac{2j_2}{k_2} \qquad (2.2.33)$$

If $R^0_{t_1}$ denotes the number of runs required to classify as defective or nondefective all the g_2 second order group-factors within a first order group-factor which is known to be defective, then

$$E(R^0_{t_1 1}) = \sum_{j_1=1}^{g_2} Eg1(R_{j_1}) P_{g_2}(j_1)$$

$$= (1-q^{k_1})^{-1} \sum_{j_1=1}^{g_2} \left\{ \frac{j_1(g_2+1)}{j_1+1} + j_1\left(1-\frac{2}{g_2}\right)\right\} \binom{g_2}{j_1} p^{*j_1}_2 (1-p^*_2)^{g_2-j_1}$$

$$= (1-q^{k_1})^{-1} \sum_{j_1=1}^{g_2} \left\{ (g_2+1) - \frac{(g_2+1)}{j_1+1} + j_1\left(1-\frac{2}{g_2}\right)\right\} \times$$

$$\binom{g_2}{j_1} p_2^{*j_1} (1-p_2^*)^{g_2-j_1}$$

$$= (1-q^{k_1})^{-1} [(g_2+1)(1-q_2^{*g_2}) + (g_2-2)p_2^* \{1-q_2^{*g_2+1} -$$

$$(g_2+1)p_2^* q_2^{*g_2}\}]$$

$$= (1-q^{k_1})^{-1} \left[(g_2+1) + g_2 p_2^* - 2p_2^* - \frac{1}{p^{2*}}(1-q_2^{*g_2+1})\right] \qquad (2.2.34)$$

where $q_2^* = 1 - p_2^*$

Let R_{t_1} be the number of runs required to classify as defective or nondefective all the $n_1 g_2$ second order group-factors within the n_1 defective first order group-factors. Then,

$$R_{t_1} = n_1 E(R^0 t_1)$$

$$= \frac{n_1}{1 - qk_1} [(g_2 + 1) + g_2 p_2^* - 2p_2^* - \frac{1}{p2^*}(1 - q_2^{*g_2+1})] \quad (2.2.35)$$

Denote by $R_{t_2}^0$ the number of runs required to classify as defective or nondefective all the k_2 factors within a defective second order group-factor. Then,

$$E(R_{t_2}^0) = \sum_{j_2=1}^{k_2} Ek_2(Rj_2) Pk_2(j_2)$$

$$= (1 - q^{k_2})^{-1} \sum_{j2}^{k_2} \{\frac{j_2(k_2+1)}{j_2+1} + j_2(1 - \frac{2}{k_2})\} \binom{k_2}{j_2} p^{j_2} (1-p)^{k_2 - j_2}$$

$$= (1 - q^{k_2})^{-1} [(k_2 + 1) + k_2 p - 2p - \frac{1}{p}(1 - q^{k_2+1})] \quad (2.2.36)$$

where $q = 1 - p$.

Let R_{t_2} be the number of runs required to classify as defective or nondefective all the $n_2 k_2$ factors within the n_2 second order group-factors found to be defective at the end of type one search steps. Then.

$$R_{t_2} = n_2 E(R^0 t_2)$$

$$= \frac{n_2}{1 - qK_2}[(k_2 + 1) + k_2 p - 2p - \frac{1}{p}(1 - q^{k_2})] \qquad (2.2.37)$$

If R denotes the total number of runs required to screen out all the defective factors from among the 'f' factors under investigation in a two-type stepwise group-screening design then,

$$R = R_1 + R_{t_1} + R_{t_2}$$

$$= 1 + f + fp + \frac{2fq^{k_2}}{k_1} - \frac{f}{k_1}(1 - q^{k_2})^{-1}(1 - q^{k_1 + k_2}) + \frac{f}{k_2} - \frac{fq^{k_2}}{k_2} + \frac{2fq^{k_2}}{k_2} - \frac{f}{k_2 p}(1 - q^{k_2 + 1}) \qquad (2.2.38)$$

3.2.3 THE R-TYPE STEP-WISE GROUP SCREENING DESIGN

1. Screening Procedure

In the initial step, first order group-factors are tested for their effects and classified as defective or nondefective. In the type one search steps, all second order group-factors within the defective first order group-factors are tested using the step-wise technique and classified as defective or nondefective. The type two search steps are used to sort out and classify as defective or nondefective all the third order group-factors. The process is continued within this way so that in type $r-1$ search steps, all r-th order group-factors within the defective $(r-1)$-th order group-factors are sorted out and classified as defective or nondefective using the step-wise group screening technique. Finally in type r search steps, all factors within the defective r-th order group-factors are sorted out and classified as defective or nondefective also using the step-wise group screening technique.

2. The expected number of Tests

Let p be the prior probability that a factor is defective and p_s^* be the probability that an s-th order group-factor is defective. Then

$$p_s^* = 1 - q^{k_s} \qquad (s = 1, 2, \ldots, r) \tag{2.2.39}$$

In the initial step we test all the g_1 first order group-factors for their effects in an experiment in

$$R_1 = g_1 + 1 \text{ tests.}$$

If n_1 is the number of first order group-factors found to be defective in the initial step, then the probability function of n_1 is

$$f(n_1) = p\,(\underline{n}_1 = n\,)$$

$$= \binom{g1}{n1} p_1^{*n_1}(\,1 - p_1^{*})^{g_1 - n_1} \qquad (\,n_1 = 0,\,1,\,\ldots g_1 \qquad (2.2.40)$$

Thus

$$E(\underline{n}_1) = g_1\, p_1^{*}$$

$$= \frac{f}{k_1}(1 - q^{k_1}) \qquad (2.2.41)$$

Let $p^*_{s+1/s}$ (s = 1, 2, …, r – 1) be the probabilities that an (s+ 1)-th order group-factor is defective given that it is within a defective s-th order group-factor. Then

$$p^*_{s+1/s} = \frac{p^{*}_{s+1}}{p^{*}_{s}}$$

$$= \frac{1 - q^{k_{s+1}}}{1 - q^{k_s}} \qquad (s = 1,2,3,..,\, r\text{-}1) \qquad (2.2.42)$$

Suppose n_{s+1} is the number of (s+1)-th order group-factors found to be defective, then the probability function of $\underline{n}_{s+1}$ given $\underline{n}_s$ is

$$f(n_{s+1}/n_s) = p(\underline{n}_{s+1} = n_{s+1}/\,\underline{n}_s = n_s)$$

$$= \binom{n_s g_{s+1}}{n_{s+1}} p^*_{s+1/s}{}^{n_{s+1}} (1 - p^*_{s+1/s})^{n_s g_{s+1} - n_{s+1}}$$

$$(s = 1, 2, .., r-1) \qquad (2.2.43)$$

Thus

$$E (\underline{n}_{s+1}) = E E (\underline{n}_{s+1}/n_s)$$

$$= g_{s+1} p^*_{s+1/s} \; E (\underline{n}_s)$$

$$= \frac{f}{k_{s+1}} (1 - q^{k_{s+1}}) \qquad (2.2.44)$$

Let $P_{g_{s+1}}(j)$ (s= 1,2,...,r-1) and $P_{g_{s+1}}$ be the probability that an s-th order group-factor contains exactly j_s defective (s+1)-th order group-factors and a defective r-th order group-factor contains exactly j_r defective factors respectively. Then

$$P_{g_{s+1}}(j_s) = (1 - q^{k_s})^{-1} \binom{g_{s+1}}{j_s} p^*_{s+1}{}^{j_s} (1 - p^*_{s+1})^{g_{s+1} - j_s}$$

$$j_s = 1,2,\ldots,g_{s+1}$$

$$s = 1,2,\ldots r-1 \qquad (2.2.45)$$

and

$$P_{k_r}(j) = (1-q^{k_r})^{-1}\binom{k_r}{j_r} p^{j_r}(1-p)^{k_r-j_r}, \quad j_r = 1,2,\ldots,k_r. \tag{2.2.46}$$

Let $E_{g_{s+1}}(R_{j_s})$ denote the expected number of tests required to classify as defective or nondefective all (s+1)-th order group-factors within a defective s-th order group-factors (s=1,2,...,r-1). Further let $E_{k_r}(R_{j_r})$ denote the expected number of tests required to classify as defective or nondefective all the k_r factors within a defective r-th order group-factors. Then

$$E_{g_{s+1}}(R_{j_s}) = \frac{j_s g_s + 1}{j_s + 1} + j_s + \frac{j_s}{j_s+1} - \frac{2j_s}{g_s+1} \qquad (j_s = 1,2,..,g_{s+1}) \tag{2.2.47}$$

and

$$E_{k_r}(R_{j_r}) = \frac{j_r k_r}{j_r+1} + j_r + \frac{j_r}{j_r+1} - \frac{2j_r}{k_r}, \quad (j_r = 1,2,\ldots,k_r) \tag{2.2.48}$$

Denote by $R^0_{t_1}$ the number of tests required to classify as defective or nondefective all the g_2 second order group-factors within a first order group-factor which is known to be defective. Then

$$E(R^0_{t_1}) = \sum_{j_1=1}^{g_2} E_{g_2}(R_{j_1})\, P_{g_2}(j)$$

$$= \sum_{j_1=1}^{g_2} \left\{ \frac{j_1(g_2+1)}{j_1+1} + j_1\left(1 - \frac{2}{g_2}\right)\right\} \frac{1}{1-q^{k_1}} \binom{g_2}{j_1} p_2^{*j_1}(1-p_2^*)^{g_2-j_1}$$

$$= (1 - q^{k_1})^{-1}[(g_2+1) + g_2 p_2^* - 2p_2^* - \frac{1}{p_2^*}(1 - q_2^{*g_2+1})]$$

where $q_2^* = 1 - p_2^*$ (2.2.49)

Let R_{t_1} be the number of tests required to classify as defective or nondefective all the $n_1 g_2$ second order group-factors within the n_1 first order group-factors found to be defective. Then

$$R_{t_1} = n_1 E(R_{t_1}^0)$$

$$= \frac{n_1}{1 - q^{k_1}}[(g_2+1) + g_2 p_2^* - 2p_2^* - \frac{1}{p_2^*}(1 - q_2^{*g_2+1})] \quad (2.2.50)$$

In general if $R_{t_s}^0$ is the number of tests required to classify as defective or nondefective all the g_{s+1} $(s+1)$-th order group factors within an s-th order group-factor which is known to be defective. then

$$E(R_{t_s}^0) = \sum_{j_1=1}^{g_{s+1}} E\, g_{s+1}(R_{j_s})\, P_{g_{s+1}}(j_s)$$

$$= (1 - q^{k_s})^{-1} \sum_{j_s=1}^{g_{s+1}} \{\frac{j_s(g_{s+1}+1)}{j_s+1} + j_s(1 - \frac{2}{g_s})\} \binom{g_{s+1}}{j_s}$$

$$\times p_{s+1}^{*j_s} q_{s+1}^{*g_{s+1}-j_s}$$

$$= (1 - q^{k_s})^{-1} \left[(g_{s+1}) + g_{s+1} p^*_{s+1} - 2p^*_{s+1} - \frac{1}{p^*_{s+1}} (1 - q^*_{s+1}{}^{g_{s+1}+1})\right]$$

(2.2.51)

where $q^*_{s+1} = 1 - p^*_{s+1}$.

Let R_{t_s} be the number of tests required to classify as defective or nondefective all the $\underline{n}_S g_{S+1}$ $(s+1)$th order group factors found to be defective. Then

$$R_{t_s} = \underline{n}_S E(R^0_{t_s})$$

$$= \frac{n_s}{1 - q^{k_s}} [(g_{s+1} + 1) + g_{s+} p^*_{s+1} - 2p^*_{s+1} - \frac{1}{p^*_{s+1}} (1 - q^*_{s+1}{}^{g_{s+1}+1})]$$

(2.2.52)

Lastly let $R^0_{t_r}$ be the number of tests required to classify as defective or nondefective all the k_r factors within an r-th order group-factor found to be defective. Then

$$E(R^0_{t_r}) = \sum_{jr=1}^{k_r} Ek_r (R_{jr} | k_r) (jr)$$

$$= \sum_{j_r=1}^{k_r} \left\{ \frac{j_r k_r}{j_r + 1} + j_r + \frac{j_r}{j_r + 1} - \frac{2j_r}{k_r} \right\} \frac{1}{1 - q^{k_r}} \binom{k_r}{j_r} p^{j_r} q^{k_r - j_r}$$

$$= (1 - q^{k_r})^{-1} \left[(k_r + 1) + k_r p - 2p - \frac{1}{p} (1 - q^{k_r + 1}) \right]$$

(2.2.52)

where $q = 1 - p$

Let R_{t_r} be the number of tests required to classify as defective or nondefective all the $\underline{n}_r k_r$ factors within the $\underline{n}_r$ r-th order group-factors found to be defective at the end of the type r subsequent steps, then

$$R_{t_r} = \underline{n}_r E(R^0_{t_r})$$

$$= \frac{n_r}{1 - q^{k_r}} \left[(k_r + 1) + k_r p - 2p - \frac{1}{p} (1 - q^{k_r + 1}) \right] \qquad (2.2.53)$$

The total number of tests required to isolate all the defective factors in an r-type step-wise group screening design, is given as

(2.2.54)

$$R = R_1 + \sum_{s=1}^{r-1} R_{t_s} + R_{t_r} .$$

Therefore the expected total number of tests required in a r-type stepwise group screening design is

$$E(R) = (R_1) + \sum_{s-1}^{r-1} E(R_{t_s}) + E(R_{t_r}) \qquad (2.2.55)$$

Now

$$E(R_{t_s}) = \frac{f}{k_s}[(g_{s+1} + 1) + g_{s+1}\, p^*_{s+1} - 2p^*_{s+1} - \frac{1}{p^*_{s+1}}(1 - q^{*g_{s+1}+1})] \qquad (2.2.56)$$

And

$$E(R_{t_r}) = \frac{f}{k_r}[(k_r + 1) + k_r p - 2p - \frac{1}{p}(1 - q^{k_r+1})] \qquad (2.2.57)$$

Therefore

$$E(R) = 1 + f + fp + \frac{f}{k_1} - \frac{f}{k_r} + \frac{2fq}{k_r} - \frac{f}{k_r p}(1 - q^{k_r+1})$$

$$+ f\sum_{s=1}^{r-1}\left[\frac{2}{k_{s+1}} - \frac{1}{k_s} - \frac{q^{k_s}}{k_{s+1}} + \frac{2q^{k_s+1}}{k_{s+1}} - \frac{(1-q^{k_s+1})-1}{k_s}(1-q^{k_s+k_{s+1}})\right] \qquad (2.2.58)$$

2.3 HWANG'S GROUP TESTING PROCEDURE

1 Introduction

In 1972 Hwang proposed a group screening procedure which assumed the knowledge of an upper bound m, of the number of defectives denoted by j. This procedure is twin to a merging algorithm suggested by Hwang and Lin[2] for merging two disjoint ordered sets by making paired comparisons. The proposed method was designed to reduce the maximum number of tests. When j is known (hence the upper bound is known), this method compared favourably with Li's method. When the probability distribution of j (not necessarily binomial) is known, then if any chosen number m, is used as the 'upper bound' in the method, we can compute the probability that $j \leq m$. With at least this probability, all the defectives will identified in not more than a specified number of tests.

2 When an Upper Bound of j is Given

Let m be the given upper bound and f the population size. We call the corresponding problem of using group testing to detect all defective members in the population, the (m,f) problem.

lemma

If a group of size 2^{α}, $\alpha \geq 0$, is defective, then we can find defective number in this group by at most α tests.

Proof

The lemma is trivially true for $\alpha = 0$. Suppose it is true for all $\alpha < \alpha'$, where $\alpha' \geq 1$. We prove the lemma for $\alpha = \alpha'$ by induction.

Theorem

(a) $R(m,f) = f$ for $f \leq 2m - 2$

(b) $R(m,f) = (\alpha + 2)m + n - 1$ for $f \geq 2m - 1$, where $\ell = 2^{\alpha} m + 2^{\alpha} n + \theta$, and n, θ are uniquely determined nonnegative integers satisfying $n < m$, $\theta < 2^{\alpha}$ (ℓ and α are defined in step 1).

Proof.

Because of the first statement in step 1, part (a) is trivially true. Part (b) is true for $2m - 1 \leq f \leq 3m - 2$ since in this case α is necessarily $f - 2m + 1$ and $R(m,f) = 2m + n - 1 = f$ which checks.

For $m = 1$, and $f = 2^{\alpha} + \theta$, $0 \leq \theta < 2^{\alpha}$, then $\ell = f - m + 1 = f$. We prove $R(1,f) = (\alpha+2) - 1 = \alpha + 1 = [\log_2 f] + 1$,

where $[x]$ denotes the largest integer $\leq x$, by induction on f. For $f = 1$, we need one test to ascertain whether this single member is defective ($m = 1$ is an upper bound). Hence $R(1,1) = [\log_2 1] + 1$. Consider the $(1, f)$ problem for $f > 1$. Take a group of 2^{α} members to test. If this group is defective, then the unique defective member will be found by α more tests according to the lemma, if not, we are left with the $(1, \theta)$ problem. But $m(1,\theta) = [\log_2 \theta] + 1$ by induction and $[\log_2 \theta] < \alpha$

Hence

$$R(1,f) = \max [1 + \alpha, 1 + R(1, \theta)] = 1 + \alpha.$$

For $m \geq 2$ and $f \geq 3m-1$, we prove (b) by induction of m+n. For the general (m,n) case, assume that (b) is true for all (m' f') cases where $m'+f' < m+$ f. according to step 1 of the algorithm,

$$R(m,n) = \max[1+ R(m, f - 2^{\alpha}), 1 + \alpha + R(m-1, f-1)].$$

For $m' = m$ and $f' = f - 2^{\alpha}$

$$l' = f' - m' + 1 \quad = n - 2^{\alpha} - m + 1$$

$$= l - 2^{\alpha}$$

$$= \begin{cases} 2^{\alpha} m + 2^{\alpha}(n-1) + \theta & \text{for } n \geq 1, \theta < 2^{\alpha} \\ 2^{\alpha-1} m + 2^{\alpha-1}(m-2) + \theta & \text{for } n = 0, \theta < 2^{\alpha-1} \\ 2^{\alpha-1} m + 2^{\alpha-1}(m-1) + (\theta - 2^{\alpha-1}) & \text{for } n = 0, \theta \geq 2^{\alpha-1} \end{cases}$$

hence by induction,

$$R(m, f - 2^{\alpha}) = \begin{cases} (\alpha+2)m + (n-1) - 1 & \text{for } n \geq 1, \quad \theta < 2^{\alpha} \\ (\alpha+1)m + (m-2) - 1 & \text{for } n = 0, \quad \theta < 2^{\alpha-1} \\ (\alpha+1)m + (m-1) - 1 & \text{for } n = 0, \quad \theta \geq 2^{\alpha-1} \end{cases}$$

and

$$1 + R(m, f - 2^{\alpha}) = \begin{cases} (\alpha+2)m + n - 2 & \text{for } n = 0, \quad \theta < 2^{\alpha-1} \\ (\alpha+2)m + n - 1 & \text{otherwise} \end{cases}$$

For $m' = m-1$ and $f' = f - 1$,

$$l' = f' - m' + 1 \quad = f - m + 1 = l \qquad \text{for } n \leq m-3$$

$$= 2^{\alpha}(m-1) + 2^{\alpha}(n+1) + \theta$$

$$= 2^{\alpha+1}(m-1)+\theta \qquad \text{for } n = m-2$$

$$= 2^{\alpha+1}(m-1)+(2^{\alpha}+\theta) \qquad \text{for } n = m-1$$

Hence by induction

$$R(m-1, f-1) = \begin{cases} (\alpha+2)(m-1)+(n+1)-1 & \text{for } n \le m-3 \\ (\alpha+3)(m-1)-1 & \text{for } m-2 \le n \le m-1 \end{cases}$$

And

$$1+\alpha+R(m-1, f-1) = \begin{cases} (\alpha+2)m+n-2 & \text{for } n = m-1 \\ (\alpha+2)m+n-1 & \text{otherwise} \end{cases}$$

But for $m \ge 2$, $p = 0$ and $n = m-1$ are mutually exclusive.

Hence

$$R(m,f) = \max[1+R(m, f-2^{\alpha}),\ 1+\alpha R(m-1, f-1)]$$

$$= (\alpha+2)m+n-1$$

Hence the proof.

3.When the Probability Distribution of j is Known

Let F(j) be the probability distribution of the number of defective members in the population. Given a probability level β, we can find a number $m\beta$ such that $F(m\beta) \geq \beta$ can be chosen as close to one as desirable.

When we use this β-level upper bound $m\beta$ in the proposed method, then we see that the probability of success in detecting all defectives in the population by $R(m\beta, f)$ tests is bounded below by β. However, if the number of defectives actually detected is less than $m\beta$, then we know with full confidence that all the defectives have been detected. Otherwise we may apply the method again with a suitable choice of m' as the upper bound for the undetected defective members and repeat the same until at some stage the number of defective members identified is less than the specified upper bound.

CHAPTER 3

RELATIVE PERFORMANCE THE GROUP SREENING DESIGNS

3.1 Introduction

We shall say that a design is more efficient than another if the expected number of tests in the former design is less than or equal to that in the latter design for all 'p' $(0 < p < 1)$ with strict inequality holding true for at least one value of 'p' (the a-priori probability of a factor to be defective.

In this section, numerical values for group screening plans which minimize the expected total number of tests are given. The values of the k_I's and min E(R) have been obtained using computer search for all classes of design. We shall use the minimum expected number of tests as the basis of comparison.

We shall have two types of comparisons namely

(1) Comparison of group screening procedures which do not assume a binomial distribution for the number of defectives (j). These include Hwang's group testing procedure and Li's group testing method. In this type of comparison, the number of defective factors is given.

(2) Comparison of group screening procedures which assume a binomial distribution for the number of defectives. These include Dorfman's procedure, Multi-stage group screening designs, Step-wise group screening designs and Sterrett's procedure. When the number of defective factors (j) is known, Hwang 's procedure could also be compared with the above procedures.

3.2 COMPARISON OF HWANG'S GROUP SCREENING PROCEDURE WITH DORFMAN'S GROUP SCREENING DESIGN AND ITS EXTENSIONS

In this section, we give comparisons of Hwang's group testing procedure with Dorfman's group screening design and its extensions by Patel (1962). For this type of comparison, we assume that the probability distribution of j, the number of defectives, is binomial with parameters (f, p), where p is the probability of any one factor being defective.

For Dorfman's method, let k be the fixed group size, then the number of tests needed is

$$R = \frac{f}{k} + p\, f \qquad (3.2.1)$$

(assuming k divides f), where p is the probability of obtaining by random selection a group of k with at least one defective member. The number of defective groups has a binomial distribution with parameters $(f / k,\ 1-(1-p)^k)$

Hence the minimum expected number of tests is

$$\min E(R) = \frac{f}{k} + f[1-(1-p)^k] \qquad (3.2.2)$$

The optimal k, is determined by minimising E(R). It is easily seen that the optimal k depends on p only, but not f.

For the Hwang's method, we use the equation below for the expected number of runs

$$E(R) = \begin{cases} f, & \text{for } f \le 2j-2 \\ (\alpha+2)j + n - 1 & \text{for } f \ge 2j-1,\ n < j. \end{cases} \qquad (3.2.3)$$

For Patel's method, the number of group-factors of various orders and their sizes can be worked out from the equations below;

$$k_1 \sim \frac{1}{p^{n/n+1}} \quad , \quad k_2 \sim \frac{1}{p^{n-1/n+1}} \quad , \ldots . \quad , \quad k_n \sim \frac{1}{p^{1/n+1}} \tag{3.2.4}$$

and

$$g_1 \sim f p^{n/n+1} \quad , \quad g_2 \sim \frac{1}{p^{1/n+1}} \quad , \ldots , \quad g_n \sim \frac{1}{p^{1/n+1}} \tag{3.2.5}$$

These two equations give approximately the values that minimise $E(R_{n+1})$. Substituting in the equation

$$E(R_{n+1}) = 1 + q_1 + \sum_{i=2}^{n+1} \prod_{j=1}^{i} g_j p_{j-1} \tag{3.2.6}$$

where $p_{j-1} = (1-q^{k_{j-1}})$ is the probability that a (j-1)-th order group-factor is defective.

We get

$$\text{Min } E(R_{n+1}) = (n+1) f p^{n/n+1} + 1 \tag{3.2.7}$$

Using the formula for E(R) given by Watson (1961), we have

$$E(R) = 1 + \frac{f}{k_1} + f(1 - q^{k_1}) \tag{3.2.8}$$

for two-stage group screening design and the formula for E(R) given by Patel (1962) as

$$E(R) = 1 + \frac{f}{k_1} + \frac{f}{k_2}(1 - q^{k_1}) + f(1 - q^{k_2}) \tag{3.2.9}$$

For a three-stage group screening design and

$$E(R) = 1 + \frac{f}{k_1} + \frac{f}{k_2}(1 - q^{K_1}) + \frac{f}{k_3}(1 - q^{k_2}) + f(1 - q^{k_3}) \tag{3.2.10}$$

for a four-stage group screening design where k_i is the size of the group-factors in the i-th stage (i = 1, 2, 3).

For the Li's group testing method, if in group of experiment the proportion of defective factors is estimated as j/f, the total number of tests to screen out j defective factors out of f variables in c cycles is constant: regardless of the number of groups or experiments into which the f factors are divided and also regardless of the distribution of the j defective factors among the different groups or experiments. The total number of tests, required to screen out j defective factors out of f factors in c test cycles, is given by

$$R_c = c\,(f j^{c-1})^{1/c} \qquad (3.2.11)$$

In this comparison we take the best value of c.

Table I demonstrates the relative performance of Hwang's group testing procedure, Dorfman's group screening method and Patel's group screening designs. As shown in the table, Hwang's group testing procedure has fewer number of tests than Dorfman's procedure for $p < 0.25$. Hwang's procedure is also superior than two-stage group screening design, for the same value of p. This is because Dorfman's group screening procedure is virtually the same as the two-stage group screening design . The table also shows that three-stage group screening design has fewer number of tests than two-stage group screening design for $p \leq 0.11$ but has more number of tests than Hwang's group testing procedure for $p \leq 0.11$. Four-stage group screening design has fewer number of tests than the three-stage group screening design for $p \leq 0.035$ but has more number of tests than Hwang's group testing procedure.

It follows then, that Hwang's group testing procedure requires fewer number of tests than Dorfman's procedure and Patel's multistage group screening designs, for small values of p.

Table III shows the relative performance of Hwang's procedure with Li's method for f = 1000. From the table, it is seen that Hwang's group testing procedure has fewer number of tests Li's group testing method for any number of defective factors.This shows the superiority of Hwang's Procedure over Li's method..

From table IV we have that three-cycle design has fewer number of tests than two-cycle design for $p < 0.9$. It is also shown that four-cycle design has fewer number of tests than three-cycle design for $p < 0.35$. This shows the superiority of four-cycle group screening design over lower cycles of Li's method.

From table I and IV , the minimum expected number of tests in s-stage designs (s = 2, 3, 4) could also be compared with the c-cycle designs (c =2,3,4). From the table we have the following deductions:

(i) For $0.001 \leq p \leq 0.006$, two-stage group screening design and two-cycle design have equal number of tests. Two-stage design has fewer number of tests than two-cycle design for $p \leq 0.3$. The two-stage design has more number of tests than three-cycle design for $p \leq 0.7$. The two-stage design has more tests than four-stage design for $p \leq 0.3$.

(ii) The three-stage group screening design has fewer number of tests than two-cycle, three-cycle and four-cycle for $p \leq 0.3$.

(iii) The four-stage group screening design also has fewer number of tests than two-cycle, three-cycle and four-cycle designs for $p \leq 0.3$.

In conclusion, two-stage design requires fewer number of tests than c-cycle group screening method (c = 2,3,4) for $0.07 < p \leq 0.3$. The three-stage and four-stage designs have fewer number of tests than c-cycle group screening method (c= 2,3,4) for $p \leq 0.3$.

3.3 COMPARISON OF HWANG'S GROUP TESTING PROCEDURE WITH STERRETT'S GROUP SCREENING DESIGN AND ITS EXTENSIONS.

In this section, we compare Hwang's group testing procedure with Sterrett's group screening design and its extensions by Patel and Manene (1987) in one-type step-wise group screening design and Manene (1987).

For Sterrett's group screening procedure, the expected number of tests is given by

$$E(R) = \frac{f}{k} \sum_{j=0}^{k} \{ p_k(j)\, E_k(j) \} \qquad (3.3.1)$$

where $E_k(j)$ is defined by equation (2.2.2). $P_k(j)$ is the probability that a pool of k samples has exactly j defective factors which is given by

$$P_k(j) = (1 - q^k)^{-1} \binom{k}{j} p^j (1-p)^{k-j} \qquad j = 1, 2, \ldots, k \qquad (3.3.2)$$

For the one-type group screening design the expected total number of runs is given by

$$E(R) = 1 + fp + \frac{2fq}{k} + f - \frac{f}{kp}\{1 - q^{k+1}\} \qquad (3.3.3)$$

For small values of 'p', the expected total number of tests is given by

$$E(R) = 1 + \frac{3fp}{2} + \frac{f}{k} - \frac{2fp}{k} + \frac{fkp}{2} \quad \text{up to order } p \qquad (3.3.4)$$

The minimum expected total number of tests is given by

$$\text{Min } E(R) = 1 + f(2p)^{1/2} + \frac{5}{12} fp \quad \text{up to order } p \qquad (3.3.5)$$

For the two-type step-wise group screening design, Rotich (1988) obtained the expression for the expected number of runs as

$$E(R) = 1 + f + fp + \frac{2fq^{k_2}}{k_1} - \frac{f}{k_1}(1-q^{k_2})^{-1}(1-q^{k_1+k_2}) + \frac{f}{k_2} - \frac{fq^{k_2}}{k_2} + \frac{2fq^{k_2}}{k_2} - \frac{f}{k_2 p}(1-q^{k_2+1}) \qquad (3.3.6)$$

For the three-type step-wise group screening design he obtained the expected number of runs as

$$E(R) \quad 1 + f + fp + \frac{2fqk_2}{k_1} - \frac{f}{k_1}(1-q^{k_2})^{-1}(1-q^{k_1+k_2}) + \frac{f}{k_2} - \frac{fqk_2}{k_2} + \frac{2fqk_3}{k_2} - \frac{f}{k_2}(1-q^{k_3})^{-1}(1-q^{k_2+k_3}) + \frac{f}{k_3} - \frac{2fq}{k_3} - \frac{fqk_3}{k_3} - \frac{f}{k_3 p}(1-q^{k_3+1}) \qquad (3.3.7)$$

In general, for the r-type he obtained the expected number of runs as

$$E(R) \quad 1 + f + fp + \frac{f}{k_1} - \frac{f}{k_r} + \frac{2fq}{k_r} - \frac{f}{k_r p}(1-q^{k_r+1}) + f\sum_{s=1}^{r-1}\left[\frac{2}{k_{s+1}} - \frac{1}{k_s} - \frac{q^{k_s+1}}{k_{s+1}} + \frac{2q^{k_s+1}}{k_s} - \frac{(1-q^{k_s+1})-1}{k_s}(1-q^{k_s+k_{s+1}})\right] \qquad (3.3.8)$$

Table II and table III show that Hwang's group testing procedure has fewer number of tests than the Sterrett's group screening procedure ,one-type step-wise and two-type step-wise procedures for $p < 0.08$, $p < 0.09$ and $p < 0.08$ respectively. From the table II we see that, Hwang's group testing procedure has fewer tests than three-type and four-type group screening designs for $p \leq 0.05$ and $p \leq 0.01$ respectively.

As shown in the tables, there is no much difference between Sterrett's procedure one-type step-wise design in terms of the expected number of tests required for all p. The two-type step-wise design has fewer tests than one-type step-wise design and Sterrett's procedure for $p < 0.14$ and $p < 0.11$ respectively. For $0.14 \leq p \leq 0.3$ the one-type and two-type step-wise designs have equal number of tests. Three-type step-wise design has fewer number of tests than the one-type , two-type step-wise designs and Sterrett's procedure for $p < 0.05$. For $0.05 \leq p \leq 0.14$, three-type step-wise design has fewer number of tests than one-type step-wise design but equal number of tests as for the two-type design. For $0.14 \leq p \leq 0.3$, one-type, two-type and three-type step-wise designs have equal number of tests. Four-type step-wise procedure requires fewer number of tests than one-type, two-type and three-type step-wise designs for $p \leq 0.01$.

In conclusion, it follows that Hwang's group testing procedure require fewer tests than Sterrett's procedure and its extensions for small values of p. Four-type step-wise design incorporates three-type, two-type and one-type step-wise group screening designs.

3.5 COMPARISON OF DORFMAN'S PROCEDURE AND ITS EXTENSIONS WITH STERRETT'S PROCEDURE AND ITS EXTENSIONS.

The minimum expected number of tests in various types of step-wise designs could also be compared with the minimum expected number of tests in a two-stage, three-stage and four-stage group screening designs and in general in an s-stage group screening design ($s \geq 2$).

From table I, II and III, we have the following inferences:

(i) Sterrett's results are a little better than Dorfman's but not much for all p.

(ii) One-type stepwise group screening design requires fewer number of te3sts than the corresponding two-stage design for $p \leq 0.3$. For $0.035 \leq p \leq 0.3$, one-type step-wise design requires fewer tests than the three-stage and the four-stage group screening designs.

(iii) The two-type step-wise design has fewer tests than two-stage, three-stage and four-stage group screening designs for $p \leq 0.3$.

(iv) Three-type step-wise design also has fewer tests than two-stage, three-stage and four-stage for $p \leq 0.3$.

(iv) Four-type step-wise design has fewer number of tests than two-stage, three-stage and four-stage designs for $p \leq 0.3$.

In conclusion, one-type step-wise design requires fewer tests than the s-stage group screening design (s=2,3,4) for $0.035 \leq p \leq 0.3$ whereas two-type, three-type and four-type step-wise designs require fewer tests than s-stage group screening designs (s=2,3,4) for $p \leq 0.3$

From tables II, III and IV, we have the following inferences:

(i) Li's group testing method has fewer number of tests than Sterrett's method for $p \leq 0.02$.

(ii) The one-type step-wise design requires fewer tests than two-cycle design for $p \leq 0.3$. For $P < 0.5$ one-type step-wise design has more tests than three-cycle design. For $p < 0.2$ one-type stepwise design requires more number of tests than four-cycle design.

(iii) The two-type step-wise design has fewer number of tests than the two-cycle,

three-cycle and four cycle group screening methods for $p \leq 0.3$

(iv) The three-type step-wise design also has fewer number of tests than two-cycle, three-cycle and four-cycle group screening methods for $p \leq 0.3$.

(v) The four-type step-wise design requires fewer tests than the two-cycle, three-cycle and four cycle designs for $p \leq 0.3$.

In conclusion therefore, we note that the one-type step-wise design requires fewer tests than the c-cycle group screening method ($c = 2,3,4$) for $0.2 < p \leq 0.3$. The two-type, three-type and four-type step-wise designs require fewer tests than c-cycle group screening method ($c = 2,3,4$) for $p \leq 0.3$).

Table I

Relative performance of Hwang's procedure with Dorfman's procedure and its extensions, for f = 1000 and specified values of p.

		Hwang's Design	Dorfman's Design		Two-stage Design		Three-stage Design			Four-stage Design			
P	j	E(R)	k	E(R)	k_1	E(R)	k_1	k_2	E(R)	k_1	k_2	k_3	E(R)
0.001	1	10	32	63	32	64	68	8	32	178	32	6	23
0.002	2	19	23	89	23	90	52	7	48	106	22	5	39
0.003	3	30	19	108	19	109	42	6	62	88	18	4	50
0.004	4	38	16	125	16	126	38	6	75	76	17	4	62
0.005	5	46	15	139	15	140	35	6	86	67	16	4	73
0.006	6	55	13	152	13	153	28	5	97	60	15	4	83
0.008	8	66	12	175	12	176	25	5	117	44	11	3	103
0.01	10	84	11	196	11	197	17	4	139	38	10	3	121
0.02	20	148	8	274	8	275	12	3	215	28	9	3	199
0.03	30	224	6	334	6	334	11	3	274	23	8	3	287
0.035	35	227	6	359	6	360	11	3	301	22	8	3	298
0.04	40	259	6	384	6	385	10	3	328				
0.05	50	298	5	426	5	427	10	3	377				
0.06	60	341	5	466	5	467	9	3	424				
0.07	70	398	4	502	4	503	9	3	468				
0.08	80	455	4	534	4	535	8	3	510				
0.09	90	517	4	564	4	565	8	3	549				
0.1	100	574	4	594	4	595	8	3	589				
0.11	110	605	4	623	4	624	8	3	623				
0.14	140	671	3	697	3	698							
0.20	200	815	3	821	3	822							
0.25	250	919	3	912	3	913							
0.3	300	1139	3	990	3	991							

Table II

Relative performance of Hwang's procedure with Step-wise procedure for f = 1000 and specified values of p and j.

		Hwang's Design	One-type Step-wise Design		Two-type Step-wise Design			Three-type Step-wise Design				Four-type Step-wise Design				
P	j	E(R)	k_1	E(R)	k_1	k_2	E(R)	k_1	k_2	k_3	E(R)	k_1	k_2	k_3	k_4	E(R)
0.001	1	10	45	47	159	13	22	299	45	7	17	437	96	21	5	16
0.003	3	30	26	81	82	9	47	131	26	5	40	182	49	13	4	38
0.005	5	46	21	106	56	7	67	92	18	4	59	121	36	11	3	57
0.007	7	65	17	127	43	7	84	69	17	4	76	92	30	10	3	74
0.01	10	84	15	153	37	6	107	58	15	4	100	69	24	8	3	98
0.02	20	148	11	221	24	5	174	31	9	3	167					
0.03	30	224	9	274	17	4	229	23	8	3	225					
0.04	40	259	8	320	15	4	279	18	7	3	277					
0.05	50	298	7	361	12	3	325	12	5	3	325					
0.06	60	341	6	398	10	3	366									
0.07	70	398	6	433	9	3	405									
0.08	80	455	5	465	9	3	442									
0.09	90	517	5	495	8	3	477									
0.1	100	574	5	524	7	3	510									
0.11	110	605	4	552	7	3	542									
0.14	140	671	4	625	4	2	625									
0.25	250	919	3	840												
0.3	300	1139	2	906												

Table III

Relative performance of Hwang's, Dorfman's, Sterrett's and Li'sprocedure for f = 1000 and specified values of p and j

		Hwang's Procedure	Dorfman's procedure		Sterrett's procedure		Li's Method
j	p	E(R)	k	E(R)	k	E(R)	E(R) (for best C)
1	0.001	10	32	63	47	40	19
3	0.003	30	19	89	30	80	47
5	0.005	46	15	125	22	100	72
7	0.007	65	14	165	20	120	94
10	0.01	84	11	196	16	140	125
20	0.02	148	8	274	11	220	212
30	0.03	224	6	334	9	270	288
40	0.04	259	6	384	8	320	351
50	0.05	298	5	426	7	350	407
60	0.06	341	5	466	7	390	460
70	0.07	398	4	502	6	420	510
80	0.08	455	4	534	6	450	557
90	0.09	517	4	564	5	480	600
100	0.1	574	4	594	5	510	632
110	0.11	605	4	623	4	540	663
140	0.14	671	3	697	4	610	748
250	0.25	919	3	912	3	840	1000
300	0.30	1139	3	990	2	900	

Table IV

Relative performance of the C-cycle group screening method for f = 1000 and specified values of j.

j	Two-cycle	Three-cycle	Four-cycle
1	63	30	23
2	89	48	38
3	110	62	51
4	127	76	64
5	141	88	75
6	155	99	86
7	167	110	97
8	179	120	107
10	200	139	127
20	283	221	213
30	346	290	288
35	374	321	324
40	400	351	
50	447	407	
60	490	460	
70	529	510	
80	566	557	
90	600	602	
100	633		
110	663		
140	748		
250	1000		

CONCLUDING REMARKS

In this project, various group screening designs have been described. We have restricted ourselves to Dorfman's procedure and its extensions, Sterrett's procedure and its extensions and Hwang's procedure. The group screening designs then have been compared. We have shown that Hwang's method is more superior than Dorfman's procedure for $p < 0.25$. It is also superior than two-stage, three-stage and four-stage group screening designs for $p < 0.25$, $p < 0.11$ and $p < 0.04$ respectively. We have also shown that four-stage group screening design incorporates three-stage and two-stage group screening designs. From table, (III) it is shown that Hwang's procedure requires fewer tests than Li's group testing method for all the prevalence rates. From table (I) and table (IV) we have concluded that two-stage design requires fewer number of tests than the c-cycle design ($c = 2,3,4$) for $0.07 < p \leq 0.3$. The three-stage and four-stage designs have fewer number of tests than the c-cycle design ($c = 2, 3, 4$) for $p \leq 0.3$.

From table (II), we have deduced that, Hwang's group testing method requires fewer tests than Sterrett's procedure, one-type, two-type, three-type and four-type stepwise group screening designs for $p < 0.08$, $p < 0.09$, $p < 0.08$, $p < 0.05$ and $p < 0.02$ respectively. Four-type step-wise design incorporates three-type, two-type and one-type step-wise group screening designs.

From table (I) and table (II), we have shown that one-type step-wise group screening design requires fewer tests than the corresponding two-stage group screening design, for all the prevalence rates of defective members. Similarly two-type, three-type and four-type step-wise group screening designs require fewer tests than their corresponding three-stage and four-stage group screening designs for all prevalence rates of defective members.

From table (II), table (III) and table (IV) we have that Li's group testing method has fewer tests than Sterrett's procedure for $p \leq 0.02$. Also we have shown

That the one-type step-wise design requires fewer tests than the C-cycle design (c= 2, 3, 4) for $0.2 < p \leq 0.3$. Two-type, three-type, four-type step-wise group screening designs have fewer tests than the c-cycle design (c= 2, 3, 4) for $P \leq 0.3$.

Group screening designs can be used in industries in sorting out defective items from non-defective ones with substantial savings in costs of inspection and time. For example in a chemical industry, the designs have been used to select the best catalyst for a chemical reaction from a large number of compounds which are possible candidates.

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