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# FUZZY IDEALS AND WEAK IDEALS IN BCK-ALGEBRAS

C. LELE, C. WU, P. WEKE AND T. MAMADOU, G. EDWARD NJOCK

ABSTRACT. In this paper, we use the notion of fuzzy point to study some basic algebraic structures such as $BCK$-algebras and ideals. Then we clarify the links between the fuzzy point approach and the classical fuzzy approach.

## 0-0 INTRODUCTION

In $[4, 7]$ some transfer theorems for fuzzy groups and fuzzy semigroups were established. In this paper, we apply those results to $BCK$-algebras. The concept of fuzzy sets was introduced by Zadeh [9]. This concept has been applied to $BCK$-algebras by Xi [8]. In this paper, given a $BCK$-algebra $(X, *, 0)$ and a fuzzy subset $A$ on $X$, we construct the set $(\tilde{X}, *)$ of all fuzzy points on $X$ and the subset $\tilde{A}$ of $\tilde{X}$. Then we establish some similarities between some properties of $A$ and $\tilde{A}$.

## 0-I PRELIMINARIES ([1, 5, 6])

An algebra $(X, *, 0)$ of type $(2, 0)$ is said to be a $BCK$-algebra if and only if for any $x, y, z$ in $X$, the following conditions hold:

$BCK$-1 $((x * y) * (x * z)) * (z * y) = 0$,

$BCK$-2 $(x * (x * y)) * y = 0$,

$BCK$-3 $x * x = 0$,

$BCK$-4 $0 * x = 0$,

$BCK$-5 $x * y = 0$ and $y * x = 0$ imply $x = y$.

If we define a binary relation$\leq$ on $X$ by

$BCK$-6 $x \leq y$ if and only if $x * y = 0$,

then $(X, \leq)$ is a partially ordered set with the least element 0.

The following properties also hold in any $BCK$-algebra ([1], [6])

(1) $x * 0 = 0$,

(2) $x * y = 0$ and $y * z = 0$ imply $x * z = 0$,

(3) $x * y = 0$ implies $(x * z) * (y * z) = 0$ and $(z * y) * (z * x) = 0$,

(4) $(x * y) * z = (x * z) * y$,

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(5) $(x * y) * x = 0$,

(6) $x * (x * (x * y)) = x * y$,

(7) $(x * y) * z = 0$ implies $(x * z) * y = 0$,

(8) $[(x * y) * (y * z)] * (x * y) = 0$,

(9) $[((x * z) * z) * (y * z)] * [(x * y) * z] = 0$,

(10) $(x * z) * (x * (x * z)) = (x * z) * z$,

(11) $[x * (y * (y * x)) * (y * (x * (y * (y * x))))] * (x * y) = 0$.

## 0-II ALGEBRAIC STRUCTURE OF THE SET OF FUZZY POINTS IN BCK-ALGEBRAS

Let $(X, *, 0)$ be a $BCK$-algebra. A fuzzy set $A$ in $X$ is a map $A : X \to [0, 1]$. If $\xi$ is the family of all fuzzy sets in $X$, $x_\lambda \in \xi$ is a fuzzy point if and only if $x_\lambda(y) = \lambda$ when $x = y$; and $x_\lambda(y) = 0$ when $x \neq y$. We denote by $\tilde{X} = \{x_\lambda | x \in X, \lambda \in (0, 1]\}$ the set of all fuzzy points on $X$ and define a binary operation on $\tilde{X}$ as follows: $x_\lambda * y_\mu = (x * y)_{min(\lambda,\mu)}$.

It is easy to verify that $(\tilde{X}, *)$ satisfies the following conditions: for any $x_\lambda, y_\mu, z_\alpha \in \tilde{X}$

$BCK$-(1') $((x_\lambda * y_\mu) * (x_\lambda * z_\alpha)) * (z_\alpha * y_\mu) = 0_{min(\lambda,\mu,\alpha)}$,

$BCK$-(2') $(x_\lambda * (x_\lambda * y_\mu)) * y_\mu = 0_{min(\lambda,\mu)}$,

$BCK$-(3') $x_\lambda * x_\lambda = 0_\lambda$,

$BCK$-(4') $0_\lambda * y_\mu = 0_{min(\lambda,\mu)}$.

*Remark0.1*: The condition $BCK$-5 is not true in $(\tilde{X}, *)$. So the partial order $\leq$ in $X$ can not be extend in $(\tilde{X}, *)$. We can also establish the following conditions: for any $x_\lambda, y_\mu, z_\alpha \in \tilde{X}$

(1') $x_\lambda * 0_\mu = X_{min(\lambda,\mu)}$,

(2') $x_\lambda * y_\mu = 0_{min(\lambda,\mu)}$ and $y_\mu * z_\alpha = 0_{min(\mu,\alpha)}$ imply $x_\lambda * z_\alpha = 0_{min(\lambda,\alpha)}$,

(3') $x_\lambda * y_\mu = 0_{min(\lambda,\mu)}$ imply $(x_\lambda * z_\alpha) * (y_\mu * z_\alpha) = 0_{min(\lambda,\mu,\alpha)}$ and $(z_\alpha * y_\mu) * (z_\alpha * x_\lambda) = 0_{min(\lambda,\mu,\alpha)}$,

(4') $(x_\lambda * y_\mu) * z_\alpha = (x_\lambda * z_\alpha) * y_\mu$,

(5') $(x_\lambda * y_\mu) * x_\lambda = 0_{min(\lambda,\mu)}$,

(6') $x_\lambda * (x_\lambda * (x_\lambda * y_\mu)) = x_\lambda * y_\mu$,

(7') $(x_\lambda * y_\mu *) * z_\alpha = 0_{min(\lambda,\mu,\alpha)}$ imply $(x_\lambda * z_\alpha) * y_\mu = 0_{min(\lambda,\mu,\alpha)}$,

(8') $[(x_\lambda * z_\alpha) * (y_\mu * z_\alpha)] * (x_\lambda * y_\mu) = 0_{min(\lambda,\mu,\alpha)}$,

(9') $[((x_\lambda * z_\alpha)) * z_\alpha) * (y_\mu * z_\alpha)] * [(x_\lambda * y_\mu) * z_\alpha] = 0_{min(\lambda,\mu,\alpha)}$,

(10') $(x_\lambda * z_\alpha) * [x_\lambda * (x_\lambda * z_\alpha)] = (x_\lambda * z_\alpha) * z_\alpha$,

(11') $\{[x_\lambda * (y_\mu * (y_\mu * x_\lambda))] * [y_\mu * (x_\lambda * (y_\mu * (y_\mu * x_\lambda)))]\} * (x_\lambda * y_\mu) = 0_{min(\lambda,\mu)}$.

We also recall that: if $A$ is a fuzzy subset of a $BCK$-algebra $X$, then we have the following: $\tilde{A} = \{x_\lambda \in \tilde{X} | A(x) \geq \lambda, \lambda \in (0,1]\}$, and for any $\lambda \in (0,1]$ $\tilde{X}_\lambda = \{x_\lambda | x \in X\}$, and $\tilde{A}_\lambda = \{x_\lambda \in \tilde{X} | A(x) \geq \lambda\}$. Hence $\tilde{X}_\lambda \subseteq \tilde{X}$, $\tilde{A} \subseteq \tilde{X}$, $\tilde{A}_\lambda \subseteq \tilde{A}$, $\tilde{A}_\lambda \subseteq \tilde{X}_\lambda$.

We can easily prove that $(\tilde{X}_\lambda, *, 0_\lambda)$ is a $BCK$-algebra.

## 1 WEAK IDEAL

**Definition 1.1** ([6])

*A non empty subset $I$ of $BCK$-algebras $X$ is called an ideal if it satisfies*

*a)* $0 \in I$,

*b)* $x * y \in I$ *and* $y \in I$ *imply* $x \in I$.

**Definition 1.2** ([6])

*A fuzzy subset $A$ of a $BCK$-algebra $X$ is a fuzzy subalgebra if and only if for any $x, y \in X$,*

$A(x * y) \geq min(A(x), A(y))$.

**Definition 1.3** *$\tilde{A}$ is a subalgebra of $\tilde{X}$ if and only if for any $x_\lambda, y_\mu \in \tilde{A}$, we have $x_\lambda, y_\mu \in \tilde{A}$.*

**Theorem 1.1** *Let $A$ be a fuzzy subset of a $BCK$-algebra $X$. Then the following conditions are equivalent:*

*1) $A$ is a fuzzy subalgebra of $X$.*

*2) for any $\lambda \in (0,1]$, $\tilde{A}_\lambda$ is a subalgebra of $\tilde{X}$.*

*3) for any $t \in (0,1]$, the $t$-level subset $A^t = \{x \in X | A(x) \geq t\}$ is a subalgebra of $X$ when $A^t \neq \emptyset$.*

*4) $\tilde{A}$ is a subalgebra of $\tilde{X}$.*

*Proof.* 1) $\Rightarrow$ 2) Let $x_\lambda, y_\lambda \in \tilde{A}_\lambda$. Since $A$ is a fuzzy subalgebra, $A(x * y) \geq min(A(x), A(y)) \geq \lambda$, then $x_\lambda * y_\mu = (x * y)_\lambda \in \tilde{A}_\lambda$.

2) $\Rightarrow$ 3) Let $x, y \in A^t$. $\tilde{A}_t$ is a subalgebra, so we have $(x * y)_t = x_t * y_t \in \tilde{A}_t$. Hence $x * y \in A^t$.

3) $\Rightarrow$ 4) Let $x_\lambda, y_\mu \in \tilde{A}$ and $t = min(\lambda, \mu)$. Then $A(x) \geq \lambda \geq t$, and $A(y) \geq \mu \geq t$, so $x, y \in A^t$. Since $A^t$ is a subalgebra, $x * y \in A^t$ so that $x_\lambda * y_\mu = (x * y)_t \in \tilde{A}$.

4) $\Rightarrow$ 1) Let $x, y \in X$ and $t = min(A(x), A(y))$. Then $x_t, y_t \in \tilde{A}$. Because $\tilde{A}$ is a subalgebra, so we have $(x * y)_t = x_t * y_t \in \tilde{A}$, hence $A(x * y) \geq t = min(A(x), A(y))$.

**Definition 1.4** *: (J.Meng [6])*

*A fuzzy subset $A$ of a $BCK$-algebra $X$ is a fuzzy ideal if and only if :*

*a) for any $x \in X$, $A(0) \geq A(x)$,*

*b) for any $x, y \in X$, $A(x) \geq min(A(x * y), A(y))$.*

Note that every fuzzy ideal of a $BCK$-algebra is a fuzzy subalgebra.([6] theorem 3-4)

**Definition 1.5** *$\tilde{A}$ is a weak ideal of $\tilde{X}$ if and only if :*

*a) For any $\nu \in Im(A)$, $0_\nu \in \tilde{A}$,*

*b) For any $x_\lambda, y_\mu \in \tilde{X}$ such that $x_\lambda * y_\mu \in \tilde{A}$ and $y_\mu \in \tilde{A}$, $X_{min(\lambda,\mu)} \in \tilde{A}$ holds.*

*Remark 1-1:* Any weak ideal $\tilde{A}$ has the following property: $x_\lambda * y_\mu = 0_{min(\lambda,\mu)}$ and $y_\mu \in \tilde{A}$ imply $x_{min(\lambda,\mu)} \in \tilde{A}$.
Clearly, let $x_\lambda, y_\mu \in \tilde{X}$ such that $x_\lambda * y_\mu = 0_{min(\lambda,\mu)}$ and $y_\mu \in \tilde{A}$.
$y_\mu \in \tilde{A}$ implies that $A(y) \geq \mu$. Let $A(y) = \alpha$. Using definition 1-5 (a), we obtain $0_\alpha \in \tilde{A}$.
So $A(0) \geq \alpha$. But $\alpha = A(y) \geq \mu \geq min(\lambda, \mu)$. So $0_{min(\lambda,\mu)} \in \tilde{A}$. Using definition 1-5 (b), we obtain $x_{min(\lambda,\mu)} \in \tilde{A}$

Now we discuss the relation between subalgebra and weak ideal. First of all, let us establish the following:

**Lemma 1.1** *: If $\tilde{A}$ is a subalgebra of $\tilde{X}$, then for any $\lambda \in Im(A)$ $0_\lambda \in \tilde{A}$.*

*Proof.* Let $\lambda \in Im(A)$ and take $x$ in $X$ such that $A(x) = \lambda$, then $x_\lambda \in \tilde{A}$. $\tilde{A}$ is a subalgebra and $BCK$-(3') imply $0_\lambda = x_\lambda * x_\lambda \in \tilde{A}$.

**Corollary** : If $A$ is a fuzzy subalgebra then for any $x \in X$, $A(0) \geq A(x)$.

**Lemma 1.2** *: Let $A$ be a fuzzy subalgebra of $X$ and $\lambda, \mu \in (0, 1]$ such that $\lambda \geq \mu$. Then*

*a) If $x_\lambda \in \tilde{A}$, then $x_\mu \in \tilde{A}$,*

*b) If $x_\lambda \in \tilde{A}$, then $0_\mu \in \tilde{A}$.*

*Proof.* a) $x_\lambda \in \tilde{A}$ implies $A(x) \geq \lambda$. Since $\lambda \geq \mu$, we obtain $A(x) \geq \mu$. So $x_\mu \in \tilde{A}$.
b) $x_\lambda \in \tilde{A}$ implies $A(x) \geq \lambda$. Since $A$ is a fuzzy subalgebra, $A(0) \geq A(x) \geq \lambda \geq \mu$ and $0_\mu \in \tilde{A}$.

**Theorem 1.2** *: Any weak ideal $\tilde{A}$ is a subalgebra.*

*Proof.* Let $x_\lambda, y_\mu \in \tilde{A}$. $y_\mu \in \tilde{A}$ implies that $A(y) \geq \mu$. Let $A(y) = \alpha$. Using definition 1.5 (a), we obtain $0_\alpha \in \tilde{A}$ such that $A(0) \geq \alpha$. But $\alpha = A(y) \geq \mu \geq min(\lambda, \mu)$. So $0_{(min\lambda,\mu)} \in \tilde{A}$. By (5'), $(x_\lambda * y_\mu) * x_\lambda = 0_{min(\lambda,\mu)}$. Using definition 1-5 (b), we obtain $x_\lambda * y_\mu \in \tilde{A}$.

**Theorem 1.3** *: Suppose that $\tilde{A}$ is a subalgebra of $\tilde{X}$. Then the following conditions are equivalent:*

*1) $A$ is a fuzzy ideal.*

*2) If $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$ with $y_\mu$ and $z_\alpha \in \tilde{A}$, then $x_{min(\lambda,\mu,\alpha)} \in \tilde{A}$.*

*3) for any $t \in (0, 1]$, the $t$-level subset $A^t = \{x \in X | A(x) \geq t\}$ is an ideal when $A^t \neq \emptyset$.*

*4) If $x_\lambda$,$y_\mu \in \tilde{A}$ and $(z_\alpha * y_\mu) * x_\lambda = 0_{min(\lambda,\mu,\alpha)}$, then $z_{min(\lambda,\mu,\alpha)} \in \tilde{A}$.*

*5) for any $x, y, z$ in $X$, the inequality $x * y \leq z$ implies $A(x) \geq min(A(y), A(z))$.*

*6) $\tilde{A}$ is a weak ideal.*

*Proof.* 1) $\Rightarrow$ 2) Let $z_\alpha, y_\mu \in \tilde{A}$ such that $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$. Since $A$ is a fuzzy ideal, we have $A(x) \geq min(A(x*y), A(y))$ and $A(x*y) \geq min(A((x*y)*z), A(z))$. So $A(x) \geq min(\lambda, \mu, \alpha)$. Hence $x_{min(\lambda,\mu,\alpha)} \in \tilde{A}$.

2) $\Rightarrow$ 3) a) Let $x \in A^t$. Then $A(x) \geq t$. Since $A$ is a fuzzy subalgebra, $A(0) \geq A(x)$. So $A(0) \geq A(x) \geq t$, and $0 \in A^t$.

b) Let $x, y$ in $X$ such that $x * y \in A^t$ and $y \in A^t$. $y \in A^t$ implies $A(y) \geq t$. Since $A$ is a fuzzy subalgebra, $A(0) \geq A(y)$. So $A(0) \geq A(y) \geq t$,then $0_t \in \tilde{A}$.

$x * y \in A^t$ implies $A(x*y) \geq t$. So $(x*y)_t \in \tilde{A}$.

Since $0_t \in \tilde{A}$ and $(x_t * y_t) * 0_t = (x*y)_t \in \tilde{A}$, using the hypothesis, we obtain $x_t \in \tilde{A}$, so $x \in A^t$.

3) $\Rightarrow$ 4) If $x_\lambda, y_\mu \in \tilde{A}$ with $(z_\alpha * y_\mu) * x_\lambda = 0_{min(\lambda,\mu,\alpha)}$, we have $(z*y)*x = 0$. Let $t = min(\lambda, \mu, \alpha)$, since $A^t$ is an ideal, $0 \in A^t$and because $x, y \in A^t$ we obtain $z \in A^t$. So $z_t = z_{min(\lambda,\mu,\alpha)} \in \tilde{A}$.

4) $\Rightarrow$ 5) Let $x, y, z$ in $X$ such that $x * y \leq z$ and $\mu = A(y), \alpha = A(z)$. Since $x * y \leq z$, we have $(x_{min(\mu,\alpha)} * y_\mu) * z_\alpha = 0_{min(\mu,\alpha)}$. Using the hyphothesis, we obtain $x_{min(\mu,\alpha)} \in \tilde{A}$. So $A(x) \geq min(\mu, \alpha) = min(A(y), A(z))$.

5) $\Rightarrow$ 6) a) By lemma 1-1, it is clear that for any $\nu \in Im(A), 0_\nu \in \tilde{A}$.

b) for $x_\lambda, y_\mu \in \tilde{X}$ such that $x_\lambda * y_\mu \in \tilde{A}$ and $y_\mu \in \tilde{A}$. We have $A(x*y) \geq min(\lambda, \mu)$ and $A(y) \geq \mu$. Since $x*(x*y) \leq y$, it follows from the hypothesis that $A(x) \geq min(A(x*y), A(y)) \geq min(\lambda, \mu)$, so that $x_{min(\mu,\alpha)} \in \tilde{A}$.

6) $\Rightarrow$ 1) a) By theorem 1-1 and the corollary, it is clear that for any $x$ in $X, A(0) \geq A(x)$.

b) Let $x, y \in X$ and $t = min(A(x*y), A(y))$. Then $x_t * y_t = (x*y)_t \in \tilde{A}$ and $y_t \in \tilde{A}$. Since $\tilde{A}$ is a weak ideal, $x_t \in \tilde{A}$. So $A(x) \geq t = min(A(x*y), A(y))$.

The following theorem gives a characterization of a weak ideal.

**Theorem 1.4** *Suppose that $A$ is a fuzzy subset of a $BCK$-algebra $X$. Then the following conditions are equivalent:*

*1) $A$ is a fuzzy ideal.*

*2) for all $x_\lambda, y_\mu \in \tilde{A}(z_\alpha * y_\mu) * x_\lambda = 0_{min(\lambda,\mu,\alpha)}$ imply $z_{min(\lambda,\mu,\alpha)} \in \tilde{A}$.*

*3) for any $t \in (0,1]$ the $t$-level subset $A^t = \{x \in X | A(x) \geq t\}$ is an ideal when $A^t \neq \emptyset$.*

*4) $\tilde{A}$ is a weak ideal.*

*Proof.* 1) $\Rightarrow$ 2) Let $x_\lambda, y_\mu \in \tilde{A}$ and $(z_\alpha * y_\mu) * x_\lambda = 0_{min(\lambda,\mu,\alpha)}$. Since $A$ is a fuzzy ideal, we have $A(0) \geq A(x) \geq \lambda \geq min(\lambda, \mu, \alpha)$. So $0_{min(\lambda,\mu,\alpha)} \in \tilde{A}$ and the proof is the same as in theorem 1-3.

2) $\Rightarrow$ 3) a) Since $A^t \neq \emptyset$, let $x \in A^t$ and $\lambda = A(x)$. By $BCK$-(4'), $(0_\lambda * x_\lambda) * x_\lambda = 0_\lambda$. Using the hypothesis, we obtain $0_\lambda \in \tilde{A}$. Hence $0 \in A^t$.

b) Let $x * y \in A^t$ and $y \in A^t$, by $BCK$-(2'), $(x_t * (x_t * y_t)) * y_t = 0_t$.

Using the hypothesis, we obtain $x_t \in \tilde{A}$. So $x \in A^t$.

3) $\Rightarrow$ 4) and 4) $\Rightarrow$ 1) follow from Theorem 1.1 and Theorem 1.3.

## 2 POSITIVE IMPLICATIVE WEAK IDEAL

**Definition 2.1** *([2]) A non empty subset $I$ of $X$ is called a positive implicative ideal if it satisfies:*

*a)* $0 \in I$,

*b) for all* $x, y, z \in X, (x * y) * z \in I$ *and* $y * z \in I$ *imply* $x * z \in I$.

**Definition 2.2** *(J.Meng [6]) A fuzzy subset $A$ of a $BCK$-algebra $X$ is a fuzzy positive implivative ideal if and only if :*

*a) for any* $x \in X, A(0) \geq A(x)$,

*b) for any* $x, y, z \in X, A(x * z) \geq min(A((x * y) * z), A(y * z))$.

**Definition 2.3** *$\tilde{A}$ is a positive implicative weak ideal of $\tilde{X}$ if and only if :*

*a) for any* $\nu \in Im(A)$, $0_\nu \in \tilde{A}$,

*b) for any* $x_\lambda, y_\mu, z_\alpha \in \tilde{X}$ *such that* $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$, *and* $y_\mu * z_\alpha \in \tilde{A}$, *we have* $(x * z)_{min(\lambda,\mu,\alpha)} \in \tilde{A}$.

The following theorem give a characterization of positive implicative weak ideal.

**Theorem 2.1** *If $\tilde{A}$ is a weak ideal (namely $A$ is a fuzzy ideal by theorem 1-4), then the following conditions are equivalent:*

*1) $A$ is a fuzzy positive implicative ideal.*

*2) for all* $x_\lambda, y_\mu \in \tilde{X}, (x_\lambda * y_\mu) * y_\mu \in \tilde{A}$ *implies* $x_\lambda * y_\mu \in \tilde{A}$.

*3) for any* $t \in (0, 1]$, *the $t$-level subset* $A^t = \{x \in X | A(x) \geq t\}$ *is a positive implicative ideal when* $A^t \neq \emptyset$.

*4) for all* $x_\lambda, y_\mu, z_\alpha \in \tilde{X}$, $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$ *implies* $(x_\lambda * z_\alpha) * (y_\mu * z_\alpha) \in \tilde{A}$.

*5) for all* $x, y, z \in X$, $A((x * z) * (y * z)) \geq A((x * y) * z)$.

*6) $\tilde{A}$ is a positive implicative weak ideal.*

Before proving theorem 3.2, we recall the following result:

**Lemma 2.1** *([5] theorem 2) Suppose that $I$ is an ideal of a $BCK$-algebra $X$. Then the following conditions are equivalent:*

*i) $I$ is positive implicative.*

*ii)* $(x * y) * y \in I$ *implies* $x * y \in I$.

*iii)* $(x * y) * z \in I$ *implies* $(x * z) * (y * z) \in I$.

*Proof.* 1) $\Rightarrow$ 2) Let $x_\lambda, y_\mu \in \tilde{X}$ and $(x_\lambda * y_\mu) * y_\mu \in \tilde{A}$. Since $A$ is fuzzy positive implicative,

$$A(x*y) \geq min(A((x*y)*y), A(y*y)) \geq min(A((x*y)*y), A(0)) \geq min(\lambda, \mu).$$

So $x_\lambda * y_\mu = (x*y)_{min(\lambda,\mu)} \in \tilde{A}$.

2) $\Rightarrow$ 3) a) Let $x \in A^t$. Then $A(x) \geq t$. Since $A$ is a fuzzy ideal, $A(0) \geq A(x)$. So $A(0) \geq A(x) \geq t$, so $0 \in A^t$.

b) If $(x*y)*y \in A^t$, then $(x_t * y_t) * y_t \in \tilde{A}$. From the hypothesis, we obtain $x_t * y_t \in \tilde{A}$. Hence $x * y \in A^t$. By lemma 2-1, $A^t$ is a positive implicative ideal.

3) $\Rightarrow$ 4) Let $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$ and $t = min(\lambda, \mu, \alpha)$. Then $(x*y)*z \in A^t$. Since $A^t$ is a positive implicative ideal, we apply lemma 2.1 and obtain $(x*z)*(y*z) \in A^t$. So $((x*z)*(y*z))_t = (x_\lambda * z_\alpha) * (y_\mu * z_\alpha) \in \tilde{A}$

4) $\Rightarrow$ 5) Let $x, y, z \in X$ and $t = A((x*y)*z)$, $((x*y)*z)_t = (x_t * y_t) * z_t \in \tilde{A}$. Using the hypothesis, we obtain $(x_t * z_t) * (y_t * z_t) = ((x*y)*(y*z))_t \in \tilde{A}$. So $A((x*z)*(y*z)) \geq t = A((x*y)*z)$

5) $\Rightarrow$ 6) a) Let $\nu \in Im(A)$. It is clear that $0_\nu \in \tilde{A}$.

b) Let $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$ and $y_\mu * z_\alpha \in \tilde{A}$. hen $A((x*y)*z) \geq min(\lambda, \mu, \alpha)$ and $A(y*z) \geq min(\mu, \alpha)$. From the hypothesis and the fact that $A$ is a fuzzy ideal, we obtain

$$A(x*z) \geq min(A((x*z)*(y*z)), A(y*z)) \geq min(A((x*y)*z), A(y*z))$$
$$\geq min(min(\lambda, \mu, \alpha), min(\mu, \alpha)) = min(\lambda, \mu, \alpha).$$

So $(x*z)_{min(\lambda,\mu,\alpha)} \in \tilde{A}$.

6) $\Rightarrow$ 1) a) let $x \in X$, it is clear that $A(0) \geq A(x)$.

b) Let $x, y, z \in X$ and $A((x*y)*z) = \beta$, $A(y*z) = \alpha$. $((x*y)*z)_{min(\beta,\alpha)} = (x_\beta * y_\alpha) * z_\alpha \in \tilde{A}$ and $y_\alpha * z_\alpha = (y*z)_\alpha \in \tilde{A}$. Since $\tilde{A}$ is a positive implicative weak ideal, we have $(x*z)_{min(\beta,\alpha)} \in \tilde{A}$. Hence $A(x*z) \geq min(\beta, \alpha) = min(A((x*y)*z), A(y*z))$.

## 3 COMMUTATIVE WEAK IDEAL

**Definition 3.1** *(Y.Jun [3])*

*A non empty subset $I$ of $X$ is called a commutative ideal if it satisfies*

*a)* $0 \in I$,

*b)* $(x*y)*z \in I$ *and* $z \in I$ *imply* $x*(y*(y*x)) \in I$.

**Definition 3.2** *(Y.L.Jun [3])*

*A fuzzy subset $A$ of a $BCK$-algebra $X$ is a fuzzy commutative ideal if and only if :*

*a) for any* $x \in X$, $A(0) \geq A(x)$,

*b) for any* $x, y, z \in X$, $A(x * (y * (y * x))) \geq min(A((x * y) * z), A(z))$.

**Definition 3.3** *$\tilde{A}$ is a commutative weak ideal of $\tilde{X}$ if and only if :*

*a) for any* $\nu \in Im(A)$, $0_\nu \in \tilde{A}$,

*b) for any* $x_\lambda, y_\mu, z_\alpha \in \tilde{X}$ *such that* $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$ *and* $z_\alpha \in \tilde{A}$, *we have*

$$x_{min(\lambda,\alpha)} * (y_\mu * (y_\mu * x_{min(\lambda,\alpha)})) \in \tilde{A}.$$

The following theorem give a characterization of a commutative weak ideal.

**Theorem 3.1** *Suppose that $\tilde{A}$ is a weak ideal, (namely $A$ is a fuzzy ideal by theorem 1.4) then the following conditions are equivarent:*

*1) $A$ is commutative.*

*2) for all* $x_\lambda, y_\mu \in \tilde{X}$ *such that* $x_\lambda * y_\mu \in \tilde{A}$, *we have* $x_{min(\lambda,\mu)} * [y_\mu * (y_\mu * x_{min(\lambda,\mu)})] \in \tilde{A}$.

*3) for any* $t \in (0, 1]$ *the $t$-level subset* $A^t = \{x \in X | A(x) \geq t\}$ *is a commutative ideal when* $A^t \neq \emptyset$.

*4) $\tilde{A}$ is commutative.*

The proof is similar to theorem 2.1 and is omitted.

## 4 IMPLICATIVE WEAK IDEAL

**Definition 4.1** *(J.Meng [6])*

*A non empty subset $I$ of $X$ is called an implicative ideal if it satisfies : for all* $x, y, z \in X$,

*a)* $0 \in X$,

*b)* $[x * (y * x)] * z \in I$ *and* $z \in I$ *imply* $x \in I$.

**Definition 4.2** *(J.Meng [6])*

*A fuzzy subset $A$ of a $BCK$-algebra $X$ is a fuzzy implicative ideal if and only if :*

*a) for any* $x \in X$, $A(0) \geq A(x)$,

*b) for any* $x, y, z \in X$, $A(x) \geq min((A(x * (y * x)) * z), A(z))$.

**Definition 4.3** *$\tilde{A}$ is an implicative weak ideal of $\tilde{X}$ if and only if :*

*a) for any* $\nu \in Im(A)$, $0_\nu \in \tilde{A}$.

*b) for any* $x_\lambda, y_\mu, z_\alpha \in \tilde{X}$, $(x_\lambda * (y_\mu * x_\lambda)) * z_\alpha \in \tilde{A}$ *and* $z_\alpha \in \tilde{A}$ *imply* $x_{min(\lambda,\mu,\alpha)} \in \tilde{A}$.

**Theorem 4.1** *Let $A$ be a fuzzy subset of a $BCK$-algebra $X$, then the following conditions are equivalent : 1) $A$ is a fuzzy implicative ideal.*

*2) $\tilde{A}$ is an implicative weak ideal.*

*Proof.* 1) $\Rightarrow$ 2) a) Let $\lambda \in Im(A)$. Suppose that $\lambda = A(x)$. Since $A$ is a fuzzy implicative ideal, we have $A(0) \geq A(x) = \lambda$. So $0_\lambda \in \tilde{A}$.

b) Let $(x_\lambda * (y_\mu * x_\lambda)) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$. Then $A((x*(y*x))*z) \geq min(\lambda, \mu, \alpha)$ and $A(z) \geq \alpha$. Since $A$ is a fuzzy implicative ideal, we have $A(x) \geq min(A((x*(y*x))*z), A(z)) \geq min(min(\lambda,\mu,\alpha),\alpha) = min(\lambda,\mu,\alpha)$. So $x_{min(\lambda,\mu,\alpha)} \in \tilde{A}$.

2) $\Rightarrow$ 1) a) Let $x \in X$ and $\lambda = A(x)$, $\lambda \in Im(A)$. Since $\tilde{A}$ is an implicative weak ideal, we have $0_\lambda \in \tilde{A}$. So $A(0) \geq \lambda = A(x)$.

b) If $x, y, z \in X$, let $A((x*(y*x))*z) = \beta$ and $A(z) = \alpha$. Then $((x*(y*x))*z)_{min(\beta,\alpha)} = (x_\beta * (y_\beta * x_\beta) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$. Since $\tilde{A}$ is an implicative weak ideal, we have $x_{min(\beta,\alpha)} \in \tilde{A}$. So $A(x) \geq min(\beta,\alpha) = min(A((x*(y*x))*z), A(z))$.

Now we describe the relation between weak ideal and implicative weak ideal.

**Theorem 4.2** *If $\tilde{A}$ is an implicative weak ideal, then $\tilde{A}$ is a weak ideal. The converse is not true in general.*

*Proof.* a) Let $\lambda \in Im(A)$, it is clear that $0_\lambda \in \tilde{A}$.

b) Let $x_\lambda * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$. Then $x_\lambda * z_\alpha = (x_\lambda * (x_\lambda * x_\lambda)) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$. Using the definition of implicative weak ideal, we obtain $x_{min(\lambda,\alpha)} \in \tilde{A}$. Hence $\tilde{A}$ is a weak ideal. Let $X = \{0,1,2,3,4\}$ be a $BCK$-algebra with a Cayley table as follows:

| * | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 0 |
| 2 | 2 | 2 | 0 | 0 | 0 |
| 3 | 3 | 3 | 3 | 0 | 0 |
| 4 | 4 | 3 | 4 | 1 | 0 |

Define a fuzzy set $A : X \to [0,1]$ by $A(0) = A(2) = 1$, $A(1) = A(3) = A(4) = 1/8$.

$\tilde{A} = \{0_\lambda | \lambda \in (0,1]\} \cup \{1_\lambda | \lambda \in (0,1/8]\} \cup \{2_\lambda | \lambda \in (0,1]\} \cup \{3_\lambda | \lambda \in (0,1/8]\} \cup \{4_\lambda | \lambda(0,1/8]\}$.

It is easy to check that $\tilde{A}$ is a weak ideal, but it is not an implicative weak ideal, because $(1_{1/2} * (3_{1/2} * 1_{1/2})) * 2_1 = (1_{1/2} * 3_{1/2}) * 2_1 = 0_{1/2} * 2_1 = 0_{1/2} \in \tilde{A}$ and $2_1 \in \tilde{A}$, but $1_{min(1/2,1)} = 1_{1/2} \notin \tilde{A}$.

**Corollary** A fuzzy implicative ideal must be a fuzzy ideal. But the converse does not hold in general.

The following theorem give a characterization of implicative weak ideal.

**Theorem 4.3** *Suppose that $\tilde{A}$ is a weak ideal (namely $A$ is fuzzy ideal by theorem 1-4), then the* *following conditions are equivarent :*

*1) $A$ is a fuzzy implicative ideal.*

*2) for all $x_\lambda, y_\mu \in \tilde{X}$, $x_\lambda * (y_\mu * x_\lambda) \in \tilde{A}$ implies $x_{min(\lambda,\mu)} \in \tilde{A}$.*

*3) for $t \in (0,1]$, the $t$-level subset $A^t = \{x \in X | A(x) \geq t\}$ is an implicative ideal when $A^t \neq \emptyset$.*

*4) $\tilde{A}$ is implicative.*

Before proving the theorem, we recall the following result :

**Lemma 4.1** *([5] Theorem 11) An ideal $I$ of a $BCK$-algebra $X$ is implicative if and only if for any $x, y, z \in X$ such that $x * (y * x) \in I$, we have $x \in I$.*

*Proof.* 1) $\Rightarrow$ 2) Let $x_\lambda * (y_\mu * x_\lambda) \in \tilde{A}$. Since $A$ is fuzzy implicative, we have

$$A(x) \geq min(A((x * (y * x)) * 0), A(0)) \geq min(\lambda, \mu). \text{So} x_{min(\lambda,\mu)} \in \tilde{A}.$$

2) $\Rightarrow$ 3) a) It is clear that $0 \in A^t$.

b) Let $x * (y * x) \in A^t$. Then $(x * (y * x))_t = x_t * (y_t * x_t) \in \tilde{A}$. Using the hypothesis, we obtain $x_t \in \tilde{A}$. So $x \in A^t$ and by Lemma 4.1, $A^t$ is implicative.

3) $\Rightarrow$ 4) a) Let $\lambda \in Im(A)$, it is clear that $0_\lambda \in \tilde{A}$.

b) If $(x_\lambda * (y_\mu * x_\lambda)) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$, let $t = min(\lambda, \mu, \alpha)$. Then $(x * (y * x)) * z \in A^t$ and $z \in A^t$. Because $A^t$ is implicative, $x \in A^t$. So $x_{min(\lambda,\mu,\alpha)} = x_t \in \tilde{A}$.

4) $\Rightarrow$ 1) Follows from Theorem 4.1.

The following theorem gives some equivalent conditions for a subalgebra $\tilde{A}$ to be an implicative weak ideal.

**Theorem 4.4** *If $\tilde{A}$ is a subalgebra (namely $A$ is a fuzzy subalgebra by theorem 1-1), the following conditions are equivalent:*

*1) $A$ is a fuzzy implicative ideal.*

*2) for any $x_\lambda, y_\mu, z_\alpha, w_t \in \tilde{X}$, $((x_\lambda * (y_\mu * x_\lambda)) * z_\alpha) * w_t = 0_{min(\lambda,\mu,\alpha,t)}$ with $z_\alpha$ and $w_t \in \tilde{A}$ imply $x_{min(\lambda,\mu,\alpha,t)} \in \tilde{A}$.*

*3) or any $x, y, z, w \in X$, $((x * (y * x)) * z) * w = 0$ imply $A(x) \geq min(A(z), A(w))$.*

*4) $\tilde{A}$ is an implicative weak ideal.*

*Proof.* 1) $\Rightarrow$ 2) Let $((x_\lambda * (y_\mu * x_\lambda)) * z_\alpha) * w_t = 0_{min(\lambda,\mu,\alpha,t)}$ with $z_\alpha$ and $w_t \in \tilde{A}$. Since $A$ is fuzzy implicative ideal, $A$ is also a fuzzy ideal, we apply Theorem 1.4 and obtain $x_{min(\lambda,\alpha,t)} * (y_\mu * x_{min(\lambda,\alpha,t)}) = x * (y * x)_{min(\lambda,\mu,\alpha,t)} \in \tilde{A}$. Using Theorem 4-3, we have $X_{min(\lambda,\mu,\alpha,t)} \in \tilde{A}$.

2) $\Rightarrow$ 3) Let $((x * (y * x)) * z) * w = 0$ and $t = min(A(z), A(w))$, then $z_t$ and $w_t \in \tilde{A}$ $((x_t * (y_t *$

$x_t)) * z_t)w_t = 0_t$. Using the hypothesis, we obtain $x_t \in \tilde{A}$. So $A(x) \geq t = min(A(z), A(w))$

3) $\Rightarrow$ 4) a) Let $\lambda \in Im(A)$, because $\tilde{A}$ is a subslgebra, we have $0_\lambda \in \tilde{A}$ (by Lemma 1.1).

b) Let $(x_\lambda * (y_\mu * x_\lambda)) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$. Since $\{(x*(y*x))*[(x*(y*x))*z]\}*z = 0$, we apply the hypothesis and obtain $A(x) \geq min(A((x*(y*x))*z), A(z)) \geq min(min(\lambda, \mu, \alpha), \alpha) = min(\lambda, \mu, \alpha)$. So $x_{min(\lambda,\mu,\alpha)} \in \tilde{A}$.

4) $\Rightarrow$ 1) Follows from Theorem 4.1.

Now we describe the relation between positive implicative weak ideal and implicative weak ideal.

**Theorem 4.5** *If $\tilde{A}$ is an implicative weak ideal, then $\tilde{A}$ is a positive implicative weak ideal.*

*Proof.* Suppose that $\tilde{A}$ is an implicative weak ideal of $\tilde{X}$ and let $x_\lambda, y_\mu, z_\alpha \in \tilde{X}$ such that $(x_\lambda * y_\mu) * z_\alpha \in \tilde{A}$ and $y_\mu * z_\alpha \in \tilde{A}$. To prove that $\tilde{A}$ is positive implicative, we need only to show that $(x * z)_{min(\lambda,\mu,\alpha)} \in \tilde{A}$. Using (8'), $\{[(x_\lambda * z_\alpha) * z_\alpha] * (y_\mu * z_\alpha)\} * [(x_\lambda * z_\alpha) * y_\mu] = 0_{min(\lambda,\mu,\alpha)}$. By (4') $(x_\lambda * z_\alpha) * y_\mu = (x_\lambda * y_m u) * z_\alpha$. Since $\tilde{A}$ is a weak ideal, we obtain $(x_{min(\lambda,\mu)} * z_\alpha) * z_\alpha \in \tilde{A}$. From $(4')$ we obtain $(x_{min(\lambda,\mu)} * z_\alpha) * [x_{min(\lambda,\mu)} * (x_{min(\lambda,\mu)} * z_\alpha)] = \{x_{min(\lambda,\mu)} * [x_{min(\lambda,\mu)} * (x_{min(\lambda,\mu)} * z_\alpha)]\} * z_\alpha = [x_{min(\lambda,\mu)} * z_\alpha] * z_\alpha$ (From (6')). It follows that $\{(x_{min(\lambda,\mu)} * z_\alpha) * [x_{min(\lambda,\mu)} * (x_{min(\lambda,\mu)} * z_\alpha)]\} * 0_{min(\lambda,\mu,\alpha)} \in \tilde{A}$. Since $\tilde{A}$ is an implicative weak ideal, we obtain $x_{min(\lambda,\mu)} * z_\alpha = (x*z)_{min(\lambda,\mu,\alpha)} \in \tilde{A}$ which completes the proof.

**Corollary** If $A$ is a fuzzy implicative ideal, then $A$ is a positive implicative ideal.

For further relations between a positive implicative weak ideal and an implicative weak ideal, we have the following:

**Theorem 4.6** *Let $\tilde{A}$ be a positive implicative weak ideal, then $\tilde{A}$ is implicative if and only if for any $x_\lambda, y_\mu \in \tilde{X}$ such that $y_\mu * (y_\mu * x_\lambda) \in \tilde{A}$, we have $x_\lambda * (x_\lambda * y_\mu) \in \tilde{A}$.*

*Proof.* Suppose that $\tilde{A}$ is an implicative weak ideal of $\tilde{X}$ and $y_\mu * (y_\mu * x_\lambda) \in \tilde{A}$. We want to show that $x_\lambda * (x_\lambda * y_\mu) \in \tilde{A}$. By (5'), $[x_\lambda * (x_\lambda * y_\mu)] * x_\lambda = 0_{min(\lambda,\mu)}$. By (3'), $y_\mu * x_\lambda) * \{y_\mu * [x_\lambda * (x_\lambda * y_\mu)]\} = 0_{min(\lambda,\mu)}$. By (3'), $\{[x_\lambda * (x_\lambda * y_\mu)] * \{y_\mu * [x_\lambda * (x_\lambda * y_\mu)]\}\} * \{[x_\lambda * (x_\lambda * y_\mu)] * (y_\mu * x_\lambda)\} = 0_{min(\lambda,\mu)}$. By (4'), $[x_\lambda * (x_\lambda * y_\mu)] * (y_\mu * x_\lambda) = [x_\lambda * (y_\mu * x_\lambda)] * (x_\lambda * y_\mu)$. By $BCK$ -(1'), $\{[x_\lambda * (y_\mu * x_\lambda)] * (x_\lambda * y_\mu)\} * [y_\mu * (y_\mu * x_\lambda)\} = 0_{min(\lambda,\mu)}$. From the fact that $\tilde{A}$ is a weak ideal we obtain $\{[x_\lambda * (x_\lambda * y_\mu)] * \{y_\mu * [x_\lambda * (x_\lambda * y_\mu)]\}\} * 0_{min(\lambda,\mu)} \in \tilde{A}$. Since $\tilde{A}$ is an implicative weak ideal, we have $x_\lambda * (x_\lambda * y_\mu) \in \tilde{A}$.

Conversely, if $[x_\lambda * (y_\mu * x_\lambda)] * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$. Since $\tilde{A}$ is a weak ideal, we obtain $x_{min(\lambda,\alpha)} * (y_\mu * x_{min(\lambda,\alpha)}) \in \tilde{A}$. Let $\beta = min(\lambda, \alpha)$ and $x_\beta * (y_\mu * x_\beta) \in \tilde{A}$ . By $BCK$-(1') $\{[y_\mu * (y_\mu * x_\beta)] * (y_\mu * x_\beta)\} * [x_\beta * (y_\mu * x_\beta)] = 0_{min(\beta,\mu)}$. Using the fact that $\tilde{A}$ is a weak ideal, we obtain $[y_\mu * (y_\mu * x_\beta)] * (y_\mu * x_\beta) \in \tilde{A}$. Since $\tilde{A}$ is positive implicative, we have $y_\mu * (y_\mu * x_\beta) \in \tilde{A}$. By applying the hypothesis, we obtain $x_\beta*(x_\beta*y_\mu)\in\tilde{A}$. ** From $BCK$-(1') $\{(x_\beta*y_\mu)*[x_\beta*(y_\mu*x_\beta)]\}*[(y_\mu*x_\beta)*y_\mu]=0_{min(\beta,\mu)}$. Also $(y_\mu*x_\beta)*y_\mu=0_{min(\beta,\mu)}$. So $(x_\beta*y_\mu)*[x_\beta*(y_\mu*x_\beta)]=0_{min(\beta,\mu)}$. Using the fact that $\tilde{A}$ is a weak ideal, we obtain $(x_\beta*y_\mu)\in\tilde{A}$. Because $x_\beta*(x_\beta*y_\mu)\in\tilde{A}$ (see **). We have $x_{min(\beta,\mu)}=x_{min(\lambda,\mu,\alpha)}\in\tilde{A}$.

**Theorem 4.7** *Let $\tilde{A}$ and $\tilde{B}$ be two weak ideals of $\tilde{X}$ such that $\tilde{A}\subseteq\tilde{B}$ and $A(0)=B(0)$. If $\tilde{A}$ is an implicative weak ideal, then $\tilde{B}$ is also an implicative weak ideal.*

To prove the Theorem 4.7, we need the following result:

**Lemma 4.2** *([5] Theorem 12) If $I$ and $J$ are two ideals of $X$ such that $I\subseteq J$ with $I$ implicative, then $J$ is also implicative.*

Using this Lemma, we can prove Theorem 4.7 as follows:

To prove that $\tilde{B}$ is implicative, it suffices to show that for any $t$ in $(0,1]$, $B^t$is an implicative ideal when $B^t\neq\emptyset$. Since $A(0)=B(0)$, it is clear that $A^t\neq\emptyset$ when $B^t\neq\emptyset$. $\tilde{A}\subseteq\tilde{B}$ implies $A^t\subseteq B^t$. since $\tilde{A}$ is implicative, $A^t$ is also implicative, using the Lemma 4.2, $B^t$ is implicative. So $\tilde{B}$ is implicative and the proof is complete.

As a consequence, for any $\lambda\in Im(A)$, if $\{0_\lambda\}$ is an implicative weak ideal, then $\tilde{A}$ is also an implicative weak ideal.

**Corollary** Let $A$ and $B$ be fuzzy ideal of $X$ such that $B\geq A$ and $B(0)=A(0)$. If $A$ is a fuzzy implicative ideal, then $B$ is also fuzzy implicative.

In the following theorem, we analyze the relation between a commutative weak ideals, positive an implicative weak ideal and implicative weak ideal.

**Theorem 4.8** *A weak ideal $\tilde{A}$ is implicative if and only if it is both commutative and implicative.*

*Proof.* Suppose that $\tilde{A}$ is an implicative weak ideal, from Theorem 4.5, we know $\tilde{A}$ is a positive implicative weak ideal. To prove that $\tilde{A}$ is a commutative weak ideal, we need only to show that $\tilde{A}$ satisfies the condition 2 of Theorem 3.1. Let $x_\lambda,y_\mu\in\tilde{X}$ such that $x_\lambda*y_\mu\in\tilde{A}$. by (5'), $\{x_\lambda*[y_\mu*(y_\mu*x_\lambda)]\}*x_\lambda=0_{min(\lambda,\mu)}$. By (3'), $(y_\mu*x_\lambda)*\{y_\mu*[x_\lambda*(y_\mu*(y_\mu*x_\lambda))]\}=0_{min(\lambda,\mu)}$. Put $t_\beta=x_\lambda*(y_\mu*(y_\mu*x_\lambda))$. We have $(y_\mu*x_\lambda)*(y_\mu*t_\beta)=0_{min(\lambda,\beta,\mu)}$ . Applying (3'), we obtain $[t_\beta*(y_\mu*t_\beta)]*[t_\beta*(y_\mu*x_\lambda)]=0_{min(\lambda,\beta,\mu)}$. But $t_\beta*(y_\mu*X_\lambda)=\{x_\lambda*(y_\mu*(y_\mu*x_\lambda))\}*(y_\mu*x_\lambda)=[x_\lambda*(y_\mu*x_\lambda)]*[y_\mu*(y_\mu*x_\lambda)]$ by (4'). From (8'), we also have $\{[x_\lambda*(y_\mu*x_\lambda)]*[y_\mu*(y_\mu*x_\lambda)]\}*(x_\lambda*y_\mu)=0_{min(\lambda,\mu)}$. Since $x_\lambda*y_\mu\in\tilde{A}$, we obtain $t_\beta*(y_\mu*x_\lambda)\in\tilde{A}$. So $t_\beta*(y_\mu*t_\beta)\in\tilde{A}$. $\tilde{A}$ is an implicative weak ideal. Hence we applying Theorem 4.3 and obtain $t_\beta=t_{min(\beta,\mu)}\in\tilde{A}$. So $t_\beta=X_\lambda*(y_\mu*(y_\mu*x_\lambda))=x_{min(\lambda,\mu)}*[y_\mu*(y_\mu*x_{min(\lambda,\mu)})]\in\tilde{A}$.

Conversely, suppose that $\tilde{A}$ is both commutative and positive implicative, we must verify that $\tilde{A}$ is implicative. Using Theorem 4.3, we need only to show that $x_\lambda * (y_\mu * x_\lambda) \in \tilde{A}$ implies $x_{min(\lambda,\mu)} \in \tilde{A}$. By $BCK$-(2') $[y_\mu * (y_\mu * x_\lambda)] * x_\lambda = 0_{min(\lambda,\mu)}$. By (3') $\{[y_\mu * (y_\mu * x_\lambda)] * (y_\mu * x_\lambda)]\} * [x_\lambda * (y_\mu * x_\lambda)] = 0_{min(\lambda,\mu)}$. Since $\tilde{A}$ is a weak ideal and $x_\lambda * (y_\mu * x_\lambda) \in \tilde{A}$, we obtain $[y_\mu * (y_\mu * x_\lambda)] * (y_\mu * x_\lambda) \in \tilde{A}$. Using Theorem 2.1, we obtain $y_\mu * (y_\mu * x_\lambda) \in \tilde{A}$. On the other hand, using $BCK$-(1') we have $\{(x_\lambda * y_\mu) * [x_\lambda * (y_\mu * x_\lambda)]\} * [(y_\mu * x_\lambda) * y_\mu] = 0_{min(\lambda,\mu)}$. Since $(y_\mu * x_\lambda) * y_\mu = 0_{min(\lambda,\mu)}$, we have $(x_\lambda * y_\mu) * [x_\lambda * (y_\mu * x_\lambda)] = 0_{min(\lambda,\mu)}$. From $x_\lambda * (y_\mu * x_\lambda) \in \tilde{A}$, we obtain $x_\lambda * y_\mu \in \tilde{A}$. Since $\tilde{A}$ is commutative, we apply Theorem 3.1 and obtain $x_\lambda * [y_\mu * (y_\mu * x_\lambda)] \in \tilde{A}$. Since $y_\mu * (y_\mu * x_\lambda) \in \tilde{A}$, we obtain $x_{min(\lambda,\mu)} \in \tilde{A}$.

**Corollary** A fuzzy ideal $A$ is implicative if and only if it is commutative and positive implicative.

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## References

[1] K.Iseki and S.Tanaka, An Introduction to the Theory of $BCK$-algebras, *Math. Japon* **23***(1987), 1-26.*

[2] Y.B.Jun, Fuzzy Commutative Ideals of $BCK$-algebras, *Fuzzy Set and Systems* **64***(1994), 401-405.*

[3] Y.B.Jun, S.M.Hong, J.Meng and X.L.Xin, Characterizations of Fuzzy Positive Implicative Ideals in $BCK$-algebras, *Math. Japon* **40(3)***(1994), 503-507.*

[4] C.LELE and Congxin Wu, Transfer Theorems for Fuzzy Semigroups, J. of Harbin Institute of Technology. (English Edition)(2000)19-23.

[5] J.Meng, On Ideals in $BCK$-algebras, *Math. Japon* **40(1)***(1994), 143-154.*

[6] J.Meng, Y.B.Jun and H.S.Kim, Fuzzy Implicative Ideals of $BCK$-algebras, *FuzzySet and Systems* **89***(1997), 243-248.*

[7] Congxin Wu and Zhenkai Qi, Transfer theorems for Fuzzy Groups, *Fuzzy mathematics* **2***(1982), 51-55.*

[8] O.G.Xi, Fuzzy $BCK$-algebras, *Math. Japon* **36***(1991), 935-942.*

[9] L.A.Zadeh, Fuzzy Sets, *Infom. And Control* **8***(1965), 338-353.*

C. LELE, C. Wu, P. Weke and T. Mamadou : Dept. of Mathematics, Harbin Institute of Technology, Harbin 150001, China
(E.mail:lele_clele@yahoo.com).
G.Edward Njock : Dept. of Mathematics. / Algebra Team, Yaounde University. P.O.Box 812 Cameroon.