PRICING EQUITY-LINKED LIFE INSURANCE POLICIES USING STOCHASTIC INTEREST RATE MODELS

BY

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DECLARATION

I declare that this is my original work and it has never been submitted to any other university for examination. The research thesis is as a result of my own work and acknowledgements have been made where I have used other peoples' ideas.

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ABSTRACT

We analyzed pricing of index-linked life insurance policies using stochastic interest rate. We analyzed insurer's risk of index linked life insurance policies using the equivalence principle and applied an insurance pricing model that builds on the framework of European put options. We looked at how changes in interest rates and changes in the prices of the reference portfolio affect the premium charged in index-linked policies. The geometric Brownian motion model which follows a log-normal process was used to forecast future prices of the reference portfolio while a Vasicek process was used o forecast the future interest rates. The analysis confirmed that the present value of premium charged was very sensitive to the processes of the reference portfolio value and interest rates.

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DEDICATION

This project is dedicated to my son Victor Kiranga.

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Chapter 1

Introduction

1.1 Background of the Study

Life Insurance Companies enter into contracts to provide a contingent payment on either the death of an individual (known as death benefit) or upon survival of an individual (known as maturity benefit), in return for a series of periodical payments called premium. These life insurance contracts are usually either designed to protect against adverse financial impacts that result from an individual's death or to provide income at a future date.

In traditional life insurance mathematics, financial markets were assumed to be deterministic. Under this assumption, the philosophy of the classical Principle of Equivalence stipulated that premiums were chosen such that incomes and losses were "balanced in the mean" i.e. the mean balance per contract converges to zero. This idea led to a pricing method called "Expectation Principle" which relied on two important ingredients namely the stochastic independence of individual lives and the Strong Law of Large Numbers.

Modern life insurance has to cope with two different kinds of risk. On the one side, there is biometric risk mostly in mortality risk which is the classical subject of life insurance mathematics. On the other side, there is financial risk which comes to life insurance by financial markets, for example by stochastic interest rates or products like unit-reference life insurance policies.

Life insurance mathematics has developed fast during the last twenty years and for many particular problems, for instance pricing, hedging and bonus theory, solutions have been developed. Nonetheless, the problem of the decomposition of gains (or risks) into biometric and financial parts has not yet been sufficiently considered, especially not with respect to the needs of modern life insurance, i.e. in the presence of stochastic financial markets.

1.2 Statement of the Problem

Life insurance companies price their products based on the deterministic (projected) values of the market interest rates for their invested assets. They base the asset performance on the performance of zero coupon bond (risk free interest rate).

In the market, the interest rate keeps on fluctuating depending on various factors (these factors are not within the scope of this paper). When the interest rate rises above the projected rates, the life office benefits and is rewarded for carrying the investment risk but when the interest rates drops below the projected rates, the life office looses and can even fail to meet it's obligation of paying the policy holders should there be a claim (either maturity benefits or death benefits).

1.3 Proposed Solutions

Use of deterministic interest rates will lead to erroneous pricing of life insurance products. This will result to either under-pricing or over-pricing of the said products. In valuation of these products, volatility in interest rates should be factored in through application of relevant stochastic, statistical or actuarial models.

1.4 General Objectives of the Study

The primary aim of this paper is to look at probabilistic pricing method, and the application of stochastic interest rates in determining the price of index-linked life insurance policies. It seeks to show the effect of stochastic interest rates in pricing life insurance policies as opposed to deterministic ones.

1.5 Specific Objectives of the Study

Stochastic interest rates are applied to price some given life insurance products. This is done and the results compared with the valuation of the same products using both deterministic interest rates and deterministic mortality rates. The study seeks to show the disparity in valuation of life insurance products under deterministic interest in comparison to stochastic rates.

1.6 Significance of the Study

The results of this study will be important to:

- Life Insurance Companies in assisting them in fair pricing of their products. This will minimize the incidences of under-pricing. This is particularly so in cases where the interest rates of their investment is below the projected levels.
- Life insurance policy holders to cushion them from the effects of over pricing. Overpricing will tend to occur when the interest rates of the invested assets are above the projected levels.

- The regulator of the insurance industry to enable realistic analysis of the life insurance products in the market. This will enable the regulator to cushion the life companies against insolvency that can occur as a result of under-pricing that can largely be driven by the market competition.

The rest of the project is organized as follows. Chapter 2 provides a brief literature review of stochastic pricing of life insurance policies. The main contributions and results of the key papers on various stochastic life insurance valuation models in the actuarial literature are presented. Chapter 3 describes the methodology. This includes the model and the assumptions regarding this model. In Chapter 4, numerical illustrations are shown for a stochastic interest rate model of pricing life insurance. Chapter 5 concludes the project.

Chapter 2

Literature Review

In this chapter, we review some results from papers which made important contributions on life insurance models. There is an extensive literature on this topic, and hence we focus on the models and their actuarial application. The theory of life contingencies evolves from single life policy valuations.

Various pricing or premium principles have been proposed for pricing life insurance contracts.

2.1 Some Definitions

There exist many different types of life insurance contracts in the market today. Four basic types of life insurance contracts are:

- **Term Life Assurance**. This contract has a fixed expiration date and pays a certain amount (known in advance) at the time the insured dies if this occurs before expiration of the contract and nothing otherwise.
- **Pure Endowment Assurance**. This contract also has a fixed expiration date and pays a predetermined amount only if the insured is alive at the expiration date of the contract.
- **Endowment Assurance**. This contract is a combination of both term life assurance and pure endowment assurance, i.e. it pays out either at the time of death of the insured or at the expiration of the contract, whatever occurs first.
- **Equity-reference and participating policies**

An equity-linked or unit-linked contract gives the customer the value of a certain reference portfolio at the payout date. In a lot of countries the contracts are offered with a guarantee. That is, the customer is guaranteed a certain minimum payout. In a participating policy the payout to the customer is tied to the return on the issuing company's own investment portfolio. In general, the contracts placed in the class of participating policies provide the customer with a guaranteed minimum payout and possibly some of the surplus that might be generated on the contract. This surplus is known as bonus once it is distributed to the customer. The guarantee element in the insurance contracts can be thought of as arising from the traditional actuarial practice for calculating premium where the value of contract benefits and hence premiums are based on assumptions of the future level of mortality rates, interest rates, and costs of handling the contract. These assumptions are set so that the company is on "the safe side" with respect to being able to honor the contract i.e. so that the reserve is large enough. Valuing the contract using more realistic assumptions, i.e. values of mortality, etc. usually yields a surplus since the assumptions used to begin with were "on the safe side". The so-called contribution principle states that this surplus must be given back to the customers (and equity holders of the company) according to the way they have contributed to it. In case that surplus is negative, the insurance company has to cover the deficit and the customer receives no bonus. It is customary that the terms of an insurance contract cannot be altered during the life of the contract and therefore the contract actually provides the customer with guaranteed benefits based on the initial assumptions (the first order basis). In some cases, the guaranteed benefits are given as an average guaranteed rate of return on the customer's stake, in others the guaranteed benefits are given through a guarantee on each year's return. The primary difference between a customer and the life insurance company is that the customer himself cannot hedge mortality risk, whereas the insurance company is assumed to be able to. The company works under the assumption that it can apply the Law of Large Numbers and diversify mortality risk away by pooling together many customers with similar characteristics such as age and gender.

2.2 Review of Related Literature

Bowers et.al. (1997) developed a pricing model based on the Equivalence principle of equating the expected Present value of benefits to the expected present value of premiums.

Bacinello (2001) analyzes the most sold life insurance contract in Italy. She takes into account mortality and suggests a contract which offers the choice among different triplets of technical rate, participation level and volatility. Paying each year a premium, the insured customer gets the guarantee to recover his initial investment accrued at a fixed rate and can possibly benefit from a bonus indexed on a reference portfolio. The pricing is achieved under the standard Black and Scholes model and assuming independence between mortality risk and financial risk.

Tanskanen and Lukkarinen (2003) consider general participating life insurance contracts. Their contract values depend on the evolution of a reference portfolio at different dates. These authors incorporate the following features: minimum interest rate guaranteed each year, right to change each year the reference portfolio, as well as possibility to surrender each year the contract. They work with constant interest rates and a constant volatility. Because there are various kinds of contracts and modeling frameworks, the pricing methodologies are diverse.

Jørgensen (2001) and also Grosen and Jørgensen (2002) showed that a life insurance contract with a minimum interest rate guarantee can be expressed in four terms, the final guarantee (equivalent to a zero-coupon bond), the European bonus option associated with a percentage of the positive performance of the company's asset portfolio, if any, a put option reference to the default risk, and finally a fourth term which is a rebate given to the policyholders in case of default prior to the maturity date.

Paul Embrechts (1997) compared Actuarial and financial pricing of Insurance products. He showed that in a sufficiently liquid insurance market, classical insurance-premium principles can be reinterpreted in a standard no-arbitrage pricing set-up. The variety of premium principles used is explained through the inherent incompleteness of the underlying risk process in so far that a whole family of equivalent
martingale measures exist.

Aase (1999) also investigated the valuation of financial contracts that are based on insurance related risk such as catastrophe insurance derivatives. He studied financial contracts based on insurance related risk.

Mette Hansen (2002) in his dissertation looked at different ways of pricing various types of insurance contracts. He looked at how valuation techniques derived from financial economics are applied to determine premiums that take into consideration the financial risk inherent in the contracts. He went further and looked at how a company should invest given that it has issued contracts with guarantees, and how competition among life insurance companies affect the decisions of the companies.

Bacinello and Ortu (1993) investigated the pricing of equity-reference life insurance contracts. The benefits of these insurance policies depend on the performance of a reference portfolio that is traded on the capital market. These contracts belong to a market containing insurance contracts that are based on both financial and insurance related risk in form of policyholders' mortality risk.

Brennan et al (1976) also investigated the pricing of equity-reference life insurance contracts. They showed that the benefits of these insurance policies depend on the performance of a reference portfolio that is traded on the capital market. These contracts belong to a market containing insurance contracts that are based on both financial and insurance related risk inform of policy holders' mortality risk.

Hong Mao et al (2004) examined the pricing of term life insurance based on the economic approach of profit maximization, and incorporating the financial approach of stochastic interest rates, investment returns, and the insolvency option, while also including actuarial modelling of mortality risk. They obtained optimal price (premium) by optimizing a stochastic objective function based on maximizing the expected net present value (NPV) of insurer profit. They analyzed numerically the influence of various parameters on optimal price, optimal expected NPV of insurer profit, and the insolvency put option. They demonstrated that optimal prices generally are most sensitive to changes in the long run equilibrium interest rate.

Lienda Noviyanti et al (2006) in their paper combined Financial and Actuarial approaches to price life insurance contract. They compared the premium of the term and the endowment insurance calculated based on constant discount function to stochastic discount function, when the discounting is presented by a stochastic differential equation as in the Hull-White model. In this case, time to maturity in financial valuation models was adjusted with a continuous random variable T(x), representing future lifetime of a life-aged-x. They quantified the premium based on the Hull-White financial valuation model and utilized sensitivity analysis to examine the differences between the constant and the stochastic interest rate. They showed that stochastic interest rate gives a fair premium value when it compared with a constant interest rate.

Chang et al[1] studied the pricing of securitization of life insurance under mortality dependence and stochastic interest rate by modelling stochastic mortality intensity and then considering other important risk factors (income, gender, age, and others) affecting mortality rate to derive the mortality probability for each policyholder. They estimated death time and applied it to design and price Collateralized Insurance Obligation under mortality dependence. Using Monte Carlo simulation they showed that the independence assumption in mortality and interest rate tends to either overestimate or underestimate the premiums.

Delbaen and Haezendonck (1989) and also Sondermann (1991) investigated a market that consists of insurance contracts based on insurance related risk. They showed how premium calculation principles for reinsurance contracts can be embedded in a no-arbitrage framework.

The important contribution of these papers lies in the construction of an analytical bridge between actuarial and financial valuation.

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[1] The publication date is not indicated.

2.3 Premium Determination Principles

In this section, we list many well-known premium principles. Let H be the minimum premium that the insurer is willing to accept in exchange for insuring the risk X.

A. ***Net Premium Principle*****:** $H[X] = EX$.

This premium principle does not load for risk. It is widely applied in the literature because actuaries often assume that risk is essentially nonexistent if the insurer sells enough identically distributed and independent policies

B. ***Expected Value Premium Principle*****:** $H[X] = (1 + \theta) EX$, **for some** $\theta > 0$.

This premium principle builds on Principle A, the Net Premium Principle, by including a proportional risk load. It is commonly used in insurance economics and in **risk theory**.

C. ***Variance Premium Principle*****:** $H[X] = EX + \alpha \mathrm{Var} X$, **for some** $\alpha > 0$.

This premium principle also builds on the Net Premium Principle by including a risk load that is proportional to the variance of the risk. B¨uhlmann [X] studied this premium principle in detail. It approximates the premium that one obtains from the principle of equivalent utility (or zero-utility).

D. ***Standard Deviation Premium Principle*****:** $H[X] = EX + \beta\sqrt{\mathrm{Var} X}$, **for some** $\beta > 0$.

This premium principle also builds on the Net Premium Principle by including a risk load that is proportional to the standard deviation of the risk. B¨uhlmann [X] also considered this premium principle and mentioned that it is used frequently in property insurance.

E. *Principle of Equivalent Utility*: $H[X]$

Solves the equation

$$u(w) = E[u(w - X + H)]$$

where u is an increasing, concave utility of wealth (of the insurer), and w is the initial wealth (of the insurer). On one side, we have the utility of the insurer who does not accept the insurance risk. On the other side, we have the expected utility of the insurer who accepts the insurance risk for a premium of H. $H[X]$ is such that the insurer is *indifferent* between not accepting and accepting the insurance risk. Thus, this premium is called the *indifference price* of the insurer. Economists also refer to this price as the *reservation price* of the insurer.

If u and w represent the utility function and wealth of a buyer of insurance, then the maximum premium that the buyer is willing to pay for coverage is the solution G of the equation below:

$$E[u(w - X)] = u(w - G).$$

The resulting premium $G[X]$ is the indifference price for the buyer of insurance, i.e indifferent between not buying and buying insurance at the premium $G[X]$.

***F. Exponential Premium Principle*: $H[X] = (1/a) \times \ln E[eaX]$, for some $a > 0$.**

This premium principle arises from the principle of equivalent utility when the utility function is exponential. Musiela and Zariphopoulou [B] adapted the Exponential Premium Principle to the problem of pricing financial securities in an incomplete market. Young and Zariphopoulou [C], Young [D], and Moore and Young [E] used this premium principle to price various insurance products in a dynamic market.

G. *Esscher Premium Principle*: $H[X] = (E[Xe^Z]/E[e^Z])$, for some random variable *Z*.

B¨uhlmann [F] derived this premium principle when he studied risk exchanges. In that case, Z is a positive multiple of the aggregate risk of the exchange market. Some authors define the Esscher Premium Principle with $Z = hX$, $h > 0$. For further background on the Esscher Premium Principle, which is based on the Esscher transform.

The Esscher Premium Principle (as given here in full generality) is also referred to as the Exponential Tilting Premium Principle.

H. *Wang's Premium Principle*: $H[X] = \int_0^\infty g\,[S_X(t)]\,dt$,

One can combine the Principle of Equivalent Utility and Wang's Premium Principle to obtain premiums based on *anticipated utility*. Young [J] showed that Wang's Premium Principle reduces to a standard deviation for location-scale families, and Wang [I] generalized her result.

Chapter 3

Methodology

3.1 Research Methodologies

Life insurance pricing model requires the actuarial equivalence principle that the present value of the expected claims should be equal to the present value of the expected insurance premiums.

The main factors affecting pricing of equity-linked life insurance policy are interest rates and mortality rates. In this analysis, we will only focus on the fluctuations of the interest rates.

3.2 Assumptions Underlying our Pricing Models

In this section, we discuss the assumptions underlying the development of the pricing models used in this study.

- In the stochastic control model the contract is a contingent-claim affected by financial risk only. All the other factors are ignored.
- Stochastic interest rates are used as discount rates that are treated as a continuous time-stochastic process.
- The insurer raises funds from policyholders, invests the funds, and pays benefits including investment income when claims occur.
- The insurer is risk neutral, where price is set according to the expectation criterion; that is, the objective of the insurance company is to maximize the expected net present value of insurer profit.
- The pricing models do not impose binding constraints from rate regulation.

- Financial markets are assumed to be perfectly competitive, frictionless and free of arbitrage opportunities.
- All consumers purchase the same unit of insurance coverage.
- All policyholders are assumed to be rational, non-satiated and share the same information.
- The Cox, Ingersoll, and Ross (1985) model for determination of the values of reference portfolio apply.
- The short-term interest rate, r, follows an Ornstein-Uhlenbeck mean reverting stochastic process.

3.3. The Equivalence Principle Model

Life insurance pricing model requires the actuarial equivalence principle that the present value of the expected claims should be equal to the present value of the expected insurance premiums.

On the other hand, in equity-linked life policies, the benefit payable consists of the greater of the value of some reference portfolio and some minimum guarantee. The reference portfolio is a portfolio formed by investing some pre-determined component of the policyholder's premium in common stocks.

Using the equivalence principle premium structure, we evaluate the present values of expected claim losses and that of insurance premiums.

Then we determine the maximum levels of constant monthly payments under the condition satisfying the present values of expected claim losses to be equal to that of insurance premiums as follows:

$$PVIP = \sum_{t=1}^{T(a)} \left[\frac{mip_t \cdot p_{a,t}}{(1+i)^t} \right]$$

$$= \sum_{t=1}^{T(a)} \left\{ \frac{\max\left[(C_t - S_t)\cdot q_{a+t}, 0\right]\cdot p_{a,t}}{(1+i)^t} \right\} = PVEL \qquad (3.1)$$

Where $PVIP$ ≡ Present value of total insurance premiums at time t=0

$PVEL$ ≡ Present value of total claim losses at time t=0

$T(a)$ ≡ The number of months that a life assured with age a will continue to pay premiums until policy maturity.

mip_t ≡ monthly insurance premiums at time t

$$mip_t = (A_t)\cdot m$$

where m =monthly premium rate

C_t ≡ Sum Assured at time t

$p_{a,t}$ ≡ The probability that an assured life of age a will survive to age $a+t$

i ≡ Expected interest rate

S_t ≡ Value of the reference portfolio at time t;

$S_t = S_0 \cdot (1+g)^t$; g = mean value of reference portfolio value growth rate

q_{a+t} = The probability that an assured life of age a will die at age $a+t$

However, after going on cover, the interest rate imposed to portfolio's performance will change depending on the fluctuations of reference interest rate (for example, the yield of monetary stability bond with 1 year's maturity) and we can expect the future reference portfolio values also to move stochastically. In this situation, if the life assured pays constant monthly premium to life office continuously according to the amount calculated by using equation (3.1) and assumptions above, the life office could suffer serious market risks during long term maturity. Therefore, it is necessary to evaluate insurer's risks considering stochastic processes of both the reference portfolio value and interest rates periodically after commencement of cover.

3.4 Considering Stochastic Models

In an ideal market, interest rates keep on changing depending on a number of market-driven factors. Similarly the price and hence the value of the assets will keep on changing in response to the market factors.

3.4.1 Stochastic Movement in Reference Portfolio Price

An equity-linked life insurance policy typically calls either for a single investment to be made in the reference portfolio at the time the policy is purchased (the single premium contract) or for regular series of periodic investments spread over the life of the contract (the periodic premium).

We can forecast future prices of the reference portfolio using geometric Brownian motion model which follows log-normal process as equation (3.2).

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma_P dz(t) \qquad (3.2)$$

where $S(t)$ ≡ Value of the reference portfolio at time t; $S(0) = S_0$

μ ≡ Expected growth rate in the value of the reference portfolio

σ_P ≡ Volatility of the reference portfolio

$z(t)$ ≡ Stochastic process of standard Brownian motion

The market value at time t of a call option on the reference portfolio exercisable at time T, is assumed to depend only on the current value of the reference portfolio, (S(t), the remaining time to maturity, T-t, the exercise price, A, and the remaining investments to be made in the reference portfolio.

In practical analysis, for forecasting reference portfolio price paths, we will use discrete form as equation (3.3). In this case we can calculate the reference portfolio price at time t as follows

$$S_t = S_{t-1} \cdot \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma_P \varepsilon_t \sqrt{\Delta t}\right] \qquad (3.3)$$

Where ε_t ≡ Standardized normal random variable; $N(0,1)$

According to the insurance contract, the life office should pay claims given by equation (3.3) at time t.

$$Loss_t = \max\left[(C_t - S_t),\ 0\right] \tag{3.4}$$

where $Loss_t$ $\equiv$ Loss at time t
C_t $\equiv$ Sum Assured at time t

In equation (3.4), we can calculate the value of C_t as follows.

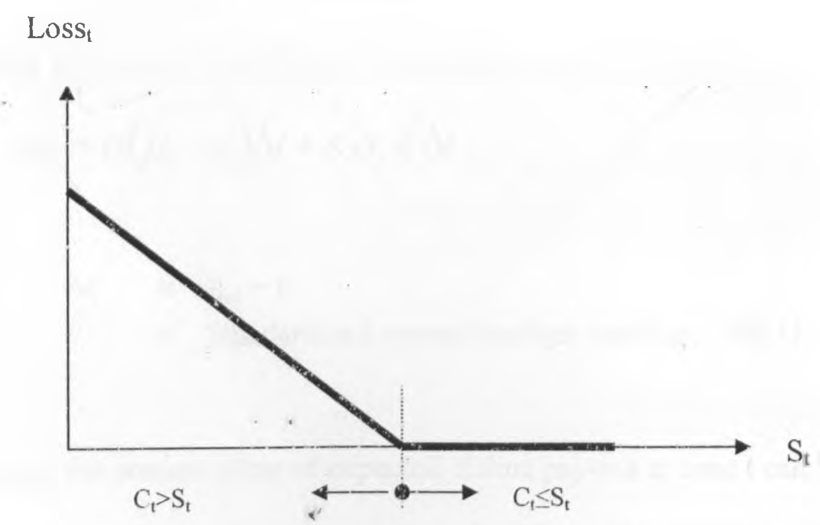
$$C_t = \left[(C_{t-1} + \Delta C)\right](1 + r_{t-1}) \tag{3.5}$$

where C_t $\equiv$ Sum Assured at time t

ΔC $\equiv$ Constant monthly increase

r_t $\equiv$ Asset interest rate at time t

Figure 1: Life insurance payoff diagram.



The present value of expected claims upon maturity at time t can be expressed as European put options with exercise price S_t, where the maturity date t is the expiry date of European put options.

3.4.2 Stochastic Movement in Interest Rates

If we assume that the reference portfolio interest rates follow Ornstein-Uhlenbeck process (or Vasicek process) under continuous time, the stochastic process can be expressed as equation (3.6).

$$dr_t = \alpha(\mu_r - r_t)dt + \sigma_r dz_t \qquad (3.6)$$

where r_t $\equiv$ Interest rate at time t
α $\equiv$ Speed of mean reversion
μ_r $\equiv$ Mean reversion level
σ_r $\equiv$ Volatility of interest rate process
z_t $\equiv$ Brownian motion term

Re-writing equation (3.6) in discrete form, we get equation (3.7)

$$\Delta r_t = \alpha(\mu_r - r_t)\Delta t + \varepsilon_t \sigma_r \sqrt{\Delta t} \qquad (3.7)$$

Where Δr_t $\equiv$ $r_{t+1} - r_t$
ε_t $\equiv$ Standardized normal random variable; $N(0,1)$

In this case, the present value of expected claims pay-out at time t can be expressed as a European put option with exercise price A_t. Where, the policy maturity date t is the expiry date of European put options.

$$PVLoss_t \equiv Put_t = \frac{E[Loss_t]}{(1+y)^t} = \frac{E\left[\max\left(C_t - S_t\right),0\right]}{(1+y)^t} \tag{3.8}$$

where $PVLoss_t$ $\equiv$ Present value of expected claims at time t

y $\equiv$ Yield rate of zero coupon bond with t months' maturity

Given that a claim occurs at time t with a probability given by $p_{a,t}\, q_{a+t}$, we could determine the present value of net insurance premium of sum assured which equal to the present value of cumulative expected claims as equation (3.9).

$$PVMIP = \sum_{t=1}^{T(a)} p_{a,t} q_{a+t} \cdot Put_t = \sum_{t=1}^{T(a)} p_{a,t} q_{a+t} \cdot PVLoss_t = PVEL \tag{3.9}$$

where $PVIP$ $\equiv$ Present value of total net insurance premium

$PVEL$ $\equiv$ Present value of total expected claims

3.4.3 The periodical premium guarantee

Let P_t for $t = 0,1,2,...,$ be the periodical premium paid at the beginning of each year that the insured is alive. Assume that the contract specifies a fixed amount of the premium, denoted by d_t, deemed to be invested in the reference portfolio.

Without guarantees, the number of units acquired at time t should therefore be equal to d_t/S_t, but different with a minimum guarantee provision, expressed by a minimum number of units guaranteed at time t. Let g_t represent this guarantee, and n_t denote the actual number of units deemed to be invested in the reference portfolio at time t. Thus,

$$n_t = \max\left[g_t, \frac{d_t}{S_t}\right], t = 0,1,2,...,T-1$$

The market value at time t of the periodical premium P_t must be equal to the value of n_t units at time t, i.e,

$$P_t = n_t S_t = d_t + g_t \max\left[S_t - k_t, 0\right] \quad (3.10)$$

with $k_t = {d_t}/{g_t}$

The time t payoff of the minimum guarantee provision, $P_t - d_t$, is then equal to the payoff of g_t call options on the reference portfolio with exercise price k_t and maturity t.

The amount of periodical premium depends on the time t value of the reference portfolio and, thus, is stochastic.

3.4.4 The benefit

If death occurs at time τ between t and $T-1$, with $t = 0,1,2,...,T-1$, the benefit $C(\tau)$ is simply the market value at time τ of the accumulated investments in the reference portfolio, i.e.,

$$C(\tau) = S_\tau \sum_{j=0}^{t} n_j \quad (3.11)$$

whereas the benefit at maturity T, due if the insured is alive, is

$$C(T) = S_T \sum_{j=0}^{T-1} n_j \quad (3.12)$$

3.4.5 Constant periodical premium

Denoting by P the constant periodical premium, we observe that the market value at time 0 of the stream of constant periodical premiums P, paid at the beginning of each year if the insured is alive, should equal the market value of the stream of time dependent periodical premiums P_t, i.e,

$$P\sum_{t=0}^{T-1} B_0(t)_t\, p_x = \sum_{t=0}^{T-1}\left[d_t B_0(t) + g_t \pi_t(k_t)\right] p_x \tag{3.13}$$

where π_t [2] is the market price at time zero of the underlying option and $\pi_0 k_0 = \max[S_0 - k_0, 0]$ and $B_t(s)$ is the market price at time t for a bond maturing at a fixed date $s \geq t$.

The right hand side represents the market value at time zero of the periodical premium payments given by expression (3.10) paid until death or the term of the contract, whatever comes first. The left hand side is simply the similar market value at time zero of the constant periodical premiums P. From this equation

P is determined as

$$P = \frac{\sum_{t=0}^{T-1}\left[d_t B_0(t) + g_t \pi_t(k_t)\right] p_x}{\sum_{t=0}^{T-1} B_0(t)_t\, p_x} \tag{3.14}$$

If, in particular, the amounts to be periodically invested in the reference portfolio d_t and the minimum guaranteed numbers of units g_t are constants, i.e., if $d_t = d$ and $g_t = g$ for all t, then

$$P = d + g\frac{\sum_{t=0}^{T-1} \pi_t(k)_t\, p_x}{\sum_{t=0}^{T-1} B_0(t)_t\, p_x} \tag{3.15}$$

with $k = d/g$.

The periodical premium for the minimum guarantee provision, $P - d$, is proportional to the ratio between the time 0 value of a portfolio of European call options on the reference portfolio, all with the same exercise price but different maturities, and the

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[2] π is the market price at time zero of the option with expiration at t.

$\pi_t(G) = S_0\phi(d_t^1(G)) - GB_0(t)\phi(d_t^2(G))$ (see Amin and Jarrow (1992) and Anna Rita Bacinello (1998) time 0 value of a portfolio of unit discount bonds with the same maturities of the options and held in the same proportions.

3.5 Comparison with the Fixed Interest Rates

The structure of the constant periodical premium contract presents some analogies, but also a fundamental difference, with respect to the fixed interest policy, in which a fixed part, *d*, of the periodical premium is deemed to be invested in a reference portfolio, but the minimum amount guaranteed, at death or maturity, is not stochastic. In our model, instead, this guarantee is expressed in units of the reference portfolio, and therefore its monetary value is unknown a priori. This fact, however, may constitute an appealing feature from the insured's point of view and, at the same time, allow him to hedge against alternative sources of economic risk such as inflation, currency devaluation, etc. The reference fund with unit price S_t could be composed of equities, as well as of units of a foreign currency, gold, silver, and so on.

In order to compare our benefit *C(t)* with the corresponding one in the fixed interest model, assume now that in case of death during the time interval $(t-1, t)$ this benefit is paid at the end of the year, i.e. at time t instead of at the time of death. The relation (3.11) is modified in the following way:

$$C(t) = \sum_{j=0}^{t-1} \max\left[gS_t, \frac{d.S_t}{S_j} \right], t = 1,2,\ldots,T. \qquad (3.16)$$

Observe that

$$C(t) \geq \max\left[gtSt, \sum_{j=0}^{t-1} \frac{d.S_t}{S_j} \right],$$

so that there is an implicit minimum benefit guaranteed at time t, given by the market value of $g.t$ units of the reference portfolio. We recall that in the Fixed interest policy the benefit, that we denote by $C^*(t)$, is instead given by

$$C^*(t) = \max\left[G_t, \sum_{j=0}^{t-1} \frac{d.S_t}{S_j} \right] \qquad (3.17)$$

where G_t represents the minimum amount guaranteed at time t, expressed in the usual unit of account.

It is also interesting to compare the periodical premium for the minimum guarantee provision in both models. As already said, in our model this market price is proportional to the time 0 value of a portfolio of European call options on one unit of the reference portfolio. We recall that for the Fixed interest policy the periodical premium, denoted by P^*, is instead given by

$$P^* = d + \frac{E^Q\left\{\sum_{t=1}^{T} \alpha_t v(t) \max\left[G_t - \sum_{j=0}^{t-1} d.\, {}^{S_t}\!/_{S_j}, 0\right]\right\}}{\sum_{t=0}^{T-1} B_0(t),\, {}_t p_x} \tag{3.18}$$

Where,

$$\alpha_t = \begin{cases} {}_{t-1}p_x(1 - p_{x+t-1}) & t = 1,2,\ldots,T-1 \\ {}_{T-1}p_x, & t = T \end{cases}$$

represents the probability that the policy expires at time t. The periodical premium for the guarantee, $P^* - d$, is then proportional to the value at time 0 of a portfolio of European put options on the accumulated investments in the reference portfolio, each one with maturity t and exercise price G_t, see Bacinello and Ortu (1994).

Based

Chapter 4

Data Analysis

4.1 Stochastic Process of the Reference Portfolio

We assumed the value of expected growth rates of reference portfolio price was 3.45% (μ=3.45%), and volatility was 3.35% (σ_p=3.35%) according to the average price index from 2005 to 2010. Several representative paths of Monte-Carlo simulation from t=0 to t=35 years are shown in Figure 2.

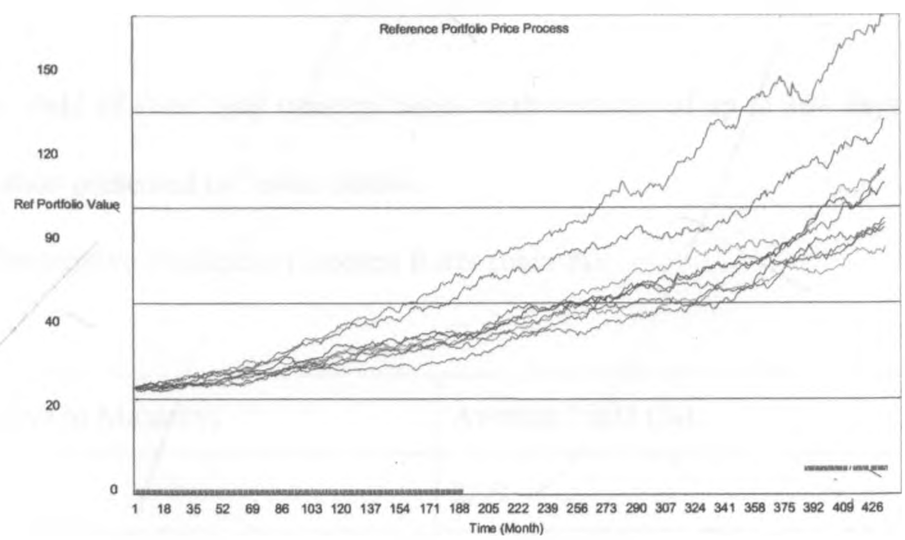


Figure 2. Stochastic process of the reference portfolio price

As shown in Figure 2, the model which uses constant value of the reference portfolio price growth rate will be constantly exposed to risk of fluctuation of price during long term maturity.

4.2 Term Structure of Interest Rates

We defined each yield of bonds as follows;

Y_{10}	=	Yield of Treasury bond with 10 years' maturity
Y_5	=	Yield of Treasury bond with 5 years' maturity
Y_3	=	Yield of Treasury bond with 3 years' maturity
Y_2	=	Yield of Treasury bond with 2 years' maturity
Y_1	=	Yield of Treasury bond with 1 year's maturity
y_{91}	=	Yield of Treasury bill with 91 days' maturity
y_{182}	=	Yield of Treasury bill with 182 days' maturity
y_{364}	=	Yield of Treasury bill with 364 days' maturity

Getting the yield of short term treasury bonds with maturity of up to 364 days, we get the information presented in Table1 below.

Table1. Descriptive Statistics of Interest Rates (unit: %)

N (No. of days to Maturity)	Average Yield (%)
91	2.99
182	3.27
364	5.1

On the other hand, getting the yield of treasury bonds with maturity of upto 30 years, we get Table 2 illustrated below.

Table 2. Descriptive Statistics of Interest Rates (unit: %)

N (yrs)	Yield			
	Min	Max	Mean	Std Dev
2	3.81	8.75	6.61	2.04
5	7.64	7.64	9.36	1.44
6	11.5	13.00	11.94	0.72
7	7.00	13.25	11.37	2.98
8	7.00	13.25	10.60	3.08
9	9.50	13.5	11.92	2.13
10	8.50	14.00	10.91	1.91
12	13.00	14.00	13.50	0.71
15	9.00	14.5	12.16	1.75
20	13.75	13.75	13.75	0.00
30	12.00	12.00	12.00	0.00

Plotting a graph of the mean yields against time will give us the trend in the interest rates as presented in Figure 3 below. As clearly observed from the figure,interest rates follow a random path and hence are said to be stochastic.

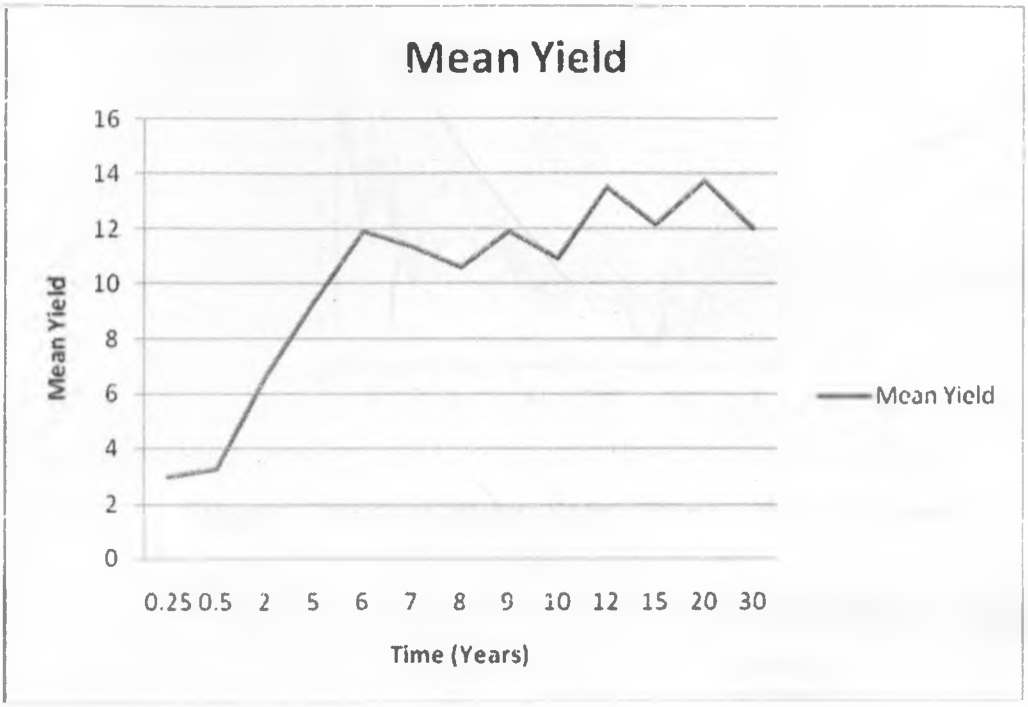


Figure 3. Trends of Interest Rates.

In this analysis, using the mean values of interest rates in both Table 1 and Table 2, we generated term structure of interest rates which adapted to build the stochastic portfolio models.

The mean value of spread between Y_{10} and y_{364} showed around 1%, and the spread showed that maximum value was 2.12% and minimum value was 0.17% while the mean value of spread between Y_{10} and Y_3 showed around 1.2%, and the spread showed that maximum value was 2.53% and minimum value was 0.18%. In Figure 5, the period of wide spread represents the peak of interest rate cycle and the period of narrow spread represents the trough of interest rate cycle.

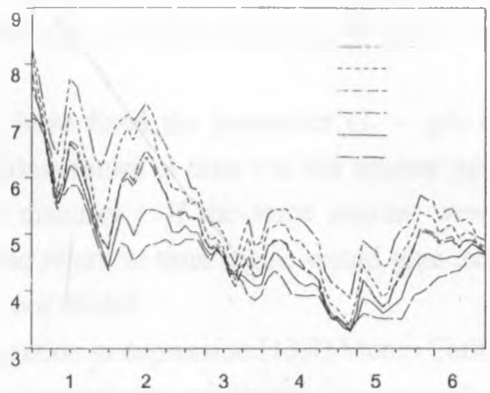


Figure 4. Trends of Interest Rates: Year 1 – Year 6 (6 years)

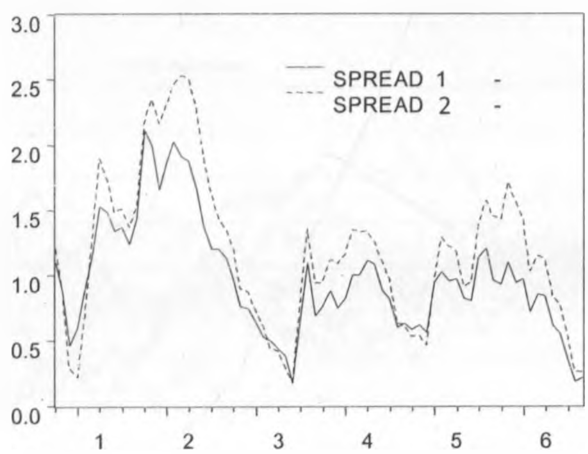


Figure 5. Interest Rates Spreads: Year 1 – Year 6 (6 years)

4.3 Numerical results

In this section, we will check analytically the behavior of P with respect to the parameters on which it depends by the sign of its partial derivatives, all in closed form. In particular we study the behavior of this premium with respect to the initial unit value of the reference portfolio S_0, the minimum number of units guaranteed at each premium payment date g, the amount d deemed to be periodically invested in the fund, the instantaneous forward rates $f_0(t)$ prevailing at time 0, the volatility parameters σ, σ_1, σ_2 (at least when they are positive) and with respect to the time 0 prices of unit discount bonds $B_0(t)$. We will also study the behavior of P with respect to the maturity T and to the survival probabilities ${}_tp_x$ (or, alternatively, to the age x of the insured at the inception of the contract).

For comparison, we have fixed the parameter $G_t = g_tS_0/B_0(t)$. This quantity can be interpreted as the riskless return at time t of the amount g_tS_0 invested at time 0 in unit discount bonds with maturity t. If the same amount were invested in the reference portfolio, its stochastic return at time t, g_tS_t, would give exactly the implicit minimum guaranteed benefit in our model.

To evaluate the expectation in expression (13.4) Monte Carlo simulations are employed. To this end we have simulated the trajectories for the standard Brownian motions in the time interval (0; T] and used them for building corresponding trajectories of the interest rates.

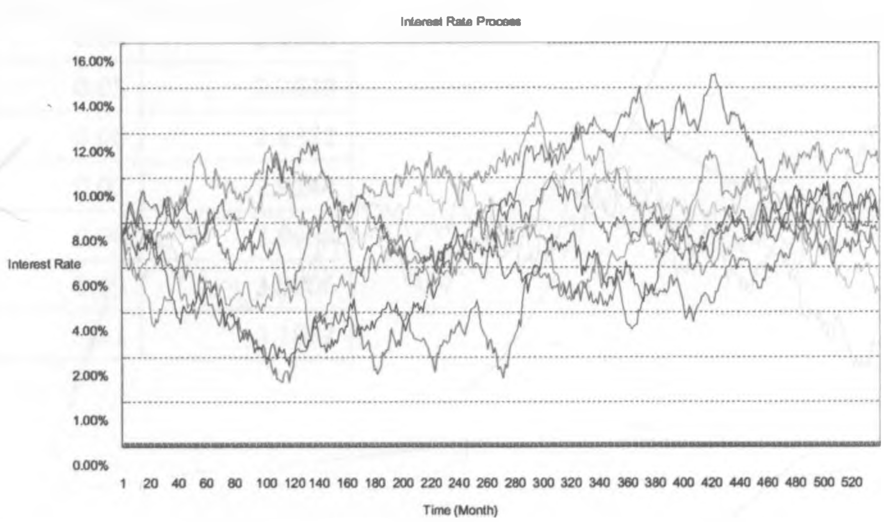


Figure 6. Simulated interest rate paths

Result of Analysis

Studying the behavior of premium with respect to the initial term structure and information presented below.

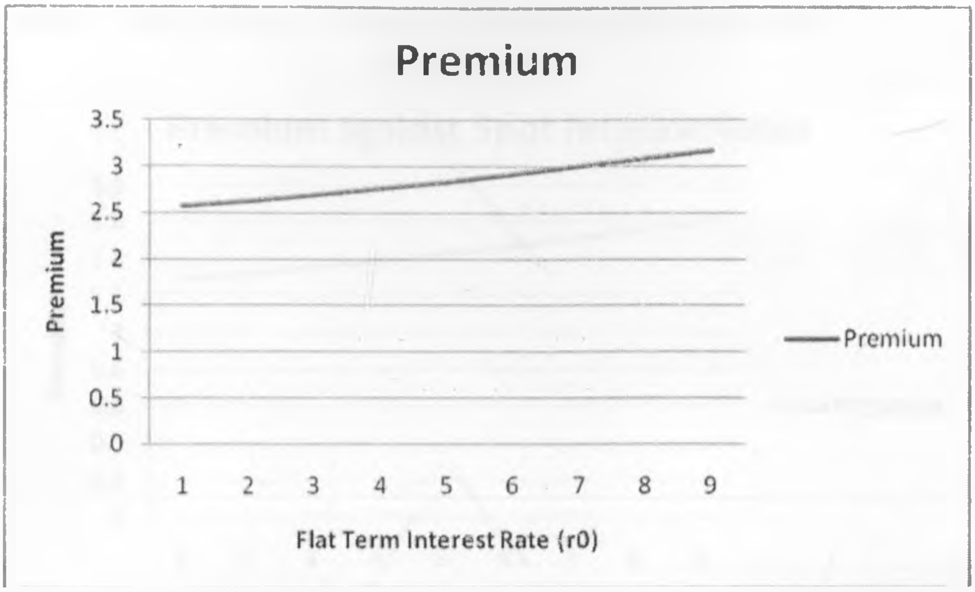
4.3.1 Flat Term Structure

a. Setting T=10, σ=6%, σ_1=3%, σ_2=2% $f_0(t)=r_0$ in equations 3.14 and 3.15, we get:

Table 4

r_0	P
0.02	2.568
0.03	2.6292
0.04	2.6946
0.05	2.7638
0.06	2.8372
0.07	2.9144
0.08	2.9956
0.09	3.0806
0.1	3.1692

Figure 7 – Case of flat term structure



This shows that premium increases with increase in the value of initial spot rate.

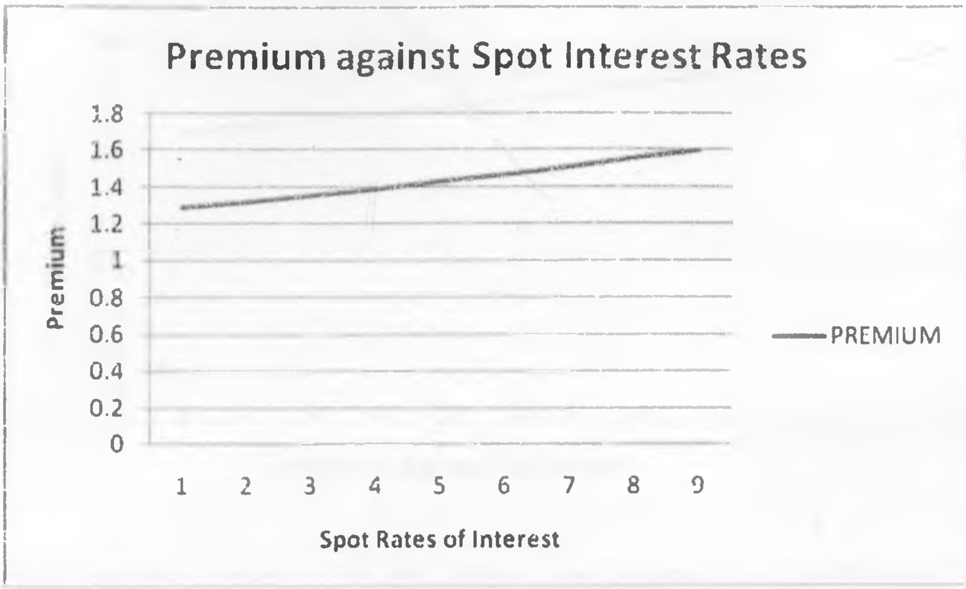
Analyzing the figures in the table above gives a correlation coefficient of 0.998252 and a covariance of 0.005013.

b. Setting T=10, σ=6%, σ_1=3%, σ_2=2% and $f_0(t)=r_0+0.01t$ in equations 3.14 and 3.15, we get:

Table 5

r_0	P
0.02	2.5862
0.03	2.6484
0.04	2.7148
0.05	2.7852
0.06	2.8596
0.07	2.9378
0.08	3.0198
0.09	3.1058
0.1	3.1952

Figure 8 shows the case for linearly increasing initial spot rates.



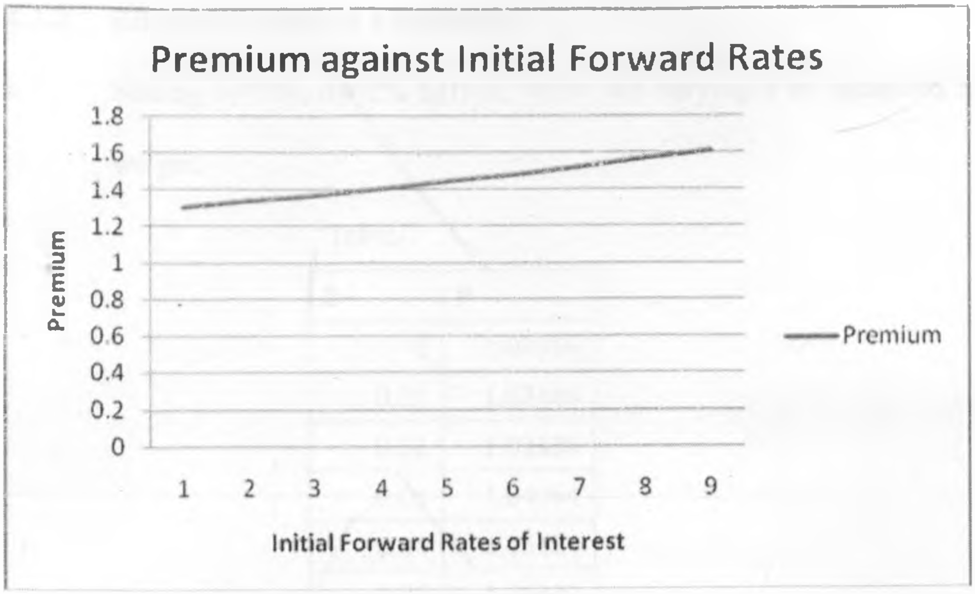
Analyzing the figures in the table above gives a correlation coefficient of 0.998328 and a covariance of 0.005079.

c. Setting T=10, σ=6%, σ_1=3%, σ_2=2% and $f_0(t)=r_0+0.02t$ in equations 3.14 and 3.15, we get:

Table 6

r_0	Premium
0.02	2.6044
0.03	2.668
0.04	2.7354
0.05	2.8068
0.06	2.882
0.07	2.9614
0.08	3.0444
0.09	3.131
0.1	3.2212

Figure 9 shows linearly decreasing initial forward rates.



Analyzing the figures in the table above gives a correlation coefficient of 0.998417 and a covariance of 0.005143.

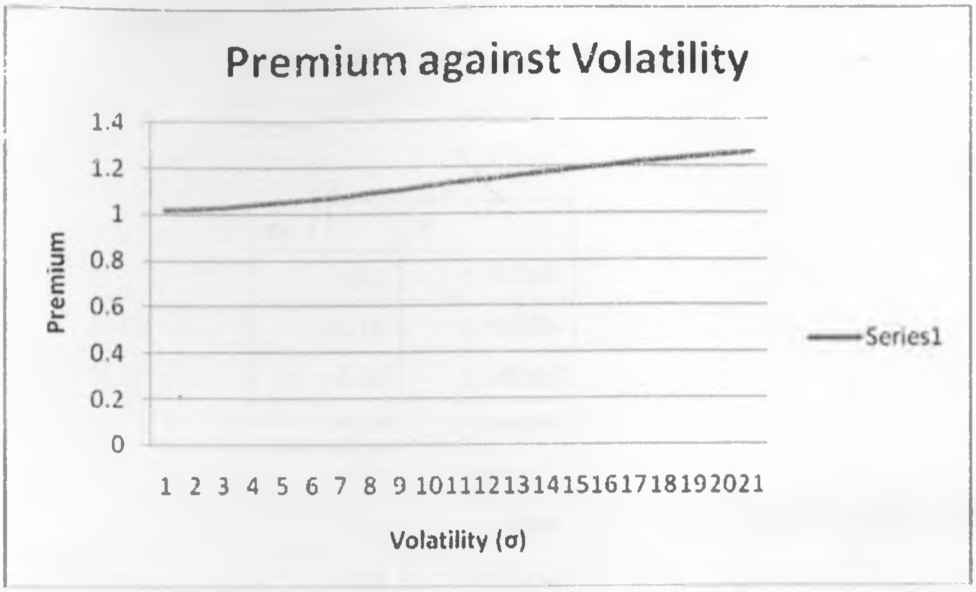
4.3.2 Effect of Volatility Parameter

a. Setting σ_1=3%, σ_2=2% $f_0(t)=r_0$ =0.04 but varying σ in equations 3.14 and 3.15, we get:

Table 7

Σ	P
0	1.02056
0.01	1.02448
0.02	1.03136
0.03	1.04064
0.04	1.05184
0.05	1.06432
0.06	1.07784
0.07	1.09192
0.08	1.10632
0.09	1.12096
0.1	1.1356
0.11	1.15008
0.12	1.16432
0.13	1.17832
0.14	1.19192
0.15	1.2052
0.16	1.21808
0.17	1.2304
0.18	1.2424
0.19	1.25384
0.2	1.26488

Figure 10 shows the effect of the volatility parameter σ on the premiums.



The graph above shows that the premium is increasing with respect to the volatility parameter σ.

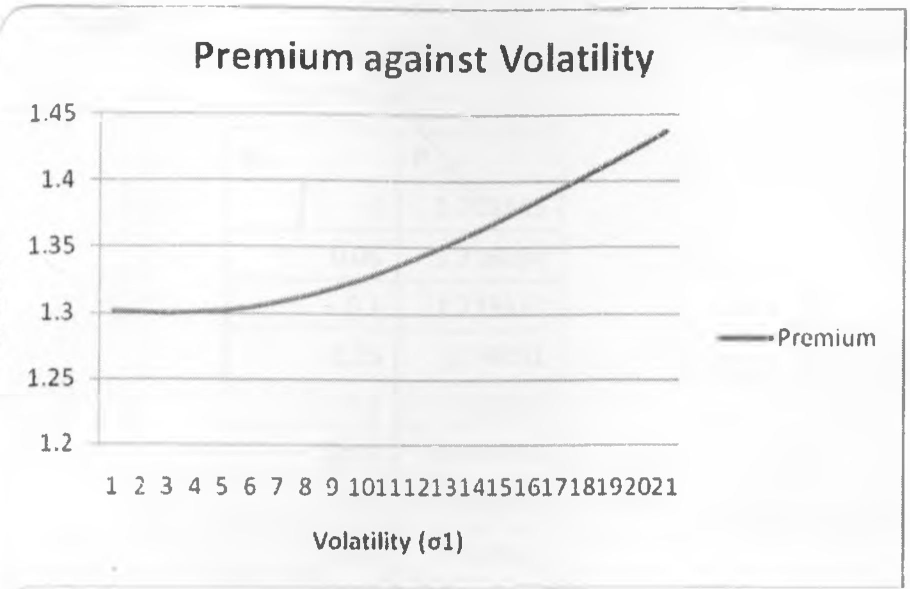
Analyzing the figures in the table above gives a correlation coefficient of 0.998112 and a covariance of 0.004803.

b. Setting σ=6%, σ_2=2% $f_0(t)=r_0$ =0.04 but varying σ_1 in equations 3.14 and 3.15, we get:

Table 8

σ_1	P
-0.2	1.56288
-0.18	1.56096
-0.16	1.56036
-0.14	1.56108
-0.12	1.56312
-0.1	1.5666
-0.08	1.57128
-0.06	1.57716
-0.04	1.58424
-0.02	1.5924
0	1.60152
0.02	1.61148
0.04	1.62216
0.06	1.63356
0.08	1.64556
0 1	1.65804
0.12	1.67088
0.14	1.68408
0.16	1.69764
0.18	1.71132
0.2	1.72512

Figure 11 shows the behavior of P with respect to the volatility parameter σ_1



The figure above shows that premium sensitive to changes in volatility. Premium increases with increase in the volatility.

Analyzing the figures in the table above gives a correlation coefficient of 0.964855 and a covariance of 0.006286.

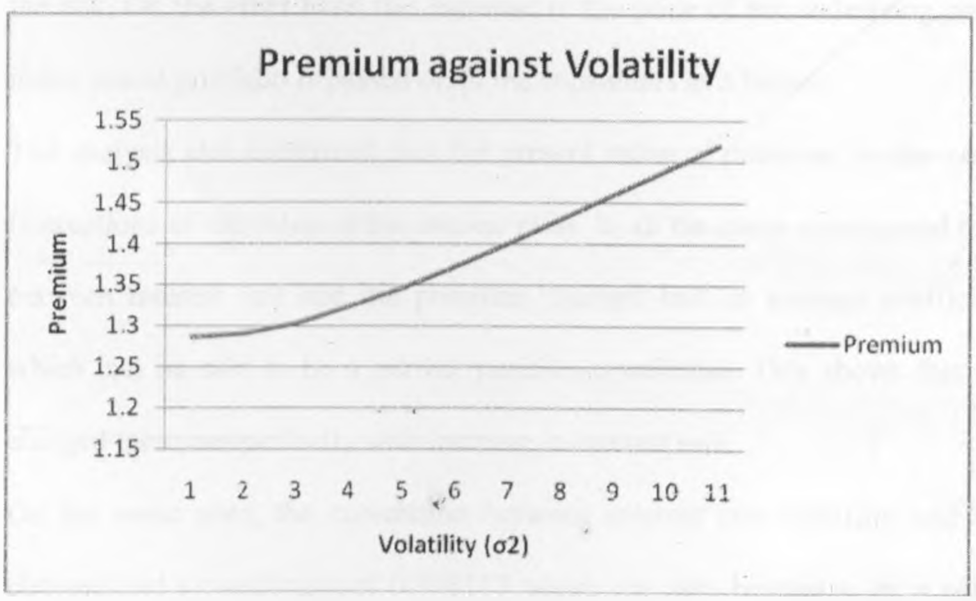
c. Setting σ=6%, σ_1=3% $f_0(t)=r_0$ =0.04 but varying σ_2 in equations 3.14 and 3.15, we get:

Table 9

σ_2	P
0	1.711843
0.05	1.718094
0.1	1.735517
0.15	1.76092
0.2	1.791909
0.25	1.826755
0.3	1.863995
0.35	1.902831
0.4	1.942465
0.45	1.982232
0.5	2.021733

Analyzing the figures in the table above gives a correlation coefficient of 0.98945 and a covariance of 0.016285.

Figure 12 shows the behavior of P with respect to the volatility parameter σ_2



Chapter 5

Conclusion

In this paper, we analyzed the risk of index linked life insurance applying insurance pricing model that uses the framework of European put options. In the concrete, we considered stochastic processes of interest rates and the reference portfolio.

From the analysis, it shows that the prices of the reference portfolio are very stochastic in nature implying that this volatility is supposed to be catered for in the pricing. It is shown that the premium charged increases with increase in the volatility of the reference portfolio to cushion the life company from any adverse effect of this. This increase in premium with increase in volatility is to cater for the uncertainity in the market performance. This is quite debatable because any positive increase in the prices of the underlying portfolio will be a gain for the life office on one hand resulting in super normal profits which according to the life office will be the reward for carrying the risk. On the other hand this increase in the price of the underlying portfolio for an index linked portfolio is passed on to the consumers as a bonus.

The analysis also confirmed that the present value of premium is also sensitive to the fluctuations of the value of the interest rates. In all the cases investigated the correlation between interest rate and the premium charged had an average coefficient of 0.998 which can be said to be a perfect positive correlation. This shows that the premium charged increases perfectly with increase in interest rate.

On the same note, the correlation between interest rate volatility and the premium charged had a coefficient of 0.998112 which can also be said to be a perfect positive correlation. This shows that the premium charged also changes perfectly with change in interest rate volatility. Just as in the volatility of the portfolio prices, volatility of the interest rate results in uncertainity hence direct variation with the price. This can be catered for by determining and applying the forward interest rates in the determination of the future premiums.

On the other hand, the covariance between premium charged and volatility was 0.004803 which can be said to be insignificant. The same case applies to covariance between interest rates and the premium charged.

Recommendations

Through this analysis, we confirmed the potential serious risk resulting from using deterministic interest rates and reference portfolio prices in pricing of life insurance policies.

We would therefore highly recommend that life offices apply stochastic interest rates in pricing all life policies. This will cushion them from the financial risk of interest rates fluctuations.

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