

A COMPARISON OF THE CLASSICAL BLACK-SCHOLES MODEL AND THE GARCH OPTION PRICING MODEL FOR CURRENCY OPTIONS

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Abstract

This paper looks at the consequences of introducing heteroscedasticity in option pricing. The analysis shows that introducing heteroscedasticity results in a better fitting of the empirical distribution of foreign exchange rates than in the Brownian model. In the Black-Scholes world the assumption is that the variance is constant, which is definitely not the case when looking at financial time series data. In this study we therefore price a European call option under a Garch model Framework using the Locally Risk Neutral Valuation Relationship. Option prices for different spot prices are calculated using simulations. We use the non-linear in mean Garch model in analyzing the Kenyan foreign exchange market.

Key words: *Heteroscedasticity, Black-Scholes, Option pricing, Garch model, Foreign exchange rates, Risk Neutral Valuation.*

1 Introduction

The foreign exchange market has become the world's largest financial market with daily trading exceeding \$1.5 trillion. This makes it the most volatile and the most liquid of all financial markets. Unlike the stock or bond markets, there is no geographic location where the transactions are bid and cleared. American and European options on foreign currencies are actively traded in both over the counter (OTC) where trading takes place via the telephone or in the electronic network as well as in exchanges. The major currencies traded include Australian Dollar, Sterling pound, Canadian dollar, Japanese yen and Euros. In the Nairobi Stock Exchange Options trading is yet to be introduced and so our research does not focus on data from the options market because that is not available. Since plans are underway to introduce a derivatives market in the Nairobi Stock Exchange, this work is intended to introduce an alternative investment for the Kenyan economy once trading in derivative securities is introduced. Foreign currency option trading has emerged as an alternative investment for many traders and investors. As an investment tool, foreign currency option trading provides both large and small investors with greater flexibility when determining the appropriate forex trading and hedging strategies to implement.

Since the derivation of an arbitrage-free and risk-neutral closed-form solution to European option pricing Black & Scholes (1973) and Merton (1973), a number of advancements and modifications to the original modeling techniques have been suggested. These attempt to account for certain behavioral patterns displayed by the underlying asset which are contrary to the assumptions that have been made in the original log-normal one-factor model. The original model is Markovian in nature and consists of a deterministic drift term (which is the continuously compounded risk free rate in the risk-neutral world) and a term that accounts for random or volatile behavior. In pricing European options that have a terminal payoff dependent on the underlying, the assumptions that are made pertain to continuous trading, transaction costs, borrowing and lending and the returns distribution of the underlying. At maturity of the option, and throughout the option's life, it is assumed that the terminal distribution of the underlying is lognormal with a constant standard deviation.

Margrabe (1978) developed the pricing equation for the value of the option (a European type) to exchange one risky asset for another. His theory is based on the Black and Scholes (1973) model for pricing a call option which assumes that the price of a risk less discount bond grows exponentially at the risk less interest rate and Merton's (1973) extension in which the discount bond's value is stochastic until maturity. Pricing foreign currency options were first done by Garman and Kohlhagen (1987) for European type currency options. Their argument was that the standard Black-Scholes option pricing model does not apply well to foreign exchange options since multiple interest rates are involved unlike in the Black-Scholes assumptions. They came up with modified valuation formulas for foreign currency options.

The assumption of normally distributed log returns has been proved to be incorrect and in addition the assumption of constant volatility is not realistic.

The family of GARCH volatility models has become an important tool kit in empirical asset pricing and financial risk management. Following the work of Engle (1982) and Bollerslev (1986), a voluminous econometric literature has developed on volatility estimation and forecasting. Important papers in empirical finance that apply GARCH models include Bollerslev (1986), Campbell and Hentschel (1992), French et al. (1987), Glosten et al. (1993), Pagan and Schwert (1990), and Schwert (1989) among others. Motivated equally by the success of GARCH models in fitting asset log returns and by the failure of deterministic volatility models in fitting option prices, researchers have extended the GARCH model into the domain of option valuation. The first attempt to provide a rigorous theoretical foundation for option pricing in the GARCH framework can be found in Duan(1990).

With reference to the application of GARCH in the option pricing area, Duan (1995) was the first to develop a risk-neutral model within the GARCH framework. He uses a model called nonlinear GARCH in mean (NGARCH-M) developed by Engle and Ng (1993). He characterized the transition between the actual and the risk-neutral probability distributions if the dynamics of the underlying equity price is given by a GARCH process, and thus established the foundation for the valuation of options under GARCH. In his model, the risk-neutral option price depends upon a leverage risk premium parameter, and the variance is negatively correlated with the past equity returns. Such a negative correlation gives rise to a negative skewness in a risk-neutral distribution. The leverage risk premium parameter is estimated directly from the empirical data since it is part of the conditional expected return of the underlying equity. Hence, Duan's model generates option prices that are consistent with the observed volatility skew. Another feature of the Duan's model (and all GARCH models) is that it is non-Markovian in nature and can explain some of the systematic biases associated with the Black-Scholes model. In particular, the option price depends on the information set generated by the past and current prices of the underlying equity and its past, current and one-step-ahead values of the conditional variance process. In his early attempt (in Duan (1990)) the Risk Neutral Valuation Relationship (RNVR) was wrongly applied to option pricing in the GARCH framework.

Posedel (2006) looked at the NGARCH-M model as an alternative to the Black-Scholes model for pricing foreign currency options for the Croatian market. In Duan (1995) and Posedel (2006) the improvement provided by the NGARCH model is that the option price is a function of the risk premium embedded in the underlying asset, which contrasts with the standard preference-free option pricing result that is obtained in the Black-Scholes model.

The goal of this study is to estimate the parameters appearing in the the currency time series dynamics using the nonlinear-in-mean, asymmetric GARCH model. The obtained parameters are then used for

simulating prices of foreign currency options under the NGARCH option pricing model. We then compare the obtained prices with the Black-Scholes ones to determine if the heteroscedasticity assumption for changes in the time series present in the NGARCH model reflects visible differences in option prices.

The paper is organized as follows. In the second part of the work we look at the Black-Scholes option pricing model and its extension for foreign currency option pricing- the Garman and Kohlhagen currency option pricing model and it's merits and demerits. In part three we then look at GARCH models and propose a model for derivative pricing under GARCH as a way of improving on some of the short falls of the Garman and Kohlhagen currency option pricing model. We develop a GARCH option pricing model under a Locally Risk Neutral Valuation Relationship (LRNVR). We also highlight some of the advantages of GARCH option pricing and its limitations. In part four we discuss the exploratory data analysis and any other analysis. The data is from daily exchange rate data for the Kenyan market from the year 1999 to 2006. We also estimate the parameters for use in pricing and we price options for different strikes and compare with those of the Black-Scholes model and finally we report our findings. we then conclude and give recommendations and finally we give a list of the references used in the text.

2 Black-Scholes Currency Model

Just like any other stock market, in a foreign exchange market, holding the basic asset (currency) is a risky business. The exchange rate varies from time to time just as the stock prices do. Foreign exchange involves two assets which pay interest and the exchange rate itself. The exchange rate means how much of the domestic currency are needed to buy one unit of foreign currency. There are therefore three instruments and processes to model- two currency cash bonds (one local and the other foreign) and the exchange rate. In order to price currency derivatives, assumptions need to be made about the evolution of the exchange rate.

The key assumption in Black-Scholes Option pricing model is lack of arbitrage opportunities and operating in a frictionless market. Arbitrage is generally defined as 'the simultaneous purchase and sale of the same securities, commodities or foreign exchange in different markets to profit from unequal prices'

Any no arbitrage argument for pricing a derivative is ultimately based on a replication strategy, which is a trading strategy that uses market instruments to 'replicate' the initial and final positions required by the derivative. If we have two strategies with the same initial position, and guaranteed final positions, then these final positions must be equal. Otherwise, by going long on the strategy with the higher final value and short on the the other we would generate an arbitrage. The arbitrage pricing theory does not depend on the real world probabilities. The pricing is done in a risk neutral world.

This is implied by the following assumptions:-

- (i) There is unlimited short selling of stock.
- (ii) There is unlimited borrowing and lending.
- (iii) There are no transaction costs or commissions.
- (iv) At time $t = 0$ there is only one market price for each security and each investor can buy and sell as many securities as desired without changing the market price. The investor is a price taker, i.e., his/her trading does not move the market.
- (v) All parties have the same access to relevant information.
- (vi) All assets are perfectly divisible and liquid.

Before looking at the valuation of currency derivatives, let us introduce some notations. The strike price which is the pre-specified fixed price is denoted by K . It is also called the exercise price. The expiry date or maturity date is denoted by T . In order to purchase an option contract, the investor needs to pay an options price (premium) to a second party at the initial date when the contract is entered into. The time t spot price of one unit of foreign currency, i.e. the exchange rate is denoted by X_t . This is the number of units of domestic currency that can be exchanged for one unit of foreign currency and X_T is the exchange rate at expiry date T . Therefore X_T is not known at time 0 and as such it gives rise to the uncertainty in our model. An option is either exercised or abandoned. The payoff of a European call option $g(X_T)$ is thus given by:-

$$\begin{aligned} g(X_T) &= (X_T - K)^+ \\ &= \max\{X_T - K, 0\} \end{aligned} \tag{1}$$

i.e.

$$g(X_T) = \begin{cases} X_T - K; & \text{if } X_T > K \quad (\text{option is exercised}), \\ 0; & \text{if } X_T \leq K \quad (\text{option is abandoned}). \end{cases}$$

Likewise, the payoff of a European put option $h(X_T)$ will be given by:-

$$\begin{aligned} h(X_T) &= (K - X_T)^+ \\ &= \max\{K - X_T, 0\} \end{aligned} \tag{2}$$

i.e.

$$h(X_T) = \begin{cases} K - X_T; & \text{if } X_T < K \quad (\text{option is exercised}), \\ 0; & \text{if } X_T \geq K \quad (\text{option is abandoned}). \end{cases}$$

Payoffs of call and put options satisfy the following relations

$$g(X_T) - h(X_T) = (X_T - K)^+ - (K - X_T)^+ = X_T - K.$$

This is the put-call parity relation of options, i.e. the price of a European put is determined by the price of a European call with the same strike price, expiry date, current price of the underlying and the properly discounted value of the strike price. Under the Black-Scholes Model we look at trading in continuous time and examine situations where asset price dynamics are driven by a Wiener process rather than by a discrete time model. The idea is that at each instant we can look at the Wiener process W_t and study its change dW_t at that instant, which, intuitively can be thought of as moving slightly up or down with equal probability.

2.1 Currency Option Valuation Formula Consider a standard European call option, whose payoff at expiry date T equals

$$C_T^X \triangleq N(X_T - K)^+ \quad (3)$$

where X_T is the spot price of the deliverable currency (i.e., the spot exchange rate at the option's expiry date), K is the strike price in units of domestic currency per unit, and $N > 0$ is the nominal value of the option, expressed in units of the underlying foreign currency. Assume $N = 1$ and consider an option to buy one unit of a foreign currency at a pre-specified price K which may be exercised at the date T only.

The arbitrage price, in units of domestic currency, of a currency European call option, is given by the risk-neutral valuation formula

$$C_T^X = e^{-r^d(T-t)} \mathbb{E}_{\mathbb{P}^*}((X_T - K)^+ | \mathcal{F}_t) \quad (4)$$

This can be further given in the following expression

$$C_T^X = X_t e^{-r^f(T-t)} N(d_1(X_t, T-t)) - K e^{-r^d(T-t)} N(d_2(X_t, T-t)),$$

where N is the standard Gaussian cumulative distribution function, \mathbb{P}^* is the risk neutral probability measure and

$$d_1(x, t) = \frac{\ln(\frac{x}{K}) + (r^d - r^f + \frac{1}{2}\sigma_X^2)(T-t)}{\sigma_X \sqrt{T-t}}$$

$$\text{and} \quad d_2(x, t) = \frac{\ln(\frac{x}{K}) + (r^d - r^f - \frac{1}{2}\sigma_X^2)(T-t)}{\sigma_X \sqrt{T-t}}$$

3 Models for Heteroskedasticity Continuous time stochastic volatility models are effective for option pricing, but can be difficult to implement. Although these models assume that volatility is observable, it's very difficult to filter a continuous volatility variable from discrete observations. Although volatility is not directly observable, it has some characteristics that are commonly seen in asset returns. First, there exist volatility clusters (i.e., volatility may be high for certain time periods and low for other periods). Second, volatility evolves over time in a continuous manner, that is, volatility jumps are rare. Third, volatility does not diverge to infinity, that is, volatility varies within some fixed range. Statistically speaking, this

means that volatility is often stationary. Fourth, volatility seems to react differently to a big price increase or a big price drop, referred to as the leverage effect. Some volatility models were proposed specifically to correct the weaknesses of the existing ones for their inability to capture the characteristics mentioned earlier. For example, the EGARCH model was developed to capture the asymmetry in volatility induced by big 'positive' and 'negative' asset returns.

3.1 Financial Time Series Conditionally heteroscedastic models for time series play an important role in today's financial risk management, which typically tries to make financial decisions based on observed discrete time exchange rate data X_t . Let $X_t, t = 0, \dots, n$ be a time series of exchange rate data. Instead of analyzing X_t which often display unit-root behavior and thus cannot be modeled as stationary, we often transform them to the so called log-returns and analyze the log returns on X_t , i.e. the series

$$P_t = \log X_t - \log X_{t-1} = \ln \left(\frac{X_t}{X_{t-1}} \right) = \ln \left(1 + \frac{X_t - X_{t-1}}{X_{t-1}} \right) = \ln(1 + R_t)$$

where

$$R_t = \frac{X_t - X_{t-1}}{X_{t-1}}$$

is the relative return. The natural logarithm of the simple gross return of an asset is called the continuously compounded return or log return. Stationarity enables parameter estimation using the entire available data set. Log-returns are approximately equal to relative price changes and appear to be in agreement with the hypothesis of stationarity, at least in the short run. By Taylor expanding the above, we see that P_t is almost equivalent to the relative return R_t . The reason we consider log returns instead of relative returns is the additivity property of log returns, which is not shared by relative returns **3.2 ARCH** Engle (1982) was the first to propose a stationary non-linear model for P_t which he termed the ARCH (Autoregressive conditionally Heteroskedastic; it means that the conditional variance of P_t evolves according to an autoregressive type process). The ARCH model is formally a regression model with the conditional volatility as the response variable and the past lags of the squared return as the covariates. The basic idea of ARCH models is that:-

- (i) the shock of an asset return is serially uncorrelated, but dependent, and
- (ii) the dependence of the shock can be described by a simple quadratic function of its lagged values.

According to the ARCH model, the conditional error distribution is normal with the conditional variance equal to a linear function of past squared errors. There is thus a tendency that extreme values will be followed by other extreme values, but of unpredictable signs.

Let the shock or innovation of an asset return at time t be a_t , where $a_t = P_t - \mu_t$ and P_t is the conditional return and μ_t is the expected return. a_t can also be looked at as the unexpected return. In its simplest

form, an ARCH model expresses the innovation return series a_t as

$$a_t = \sigma_t \varepsilon_t \quad \text{where} \quad \varepsilon_t \sim N(0, 1)$$

and σ_t satisfies

$$\sigma_t^2 = \omega + \sum_{i=1}^p \alpha_i a_{t-i}^2. \quad (5)$$

where $\alpha_1, \dots, \alpha_p$ and ω are constant parameters. The predictable volatility is dependent on past news. The conditional variance is modeled as a function of lagged a 's. The effect of a return shock i periods ago ($i \leq p$) on current volatility is governed by the parameter α_i . Normally, we would expect that $\alpha_i < \alpha_j$ for $i > j$ i.e. the older the news the less effect it has on the current volatility. In an ARCH(p) model, old news which arrived at the market p periods ago has no effect at all on current volatility

3.3 GARCH Bollerslev (1986) generalizes the ARCH(p) model to the GARCH(p,q) such that:-

$$a_t = \sigma_t \varepsilon_t \quad \text{where} \quad \varepsilon_t \sim N(0, 1)$$

$$\sigma_t^2 = \omega + \sum_{i=1}^p \alpha_i a_{t-i}^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2. \quad (6)$$

where $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q$ and ω are constant parameters. The GARCH model is an infinite order ARCH model. In the GARCH model the effect of a return shock on current volatility declines geometrically over time. The main idea is that σ_t^2 , the conditional variance of P_t , given the information available up to time $t - 1$ has an autoregressive structure and is positively correlated to it's own recent past and the recent values of the squared innovations a^2 . This captures the idea pf volatility (conditional variance) being "persistent": large (small) values of a_t^2 are likely to be followed by large (small) values.

In its standard version, the GARCH model is assumed to be driven by normally distributed innovations. Conditions on α 's and β 's need to be imposed for equation (7) to be well defined. Empirically, the family of GARCH models has been successful. Of these models GARCH(1,1) is preferred in most cases (see survey by Bolleslev et al (1992)).

In the basic GARCH model (7), since only squared residuals a_{t-j}^2 enter the equation, the signs of the residuals or shocks have no effects on conditional volatility. However, a stylized fact of financial volatility is that bad news (negative shocks) tends to have a larger impact on volatility than good news (positive shocks). Black (1976) attributes this effect to the fact that bad news tends to drive down the stock price, thus increasing the leverage (i.e., the debt-equity ratio) of the stock and causing the stock to be more volatile. Based on this conjecture, the asymmetric news impact is usually referred to as the leverage effect. One of the main reasons to consider the GARCH extension is that by allowing past volatilities to affect the present volatility in (7), a more parsimonious model may result.

For example if a_t is a GARCH(1,1) then

$$\sigma_t^2 = \omega + \alpha a_{t-1}^2 + \beta \sigma_{t-1}^2$$

When $\alpha_1 + \beta_1 = 1$, the underlying process a_t is no longer stationary and we have an *integrated* GARCH(1,1) also known as the IGARCH(1,1) model. One of the interpretations of the IGARCH(1,1) model is that the volatility is persistent i.e it is not covariance stationary. Although the IGARCH(1,1) model bears a strong resemblance to the random walk model, precautions need to be taken for such an analogy since the autocorrelations still decay exponentially which is in contrast with the autocorrelations for a random walk model for which the autocorrelations are approximately equal to 1. In equation (7), only the squares or magnitudes of a_{t-i} and σ_{t-j} affect the current volatility of σ_t^2 . This is often unrealistic since the market reacts differently to bad news than to good news. There is a certain amount of asymmetry that cannot be explained by equation (7). Owing to the considerations above, there are many generalizations of the GARCH model to capture some of these phenomena. There are FIGARCH, GJR-GARCH, TGARCH, EGARCH, and NGARCH models, just to name a few. The Gaussian assumption of ε_t is not crucial. One may relax it to allow for more heavy-tailed distributions, such as a t-distribution. Of course, having the Gaussian assumption facilitates the estimation process.

3.4 The GARCH Option Pricing Model The seminal work of Black and Scholes (1973) on option pricing lays the foundation for the modern option pricing theory. One of the most significant theoretical implications of the Black-Scholes theory is the irrelevance of the underlying assets risk premium. Specifically, it stipulates that an options value is a direct function of the underlying assets volatility but is not a function of the expected return. This implies that an options value is independent of the relative proportions of systematic and non-systematic risks as long as the total asset risk remains the same. Although this conclusion logically follows from a continuous hedging argument within the BlackScholes framework, it seems at odds with a simple economic intuition: a higher expected return (due to a bigger systematic risk) should increase the chance for a call option to finish in the money keeping other variables fixed.

The recently developed GARCH option pricing theory by Duan(1995), however, offers a drastically different theoretical predication on the effect of systematic risk. The GARCH framework allows for fat tails in returns as well a correlation between the lagged exchange rate returns and the conditional variance, both of which will affect the value of the foreign currency options. In contrast to the Black-Scholes theory and other generalizations, an options value is a direct function of the stocks expected return (via the risk premium λ_t). Although it is intuitively clear that λ_t can be related to the systematic risk of the underlying asset, a formal relationship in the GARCH framework was first derived in Duan and Wei (2005). In addition to the assumptions needed for the GARCH option pricing model, they assumed the resulting stochastic discount factor follows a one-factor (the market portfolio return) linear structure. They were able to express the risk premium per unit of return standard deviation for the $j - th$ asset as

$$\lambda_{jt} = \frac{c_t \beta_{jt} \sigma_{mt}}{\sigma_{jt}}$$

where

$$\beta_{jt} = cov^p \left(\ln \frac{s_t}{s_{t-1}}, \ln \frac{m_t}{m_{t-1}} | \mathcal{F}_{t-1} \right) / \sigma_{mt}^2$$

and $\ln \frac{m_t}{m_{t-1}}$ and σ_{mt}^2 are log return and variance of the market portfolio and c_t is constant across different assets and can be identified using the market portfolio because its own systematic risk equals one by definition. This risk premium can be explicitly tied to the systematic risk β_{jt} . The difference across assets only comes from the difference in the systematic risk as measured relative to the market portfolio. Since the total risk premium is $\lambda_{jt}\sigma_{jt} = c_t\beta_{jt}\sigma_{mt}$ it is seen that an asset with a larger systematic risk will have a higher total risk premium, a result that is consistent with the standard asset pricing theory.

We use a nonlinear asymmetric GARCH(1,1) process of Engle and Ng (1993) also known as the NGARCH. Consider a discrete time economy and let X_t be the KES/EURO exchange rate at time t and σ^2 be the conditional variance given information at date t of the log return per day. given that the log returns $P_t = r_t + a_t$ where $a_t = \sigma_t\varepsilon_t$ and r_t is the constant one period risk free rate of return.

It's one period rate of return $P_t = \ln(X_t/X_{t-1})$ is assumed to be conditionally log normally distributed under probability measure \mathbb{P} , that is

$$P_t = \ln \left(\frac{X_t}{X_{t-1}} \right) = r_t + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + \sigma_t\varepsilon_t. \quad (7)$$

For a currency option the drift term $r_t = r^d - r^f$ where r^d and r^f denote, respectively, the constant one period risk less domestic and foreign interest rates. Therefore the log return's evolution dynamics are given by:-

$$\begin{aligned} P_t &= \ln \left(\frac{X_t}{X_{t-1}} \right) = r^d - r^f + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + \sigma_t\varepsilon_t \\ &= \ln \left(\frac{X_t}{X_{t-1}} \right) = r^d - r^f + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + a_t \quad \text{and} \end{aligned} \quad (8)$$

$$\varepsilon_t \stackrel{\mathbb{P}}{\sim} N(0, 1)$$

The risk less rate of return is given by $(r^d - r^f)$ and λ is the unit risk premium of an asset, in this case, the reward for investing in the foreign currency. The value of the premium influences the conditional variance process. we also assume that a_t follows a NGARCH(1,1). Formally

$$a_t \stackrel{\mathbb{P}}{\sim} N(0, \sigma_t^2) \quad \text{and} \quad (9)$$

$$\sigma_t^2 = \omega + \alpha\sigma_{t-1}^2(\varepsilon_{t-1} - \theta)^2 + \beta\sigma_{t-1}^2$$

The parameter θ is used to capture the relationship between the level of conditional variance and the return on the risky asset; A positive value (respectively negative) of θ implies a negative (respectively positive) relation between the return and the conditional variance. The parameters α and β are also

called persistence parameters. The GARCH process defined in (10) reduces to a standard homoscedastic log normal process in the Black-Scholes model if the two persistence parameters α and β are equated to zero. This ensures that the Black-Scholes model is a special case. In order to ensure the non negativity and stationarity of the variance process σ_t^2 ,

$$\omega > 0, \alpha \geq 0, \beta \geq 0 \quad \text{and} \quad \alpha(1 + \theta^2) + \beta < 1.$$

One of the advantages of the GARCH model for risk management is the possibility of one-day-ahead forecasting of various quantities relevant for investors. σ_t^2 i.e today's variance is known at the end of time $t - 1$ i.e. yesterday. Based on the information available at time $t - 1$, the conditional mean and variance are given by:-

$$\begin{aligned} E^{\mathbb{P}} [P_t] &= r^d - r^f + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 \\ \text{var}^{\mathbb{P}} [P_t] &= \sigma_t^2 \end{aligned} \tag{10}$$

To ensure stationarity of the variance process (σ_t^2), the unconditional variance is given by

$$\sigma^2 \equiv E [\sigma_t^2] = \frac{\omega}{1 - \alpha(1 + \theta^2) - \beta} \tag{11}$$

The expression $(1 - \alpha(1 + \theta^2) - \beta)$ is the persistence of the model. If the value of the persistence is close to 1 then the shocks in the market persist for a long time. In that case the time series is said to have a long memory. on the other hand small values of the persistence, shocks die out quickly. From equation (11) we see that the forecast of the variance is directly given by the model with σ_t^2 . Observing the forecasts for daily returns' variance for k periods ahead and using the recursive specification of the nonlinear assymetric GARCH given in (10) we see that

$$E [\sigma_{t+k}^2] = \sigma^2 + [\beta + \alpha(1 + \theta^2)]^{k-1} (\sigma_t^2 - \sigma^2) \tag{12}$$

Where σ^2 defined with $E [\sigma_{t+k}^2]$, represents the expected value of the future variance for horizon k

In Duan (1995), pricing derivatives in a GARCH-model setting is made possible by switching from a GARCH-family model under the \mathbb{P} measure to what he calls a 'locally risk-neutral measure' \mathbb{P}^* . A pricing measure \mathbb{P}^* is said to satisfy the *locally risk-neutral valuation relationship* (LRNVR) if the domestic measure \mathbb{P}^* is mutually absolutely continuous with respect to measure \mathbb{P} , $\frac{X_t}{X_{t-1}}|\mathcal{F}^{t-1}$ distributes log normally (under \mathbb{P}^*),

$$E^{\mathbb{P}^*} \left(\frac{X_t}{X_{t-1}} | \mathcal{F}_{t-1} \right) = e^r$$

and

$$\text{var}^{\mathbb{P}^*} \left(\ln \frac{X_t}{X_{t-1}} | \mathcal{F}_{t-1} \right) = \text{var}^{\mathbb{P}} \left(\ln \frac{X_t}{X_{t-1}} | \mathcal{F}_{t-1} \right)$$

almost surely with respect to measure \mathbb{P} . In the definition above the conditional variances under both measures are required to be equal. This is desirable because one can observe and hence estimate the conditional variance under \mathbb{P} . This property and the fact that the conditional mean can be replaced by a risk free rate yield a well specified model that does not locally depend on preferences. In the homoscedastic case, the conditional variances become the same constant and the LRNVR reduces to the conventional risk-neutral valuation relationship (see Duan 1995).

Under this measure, the dynamics of the return of a risky asset becomes

$$\ln\left(\frac{X_t}{X_{t-1}}\right) = r^d - r^f - \frac{1}{2}\sigma_t^2 + \sigma_t\xi_t$$

$$\sigma_t^2 = \omega + \alpha\sigma_{t-1}^2(\xi_t - \lambda - \theta)^2 + \beta\sigma_{t-1}^2 \quad (13)$$

$$\xi_t \stackrel{\mathbb{P}^*}{\sim} N(0, 1).$$

Where $\xi_t = \varepsilon_t + \lambda$. It is seen that the risk premium λ plays a critical role in the GARCH option pricing model. With regard to equation (3.33) we can price a European type currency option using the formula

$$C_t^{FX} = E^{\mathbb{P}^*} \left[\exp\left(-\sum_{s=t}^T r_s^d\right) (X_T - K)^+ | \mathcal{F}_{t-1} \right]$$

$$= E^{\mathbb{P}^*} \left[\exp(-r^d(T-t)) \max(X_T - K, 0) | \mathcal{F}_{t-1} \right] \quad (14)$$

Since there is no closed form solution for (15) we resort to simulations to recursively apply the dynamics in (14) in order to generate values for X_T . the option prices are expressed in terms of days to maturity and using equation (15) the option price obtained by simulations is given by:

$$C_t^{FX} = \exp(-r^d\tau) \frac{1}{N} \sum_{i=1}^N \max\{X_{T,i} - k, 0\} \quad (15)$$

where τ represents days to maturity and $N = 50,000$. For the i th simulation we therefore have

$$X_{T,i} = X_0 \exp\left(\sum_{j=1}^{\tau} P_{i,j}\right) \quad i = 1, 2, \dots, 50,000 \quad (16)$$

4 Application on Pricing Currency Options in the Kenyan Market

For the Kenyan market, trading in derivative securities is yet to be introduced. There is however a lot of trade involving foreign currencies. The data¹ used in this study is the daily KES/EURO exchange rates for the period of 1999 to 2006. The data has been analyzed using S+ and R.

4.1 Exploratory Data Analysis:

¹We would like to thank the Central Bank of Kenya and specifically Mr. Isaiah Mana for assisting in obtaining this data.

Let P_{t+1} be the log return of an asset at time $t + 1$. The basic idea behind volatility study is that the series P_t is either serially uncorrelated or with minor lower order serial correlations, but it is a dependent series.

Figure (1) shows a graph of the actual exchange rate series. From the actual series we can see the underlying stochastic process. We also have the normal QQ plot for these values. We can see from that that we have a heavy tailed distribution.

Figure 1: The KES/EURO exchange rate dynamics and daily log returns for the period 1999-2006

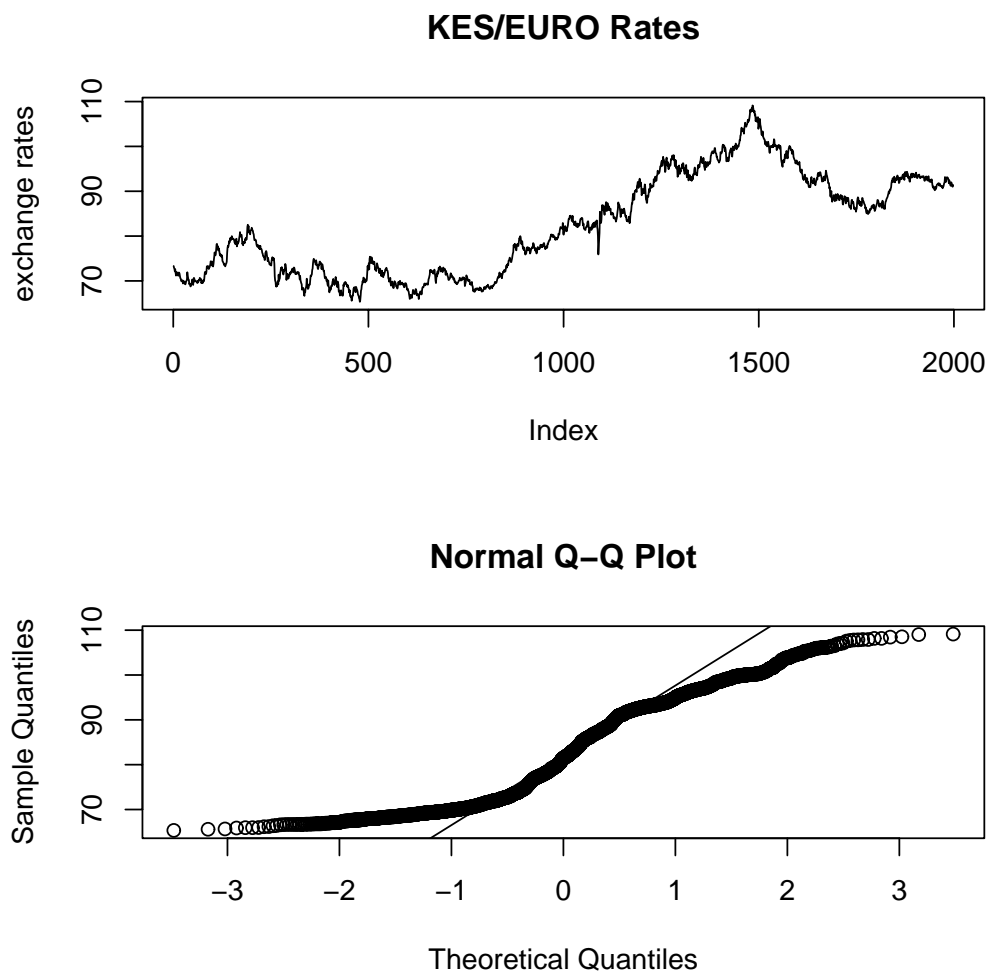
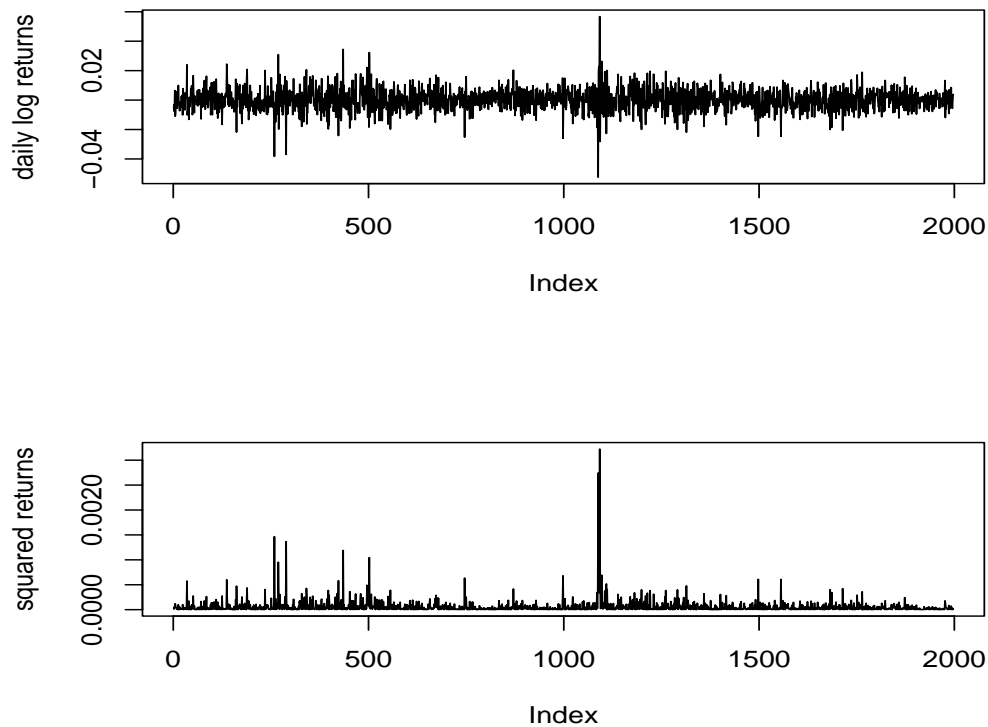


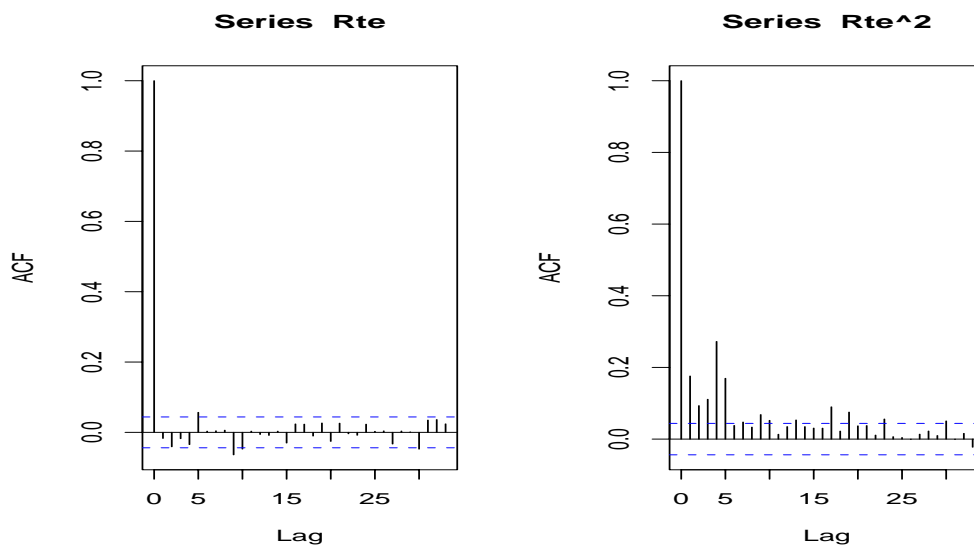
Figure (2) is a plot of the log returns and this plot of the log-return series displays the typical “stylized facts” present in financial log-return series. We also have a plot of the squared innovations series. From the graph it can be noted that some periods have different volatilities. For purposes of estimating the variability of returns, the variable representing the squared innovation returns is very important.

Figure 2: log return series and squared innovations for the period 1999-2006



From figure (3) we see the log-return series is uncorrelated with the exception of lag 1 (typically, log-return series are uncorrelated with the exception of the first few lags). However as can be seen in the second graph to the right, the squared series is strongly auto-correlated even for large lags but as the lags increase, the values decay to zero.

Figure 3: acf of the log-returns and the squared log-returns



4.2 Parameter Estimation

The non-observable variables have to be estimated alongside the other model parameters. Therefore we need to estimate σ_t^2 together with $\omega, \alpha, \beta, \lambda$ and θ . We therefore fit an NGARCH² model to the exchange rate returns and obtain the non-observable variables. The table below shows the values of the estimated parameters³ after fitting the model. From these results we see that λ is insignificant and as such we

Table 1: Estimated Coefficients

Parameter	Value	Std.Error	t value	Pr(> t)
$\hat{\lambda}$	3.412×10^{-2}	1.429×10^{-1}	0.2388	4.451×10^{-1}
$\hat{\omega}$	2.295×10^{-6}	4.880×10^{-7}	4.7030	1.370×10^{-6}
$\hat{\alpha}$	6.332×10^{-2}	8.318×10^{-3}	7.6122	2.065×10^{-14}
$\hat{\beta}$	8.958×10^{-1}	1.377×10^{-2}	65.0642	0
$\hat{\theta}$	-1.339×10^{-1}	6.073×10^{-2}	-2.2049	1.379×10^{-2}
$\hat{\alpha}(1 + \hat{\theta}^2) + \hat{\beta}$	0.96025			

exclude it from any further analysis.

4.3 Option pricing

The Garch option prices are found using relations 14, 15, 16 and 17 and the Black-Scholes prices are obtained using the closed form solution in equation 4

Table (2) shows a comparison of the Black-Scholes option prices and the GARCH option prices We see that the GARCH⁴ option prices are slightly higher than the Classical Black-Scholes prices especially for options that are at the money. The same can be done for options with different maturities. The red graph shows the garch option price and the black one is the Classical Black-Scholes price.

5 Conclusions and Recommendations

In this section we give a conclusion of our findings and recommendations for further research.

5.1 Conclusions

There is a lot of foreign trade and other foreign currency activities in the Kenyan market and therefore the Kenyan investor or any investor in this market will be interested in any initiative that can hedge against exchange rate risk. We can see that the Black-Scholes model under prices options that are at the money even for the Kenyan market. This is in line with what is in literature from research previously carried out. Therefore pricing under the GARCH framework can be used to arrive at more realistic prices.

²The author would like to thank Dr. T. Achia for helping with the S+ Finmetrics.

³the parameters were obtained using S+FinMetrics

⁴The author would like to thank Mr. Mwaniki for all his help in writing the R scripts for the pricing formulas.

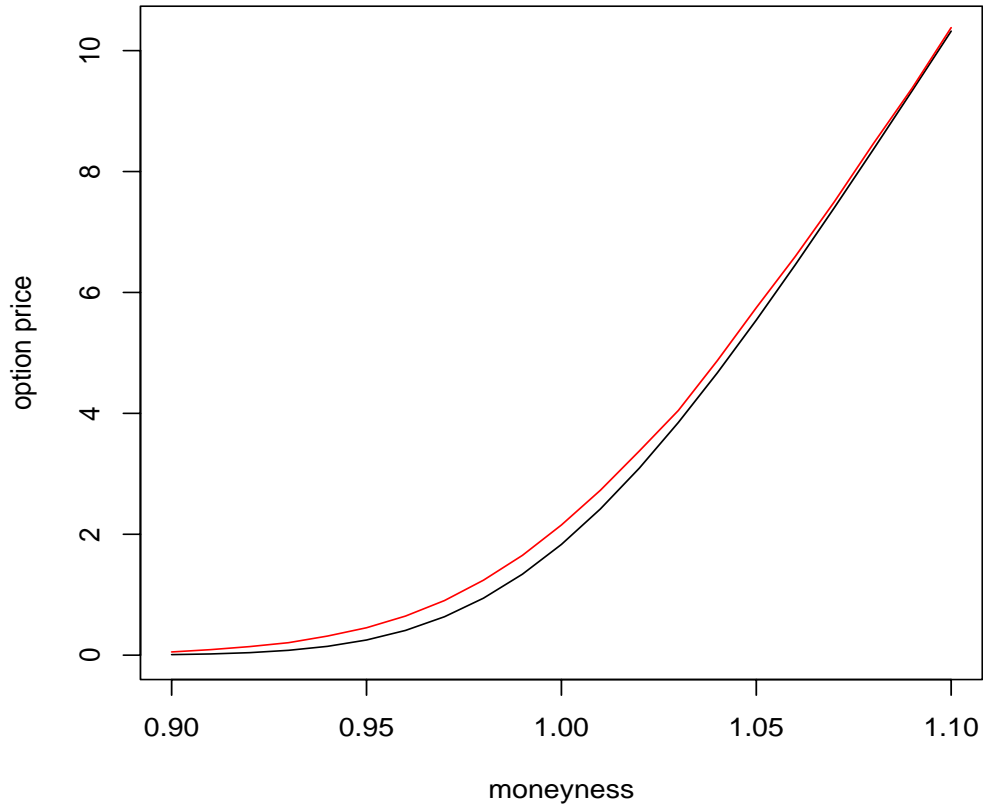
Table 2: Prices from the BSM Model and the Garch Model for 30 days to maturity, $r^d = 0.07$, $r^f = 0.04$ and $k = 100$

X_0	BSM price	Garch Price
90	0.009189504	0.05331825
91	0.020205026	0.09131948
92	0.041563204	0.1417519
93	0.080263073	0.206865
94	0.146002832	0.3163376
95	0.251037385	0.4536249
96	0.409406439	0.6468754
97	0.635526694	0.9032696
98	0.942355809	1.240148
99	1.339518585	1.651104
100	1.831851619	2.153091
101	2.418731438	2.727791
102	3.094332914	3.377726
103	3.848705571	4.047301
104	4.669353652	4.871412
105	5.542925885	5.747278
106	6.456669565	6.594491
107	7.399437914	7.497247
108	8.362195972	8.457324
109	9.338094716	9.380539
110	10.322249380	10.38023

A pricing model that is homoscedastic or holds the standard deviation constant may not reflect the true dynamics of the market since financial time series data has a heavy tailed distribution. Unlike the preference free Black- scholes model, pricing in the GARCH model framework incorporates the risk premium in its formulation. Though for this market, the risk premium parameter obtained from the fitted model, turned out to be insignificant, it can not be concluded that it is altogether not important. Once these derivative instruments are introduced in the market, a more in depth analysis can be carried out and this can be verified.

We are therefore suggesting a nonlinear model for option pricing for this market in line with the findings in this study as this would be more indicative of the realities in the market as opposed to the homoscedas-

Figure 4: Plot of option prices from both models



tic case.

5.2 Recommendations

For further research we recommend use of other stochastic volatility models or other statistical models to investigate the dynamics of exchange rate returns.

Further research could also look at modeling exchange rates when they follow a jump-diffusion process to look into cases where the market experiences jumps at various stages.

More work can also be done to look at the pricing of American type options, cross- currency options and other exotic options under the heteroscedastic framework.

Further research could also look at possibilities of incorporating term structure models and having the interest rates changing over time.

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