CURVATURE TENSORS AND THEIR RELATIVISTIC SIGNIFICANCE (II)

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Summary: In this paper we have defined the curvature tensors and their properties are studied.

1. Introduction: In the n-dimensional space V_n, the tensors

$$(1.1)\qquad C(X, Y, Z, T)=R(X, Y, Z, T)-\frac{R}{n(n-1)}[g(X, T)g(Y, Z)-g(Y, T)g(X, Z)],$$

$$(1.2)\qquad L(X, Y, Z, T)=R(X, Y, Z, T)-\frac{1}{n-2}[g(Y, Z)\operatorname{Ric}(X, T)-g(X, Z)\operatorname{Ric}(Y, T)+g(X, T)\operatorname{Ric}(Y, Z)-g(Y, T)\operatorname{Ric}(X, Z)],$$

and

$$(1.3)\qquad V(X, Y, Z, T)=R(X, Y, Z, T)-\frac{1}{n-2}[g(X, T)\operatorname{Ric}(Y, Z)-g(Y, T)\operatorname{Ric}(X, Z)+g(Y, Z)\operatorname{Ric}(X, T)-g(X, Z)\operatorname{Ric}(Y, T)]+\frac{R}{(n-1)(n-2)}[g(X, T)g(Y, Z)-g(Y, T)g(X, Z)],$$

are called *concircular, conharmonic* and *conformal curvature tensor* respectively [1]. These satisfy the symmetric and skew symmetric as well as the cyclic property possessed by curvature tensor $R(X, Y, Z, T)$.

The projective curvature tensor is given by

$$(1.4)\qquad W(X, Y, Z, T)=R(X, Y, Z, T)+\frac{1}{n-1}[g(X, Z)\operatorname{Ric}(Y, T)-g(X, T)\operatorname{Ric}(Y, Z)].$$

We shall now define the curvature tensors and study their properties.

2. Curvature Tensors

Definition (2.1): We define the tensors

$$(2.1)\qquad W^*(X, Y, Z, T) \overset{\text{def}}{=} R(X, Y, Z, T) - \frac{1}{(n-1)}[g(X, Z)\,\mathrm{Ric}(Y, T) - g(X, T)\,\mathrm{Ric}(Y, Z)],$$

$$(2.2)\text{a}\qquad W_1(X, Y, Z, T) \overset{\text{def}}{=} R(X, Y, Z, T) + \frac{1}{n-1}[g(X, T)\,\mathrm{Ric}(Y, Z) - g(Y, T)\,\mathrm{Ric}(X, Z)],$$

and

$$(2.2)\text{b}\qquad W_1^*(X, Y, Z, T) \overset{\text{def}}{=} R(X, Y, Z, T) - \frac{1}{n-1}[g(X, T)\,\mathrm{Ric}(Y, Z) - g(Y, T)\,\mathrm{Ric}(X, Z)].$$

From the equations (1.1) to (2.2)b, it is clear that for an empty gravitational field characterized by $\mathrm{Ric}(X, Y)=0$, the eight fourth rank tensors are identical.

In the space V_n, from equations (1.1), (1.2) and (1.3) we have

$$(2.3)\qquad V(X, Y, Z, T) = L(X, Y, Z, T) + \frac{n}{n-2}[R(X, Y, Z, T) - C(X, Y, Z, T)],$$

which for V_4 reduces to

$$(2.4)\qquad V(X, Y, Z, T) = L(X, Y, Z, T) + 2R(X, Y, Z, T) - 2C(X, Y, Z, T).$$

We notice that (2.1) is skew-symmetric in Z, T and it also satisfies

$$(2.5)\qquad W^*(X, Y, Z, T) + W^*(Y, Z, X, T) + W^*(Z, X, Y, T) = 0.$$

Breaking $W^*(X, Y, Z, T)$ in two parts viz.

$$G(X, Y, Z, T) = \frac{1}{2}[W^*(X, Y, Z, T) - W^*(Y, X, Z, T)],$$

and

$$H(X, Y, Z, T) = \frac{1}{2}[W^*(X, Y, Z, T) + W^*(Y, X, Z, T)],$$

which are respectively skew-symmetric and symmetric in X, Y. From (2.1) it follows that

$$(2.6)\qquad G(X, Y, Z, T) = R(X, Y, Z, T) - \frac{1}{2(n-1)}[g(X, Z)\,\mathrm{Ric}\,(Y, T) - g(X, T)\,\mathrm{Ric}\,(Y, Z) - g(Y, Z)\,\mathrm{Ric}\,(X, T) + g(Y, T)\,\mathrm{Ric}\,(X, Z)]\,,$$

and

$$(2.7)\qquad H(X, Y, Z, T) = \frac{1}{2(n-1)}[g(X, T)\,\mathrm{Ric}\,(Y, Z) - g(X, Z)\,\mathrm{Ric}\,(Y, T) - g(Y, Z)\,\mathrm{Ric}\,(X, T) + g(Y, T)\,\mathrm{Ric}\,(X, Z)]\,.$$

It can be seen from (2.6) that $G(X, Y, Z, T)$ possesses all the symmetric and skew-symmetric properties of $R(X, Y, Z, T)$ as well as the cyclic property [2].

From equations (1.3) and (2.6) we get

$$(2.8)\qquad G(X, Y, Z, T) = \frac{1}{2(n-1)}\Big[(3n-4)R(X, Y, Z, T) - (n-2)V(X, Y, Z, T) - \frac{R}{(n-1)}\{g(X, Z)g(Y, T) - g(X, T)g(Y, Z)\}\Big]\,,$$

which for an electromagnetic field in V_4 becomes

$$(2.9)\qquad 3G(X, Y, Z, T) = 4R(X, Y, Z, T) - V(X, Y, Z, T)\,.$$

From equations (1.2) and (2.6), for V_4, we have

$$(2.10)\qquad 3G(X, Y, Z, T) = 4R(X, Y, Z, T) - L(X, Y, Z, T)\,.$$

Thus equation (2.9) is the consequence of (2.10) for a space of vanishing scalar curvature.

Considering $W_1(X, Y, Z, T)$ and $W_1^*(X, Y, Z, T)$, we notice that these are skew-symmetric in X, Y and satisfy the cyclic property. We break both into two parts which are respectively skew-symmetric and symmetric in Z, T.

$$A(X, Y, Z, T) = \frac{1}{2}[W_1(X, Y, Z, T) - W_1(X, Y, T, Z)]\,,$$

$$B(X, Y, Z, T) = \frac{1}{2}[W_1(X, Y, Z, T) + W_1(X, Y, T, Z)]\,,$$

and

$$M(X, Y, Z, T) = \frac{1}{2}[W_1^*(X, Y, Z, T) - W_1^*(X, Y, T, Z)]\,,$$

$$N(X, Y, Z, T) = \frac{1}{2}[W_1^*(X, Y, Z, T) + W_1^*(X, Y, T, Z)]\,,$$

From (2.2)a it follows that

$$A(X, Y, Z, T)=R(X, Y, Z, T)+\frac{1}{2(n-1)}[g(X, T)\,\mathrm{Ric}(Y, Z) - g(Y, T)\,\mathrm{Ric}(X, Z)-g(X, Z)\,\mathrm{Ric}(Y, T) + g(Y, Z)\,\mathrm{Ric}(X, T)]\,, \tag{2.11}$$

and

$$B(X, Y, Z, T)=\frac{1}{2(n-1)}[g(X, T)\,\mathrm{Ric}(Y, Z)-g(Y, T)\,\mathrm{Ric}(X, Z) + g(X, Z)\,\mathrm{Ric}(Y, T)-g(Y, Z)\,\mathrm{Ric}(X, T)]\,. \tag{2.12}$$

$A(X, Y, Z, T)$ satisfies symmetric, skew-symmetric and the cyclic properties as satisfied by $R(X, Y, Z, T)$.

From (1.3) and (2.11), for an electromagnetic field in V_4, we have

$$3A(X, Y, Z, T)=4R(X, Y, Z, T)-V(X, Y, Z, T)\,. \tag{2.13}$$

Equations (1.2) and (2.11) in V_4 yield

$$3A(X, Y, Z, T)=4R(X, Y, Z, T)-L(X, Y, Z, T)\,. \tag{2.14}$$

From equation (2.2)b, we have,

$$M(X, Y, Z, T)=R(X, Y, Z, T)+\frac{1}{2(n-1)}[g(X, Z)\,\mathrm{Ric}(Y, T) - g(X, T)\,\mathrm{Ric}(Y, Z)+g(Y, T)\,\mathrm{Ric}(X, Z) - g(Y, Z)\,\mathrm{Ric}(X, T)]\,, \tag{2.15}$$

and

$$N(X, Y, Z, T)=\frac{1}{2(n-1)}[g(Y, T)\,\mathrm{Ric}(X, Z)-g(X, T)\,\mathrm{Ric}(Y, Z) + g(Y, Z)\,\mathrm{Ric}(X, T)-g(X, Z)\,\mathrm{Ric}(Y, T)]\,. \tag{2.16}$$

For the electromagnetic field in V_4, from (1.3) and (2.15) we have

$$3M(X, Y, Z, T)=V(X, Y, Z, T)+2R(X, Y, Z, T)\,. \tag{2.17}$$

Also from (1.2) and (2.15) for V_4, we have

$$3M(X, Y, Z, T)=L(X, Y, Z, T)+2R(X, Y, Z, T)\,. \tag{2.18}$$

$M(X, Y, Z, T)$ satisfies symmetric, skew-symmetric as well as the cyclic property as satisfied by $R(X, Y, Z, T)$.

The vector

$$\theta_i=\frac{g_{ij}\epsilon^{jklm}R^p_k R_{pl;m}}{\sqrt{-g}\,R_{ab}R^{ab}}\,, \tag{2.19}$$

is called the complexion vector of a non-null electromagnetic field with no matter by *Misner and Wheeler* [3] and its vanishing implies that field is purely electrical. A semi-colon stands for covariant differentiation.

Interchanging the dummy indices l, m (2.19) can be written as

$$\theta_i = \frac{g_{ij}\epsilon^{jkml} R^p_k R_{pm;l}}{\sqrt{-g} R_{ab} R^{ah}} , \tag{2.20}$$

$$= - \frac{g_{ij}\epsilon^{iklm} R^p_k R_{pm;l}}{\sqrt{-g} R_{ab} R^{ab}} .$$

By setting $W^{*h}\ pm;h=0$, we get

$$R_{pm;l} = R_{pl;m} , \tag{2.21}$$

which on substitution in (2.20) implies that $\theta_i = 0$. Thus the vanishing of the divergence of the Weyl* curvature tensor in an electromagnetic field implies a purely electric field.

It is seen that we can't get a purely electric field with the help of $W_{1\,hijk}$ and $W^*_{1\,hijk}$.

From equation (1.1) on contracting, we get

$$C_i{}^j = \left(R_{ij} - \frac{R}{n} g_{ij} \right) , \tag{2.22}$$

an *Einstein* tensor.

By contracting W^*_{hijk}, defined by (2.1), we get

$$W^*_{hk} = g^{ij} W^*_{hijk} = \left(\frac{n-2}{n-1}\right)\left(R_{hk} + \frac{R}{n-2} g_{hk} \right) , \tag{2.23}$$

and

$$W^* = g^{hk} W^*_{hk} = \left(\frac{n-2}{n-1}\right)\left\{ R + \frac{nR}{n-2} \right\} = 2R .$$

The scalar invariant of second degree in W^*_{hk} is given by

$$W^*_2 = W^*_{hk} W^{*hk} = \left(\frac{n-2}{n-1}\right)^2 \left\{ R_2 + \frac{2R^2}{n-2} + \frac{nR^2}{(n-2)^2} \right\} . \tag{2.24}$$

From (2.6), on contracting, we have

$$G_{hk} = \frac{2n-3}{2(n-1)} \left[R_{hk} + \frac{R}{(2n-3)} g_{hk} \right] , \tag{2.25}$$

and

$$G = g^{hk} G_{hk} = \frac{3}{2} R .$$

Similarly from (2.7) we have,

$$H_{hk}=\frac{-n}{2(n-1)}\left[R_{hk}-\frac{R}{n}g_{hk}\right], \tag{2.26}$$

and the scalar $H=0$.

Contracting $W_{1\,hijk}$, A_{hijk} and B_{hijk}, we find that the results obtained are exactly similar to equations (2.23) to (2.26).

In the similar manner, on contracting $W^*_{1\,hijk}$ we have

$$W^*_{1\,hk}=\frac{n}{n-1}\left(R_{hk}-\frac{R}{n}g_{hk}\right), \tag{2.27}$$

and

$$W^*_1=g^{hk}W^*_{1\,hk}=0\ ,$$

This scalar invariant of second degree in $W^*_{1\,hk}$ is given by

$$(W^*_1)_2=W^*_{1\,hk}W^{*hk}_1=\frac{n^2}{(n-1)^2}\left(R_2-\frac{R^2}{n}\right). \tag{2.28}$$

From (2.27), we have

$$W^*_{1\,hk}R^{hk}=\frac{n}{(n-1)}\left(R_2-\frac{R^2}{n}\right). \tag{2.29}$$

Hence

$$W^*_{1\,hk}W^{*hk}_1=\left(\frac{n}{n-1}\right)W^*_{1\,hk}R^{hk}\ . \tag{2.30}$$

From (2.15) we notice that contracted M_{ij} vanishes identically for an *Einstein* space. This enables us to extend the *Pirani* formation of gravitational waves to the *Einstein* space with the help of M_{hijk}.

For an *Einstein* space M_{hijk}, $W^*_{1\,hijk}$, W_{hijk} and V_{hijk} are identically equal. Also we can show that the vanishing of symmetric part N_{hijk} is the necessary and sufficient condition for a space to be an *Einstein* space.

We can also obtain the *Rainich* [4] type of conditions for the existence of the non-null electrovac universe with the help of $W^*_{1\,hijk}$ similar to those obtained by W_{hijk} [2].

From the above discussion it is seen that except the vanishing of the complexion vector, the tensor $W^*_{1\,hijk}$ behaves exactly in the same manner as W_{hijk} and their symmetric as well as skew symmetric parts have same properties. Similar thing happens with W^*_{hijk} and $W_{1\,hijk}$.

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