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MATHEMATICAL MODELS IN PORTFOLIO SELECTION: THE CASE OF THE EMERGING NAIROBI STOCK EXCHANGE

By

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DECLARATION

DECLARATION BY THE CANDIDATE

This project is my original work and has never been presented for any postgraduate course in any other University to the best of my knowledge.

Signature: ....

ODUOR DAVID OCHIENG **Date**...26|11|2010...

DECLARATION BY THE SUPERVISOR

This project has been submitted for examination with my approval as a university supervisor.

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ABSTRACT

A GREAT deal of theoretical and empirical work has been carried out in recent years, on the use of mathematical models as an aid to the selection of investment portfolios containing equities. Some of the models have been advanced include Harry Markowitz (1952) portfolio theory: capital asset pricing model (CAPM) whose critics have argued its limitation in application though thought to be the best model available and Stephen Ross (1976) Arbitrage pricing theory (APT).

The arbitrage pricing model attempts to capture the limitations of CAPM and recognizes the sensitivities to a number of factors that influences their return.

After a discussion of the background to portfolio selection models, the paper discusses work done on the manner in which share returns move over time. Next the paper examines the principles behind various portfolio selection models, discussing in some detail the work of Markowitz, Sharpe and developments by Stephen Ross. Employing the data from the Nairobi Stock Exchange between Jan 2004 and Dec 2008 under the light of the methodology proposed, the research investigates the relationship between the stock returns and measured risk for each model.

This study therefore attempts to engage elements of Modern Portfolio Theory, Capital Asset Pricing Model and Arbitrage Pricing Theory to discover how each can be used in the process of portfolio selection.

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Contents

CHAPTER ONE: INTRODUCTION

1.1) Background

Today stock market is an important pillar of each economy. In between, portfolio selection is concerned with an individual who is trying to allocate one's wealth among alternative securities such that the investment goal can be achieved. Having many companies to select, portfolio selection becomes more and more sophisticated.

Selecting right security and right mix of security for the portfolio is considered as one of the most important tasks for the investor. I addition there are numerous models and techniques which help the investor in selecting a favorable portfolio. However each has its own advantages and disadvantages. This study explains Modern Portfolio Theory, Capital Asset Pricing Model and Arbitrage Pricing Theory. it involves introduction to the basis of risk that one must understand when combining different assets.

Once this risk and other terms such as return, covariance's, and coefficient of determination plus co-relation coefficient are understood we increase our understanding of not only why you should diversify your portfolio but also how you should diversify.

Much of modern theory of portfolio selection originates from the pioneer work of Markowitz who in 1952 first provided a reasonable analytic frame work for an investor to choose between a small number of efficiently diversified portfolios. The return anticipated from a portfolio is not the only criterion for selection, as the uncertainty of achieving this return must also be considered. Markowitz took the investor's belief about individual shares and suggested the use of mathematical technique of quadratic programming to compute those portfolios having highest possible return on each different level of uncertainty. The actual method put forward by Markowitz was not, however a very practical one. It required a large amount of information on the investor's belief about the relationship between different shares analyzed and, unless there are a few of these, the calculations required are extensive. The amount of computation involved in applying the Markowitz approach was later vastly reduced by the simplified single index models of Sharpe, published in 1963 and 1967, where the difficulties are largely overcome by relating the price changes of each share to an index of the market rather than to all other shares.

The simplifications made the approach relatively cheap and easy to implement, and with them a new era in portfolio selection seemed to have arrived. However the Arbitrage pricing theory is based on the idea that an asset's returns can be predicted using the relationship between that same asset and many common risk factors. Created in 1976 by Stephen Ross, this theory predicts a relationship between the returns of a portfolio and the returns of a single asset through a linear combination of many independent macro-economic variables. It is often viewed as an alternative to the capital asset pricing model (CAPM), since the APT has more flexible assumption requirements. Whereas the CAPM formula requires the market's expected return, APT uses the risky asset's expected return and the risk premium of a number of macro-economic factors.

1.2) Statement of problem

Portfolio selection is concerned with the problem of how to generate a portfolio to give the maximum possible return whilst, at the same time, minimizing the risk of poor performance. The return from either a single equity share or from a whole portfolio will always be uncertain and, subject to obtaining a satisfactory expected return, this uncertainty needs to be made as small as possible. Only in exceptional circumstances does it make sense to prefer an increased amount of uncertainty. Unfortunately, there is no straightforward way of assessing directly the future uncertainty of the return from a holding, but empirical evidence suggests that there is enough stability in the measures of historical variation (such as volatility) for these to be used as reasonable indicators of uncertainty. This uncertainty of return is usually equated with an estimate of the variation of future returns, measured as a standard deviation (SD) in the case of Markowitz model and beta in the case of CAPM and factor sensitivity for APT. Proper diversification enables this uncertainty to be varied within limits for any given level of expected return, and a portfolio with the lowest possible uncertainty for its expected return is said to be efficient.

1.3) Aims and Objectives

The aim of this study is to capture the essence of portfolio selection based on Kenya's stock market that is the NSE (Nairobi Stock Exchange). The aim constituted of four objectives:

- To better understand how project portfolio selection works in the academic and practical field of portfolio selection.
- Help an investor identify, design an optimal portfolio with numerous possibilities available using data from the NSE.
- To propose a combination of securities best fits an investor, emphasis the importance of diversification and conclude by making recommendations based on the study findings on the theory and practice of the models.
- Define particular risks involved in the investment of on securities and their particular influence on investment decision

1.4) Significance of the study

Though financial and capital markets have registered a substantial growth in East Africa over the past years still not much information is available for concrete investment decision. The study thus significantly seeks to highlight which combinations would provide an optimal portfolio. Therefore the study findings of this research will:

- Enable investors and investment practitioners make informed based on the risks involved and historical trends of equity returns.
- Use the information of the study to improve on their portfolio selection strategies.

CHAPTER TWO: LITERATURE REVIEW

2.1) MODERN PORTFOLIO THEORY

2.1.1) REVIEW OF THEORETICAL LITERATURE

The foundation of modern portfolio theory (MPT) was introduced by Harry Markowitz in 1952. Modern portfolio theory (commonly referred as mean variance analysis) established a whole new terminology which became a norm among investment managers. (Gupta, Francis Markowitz, Fabozzi, Frank. 2002). The goal of MPT is to choose a collection, or *portfolio*, of assets that holds a lower collective risk for a given expected return than any individual asset. This is possible because the prices of different assets do not move exactly the same. This means that an asset should not be chosen based only on its own merits, but rather based on how it performs and changes in value relative to every other asset in the portfolio. In MPT, the returns of individual assets are assumed to be normally distributed random variables. The return of the entire portfolio, containing n assets, is a linear combination of the returns of the individual assets

$$R_p = w_1R_1 + w_2R_2 + \dots\dots + w_nR_n = \sum_{i=1}^{n} w_iR_i$$

Where w_i is the relative amount invested in asset i and R_i the expected return of that asset. The return of the portfolio is also a random variable (Montgomery & Runger, 2006). From basic probability theory, the expected value and standard deviation of the portfolio return R_p are:

$$E(R_p) = \sum_{i=1}^{n} W_iE(R_i)$$

And

$$\sigma_p = \sqrt{w_1^2Var(R_1) + w_2^2VarR_2 + \dots\dots + w_n^2VarR_n + 2\sum_{i<j}\sum w_iw_jCov(R_iR_j)}$$

Another assumption in MPT is that the portfolio's standard deviation, σP, is an adequate measure of risk. In reality, there are many different kinds of risk, for example credit risk, market risk and operational risk, and some argue that the standard deviation does not fully capture all types of risk (Morien, 2005). Note that the individual asset returns are not assumed to be independent, which is why $2\sum_{i<j}\sum w_i w_j Cov(R_i R_j)$ appears in the formula for σP.

The goal of MPT is to maximize the expected portfolio return $E(R_p)$ for a given level of risk (σP), or to minimize risk (σP) for a given level of return $E(R_p)$. Here, another assumption is introduced; namely that investors are rational and risk averse. This means that investors consider expected return a desirable thing and variance of return, or risk, an undesirable thing (Markowitz, 1952).

2.1.2) CRITICAL ANALYSIS OF THE REVIEW

Since the introduction of MPT, in particular during recessions, investors and academics have increasingly started to question and criticize the theory.

One point of criticism lies in the use of historical data when estimating returns, standard deviations and correlations. Although MPT is not concerned with the process of estimating these variables, in practice, they usually are estimated by quantitatively analyzing historical data (Fabozzio, et al., 2002). Another issue when using historical data is that it might lack information about highly improbable but fatal future events. This point of criticism is strongly advocated by Nassim Nicholas Taleb, author of *The Black Swan*. He points out that quantitative methods are unable to predict and protect investors from these events (Webb, 2008).

Another point frequently criticized is the assumption that the returns follow a normal distribution. If this was true, the actual return would fall one standard deviation or more below the expected return once every six years. Similarly, the actual return would fall two standard deviations or more below the expected once every 44 years (Jensen, 2007). This stands in stark contrast to reality. In recent years, actual returns have fallen six, seven and even eight standard deviations below the expected, something that would be virtually impossible if the returns were normally distributed.

A third point of criticism lies in the assumption that investors are rational and risk averse. Critics point out that investors are emotionally driven (Maehl, 2008). This can lead to investors making irrational financial decisions based on rumors and hunches (Morien, 2005). In other situations investors can be driven be psychological aspects. Psychologists have proven that a loss leaves deeper impressions than an equivalent gain does (Maehl, 2008). Some investors will therefore fail to sell an asset to avoid realizing a loss.

The assumption that the correlation between any two assets is static over time is also criticized. The correlation depends on the underlying assets and how they relate to each other. These relationships are dynamic and major events such as general market crashes can substantially change the correlations between assets. Empirical studies also show that during financial crisis assets tends to become positively correlated, moving down at the same time. This means that MPT fails to protect from the risks associated with financial crisis (Chesnay and Jondeau, 2001).

Among other commonly criticized assumptions the one that there are no transaction costs in buying and selling securities is well known and understandably incorrect. In reality an investor pays brokerage, taxes and other transactional fees. This leads to a different composition of the optimal portfolio than the one given by MPT (Mantegna and Stanley, 2000).

2.2) CAPITAL ASSET PRICING MODEL

2.2.1) REVIEW OF THEORETICAL LITERATURE

No matter how much we diversify our investments, it's impossible to get rid of all the risk. As investors, we deserve a rate of return that compensates us for taking on risk. The capital asset pricing model (CAPM) helps us to calculate investment risk and what return on investment we should expect. The model was the work of financial economist (and, later, Nobel laureate in economics) William Sharpe, set out in his 1970 book "Portfolio Theory And Capital Markets". His model starts with the idea that individual investment contains two types of risk:

Systematic Risk - These are market risks that cannot be diversified away. Examples of systematic risks include interest rates and recession

Unsystematic Risk - This risk is specific to individual stocks and can be diversified away as the investor increases the number of stocks in his or her portfolio. In more technical terms, it represents the component of a stock's return that is not correlated with general market moves.

The trouble is that diversification still doesn't solve the problem of systematic risk; even a portfolio of all the shares in the stock market can't eliminate that risk. Therefore, when calculating a deserved return, systematic risk is what plagues investors most. CAPM, therefore, evolved as a way to measure this systematic risk. Sharpe found that the return on an individual stock, or a portfolio of stocks, should equal its cost of capital. The standard formula remains the CAPM, which describes the relationship between risk and expected return.

$$E(R_i) = R_f + \beta_i\{E(R_m) - R_f\}$$

Where:

- $E(R_i)$ is the expected return on the capital asset
- R_f the risk free rate of interest
- β_i the sensitivity of the expected excess returns to the expected market returns
- $E(R_m)$ Expected market return
- $\{E(R_m) - R_f\}$ -Equity market premium

CAPM's starting point is the risk-free rate. To this is added a premium that equity investors demand to compensate them for the extra risk they accept. This equity market premium consists of the expected return from the market as a whole less the risk-free rate of return. The equity risk premium is multiplied by a coefficient that Sharpe called "beta". According to CAPM, beta is the only relevant measure of a stock's risk. It measures a stock's relative volatility that is, it shows how much the price of a particular stock jumps up and down compared with how much the stock market as a whole jumps up and down.

Beta is found by statistical analysis of individual, daily share price returns, in comparison with the market's daily returns over precisely the same period.

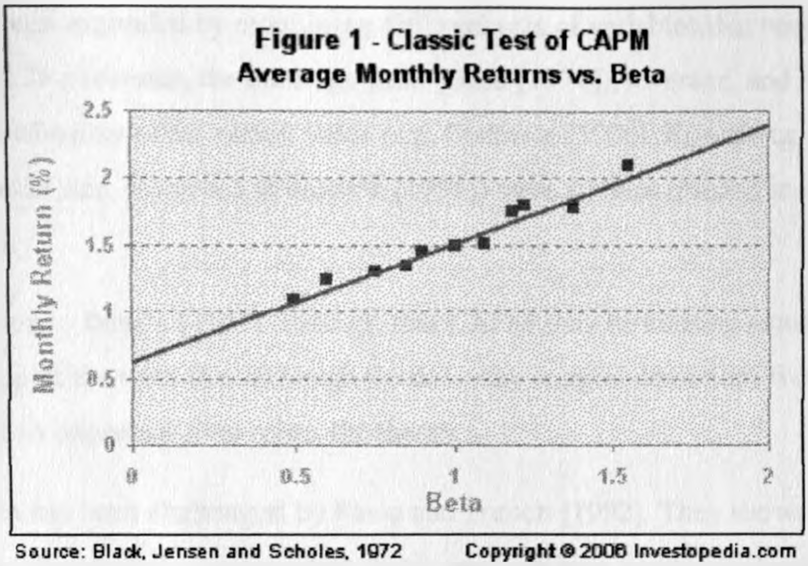
What this shows is that a riskier investment should earn a premium over the risk-free rate - the amount over the risk-free rate is calculated by the equity market premium multiplied by its beta. In other words, it's possible, by knowing the individual parts of the CAPM, to gauge whether or not the current price of a stock is consistent with its likely return. The theory says that the only reason an investor should earn more, on average, by investing in one stock rather than another is that one stock is riskier.

2.2.2) REVIEW OF EMPIRICAL AND CRITICAL LITERATURE

Since its introduction in early 1960s, CAPM has been one of the most challenging topics in financial economics. The theory itself has been criticized for more than 30 years and has created a great academic debate about its usefulness and validity. In general, the empirical testing of CAPM has two broad purposes. (i) to test whether or not the theories should be rejected (ii) to provide information that can aid financial decisions. To accomplish (i) tests are conducted which could potentially at least reject the model. The model passes the test if it is not possible to reject the hypothesis that it is true. Methods of statistical analysis need to be applied in order to draw reliable conclusions on whether the model is supported by the data. To accomplish (ii) the empirical work uses the theory as a vehicle for organizing and interpreting the data without seeking ways of rejecting the theory. This kind of approach is found in the area of portfolio decision-making, in particular with regards to the selection of assets to the bought or sold.

Classic support of the theory

In its simple form, the CAPM predicts that the expected return on an asset above the risk-free rate is linearly related to the non-diversifiable risk, which is measured by the asset's beta. In their classic 1972 study titled "The Capital Asset Pricing Model: Some Empirical Tests", financial economists Fischer Black, Michael C. Jensen and Myron Scholes confirmed a linear relationship between the financial returns of stock portfolios and their betas.



They studied the price movements of the stocks on the New York Stock Exchange between 1931 and 1965.

Another classic empirical study that supports the theory is that of Fama and McBeth [1973]; they examined whether there is a positive linear relation between average returns and beta. Moreover, the authors investigated whether the squared value of beta and the volatility of asset returns can explain the residual variation in average returns across assets that are not explained by beta alone.

Challenges to the validity of the theory

In the early 1980s several studies suggested that there were deviations from the linear CAPM riskreturn trade-off due to other variables that affect this tradeoff. The purpose of the above studies was to find the components that CAPM was missing in explaining the risk-return trade-off and to identify the variables that created those deviations.

Banz [1981] tested the CAPM by checking whether the size of firms can explain the residual variation in average returns across assets that remain unexplained by the CAPM's beta. He challenged the CAPM by demonstrating that firm size does explain the cross sectional-variation in average returns on a particular collection of assets better than beta. The author concluded that the average returns on stocks of small firms (those with low market values of equity) were higher than the average returns on stocks of large firms (those with high market values of equity). This finding has become known as the size effect.

The research has been expanded by examining different sets of variables that might affect the riskreturn tradeoff. In particular, the earnings yield (Basu [1977]), leverage, and the ratio of a firm's book value of equity to its market value (e.g. Stattman [1980], Rosenberg, Reid and Lanstein [1983] and Chan, Hamao, Lakonishok [1991]) have all been utilized in testing the validity of CAPM.

The general reaction to Banz's [1981] findings, that CAPM may be missing some aspects of reality, was to support the view that although the data may suggest deviations from CAPM, these deviations are not so important as to reject the theory.

However, this idea has been challenged by Fama and French [1992]. They showed that Banz's findings might be economically so important that it raises serious questions about the validity of the CAPM. Fama and French [1992] used the same procedure as Fama and McBeth [1973] but arrived at very different conclusions. Fama and McBeth find a positive relation between return and risk while Fama and French find no relation at all.

The academic debate continues

The Fama and French [1992] study has itself been criticized. In general the studies responding to the Fama and French challenge by and large take a closer look at the data used in the study. Kothari, Shaken and Sloan [1995] argue that Fama and French's [1992] findings depend essentially on how the statistical findings are interpreted.

Amihudm, Christensen and Mendelson [1992] and Black [1993] support the view that the data are too noisy to invalidate the CAPM. In fact, they show that when a more efficient statistical method is used, the estimated relation between average return and beta is positive and significant. Black [1993] suggests that the size effect noted by Banz [1981] could simply be a sample period effect i.e. the size effect is observed in some periods and not in others.

Despite the above criticisms, the general reaction to the Fama and French [1992] findings has been to focus on alternative asset pricing models. Jagannathan and Wang [1993] argue that this may not be necessary. Instead they show that the lack of empirical support for the CAPM may be due to the inappropriateness of basic assumptions made to facilitate the empirical analysis. For example, most empirical tests of the CAPM assume that the return on broad stock market indices is a good proxy for the return on the market portfolio of all assets in the economy. However, these types of market indexes do not capture all assets in the economy such as human capital.

Other empirical evidence on stock returns is based on the argument that the volatility of stock returns is constantly changing. When one considers a time-varying return distribution, one must refer to the conditional mean, variance, and covariance that change depending on currently available information. In contrast, the usual estimates of return, variance, and average squared deviations over a sample period, provide an unconditional estimate because they treat variance as constant over time. The most widely used model to estimate the conditional (hence time-varying) variance of stocks and stock index returns is the generalized autoregressive conditional heteroscedacity (GARCH) model pioneered by Robert.F.Engle.

All the models above aim to improve the empirical testing of CAPM. There have also been numerous modifications to the models and whether the earliest or the subsequent alternative models validate or not the CAPM is yet to be determined.

Although it is difficult to predict from beta how individual stocks might react to particular movements, investors can probably safely deduce that a portfolio of high-beta stocks will move more than the market in either direction, or a portfolio of low-beta stocks will move less than the market. The capital asset pricing model is by no means a perfect theory. But the spirit of CAPM is correct. It provides a usable measure of risk that helps investors determine what return they deserve for putting their money at risk.

2.3) ARBITRAGE PRICING THEORY

2.3.1) REVIEW OF THEORETICAL LITERATURE

The major criticism of the CAPM is that it uses only a single factor in determining the return of a portfolio, namely the beta of the portfolio. In other words, the non-diversifiable risk of the portfolio (in relation to the market risk) is the sole determinant of its return. No other factors will have any effect on the portfolio's return. To address this criticism of the CAPM, a new model has been developed based on the arbitrage pricing theory (APT). The theory was initiated by the economist **Stephen Ross** in 1976.

Similar to the CAPM, the APT assumes that there is a relationship between the risk and return of a portfolio. However, compared to the CAPM, the APT has fewer assumptions. The following assumptions are required for the CAPM but not for the APT:

(a) A single-period investment horizon

(b) Borrowing and lending at the risk-free rate

(c) Investors are mean-variance optimizer

The APT is based on the concept of arbitrage (or law of one price), which states that any two identical investments cannot be sold at a different price. In other words, the theory states that market forces will adjust to eliminate any arbitrage opportunities, where a zero investment portfolio can be created to yield a risk-free profit. Unlike the CAPM, the APT does not assume that the market risk is the only factor that influences the return of a portfolio. The APT recognizes that several other factors (or risks) can influence the return of a portfolio.

The APT preserves the linear relationship between risk and return of the CAPM but abandons the single measure of risk by the beta of the portfolio. The APT model is a multiple factor model, which uses factors such as the inflation rate, the growth rate of the economy, the slope of the yield curve, etc. in addition to the beta of the portfolio in determining the return of the portfolio. Just as in the case with the CAPM, the APT can also be modified to determine the return of an individual investment. The formula of the APT can be presented as follows:

$$E(r_i) = r_{rf} + \beta_1\left[E(r_1) - r_{rf}\right] + \beta_2\left[E(r_2) - r_{rf}\right] + \ldots + \beta_n\left[E(r_n) - r_{rf}\right]$$

Where 1, 2, ..., n represent the different factors that have impact over an investment's return.

2.3.2) REVIEW OF EMPIRICAL AND CRITICAL LITERATURE

The Arbitrage Pricing Theory (APT) of Ross (1976) provides a theoretical framework to determine the expected returns on stocks, but it does not specify the number of factors nor their identity. The problem with the APT is that the factors are not well-specified ex-ante. Some research had been conducted to determine the appropriate factors that should be included in the model. However, there is no consensus on what the factors should be. Hence, the implementation of this model follows two avenues: factors can be extracted by means of statistical procedures, such as factor analysis or principal component analysis, or be pre-specified using mainly macro-economic variables.

Statistical APT

The first test of the APT is conducted by Gehr (1975) who applies factor analysis to U.S. stock returns. This approach is further developed by Roll and Ross (1980) who report a five factor structure of which two are priced after cross-sectional testing. In a closely related paper, Chen (1983) assumes *a priori* a five-factor structure and finds that the factors change over time.

Factor analysis has been criticized for many reasons: the factors are not selected in the same order between two different samples, their sign is not reliable and they have scaling problems (Elton and Gruber, 1995). Additional problems occur when implementing the APT using factor analysis. The number of factors extracted and priced increases with the number of stocks in the sample (Dhrymes *et al.*, 1984) and the length of the time series (Dhrymes *et al.*, 1985). Further, the estimates of the risk premia are sensitive to seasonality (Cho and Taylor, 1987; Gültekin and Gültekin, 1987) and to the choice of the criteria used to construct portfolios (Lehman and Modest, 1988). They also suffer from the standard error-invariables problem.

To address these criticisms, Chamberlain and Rothschild (1983) develop an alternative methodology: asymptotic principal component analysis. However, this technique also has several drawbacks: the number of factors increases with the number of stocks included in the analysis (Trzcinka, 1986), this procedure overestimates the number of factors (Brown, 1989) and the estimates are biased unless a very large number of assets are considered (Grinblatt and Titman, 1985). Connor and Korajczyk (1986, 1988) propose an alternative procedure that yields more robust estimates, but also requires a very large number of assets. Formal comparisons of factor analysis and principal component analysis are provided by Shukla and Trzcinka (1990) and Huang and Jo (1995). They do not find a clear dominance of either technique.

Macro-economic APT

Chen (1983) is the first author to suggest giving an economic interpretation to statistical factors. The idea is that firms' expected cash flows and discount rates, and hence expected returns, are sensitive to various macro-economic influences. In a widely quoted paper, Chen *et al.* (1986) use a six-factor model consisting of market index returns, changes in expected inflation, unexpected inflation, industrial production, the risk premium and the term structure premium. They find that the last three variables are significant determinants of U.S. stock returns. Chan *et al.* (1985) show that the size effect no longer exists in that model because is captured by the risk premium.

Using an alternative technique based on the generalized method of moments, Zhou (1999) confirms that four out of the six macro-economic variables used by Chen *et al.* (1986) are relevant to explain U.S. stock returns. Other authors estimate the APT equilibrium relationship using non-linear seemingly unrelated regressions. They find that other variable, such as real final sales, the budget deficit and nonfarm employment, are also important in explaining stock returns.

As is the case for the statistical implementations, the macroeconomic models also have some important drawbacks. The factor structure is not robust to the portfolio formation criteria (Clare and Thomas, 1994), it changes over time (Chen *et al.*, 1986) and it suffers from the standard error-in-variables problem.

Comparison between the statistical and the macro-economic APT

Determining which model provides the best description of stock returns is a crucial question. Given the variety of methods that have been used in the literature, it is difficult to compare the results of the various studies and hence no clear-cut conclusion about the superiority of one model over the other can be drawn. Based on different samples, Chen and Jordan (1993) find contradictory results: in-sample tests prove the net superiority of a statistical model extracted from a factor analysis over the classical macro-economic relation, whereas out-of-sample results are mitigated and slightly in favour of the macro-economic model. The superiority of the statistical APT over the macro-economic APT in explaining U.S. stock returns is confirmed by Connor (1995) and Chan *et al.* (1998)3. On the other hand, Shafiqur *et al.* (1998) and Groenewold and Fraser (1997) provide evidence in favour of the macro-economic APT.

CHAPTER THREE: METHODOLOGY

3.1) Sample selection Nairobi Stock Exchange is Africa's fourth largest stock exchange in terms of trading volumes, and fifth in terms of market capitalization as a percentage of GDP. The Exchange was established in 1954.The NSE, like many other emerging markets, suffers from the lack of liquidity in the market (averaging 4% in 1996).

This study covers the period from January 2004 to December 2008. This time period was chosen because it is characterized by intense return volatility with historically high and low returns for the Nairobi stock market.

For the purpose of the study, 10 stocks were selected from the pool of securities. The selection was made on the basis of the trading volume and excludes stocks that were traded irregularly or had small trading volumes. The study utilizes monthly stock returns and yearly stock returns of the 10 securities. Furthermore, the 91 day Treasury bill is used as the proxy for the risk-free asset. The sample of five stocks is analyzed using the three models MPT, CAPM (factor model) and APT (2-factor analysis).

3.2) Sample size

CATEGORY	POPULATION	SAMPLE	COMPANY
1.AGRICULTURAL	3	2	Sasini Ltd and Kakuzi ord
2.COMMERCIAL AND SERVICES	12	2	Kenya Airways Ltd and Nation Media
3.FINANCE AND INVESTMENT	15	2	Kenya Commercial bank and Barclays bank
4.INDUSTRIAL AND ALLIED	17	2	Mumias sugar Co ltd and East African Breweries
5. ALTERNATIVE INVESTMENT MARKET SEGMENT	7	2	Express Ltd and Eaagads ltd
TOTAL	54	10	

3.3) Markowitz Portfolio Theory- MPT

We will consider the model developed by Markowitz and his work on mean-variance analysis. He states that the expected return (mean) and variance of returns of a portfolio are the whole criteria for portfolio selection. These two parameters can be used as a possible hypothesis about actual behaviour and a maxim for how investors ought to act.

It is essential to understand the intimates of Markowitz model. It is not all about offering a good model for investing in high return assets. It might be interesting to know that whole the model is based on an economic fact, "Expected Utility". In economic term the concept of utility is based on the fact that different consumers have different desires and they can be satisfied in different ways. In behavioural finance we can explain it so; Investors are seeking to maximize utility. Consequently if all investors are seeking to maximize the utility, so all of them must behave in essentially the same way! Which this consistency in behaviour can suggest a very specific statement about their aggregate behaviour. It helps us to reach some description for future actions. Every model or theory is based on some assumption, basically some simplification tools. Markowitz model relies on the following assumptions;

- Investors seek to maximize the expected return of total wealth.
- All investors have the same expected single period investment horizon.
- All investors are risk-adverse, that is they will only accept a higher risk if they are compensated with a higher expected return.
- Investors base their investment decisions on the expected return and risk.
- All markets are perfectly efficient.

By having these assumptions in mind, we will go through some concepts and terminologies that will make us understand the model constructed in further part of this paper.

Risk and Reward (Mean and Variance Analysis)

Markowitz model relies on balancing risk and return, and it is important to understand the role of consumer's preferences in this balance. There are different methods to calculate risk and return and the choice of these methods can change the result of our calculations dramatically. The following sections describe these methods in brief and we motivate our choice by mathematical proof.

By assumption for the Markowitz model, investors are risk averse. Assuming equal returns, the investor prefers the one with less risk, which implies that an investor who seeks higher return must also accept the higher risk. There is no exact formula or definition for this and it is totally dependent on individual risk aversion characteristics of the investor.

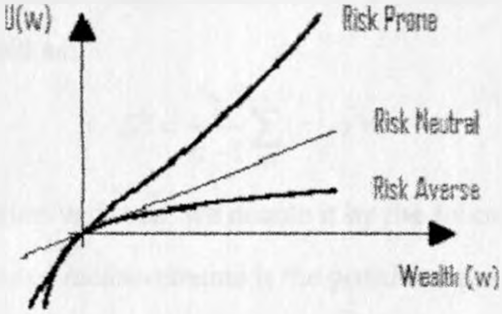


figure 2 1

Utility curve of investors with different preference

A further assumption is that risk and return preferences of an investor can be described via a quadratic utility function. This means when plotted on a graph, your utility function is a curve with decreasing slope, for larger risk. Where *w* is an indicator for wealth and *U* is a quadratic utility function. We have,

$$U(w) = w - w^2$$

A consumer's utility is hard to measure. However, for the usual person, utility increased with wealth but at a decreasing rate.

Risk aversion can be determined through defining the risk premium, which by Markowitz defined to be the maximum amount that an individual is prepared to give up to avoid uncertainty. It is calculated as the difference between the utility of the expected wealth and the expected utility of the wealth.

$$U[E(w)] - E[U(w)]$$

This allows us to determine the characteristic of the behaviour of the investor regarding risk;

- If $U[E(w)] > E[U(w)]$ then the utility function is concave and the individual is risk averse;
- If $U[E(w)] = E[U(w)]$ then the utility function is linear and the individual is risk neutral;
- If $U[E(w)] < E[U(w)]$ then the utility function is convex and the individual is risk seeking.

Figure 2.1 gives a graphical interpretation of what was stated above.

Variance and Standard Deviation

Variance of a sample of measurements $r_1, r_2, \ldots r_n$ is the sum of the square of the differences between the measurements and their mean, divided by $n-1$, where n denoted the sample size. The sample variance is denoted as:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \bar{r})^2$$

When referring to the population variance, we denote it by the following symbol σ^2. The standard deviation of a sample of measurements is the positive square root of the variance, which can be denoted as:

The population standard deviation is denoted by

$$s = \sqrt{s^2}$$

In portfolio theory the standard deviation measures how much the return of a portfolio or the stock moves around the average return. This is considered as a measure of risk.

Mathematics of the Markowitz Model

In the theory part we introduced some basic definitions and building blocks of Markowitz model, risk and return. Markowitz model makes it possible to construct a portfolio with different combinations where short sales and lending or borrowing might be allowed, or not. The case might be the best alternative to consider for the purpose of our paper, which is clarifying the construction of a portfolio when short sales are allowed and riskless borrowing and lending is possible. The Markowitz model is all about maximizing return and minimizing risk, but simultaneously.

The investor preferences are the most important parameter which is hidden in the balancing of the two parameters of Markowitz model. We should be able to reach a single portfolio of risky assets with the least possible risk that is preferred to all other portfolios with the same level of return. Let's consider the following coordinate system of expected return and standard deviation of return. It will help us to plot all combinations of investments available to us. Some investments are riskless and some are risky. Our optimal portfolio will be somewhere on the ray connecting risk free investments R_F to our risky portfolio and where the ray becomes tangent to our set of risky portfolios or efficient set it has the highest possible slope, in Figure 3.1 this point is showed by B. Different points on the ray between tangent point and interception with expected return coordinate represents combination of different amounts possible to lend or borrow to combine with our optimal risky portfolio on intersection of tangent line and efficient set.

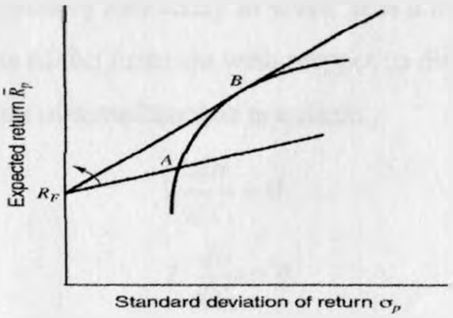


figure 3 1

Combinations of the risk less asset in a risky portfolio

As we mentioned above, the ray discussed has the greatest slope. It can help us to determine the ray. The slope is simply the return on the portfolio, R_P minus risk-free rate divided by standard deviation of the portfolio σ_p. Our task is to determine the portfolio with greatest ratio of excess return to standard deviation θ. In mathematical terms we should maximize θ.

$$\theta = \frac{\overline{R_p} - R_f}{\sigma_p}$$

This function is subject to the constraint,

$$\sum_{i=1}^{n} X_i = 1$$

Where X_i s are the samples members, also can be random variables. The above constrained problem can be solved by Lagrangian multipliers. We consider an alternative solution, by substituting the constraint in the objective function, where it will become maximized as in unconstrained problem. By writing R_F as R_F times one,

$$R_f = 1R_f = \left(\sum_{i=1}^{n} X_i\right) R_f = \sum_{i=1}^{n} (X_i R_f)$$

By stating the expected return and standard deviation of the expected return in the general form we get,

$$\theta = \frac{\sum_{i=1}^{n} X_i\left(\overline{R}_i - R_f\right)}{\left[\sum_{i=1}^{n} X_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} X_iX_j\sigma_{ij}\right]^{\frac{1}{2}}}$$

Now we have the problem constructed and ready to solve. It is a maximization problem and solved by getting the derivatives of the function with respect to different variables and equating them to zero. It gives us a system of simultaneous equations,

$$1.\ \frac{d\theta}{dX_1} = 0$$

$$2.\ \frac{d\theta}{dX_2} = 0$$

$$\vdots$$

$$N.\ \frac{d\theta}{dX_N} = 0$$

In order to solve the maximization problem we need to take derivatives of the ratio θ. We rewrite θ in the following form;

$$\theta = \left[\sum_{i=1}^{n} X_i\left(\overline{R}_i - R_f\right)\right]\left[\sum_{i=1}^{n} X_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} X_iX_j\sigma_{ij}\right]^{-\frac{1}{2}}$$

As it is written above, the ratio consists of multiplication of two functions. To derivate this ratio we need to use Product Rule and as the second term suggests where it has power -1/2 another rule of calculus, the Chain Rule must be applied. After applying the chain rule, we use product rule and we get,

$$\frac{d\theta}{dX_k} = \left[\sum_{i=1}^{n} X_i\left(\overline{R}_i - R_f\right)\right]\left[\left(-\frac{1}{2}\right)\left(\sum_{i=1}^{n} X_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} X_iX_j\sigma_{ij}\right)^{-\frac{3}{2}} \times \left(2X_k\sigma_k^2 + 2\sum_{\substack{j=1 \\ j\neq k}}^{n} X_j\sigma_{kj}\right)\right]$$

$$+\left[\sum_{i=1}^{n} X_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} X_iX_j\sigma_{ij}\right]^{-\frac{1}{2}} \times\left[\left(\overline{R}_k - R_f\right)\right] = 0$$

If we multiply the derivative by

$$\left(\sum_{i=1}^{n} X_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} X_iX_j\sigma_{ij}\right)^{\frac{1}{2}}$$

And rearrange, then;

$$-\left[\frac{\sum_{i=1}^{n} X_i\left(\overline{R_i}-R_f\right)}{\sum_{i=1}^{n} X_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} X_iX_j\sigma_{ij}}\right]\left(X_k\sigma_k^2 + \sum_{\substack{i=1\\ j\neq k}}^{n} X_j\sigma_{kj}\right)+\left[\left(\overline{R_k}-R_f\right)\right]=0$$

Where we define λ as Lagrange multiplier,

$$\frac{\sum_{i=1}^{n} X_i\left(\overline{R_i}-R_f\right)}{\sum_{i=1}^{n} X_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} X_iX_j\sigma_{ij}}$$

Yields,

$$-\lambda\left(X_k\sigma_k^2 + \sum_{\substack{i=1\\ j\neq k}}^{n} X_j\sigma_{kj}\right)+\left[\left(\overline{R_k}-R_f\right)\right]=0$$

By multiplication,

$$-\left(\lambda X_k\sigma_k^2 + \sum_{\substack{i=1\\ j\neq k}}^{n} \lambda X_j\sigma_{kj}\right)+\left[\left(\overline{R_k}-R_f\right)\right]=0$$

Now by extension

$$\frac{d\theta}{dX_i} = -\left(\lambda X_1\sigma_{1i} + \lambda X_2\sigma_{2i} + \ldots + \lambda X_i\sigma_i^2 + \ldots + \lambda X_{N-1}\sigma_{N-1i} + \lambda X_N\sigma_{Ni}\right) + \overline{R_i} - R_f = 0$$

We use a mathematical trick, where we define a new variable $Z_k = \lambda X_k$ .The X_k are fraction to invest in each security, and Z_k are proportional to this fraction. In order to simplify we substitute Z_k for λX_k and move variance covariance terms to the left,

$$\overline{R_i} - R_f = Z_1\sigma_{1i} + Z_2\sigma_{2i} + \ldots + Z_i\sigma_i^2 + \ldots + Z_{N-1}\sigma_{N-1i} + Z_N\sigma_{Ni}$$

The solution of the above statement involves solving the following system of simultaneous equations,

$$\overline{R_1} - R_f = Z_1\sigma^2_1 + Z_2\sigma_{12} + \ldots + Z_3\sigma_{13} + \ldots + Z_N\sigma_{1N}$$
$$\overline{R_2} - R_f = Z_1\sigma_{12} + Z_2\sigma^2_2 + \ldots + Z_3\sigma_{23} + \ldots + Z_N\sigma_{2N}$$
$$\overline{R_3} - R_f = Z_1\sigma_{13} + Z_2\sigma_{23} + \ldots + Z_3\sigma^3_3 + \ldots + Z_N\sigma_{3N}$$
$$\vdots$$
$$\overline{R_N} - R_f = Z_1\sigma_{1N} + Z_2\sigma_{2N} + \ldots + Z_3\sigma_{3N} + \ldots + Z_N\sigma^2_N$$

Now we have N equations with N unknowns. By solving for Zs we can get X_k, which are the optimum proportions to invest in stock k,

$$X_k = Z_k \Big/ \sum_{i=1}^{n} Z_i$$

Up to here we calculate the weights for the general form, where short sales are allowed and lending and borrowing is possible.

Diversification

Diversification is a risk management technique that mixes a wide variety of investments within a portfolio. It is done to minimize the impact of any security on the overall portfolio performance. In order to have a diversified portfolio it is important that the assets chosen to be included in a portfolio do not have a perfect correlation, or a correlation coefficient of one. Diversification reduces the risk on a portfolio, but not necessarily the return, and though it is referred as the only free lunch in finance. Diversification can be loosely measured by some statistical measurement, correlation. It has a range from negative one to one and measures the degree to which the various asset in a portfolio can be expected to perform in a similar fashion or not. The total risk of a portfolio is the result of summation of systematic and unsystematic risks. On average, the total risk of a diversified portfolio tends to diminish as more randomly selected common stocks are added to the portfolio.

Diversification in Markowitz model

Markowitz model suggests that it is possible to reduce the level of risk below the undiversifiable risk. We categorized Markowitz diversification on three basic interrelated concepts,

The Weights Sum to One: The first concept requires that the weights of the assets in the portfolio sum to 100%. Simply the investment weights are a decision variable, which is the main task for portfolio manager to determine them.

$$\sum_{i=1}^{n} X_i = 1$$

Where x represents weights or participation level of asset i in a portfolio that contains N assets

A Portfolio's Expected Return: It is the weighted average of the expected returns of the assets that make up the portfolio, the portfolio's expected rate of return for N-assets portfolio is,

$$E(R_p) = \sum_{i=1}^{n} x_i E(R_i)$$

Where $E(R_i)$ is the security analysts forecast for expected rate of return from the ith asset.

The Objective: Investment weights chosen by portfolio managers should add up to an efficient portfolio which is:

- The maximum expected return in its risk-class, or, conversely.
- The minimum risk at its level of expected return.

The set of all efficient portfolios is called efficient frontier. This is the maximum return at each level of risk. The efficient frontier dominates all other investment opportunities.

Portfolio Risk: The risk of the portfolio, or its variance should be broken into two parts, the variance which represents the individual risks and interaction between N candidate assets. This equation (double summation) represents the variance-covariance matrix and can be expanded and written in matrix form.

$$VAR(R_p) = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}$$

Where $\sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$ and ρ_{ij} is correlation coefficient between assets i and j. In order to have a portfolio well-diversified according to Markowitz, the assets included in the portfolio should have low enough correlations between their rates of returns. As shown in the figure 3.2, a portfolio with correlation coefficient equal to zero gives the same level of return, but with a lower risk level, than a portfolio which the assets including it have a correlation coefficient of one. If an investment or portfolio manager achieves to include securities whose rates of return have low enough correlation, according to Markowitz, he or she can reduce a portfolio's risk below the undiversifiable level.

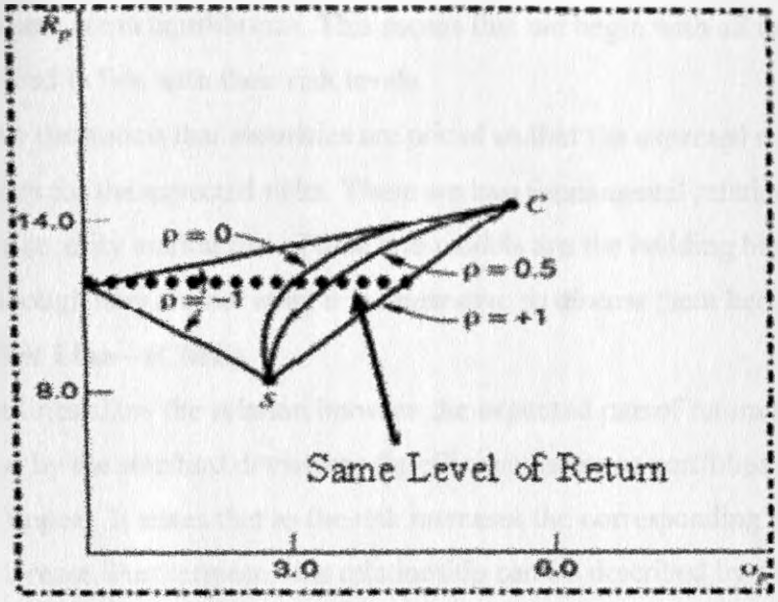


figure 3 2

Relationship between expected return and standard deviation of return for various correlation coefficients

3.4) The Capital Asset Pricing Model – (CAPM)

Because capital market theory builds on the Markowitz portfolio model, it requires the same assumptions, along with some additional ones:

1. All investors are Markowitz efficient investors who want to target points on the efficient frontier. The exact location on the efficient frontier and, therefore, the specific portfolio selected will depend on the individual investor's risk-return utility function.
2. Investors can borrow or lend any amount of money at the risk-free rate of return *(RFR)*.
3. All investors have homogeneous expectations; that is, they estimate identical probability distributions for future rates of return.
4. All investors have the same one-period time horizon such as one month, six months, or one year. The *model will be developed for a single hypothetical period, and its results could be affected by a different* assumption.
5. All investments are infinitely divisible, which means that it is possible to buy or sell fractional shares of any asset or portfolio. This assumption allows us to discuss investment alternatives as continuous curves.
6. There are no taxes or transaction costs involved in buying or selling assets.
7. There is no inflation or any change in interest rates, or inflation is fully anticipated.

8. Capital markets are in equilibrium. This means that we begin with all investments properly priced in line with their risk levels.

The CAPM conveys the notion that securities are priced so that the expected returns will compensate investors for the expected risks. There are two fundamental relationships: the capital market line and the security market line. These two models are the building blocks for deriving the CAPM. Even though they are not new, it is illustrative to discuss them here briefly.

The Capital Market Line – (CML)

The capital market lines show the relation between the expected rate of return and the risk of return (as measured by the standard deviation) for efficient assets or portfolios of assets. The line has great intuitive appeal. It states that as the risk increases the corresponding expected ratio of return must also increase. Furthermore, this relationship can be described by a straight line if risks measured by standard deviation. In mathematical terms the capital market line states that

$$\bar{r} = r_f + \frac{\bar{r}_m - r_f}{\sigma_m}$$

Where $\bar{r}_m$ and σ_m are the expected values and standard deviation of the rate of return of an arbitrary efficient asset. The slope of the CML is

$$k = \frac{\bar{r}_m - r_f}{\sigma_m}$$

And this value is frequently called the price of risk. It tells by how much the expected rate of return must increase if the standard deviation of that rate increases by one unit.

The CML line is derived by drawing a tangent line from the intercept point of the efficient frontier (or the optimal portfolio) to the point where the expected return equals the risk-free rate R_F.

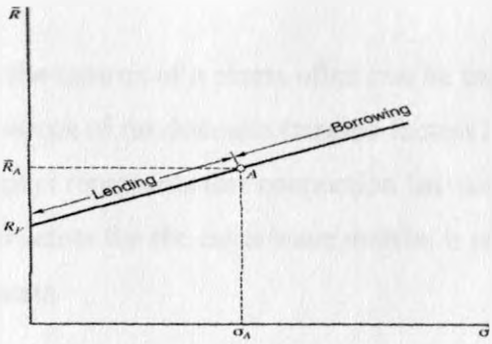


figure 3 3

The capital market line does not show how the expected rate of return of an individual asset relates to its individual risk. This relation is expressed by the capital asset pricing model.

The Security Market Line – (SML)

The Security Market Line is based on the CAPM model, where one believes that the correct measure of risk (systematic risk) is based on the market and called Beta. This means that the SML line is graphed by the CAPM equation.

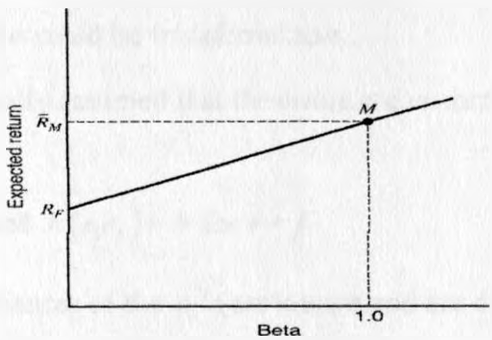


figure 3 4

The Security Market Line

Here in the graph we can see that as the expected return increases so does the risk (Beta). The SML line is based on the risk free rate R_F. We can then also see that since R_F is risk free it has a zero beta. When you go to the right of the graph, you will come to the market portfolio (M). The market portfolio is a hypothetical portfolio, consisting of all the securities that are available for an investor. That is why we have a beta of 1. The markets risk premium is determined by the slope of the SML line.

3.5) FACTOR MODELS

The randomness displayed by the returns of n assets often can be traced back to a smaller number of underlying basic sources of randomness (termed factors). That influence the individual returns. A factor model represents this connection between factors and individual returns leads to a simplified structure for the covariance matrix, it provides an important insight into the relationship among assets.

3.5.1) Single Factor Model

It assumes the security returns are correlated for only one reason, each security responds to a single factor such as the stock market average rate of return for the period). We assume that the rates of return and the factor are related by the following equation.

$$r_i = a_i + b_i f + e_i$$

For i=1,2…..n

In this equation, the a_i's and b_i's are fixed constants. The e_i's are random quantities which represent errors, without loss of generality. It can be assumed that the errors have zero mean that is $E(e_i) = 0$. Since any nonzero could be transferred to a_i.

In addition, however, it is usually assumed that the errors are uncorrelated with f and with each other that is

$E\left[(f - \bar{f})e_i\right] = 0$ for each i and $E(e_i e_j) = 0$ for $i \neq j$

It is also assumed that the variances of the e_i's are known and are denoted by $\sigma_{e_i}^2$. An individual factor model can be viewed graphically as defining a linear fit to (potential) data. Imagine that several independent observations are made for of both the rate of return and the factor. Since both are random quantities, the points are likely to be scattered. A straight line is fitted through these points in such a way that the average value of the error, as measured by the vertical distance from a point to the line is zero.

When applied to a group of assets, the fitting process is carried out for each asset separately. As a result we obtain for each asset i an a_i and b_i. The a_i's are termed intercepts because is the intercept because is the intercept of the line for asset i and the vertical axis. The b_i's are termed factor loadings because they measure the sensitivity of the return to the factor.

If an historical record of asset returns are available parameters of single factor models can be estimated by actually fitting straight lines. If we agree to use a single factor model, then the standard parameters for mean variance analysis can be determined directly from that model. We calculate

$$\bar{r}_i = a_i + b_i \bar{f}$$

$$\sigma_i^2 = b_i^2 \sigma_f^2 + \sigma_{e_i}^2$$

$$\sigma_{ij} = b_i b_j \sigma_f^2 \quad i \neq j$$

$$b_i = \frac{Cov(r_i, f)}{\sigma_f^2}$$

The equations reveal the primary advantage of factor model. In the usual representation of asset returns , at total of 2n+n(n-1)/2 parameters are required to specify means, variances and co variances .In a single factor model only the a_i's , b_i's, $\sigma_{e_i}^2$'s, f and σ_f^2 are required a total of just 3n+2 parameters.

3.5.2)The capm as a factor model

The CAPM can be derived as a special case of a single factor model. This view adds considerable insight to the CAPM development. Let us hypothesize a single factor model for stock returns, with the factor being the market rate of return r_m . For convenience we can subtract the constant r_f from this factor and also from the rate of return r_i, thereby expressing the model in terms of excess returns $r_i - r_f$ and $r_m - r_f$ The factor model then becomes

$$r_i - r_f = \alpha_i + \beta_i \left(r_m - r_f\right) + \varepsilon_i \qquad (2.1)$$

It is conventional to use the notation α_i and β_i for the coefficients of this special model rather than the a_i's and b_i 's that are being used more generally. Again it is assumed that $E\left(\varepsilon_i\right) = 0$ and that ε_i is uncorrelated with the market return (the factor) and with other ε_j's.

The characteristic line corresponding to the above equation is formed by putting $\varepsilon_i = 0$ that is the line

$$r_i - r_f = \alpha_i + \beta_i\left(r_m - r_f\right)$$

Drawn on a diagram of r_i versus r_m. The expected value of this equation is

$$\bar{r}_i - r_f = \alpha_i + \beta_i\left(\bar{r}_m - r_f\right)$$

Which is identical to the CAPM except for the presence of α_i. The CAPM predicts that α_i=0 .The value of β_i in this model can be calculated directly we take the covariance of both sides of (2.1) with r_m. This produces

$$\sigma_{im} = \beta_i\sigma_m^2 \text{ And hence}$$

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$$

This is exactly the same expression that holds for the β_i used in the CAPM. The characteristic line is in a sense more general than the CAPM because it allows α_i to be nonzero. From the CAPM view point α_i can be regarded as a measure of the amount that asset i is mispriced. A stock with a positive α_i is according to this view performing better than it should.

Note , however that the single factor model that leads to the CAPM formular is not equivalent to the general model underlying the CAPM, since the general model is based on an arbitrary covariance matrix, but assumes that the market is efficient. The single factor model has a very simple covariance structure, but makes no assumption about efficiency.

3.5.3) Multifactor Models

The proceeding development can be extended to include more than one factor, the model for the rate of return of asset i would have the form.

$$r_i = a_i + b_{1i}f_1 + b_{2i}f_2 + e_i$$

Again the constant a_i is called the intercept b_{1i} and b_{2i} are the factor loadings. The factors f_1 and f_2 and the error e_i are random variables.

It is assumed that the expected value of the error is zero, and that the error is uncorrelated with the two factors and with the errors of the other assets. However it is not assumed that the two factors are uncorrelated with each other. In the case of two factor model we easily derive the following values for the expected rates of return and the co variances:

$$\bar{r}_i = a_i + b_{1i}\bar{f}_1 + b_{2i}f_2$$

$$Cov(r_i r_j) = \left\{ b_{1i}b_{1j}\sigma_{f_1}^2 + \left(b_{1i}b_{2j} + b_{2i}b_{1i}\right)Cov(f_1, f_2) + b_{2i}b_{2j}\sigma_{f_2}^2 \right\} i \neq j$$

$$Cov(r_i r_j) = \left\{ b^2{}_{1i}\sigma_{f_1}^2 + 2b_{1i}b_{2i}Cov(f_1, f_2) + b^2{}_{2i}\sigma_{f_2}^2 + \sigma_{\varepsilon_i}^2 \right\} i = j$$

The b_{1i} 's and b_{2i} 's can be obtained by forming the covariance of r_i with f_1 and f_2 leading to

$$Cov(r_i, f_1) = b_{1i}\sigma_{f_1}^2 + b_{2i}\sigma_{f_1 f_2}$$

$$Cov(r_i, f_1) = b_{1i}\sigma_{f_1 f_2} + b_{2i}\sigma_{f_2}^2$$

These give two equations that can be solved for the two unknowns b_{1i} and b_{2i}.

A two factor model is often an improvement of a single factor model. For example, suppose a single-factor model were proposed and then determined by fitting data. It might be found that the resulting error terms are large and that they exhibit correlation with the factor and with each other. In this case the single factor model is not a good representation of the actual returns structure. A two factor model may lead to smaller error terms and these terms may exhibit the assumed correlation properties. The two factor model will still be much simpler than a full unstructured covariance matrix.

3.6)Arbitrage Pricing Theory-APT

The APT has recently attracted considerable attention as a testable alternative to capital asset pricing model. The APT states that, under certain assumptions, the single period expected return on any risky asset is approximately linearly related to its associated factor loadings (i.e., systematic risks) as shown below,

$$\overline{R}_i = E\left(\overline{R}_i\right) + b_{i1}\overline{F}_1 + \ldots + b_{ik}\overline{F}_k + \overline{\varepsilon}_k \quad \ldots\ldots\ldots\ldots\ldots (3.1)$$

Where $\overline{R_i}$ is the random rate of return on the *i*th asset, $E(\overline{R_i})$ is the expected rate of return on the *i*th asset, b_{ik} is the sensitivity of the *i*th asset's returns to the *k*th factor, $\overline{F_k}$ is the mean zero *k*th factor common to the returns of all assets under considerations, $\overline{\varepsilon_k}$ is random zero mean noise term for the *i*th asset.

The APT is derived under the usual assumptions of perfectly competitive and frictionless capital markets. Furthermore, individuals are assumed to have homogeneous beliefs that the random returns for the set of assets being considered are governed by the linear *k*-factor model given in Equation (3.1). The theory requires that the number of assets under consideration, *n*, be much larger than the number of factors, *k*, and that the noise term $\overline{\varepsilon_k}$ be the unsystematic risk component for the *i*th asset. It must be independent of all factors and all error terms for other assets. The basic idea of APT is that in equilibrium all portfolios that can be selected from among the set of assets under consideration and that satisfy the conditions of (a) using no wealth and (b) having no risk must earn no return on average. These portfolios are called arbitrage portfolios. To see how they can be constructed, let *wi* be the wealth invested in the *i*th asset as a percentage of an individual's total invested wealth. To form an arbitrage portfolio that requires no change in wealth, the usual course of action would be to sell some assets and use the proceeds to buy others. Thus the zero change in wealth is written as

$$\sum_{i=1}^{n} w_i = 0 \quad \text{..............} (3.2)$$

If there are n assets in the arbitrage portfolio, then the additional portfolio return gained is

$$\overline{R_p} = \sum_{i=1}^{n} w_i \overline{R_i}$$

$$= \sum_i w_i E(R_i) + \sum_{i=1}^{n} w_i b_{i1} \overline{F_1} + \ldots + \sum_i w_i b_{ik} \overline{F_k} + \sum_i w_i \overline{\varepsilon_i} \quad \text{..........} (3.3)$$

To obtain a riskless arbitrage portfolio it is necessary to eliminate both diversifiable (i.e., unsystematic or idiosyncratic) and undiversifiable (i.e., systematic) risks. This can be done by meeting three conditions: (1) selecting percentage changes in investment ratios *wi* , that are small, (2) diversifying across a large number of assets, and (3) choosing changes *wi* , so that for each factor, *k*, the weighted sum of the systematic risk components, b_k , is zero. These conditions can be written as follows,

$w_i \approx 1/n$ .... (3.4 a)

n Chosen to be a large number.......... (3.4 b)

$\sum_i w_i b_{ik} = 0$ For each factor.......... (3.4 c)

Because the error terms, $\overline{\varepsilon_k}$ are independent, the law of large numbers guarantees that a weighted average of many of them will approach to zero in the limit as n becomes large. In other words, costless diversification eliminates the last term i.e., idiosyncratic risk in Equation (3.1). Thus we are left with

$$\overline{R_p} = \sum_i w_i E\left(\overline{R_i}\right) + \sum_i w_i b_{i1} \overline{F_1} + \ldots + \sum_i w_i b_{ik} \overline{F_k} \quad .... (3.5)$$

Since we have chosen the weighted average of the systematic risk components for each factor to be equal to zero $\left(\sum_i w_i b_{ik} = 0\right)$ this eliminates all systematic risk. This can be considered as selecting an arbitrage portfolio with zero beta in each factor. Consequently, the return on the arbitrage portfolio becomes a constant because of the choice of weights has eliminated all uncertainty. Therefore Equation (3.3) can be written as,

$$R_p = \sum_i w_i E\left(\overline{R_i}\right) \quad .... (3.6)$$

Since the arbitrage portfolio is so constructed, that it has no risk and requires no new wealth. If the return on the arbitrage portfolio were not zero, then it would be possible to achieve an infinite rate of return with no capital requirements and no risk. Such an opportunity is clearly impossible if the market is to be in equilibrium. In fact, if the individual investor is in equilibrium, then the return on any and all arbitrage portfolios must be zero. This can be expressed as,

$$R_p = \sum_i w_i E\left(\overline{R_i}\right) = 0 \quad .... (3.7)$$

From no wealth constraint represented by Equation (3.2), any orthogonal vector to this constraint vector can be formed as given below

$$\left(\sum_i w\right) . e = 0 \quad .... (3.8)$$

And to each of the coefficient vectors from Equation (3.4c), i.e.

$\sum_i w_i b_{ik} = 0$ for each k and must also be orthogonal to the vector of expected returns, Equation (3.7), i.e.,

$$\sum_i w_i E\left(\overline{R}_i\right) = 0$$

Thus the expected return vector can be written as a linear combination of the constant vector and the coefficient vectors. That is, there must exist a set of k + 1 coefficients, $\lambda_0 + \lambda_1 + \ldots \lambda_k$ such that

$$E\left(\overline{R}_i\right) = \lambda_0 + \lambda_1 b_{i1} + \ldots \lambda_k b_{ik} \quad \text{..........} (3.9)$$

Since b_{ik} are the sensitivities of the returns on the *i*th security to the *k*th factor. If there is a riskless asset with a riskless rate of return, R_f, then $b_{ok} = 0$ and $R_f = \lambda_0$. Hence Equation (3.9) can be rewritten in excess returns form as follows,

$$E(R_i) - R_f = \lambda_1 b_{i1} + \ldots \lambda_k b_{ik} \quad \text{..........} (4.0)$$

The arbitrage pricing relationship (4.0) says that the arbitrage pricing relationship is linear and λ represents the risk premium (i.e., the price of risk), in equilibrium, for the *k*th factor. Now rewrite Equation (4.0) as

$$E(R_i) - R_f = \left[\overline{\delta_k} - R_f\right] b_{ik}$$

Where $\overline{\delta_k}$ is the expected return on a portfolio with unit sensitivity to the *k*th actor and zero sensitivity to all other factors. Therefore the risk premium, λ_k is equal to the difference between the expectation of a portfolio that has unit response to the *k*th factor and zero response to the other factors and the risk rate, R_f .

Thus the APT model is represented by following equation,

$$E(R_i) - R_f = \left[\overline{\delta_k} - R_f\right] b_{1k} + \ldots + \left[\overline{\delta_k} - R_f\right] b_{ik} \quad \text{..........} (4.2)$$

The Equation (4.2) represents a linear regression equation and coefficients, b_{ik} are defined in exactly the same way as beta in the capital asset pricing model.

CHAPTER 4: DATA, DATA ANALYSIS AND RESULTS

The study uses stock returns data from 10 companies (Kakuzi Ord, Sasini tea and coffee, Kenya Aiways, Nation Media, Barclays Bank, Kenya Commercial Bank, Mumias Sugar, EABL, Eaagads Ltd and Express Ltd) listed on the NSE for the period of 5 years 2004 to 2008. The data are obtained from the NSE, library and Data centre. The NSE 20 Share index is used as a proxy for the market portfolio. This index is a market value weighted index, is comprised of the 20 most highly capitalized shares of the main market, and reflects general trends of the Kenyan .stock market. The data were analyzed with the help of Microsoft Excel and SPSS (Statistical Package for Social Sciences).the observed stock prices were used to calculate the annul rates of return using the formula:

$$R_{it} = \frac{P_1 - P_0 + D}{P_0}$$

Where

R_{it} -Return of the stock I for the period t

P_1 -Market price of the stock at the end of the period

P_0 -Market price of the stock at the beginning of the period.

D - Cash dividend paid for the period

4.1) MPT

Using the co variances and variances of the stocks, I was able to come up with a covariance matrix of the returns of the selected stocks. Therefore I was able to get my Z values for each stock by solving the matrix below:

$$\begin{bmatrix} 0.1864 & 0.10 & 0.45 & 0.0082 & 0.057 & 0.054 & 0.22 & -0.097 & -0.082 & 0.014 \\ 0.10 & 0.3281 & 0.53 & 0.0073 & 0.13 & 0.29 & 0.40 & -0.086 & -0.15 & 0.15 \\ 0.45 & 0.53 & 2.0447 & 0.025 & 0.23 & 0.45 & 0.92 & -0.34 & -0.27 & 0.23 \\ 0.0082 & 0.0073 & 0.025 & 0.0477 & 0.0027 & 0.058 & 0.11 & 0.044 & 0.037 & 0.015 \\ 0.057 & 0.129 & 0.228 & 0.027 & 0.1716 & 0.165 & 0.27 & 0.02 & -0.166 & 0.04 \\ 0.054 & 0.286 & 0.45 & 0.058 & 0.165 & 0.5238 & 0.532 & 0.0059 & -0.082 & 0.211 \\ 0.22 & 0.40 & 0.92 & 0.11 & 0.27 & 0.532 & 0.0921 & -0.032 & -0.166 & 0.226 \\ -0.097 & -0.086 & -0.34 & 0.044 & 0.02 & 0.0059 & -0.032 & 0.1884 & -0.045 & -0.03 \\ -0.082 & -0.15 & -0.27 & 0.037 & -0.166 & -0.082 & -0.166 & 0.045 & 0.4663 & 0.0167 \\ 0.0137 & 0.15 & 0.23 & 0.015 & 0.04 & 0.211 & 0.226 & -0.03 & 0.0167 & 0.1723 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \\ Z_7 \\ Z_8 \\ Z_9 \\ Z_{10} \end{bmatrix} = \begin{bmatrix} 0.05676 \\ -0.0203 \\ 0.7915 \\ 0.0749 \\ -0.1412 \\ 0.0439 \\ 0.5203 \\ -0.064 \\ 0.2009 \\ 0.14606 \end{bmatrix}$$

Co-variance matrix of the stocks

Using the formular

$$X_k = Z_k \Big/ \sum_{i=1}^{n} Z_i$$

To compute the values of X_i s, which represent the weights of each stock and according to Markowitz these weights enables one to get an efficient portfolio for a given return. This implies that for this sample of assets one should invest a proportion of:

Proportion of wealth	**Stock**
0.5892	**Kakuzi Ord**
0.5358	**Sasini Tea and Coffee**
-0.2832	**Kenya Airways**
-0.7510	**Nation Media**
1.0676	**Barclays Bank**
0.3296	**Kenya Commercial Bank**
-0.370	**Mumias Sugar Co**
-0.0457	**East African Breweries ltd**
0.4644	**Eaagads ltd**
-0.4460	**Express ltd**
$\sum_{1=1}^{10} = 1$	

For each stock for a given total amount of wealth invested on the portfolio. Notice the negative proportions in Kenya Airways, Nation media, Mumias sugar, EABL and Express Ltd. This means that investors should short sale those particular stocks.

4.2) CAPM

The securities alpha and beta can be estimated by a statistical model that describes realized excess returns through time:

$$r_i - r_f = \alpha + \beta_i(r_m - r_f) + \varepsilon$$

This model can be estimated by regression. I have provided results of the 10 stocks from 2004-2008 time periods which are summarized below.

STOCK	Mean	SD	α	SE_α	β	SE_β
Kakuzi Ord	0.1318	0.4317	-0.63	0.22	1.107	1.05
Sasini Tea and Coffee	0.0546	0.5728	-0.202	0.28	1.687	1.340
Kenya Airways	*0.8665*	1.4309	0.299	0.661	4.573	3.157
Nation Media	0.1499	0.2184	-0.09	0.081	0.874	0.388
Barclays Bank	-0.0662	0.4142	-0.246	0.219	0.97	1.045
Kenya Commercial Bank	0.1190	0.7237	-0.251	0.290	2.733	1.385
Mumias Sugar Co	0.5953	1.0450	0.21	0.291	4.628	1.391
East African Breweries Ltd	0.0103	0.4340	-0.067	0.262	0.018	1.253
Eaagads Ltd	0.2759	0.6829	0.133	0.408	0.625	1.950

Express Ltd	0.2211	0.4150	0.004	0.199	1.320	0.952
MARKET	**0.1828**	**0.1967**	**0**		**1**	

Table of stocks Alpha and Beta value 1

The results indicate that the beta of the stocks varied between 0.018 to 4.628 over the examined period. The methodology required the estimation of alpha and betas for individual stocks by using observations on rates of return for a sequence of 5 years. Useful remarks can be derived from the results of this procedure, for the assets used in this study.

The CAPM theory indicates that higher risk (beta) is associated with a higher level of return. However, the results of the study do not support this hypothesis. The beta of the 10 stocks does not clearly indicate that higher beta portfolios are related with higher returns. Mumias sugar for example, the highest beta stock, yields second highest returns. In contrast, EABL, the lowest beta stock produces the second least return. However based on the result, the general trend seems to support the theory despite the fact that Kenya Airways and Mumias sugar have abnormally high estimated beta values. These contradicting results can be partially explained by the significant fluctuations of stock returns over the period examined this can be due to the Global financial crisis of 2008 and the post-election violence witnessed in 2007.

In the estimation of SML, the CAPM's prediction for α is that it should be equal to zero when true expected returns are used. The calculated value of the intercept is small but it is not significantly different from zero especially for Nation media, EABL and Express ltd stocks. Therefore α measures approximately, how much the performance of the stocks has deviated from the theoretical values of zero. A positive value of α presumably implies that the stock did better than the CAPM prediction (but of course we recognize that approximations are introduced by the use of a finite amount of data to estimate the important quantities). In terms of α Kenya Airways, Mumias sugar, Eaagads ltd and Express ltd appear to have performed better than the CAPM.

4.3) APT

Multifactor models assume that stocks tend to move up and down together because they are responding to two or more factors. In this instance interest rates and inflation rate.

TOCK	**Mean**	**SD**	a	SE_a	b_{i1}	$SE_{b_{i1}}$	b_{i2}	$SE_{b_{i2}}$
akuzi ord	0.1318	2.1649	1.036	6.04	-10.75	1.937	-0.733	6.898
asini Tea and Coffee	0.0546	8.566	3.599	1.911	-13.024	6.128	-33.520	21.822
enya Airways	*0.8665*	8.6220	7.942	1.872	-39.393	6.002	-52.815	21.874
ation Media	0.1499	5.6074	0.875	1.251	-1.934	4.012	-7.631	14.284
Barclays Bank	-0.0662	9.2863	0.624	2.071	-6.513	6.642	-2.342	23.654
Kenya Commercial Bank	0.1190	11.2432	5.292	2.508	-13.019	8.042	-55.236	28.639
Mumias Sugar Co	0.5953	13.4983	7.070	3.007	-25.698	9.533	-59.025	34.341
East African Breweries Ltd	0.0103	10.1960	-0.887	2.274	6.434	7.292	5.185	25.971
Eaagads Ltd	0.2759	14.6883	1.939	3.276	5.148	10.506	-27.58	37.414

xpress Ltd	0.2211	3.6546	3.655	0.815	-6.202	2.614	-39.233	9.309

Table of stocks Expected return and Factor loadings

In this instance interest rates and inflation rate. The rates of return in a 2 factor model for a period of 5 years for the 10 securities is given by

$$r_{it} = a_i + b_{i1} f_{it} + b_{i2} f_{2t} + \varepsilon_{it}$$

Where

a_i -the expected level of return for stock i.

f_{it} and f_{2t} - are factors that have persuasive influence on security returns which in our case are inflation and interest rate.

b_{i1} and b_{i2} - the sensitivities of security I returns on this factors; Inflation rate and interest rate.

ε_{it} - is a random error term.

Expected returns- an investor who wants to invest in these securities will therefore rank them according to the one with highest expected return which is Kenya Airways followed closely by Mumias sugar up to the least which is EBL with an expected return of -0.887.

The Standard deviation-the two factor model gives variances of a given security as.

$$\sigma_i^2 = b_{i1}^2 \sigma_{f_1}^2 + b^2_{i2} \sigma_{f_2}^2 + 2b_{i1} b_{i2} Cov(f_1 f_2) + \sigma_{\varepsilon_i}^2$$

Therefore SD

$$\sigma_i = \sqrt{\sigma_i^2}$$

From the above table it can be seen that Eaagads stocks have the highest risk while Kakuzi is considered the least in terms of risk.

CHAPTER FIVE: CONCLUSIONS AND RECOMMENDATIONS

5.1) Concluding remarks

The paper examined the validity and practicality of the Mathematical models- MPT, CAPM and APT in the Kenyan stock market. The study used stock returns from 10 companies listed on the Nairobi stock exchange from January 2004 to December 2008.

The basic Markowitz portfolio model derived the expected rate of return for the stocks and a measure of expected risk, which is the standard deviation of expected rate of return. Markowitz shows that the expected rate of return of a portfolio is the weighted average of the expected return for the individual investments in the portfolio. The standard deviation of a portfolio is a function not only of the standard deviations for the individual investments but *also* of the covariance between the rates of return for all the pairs of assets in the portfolio. In a large portfolio, these co variances are the important factors.

Different weights or amounts of a portfolio held in various assets yield a curve of potential combinations. Correlation coefficients among assets are the critical factor you must consider when selecting investments because you can maintain your rate of return while reducing the risk level of your portfolio by combining assets or portfolios that have low positive or negative correlation.

Assuming numerous assets and a multitude of combination curves, the efficient frontier is the envelope curve that encompasses all of the best combinations. It defines the set of portfolios that has the highest expected return for each given level of risk or the minimum risk for each given level of return. From the sample set I studied was able to compute the assets weights which produces the efficient frontier. Hence assuming that the market is made up of the 10 stocks, one should invest their wealth according to the proportions given.

Using the MPT I was able to come up with an efficient portfolio of the 10 stocks by calculating the optimal weights from the covariance matrix I constructed.

The findings of the article are not supportive of the theory's basic hypothesis that higher risk (beta) is associated with a higher level of return. The CAPM's prediction for the intercept is that it should be equal to zero and the slope should equal the excess returns on the market portfolio. In summary, while the empirical work is not fully a satisfactory test of the CAPM, it produces results that that are consistent with what we would expect from a test of the CAPM. Furthermore the results are produced with respect to observable variables (market proxies)

The beta of the portfolio is the weighted average of the individual asset betas where the weights are the portfolio weights. So we can think of constructing a portfolio with whatever beta we want. All the information we need is the betas of the underlying asset. If we construct a portfolio with minimum market (or systematic) risk, then we would chose an appropriate combinations of securities and weights that deliver a minimum portfolio beta for a given return.

Therefore for the process of portfolio selection the investor should consider stocks according to the highest α and lowest β values.

Using the method suggested by Fama and McBeth (1973), this research has examined the stock price behaviour in an emerging stock market, the NSE. The findings may be affected from thin data base of the stock profile, since only 10 stocks are investigated. However the general trend obtained may be persuasive. On the basis of available assumption that stock prices are influenced by two macro-economics factors; inflation rate and interest rate and within the scope of this papers methodology, the findings does not confirm that the stock price conform to the inspiration of the APT. this is clearly seen from the abnormally high expected returns and standard deviation values. Despite these problems an insight on portfolio selection using the model was achieved by considering stock with the highest expected return and lowest standard deviation.

5.2) Recommendations

- We have discussed and illustrated the use of MPT to solve for the set of all possible portfolios that are efficient. The solution technique discussed is feasible and has been used to solve problems. However, the technique requires gigantic amount of input data and large amounts of computational time. Furthermore the input data are in a form in which security analysts and portfolio managers cannot easily relate. For this reason it is difficult to get estimates of the input data or get practioners to relate to the final input. The study therefore recommends a simplification of the number and type of input requirements for portfolio selection and in turn the simplification of computation.
- The capital asset pricing model has been derived under a set of very restrictive assumptions. The test of a model is how well it describes reality. It is therefore important other forms of the general equilibrium that exists under less restrictive assumptions. Even if the standard CAPM model explains the behavior of the security, it obviously does not explain the behavior of individual investors. This necessitates the build up of versions of CAPM that will cater for this.
- As APT advocates or more sophisticated multifactor beta which includes a large number of other inputs besides market index to measure risk. Despite this paper making the assumption of 2 factor model, a greater number of factors should be included. It is therefore future studies should incorporate between 3to15 factors.

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