ECONOMIC OPTIMISATION OF ALTERNATE ROUTES

IN THE EAST AFRICAN TELEPHONE TRUNK NETWORK

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1975

This thesis is my original work and has not been presented for a degree in any other University.

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This thesis has been submitted for examination with our approval as University supervisors.

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CONTENTS

SUMMARY

This dissertation discusses our economic method of optimising a telephone network with alternate routes by use of a computer. The method chosen uses a mathematical programming approach in terms of junction circuits and traffic carrying and overflowing properties of the chains involved in routing traffic between the exchanges. A mathematical formulation is given and the solution by means of a convex programming technique is also given. The use of this method has been illustrated by the three node trunk network presented for study.

LIST OF ABBREVIATIONS AND SYMBOLS

N	- sources
n	- devices
λ_j	- intensity of increase
μ_j	- intensity of decrease
E_j	- state system when j sources are busy
P(j.t)	- probability that there are j simultaneous calls at time t
j	- number of simultaneous calls/momentary state of the system
$\bar{t}$	- mean holding time
W(j)	- ability of the system to accept a new call when state j prevails
P(j)	- the probabilities of state
E	- time congestion
B	- call congestion
y(j)	- call intensity of the source when the system is in state j.
A_c	- traffic carried
A_o	- traffic offered
g	- number of inlet groups
k	- availability (hunting capacity of a switch)

CHAPTER 1

1.0 INTRODUCTION

The fundamental problem which besets people concerned with the design of communication networks is how to provide a network which collectively is:

1. Of sufficient routing capability to allow any two users, from a large number of subscribers connected to an exchange, to be connected with a high probability of success.
2. Economical in its use of transmission facilities and switching centres.
3. Capable of surviving extensive network or man-made damage, such as lightning or breaking of an overhead wire pole respectively.
4. Adaptable to changing traffic patterns and overload situations.
5. Capable of being engineered and implemented in small sections over a period of years by many different people.

These factors, in addition to the very large development programmes East African Posts and Telecommunications Corporation (E.A.P & T) has to implement motivated the preparation of this dissertation. We consider here, the design of alternative routes in the East African communication network and a computer programme written for the economic optimisation of the alternative routing networks.

In the East African network, all traffic is routed through the final routes, that is, via tandem exchanges, and hence direct routes between two exchanges will have to be determined as considered below. Then the existing routes will be used as the alternate routes, where this will be justified.

E.A.P & T will provide the basis on which the direct and alternate routes will be determined. In brief, the information will consist of:

(i) the minimum traffic that justifies provision of a direct and an alternate route.
(ii) the traffic between the two points to be considered.
(iii) the grade of service required.
(iv) the cost for providing and maintaining the route.
(v) the maximum capacity of the proposed routes.

The constant cost per route and the cost for providing each circuit will vary each time the maximum capacity of a route is reached and an extension or a larger capacity route is required. There will be a discontinuity in the cost function at this point as it will be considered in Chapter 4.

CHAPTER 2

2.0 GENERAL TRAFFIC THEORY

Traffic theory is applied to practical cases on the assumption of statistical equilibrium (see Appendix 2) which implies that the process is assumed to be stationary statistically. An assumption we make throughout in this dissertation. The general form of traffic theory is first presented and special cases are then derived from this general form.

The mathematical description of the traffic processes is based on the Theory of Stochastic Processes of the type known as the Birth and Death Process as applied to the study of practical problems of congestion in telephone systems (see Appendix 1).

For purposes of telephone traffic engineering this process, the "traffic process", describes the number of busy devices or single sources as a function of time. This leads to the general equations of state (see Appendix 1.1.2) from which, in the limit $t \to \infty$, one can derive the state probabilities for the system. It is found that from these equilibrium equations, one can set up abbreviated expressions for the state probabilities which well cover the most general cases within traffic theory.

2.1 Assumptions

(i) The system changes only through transitions from states to their next neighbours (from E_j to E_{j+1} or to E_{j-1} if $0 < j < N$, but for end cases from E_0 to E_1, and from E_N to E_{N-1} only).

(ii) The principle of statistical equilibrium holds the stationarity assumption.

(iii) Holding times of the individual occupations and time intervals between successive calls have an exponential distribution.

(iv) Duration of conversations, as well as, the individual traffic sources are mutually independent.

(v) There are probability distributions for unsuccessful calls.

2.2 The Traffic Theory

By assuming an exponential distribution for time intervals or durations, it is implied that one has the frequency function (see Appendix 3).

$$f(t) = \alpha \exp(-\alpha t) \qquad 0 \leqslant t \leqslant \infty \qquad (2.1)$$

The average time interval or duration is

$$\bar{t} = \int_{t=0}^{\infty} t.f(t)dt$$

$$= \int_{t=0}^{\infty} t\alpha \exp(-\alpha t)dt$$

$$= \frac{t\alpha}{-\alpha} \exp(-\alpha t) \Big|_{t=0}^{\infty} - \int_{t=0}^{\infty} - \frac{\alpha}{\alpha} \exp(-\alpha t)dt$$

$$= - \frac{1}{\alpha} \exp(-\alpha t) \Big|_{t=0}^{\infty}$$

$$= - \frac{1}{\alpha}(0-1) = \frac{1}{\alpha} \qquad (2.2)$$

This determines the birth or death coefficient ($\alpha = \lambda$ or $\alpha = \mu$ respectively) as the reciprocal of the average holding time $\bar{t}$.

The probability that there is no change (no arrival or termination of call) during the time interval measured from the occurance of the last change is

$$P(0,t) = \exp(-\alpha t) \qquad (2.3)$$

This is equivalent to the event that the interval between two consecutive changes is larger than t. In fact, the interval in which no change occurs must lie within the length of time between two changes.

If T is a random variable representing the length of the time interval (or duration) between two changes, then the probability

$$P(T > t) = \exp(-\alpha t) \tag{2.4}$$

and the probability that T is at most equal to t is

$$P(T \leqslant t) = 1 - \exp(-\alpha t) = F(t) \tag{2.5}$$

which is the same as the probability of one or more changes in the time interval of length t. This is also the distribution function known as the negative exponential distribution. The probability that no change occurs in time interval t is the inverse distribution function.

$$\Phi(t) = 1 - F(t) = \exp(-\alpha t) \tag{2.6}$$

The probability that t terminates in the interval $(t, t + \Delta t)$ is

$$P(t, t + \Delta t) = \int_{t}^{t+\Delta t} f(\tau)d\tau = F(t + \Delta t) - F(t)$$

$$= \Phi(t) - \Phi(t + \Delta t)$$

$$= \exp(-\alpha t) - \exp\left[-\alpha(t + \Delta t)\right] \tag{2.7}$$

The probability that the duration is at least t is

$$P(t) = \Phi(t) = \exp(-\alpha t) \tag{2.8}$$

But $\quad P(t, t + \Delta t) = P(t)\, P\left(\frac{t + \Delta t}{t}\right)$ (2.9)

where $P\left(\frac{t + \Delta t}{t}\right)$ is the conditional probability. (2.10)

Such that the conditional probability that a duration which still exists at t ceases in $(t, t + \Delta t)$ is

$$P(\Delta t) = P\left(\frac{t + \Delta t}{t}\right) = \frac{P(t, t + \Delta t)}{P(t)}$$

$$= \frac{\exp(-\alpha t) - \exp\left[-\alpha(t + \Delta t)\right]}{\exp(-\alpha t)}$$

$$= 1 - \exp(-\alpha\Delta t). \tag{2.11}$$

Now

$$\mathrm{Exp}(-\alpha\Delta\ t) = 1 - \alpha\Delta t + \frac{(\alpha\Delta t)^2}{2!} - \frac{(\alpha\Delta\ t)^3}{3!} + \ldots \qquad (2.12)$$

and to a first order approximation for Δt small,

$$\exp(-\alpha\Delta t) \simeq 1 - \alpha\Delta t \qquad (2.13)$$

such that

$$P(\Delta t) \simeq 1 - (1-\alpha\Delta t) = \alpha\Delta t \qquad (2.14)$$

$$\Delta t \to 0$$

Thus $P(\Delta t)$ is proportional to the intensity of change and to the time interval.

The assumption of exponential distribution implies

(i) $P(\Delta t)$ will be independent of t.

(ii) The process assumes that only one event can take place at a time.

This confirms that the assumptions made were reasonable. Equation (2.14) is also defined in Appendix 1. 3.

2.2.1 Stationary Conditions

The momentary state of a system is described by the number of simultaneously engaged sources, which is synonymous to the number of engaged devices. The momentary states of the system are denoted by j and the probabilities of state are denoted by P(j), (See Appendix 2).

By the principle of statistical equilibrium the system must change from j-1 to j as often as from j to j-1. Otherwise the system would not retain its equilibrium, but tend to either j=0 or j=maximum.

Let the intensity of increase be λ_j and the intensity of decrease be μ_j, when the system is in state j, and with the assumption of exponential description of intervals between calls and of holding times. Then for statistical equilibrium

probability of increase = probability of decrease

such that

$$\lambda_{j-1} P(j-1) = \mu_j\, P(j) \qquad (2.15)$$

and since

$$\sum_{j} P(j) = 1 \qquad (2.16)$$

$$P(j) = \frac{\lambda_{j-1} P(j-1)}{\mu_j} \qquad (2.17)$$

Consider the system in j-1 state. Then

$$\lambda_{j-2}P(j-2) = \mu_{j-1}P(j-1) \qquad (2.18)$$

$$P(j-1)= \frac{\lambda_{j-2} P(j-2)}{\mu_{j-1}} \qquad (2.19)$$

and

$$P(j) = \frac{\lambda_{j-1}\lambda_{j-2} P(j-2)}{\mu_j \mu_{j-1}} \qquad (2.20)$$

By induction

$$P(j) = \frac{\lambda_{j-1}\lambda_{j-2}\lambda_{j-3} \ldots\ldots \lambda_{j-j} P(0)}{\mu_j \mu_{j-1} \ldots\ldots \mu_1} \qquad (2.21)$$

so that

$$P(j) = \frac{\prod_{\rho=0}^{j-1} \lambda_\rho}{\prod_{\rho=1}^{j} \mu_\rho} P(0) \qquad (2.22)$$

This is the general solution for a traffic process in statistical equilibrium with exponential distribution of changes.

2.3 Special Cases

2.3.1 Call Intensity

Consider a single source whose call intervals have an exponential distribution

$$f(t) = \alpha\exp(-\alpha t). \qquad (2.23)$$

The conditional probability that the source makes a new call in the interval $(t, t + \Delta t)$ while there is already another call existing is

$$P(\Delta t) = 1 -\exp(-\alpha\Delta t) \qquad (2.24)$$

and for small Δt

$$P(\Delta t) \simeq \alpha. \Delta t \qquad (2.25)$$

Now consider x independent, free sources with exponential distribution of call intervals, the probability that ν of x sources make calls in

$(t, t + \Delta t)$ is then

$$P_\nu(\Delta t) = \binom{x}{\nu} \left[1-\exp(\alpha\Delta t)\right] \exp(-\alpha\Delta t(x-\nu)) \qquad (2.26)$$

The expected number of calls in the interval is

$$E(\nu) = x\left[1-\exp(-\alpha\Delta t)\right] \qquad (2.27)$$

and for small Δt

$$E(\nu) = x.\alpha\Delta t. \qquad (2.28)$$

Consequently the call intensity with x free sources is

$$y_x = x.\alpha \qquad (2.29)$$

Then the following assumptions (See. 2.3.1.1) which result in traffic distributions of known types, are made for the call intensity $y(j)$:

$y(j)$ is the call intensity of the sources when the system is in state j, and it is related to the birth coefficients by the assumption

$$\lambda_j = y(j).W(j) \qquad (2.30)$$

where $W(j)$ is the ability of the system to accept a new call when state j prevails.

$W(j) = 1$ denotes the conditions that a new call can always seize a devise in the systems.

$W(j) = 0$ denotes the condition the system cannot accept a new call.

$0< W(j)< 1$ denotes the condition that only certain calls can be accepted by the system

2.3.1.1 Assumptions

(i) $$y(j) = (N-j).\alpha \qquad (2.31)$$

Thus the call intensity decreasing with increased number of occupations, the resultant traffic distribution are the Bernouilli and Engset distributions.

(ii) $$y(j) = y \qquad (2.32)$$

Thus the call intensity being independent of the number of occupations, the resultant traffic distributions are the Erlang and Poisson distributions.

$$\text{(iii)} \quad y(j) = a(\gamma + j) \tag{2.33}$$

Thus the call intensity increasing with the number of occupations, the resultant traffic distributions are Negative Binomial Distributions.

These different distributions are dealt with in Appendix 4. Only the loss systems will be considered since they will be used in alternate routing problems in later chapters.

2.3.2 Termination

By assuming an exponential distribution of holding times, the duration of the individual occupation is described by

$$f(t) = \frac{1}{\bar{t}} \exp(- \frac{t}{\bar{t}}) \tag{2.34}$$

where $$\bar{t} = \int_0^\infty t.f(t)\, dt = \frac{1}{\alpha} \tag{2.35}$$

and the probability of termination of an occupation in $(t, t + \Delta t)$ is

$$P(\Delta t) = 1 - \exp(- \frac{\Delta t}{\bar{t}}) \tag{2.36}$$

and that it does not terminate

$$Q(\Delta t) = 1\text{-}P(\Delta t) = \exp(- \frac{\Delta t}{\bar{t}}) \tag{2.37}$$

If there are j independent occupations with the same holding time distribution, the probability that exactly ν of them terminate in $(t, t + \Delta t)$ is

$$P_\nu(\Delta t) = \binom{j}{\nu}.(1\text{-}\exp(-\frac{\Delta t}{\bar{t}}))^\nu \, . \, \exp(- \frac{\Delta t}{\bar{t}} (j - \nu)) \tag{2.38}$$

for $\nu = 0$

$$P_0(\Delta t) = \exp(- \frac{\Delta t}{\bar{t}} \, . \, j) \tag{2.39}$$

The probability that at least one ooaupation terminates is then

$$Q_0(\Delta t) = 1\text{-}P_0(\Delta t) = 1\text{-}\exp(-j. \frac{\Delta t}{\bar{t}}) \tag{2.40}$$

For small Δt

$$Q_o(\Delta t) = j.\frac{\Delta t}{\bar{t}} \tag{2.41}$$

The expected number of terminations in $(t, t + \Delta t)$ is

$$\xi(\nu) = \sum_{\nu=o}^{j} \quad .\binom{j}{\nu}.(1-\exp(-\frac{\Delta t}{\bar{t}}))^{\nu} \exp(-(j-\nu)\frac{\Delta t}{\bar{t}}) \tag{2.42}$$

$$\xi(\nu) = j.(1-\exp(-\frac{\Delta t}{\bar{t}})) \tag{2.43}$$

and for small Δt can be written as

$$\xi(\nu) \approx j.\frac{\Delta t}{\bar{t}} \tag{2.44}$$

Hence the intensity of termination for j exponentially distributed and independent occupations is

$$\mu_j = \frac{j}{\bar{t}} \tag{2.45}$$

For statistical equilibrium equation (2.22) can then be written as

$$P(j) = \frac{\prod_{\rho=o}^{j-1} \lambda_\rho \cdot \bar{t}^{j}}{j!} P(0) \qquad 0 < j \leq n \tag{2.46}$$

2.4 Some General Concepts and Definitions

2.4.1 Time Congestion (E)

Time congestion is the proportion of the time during which congestion prevails. It is also defined as the probability that all available devices are engaged.

$$E = \sum_{j>n} P(j) \tag{2.47}$$

$j > n$ denotes all states when congestion prevails.

2.4.2 Call Congestion (B)

Call congestion is the proportion of unsuccessful calls. It is also defined as the probability of an unsuccessful call.

$$B = \frac{\sum_{j>n} P(j).y(j)}{\sum_{\text{all } j} P(j)\ y(j)} \tag{2.48}$$

It is the expected number of calls when congestion prevails divided by the total number of expected calls. Calculated per time unit.

2.4.3 Traffic Carried (A_c)

Traffic carried is the mean number of simultaneous occupations.

$$A_c = \sum_{j=1}^{n} j\ P(j) \tag{2.49}$$

where n is the maximum number of simultaneous occupations.

The terminating rate is given by $\mu_j = \frac{j}{\bar{t}}$ and from statistical equilibrium (See 2.1.2.4)

$$\mu_j P(j) = \lambda_{j-1}\ P(j-1) \tag{2.50}$$

$$jP(j) = \bar{t}.\lambda_{j-1}\ P(j-1) \tag{2.51}$$

from which

$$A_c = \sum_{j=1}^{n} \bar{t}.\lambda_{j-1}\ P(j-1) \tag{2.52}$$

$$A_c = \bar{t} \sum_{j=0}^{n-1} \lambda_j\ P(j) \tag{2.53}$$

Thus the traffic carried is measurable,

2.4.4 Traffic Offered (A_o)

Traffic offered is the mean number of simultaneous occupations offered to the system (whether accepted or not).

$$A_o = \sum_{\text{all } j} \bar{t}.y(j).P(j) \tag{2.54}$$

Traffic offered cannot be measured, it is dependent exclusively on the theoretical model used.

2.4.5 Difference Between Traffic Offered and Traffic Carried

Traffic Offered

$$A_o = \sum_{j=0}^{r} \bar{t}.y(j).P(j) \tag{2.55}$$

and Traffic Carried

$$A_c = \sum_{j=0}^{n-1} \bar{t}.y(j).W(j)P(j) \tag{2.56}$$

$n < r$ and n is the maximum number of simultaneous occupations.

$$A_o-A_c = \sum_{j=0}^{n-1} \bar{t}.y(j)P(j)\left[1-W(j)\right]+ \sum_{j=n}^{r} \bar{t}.y(j).P(j) \tag{2.57}$$

or

$$A_o-A_c = \sum_{j=n}^{r} \bar{t}.y(j)P(j) \quad \text{if } W(j) = 1 \text{ for } j<n \tag{2.58}$$

This difference can also be defined as

$$A_o-A_c = \bar{t} \sum_{j=0}^{r} y(j)\ P(j)\left[1-W(j)\right] \tag{2.59}$$

such that $A_o-A_c > 0$ only if $W(j) < 1$ occurs in the system.

In which case $A_o > A_c$ for a loss system, and $A_o=A_c$ for a delay system provided that all queuing calls wait until served.

These concepts and definitions are going to be used in the Chapters that will follow.

CHAPTER 3

3.0 GRADING

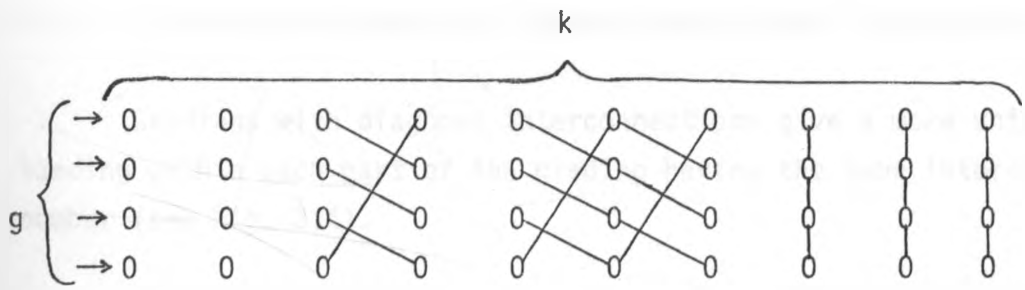
3.1 General

Grading is a method of connecting level multiples of a number of selectors together, so that an inlet group of selections is given access to individual trunks on the early choices but on the later choices, shares access to trunks with other groups.

Due to the commoning of outlets, the total number of trunks to which the grading has access is less than the maximum possible number of trunks.

Usually, a grading is arranged in such a manner that a selector hunts for a free outlet in a sequential manner. This is sequential hunting. Grading for random hunting will not be considered as this does not include cases which occur in alternate routing.

The main object of a grading is to increase the efficiency of the later devices by giving access to them from two or more inlet groups, However, a full availability group is more efficient than a corresponding graded group, the limited access imposes limitations in trunking efficiency. A typical grading is illustrated in Fig. 3.1.



g is the number of inlet groups
k is the availability (hunting capacity of a switch)

Fig. 3.1 A Typical Grading

3.2 Grading for Sequential Hunting

Grading for sequential hunting are usually designed either with a degree of interconnection which rises along the direction of hunting, known as the B.P.O. or O'Dell type, or alternatively designed as homogeneous gradings.

The O'Dell type of grading in its original and simplest form has only straight interconnections, i.e. outlets in the same hunting position are interconnected between the nearest inlet groups, (see Fig. 3.2).

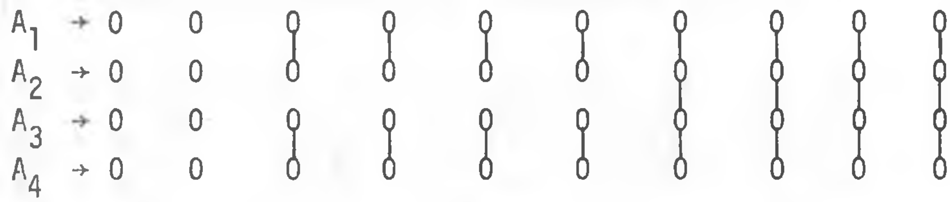


Fig. 3.2 Straight Multiple with an Increasing Interconnection Number.

An improvement is obtained if the interconnections are arranged for all combinations of switching groups, (see Fig. 3.3).

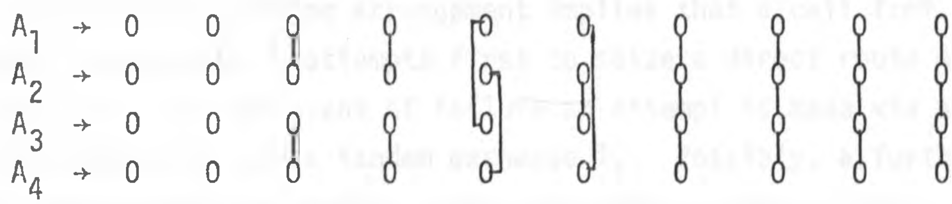


Fig. 3.3 More Interconnection Combinations Between Inlet Groups.

Gradings with diagonal interconnections give a more uniform loading within each part of the grading having the same interconnection number (see Fig. 3.4).

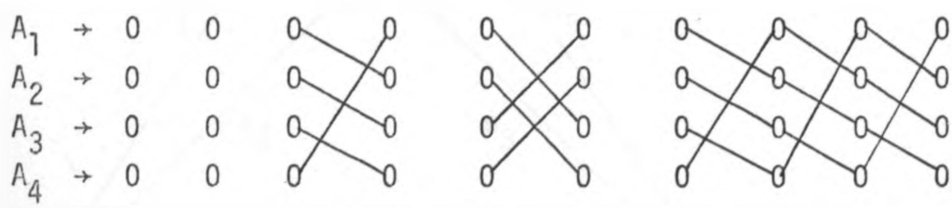


Fig. 3.4 Diagonal Interconnections

Examples of homogeneous gradings with either straight interconnections (Fig. 3.5) or diagonal interconnections (Fig. 3.6) are given below.

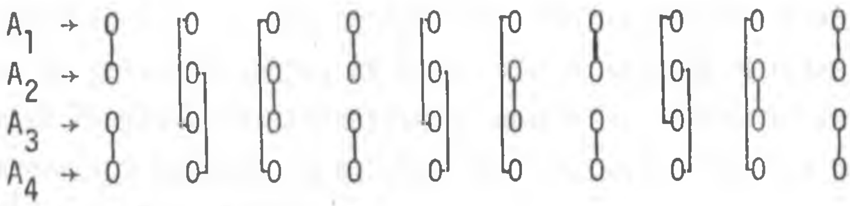


Fig. 3.5 Straight Interconnections Homogeneous Grading .

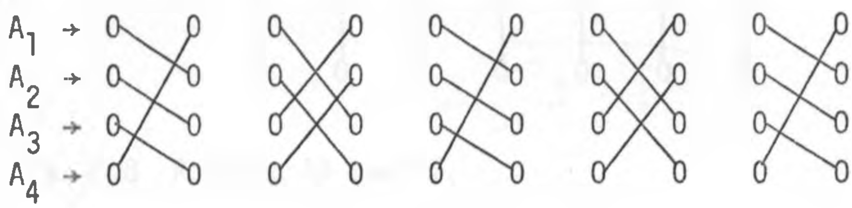


Fig. 3.6 Homogeneous Grading Diagonal Interconnections.

3.3 Gradings for Alternate Routes

An alternate routing arrangement implies that a call from exchange A to exchange B attempts first to seize a direct route AB (see Fig. 3.7). In the event of failure an attempt is made via an alternate route AT_1B and a tandem exchange T_1. Possibly, a further attempt may be made via another tandem exchange T_2 and so forth.

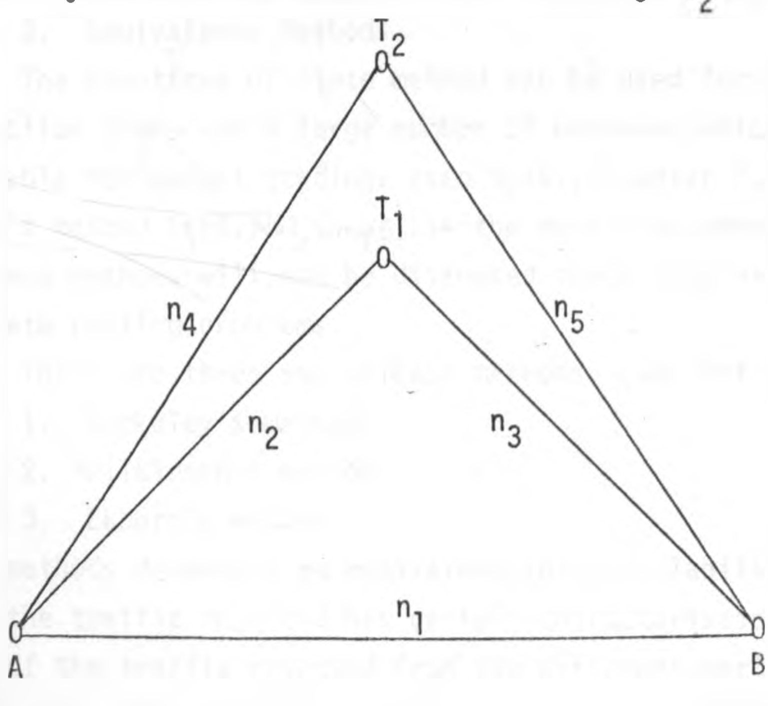


Fig. 3.7 Alternate Routing Arrangement

The direct route AB carries only the traffic flow initiated in A and destined for B. The tandem route AT_1 carries also the traffic which is to pass the tandem exchange T_1. Route T_1B carries the traffic to B which normally or alternatively passes T_1. For the case of AB, AT_1 the grading pattern is as shown in Fig. 3.8. AT_1 may be both first choice and overflow traffic.

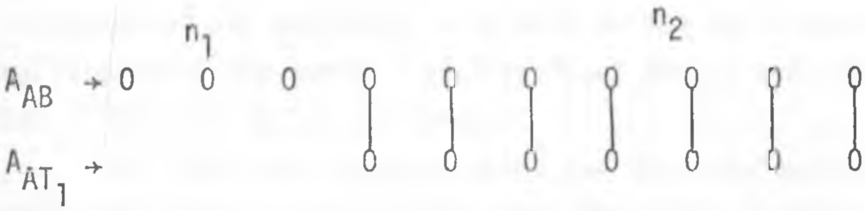


Fig. 3.8 Routing AB and AT_1

Alternate routings are thus a special type of grading with sequential hunting in which the inlet groups have different hunting capacities.

3.4 Methods of Computing Gradings with Sequential Hunting

Generally, gradings can be calculated by one of the following methods:

1. Equations of State
2. Weighting Methods
3. Equivalence Methods

The Equations of State method can be used for very small gradings, In practice there are a large number of unknowns which make the method unsuitable for normal gradings (see Syski, Chapter 7, Section 2). O'Dell"s method (Ref. No 1, Chap.5) is the most used among Weighting Methods, but these methods will not be discussed since they will not be used in alternate routing problems.

There are three equivalence methods. (see Ref. No. 1, Chapter 5):

1. Berkeley's method
2. Wilkinson's method
3. Ekberg's method

These methods determine an equivalent full availability group from which the traffic rejected has certain characteristics similar to those of the traffic rejected from the different parts of the grading. In this way, the properties of the sequentially limited grading are reduced to an equivalent full availability group. Wilkinson's method will be dealt with in detail (Ref. No. 2)

3.4.1 Wilkinson's Method

3.4.1.1 Wilkinson's Equivalent Random Theory

The overflow traffic from one or more first choice groups which are offered independent random traffic is described by two parameters, its mean and its variance. The mean of the total overflow traffic is calculated as the sum of the individual means, and the variance as the sum of the individual variances.

All the first choice groups are then replaced by an equivalent full availability group, from which the total overflow traffic has the same mean and the same variance as the total overflow traffic from the individual groups. From these two conditions the number of functions (trunks or circuits), and the (random) traffic offered to the equivalent group are determined. The complex overflow system is thus reduced to a full availability group consisting of the equivalent group and the common overflow group.

The equivalent group is determined in the following way:

1. The mean M_ν and the variances V_ν of the individual overflow traffics from first choice groups are calculated from (see ref. No. 2, Appendix II).

$$M_\nu = A_\nu \cdot E_{n_\nu}(A_\nu) \tag{3.1}$$

$$V_\nu = M_\nu \left[1 - M_\nu + \frac{A_\nu}{n_\nu + 1 + M_\nu - A_\nu}\right] \tag{3.2}$$

where $E_{n_\nu}(A_\nu)$ is the Erlang Loss Formula, n_ν is the number of functions and A_ν is the traffic offered to the νth first choice group. For a "first routed traffic" i.e. a random traffic offered to the common group, we have (see Elldin and Lind Chapter 3, ref. No. 1).

$$M_\nu = V_\nu = A_\nu \tag{3.3}$$

Since for random traffic input a Poissonian process is assumed when the number of subscribers initiating calls is large.

2. The mean M and variance V of the total overflow traffic is obtained from

$$M = \sum_\nu M_\nu \tag{3.4}$$

$$V = \sum_{\nu} V_{\nu} \tag{3.5}$$

3. The number of junctions n in the equivalent group and the traffic A offered to the equivalent group are calculated from the system

$$M = AE_n(A) \tag{3.6}$$

$$V = M\left[1-M+\frac{A}{n+1+M-A}\right] \tag{3.7}$$

From the equivalent scheme various congestion estimates may be formed to determine the number of junctions, m, in the common overflow group. The traffic rejected from the whole system is estimated as

$$A.E_{n+m}(A) \tag{3.8}$$

and this defines

1. the overall average congestion as

$$E = \frac{A.E_{n+m}(A)}{\sum_{\nu} A_{\nu}} \tag{3.9}$$

2. the average congestion for traffic offered to the overflow group as

$$E_t = \frac{A.E_{n+m}(A)}{M} = \frac{E_{n+m}(A)}{E_n(A)} \tag{3.10}$$

3. It is assumed that the individual congestion is approximately proportional to the degree of degeneration, i.e.,

$$E_{\nu} = k\,\frac{V_{\nu}}{M_{\nu}} \tag{3.11}$$

where k is a constant, determined by the condition

$$AE_{n+m}(A) = \sum_{\nu} M_{\nu}E_{\nu} = \sum_{\nu} kV_{\nu} = kV \tag{3.12}$$

giving

$$k = \frac{A\,E_{n+m}(A)}{V} = \frac{M}{V}\,E_t \tag{3.13}$$

and

$$\nu = \frac{\dfrac{V_\nu}{M_\nu}}{\dfrac{V}{M}} \; \frac{E_{n+m}(A)}{E_n(A)} \tag{3.14}$$

giving

$$E_\nu = \frac{V_\nu}{V} \; \frac{A \, E_{n+m}(A)}{M_\nu} \tag{3.15}$$

<u>3.4.1.2 Example</u>

Given a grading shown in Fig. 3.9 one first forms

$$M_\nu = A_\nu \cdot E_{n_\nu}(A_\nu) \tag{3.16}$$

$$V_\nu = M_\nu \left[1 - M_\nu + \frac{A_\nu}{n_\nu + 1 + M_\nu - A_\nu} \right] \tag{3.17}$$

(ν = 1, 2, 3 and 4)

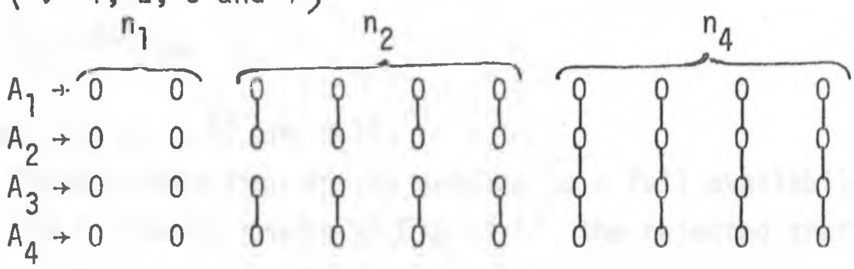


Fig. 3.9 Grading

The equivalent full availability group common to groups 1 and 2 after n_1 devices is then determined from the conditions (see Fig. 3.10).

$$M_{12} = M_1 + M_2 \tag{3.18}$$

$$V_{12} = V_1 + V_2 \tag{3.19}$$

from which the equivalent traffic A_{12} and the equivalent number of trunks n_{12} are determined, (Fig. 3.11).

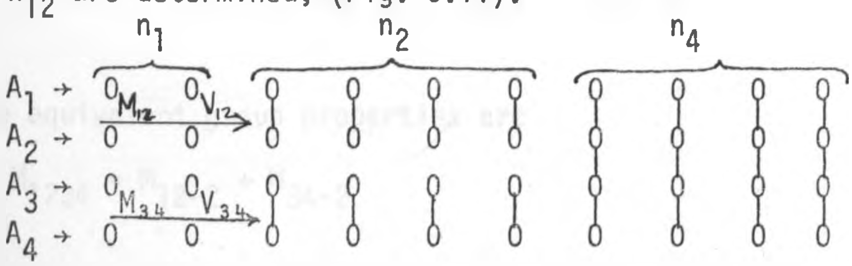


Fig. 3.10

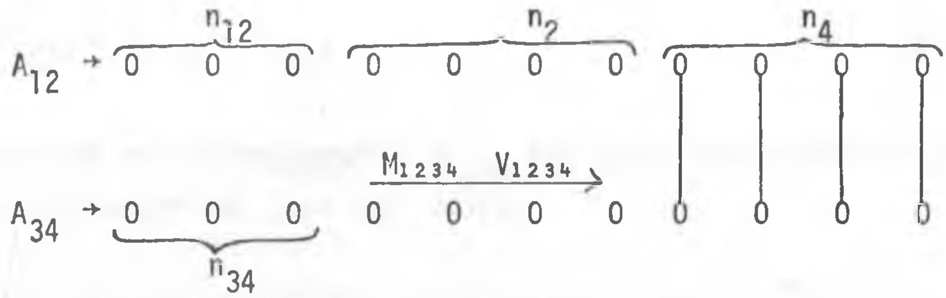


Fig. 3.11 Equivalent Full Availability Group Common to Groups 1 and 2 and Groups 3 and 4 after n_1 devices.

The equivalent full availability group common to groups 3 and 4 after n_1 devices is also determined in the same manner obtaining

$$M_{34} = M_3 + M_4 \tag{3.20}$$

$$V_{34} = V_3 + V_4 \tag{3.21}$$

A_{34} and n_{34}

as shown in Fig. 3.10 and 3.11.

Further reduction of the grading to a full availability group offers the n_4 common trunks in Fig. 3.11, the rejected traffics

$$M_{12-2} = A_{12}\, E_{n_{12}+n_2}(A_{12}) \tag{3.22}$$

$$M_{34-2} = A_{34}\, E_{n_{34}+n_2}(A_{34}) \tag{3.23}$$

whose variances are

$$V_{12-2} = M_{12-2}\left[1 - M_{12-2} + \frac{A_{12}}{n_{12}+n_2+1+M_{12-2} - A_{12}}\right] \tag{3.24}$$

$$V_{34-2} = M_{34-2}\left[1 - M_{34-2} + \frac{A_{34}}{n_{34}+n_2+1+M_{34-2} - A_{34}}\right] \tag{3.25}$$

and the equivalent group properties are

$$M_{1234} = M_{12-2} + M_{34-2} \tag{3.26}$$

$$V_{1234} = V_{12-2} + V_{34-2} \tag{3.27}$$

from which the equivalent traffic A_{1234} and equivalent number of trunks n_{1234} can be determined (see Fig. 3.12).

$$A_{1234} \quad \overbrace{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0}^{n_{1234}} \quad \overbrace{0 \quad 0 \quad 0 \quad 0}^{n_4}$$

Fig. 3.12 The Equivalent Full Availability Group

The grading has now been reduced to the full availability group. The total traffic rejected from this group is now

$$A_{1234}\, E_{n_{1234} + n_4}(A_{1234}) \tag{3.28}$$

and the resulting congestion for all traffic to the grading (overall average congestion) is

$$\bar{E} = \frac{A_{1234} E_{n_{1234}+n_4}(A_{1234})}{A_1 + A_2 + A_3 + A_4} \tag{3.29}$$

while the congestion for the individual inlet groups of traffic are determined as shown below.

$$E_1 = \frac{V_1}{V_{1234}} \cdot \frac{A_{1234} E_{n_{1234}+n_4}(A_{1234})}{M_1} \tag{3.30}$$

$$E_2 = \frac{V_2}{V_{1234}} \quad \frac{A_{1234} E_{n_{1234}+n_4}(A_{1234})}{M_2} \tag{3.31}$$

$$E_3 = \frac{V_3}{V_{1234}} \quad \frac{A_{1234} E_{n_{1234}+n_4}(A_{1234})}{M_3} \tag{3.32}$$

$$E_4 = \frac{V_4}{V_{1234}} \quad \frac{A_{1234} E_{n_{1234}+n_4}(A_{1234})}{M_4} \tag{3.33}$$

CHAPTER 4

4.0 ALTERNATE ROUTING

4.1 General Principles of Alternate Routing

Generally, alternate routing of telephone traffic is a procedure whereby a parcel of traffic is provided a second, third or more choice route to its destination, see Fig. 4.0. A parcel of traffic originating at A and terminating at B is provided with a second choice route via C. The second choice route can also be used by one or more other parcels of traffic, for instance, traffic originating at A and destined to D can also be provided with a second choice route via C. As a result the route AC serves as a second choice route to the two parcels of traffic:

(i) originating at A and destined to B

(ii) originating at A and destined to D

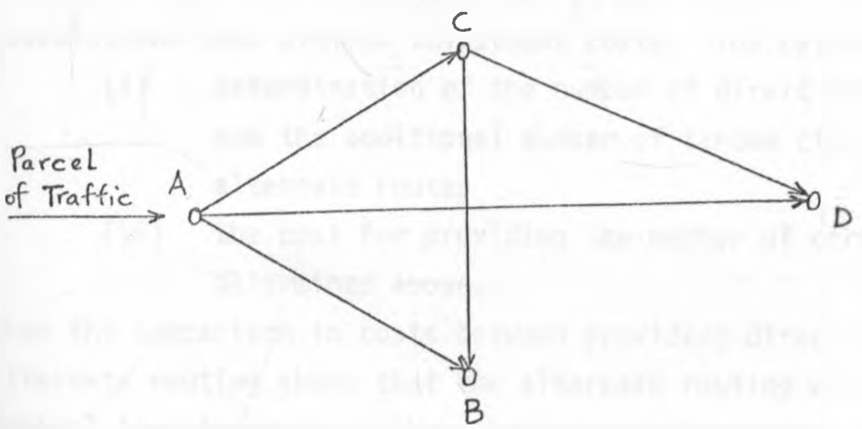


Fig. 4.0 Network Illustrating Alternate Routing

The use in common of the second choice route by the several parcels of traffic results in an overall requirement of trunk routes to the particular destination which is less than would be the case if each parcel of traffic had sole access to its own group of routes. This is easily realized in graded multiple arrangements, see Fig. 4.1. If parcels of traffic, p, q and r had each access to its own group of trunk routes, the total number of trunks would have been more than those specified in Fig. 4.1. While the efficiency per trunk would not very much be less than that required for the single trunk group. Hence, alternate routing is introduced in order to reduce the total cost of a telephone network, including cables and switches.

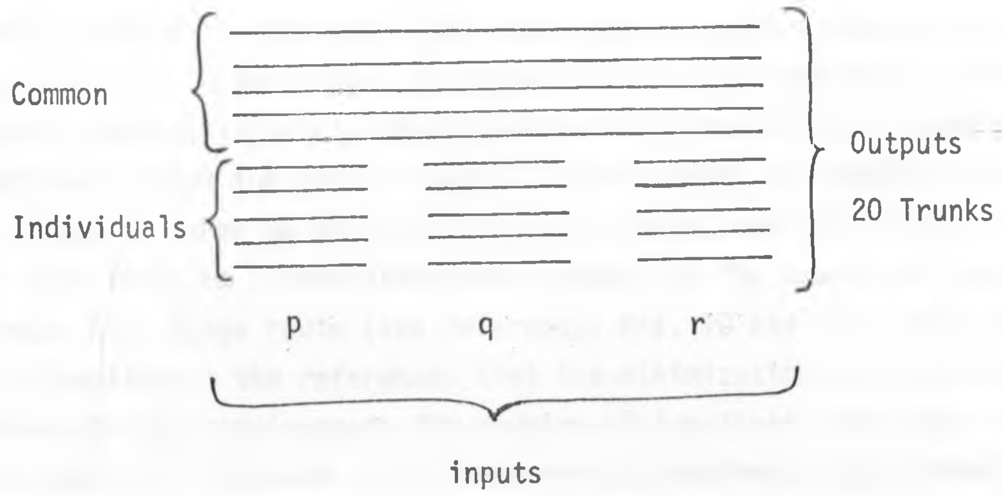


Fig. 4.1 Graded Multiple Arrangement

To develop a network with a given number of exchanges placed at given points, alternate choices can be provided on routes for which calculations show minimum investment costs. The calculations involve:

(i) determination of the number of direct circuits and the additional number of tandem circuits on alternate routes

(ii) the cost for providing the number of circuits determined above.

When the comparison in costs between providing direct routing and alternate routing shows that the alternate routing will provide the optimal investment cost, then alternate routing should be introduced.

4.2 Optimization of the Number of Circuits

Determination of the exact number of direct circuits and additional circuits required on tandem routes is in principle a problem of arriving at the most economical system.

The numbers can be estimated separately for each routing case. When all cases have been considered, they provide an estimate of the total cost of the network. Since the different routing cases interfere in the alternate routes, certain recalculations may be necessary.

However, this method of optimization which may be referred to as the Constant Background Traffic model assumes that traffic other than that originated at point O^k and going to point D^k is background traffic which is constant. Thus when optimising procedures are used, any variation in the number of circuits on a route shared by several traffic parcels is solely due to variation in the O^k to D^k traffic overflowing from the direct route. In this model the number of junctions on a link is taken to be a continuous variable, and the minimization of the cost function is performed with respect to the number of junctions on each high usage route (see references Nos. 10 and 11). But, there is no mention in the references that the minimization is subject to the non-negativity requirements for numbers of junctions, and when this requirement is included in the mathematical analysis, the optimality conditions are incomplete, (see reference No. 12).

The method of optimization that will be used in this dissertation is based on Berry's model, (see reference No. 8 and Chapter 5). This method assumes that traffic is characterised by its mean and its variance. Berry showed that a general formula for the number of junctions on a given link i can be expressed as

$$n_i = \hat{x}_i + A_i \left[\frac{(M_i-\hat{x}_i)^2 - v_i}{(M_i-\hat{x}_i)^2 - (M_i-\hat{x}_i) + v_i} - \frac{M_i^2 + V_i}{M_i^2 - M_i + V_i} \right]$$

where

$\hat{x}_i$ is the total flow on link i

A_i the equivalent random traffic producing M_i and V_i (mean and variance) on link i, is given by the Rapp formula (reference No. 4):

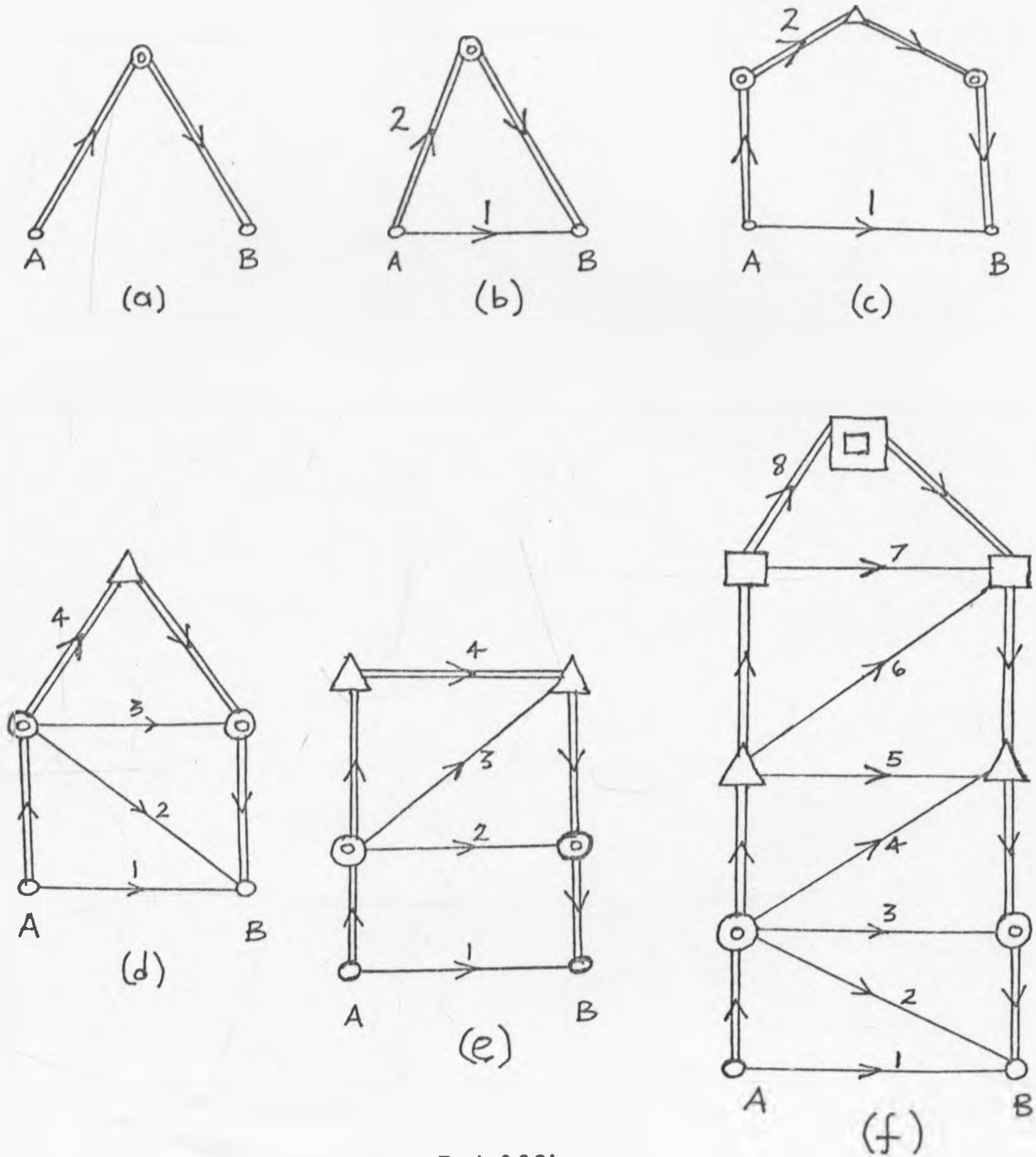
$$A_i = V_i + \frac{3V_i}{M_i} \left(\frac{V_i}{M_i} - 1\right)$$

and the variance of the overflow traffic from link i is given by

$$v_i = ((M_i-\hat{x}_i)(3-(M_i-\hat{x}_i) + ((M_i-\hat{x}_i-3)^2 + 12A_i)^{\frac{1}{2}}))/6$$

For a direct link it is assumed that the offered traffic is Poissonian and hence the equivalent random traffic is equal to the offered traffic. The aim of Berry's model is to obtain a System Optimised Chain Flow Pattern[12]. Berry's model is discussed in detail in Chapter 5.

Fig. 4.2 shows the various routing patterns which may arise between two exchanges, the figures indicate the order of selection.



End Office

Group Switching Centre

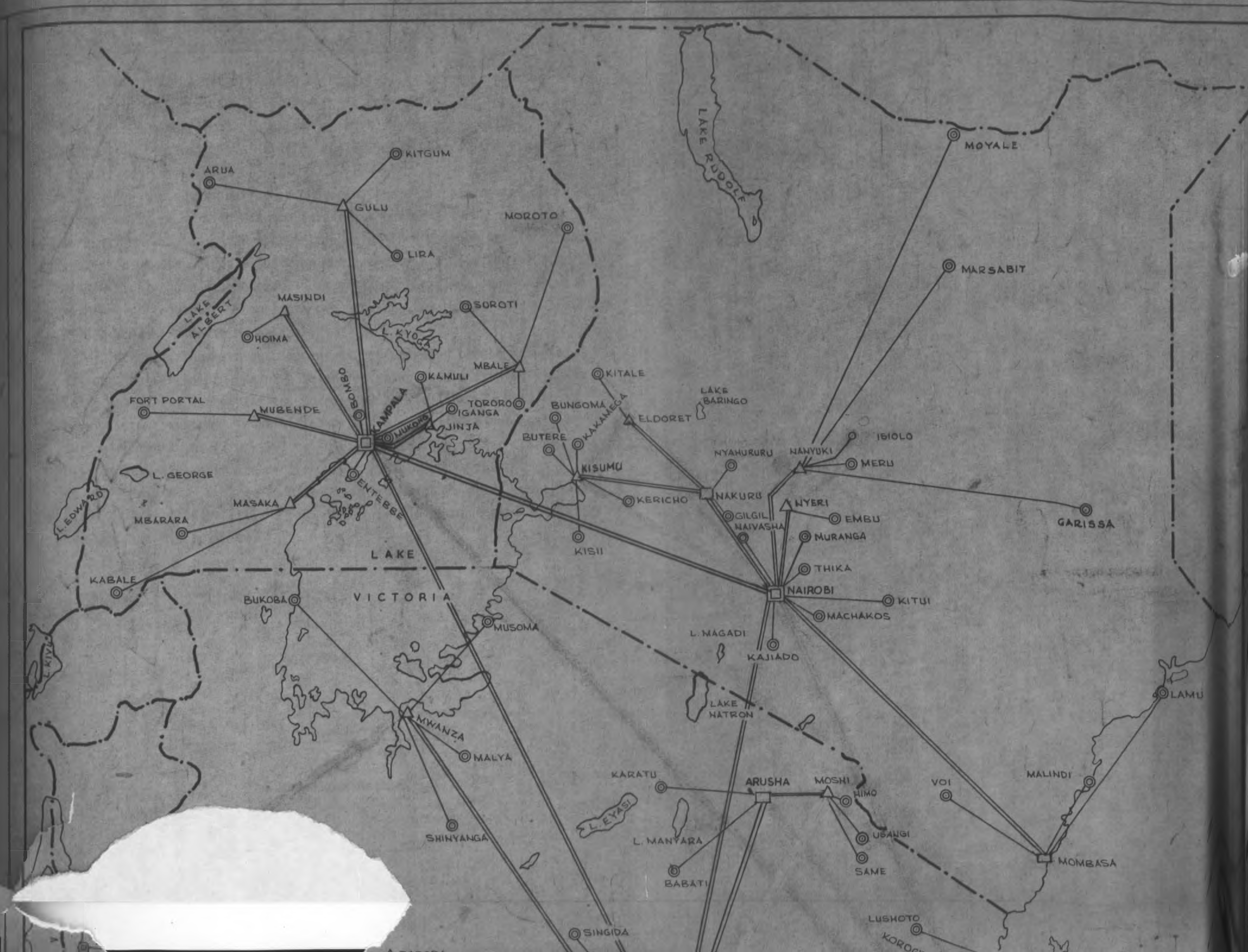
Area Switching Centre

District Switching Centre

National Switching Centre

Fig. 4.2 Various Routing Patterns Between Two Exchanges

EAST AFRICA


LAKE RUDOLF
MOYALE
MARSABIT
KITGUM
ARUA
GULU
LIRA
MOROTO
MASINDI
HOIMA
LAKE ALBERT
L. KYOGA
SOROTI
MBALE
KAMULI
TORORO
IGANGA
JINJA
MUKONO
KAMPALA
BOMBO
FORT PORTAL
MUBENDE
L. GEORGE
L. EDWARD
ENTEBBE
MASAKA
MBARARA
KABALE
KITALE
LAKE BARINGO
BUNGOMA
KAKAMEGA
ELDORET
BUTERE
KISUMU
KERICHO
KISII
NYAHURURU
NAKURU
GILGIL
NAIVASHA
NANYUKI
ISIOLO
MERU
NYERI
EMBU
MURANGA
THIKA
GARISSA
NAIROBI
KITUI
MACHAKOS
L. MAGADI
KAJIADO
LAKE NATRON
LAKE VICTORIA
BUKOBA
MUSOMA
MWANZA
MALYA
SHINYANGA
L. KIVU
KARATU
ARUSHA
MOSHI
HIMO
L. EYASI
L. MANYARA
BABATI
USANGI
SAME
VOI
MALINDI
LAMU
MOMBASA
LUSHOTO
SINGIDA

4.3 The East African Routing Network

The routing network shown in Fig. 4.3 is the trunk routing network in East Africa, This network is developing very rapidly, and as a result more direct routes are going to be required between the various centres, with the existing routes being used as the final routes.

For the determination of providing the alternate routes economically (i.e. with minimum investment costs) a three node network will be adequate for most of the cases in question, for instance, provision of direct routes between Mbeya and Iringa, see Fig. 4.3, with a final route via Dodoma tandem exchange, or Jinja and Mbale, with a final route via Kampala tandem exchange, or Malindi and Nairobi with a final route via Mombasa.

However, the main development is concentrated on the Kampala, Nairobi, Dar-es-Salaam, Kampala triangle, in which a microwave link is being installed, except for the Dodoma to Kampala link which is a troposcatter system. These cities have 4-wire switched national switching exchanges (Trunk Switching Units). Such that the transmission losses between the centres are very low. This then conveniently forms a three-node mesh network which is going to form the study case in this paper, see Fig. 4.4. The links connecting the cities are numbered one to three.

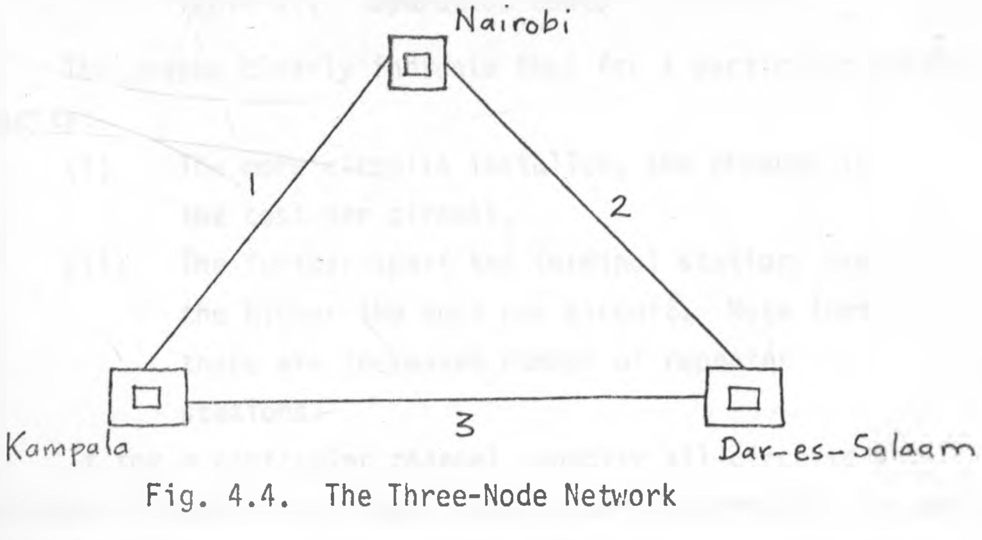


Fig. 4.4. The Three-Node Network

4.3.1 The Costing of the Network

E.A.P.& T. Corporation which is interested in this study has provided the figures that form the basis of our cost calculations. For the radio routes, graphs are given in Fig. 4.6 to Fig. 4.21 which provide equipment and installation costs per circuit for specified number of radio circuits. The operation costs are calculated from the percentages of investment costs as given in Table 4.1 below. However, the equipment, installation and operation costs for multiplexing and switching equipment has been given by E.A.P. & T (Kenya shillings) as Kshs.12,000 per circuit.

Project	Operation Costs as a Percentage of Investment
Exchanges	3%
Multiplex Equipment	0.8%
Other Transmission Equipment	3.3%
U.H.F.	5%
Junction Cable	0.9%
Coaxial Cable	0.8%
Housing	2.5%

Table 4.1 Operation Costs

The graphs clearly indicate that for a particular channel capacity:

(i) The more circuits installed, the cheaper is the cost per circuit.

(ii) The further apart the terminal stations are the higher the cost per circuit. Note that there are increased number of repeater stations.

If for a particular channel capacity all circuits are installed to start with, then the cost per circuit will be very low. But this cost includes the constant and variable costs. Thus,

$$c_\ell = c_c + c_v \qquad (4.1)$$

where

c_ℓ - the cost per circuit

c_c - the constant cost per circuit

c_v - the variable cost per circuit

For 300- and 960- channel systems considered, it is assumed that to provide 50 circuits or less, the cost per circuit will be constant, based on the cost per circuit when 50 circuits are provided. As a result an average cost per circuit is calculated based on the cost per circuit when 50 circuits are provided, and when all the circuits are provided. The relevant graphs to the distances between the cities are interpolated to provide this information (see Fig. 4.10A, 4.12, 4.13, 4.17, 4.19, 4.20, 4.21).

The total cost, C, is then a function of the cost per circuit, and the number of circuits provided on both the direct and alternate or overflow routes. This is represented mathematically in equation (4.2) below

$$C = \sum_{k=1}^{N} c_k n_k + \sum_{i=1}^{\hat{N}} \hat{c}_i \hat{n}_i \qquad (4.2)$$

where

C - the total cost of the network

c_k - the cost per circuit on the direct route

n_k - the number of circuits on the direct route

$\hat{c}_i$ - the cost per circuit on the overflow route

$\hat{n}_i$ - the number of circuits on the overflow route

N - the maximum number of direct routes

$\hat{N}$ - the maximum number of overflow routes

k - the link number on the direct route

i - the link number on the overflow route

A graph of the total cost, C, as a function of the number of circuits installed would be of the form shown in Fig. 4.5.

The points of discontinuity in cost represent the maximum capacity in number of circuits that a particular route may have been planned to accommodate. When this maximum number of circuits is being approached, an extension is normally planned, in advance, for a further, say, ten year period. As the costing process for a

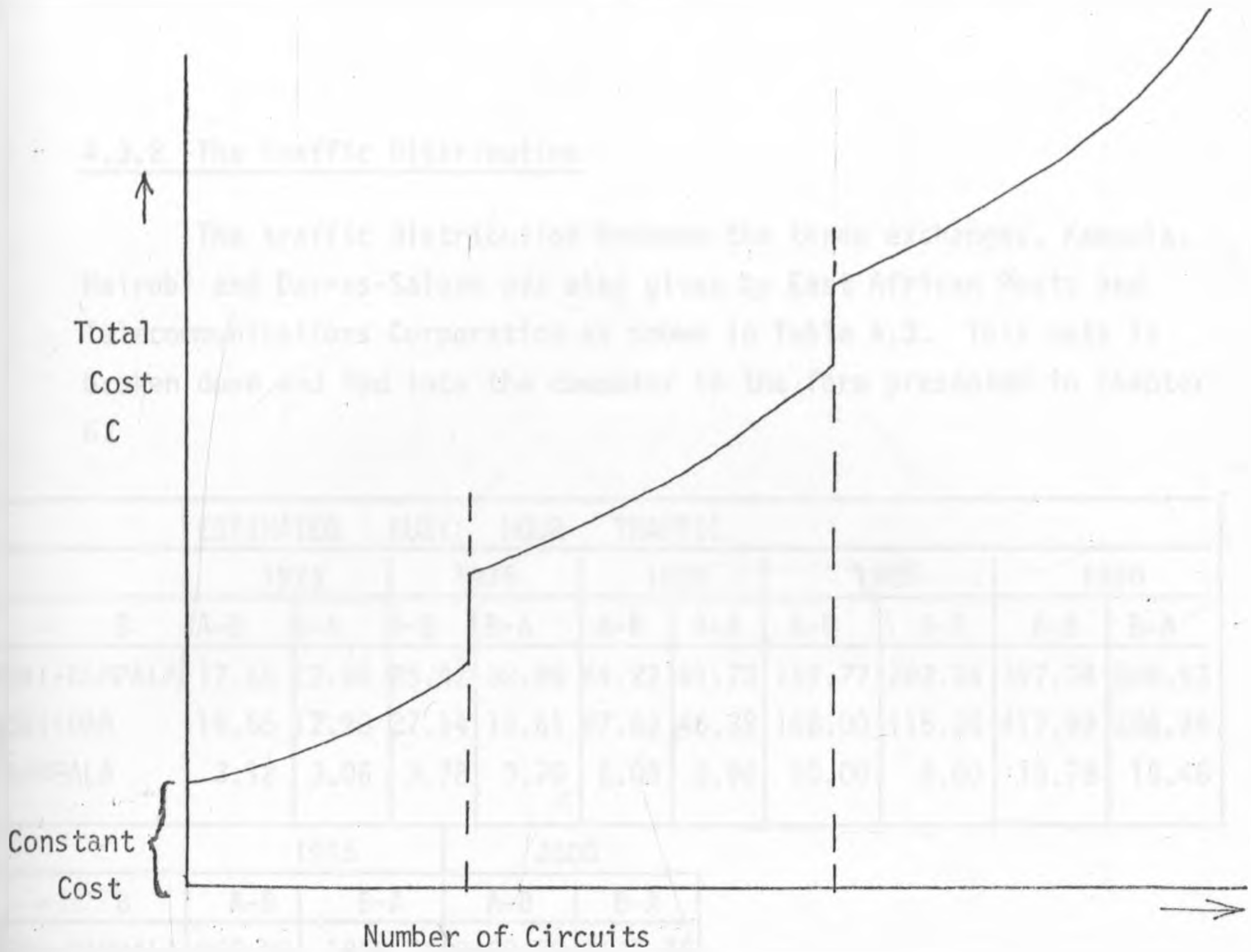


Fig. 4.5 Total Cost as a Function of Number of Circuits

system with a higher maximum capacity to cater for the period in question is being considered, a new constant cost is consequently determined. This follows, as a result of the constant costs involved in the investment.

The costs per circuit calculated from the graphs, Table 4.1 and considering the multiplexing and switching costs are given in Table 4.2.

Direct Routes		Alternate Routes	
Link No.	Cost Per Circuit Kshs.	Link No.	Cost Per Circuit Kshs.
1	37570	1	37570
2	54470	2	54470
3	78730	3	78730

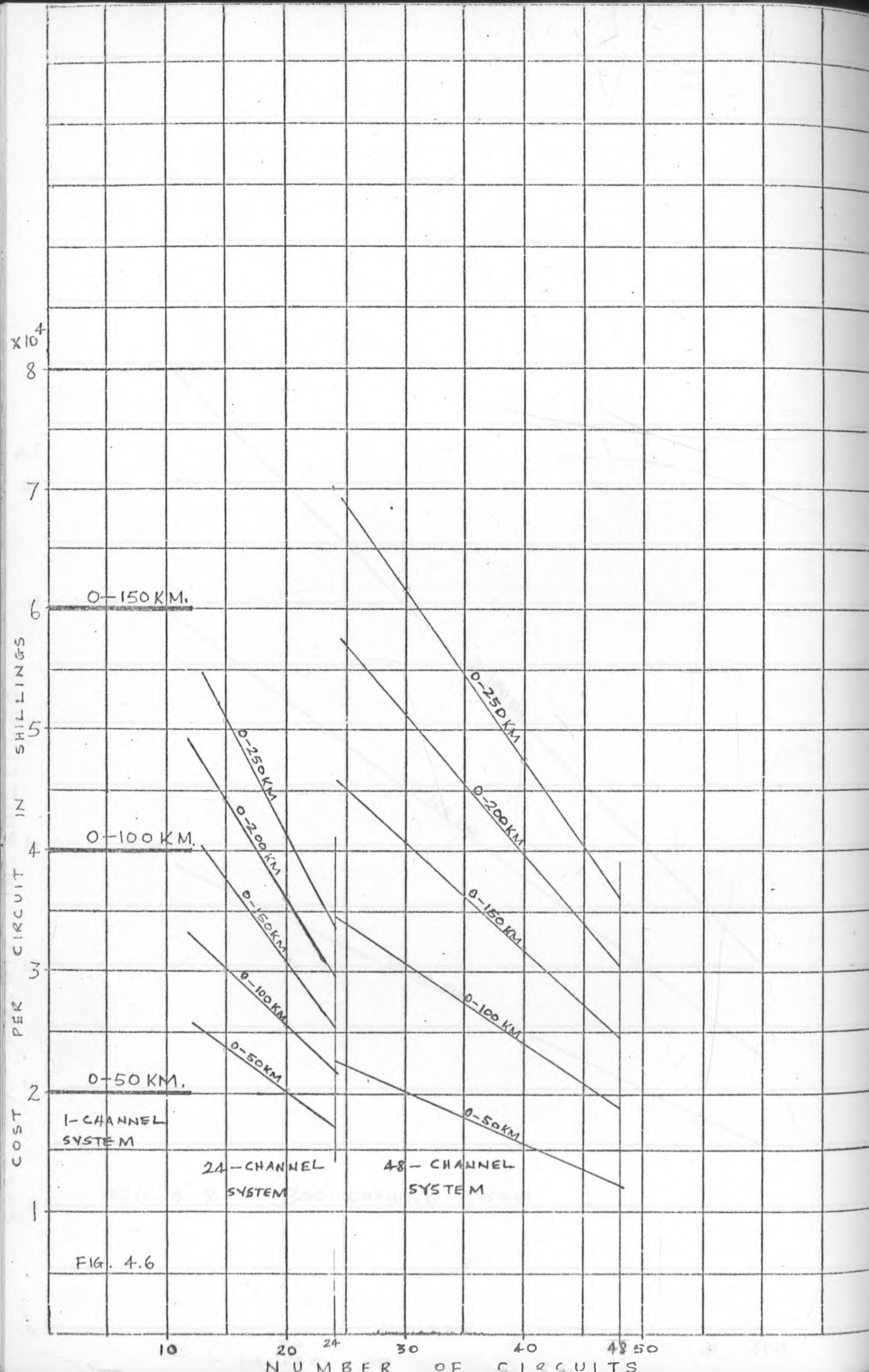
Table 4.2 Costs Per Circuit

4.3.2 The Traffic Distribution

The traffic distribution between the three exchanges, Kampala, Nairobi and Dar-es-Salaam was also given by East African Posts and Telecommunications Corporation as shown in Table 4.3. This data is broken down and fed into the computer in the form presented in Chapter 6.

	ESTIMATED BUSY HOUR TRAFFIC													
	1973		1975		1980		1985		1990		1995		2000	
A - B	A-B	B-A	A-B	B-A	A-B	B-A	A-B	B-A	A-B	B-A	A-B	B-A	A-B	B-A
NAIROBI-KAMPALA	17.65	22.81	25.42	32.85	64.22	81.73	159.77	203.34	397.34	505.93	988.98	1258.74	2460.59	3131.75
NAIROBI-DAR	18.85	12.93	27.14	18.61	67.52	46.32	168.00	115.26	417.99	286.76	103994	713.45	2587.39	1775.06
DAR-KAMPALA	3.12	3.06	3.78	3.70	6.08	5.96	10.00	9.60	15.78	15.48	25.43	24.94	40.97	40.18

Table 4.3 The Traffic Distribution



x10^4
8
7
6
5
4
3
2
1
COST PER CIRCUIT IN SHILLINGS
0-150 KM.
0-100 KM.
0-50 KM.
1-CHANNEL SYSTEM
0-250 KM
0-200 KM
0-150 KM
0-100 KM
0-50 KM
24-CHANNEL SYSTEM
48-CHANNEL SYSTEM
10
20
24
30
40
48
50
NUMBER OF CIRCUITS

FIG. 4.6

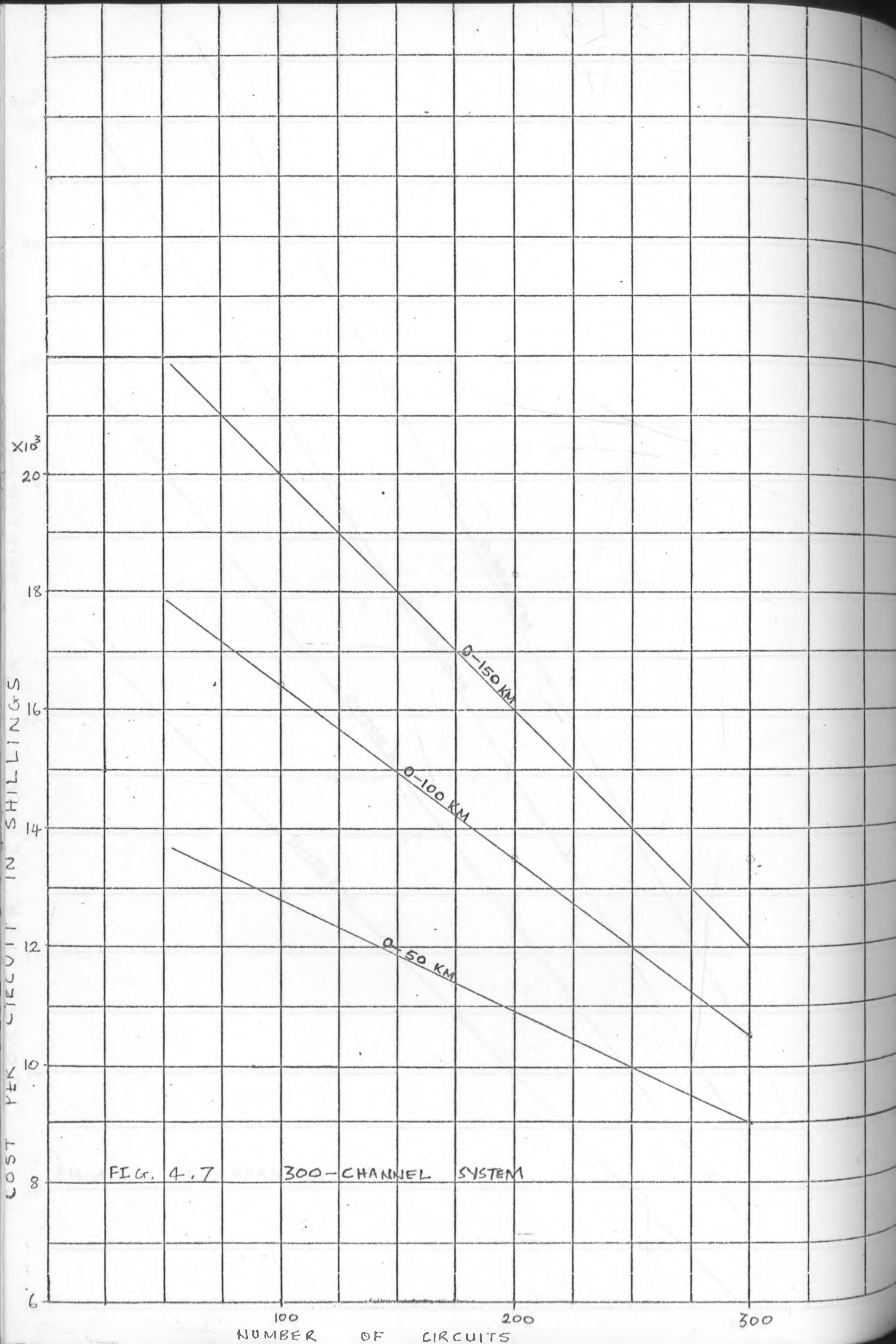


FIG. 4.7 300-CHANNEL SYSTEM

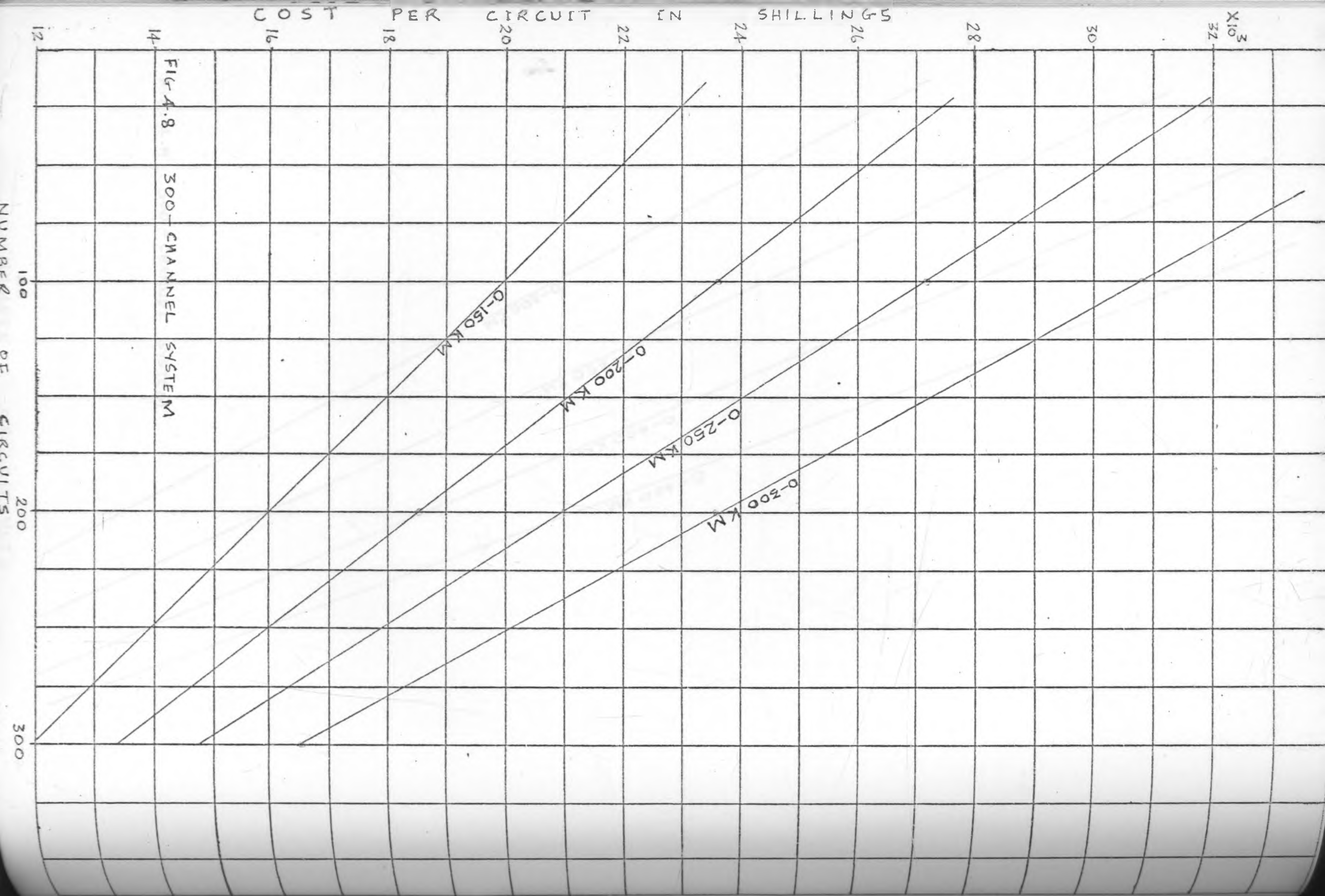


FIG 4.8 300-CHANNEL SYSTEM

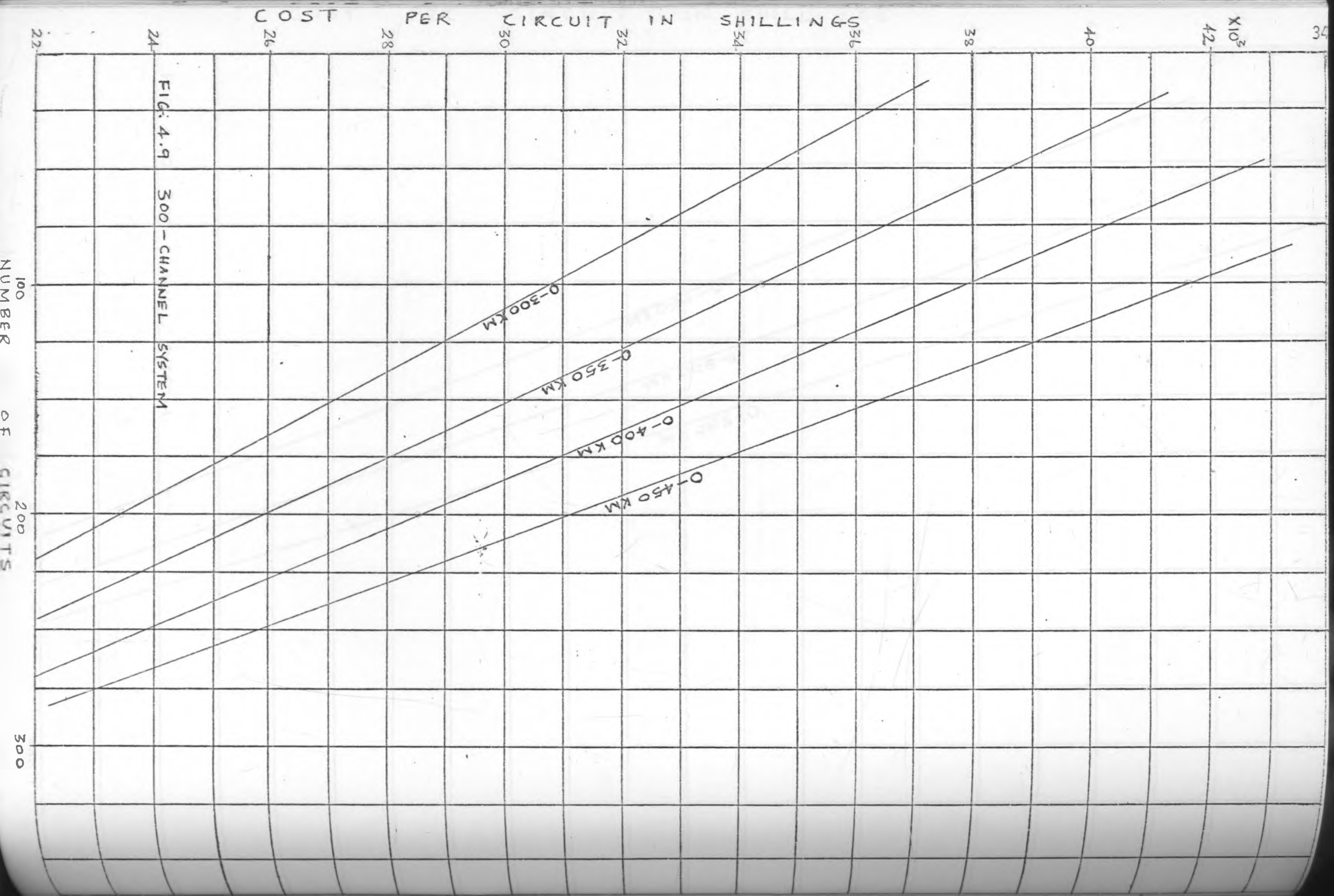


FIG. 4.9 300-CHANNEL SYSTEM

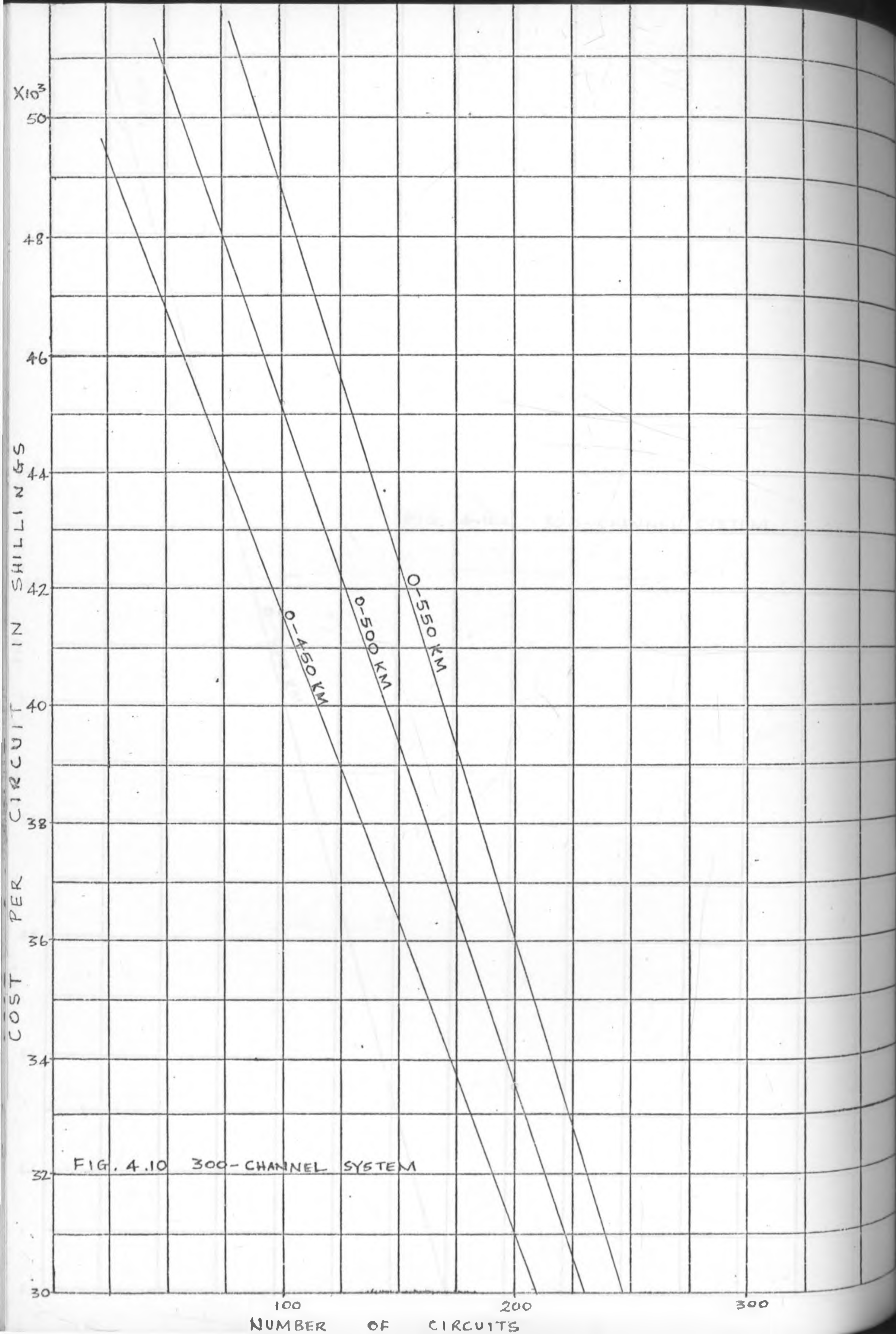


FIG. 4.10 300-CHANNEL SYSTEM

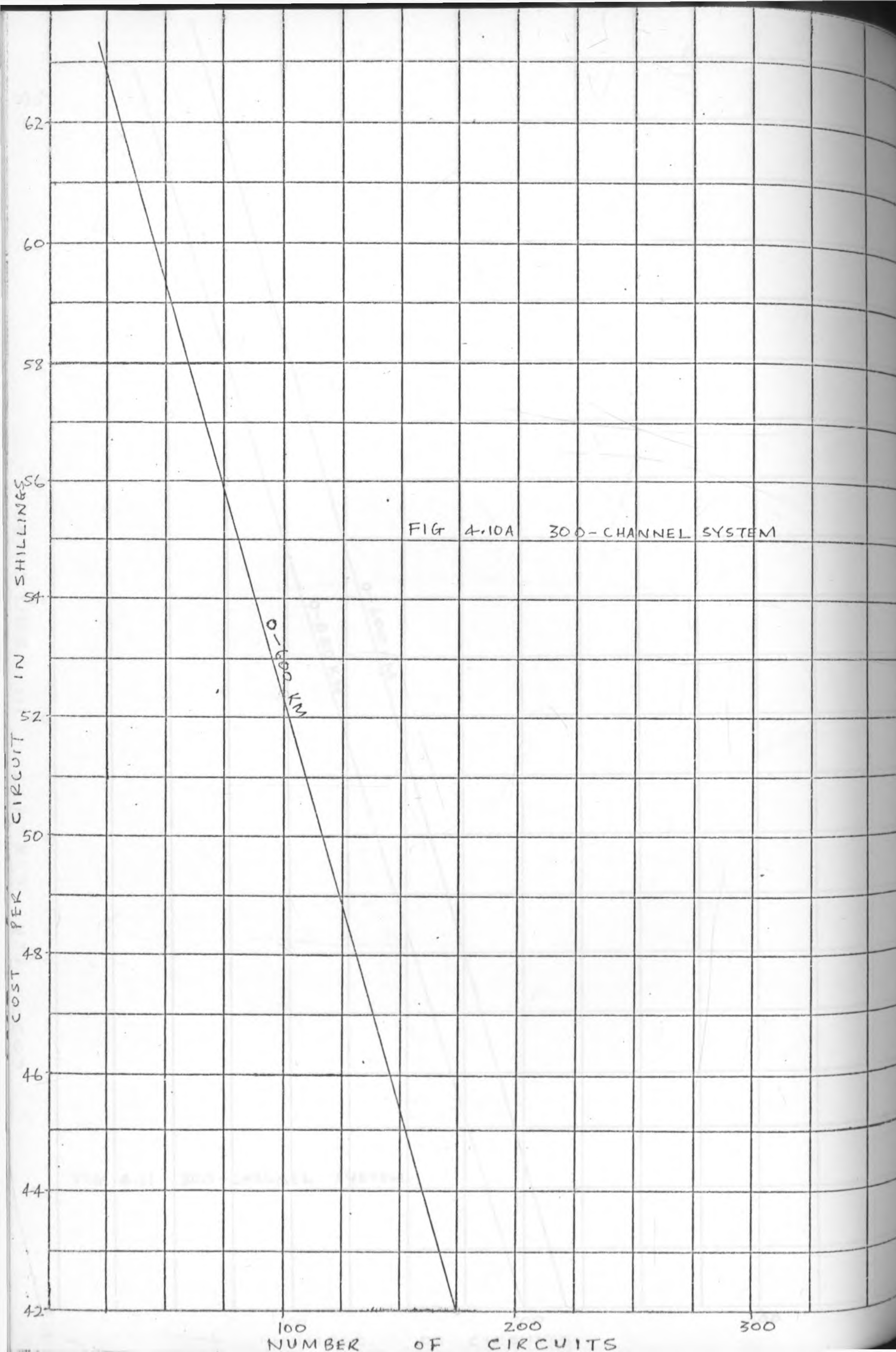


FIG 4.10A 300-CHANNEL SYSTEM

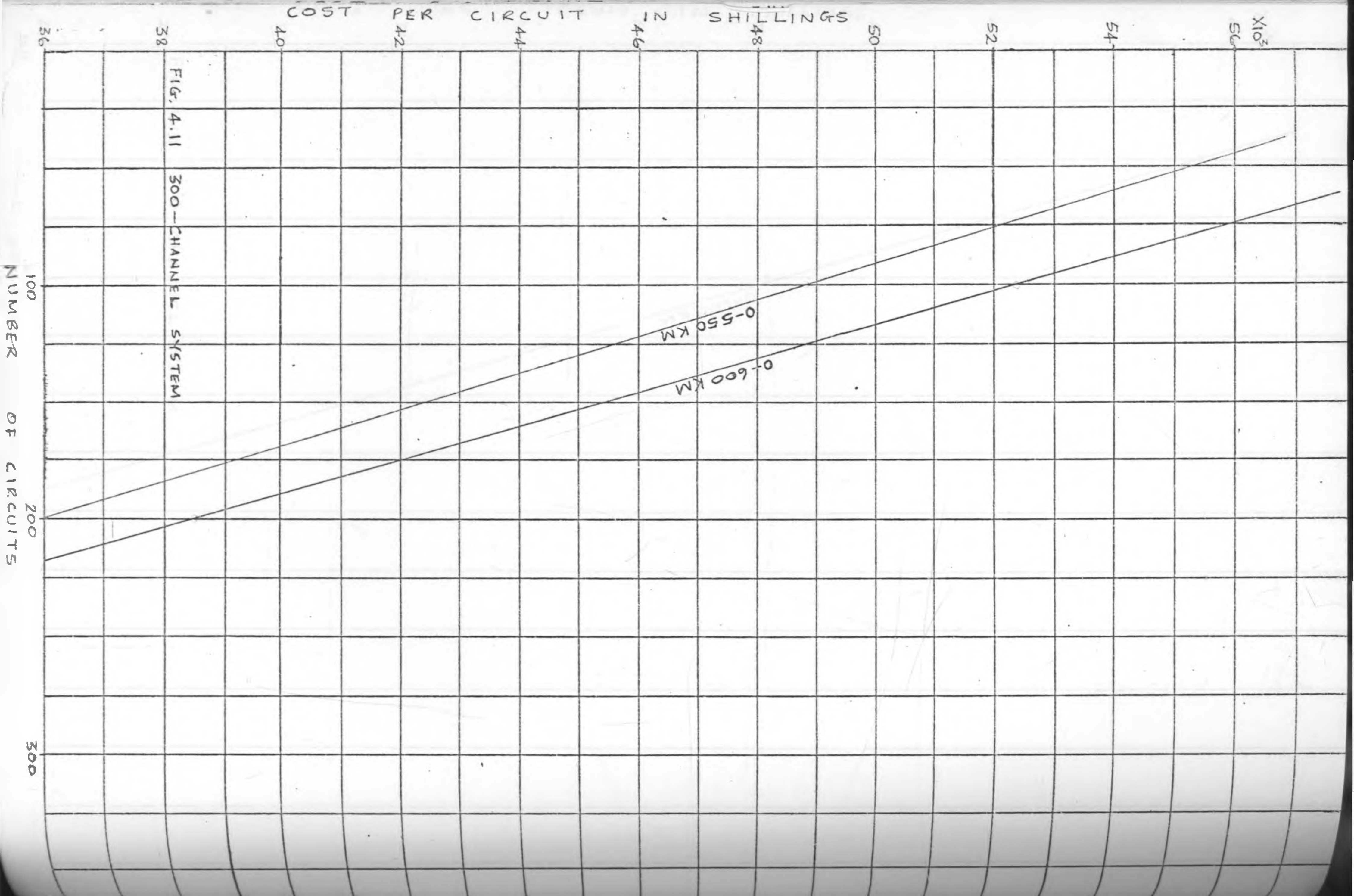


FIG. 4.11 300-CHANNEL SYSTEM

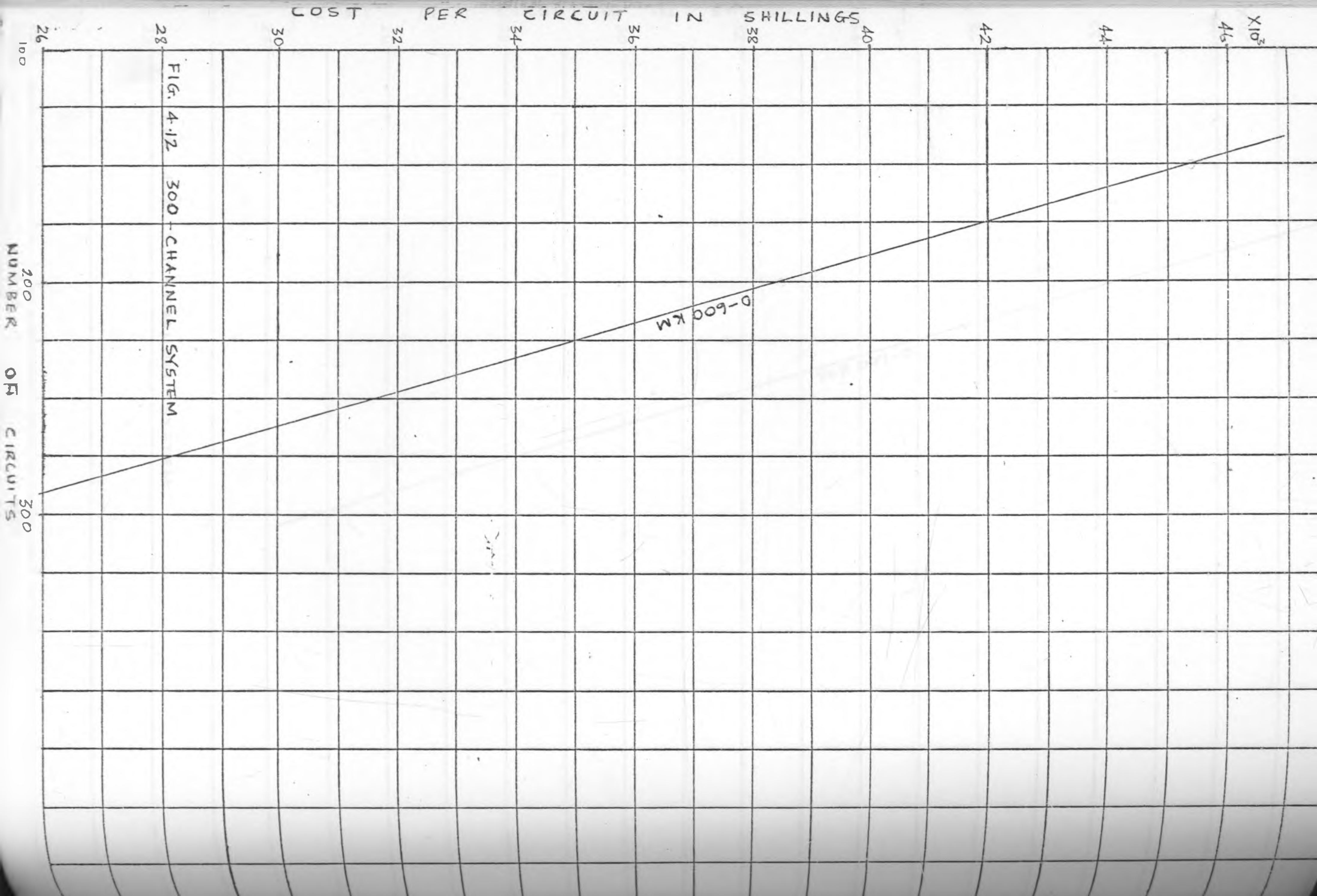


FIG. 4.12 300-CHANNEL SYSTEM

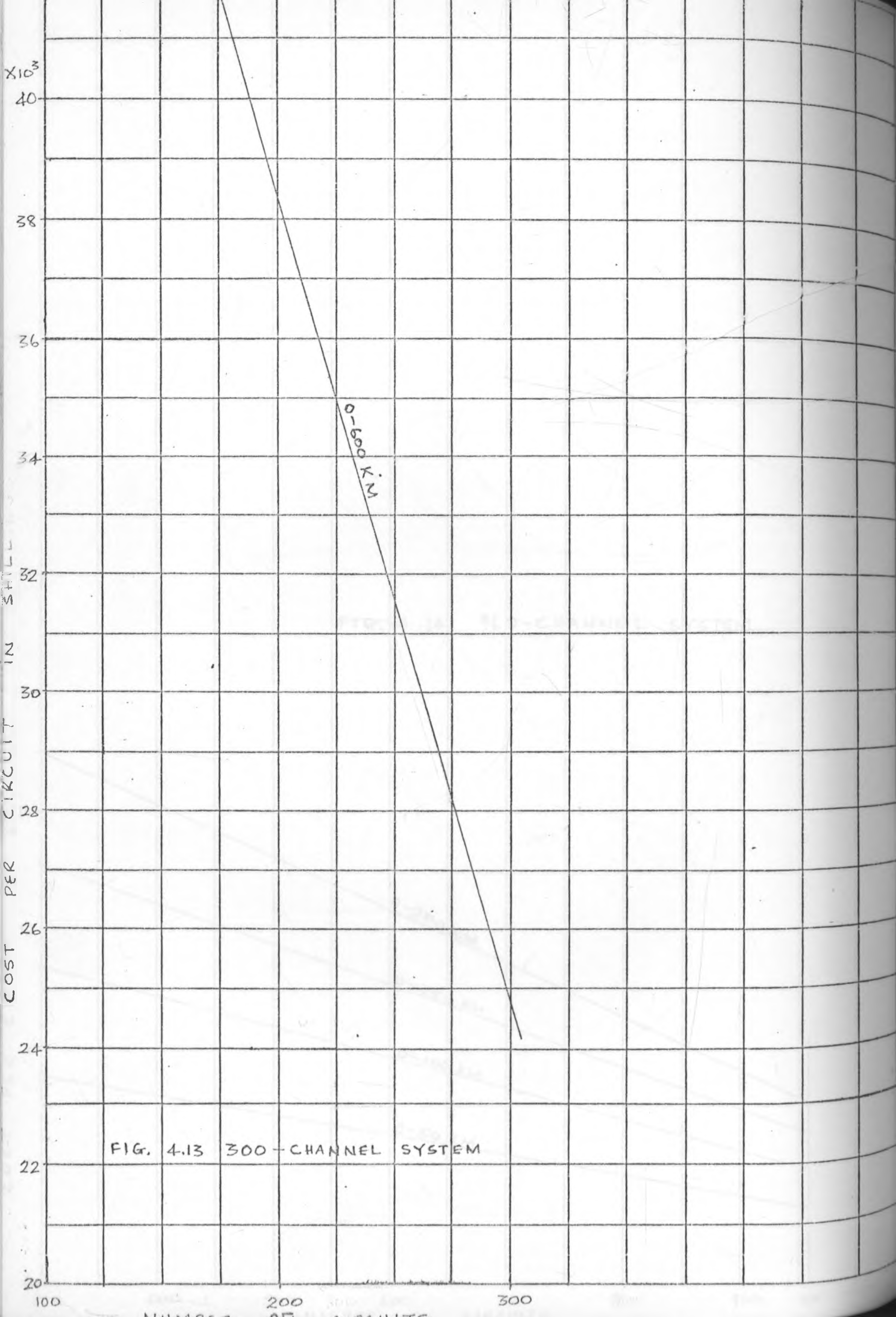


FIG. 4.13 300-CHANNEL SYSTEM

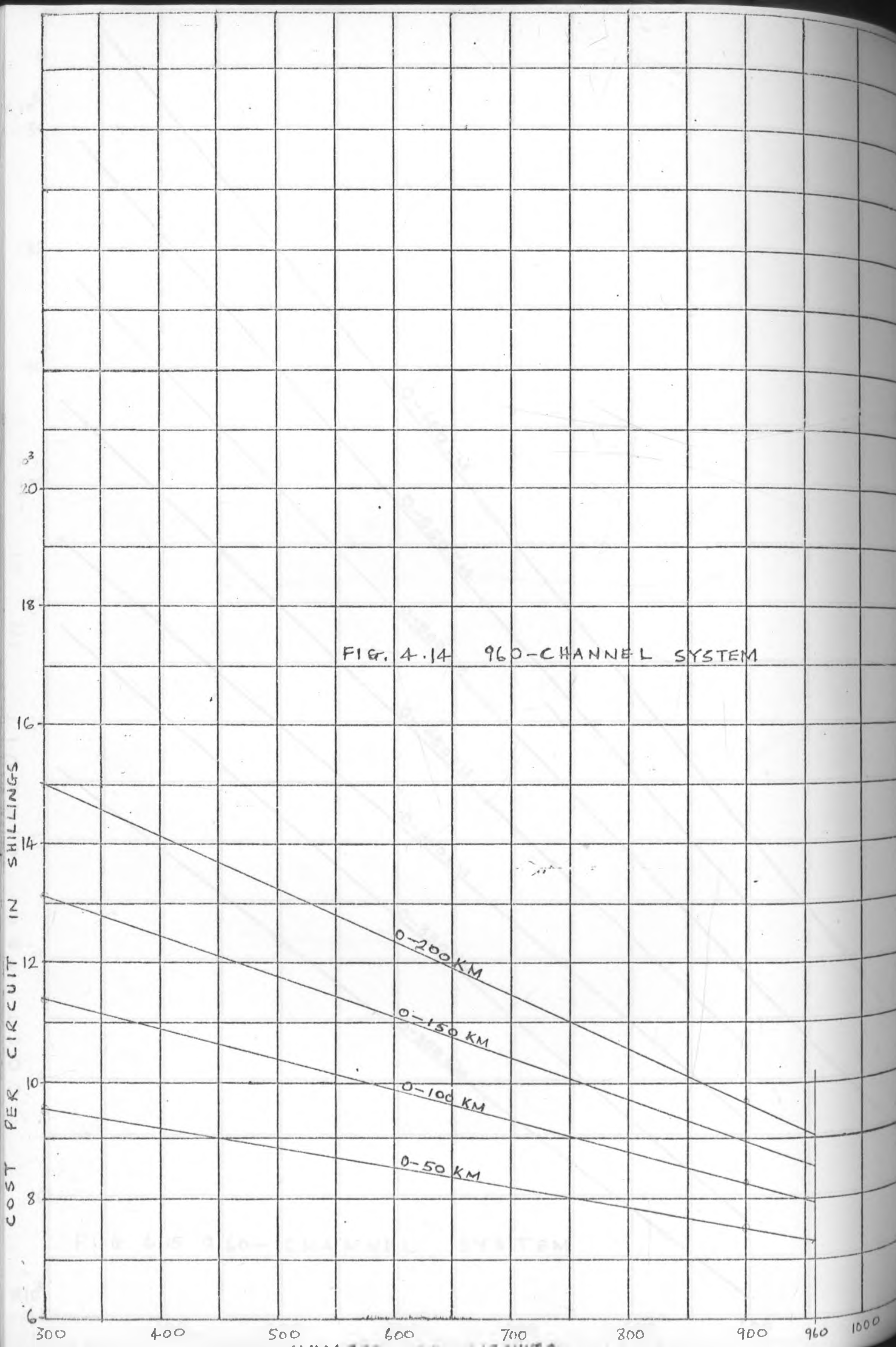


FIG. 4.14 960-CHANNEL SYSTEM

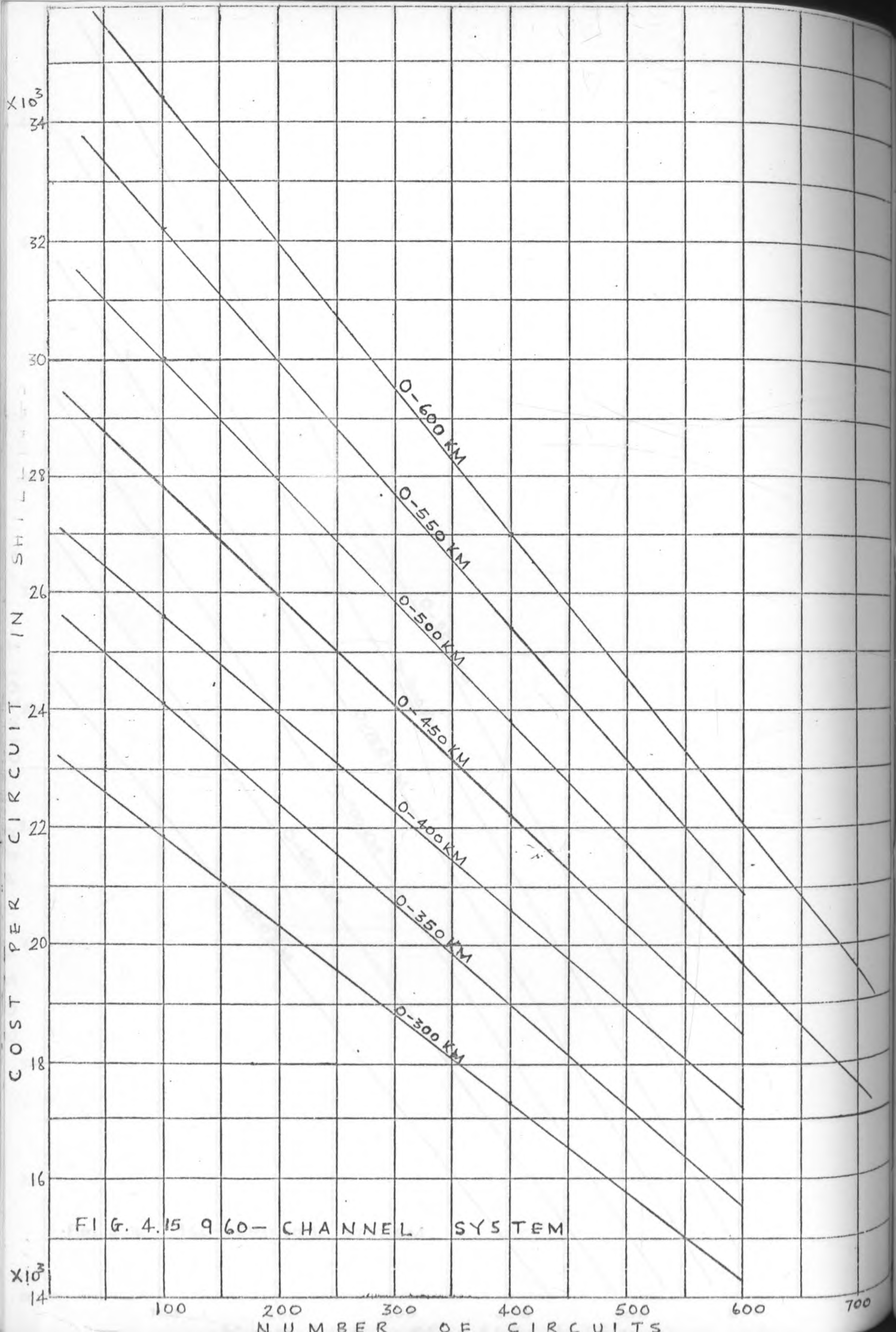


FIG. 4.15 960-CHANNEL SYSTEM

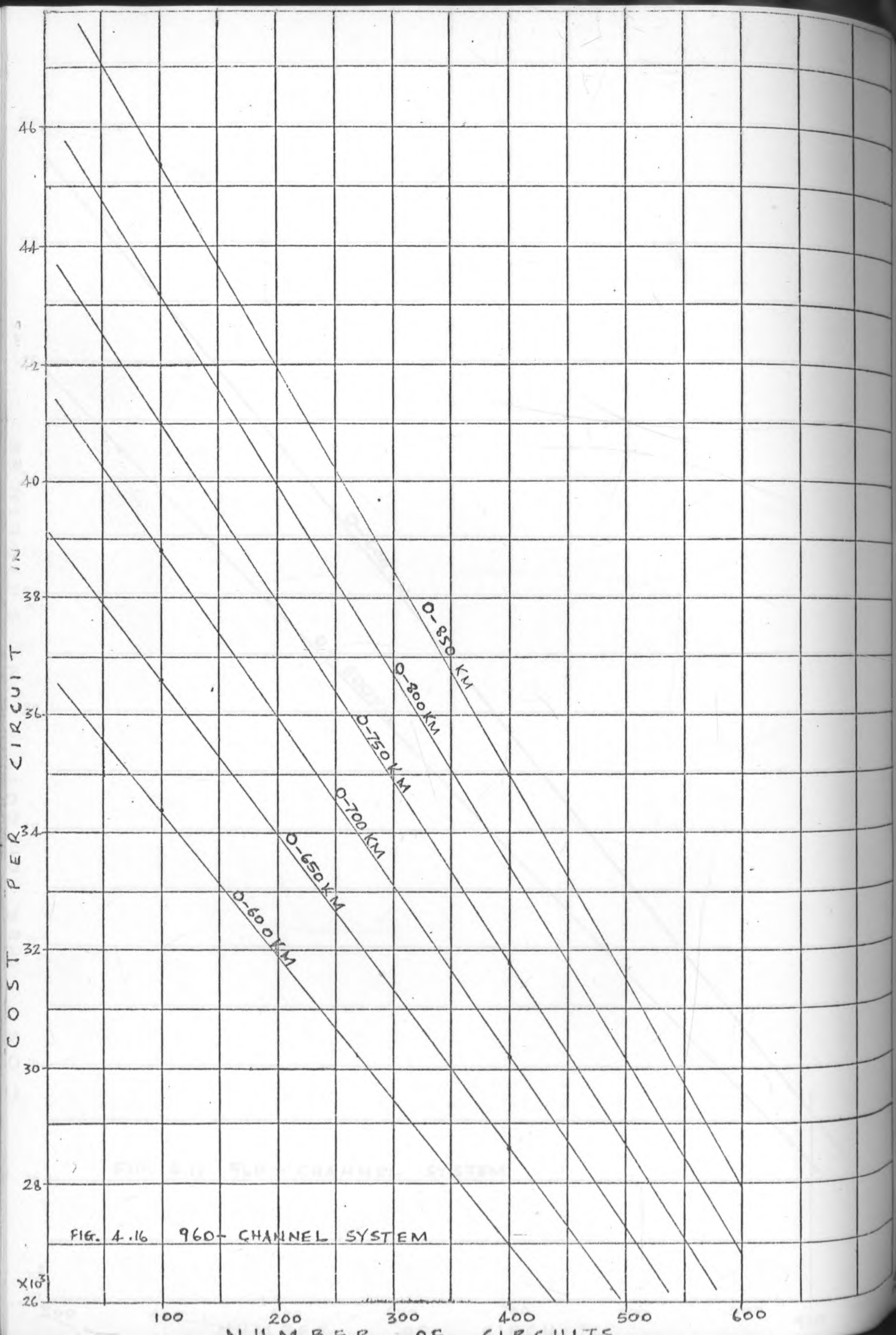


FIG. 4.16 960- CHANNEL SYSTEM

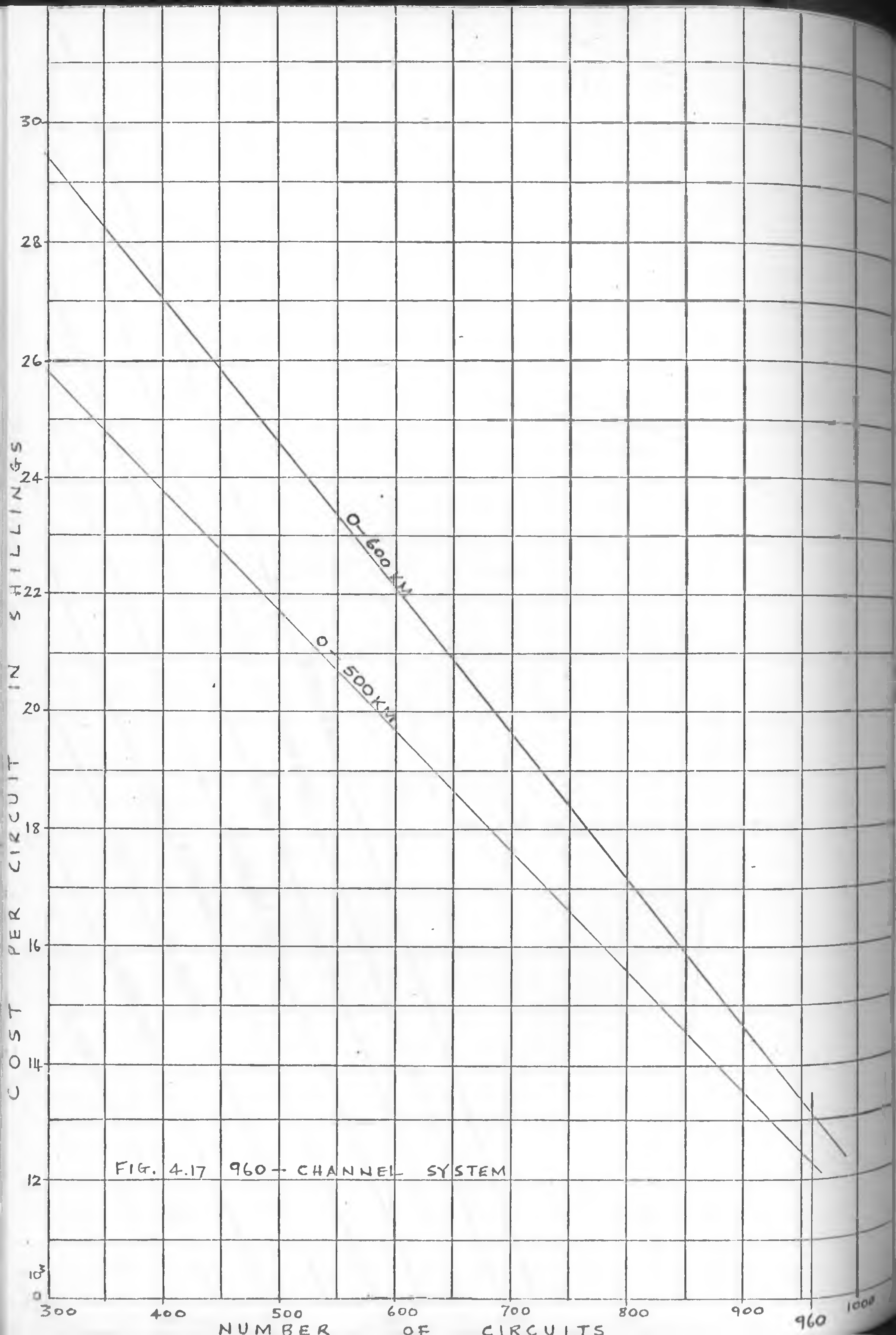


FIG. 4.17 960-CHANNEL SYSTEM

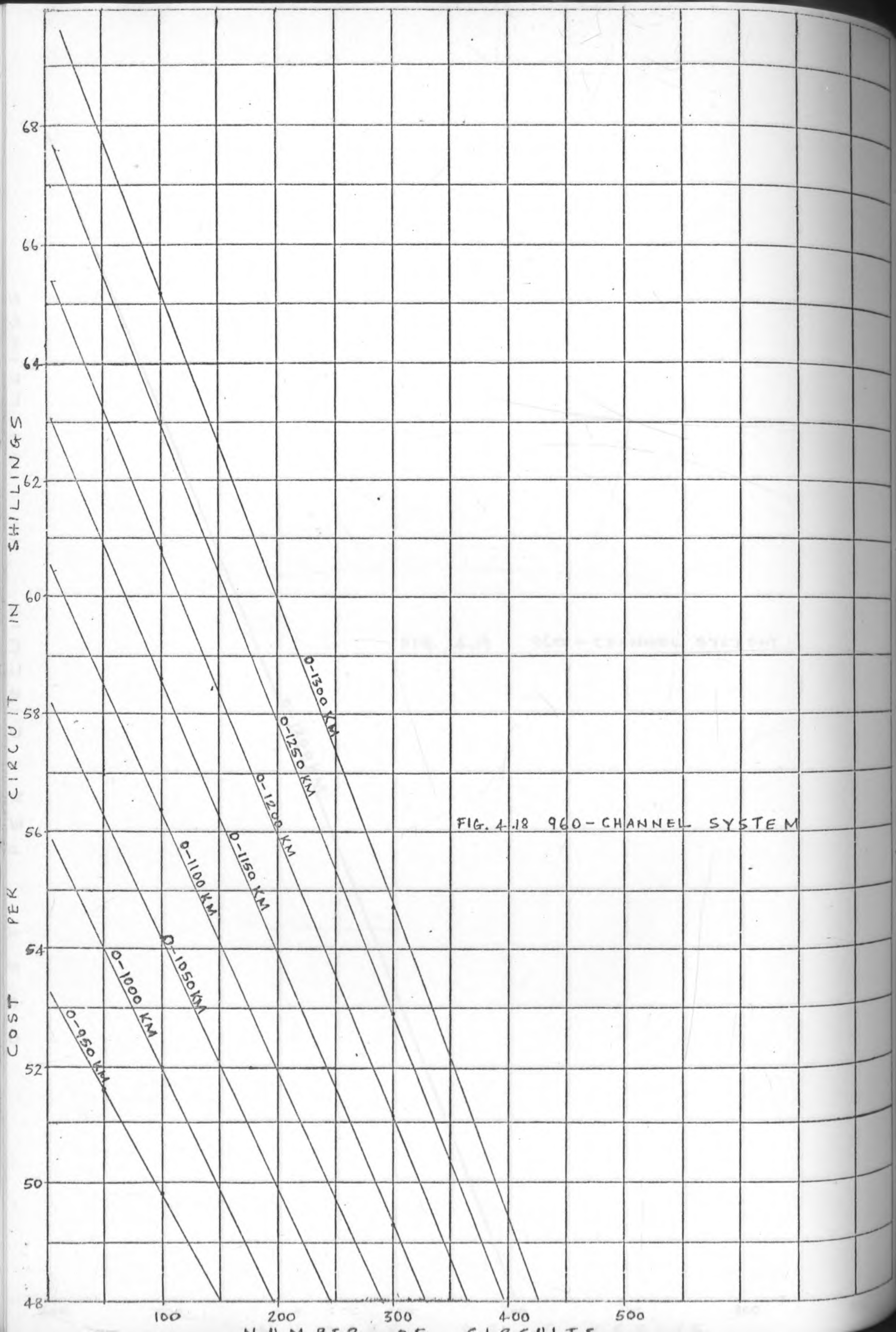


FIG. 4.18 960-CHANNEL SYSTEM

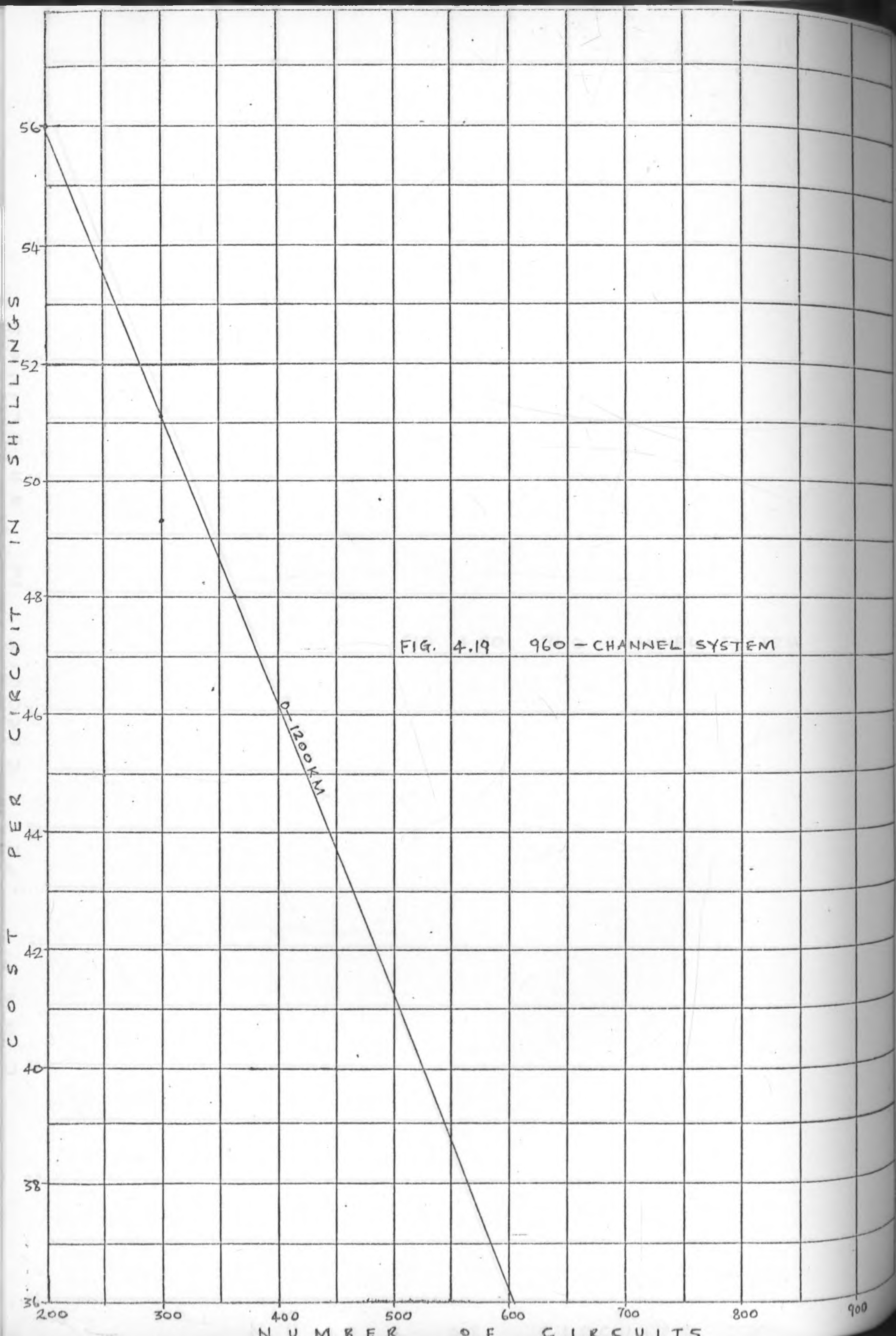


FIG. 4.19 960-CHANNEL SYSTEM

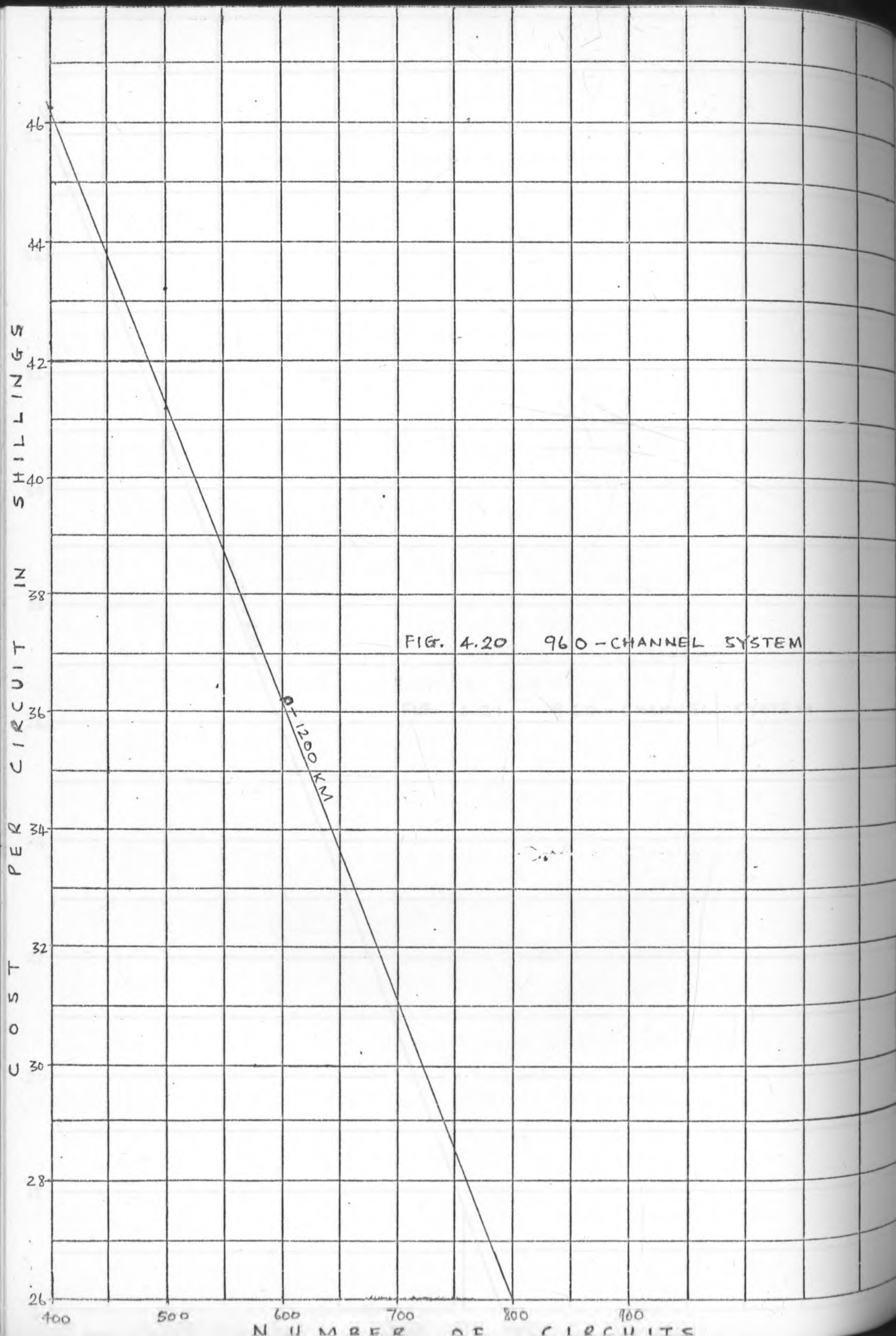


FIG. 4.20 960-CHANNEL SYSTEM

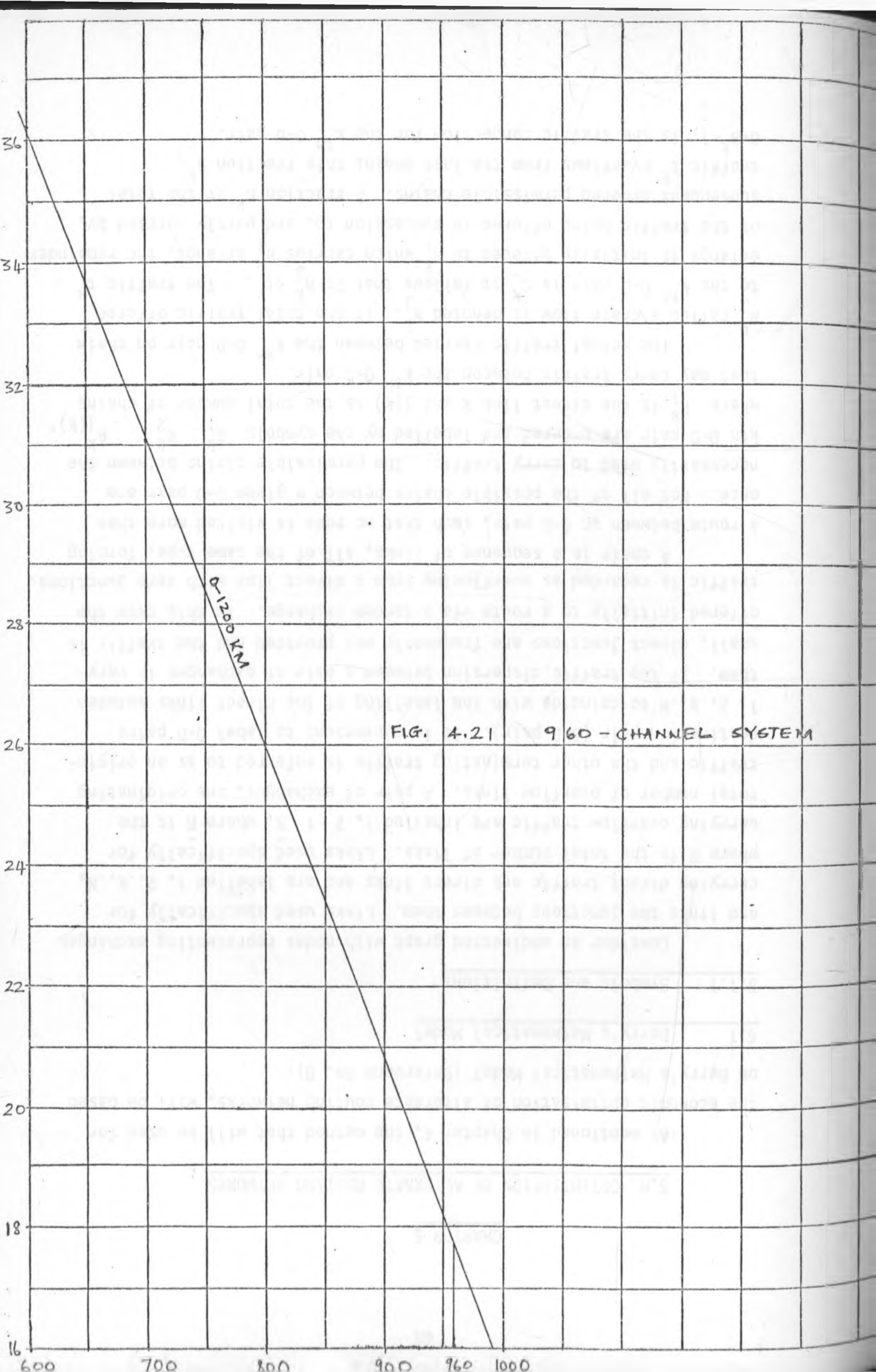


FIG. 4.21 960 - CHANNEL SYSTEM

CHAPTER 5

5.0 OPTIMISATION OF ALTERNATE ROUTING NETWORKS

As mentioned in Chapter 4, the method that will be used for the economic optimisation of alternate routing networks, will be based on Berry's Mathematical Model (Reference No. 8).

5.1 Berry's Mathematical Model

5.1.1 Symbols and Definitions

Consider an undirected graph with nodes representing exchanges and links the junctions between them. Links used specifically for carrying direct traffic are direct links and are labelled 1, 2..k..N, where N is the total number of links. Links used specifically for carrying overflow traffic are labelled 1, 2 i $\hat{N}$, where $\hat{N}$ is the total number of overflow links. A pair of exchanges, one originating traffic and the other terminating traffic is referred to as an origin-destination pair (O-D pair). It is convenient to label O-D pairs 1, 2..k..N to coincide with the labelling of the direct links between them. If the traffic dispersion between a pair of exchanges is very small, direct junctions are frequently not provided and the traffic is offered initially to a route via a tandem exchange. In this case the traffic is regarded as overflowing from a direct link with zero junctions.

A chain is a sequence of links, all of the same type, forming a route between an O-D pair, such that no node is visited more than once. Not all of the possible chains between a given O-D pair are necessarily used to carry traffic. The permissible chains between the kth O-D pair are ordered and labelled by the symbols R_1^k, R_2^k $R_{j(k)}^k$, where R_1^k is the direct link k and j(k) is the total number of chains that may carry traffic between the k^{th} O-D pair.

The actual traffic carried between the k^{th} O-D pair on chain R_j^k called a chain flow is denoted h_j^k. If the total traffic offered to the k^{th} O-D pair is t^k it follows that $0< h_j^k <t^k$, The traffic t^k erlangs is initially offered to R_1^k which carries h_1^k erlangs, the remainder of the traffic being offered in succession to, and partly carried by, subsequent ordered permissible chains. A fraction B^k of the total traffic t^k overflows from the last chain; this fraction B^k, $0<B^k<1$, is the traffic congestion for the k^{th} O-D pair.

If we let

$$a^k_{ij} = \begin{cases} 1 & \text{if link i is on chain } R^k_j \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

then the total flow on link i, resulting from the various chain flows h^k_j can be obtained as follows:

$$x_i = \sum_k a^k_{i1} h^k_1 = a^i_{i1} h^i_1 = h^i_1 \tag{5.2}$$

and

$$\hat{x}_i = \sum_k \sum_j a^k_{ij}\, h^k_j$$

where Σ denotes summation from k = 1 to N.

and Σ_j^k denotes summation from j = 2 to j(k).

The mean and variance of the overflow traffic offered to chain R^k_j, by the originating exchange for the k^{th} O-D pair are denoted M^k_j and V^k_j respectively. The mean and variance of the overflow traffic offered to overflow link i, from all O-D pairs using the link, are denoted M_i and V_i respectively. The equivalent random traffic that would produce the overflow M_i and V_i when offered to N_i equivalent junctions is denoted A_i.

In addition to the above notation, the following symbols will be used for direct link k:

- c_k cost per junction
- n_k number of junctions
- x_k traffic carried ($= h^k_1$)
- e_k circuit efficiency (the average number of erlangs carried per junction)

The value of e_k is obtained from

$$e_k = \begin{cases} \dfrac{x_k}{n_k} & \text{provided } h^k_1 \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

Corresponding symbols $\hat{c}_1$, $\hat{n}_1$, $\hat{x}_1$ and $\hat{e}_1$ refer to the overflow link i.

5.1.2 Assumptions

The main assumptions of the model are:

1. Traffic throughout the network is routed under full availability conditions.
2. The traffic initiated at an origin is considered to be Poissonian. This assumption of randomness is generally accepted as being valid when the number of subscribers initiating the calls is large.
3. In addition to assuming independence of traffic initiated at the various origins throughout the network, it is assumed that the chain flows from different O-D pairs are statistically independent.
4. Traffic is completely described by its mean and variance.
5. If $(R^k_{j_1} \ldots\ldots R^k_{j_p} | j_1 < \ldots < j_p)$ is the set of all ordered chains, between the k^{th} O-D pair, which share a common link i, then the traffic offered to link i by the k^{th} O-D pair is considered to have a mean M^k_j and a variance V^k_j.
6. For each O-D pair the total traffic may be split into arbitrarily assigned chain flows such that the overall traffic congestions are equal to the specified values, and that when this has been done it is possible to dimension the network so that these overall traffic congestions are in fact achieved.

Assumptions (3) and (5) permit the neglect of certain correlation effects and allow the mean and variance of the total traffic offered to a particular link to be obtained from the respective additions of the means and variances of the individual traffic streams offered to the link.

5.1.3 The Dimensioning Method

Consider an alternate routing network in which each originating exchanges has access to each of its destination exchanges by means of a number of routes. Suppose the traffic dispersions, t^k, and the chain flows, h^k_j are known. It is then apparent that for the k^{th} O-D pair the mean traffic overflowing from R^k_j is given by

$$M^k_{j+1} = t^k - \sum_{\ell=1}^{j} h^k_\ell \qquad j = 1, 2, \ldots. j(k) \qquad (5.5)$$

From assumption (2),

$$M_1^k = V_1^k = t^k \tag{5.6}$$

In order to obtain values for V_j^k, $j > 1$, each O-D pair is considered independently using Wilkinson's Equivalence method (see 3.1.4.1). For each chain R_j^k imagine an equivalent single link which overflows traffic with mean M_{j+1}^k and variance V_{j+1}^k when offered traffic with mean M_j^k and variance V_j^k. Now suppose that A_j^k is the equivalent random traffic which overflows M_j^k erlangs with a variance V_j^k from a certain number of equivalent junctions $\bar{n}$. The values of A_j^k and $\bar{n}$ could be calculated from equations 3.6 and 3.7. As the equations involve excessive computation, the simpler approximate formula due to Rapp (see reference No. 4) is used to obtain A_j^k.

$$A_j^k = V_j^k + 3\,\frac{V_j^k}{M_j^k}\,\left(\frac{V_j^k}{M_j^k} - 1\right),\ j = 1,\ 2....j(k) \tag{5.7}$$

Now consider the A_j^k erlangs to be offered to the equivalent single link which overflows traffic with mean M_{j+1}^k and a variance V_{j+1}^k. Then

$$A_j^k = V_{j+1}^k + 3\,\frac{V_{j+1}^k}{M_{j+1}}\,\left(\frac{V_{j+1}^k}{M_{j+1}} - 1\right),\ j=1\ \ j(k) \tag{5.8}$$

As

$$A_1^k \equiv t^k$$

the occurance of A_j^k in both equations 5.7 and 5.8 reveals that

$$A_1^k = A_2^k = \ldots = A_{j(k)}^k = t^k \text{ for all } k \tag{5.9}$$

Thus it is now possible to obtain the value of the chain overflow variances, V_j^k, as positive roots of the quadratic equations:

$$3(V_{j+1}^k)^2 + \left[(M_{j+1}^k)^2 - 3M_{j+1}^k\right] V_{j+1}^k - t^k (M_{j+1}^k)^2 = 0 \tag{5.10}$$

$j = 1,\ 2.....j(k)$

$M_{j(k)+1}^k$ is the traffic lost from the system.

The mean M_i and the variance V_i of the traffic offered to each overflow link i is obtained as in equations (3.4) and (3.5) modified by the condition in equation (5.1), thus

$$M_i = \sum_{kj}\sum a^k_{ij} M^k_j$$

$$V_i = \sum_{kj}\sum a^k_{ij} V^k_j \tag{5.11}$$

The equivalent random traffic, A_i, is obtained from Rapp's formula (reference No. 4)

$$A_i = V_i + 3 \frac{V_i}{M_i} \left(\frac{V_i}{M_i} - 1\right) \tag{5.12}$$

and the number of equivalent junctions is given by

$$N_i = \frac{A_i\left(M_i + \frac{V_i}{M_i}\right)}{M_i + \frac{V_i}{M_i} - 1} - M_i - 1 \tag{5.13}$$

Considering A_i erlangs to be offered to $N_i + \hat{n}_i$ junctions with an overflow mean $M_i - \hat{x}_i$, where

$$\hat{x}_i = \sum_{kj}\sum a^k_{ij}\, h^k_j \tag{5.14}$$

then from equation (5.12) the link overflow variance v_i can be written as

$$v_i = (m_i - \hat{x}_i)\, \frac{(3-(M_i - \hat{x}_i) + \left[(M_i - \hat{x}_i - 3)^2 + 12A\right]^{\frac{1}{2}})}{6} \tag{5.15}$$

and the required number of functions for the overflow link: if finally given by (reference No. 4)

$$\hat{n}_i = \hat{x}_i + A_i\left(\frac{(M_i - \hat{x}_i)^2 + v_i}{(M_i - \hat{x}_i - 1)(M_i - \hat{x}_i) + V_i} - \frac{(M_i)^2 + V_i}{(M_i - 1)M_i + V_i}\right) \tag{5.16}$$

The required number of junction for direct link k is calculated from (Reference No. 4).

$$n_k = t^k - M^k_2 + t^k\left(\frac{(M^k_2)^2 + V^k_2}{(M^k_2 - 1)M^k_2 + V^k_2} - \frac{t^k + 1}{t^k}\right) \tag{5.17}$$

5.1. Formulation of the Model as a Mathematical Programme

The total junction cost, C, for the network can be expressed in terms of the number of direct functions n_k and the number of overflow junctions $\hat{n}_i$:

$$C = \sum_{k=1}^{N} c_k n_k + \sum_{i=1}^{\hat{N}} \hat{c}_i \hat{n}_i \qquad (5.18)$$

In contrast to this a non-linear expression will be established for the total cost C in terms of the chain flows h_j^k.

As the average cost per erlang on direct link k is $\frac{c_k}{e_k}$, the total cost for all direct junctions is

$$\sum_{k=1}^{N} \frac{c_k h_1^k}{e_k} \qquad (5.19)$$

Considering the overflow links, the average cost per erlang on overflow link i is $\frac{\hat{c}_i}{\hat{e}_i}$, giving

$$\sum_{i=1}^{\hat{N}} \frac{a_{ij}^k \hat{c}_i}{\hat{e}_i} \qquad (5.20)$$

as the average cost per erlang for chain R_j^k. It follows that the cost for chain flow h_j^k is

$$\sum_i \frac{a_{ij}^k \hat{c}_i h_j^k}{\hat{e}_i} \qquad (5.21)$$

and the cost for all routes between O-D pair k is

$$\sum_j \sum_i \frac{a_{ij}^k \hat{c}_i h_j^k}{\hat{e}_i} \qquad (5.22)$$

where Σ_j denotes summation from j=2 to j(k).

Finally, the total network cost is given by

$$C = \sum_k \frac{c_k h_1^k}{e_k} + \sum_k \sum_j \sum_i \frac{a_{ij}^k h_j^k \hat{c}_i}{\hat{e}_i} \qquad (5.23)$$

where the traffic efficiencies e_k and $\hat{e}_i$ are uniquely determined functions of the chain flows, h_j^k.

Conservation considerations for the chain flows give the constraint equations

$$\sum_j h_j^k = b^k, \text{ for all } k \tag{5.24}$$

where

$$b^k = t^k(1 - B^k) \tag{5.25}$$

and $$h_j^k > 0 \text{ for all } j \text{ and } k. \tag{5.26}$$

Minimising the total cost C given by equation (5.23) subject to the constraints of equations (5.24) and (5.26) constitutes a non-linear mathematical programme whose solution is considered in 5.1.5. The solution to the mathematical programme is a set of optimal chain flows. These chain flows are used to calculate the optimal numbers of junctions on each link, by applying the method in 5.1.5.

Case Study

Example: Formulation of the Model on a Routing Pattern in East Africa.

The routing pattern on which this model is going to be applied in East Africa is shown in Fig. 5.1.

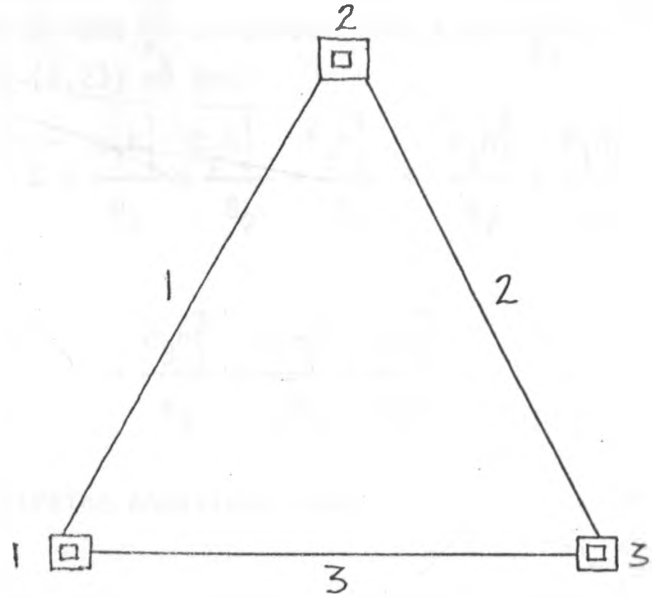


Fig. 5.1. Routing Pattern

Let the traffic dispersion values be t^1, t^2, t^3 and the trunk costs per circuit be c_1, c_2. c_3 for direct links and $\hat{c}_1$, $\hat{c}_2$, $\hat{c}_3$ for overflow links. Chains R_j^k and the values of a_{ij}^k can be obtained from the link-chain matrix.

LINK-CHAIN MATRIX

CHAIN

L	1	1	0	0	1	0	1
I N	2	0	1	1	0	0	1
K	3	0	1	0	1	1	0
		R_1^1	R_2^1	R_1^2	R_2^2	R_1^3	R_2^3

The chains are ordered R_1^1, R_2^1, R_1^2, R_2^2, R_1^3, R_2^3. The link numbers refer to direct links for R_1^1, R_1^2. R_1^3, otherwise they refer to overflow links. There are two possible chains between each exchange pair and these chains are all permitted, that is $j(k) = 2$ for all k.

The total junction cost, C, expressed in terms of the number of direct junctions n_k, and the number of overflow junctions $\hat{n}_i$ is :

$$C = c_1n_1+\hat{c}_2\hat{n}_2+\hat{c}_3\hat{n}_3+c_2n_2+\hat{c}_1\hat{n}_1+\hat{c}_3\hat{n}_3+c_3n_3+\hat{c}_1\hat{n}_1+\hat{c}_2\hat{n}_2 \qquad (5.27)$$

Expressing this cost in terms of the chain flow h_j^k, and in terms of cost per erlang $\frac{c_k}{e_k}$ on direct link k and $\frac{\hat{c}_i}{\hat{e}_i}$ on overflow link; that is equation (5.23) we get:

$$C = \frac{c_1h_1^1}{e_1} + \frac{\hat{c}_2h_2^1}{\hat{e}_2} + \frac{\hat{c}_3h_2^1}{\hat{e}_3} + \frac{c_2h_1^2}{e_2} + \frac{\hat{c}_1h_2^2}{\hat{e}_1} + \frac{\hat{c}_3h_2^2}{\hat{e}_3}$$

$$+ \frac{c_3h_1^3}{e_3} + \frac{c_1h_2^3}{\hat{e}_1} + \frac{c_2h_2^3}{\hat{e}_2} \qquad (5.28)$$

The constraint equations are:

$$\left.\begin{aligned} h_1^1 + h_2^1 &= t^1(1-B^1) \\ h_1^2 + h_2^2 &= t^2(1-B^2) \\ h_1^3 + h_2^3 &= t^3(1-B^3) \end{aligned}\right\} \qquad (5.29)$$

$$\left.\begin{aligned} h_1^1 &> 0 \\ h_1^2 &> 0 \\ h_1^3 &> 0 \\ h_2^1 &> 0 \\ h_2^2 &> 0 \\ h_2^3 &> 0 \end{aligned}\right\} \qquad (5.30)$$

5.1.5 Solution of the Mathematical Programme

The mathematical programme is solved for the routing pattern (Fig. 5.1), using a gradient projection method based on Rosen's technique (see reference No. 5), Chapter 12). The determination of the gradient is discussed in detail in Appendix 5.

As the total network cost, C, is to be minimized, the projection of the gradient,$-\nabla C$, onto the constraint set defined by equations (5.29) and (5.30) provides the direction of best improvement from a given feasible point. But the approach taken here is to project $-\nabla C$ onto the set S, defined as the intersection of only the three equality constraints given by equation (5.29). As the constraint equations are independent, S has dimension three.

If the chain flows:

$$h_1^1,\ h_2^1,\ h_3^1,\ h_1^2,\ h_2^2,\ h_3^2,\ h_1^3,\ h_2^3,\ h_3^3.$$

where h_3^1, h_3^2 and h_3^3 represent the traffic lost from the system, are denoted by:

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7,\ x_8,\ x_9$$

respectively, and the traffic congestions B^k are specified the same for all k, $B^k = 0.002$, denoted B. Then the constraint equations become:

$$\begin{aligned} x_1 + x_2 &= t^1(1-B) \\ x_4 + x_5 &= t^2(1-B) \\ x_7 + x_8 &= t^3(1-B) \end{aligned} \tag{5.31}$$

such that

$$\begin{aligned} x_1 &= -x_2 + t^1(1-B) \\ x_4 &= -x_5 + t^2(1-B) \\ x_7 &= -x_8 + t^3(1-B) \end{aligned} \tag{5.32}$$

giving a basis of three independent vectors. The basic vectors are

$$q_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad q_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \tag{5.33}$$

and the vectors which form a basis for the three dimensional orthogonal complement of S are

$$P_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \tag{5.34}$$

If $\vec{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$ is a point belonging to S, and $\vec{a} = (a_1, a_2, \ldots\ldots a_9)$ is such that the vector $\vec{a} - \vec{x}$ is in the direction of $-\nabla\vec{C}$, thus

$$-\nabla\vec{C} = \vec{a} - \vec{x}$$

then

$$\vec{a} = (x_1 - \frac{\partial C}{\partial x_1}, x_2 - \frac{\partial C}{\partial x_2}, x_3 - \frac{\partial C}{\partial x_3} \quad \ldots \tag{5.35}$$

$$x_7 - \frac{\partial C}{\partial x_7}, x_8 - \frac{\partial C}{\partial x_8}, x_9 - \frac{\partial C}{\partial x_9}) \tag{5.36}$$

then the point

$$\vec{y} = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \vec{a} \tag{5.37}$$

satisfying the constraints is the orthogonal projection of $\vec{a}$ onto S giving (reference No. 8)

$$\begin{aligned} 2\lambda_1 + a_1 + a_2 &= t^1(1-B) \\ 2\lambda_2 + a_4 + a_5 &= t^2(1-B) \\ 2\lambda_3 + a_7 + a_8 &= t^3(1-B) \end{aligned} \tag{5.38}$$

such that

$$\begin{aligned} \lambda_1 &= \frac{t^1(1-B) - a_1 - a_2}{2} \\ \lambda_2 &= \frac{t^2(1-B) - a_4 - a_5}{2} \\ \lambda_3 &= \frac{t^3(1-B) - a_7 - a_8}{2} \end{aligned} \tag{5.39}$$

Substituting values of $\vec{a}$ and $t^k(1-B)$ from equations (5.31) and (5.36) into equation (5.39) we obtain:

$$\lambda_1 = \frac{x_1+x_2-(x_1-\frac{\partial C}{\partial x_1})-(x_2-\frac{\partial C}{\partial x_2})}{2} = \frac{\frac{\partial C}{\partial x_1}+\frac{\partial C}{\partial x_2}}{2}$$

$$\lambda_2 = \frac{x_4+x_5 - (x_4-\frac{\partial C}{\partial x_4})-(x_5-\frac{\partial C}{\partial x_5})}{2} = \frac{\frac{\partial C}{\partial x_4}+\frac{\partial C}{\partial x_5}}{2} \tag{5.40}$$

$$\lambda_3 = \frac{x_7+x_8-(x_7-\frac{\partial C}{\partial x_7})-(x_8-\frac{\partial C}{\partial x_8})}{2} = \frac{\frac{\partial C}{\partial x_7}+\frac{\partial C}{\partial x_8}}{2}$$

$$\lambda_1 p_1+\lambda_2 p_2+\lambda_3 p_3 = \frac{\frac{\partial C}{\partial x_1} \quad \frac{\partial C}{\partial x_2}}{2}\begin{bmatrix}1\\1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} + \frac{\frac{\partial C}{\partial x_4}+\frac{\partial C}{\partial x_5}}{2}\begin{bmatrix}0\\0\\0\\1\\1\\0\\0\\0\\0\end{bmatrix} + \frac{\frac{\partial C}{\partial x_7} \quad \frac{\partial C}{\partial x_8}}{2}\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\1\\0\end{bmatrix}$$

$$= \begin{bmatrix} \frac{\frac{\partial C}{\partial x_1}+\frac{\partial C}{\partial x_2}}{2} \\[2ex] \frac{\frac{\partial C}{\partial x_1}+\frac{\partial C}{\partial x_2}}{2} \\ 0 \\ \frac{\frac{\partial C}{\partial x_4}+\frac{\partial C}{\partial x_5}}{2} \\ 0 \\ \frac{\frac{\partial C}{\partial x_7}+\frac{\partial C}{\partial x_8}}{2} \\[2ex] \frac{\frac{\partial C}{\partial x_7} \quad \frac{\partial C}{\partial x_8}}{2} \\ 0 \end{bmatrix} \qquad \frac{\frac{\partial C}{x_4}+\frac{\partial C}{x_5}}{2} \tag{5.41}$$

and $\vec{y} = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \vec{a}$ is

$$\vec{y} = \begin{bmatrix} \dfrac{\frac{\partial C}{\partial x_1} + \frac{\partial C}{\partial x_2}}{2} + x_1 - \dfrac{\partial C}{\partial x_1} \\[2ex] \dfrac{\frac{\partial C}{\partial x_1} + \frac{\partial C}{\partial x_2}}{2} + x_2 - \dfrac{\partial C}{\partial x_2} \\[2ex] 0 \quad + x_3 - \dfrac{\partial C}{\partial x_3} \\[2ex] \dfrac{\frac{\partial C}{\partial x_4} + \frac{\partial C}{\partial x_5}}{2} + x_4 - \dfrac{\partial C}{\partial x_4} \\[2ex] \dfrac{\frac{\partial C}{\partial x_4} + \frac{\partial C}{\partial x_5}}{2} + x_5 - \dfrac{\partial C}{\partial x_5} \\[2ex] 0 \quad + x_6 - \dfrac{\partial C}{\partial x_6} \\[2ex] \dfrac{\frac{\partial C}{\partial x_7} + \frac{\partial C}{\partial x_8}}{2} + x_7 - \dfrac{\partial C}{\partial x_7} \\[2ex] \dfrac{\frac{\partial C}{\partial x_7} + \frac{\partial C}{\partial x_8}}{2} + x_8 - \dfrac{\partial C}{\partial x_8} \\[2ex] 0 \quad + x_9 - \dfrac{\partial C}{\partial x_9} \end{bmatrix} \qquad (5.42)$$

which reduces to equation (5.43)

$$\vec{y} = \begin{bmatrix} x_1 - \dfrac{\dfrac{\partial C}{\partial x_1} - \dfrac{\partial C}{\partial x_2}}{2} \\ \\ x_2 - \dfrac{\dfrac{\partial C}{\partial x_2} - \dfrac{\partial C}{\partial x_1}}{2} \\ \\ x_3 - \dfrac{\partial C}{\partial x_3} \\ \\ x_4 - \dfrac{\dfrac{\partial C}{\partial x_4} - \dfrac{\partial C}{\partial x_5}}{2} \\ \\ x_5 - \dfrac{\dfrac{\partial C}{\partial x_5} - \dfrac{\partial C}{\partial x_4}}{2} \\ \\ x_6 - \dfrac{\partial C}{\partial x_6} \\ \\ x_7 - \dfrac{\dfrac{\partial C}{\partial x_7} - \dfrac{\partial C}{\partial x_8}}{2} \\ \\ x_8 - \dfrac{\dfrac{\partial C}{\partial x_8} - \dfrac{\partial C}{\partial x_7}}{2} \\ \\ x_9 - \dfrac{\partial C}{\partial x_9} \end{bmatrix} \tag{5.43}$$

and the projection vector $\vec{v} = \vec{y} - \vec{x}$

$$\vec{V} = - \begin{bmatrix} \dfrac{\dfrac{\partial C}{\partial x_1} - \dfrac{\partial C}{\partial x_2}}{2} \\[2ex] \dfrac{\dfrac{\partial C}{\partial x_2} - \dfrac{\partial C}{\partial x_1}}{2} \\[2ex] \dfrac{\partial C}{\partial x_3} \\[2ex] \dfrac{\dfrac{\partial C}{\partial x_4} - \dfrac{\partial C}{\partial x_5}}{2} \\[2ex] \dfrac{\dfrac{\partial C}{\partial x_5} - \dfrac{\partial C}{\partial x_4}}{2} \\[2ex] \dfrac{\partial C}{\partial x_6} \\[2ex] \dfrac{\dfrac{\partial C}{\partial x_7} - \dfrac{\partial C}{\partial x_8}}{} \\[2ex] \dfrac{\dfrac{\partial C}{\partial x_8} - \dfrac{\partial C}{\partial x_7}}{2} \\[2ex] \dfrac{\partial C}{\partial x_9} \end{bmatrix} \tag{5.44}$$

The search procedure used is slightly different from that described in Berry's mathematical model, (reference No. 8). Berry first determines the maximum step length, and then searches for a minimum point within the step length. On the other hand we pick a very small arbitrary step length and try to determine whether this is a minimum point. The procedure used is described below.

At a non-optimal point, $\vec{x}$, where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} \tag{5.45}$$

the projection vector is non-zero, $\vec{v} \neq 0$, it may be positive or negative. It is important at this stage to note that the point, $\vec{x}$, and the projection vector, $\vec{v}$, are two different points.

If the projection vector, $\vec{v}$, is positive, then the point, $\vec{x}$, is negative with respect to $\vec{v}$, and $\vec{x}$ is increased in the direction of the projection vector, $\vec{v}$. If the projection vector, $\vec{v}$, is negative, then the point, $\vec{x}$, is positive with respect to $\vec{v}$, and $\vec{x}$ has to be decreased in the direction of the projection vector, $\vec{v}$.

The increase or decrease to the point, $\vec{x}$, is a function of the projection vector, $\vec{v}$, and an arbitrary step, t, chosen thus

$$\vec{z} = t\,\vec{v} \tag{5.46}$$

where

$\vec{z}$ is an increase or decrease depending on $\vec{v}$. Then a new point is established in equation (5.47) satisfying the non-negativity constraints of equation (5.30)

$$\vec{x}_1 = \vec{x} + \vec{z} \tag{5.47}$$

With the initial given values of traffic, $\vec{x}$, the total cost is calculated as described in section 5.1.5. and stored. After a new point, $\vec{x}_1$, has been established as in equation (5.47), the total cost is recalculated and compared with the initial cost. If the new cost is less than the initial cost, then the initial cost is disregarded and the new cost is now stored. A new point $\vec{x}_2$ has to be established and a corresponding total cost calculated accordingly.

When the new total cost is more than the stored total cost, then the minimum cost is between the two costs. It is therefore necessary to reduce the incremental/decremental step length, t, to determine the exact minimum cost. If none is found then obviously the stored cost is the minimum cost.

The number of circuits are calculated as described in 5.1.3. However, these number of circuits do not take into account the direction of traffic. They represent the optimised number of circuits required on a particular link. To determine the number of circuits to be provided in either direction between two points linked directly, the following procedure is adopted.

Consider traffic originating at point A and terminating at point B and vice-versa. Offered traffic T, on the direct link from point A to point B is calculated as a ratio of the optimised traffic on the direct route between point A and point B as calculated above, to the optimised traffic on the direct route between A and point B and the optimised traffic on the overflow route between points A and B, mulitplied by the actual traffic offered. Thus if x_1 represents optimised traffic on the direct link and x_2 represents optimised traffic on the overflow link, A_{AB} is traffic originating at point A and terminating at point B, then of the originating traffic, the traffic that is offered on the direct link is

$$T_1 = \left(\frac{x_1}{x_1 + x_2}\right)A_{AB} \tag{5.48}$$

and T_2, the traffic offered as the overflow link is

$$T_2 = \left(\frac{x_2}{x_1 + x_2}\right)A_{AB} \tag{5.49}$$

As the grade of service between the two points A and B will remain constant, it is now possible to apply Erlang's Loss Formula (reference No. 1), and calculate the number of circuits in direction A to B on the direct and overflow routes. This procedure is repeated on the whole network. When all the circuits have been calculated, it is now possible to compile a total number of unidirectional circuits required from point A to point B.

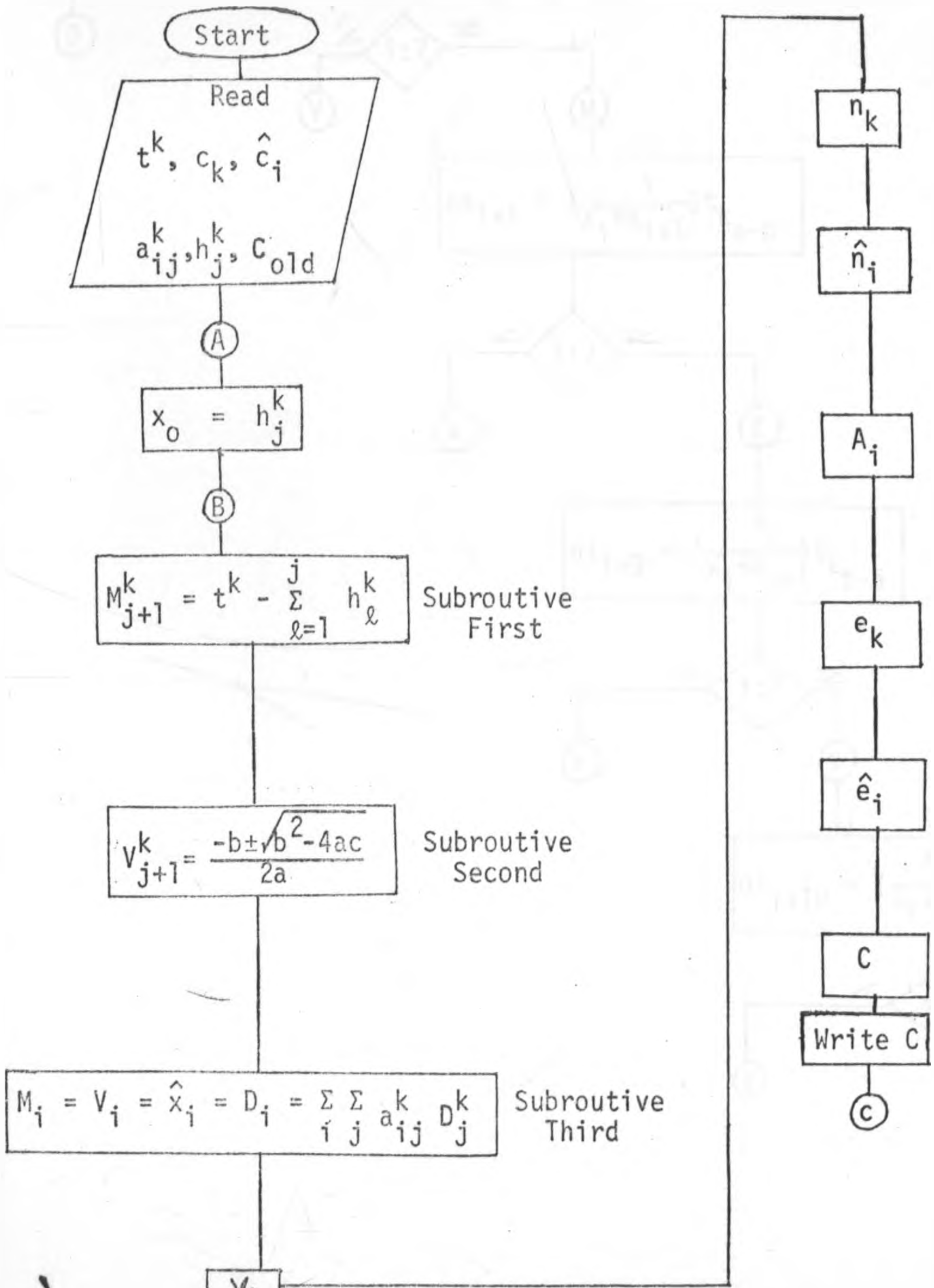
CHAPTER 6

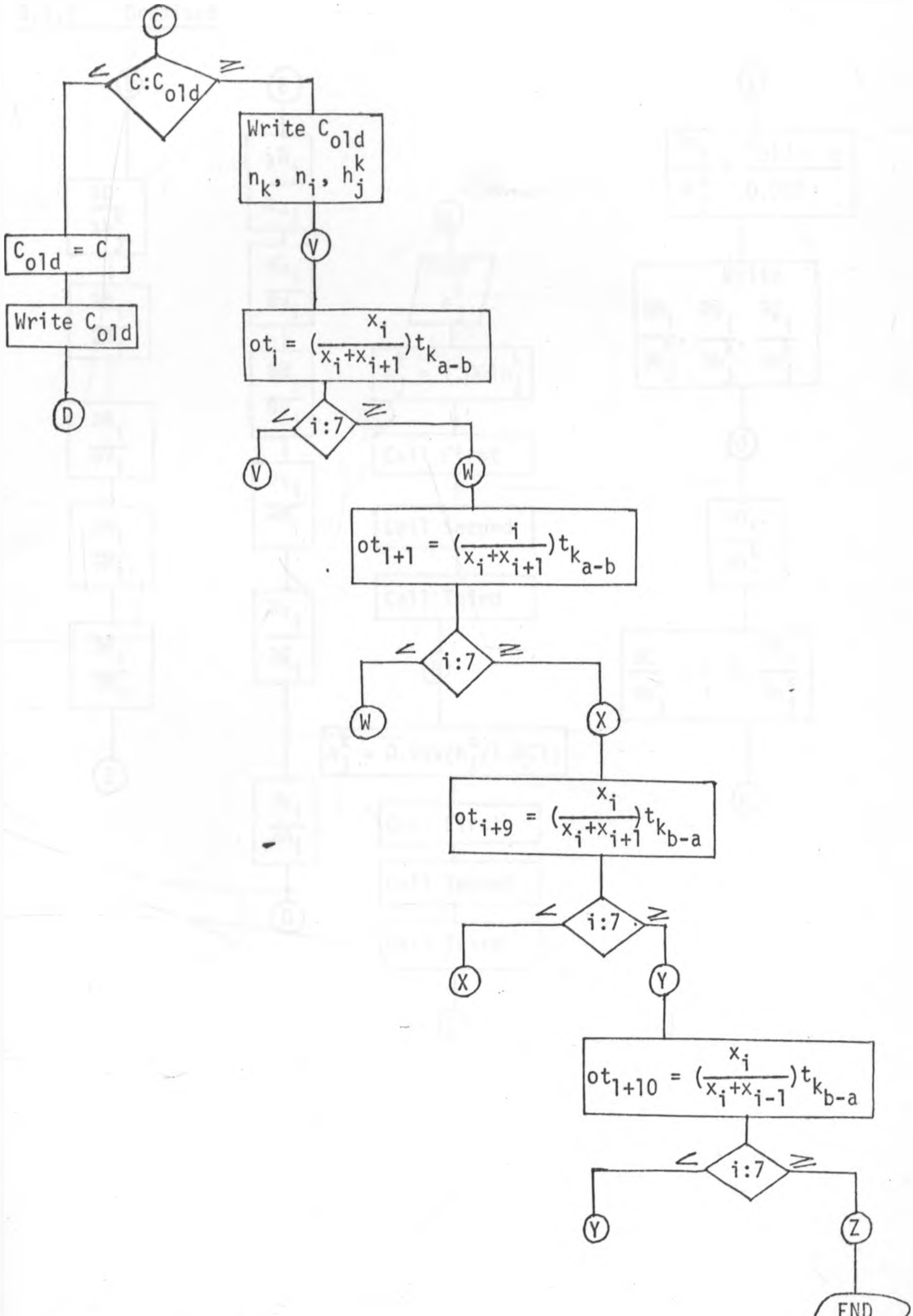
6.0 THE COMPUTER PROGRAMME

The computer programme is based on the optimization method discussed in Chapter 5. A Flow Chart describing the various operations is presented below:

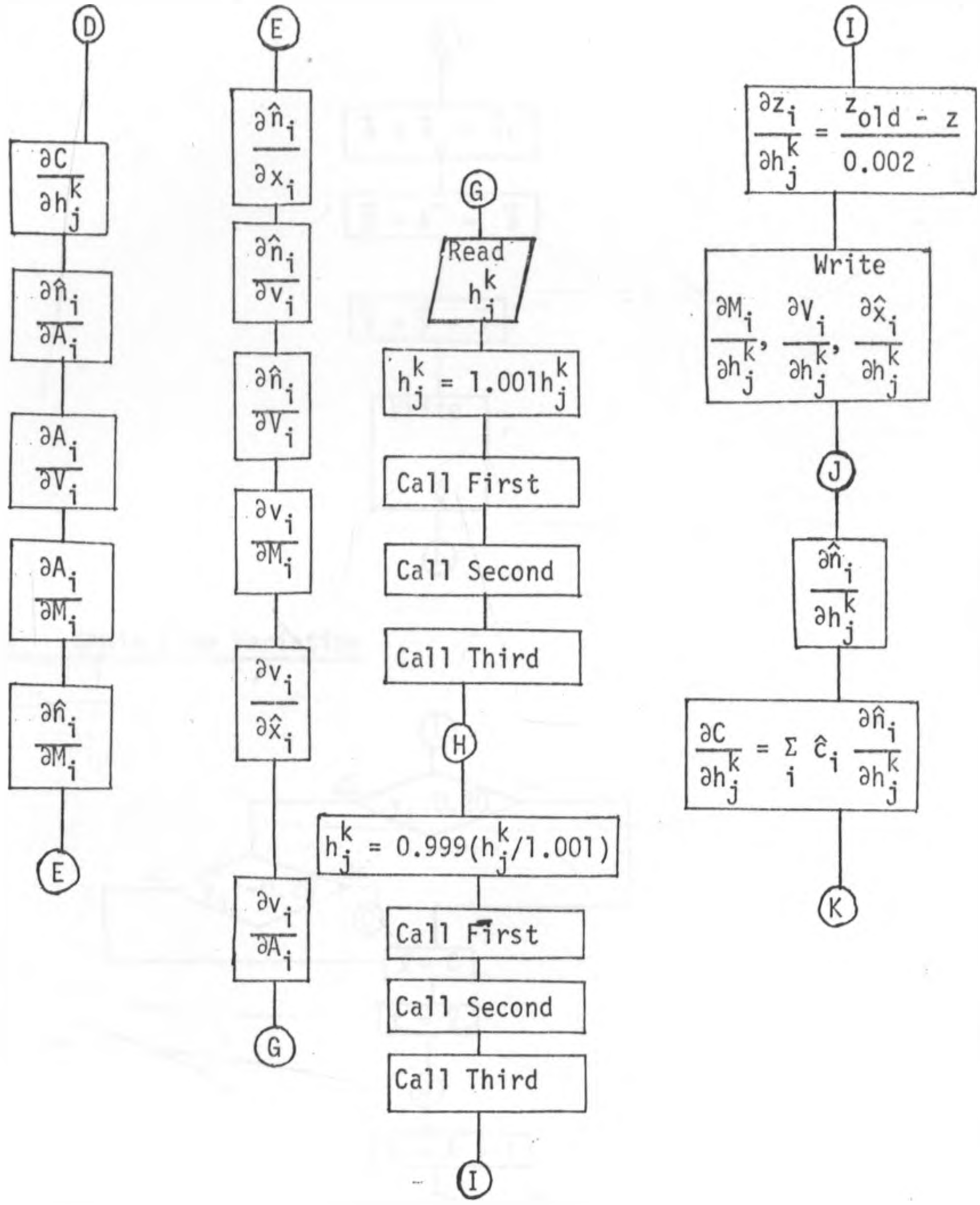
6.1 The Flow Chart

6.1.1 The Method of Dimensioning

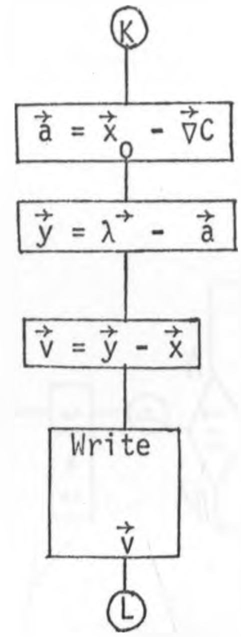



C
C:C_old
<
≥
Write C_old n_k, n_i, h_j^k
C_old = C
Write C_old
D
V
ot_i = (x_i / (x_i + x_i+1)) t_k a-b
i:7
V
W
ot_1+1 = (i / (x_i + x_i+1)) t_k a-b
i:7
W
X
ot_i+9 = (x_i / (x_i + x_i+1)) t_k b-a
i:7
X
Y
ot_1+10 = (x_i / (x_i + x_i-1)) t_k b-a
i:7
Y
Z
END

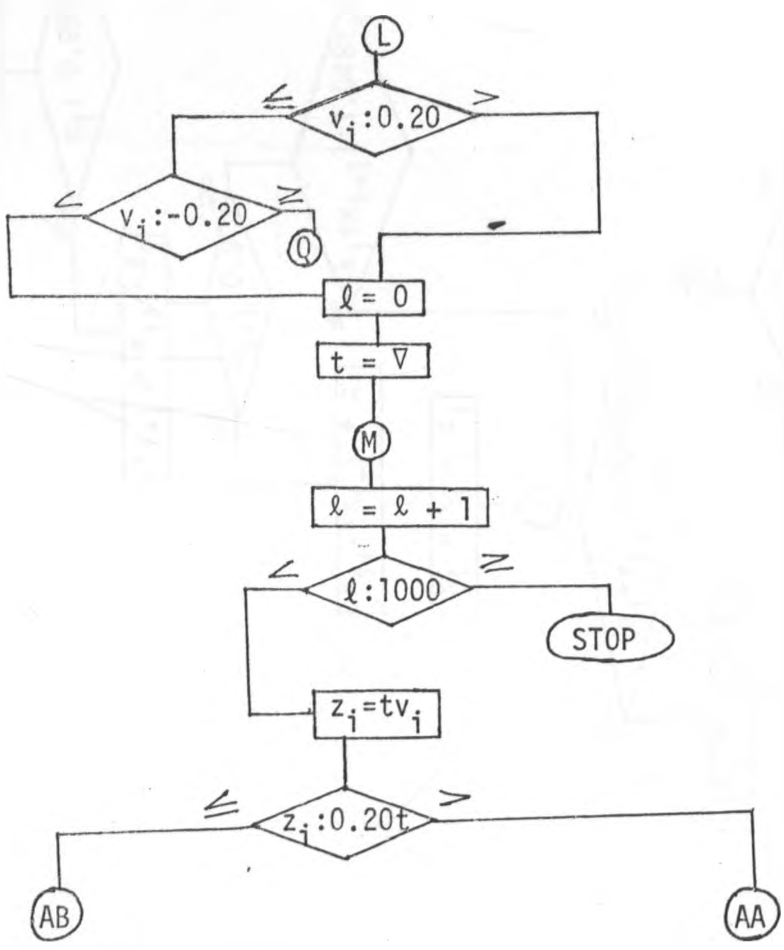
6.1.2 Gradient

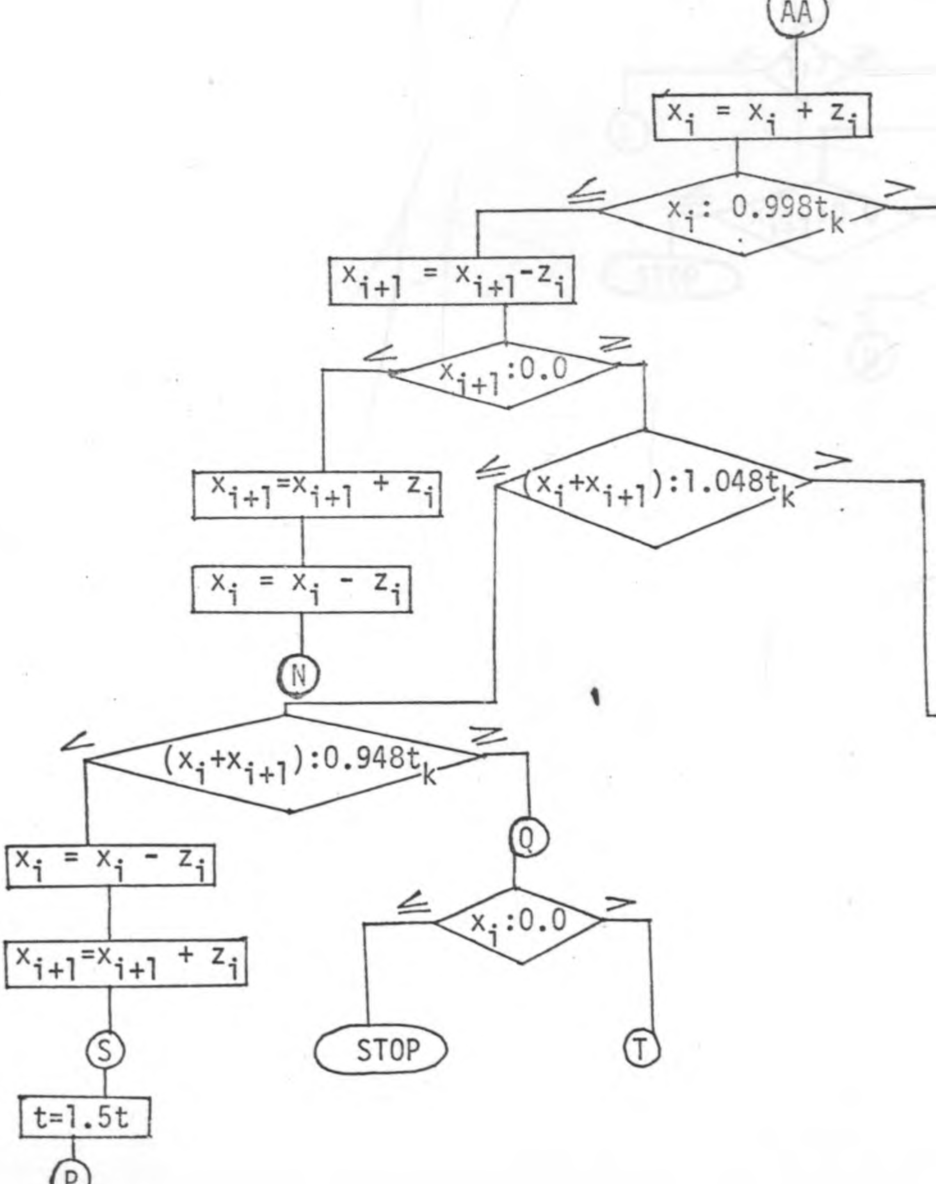


6.1.3 Gradient Projection

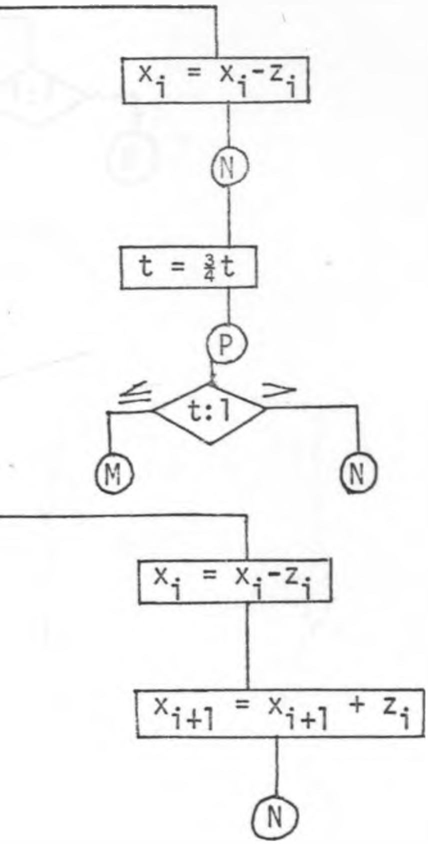


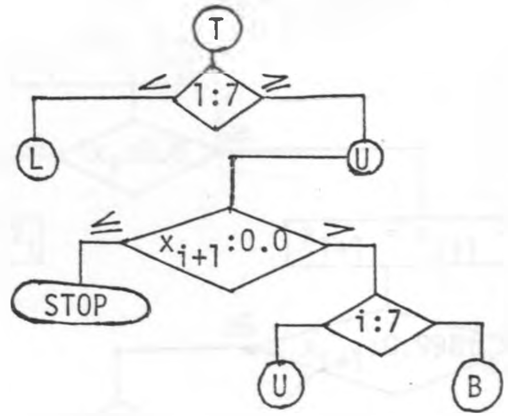
6.1.4 Chain Flow Variation

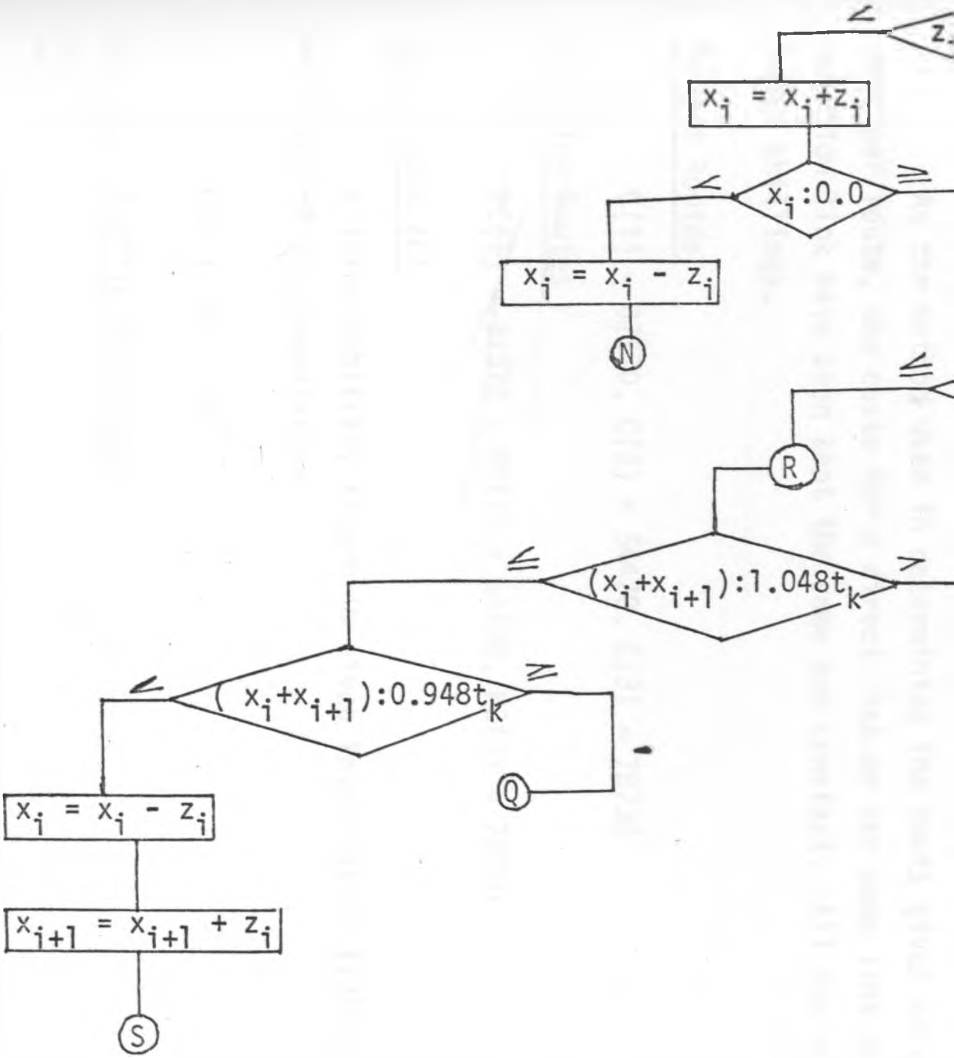




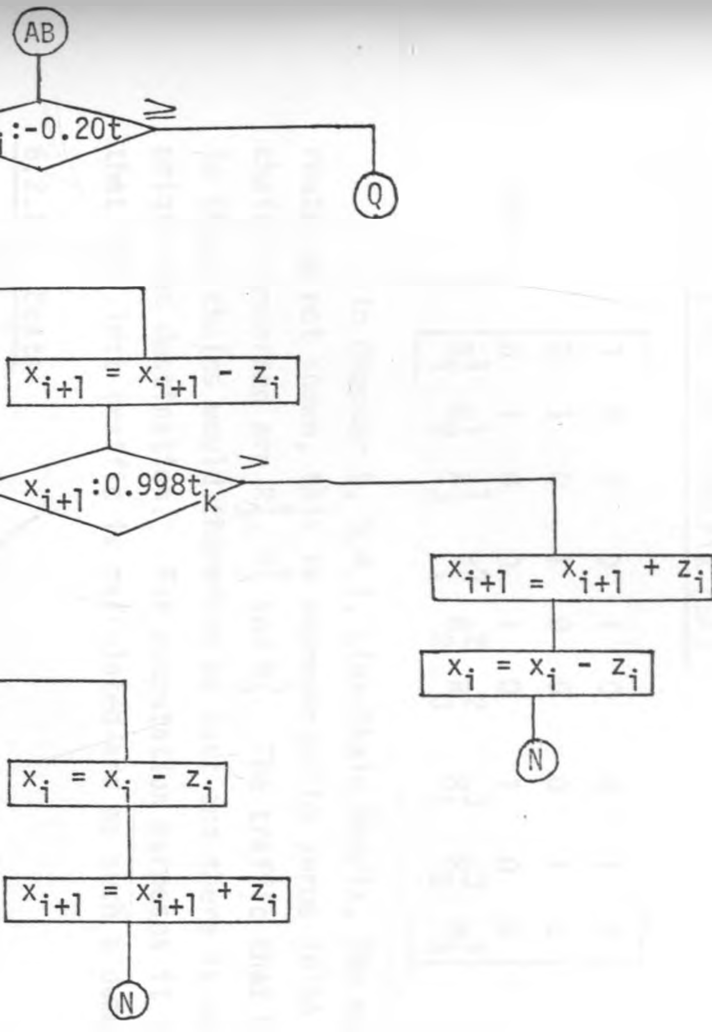
AA
x_i = x_i + z_i
x_i : 0.998t_k
≤
>
x_{i+1} = x_{i+1} - z_i
x_{i+1} : 0.0
<
≥
x_{i+1} = x_{i+1} + z_i
x_i = x_i - z_i
(x_i + x_{i+1}) : 1.048t_k
≤
>
N
(x_i + x_{i+1}) : 0.948t_k
<
≥
x_i = x_i - z_i
x_{i+1} = x_{i+1} + z_i
S
t=1.5t
P
Q
x_i : 0.0
≤
>
STOP
T


$x_i = x_i - z_i$
N
$t = \frac{3}{4}t$
P
$\leq$
t:1
>
M
N
$x_i = x_i - z_i$
$x_{i+1} = x_{i+1} + z_i$
N


T
<
1:7
≥
L
U
≤
x i+1 :0.0
>
STOP
i:7
U
B



<
x_i = x_i+z_i
<
x_i:0.0
≥
x_i = x_i - z_i
N
≤
R
≤
(x_i+x_i+1):1.048t_k
>
<
(x_i+x_i+1):0.948t_k
≥
Q
x_i = x_i - z_i
x_i+1 = x_i+1 + z_i
S


AB
:-0.20t
≧
Q
$x_{i+1} = x_{i+1} - z_i$
$x_{i+1}:0.998t_k$
>
$x_{i+1} = x_{i+1} + z_i$
$x_i = x_i - z_i$
N
$x_i = x_i - z_i$
$x_{i+1} = x_{i+1} + z_i$
N

6.2 Data

6.2.1 Link Chain Matrix (SA)

$$SA = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ R_1^1 & R_2^1 & R_3^1 & R_1^2 & R_2^2 & R_3^2 & R_1^3 & R_2^3 & R_3^3 \end{bmatrix}$$

In Chapter 5, 5.4.1, Link-Chain Matrix, the second overflow route is not shown, this is represented by zeros in SA above, the chains concerned are R_3^1, R_3^2 and R_3^3. The traffic that would overflow to these chains would therefore be lost and there is no link between origin and destination. For computation purposes it is important that this lost traffic is calculated and as such a dummy route is shown.

6.2.2 Costs

As the method used in determining the costs gives an average cost per route, the costs for a direct link or the same link used as an overflow link have been kept the same and constant. All the costs are in Kenya shillings.

Direct Routes

$C(1) = 37570$, $C(2) = 54470$, $C(3) = 78730$

Overflow Routes

$PC(1) = 37570$, $PC(2) = 54470$, $PC(3) = 78730$

Total Cost (C)

A large arbitrary figure is given initially to enable starting of the computation.

$$C = 1.245 \times 10^9$$

6.2.3 Traffic in Erlangs

1975

ABT(1) = 0025.42, ABT(2) = 0027.14, ABT(3) = 0003.78

BAT(1) = 0032.85, BAT(2) = 0018.61, BAT(3) = 0003.70

T(1) = 0058.27, T(2) = 0045.75, T(3) = 0007.48

40% Traffic Flow on Direct Route

H(1,1) = 0023.26, H(2,1) = 0034.89, H(3.1) = 0000.12

H(1,2) = 0018.26, H(2,2) = 0027.40, H(3.2) = 0000.09

H(1,3) = 0002.99, H(2,3) = 0004.48, H(3,3) = 0000.01

60% Traffic Flow on Direct Route

H(1,1) = 0034.89, H(2,1) = 0023.26, H(3,1) = 0000.12

H(1,2) = 0027.40, H(2,2) = 0018.26, H(3.2) = 0000.09

H(1,3) = 0004.48, H(2,3) = 0002.99, H(3,3) = 0000.01

80% Traffic Flow on Direct Route

H(1,1) = 0046.52, H(2,1) = 0011.63, H(3,1) = 0000.12

H(1,2) = 0036.52, H(2,2) = 0009.14, H(3,2) = 0000.09

H(1,3) = 0005.98, H(2,3) = 0001.49, H(3,3) = 0000.01

1980

ABT(1) = 0064.22, ABT(2) = 0067.52, ABT(3) = 0006.08

BAT(1) = 0081.73, BAT(2) = 0046.32, BAT(3) = 0005.96

T(1) = 0145.95, T(2) = 0113.84, T(3) = 0012.04

40% Traffic Flow on Direct Route

H(1,1) = 0058.26, H(2,1) = 0087.40, H(3,1) = 0000.29

H(1,2) = 0045.44, H(2,2) = 0068.17, H(3,2) = 0000.23

H(1,3) = 0004.80, H(2,3) = 0007.22, H(3,3) = 0000.02

60% Traffic Flow on Direct Route

H(1,1) = 0087.40, H(2,1) = 0058.26, H(3.1) = 0000.29

H(1,2) = 0068.17, H(2,2) = 0045.44, H(3,2) = 0000.23

H(1,3) = 0007.22, H(2,3) = 0004.80, H(3,3) = 0000.02

80% Traffic Flow on Direct Route

H(1,1) = 0116.52, H(2,1) = 0029.14, H(3,1) = 0000.29

H(1,2) = 0090.88, H(2,2) = 0022.73, H(3.2) = 0000.23

H(1,3) = 0009.60, H(2,3) = 0002.42, H(3,3) = 0000.20

1985

ABT(1) = 0159.77, ABT(2) = 0168.00, ABT(3) = 0010.00

BAT(1) = 0203.34, BAT(2) = 0115.26, BAT(3) = 0009.60

T(1) = 0363.11, T(2) = 0283.26, T(3) = 0019.60

40% Traffic Flow on Direct Route

H(1,1) = 0144.95, H(2,1) = 0217.43, H(3.1) = 0000.73

H(1,2) = 0113.08, H(2,2) = 0169.61, H(3,2) = 0000.57

H(1,3) = 0007.82, H(2,3) = 0011.74, H(3,3) = 0000.04

60% Traffic Flow on Direct Route

H(1,1) = 0217.43, H(2,1) = 0144.95, H(3,1) = 0000.73

H(1,2) = 0169.61, H(2,2) = 0113.08, H(3,2) = 0000.57

H(1,3) = 0011.74, H(2,3) = 0007.82, H(3,3) = 0000.04

80% Traffic Flow on Direct Route

H(1,1) = 0289.90, H(2,1) = 0072.48, H(3,1) = 0000.73

H(1,2) = 0226.16, H(2,2) = 0056.53, H(3,2) = 0000.57

H(1,3) = 0015.64, H(2,3) = 0003.92, H(3,3) = 0000.04

1990

ABT(1) = 0397.50, ABT(2) = 0417.99, ABT(3) = 0015.78

BAT(1) = 0505.93, BAT(2) = 0286.76, BAT(3) = 0015.48

T(1) = 0903.43, T(2) = 0704.75, T(3) = 0031.26

<u>40% Traffic Flow on Direct Route</u>

H(1,1) = 0360.65, H(2,1) = 0540.97, H(3,1) = 0001.81

H(1,2) = 0281.34, H(2,2) = 0422.00, H(3,2) = 0001.41

H(1,3) = 0012.48, H(2,3) = 0018.72, H(3,3) = 0000.06

<u>60% Traffic Flow on Direct Route</u>

H(1,1) = 0540.97, H(2,1) = 0360.65, H(3,1) = 0001.81

H(1,2) = 0422.00, H(2,2) = 0281.34, H(3.2) = 0001.41

H(1,3) = 0018.72, H(2,3) = 0012.48, H(3,3) = 0000.06

<u>80% Traffic Flow on Direct Route</u>

H(1,1) = 0721.30, H(2,1) = 0180.32, H(3,1) = 0001.81

H(1,2) = 0562.68, H(2,2) = 0140.66, H(3,2) = 0001.41

H(1,3) = 0024.96, H(2,3) = 0006.24, H(3,3) = 0000.06

<u>1990 HYPOTHETICAL</u>

ABT(1) = 0397.50, ABT(2) = 0417.99, ABT(3) = 0400.00

BAT(1) = 0505.93, BAT(2) = 0286.76, BAT(3) = 0350.05

T(1) = 0903.43, T(2) = 0704.75, T(3) = 0750.05

<u>60% Traffic Flow on Direct Route</u>

H(1,1) = 0540.97. H(2,1) = 0360.65, H(3,1) = 0001.81

H(1,2) = 0422.00, H(2,2) = 0281.34, H(3,2) = 0001.41

H(1,3) = 0447.13, H(2,3) = 0301.42, H(3,3) = 0001.50

<u>1995</u>

ABT(1) = 0988.98, ABT(2) = 1039.94, ABT(3) = 0025.43

BAT(1) = 1258.74, BAT(2) = 0713.45, BAT(3) = 0024.94

T(1) = 2247.72, T(2) = 1753.39, T(3) = 0050.37

40% Traffic Flow on Direct Route

H(1,1) = 0897.29, H(2,1) = 1345.93, H(3,1) = 0004.50

H(1,2) = 0699.95, H(2,2) = 1049.93, H(3,2) = 0003.51

H(1,3) = 0020.11, H(2,3) = 0030.16, H(3,3) = 0000.10

60% Traffic Flow on Direct Route

H(1,1) = 1345.93, H(2,1) = 0897.29, H(3,1) = 0004.50

H(1,2) = 1049.93, H(2,2) = 0699.95, H(3,2) = 0003.51

H(1,3) = 0030.16, H(2,3) = 0020.18, H(3,3) = 0000.10

80% Traffic Flow on Direct Route

H(1,1) = 1794.58, H(2,1) = 0448.64, H(3,1) = 0004.50

H(1,2) = 1399.90, H(2,2) = 0349.98, H(3,2) = 0003.51

H(1,3) = 0040.32, H(2,3) = 0010.05, H(3,3) = 0000.10

2000

ABT(1) = 2460.59, ABT(2) = 2587.39, ABT(3) = 0040.97

BAT(1) = 3131.75, BAT(2) = 1775.06, BAT(3) = 0040.18

T(1) = 5592.34, T(2) = 4362.45, T(3) = 0081.15

40% Traffic Flow on Direct Route

H(1,1) = 2232.46, H(2,1) = 3348.70, H(3,1) = 0011.18

H(1,2) = 1741.49, H(2,2) = 2612.24, H(3,2) = 0008.72

H(1,3) = 0032.40, H(2,3) = 0048.59, H(3,3) = 0000.16

60% Traffic Flow on Direct Route

H(1,1) = 3348.70, H(2,1) = 2232.46, H(3,1) = 0011.18

H(1,2) = 2612.24, H(2,2) = 1741.49, H(3,2) = 0008.72

H(1,3) = 0048.59, H(2,3) = 0032.40, H(3,3) = 0000.16

80% Traffic Flow on Direct Route

H(1,1) = 4464.92, H(2,1) = 1116.24, H(3,1) = 0011.18

H(1,2) = 3482.98, H(2,2) = 870.75, H(3,2) = 0008.72

H(1,3) = 0064.80, H(2,3) = 0016.19, H(3,3) = 0000.16

6.3 The Programme

The programme is attached at the end of the appendices.

6.4 Results

1980

40% Traffic Flow on Direct Route

Step Length: STPT = 0.000000001

Total Cost: C = shs.26,714,892.12

Optimal Traffic in Erlangs:

	Direct Route	Alternate Route
Nairobi-Kampala	24.35	39.87
Nairobi-Dar-es-Salaam	25.21	42.31
Dar-es-Salaam-Kampala	0.90	5.18
Kampala-Nairobi	30.99	50.74
Dar-es-Salaam-Nairobi	17.29	29.03
Kampala-Dar-es-Salaam	0.88	5.08

60% Traffic Flow on Direct Route

Step Length: STPT = 0.000000001

Total Cost: C = shs.22,698,114.53

Optimal Traffic in Erlangs:

	Direct Route	Alternate Route
Nairobi-Kampala	36.40	27.82
Nairobi-Dar-es-Salaam	37.64	29.88
Dar-es-Salaam-Kampala	1.21	4.87
Kampala-Nairobi	46.33	35.40
Dar-es-Salaam-Nairobi	25.82	20.50
Kampala-Dar-es-Salaam	1.18	4.78

80% Traffic Flow on Direct Route

Step Length STPT = 0.000000001

Total Cost: C = shs.18,670,984.19

Optimal Traffic in Erlangs:

	Direct Route	Alternate Route
Nairobi-Kampala	48.96	15.26
Nairobi-Dar-es-Salaam	50.76	16.76
Dar-es-Salaam-Kampala	2.09	3.99
Kampala-Nairobi	62.31	19.42
Dar-es-Salaam-Nairobi	34.82	11.50
Kampala-Dar-es-Salaam	2.05	3.91

1985

40% Traffic Flow on Direct Route

Step Length: STPT = 0.000000001

Total Cost: C = shs.64,117,112.61

Optimal Traffic in Erlangs:

	Direct Route	Alternate Route
Nairobi-Kampala	53.74	106.03
Nairobi-Dar-es-Salaam	53.49	114.51
Dar-es-Salaam-Kampala	0.27	9.73
Kampala-Nairobi	68.39	134.95
Dar-es-Salaam-Nairobi	36.70	78.56
Kampala-Dar-es-Salaam	0.26	9.34

80% Traffic Flow on Direct Route

Step Length: STPT = 0.000000001

Total Cost: C = shs43,818,374.44

Optimal Traffic in Erlangs

	Direct Route	Alternate Route
Nairobi-Kampala	113.81	45.96
Nairobi-Dar-es-Salaam	115.53	52.47
Dar-es-Salaam-Kampala	1.14	8.86
Kampala-Nairobi	144.84	58.50
Dar-es-Salaam-Nairobi	79.26	36.00
Kampala-Dar-es-Salaam	1.10	8.50

1990

40% Traffic Flow on Direct Route

Step Length: STPT = 0.000000001

Total Cost: C = shs.155,897,624.08

Optimal Traffic in Erlangs

	Direct Route	Alternate Route
Nairobi-Kampala	79.93	317.57
Nairobi-Dar-es-Salaam	60.61	357.38
Dar-es-Salaam-Kampala	1.20	14.58
Kampala-Nairobi	101.73	404.20
Dar-es-Salaam-Nairobi	41.58	245.18
Kampala-Dar-es-Salaam	1.18	14.30

60% Traffic Flow on Direct Route

Step Length: STPT = 0.000000001

Total Cost: C = shs.130,349,682.66

Optimal Traffic in Erlangs:

	Direct Route	Alternate Route
Nairobi-Kampala	130.74	266.76
Nairobi-Dar-es-Salaam	105.53	312.46
Dar-es-Salaam-Kampala	0.19	15.59
Kampala-Nairobi	166.40	339.53
Dar-es-Salaam-Nairobi	72.40	214.36
Kampala-Dar-es-Salaam	0.18	15.30

80% Traffic Flow on Direct Route

Step Length: STPT = 0.000000001

Total Cost: C = shs.104,882,464.68

Optimal Traffic in Erlangs

	Direct Route	Alternate Route
Nairobi-Kampala	234.83	162.67
Nairobi-Dar-es-Salaam	222.28	195.71
Dar-es-Salaam-Kampala	3.07	12.71
Kampala-Nairobi	298.88	207.05
Dar-es-Salaam-Nairobi	152.49	134.27
Kampala-Dar-es-Salaam	3.01	12.47

80% Traffic Flow on Direct Route

Step Length: STPT = 0.0000000005

1980

Total Cost: C = shs.18,670,984.19

Optimal Traffic in Erlangs

	Direct Route	Alternate Route
Nairobi-Kampala	50.17	14.05
Nairobi-Dar-es-Salaam	52.39	15.13
Dar-es-Salaam-Kampala	3.47	2.61
Kampala-Nairobi	63.85	17.88
Dar-es-Salaam-Nairobi	35.94	10.38
Kampala-Dar-es-Salaam	3.40	2.56

1985

Total Cost: C = shs.43,818,374.44

Optimal Traffic in Erlangs

	Direct Route	Alternate Route
Nairobi-Kampala	120.81	38.96
Nairobi-Dar-es-Salaam	124.97	43.03
Dar-es-Salaam-Kampala	1.91	8.09
Kampala-Nairobi	153.76	49.58
Dar-es-Salaam-Nairobi	85.74	29.52
Kampala-Dar-es-Salaam	1.83	7.77

1990

Total Cost: C = shs.104,882,464.68

Optimal Traffic in Erlangs:

	Direct Route	Alternate Route
Nairobi-Kampala	276.42	121.08
Nairobi-Dar-es-Salaam	278.34	139.65
Dar-es-Salaam-Kampala	1.30	14.48
Kampala-Nairobi	351.82	154.11
Dar-es-Salaam-Nairobi	190.95	95.81
Kampala-Dar-es-Salaam	1.28	14.20

1995

Total Cost: C = shs.254,494.830.37

Optimal Traffic in Erlangs

	Direct Route	Alternate Route
Nairobi-Kampala	564.79	424.19
Nairobi-Dar-es-Salaam	20.23	19.71
Dar-es-Salaam-Kampala	0.84	24.59
Kampala-Nairobi	147.76	110.98
Dar-es-Salaam-Nairobi	361.39	352.06
Kampala-Dar-es-Salaam	0.83	24.11

2000

Total Cost: C = shs.623,114,947.98

Optimal Traffic in Erlangs

	Direct Route	Alternate Route
Nairobi-Kampala	160.53	300.07
Nairobi-Dar-es-Salaam	129.95	457.44
Dar-es-Salaam-Kampala	2.50	38.47
Kampala-Nairobi	45.92	85.83
Dar-es-Salaam-Nairobi	171.47	603.59
Kampala-Dar-es-Salaam	2.45	37.73

1990 Hypothetical

Step Length: STPT = 0.0001

Total Cost: C = shs.170,575,972.20

Opticals traffic in Erlangs

	Direct Route	Alternate Route
Nairobi-Kampala	49.49	348.01
Nairobi-Dar-es-Salaam	74.02	343.97
Dar-es-Salaam-Kampala	7.43	392.57
Kampala-Nairobi	62.99	442.94
Dar-es-Salaam-Nairobi	50.78	235.98
Kampala-Dar-es-Salaam	6.50	343.55

CHAPTER 7

7.0 COMMENTS AND SUMMARY OF CONCLUSIONS

The object of this dissertation was to determine a method of economically optimising a telephone network which has direct and alternate routes, with emphasis on trunk networks by use of a computer. This has been done as explained in the preceding chapters, by deciding to use a suitable telephone traffic mathematical model from which a search procedure and hence a computer programme were developed. It should be noted that this is the first attempt in East Africa to provide a method of economic optimisation of a telephone network. To date there are no alternate routes provided in the network, and there has been no way to examine the economic viability of the routes being provided. The method used here may not be the only one available, but its suitability was based on its non-negativity constraints, flexibility of the grade of service, system optimising chain flow pattern, and that the traffic is completely described in its mean and variance. It should also be noted that political parameters have not been considered in the otpimisation method.

The computer programme results show that the traffic flow on the Dar-es-Salaam-Kampala direct route was the first variable to break the non-negativity constraints, and the step-length had to be reduced to maintain the non-negativity constraints (a typical example is under the Chain Flow Variation in the computer programme attached). This is an important point in the search procedure because once the first variable to break the non-negativity constraints is found, the traffic flow on the chain corresponding to the variable is transferred to another chain between the same exchange pair, while maintaining the conservation requirements.

Having transferred the traffic flow to another chain between the same exchange pair, the chain which was considered as a direct route between Kampala and Dar-es-Salaam can now only serve as an alternate route to traffic flow on other chains. Therefore in the chain matrix, the direct route which was initially represented by digit one (1), is now represented by digit zero (o). In practice this means that a direct link between Kampala and Dar-es-Salaam is uneconomic. If a direct route is not economically justifiable, then it should not be provided.

It should be eliminated in which case it can not be used as an alternate route either. It is therefore concluded that an economic route between Kampala and Dar-es-Salaam would be to use the chain or alternate route through Nairobi. The traffic flow between Nairobi and Kampala, and Nairobi and Dar-es-Salaam would be routed on the direct routes respectively. Examination of the input data would also lead to the same conclusion that a direct link between Kampala and Dar-es-Salaam is uneconomic, since the traffic flow is very small indeed in comparison to the traffic flow on the other routes, and yet if it was transferred to the other routes it would not make any appreciable difference in the percentage increase in circuit provision.

There are other observations made on the study case results generally. The percentage of traffic flow on the direct route was chosen arbitrarily, and from the results, the higher percentage of traffic flow on the direct route, the less the total cost. From the cost function it can be concluded that an alternate route is not economically viable, this however has been confirmed by the non-negativity constraints which determined the exact chain flow to be considered. Examination of the projection vector shows that no optimal point was obtained, not even within the limits acceptable for practical purposes. This was expected since the network could not maintain the conservation requirements. However, on eliminating the direct route between Kampala and Dar-es-Salaam it was not necessary to re-compute traffic to obtain an optimal point, since the number of circuits could be calculated easily using the available Erlang's Loss Traffic Tables, and the remaining network was too simple to require the use of a computer.

In the example considered, the grade of service was taken to be the same on all routes for simplicity, but in practice, this is not always so, and provision for different grade of service for various routes can be made easily (see equations 5.29). In the search procedure, if the arbitrary step length is too large, this is reflected in the results by the traffic and number of circuits calculated not having any direct relationship with the input data. As the step length is reduced the output data begins to be more sensible. But as the step length is further reduced, there is some little change in the number of circuits on a particular link, but not in the total cost of the network.

Just to compare and realize the validity of this method, hypothetical input data has been used for 1990, and here an economical viable route has been established.

To illustrate the method the input data was fed into the computer and an estimated time of ten minutes made for running the programme. The results were examined and from the gradient projection and step length, it was possible to determine in which direction to move the step length. In the example it was possible to minimize the total cost only up to a certain point and then the increase in the traffic parameters became too large and in turn gave very large total cost. It was possible to see that the input data was far too distant from the optimal point. Therefore data which corresponded to a minimum total cost was chosen as the next starting input data and the step length reduced accordingly. This procedure was repeated several times until the final results were obtained (the attached programme). In this results the projection vector could not be reduced to zero, but it had reduced to a minimum, which was accepted as close to zero as possible particularly when comparing the initial and final results. It should be noted here that it is not always possible to reduce the projection vector, down to zero, and minimum values are accepted to be as close to zero as practicable.

If an estimated computing time is not given, and the arbitrary step length was too small, it could take a very long time to ever come down to reasonable figures,and this is very extravagant to the machine both economically and time consumption. It is therefore essential that an estimated computing time is given so that the programme is stopped and examined. Then a realistic step length can be determined which will speed the search for a minimum. It was also noted that the projection vector had a direct proportional relation with the total cost. The higher the total cost the higher the projection vector and the lower the total cost the lower the projection vector.

Examining the attached programme, which illustrates the hypothetical case, it is clear how the search for the minimum total cost is arrived at; by a number of loops through the main programme. Every time a total cost has been calculated and it is less than a previous total cost, the input data has to be varied and a new total cost be recalculated. This is repeated until a new total cost is higher than the previous total cost. Then the minimum cost can be searched around the previous total cost.

It is also possible that there may be more than one minimum point. In this case you would not stop the programme once you determine one minimum point, but you would examine on either side whether it is possible for you to rise to a maximum and begin coming down to another minimum. If this is lower, then you have to disregard the first one and follow this until you can determine that it is the minimum. However, the examination of other minimum points was not done in the hypothetical case.

It is important to note that for the hypothetical data, sixty per cent (60%) of traffic flow on the direct route was chosen arbitrarily as the input data. The search procedure then determined the minimum total cost, determining the economic viability of the direct route between Kampala and Dar-es-Salaam.

In theory, at the optimal point, when the projection vector is zero or as near zero as practicable, for each exchange pair, the marginal cost per erlang is the same for the direct circuit and each of its alternate overflow chains.

The last set of output data is the unidirectional traffic offered on each route, (direct and alternate). It is from these data and knowledge of the grade of service, that one used the Erlang Loss Traffic Tables to determine the number of circuits to be provided in a particular direction on a particular route.

As mentioned before, the provision of circuits between exchanges, to meet the present and future traffic demands, is an important economic consideration. This has been an approach to give an economic optimisation of alternate routing networks, in a telephone network, in terms of the circuit costs. The method optimises the network as a whole, and allows specification at will, of grades of service between each exchange pair. The practicability of the method has been illustrated by the case study and the hypothetical data. This method of optimisation, when restricted or limited availability has been considered, is proposed to be applied to the East African Posts and Telecommunications trunk and local multi-exchange networks.

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January, 1975

T. O. Wandera

APPENDIX I

1.0 THE BIRTH AND DEATH PROCESS

1.1 Transition Probabilities

Consider a system consisting of N sources and R devices. The system is in state E_j when j sources are busy. The number of possible states is N+1 (including 0), where N may be ∞.

Let P(j t) be the probability that the system is in state E_j at time t.

When a new call arrives or a call in progress terminates, the system changes its state. In order to find the distribution P(j,t) it is necessary to have advance knowledge concerning the incoming traffic and properties of the terminating calls.

The changes of the state of the system are studied using the Theory of Markov Processes as a general mathematical model of the actual situation. The Markov Process is a stochastic process in which the future development depends on the present state, but not on the past history of the system.

A Markov Process is characterised by its transition probabilities:

$$P(Y(t) = j|Y(\tau) = i) = P(i, \tau, j, t) \tag{A1.1}$$

which give conditional probability of finding the system at time t in state E_j, given that at a previous time (instant) τ, the state was E_i, where $\tau < t$.

These transition probabilities have to be consistent and satisfy the Chapman-Kolmogorov equation

$$P(i, \tau;j, t+h) = \sum_{v} P(i, \tau;v, t)P(v,t;j, t+h) \tag{A1.2}$$

which is based on the following reasoning. Consider three moments (see sketch below $\tau < t < t + h$)

E_i E_v E_j

time

τ t h t+h

Suppose that at time τ the system is in state E_i, at t, E_v, then the Chapman-Kolmogorov equation reads:

> the probability that there are j busy sources at time t+th given that there were i busy sources at time τ, equals the sum of the probability of transition from i busy sources into j busy sources occuring through intermediary situations of v busy sources at a certain moment t.

For a general stochastic process the probability of transition from state E_v at time t to state E_j at time t+h would depend on state E_i at time τ, i.e. $P(i, \tau; v, t; j, t+h)$. E_i, E_v, E_j being past, present and future states respectively, the past then has an influence on the future.

The transition probabilities in a Markov process relates the present state with the future state, i.e. $P(v, t; j, t+h)$.

Restricting to a temporarily homogenous Markov process $P(i, \tau; j, t)$ depends only on the difference $(t - \tau)$ and is independent of τ.

The transition probabilities $P(i; j, t)$ are now the same for all intervals of length $(t - \tau)$. Putting $\tau = 0$, t becomes a length of an interval arbitrarily placed.

Then for a time-homogenous process, the Chapman-Kolmogorov equation becomes

$$P(i; j, t+h) = \sum_v P(i; v, t)\, P(v; j, h) \tag{A1.3}$$

$P(i; j, t+h)$ is the transition probability that the system is at time t+h in state E_j, given that at time zero it was in state E_i.

$P(i; v, t)$ represents the transition probability from a state E_i at time zero to state E_v at time t,

$P(v; j, h)$ is the conditional probability that if the system is in state E_v at t, it will change into state E_j during h.

Suppose initial state is known E_i. Then $P(i; j, t)$ may be regarded as the absolute probability $P(j, t)$ i.e. E_j at t. Then the Chapman-Kolmogorov equation becomes

$$P(j, t+h) = \sum_v P(v, t)\, P(v; j, h) \tag{A1.4}$$

where

$P(v, t)$ - absolute probability of E_v at t

$P(v; j, h)$- conditional probability of transition.

By this equation the probability of the system being in state E_j at time t+h is obtained, if the transition probabilities from other states E_v to state E_j are known.

To determine the transition probability $P(v; j, h)$ i.e. the probability that the system changes during interval h from state E_v to state E_j; it is assumed that the time-dependent Markov Process-the Birth and Death Process-will describe the physical situation satisfactorily.

1.2 The Birth and Death Process

Assumptions

1. The system changes only through transitions from states to their next neighbours (from E_j to E_{j+1} or E_{j-1} if $0 < j < N$ but from E_0 to E_1 and E_N to E_{N-1} only)

2. If at any time t the system is in state E_j, then the conditional probability that during (t, t+h) the transition from E_j to E_{j+1} (if $j < N$) occurs is

$$P(Y(t+h) = j+1 | Y(t) = j) = \lambda_j h + 0h \qquad (A1.5)$$

where λ_j is a non-negative constant depending on j. (See Appendix 1.1.3).

3. If at any time t the system is in state j, then the conditional probability that during (t, t+h) the transition from E_j to E_{j-1} (if $j > 0$) occurs is

$$P(Y(t+h) = j-1 \mid Y(t) = j) = \mu_j h + 0h \qquad (A1.6)$$

where μ_j is a non-negative constant depending on j.

4. The probability of more than one transition during (t, t+h) is 0h.

The change from E_j to E_{j+1} is interpreted as an arrival of a new call to the system, whereas the change of E_j to E_{j-1} represents the termination of a particular call already in progress. Coefficients λ_j and μ_j represent incoming calls (birth) and terminating calls (death) respectively. λ_j could be called the average rate of growth at the time when the population size is j, and μ_j the average rate of decay. See below for a graphical representation.

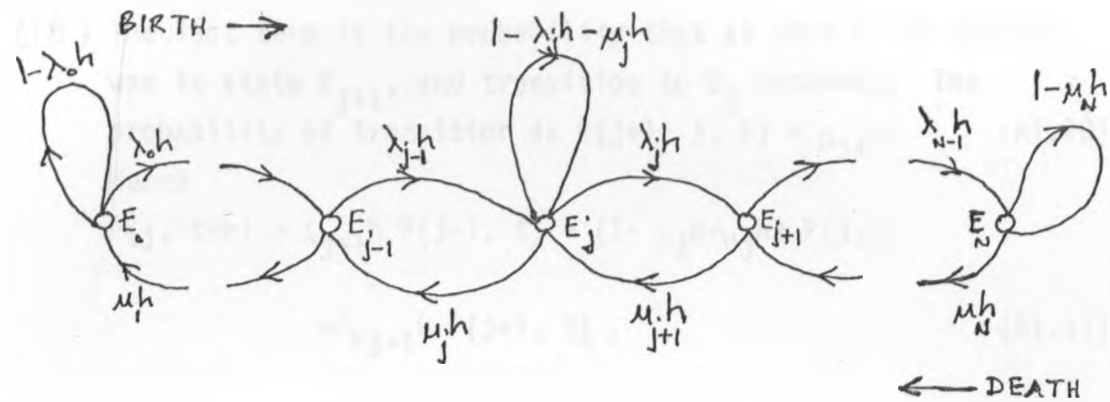


Illustration of the Birth and Death Process

States are junction points on the graph, and the probabilities are written beside the corresponding times.

The assumptions imply that in a small interval h only one event takes place, and the probability of this is proportional to the length h of the interval, and depends only on state E_v of the system at the beginning of the interval h. Consequently, for transition probabilities P(v; j, h) it is enough to state only the length h of the time interval, instead of noting both end points.

The only values permitted for v are now j-1, j and j+1, then

$$P(j,\ t+h) = P(j-1,t)\ P(j-1;\ j,h) + P(j,t)\ P(j;j,h)$$

$$+ P(j+1,\ t)\ P(j+1;\ j,\ h) \qquad \text{(A1.7)}$$

The equation expresses the sum of the probabilities of three mutually exclusive events by which the system can arrive to state E_j at time t+h namely:

(i) The first term is the probability that at time t the system was in state E_{j-1}, with the probability P(j-1, t), and a transition to E_j occurred. The transition probability $P(j-1;\ j,\ h) = \lambda_{j-1}h$. (A1.8)

(ii) The second term is the probability that at time t, the system was in state E_j and during h no change occurred. The probability that there has been no change is

$$P(j; j, h) = 1 - \lambda_j h - \mu_j h \qquad (A1.9)$$

(iii) The last term is the probability that at time t the system was in state E_{j+1}, and transition to E_j occurred. The probability of transition is $P(j+1; j, h) = \mu_{j+1}h$ (A1.10)

Hence

$$P(j, t+h) = \lambda_{j-1}h\, P(j-1, t) + (1 - \lambda_j h - \mu_j h)\, P(j,t) + \mu_{j+1}h\, P(j+1, t) \qquad (A1.11)$$

Rewriting the relation in the form:

$$\frac{P(j, t+h) - P(j,t)}{h} = \lambda_{j-1}P(j-1, t) - (\lambda_j + \mu_j)\, P(j,t) + \mu_{j+1}\, P(j+1, t) \qquad (A1.12)$$

As h tends to zero $(h \to 0)$

$$\frac{P(j, t+h) - P(j,t)}{h} \to \frac{d\,P(j,t)}{dt} \qquad (A1.13)$$

Then

$$\frac{d\,P(j,t)}{dt} = \lambda_{j-1}\, P(j-1,t) - (\lambda_j + \mu_j)\, P(j,t) + \mu_{j+1}\, P(j+1, t) \qquad 1 < j < N \qquad (A1.14)$$

If at a time t = 0 the system is in state E_i, the initial conditions are:

$$P(i,0) = 1 \qquad P(j,0) = 0 \qquad j \neq i \qquad (A1.15)$$

For boundary conditions i.e. j = 0 and j = N

$$\frac{dP(0,t)}{dt} = -\lambda_0 P(0,t) + \mu_1\, P(1,t) \qquad (A1.16)$$

$$\frac{dP(N,t)}{dt} = \lambda_{N-1} P(N-1), t) - \mu_N P(N,t) \quad \text{(A1.17)}$$

The Birth and Death equation, also known as the Equation of State is the equation (A1.14) rewritten

$$\frac{dP(j,t)}{dt} = \lambda_{j-1} P(j-1 ,t) - (\lambda_j + \mu_j) P(j, t) + \mu_{j+1} P(j+1,t) \quad \text{(A1.18)}$$

The equation is derived from the Chapman-Kolmogorov equation under suitable restrictions and for an arbitrary initial state E_i.

When the number of states is finite

$$\sum_{j=0}^{N} P(j,t) = 1 \quad \text{(A1.19)}$$

For the infinite number of states, the Birth and Death process may become a quasi-process, so the transition probability do not add to unity

$$\sum_{j=0}^{N} P(j,t) < 1 \qquad N = \infty \quad \text{(A1.20)}$$

This case presents only theoretical interest and has no traffic applications.

1.3 Definition of the Non-Negative Constants λ_j, μ_j

It is assumed that for each state i in the Birth and Death Process, there is a non-negative continuous function $q_i(s)$ defined by the limit (uniform in (s)).

$$\lim_{h \to 0} \frac{1-P\ (i,s;i,\ s+h)}{h} = q_i(s) \quad \text{(A1.21)}$$

and such that

$$q_i(s) = \sum_{j \neq i} q_{ij}(s) \quad \text{(A1.22)}$$

Equivalent definitions are

$$\frac{\partial P(i,s;i,t)}{\partial t}\Big|_{t=s} = - q_i(s), \frac{\partial P(i,s,j,t)}{\partial t}\Big|_{t=s} = q_{ij}(s) \quad \text{(A1.23)}$$

These functions admit the probabilistic interpretation that the probability of transition from state i to state j during the time interval $(t, t + \Delta t)$ is

$$P(i,t; j, t + \Delta t) = q_{ij}(t)\Delta t + 0\Delta t \qquad i \neq j \qquad \text{(A1.24)}$$

whereas the probability of transition from i to some other state during the time interval $(t, t + \Delta t)$ is

$$1 - P(i, t; i, t + \Delta t) = -q_i(t)\Delta t + 0\Delta t \qquad \text{(A1.25)}$$

Hence the probability of transitions within Δt are asymptotically proportional to the length Δt (in the time homogenous process, the proportionality factor is constant).

APPENDIX 2

2.0 STATISTICAL EQUILIBRIUM

2.1 Definitions

A simple Markov chain is a time-homogeneous (stationary) Markov process for which the variable j takes values in a discrete space I with a finite or infinite number of elements.

A set Z of states E_j is closed if no transition is possible from any state in Z to any state not belonging to Z.

A chain is irreducible if the only closed set in the chain is a set of all states. The absorbing state is a closed set composed of a single state.

The state E_j is persistent if the return to E_j is certain. If this return is uncertain, the state E_j is transient.

A persistent state to which the first return occurs after infinitely long time is a null state.

Another classification divides all states into periodic and aperiodic states. The state E_j is periodic with period k, where k is an integer larger than unity, if the return to E_j occurs only in k, 2k..... steps.

A persistent state which is neither null nor periodic is ergodic.

For irreducible chains, the classification of states can be represented as shown below:

	Persistent		Transient
	Non-null	Null	
Aperiodic	Ergodic		
Periodic			

Table A2.1 Classification of States

In an irreducible Markov chain every state can be reached from every other state and this implies that every state of the system is of the same character: either transient or persistent null or persistent non-null.

Furthermore, in every chain the persistent states can be divided into closed sets so that it is impossible to reach from any state in a given set the states belonging to other sets.

In addition to closed sets, the chain may contain transient states from which states of the closed sets can be reached.

Also a finite chain can contain no null states, and it is impossible that all its states are transient.

These properties are related to the structure of graphs in Fig. A2.1.

The chain is irreducible if the graph does not consist of isolated parts which make it impossible to go from junction points in any part to junction points in other parts along lines of the graph, in the direction of arrows.

A state is aperiodic if the number of lines joining two consecutive junctions, in any possible closed serties of lines with arrows of the same orientation, have unity as the greatest common diviser.

A junction corresponds to the transient state if the graph consists of parts connected so that it is possible to pass from one junction in a certain part to junctions in the isolated part, but not vice versa.

The ergodic chain is represented by a graph on which it is always possible to pass from any junction to any other junction.

2.2 Statistical Equilibrium

If the state E_j is either transient or a persistent-null state then:

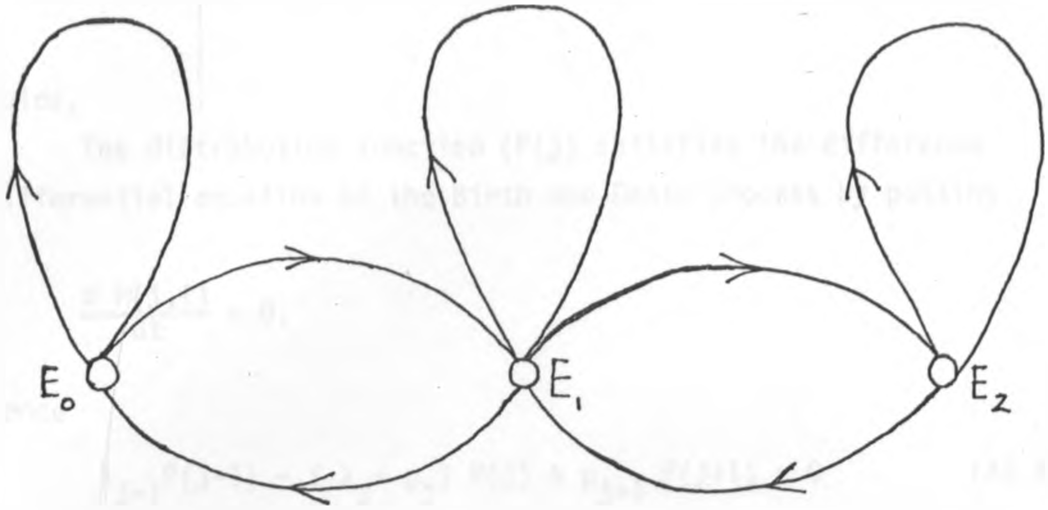
$$\lim_{t \to \infty} P(i, j, t) = 0 \tag{A2.1}$$

If the state E_i and E_j are ergodic then for every pair i, j:

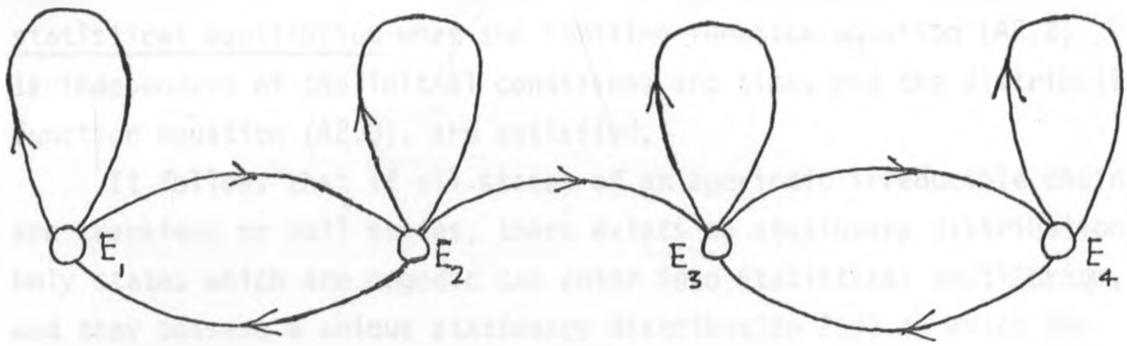
$$\lim_{t \to \infty} P(i, j, t) = P(j) \tag{A2.2}$$

The limiting function is independent of the initial conditions and time, and it is also a distribution function, since the relation

$$\sum_{j=0}^{N} P(j) = 1 \tag{A2.3}$$



All States Ergodic
Irreducible Chain



State E_2 Transient
States E_3 and E_4 Form Closed Set

Periodicity

Fig. A2.1 Various Types of State

holds.

The distribution function (P(j) satisfies the difference differential equation of the Birth and Death process by putting

$$\frac{d\,P(j,t)}{dt} = 0.$$

Hence

$$\lambda_{j-1}P(j-1) - (\lambda_j + \mu_j)\,P(j) + \mu_{j+1}\,P(j+1) = 0 \qquad \text{(A2.4)}$$

$$-\lambda_0 P(0) + \mu_1 P(1) = 0 \qquad \text{(A2.5)}$$

$$\lambda_{N-1}\,P(N-1) - \mu_N P(N) = 0 \qquad \text{(A2.6)}$$

P(j) is the stationary distribution, and the process enters into statistical equilibrium when the limiting function equation (A2.2) is independent of the initial conditions and time, and the distribution function equation (A2.3), are satisfied.

It follows that if all states of an aperiodic irreducible chain are transient or null states, there exists no stationary distribution. Only states which are ergodic can enter into statistical equilibrium, and they possess a unique stationary distribution P(j) to which the distribution P(j, t) tends. Always so for finite chain.

The limit

$$\underset{t \to \infty}{\Sigma} \quad P(i;\, j,\, t) = P(j) \qquad \text{(A2.7)}$$

indicated that eventually the influence of the initial state disappears and steady state conditions are reached. Thus the probability of the system being in any state E_j is the same at the end of any time interval, in other words, the probability of the system is independent of the time at which the system is examined.

APPENDIX 3

3.0 EXPONENTIAL HOLDING TIME

Consider a single telephone line, if at an arbitrary moment t the line is busy, the probability of a change in state during Δt depends on how long the conversation has been going on. In fact the longer the conversation is in progress, the more likely is its termination. Thus, the past has an influence on the future and the process is not a Markov process. It becomes a hereditary process.

Consider a single source, which can be either busy or idle, so N=1 and j can take values 0 and 1 only. Suppose that the system is initially in state E_1 and only the transition from E_1 to E_0 is possible. This means that initially the source is busy and after a certain time the conversation ends. The duration of state E_1 is there interpreted as holding time.

A process of this sort in which only deaths occur is a Death process. So that referring to the Equation of State

$$\frac{dP(1,t)}{dt} = -\mu P(1, t) \tag{A3.1}$$

where P(1, t) is the probability that a holding time is at least t. Note that

$$\frac{d(\exp(x))}{dx} = \exp(x) \tag{A3.2}$$

and

$$\frac{d(\exp(ax))}{dx} = a \exp(ax) \tag{A3.3}$$

Hence for

$$\frac{d P(1, t)}{dt} = -\mu P(1, t) \tag{A3.4}$$

the solution is

$$P(1, t) = \exp(-\mu t) \tag{A3.5}$$

The differential coefficient $\frac{-d P(1, t)}{dt}$ is the probability that the holding time lies between t and $t + \Delta t$, as $\Delta t \to 0$. So that

$$\frac{-d P(1, t)}{dt} = \mu \exp(-\mu t) \tag{A3.6}$$

where $\exp(-\mu t)$ is the distribution of holding times known as an <u>exponential distribution of holding times.</u>

APPENDIX 4

4.0 TRAFFIC DISTRIBUTIONS FOR FULL AVAILABILITY GROUP, LOSS SYSTEM

Consider a full availability group with N sources and n devices in a loss system

O— O— O— O— O— O— O— 0 0 0 0

N Sources n devices

where N and n may be finite or infinite. Also $N \gtreqless n$.

The system can have j simultaneous occupations (j), where

$$0 \leq j \leq \min(n, N) = r \qquad (A4.1)$$

Such that in the expression (see 2.3.1)

$$\lambda_j = y(j).W(j) \qquad (A4.2)$$

we have

$$W(j) = 1 \text{ for } 0 \leq j < r \qquad (A4.3)$$

$$W(j) = 0 \text{ for } j \geq r$$

If $N \leq n$, no calls are rejected since the sources cannot produce more than N simultaneous occupations.

In all distributions it is assumed that the termination of occupations is expressed as below:

$$\mu_j = \frac{j}{\bar{t}} \qquad (A4.5)$$

μ_j is the terminating rate.

j is the number of simultaneous calls.

$\bar{t}$ is the mean holding time.

The assumptions regarding call intensity are considered separately for each distribution.

4.1 Bernouilli Distribution

4.1.1 Assumptions

1. $N \leq n$

2. $\mu_j = \frac{j}{\bar{t}}$ (A4.6)

3. $\lambda_j = y(j).W(j)$

$$y(j) = (N-j)\alpha \ , \ W(j) = 1 \text{ for } 0 \leq j \leq N \quad \text{(A4.7)}$$

where α is the call intensity per source when it is free. Clearly $y(j)$ is proportional to the number of free sources, $(N-j)$.

4.1.2 The Bernouilli Distribution

By assuming statistical equilibrium conditions (see 2. 2.1)

$$\lambda_{j-1} P(j-1) = \mu_j P(j)$$

$$(N-j+1)\alpha \ P(j-1) = \frac{j}{\bar{t}} P(j)$$

$$P(j) = \frac{(N-j+1)\alpha\bar{t}}{j} . P(j-1)$$

$$P(j) = \frac{(N-j+1)\beta}{j} P(j-1) \quad \text{(A4.8)}$$

where $\beta = \alpha\bar{t}$ is the traffic per source when free.
By recursion from $P(j)$ to $P(0)$

$$P(j) = \frac{(N-j+1)(N-j+2) \ ..... \ (N-1)N}{j(j-1) \ ...... \ 2 \ . \ 1} \beta^j \ P(0) \quad \text{(A4.9)}$$

which is precisely expressed as

$$P(j) = \frac{N!}{(N-j)! \ j!} \ \beta^j P(0) \quad \text{(A4.10)}$$

or $$P(j) = \binom{N}{j} \beta^j \ P(0) \quad \text{(A4.11)}$$

from

$$\sum_{j=0}^{N} P(j) = 1 \qquad \text{(see 2. 2.1)} \quad \text{(A4.12)}$$

$$\sum_{j=0}^{N} \binom{N}{j} . \beta^j P(0) = 1 \qquad (A4.13)$$

Re-writing the expression as

$$(1+ \ )^N.P(0) = 1$$

$$P(0) = \frac{1}{(1+\beta\ )^N} \qquad (A4.14)$$

$$\therefore P(j) = \binom{N}{j} \frac{\beta^j}{(1+\beta\ )^N} \qquad (A4.15)$$

can be written as

$$P_{(j)} = \binom{N}{j} \left(\frac{\beta}{1+\ \beta}\right)^j \quad \left(1 - \frac{\beta}{1+\beta}\right)^{N-j} \qquad (A4.16)$$

and this is the Bernouilli Distribution.
Or can be written as

$$P_{(j)} = \binom{N}{j} a^j (1-a)^{N-j} \qquad (A4.17)$$

where $\quad a = \frac{\beta}{1+B} = \frac{\alpha\bar{t}}{1+\alpha t} \qquad (A4.18)$

4.2 Engset Distribution

4.2.1 Assumptions

1. $N > n$
2. $\mu_j = \frac{j}{\bar{t}}$ (A4.19)
3. $\lambda_j = y(j).W(j)$

where

$y(j) = (N-j)\alpha$ (see 4.1.1)

$W(j) = 1$ for $0 \leqslant j < n$

4.2.2 The Engset Distribution

Assuming statistical equilibrium

$\lambda_{j-1}\, P(j-1) = \mu_j\, P(j)$

$$P(j) = \frac{(N-j+1)\alpha\,\bar{t}}{j}\; P(j-1) \qquad \text{(A4.20)}$$

$$P(j) = \frac{(N-j+1)\beta}{j}\quad P(j-1) \quad \text{for } 0 \leq j \leq n \qquad \text{(A4.21)}$$

By recursion from P(j) to P(0)

$$P(j) = \frac{(N-j+1)(N-j+2)\;.....\;(N-1)\;N}{j(j-1)\;.........\;2\;\;\;1}\;\beta^j\;(P(0) \qquad \text{(A4.22)}$$

for $0 \leq j \leq n$

and since $\sum_{j=0}^{n} P(j) = 1$ (A4.23)

$$P(0) = \frac{1}{\sum_{j=0}^{n} \binom{N}{j} \beta^j} \qquad \text{(A4.24)}$$

$$\therefore \quad P(j) = \frac{\binom{n}{j}\beta^j}{\sum_{\nu=0}^{n} \binom{N}{\nu} \beta^\nu} \qquad 0 \quad j \quad n \qquad \text{(A4.25)}$$

and this is the Engset Distribution.

4.3 Erlang Distribution

4.3.1 Assumptions

1. $N = \infty$
2. n finite
3. $\mu_j = \frac{j}{\bar{t}}$ (A4.26)

4. $\lambda_j = y(j).W(j)$

$y(j) = y$ (a constant)

$W(j) = 1$ for $0 \leq j < n$

4.3.2 The Erlang Distribution

Assuming statistical equilibrium

$$\lambda_{j-1} P(j-1) = \mu_j P(j)$$

$$P(j) = \frac{y\bar{t}}{j} P(j-1) \tag{A4.27}$$

By recursion from P(j) to P(0)

$$P(j) = \frac{A.A.A........ \quad A.A}{j(j-1)(j-2).... \quad 2 \ 1} P(0) \tag{A4.28}$$

$$P(j) = \frac{A^j}{j!} P(0) \qquad \text{for } 0 \leq j \leq n \tag{A4.29}$$

where $A = y.\bar{t}$

and since

$$\sum_{j=0}^{n} P(j) = 1 \tag{A4.30}$$

$$P(0) = \frac{1}{\sum_{j=0}^{n} \frac{A^j}{j!}} \tag{A4.31}$$

and hence

$$P(j) = \frac{\frac{A^j}{j!}}{\sum_{\nu=0}^{n} \frac{A^\nu}{\nu!}} \qquad \text{for } 0 \quad j \quad n \tag{A4.32}$$

and this is the Erlang Distribution.

4.4 Poisson Distribution

4.4.1 Assumptions

1. $N = \infty$
2 $n = \infty$
3. $\mu_j = \frac{j}{\bar{t}}$ (A4.33)
4. $\lambda_j = y(j).W(j)$

$y(j) = y$

$w(j) = 1$ for all $j. 0 \leq j \leq \infty$

4.4.2 The Poisson Distribution

Assuming statistical equilibrium

$$\lambda_{j-1} P(j-1) = \mu_j P(j) \qquad \text{(A4.34)}$$

$$P(j) = \frac{A}{j} P(j-1) \qquad \text{(A4.35)}$$

where $A = y, \bar{t}$.

By recursion from $P(j)$ to $P(0)$

$$P(j) = \frac{A.A. \ldots\ldots\ldots\ldots\ldots\ldots A.A.}{j\ (j-1) \qquad\qquad\qquad 2.1.} P(0)$$

$$P(j) = \frac{A^j}{j!} P(0) \qquad \text{(A4.36)}$$

and since

$$\sum_{P=0}^{\infty} P(j) = 1 \qquad \text{(A4.37)}$$

$$P(0) = \frac{1}{\sum_{j=0}^{\infty} \frac{A^j}{j!}} = \frac{1}{\exp(A)} \qquad \text{(A4.38)}$$

thus $P(0) = \exp(-A)$ (A4.39)

and $P(j) = \frac{A^j}{j!} \exp(-A) \qquad (n = N = \infty)$ (A4.40) and this is the Poisson Distribution. Erlang distribution can be regarded as a Poisson distribution truncated at n.

4.5 Negative Binomial Distribution

4.5.1 Assumptions

1. $N = \infty$
2. $n = \infty$ (A4.41)
3. $\mu_j = \frac{j}{\bar{t}}$
4. $\lambda_j = y(j).W(j)$

$y(j) = a(\gamma + j)$

$W(j) = 1$ for all j $0 \leq j \leq \infty$

4.5.2 The Negative Bionomial Distribution

Assuming statistical equilibrium

$$\lambda_{j-1} P(j-1) = \mu_j P(j) \quad \text{(A4.42)}$$

$$P(j) = \frac{\lambda_{j-1}}{\mu_j} P(j-1) = \frac{a\bar{t}(\gamma + j-1)}{j} P(j-1) \quad \text{(A4.43)}$$

By recursion from P(j) to P(0).

$$P(j) = \frac{a\bar{t}(\gamma + j-1)\ a\bar{t}(\gamma + j-2) \ldots\ldots a\bar{t}(\gamma + 1) a\bar{t}\gamma}{j \ldots (j-1) \ldots\ldots 2 \quad 1} P(0)$$

(A4.44)

and letting $b = a\bar{t}$

$$P(j) = b^j \frac{(\gamma+j-1)!}{(\gamma-1)!\ j!} P(0) \quad \text{(A4.45)}$$

$$P(j) = b^j \binom{\gamma+j-1}{j} P(0) \tag{A4.46}$$

$$P(j) = b^j (-1)^j \binom{-\gamma}{j} P(0) \tag{A4.47}$$

$$P(j) = (-b)^j \binom{-\gamma}{j} P(0) \tag{A4.48}$$

and since

$$\sum_{j=0}^{\infty} P(j) = 1 \tag{A4.49}$$

$$\sum_{j=0}^{\infty} (-b)^j \binom{-\gamma}{j} P(0) = 1 \tag{A4.50}$$

$$P(0) = \frac{1}{\sum_{j=0}^{\infty} (-b)^j \binom{-\gamma}{j}} \tag{A4.51}$$

$$P(0) = \frac{1}{(1-b)^{-\gamma}} = (1-b)^{\gamma} \tag{A4.52}$$

$$P(j) = (-b)^j \binom{-\gamma}{j} (1-b)^{\gamma} \qquad 0 \leq j \leq \infty \tag{A4.53}$$

and this is the Negative Binomial Distribution.

4.6 Truncated Negative Binomial Distribution

4.6.1 Assumptions

1. $N = \infty$
2. $n = \text{finite}$ (A4.54)
3. $\mu_j = \frac{j}{t}$
4. $\lambda_j = y(j).W(j)$

$y(j) = a(\gamma + j)$

$W(j) = 1$ for $0 \leq j < n$

4.6.2 The Truncated Negative Binomial Distribution

Assuming statistical equilibrium

$$\lambda_{j-1} P(j-1) = \mu_j P(j) \qquad (A4.55)$$

$$P(j) = \frac{a(\gamma + j - 1)\bar{t}}{j} P(j-1) \qquad (A4.56)$$

By recursion from P(j) to P(0).

$$P(j) = (a\bar{t})^j \frac{(\gamma + j - 1)}{(\gamma - 1)! \quad j!} P(0) \qquad (A4.57)$$

let $b = a\bar{t}$

$$P(j) = b^j \binom{\gamma + j - 1}{j} P(0) \qquad (A4.58)$$

$$P(j) = (-b)^j \binom{-\gamma}{j} P(0) \qquad (A4.59)$$

and since

$$\sum_{j=0}^{n} P(j) = 1 \qquad (A4.60)$$

$$P(0) = \frac{1}{\sum_{j=0}^{n} (-b)^j \binom{-\gamma}{j}} \qquad (A4.61)$$

hence

$$P(j) = \frac{(-b)^j \binom{-\gamma}{j}}{\sum_{\nu=0}^{n} (-b)^\nu \binom{-\gamma}{\nu}} \quad 0 \quad j < n \qquad (A4.62)$$

and this is the Truncated Negative Binomial Distribution.

APPENDIX 5

5.0. DETERMINATION OF THE GRADIENT

The gradient $\vec{\nabla}C$ of the cost function equation (5.28) is the vector

$$\left(\frac{\partial C}{\partial h_1^1}, \frac{\partial C}{\partial h_1^2}, \frac{\partial C}{\partial h_1^3}, \frac{\partial C}{\partial h_2^1}, \frac{\partial C}{\partial h_2^2}, \frac{\partial C}{\partial h_2^3}, \frac{\partial C}{\partial h_3^1}, \frac{\partial C}{\partial h_3^2}, \frac{\partial C}{\partial h_3^3} \right) \qquad \text{(A5.1)}$$

The first three components which relate specifically to chain flows on the direct junctions are obtained by direct differentiation giving (see reference No. 8)

$$\frac{\partial C}{\partial h^k} = C_k\left(1 + \frac{ft^k}{(g)^2}\right) \qquad \text{(A5.2)}$$

where

$$f = 5 + \frac{(t^k - h_1^k - 3)}{\left[(t^k - h_1^k - 3)^2 + 12t^k\right]^{\frac{1}{2}}} \qquad \text{(A5.3)}$$

$$g = 5(t^k - h_1^k)-3 + \left[(t^k - h_1^k -3)^2 + 12t^k\right]^{\frac{1}{2}} \qquad \text{(A5.4)}$$

For the components which relate to the chain flows on overflow junctions h_j^k for $j > 1$, we refer to equation (5.18) and obtain the components

$$\frac{\partial C}{\partial h_j^k} = \sum_{i=1}^{3} \hat{C}_i \frac{\partial \hat{n}_i}{\partial h_j^k} \qquad \text{for } j > 1 \qquad \text{(A5.5)}$$

From equations (5.11), (5.12), (5.14), (5.15) (5.16)

$$n_i = \hat{x}_i + A_i\left(\frac{(M_i-\hat{x}_i)^2 + v_i}{(M_i-\hat{x}_i-1)(M_i-\hat{x}_i) + v_i} - \frac{(M_i)^2 + V_i}{(M_i-1)(M_i) + V_i}\right)$$

$$A_i = v_i + 3\frac{v_i}{M_i}\left(\frac{v_i}{M_i} - 1\right) \qquad \text{(A5.6)}$$

$$v_i = \frac{(M_i-\hat{x}_i)\left[3-(M_i-\hat{x}_i) + \left[(M_i-\hat{x}_i-3) + 12A_i\right]^{\frac{1}{2}}\right]}{6}$$

$$\hat{x}_i = \sum_k \sum_j a^k_{ij} h^k_j$$

$$M_i = \sum_k \sum_j a^k_{ij} M^k_j$$

$$V_i = \sum_k \sum_j a^k_{ij} V^k_j$$

Further examination of functional dependence gives

$$\frac{\partial \hat{n}_i}{\partial h^k_j} = \frac{\partial \hat{n}_i}{\partial A_i} \cdot \frac{\partial A_i}{\partial h^k_j} + \frac{\partial \hat{n}_i}{\partial M_i} \cdot \frac{\partial M_i}{\partial h^k_j} + \frac{\partial \hat{n}_i}{\partial x_i} \cdot \frac{\partial \hat{x}_i}{\partial h^k_j} + \frac{\partial \hat{n}_i}{\partial v_i} \cdot \frac{\partial v_i}{\partial h^k_j} + \frac{\partial \hat{n}_i}{\partial v_i} \cdot \frac{\partial v_i}{\partial h^k_j} \qquad \text{(A5.7)}$$

where

$$\frac{\partial A_i}{\partial h^k_j} = \frac{\partial A_i}{\partial V_i} \cdot \frac{\partial V_i}{\partial h^k_j} + \frac{\partial A_i}{\partial M_i} \cdot \frac{\partial M_i}{\partial h^k_j} \qquad \text{(A5.8)}$$

$$\frac{\partial v_i}{\partial h^k_j} = \frac{\partial v_i}{\partial M_i} \cdot \frac{\partial M_i}{\partial h^k_j} + \frac{\partial v_i}{\partial \hat{x}_i} \cdot \frac{\partial \hat{x}_i}{\partial h^k_j} + \frac{\partial v_i}{\partial A_i} \cdot \frac{\partial A_i}{\partial h^k_j} \qquad \text{(A5.9)}$$

substituting equation (A5.8) into equation (A5.9) gives

$$\frac{\partial v_i}{\partial h_j^k} = \frac{\partial v_i}{\partial M_i} \cdot \frac{\partial M_i}{\partial h_j^k} + \frac{\partial v_i}{\partial \hat{x}_i} \cdot \frac{\partial \hat{x}_i}{\partial h_j^k} + \frac{\partial v_i}{\partial A_i} \left(\frac{\partial A_i}{\partial v_i} \cdot \frac{\partial v_i}{\partial h_j^k} + \frac{\partial A_i}{\partial V_i} \cdot \frac{\partial M_i}{\partial h_j^k} \right) \tag{A5.10}$$

such that further substitutions of equations (A5.8) and (A5.10) into equation (A5.7) yields

$$\frac{\partial \hat{n}_i}{\partial h_j^k} = \frac{\partial \hat{n}_i}{\partial A_i} \left(\frac{\partial A_i}{\partial v_i} \cdot \frac{\partial V_i}{\partial h_j^k} + \frac{\partial A_i}{\partial M_i} \cdot \frac{\partial M_i}{\partial h_j^k} \right) + \frac{\partial \hat{n}_i}{\partial M_i} \cdot \frac{\partial \hat{x}_i}{\partial h_j^k} + \frac{\partial \hat{n}_i}{\partial x_i} \cdot \frac{\partial \hat{x}_i}{\partial h_j^k}$$

$$+ \frac{\partial \hat{n}_i}{\partial v_i} \left(\frac{\partial v_i}{\partial M_i} \cdot \frac{\partial M_i}{\partial h_j^k} + \frac{\partial v_i}{\partial \hat{x}_i} \cdot \frac{\partial \hat{x}_i}{\partial h_j^k} + \frac{\partial v_i}{\partial A_i} \left(\frac{\partial A_i}{\partial V_i} \cdot \frac{\partial V_i}{\partial h_j^k} + \frac{\partial A_i}{\partial M_i} \cdot \frac{\partial M_i}{\partial h_j^k} \right) \right)$$

$$+ \frac{\partial \hat{n}_i}{\partial V_i} \cdot \frac{\partial V_i}{\partial h_j^k} \tag{A5.11}$$

where

$$\frac{\partial \hat{n}_i}{\partial A_i} = \frac{(M_i - \hat{x}_i)^2 + v_i}{(M_i - \hat{x}_i - 1)(M_i - \hat{x}_i) + v_i} - \frac{(M_i)^2 + V_i}{(M_i - 1)(M_i) + V_i} \tag{A5.12}$$

$$\frac{\partial A_i}{\partial V_i} = 1 + \frac{6V_i}{M_i} - \frac{3}{M_i} \tag{A5.13}$$

$$\frac{\partial A_i}{\partial M_i} = \frac{3V_i}{(M_i)^2} - \frac{6(V_i)^2}{(M_i)^3} \tag{A5.14}$$

$$\frac{\partial \hat{n}_i}{\partial M_i} = A_i \left(\frac{2\left[(M_i-\hat{x}_i-1)(M_i-\hat{x}_i) + v_i\right]\left[M_i-\hat{x}_i\right] - \left[(M_i-\hat{x}_i)^2 + v_i\right]\left(\left[M_i-\hat{x}_i\right] 2-1\right)}{\left[(M_i - \hat{x}_i - 1)(M_i - \hat{x}_i) + v_i\right]^2} \right.$$

$$\left. - \frac{2\left[(M_i-1)M_i + V_i\right]\left[M_i\right] - \left[(M_i)^2 + V_i\right]\left[2M_i - 1\right]}{\left[(M_i - 1)(M_i) + V_i\right]^2} \right) \tag{A5.15}$$

$$\frac{\partial\hat{n}_i}{\partial x_i} = 1+A_i\left(\frac{-2\left[(M_i-\hat{x}_i-1)(M_i-\hat{x}_i)+V_i\right]\left[M_i-\hat{x}_i\right] - \left[2\hat{x}_i-2M_i+1\right]\left[(M_i-\hat{x}_i)^2+v_i\right]}{\left[(M_i-\hat{x}_i-1)(M_i-\hat{x}_i)+v_i\right]^2}\right)$$

(A5.16)

$$\frac{\partial\hat{n}_i}{\partial v_i} = A_i\left(\frac{\left[(M_i-\hat{x}_i-1)(M_i-\hat{x}_i)+v_i\right] - \left[(M_i-\hat{x}_i)^2+v_i\right]}{\left[(M_i-\hat{x}_i-1)(M_i-\hat{x}_i)+v_i\right]^2}\right) \qquad \text{(A5.17)}$$

$$\frac{\partial\hat{n}_1}{\partial v_i} = -A_i\left(\frac{\left[(M_i-1)(M_i)+V_i\right] - \left[(M)^2+V_i\right]}{\left[(M_i-1)(M_i)+v_i\right]^2}\right) \qquad \text{(A5.18)}$$

Rewriting equation (5.15) as

$$v_i = \frac{3(M_i-\hat{x}_i)-(M_i-\hat{x}_i)^2+\left[(M_i-\hat{x}_i-\ )^2(M_i-\hat{x}_i)^2+12A_i(M_i-\hat{x}_i)^2\right]^{\frac{1}{2}}}{6}$$

$$v_i = \frac{3(M_i-\hat{x}_i)-(M_i-\hat{x}_i)^2}{6}$$

$$+ \frac{(\left[(M_i)^2-2\hat{x}_iM_i-6M_i+6\hat{x}_i+(\hat{x}_i)^2+9\right]\left[(M_i)^2\right.}{6}$$

$$- \frac{\left.2\hat{x}_iM_i+(\hat{x}_i)^2\right]+12A_i(M_i-\hat{x}_i)^2)^{\frac{1}{2}}}{6} \qquad \text{(A5.19)}$$

gives

$$\frac{\partial v_i}{\partial M_i} = (3-2(M_i-\hat{x}_i)+\tfrac{1}{2}(\left[(M_i)^2-2\hat{x}_iM_i-6M_i+6\hat{x}_i+(\hat{x}_i)^2+9\right]$$

$$\left[(M_i)^2-2(\hat{x}_i)M_i+(\hat{x}_i)^2\right]+12A_i(M_i-\hat{x}_i)^2)^{\frac{1}{2}}(4(M_i)^3$$

$$-6\hat{x}_i(M_i)^2+2(\hat{x}_i)^2M_i-6\hat{x}_i(M_i)^2+8\hat{x}_iM_i-2(\hat{x}_i)^3-18(M_i)^2$$

$$+24\hat{x}_iM_i-6(\hat{x}_i)^2+12\hat{x}_iM_i-12(\hat{x}_i)^2+2(\hat{x}_i)^2M_i-2(\hat{x}_i)^3$$

$$+18M_i-18\hat{x}_i+24A_i(M_i-\hat{x}_i)))/6 \qquad \text{(A5.20)}$$

$$\frac{\partial v_i}{\partial \hat{x}_i} = (-3+2(M_i-\hat{x}_i)+\tfrac{1}{2}(\left[(M_i)^2-2\hat{x}_iM_i-6M_i+6\hat{x}_i+(\hat{x}_i)^2+9\right]\left[(M_i)^2\right.$$

$$\left.-2\hat{x}_iM_i+(\hat{x}_i)^2\right]+12A_i(M_i-\hat{x}_i)^2)^{-\frac{1}{2}}(-2(M_i)^3+2(M_i)^2\hat{x}_i$$

$$-2(M_i)^3+8(M_i)^2\hat{x}_i-6(M_i)(\hat{x}_i)^2+12(M_i)^2-12M_i\hat{x}_i+6(M_i)^2-24M_i\hat{x}$$

$$+18(\hat{x}_i)^2+(M_i)^2\hat{x}_i-6M_i(\hat{x}_i)^2+4(\hat{x}_i)^3-18M_i+18\hat{x}_i-24A_i(M_i-\hat{x}_i)))/$$

(A5.21)

$$\frac{\partial v_i}{\partial A_i} = (\tfrac{1}{2}\left[12(M_i-\hat{x}_i)^2\right](\left[(M_i)^2-2\hat{x}M_i-6M_i+6\hat{x}_i+(\hat{x}_i)^2+9\right]\left[(M_i)^2\right.$$

$$\left.-2\hat{x}_iM_i+(\hat{x}_i)^2\right]+12A_i(M_i-\hat{x}_i)^2)^{-\frac{1}{2}})/6 \qquad \text{(A5.22)}$$

The rest of the partial derivatives at point h_j^k are approximated by using a very small finite value Δh_j^k in the definition of the derivatives (see reference No. 6 Chapter 9.3). Thus

$$f'(h_j^k) = \frac{f(h_j^k+\Delta h_j^k) - f(h_j^k-\Delta h_j^k)}{2\Delta h_j^k} \qquad \text{(A5.23)}$$

Such that for every value of h_j^k we determine $h_j^k+\Delta h_j^k$ and $h_j^k-\Delta h_j^k$. For each of these new values we calculate M_i, V_i and $\hat{x}_i$. Then the partial derivatives are accordingly defined as:

$$\frac{\partial M_i}{\partial h_j^k} = \frac{M_i^+ - M_i^-}{2\Delta h_j^k}$$

$$\frac{\partial V_i}{\partial h_j^k} = \frac{V_i^+ - V_i^-}{2\Delta h_j^k} \qquad \text{(A5.24)}$$

$$\frac{\partial \hat{x}_i}{\partial h_j^k} = \frac{\hat{x}_i^+ - \hat{x}_i^-}{2\Delta h_j^k}$$

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