FROM THE CLASSICAL BETA DISTRIBUTION TO GENERALIZED BETA DISTRIBUTIONS

A project submitted to the School of Mathematics, University of Nairobi in partial fulfillment of the requirements for the degree of Master of Science in Statistics.

By

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Declaration

This project is my original work and has not been presented for a degree in any other University

Signature ________________________

Otieno Jacob

This project has been submitted for examination with my approval as the University Supervisor

Signature ________________________

Prof. J.A.M Ottieno

Dedication

I dedicate this project to my wife Lilian; sons Benjamin, Vincent, John; daughter Joy and friends. A special feeling of gratitude to my loving Parents, Justus and Margaret, my aunt Wilkista, my Grandmother Nereah, and my late Uncle Edwin for their support and encouragement.

Acknowledgement

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Table of Contents

List of Figures

Executive Summary

This Master's project considers on one hand, the construction of the two parameter classical beta distribution on the domain (0,1) and its generalization and on the other hand, the construction of the extended two parameter beta distribution on the domain $(0,\infty)$ and its generalization. It provides a comprehensive mathematical treatment of these two parameter beta distributions, their constructions, properties, shapes and special cases.

There are various ways of constructing distributions in general. Four ways of constructing the classical beta distribution considered in this work are constructions from,

i. The definition of the beta function;
ii. A Poisson process;
iii. The transformation of a ratio of two independent gamma variables; and
iv. Order statistic.

Properties derived include the expressions of the r^{th} moments, first four moments, mode, coefficient of skewness and coefficient of kurtosis. Special cases such as Power, Uniform, Arcsine, Triangular shaped, Parabolic shaped and Wigner Semicircle distributions are given. By the transformation technique a two parameter inverted beta distribution was obtained and its special cases such as Lomax (Pareto II) and Log-logistic (Fisk) distributions were constructed.

The work also looked at three methods of generalizing the classical beta distribution i.e. through:

1. Transformation technique
2. Generator Approach
3. Use of Special functions

The transformation technique resulted into the following:

i. Three parameter beta distributions of both first and second kinds. In particular, the Libby-Novick and McDonald's distributions.
ii. Generalized four parameter beta distributions of the first and second kinds. Particular cases are the McDonald four parameter beta distribution of the first kind with its special

cases, the McDonald four parameter beta distribution of the second kind with its special cases and the four parameter generalized beta distribution.

iii. The five parameter generalized beta distribution due to McDonald and Xu (1995).

Generalized distributions based on generator approach due to the work of Eugene et al. (2002) are classified into:

- Beta generated distributions
- Exponentiated generated distributions
- Generalized beta generated distributions

The generalized beta distributions based on the use of special functions considered includes the Confluent hypergeometric distribution and the Gauss Hypergeometric distribution from the Confluent hypergeometric function and the Gauss Hypergeometric function respectively. Other distributions are based on the Appell function and the Bessel function.

Tables showing special cases of beta distributions and their generalizations are given in the appendix.

1 Chapter I: General Introduction

1.1 Introduction

In statistical distributions, a vast activity has been observed in generalizing classical distributions by adding more parameters to make them more flexible in analyzing empirical data. Various methods have been developed to achieve this purpose. Among such methods are the transformation technique, generator approach, mixture (compounding) and use of special functions. In this work we examine the case of classical beta distribution. The term generalized used in this work has the notion of adding more parameters to the existing distribution in order to give a more universal distribution as opposed to the well known existing generalized distributions.

Historically, the work on beta distributions started as early as 1676 when Sir Isaac Newton wrote a letter to Henry Oldenberg as given by Dutka, J., (1981). In his letter dated 24th October, 1676, Newton evaluated $\int_0^x y dx$ as a series. The particular case of the result was applicable to the evaluation of the classical beta distribution. The two parameter classical beta distributions are among the celebrated Pearson system of statistical distributions derived by Karl Pearson in the 1890s in connection with his work on evolution.

A handbook of beta distribution and its applications was published in 2004 edited by Gupta and Nadarajah. The book enumerates the properties of beta distributions and related mathematical notions. It demonstrates the applications in the fields of economics, quality control, soil science and biomedicine. It further discusses the centrality of beta distributions in Bayesian inference, the beta-binomial model and applications of the beta-binomial distribution. However, immediately after that publication, further work has been done in that area. There is therefore a need to re-examine the works on beta distribution.

1.2 Problem Statement

Recent developments focus on new techniques for building meaningful distributions, a remarkable development has been observed in generalizing the classical beta distribution. There has been a growing interest in the applications of these distributions in the analysis of empirical data and for many statistical procedures. However, their applications are limited in important ways as they could not fit data very well. The main trend is therefore to add more parameters to make them more flexible in order to fit the existing empirical data.

1.3 Objective

The objective of this project is to review various methods that have been used to construct the beta distribution from the classical to generalized beta distributions. The specific objectives are:

- to study the classical two parameter beta distribution within $(0,1)$ domain and $(0,\infty)$ domain
- to generalize the two parameter beta distributions using
    - i. transformation technique
    - ii. generator approach
    - iii. some special functions

1.4 Literature Review

This section reviews various methods that have been used in constructing beta distributions. There are three classes of generalized beta distributions:

i. Those based on transformations

The generalized beta distributions based on transformation has largely been contributed by McDonald (1984). McDonald and his associates contributed in the development of the generalized beta distributions as an income distribution and in unifying the various research activities in closely related fields.

McDonald and Xu (1995) introduced a five parameter beta distribution which nest the generalized beta and gamma distributions and include more than thirty distributions as limiting or special cases.

McDonald and Richards (1987*a*, *b*) presented a four parameter generalized beta distribution which includes as special cases three and two parameter beta, generalized gamma, Weibull, power function, Pareto, lognormal, half-normal, uniform and others.

Libby and Novick (1982) proposed a three parameter generalized beta distribution for utility fitting.

The figure 1.1 below illustrates the transformation from the two parameter classical beta distribution to the five parameters generalized beta distribution where the transformations are given as follows:

> $\mathbf{T_1}$: is the transformation changing a beta type I (classical) distribution to a two parameter beta type II (Inverted Beta) distribution
>
> $\mathbf{T_2}$: transforms a 2 parameter to 3 parameter beta distributions
>
> $\mathbf{T_3}$: transforms a 3 parameter to a 4 parameter beta distributions
>
> $\mathbf{T_4}$: transforms a 4 parameter to 5 parameter beta distributions
>
> **T**i's are defined in the coming chapters

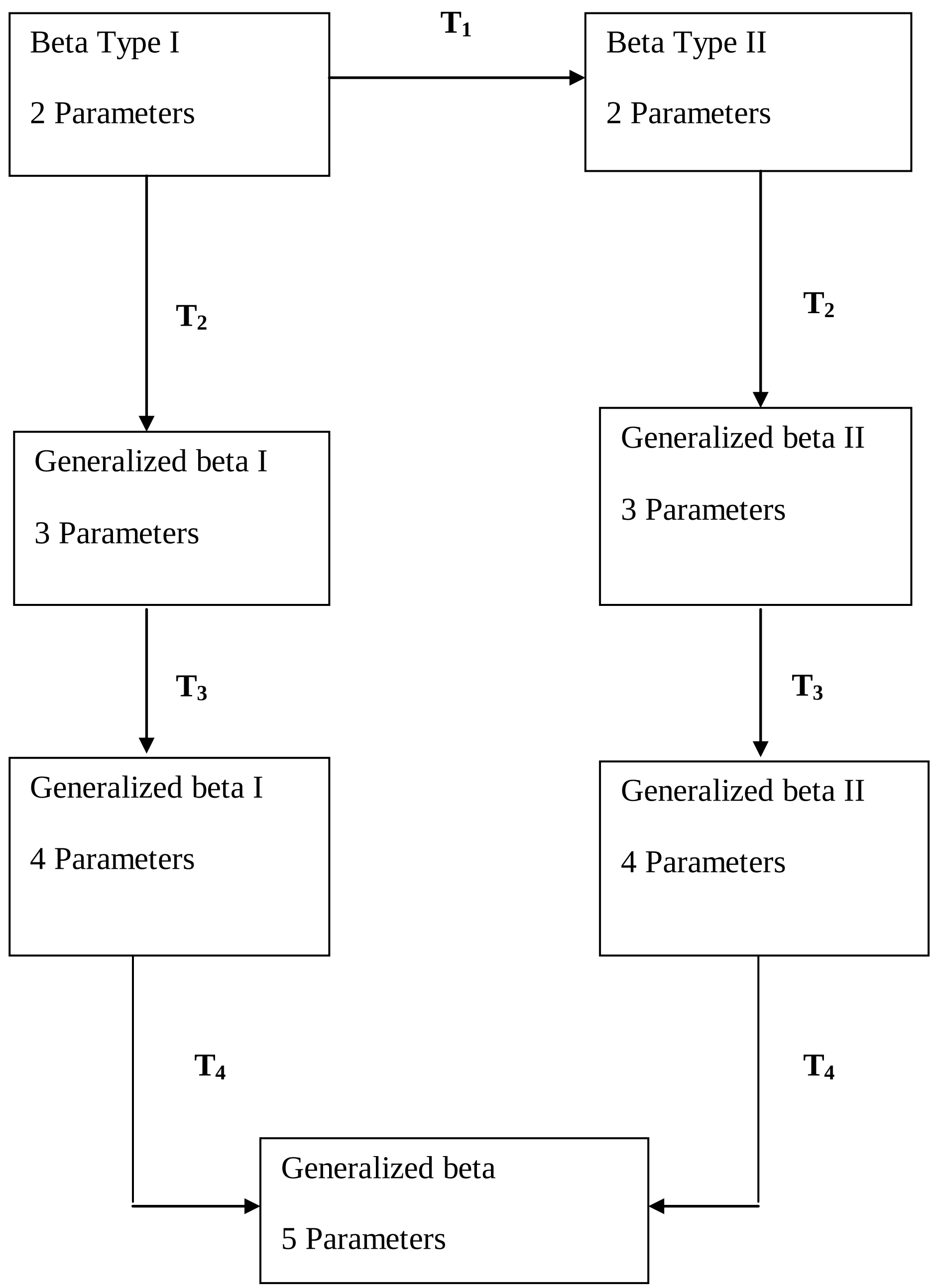


Fig.1. 1: General Framework based on transformation

ii. Generator approach based on CDF

The latest work involving the class of generalized beta distributions was introduced by Eugene et.al (2002) that defined the beta-normal distribution and studied some of its properties. Since then, several authors have been studying particular cases of this class of distributions. An advantage of the beta-normal distribution over the normal distribution is that the beta normal can be unimodal and bimodal.

The bimodal region for the beta-normal distribution was studied by Famoye et. al (2003). Gupta and Nadarajah (2004) later derived a more general expression for the n^{th} moment of the beta-normal distribution.

Nadarajah and Kotz (2004) defined the beta-Gumbel distribution, which has greater tail flexibility than the Gumbel distribution.

Nadarajah and Gupta (2004) defined the beta-Frèchet distribution and Barreto-Souza et.al. (2008) presented some additional mathematical properties.

Nadarajah and Kotz (2006) defined the beta-exponential distribution and gave a detailed presentation.

Famoye et.al (2005) defined the beta-Weibull and Lee et.al (2007) made some applications to censored data.

Kong et.al (2007) proposed the beta-gamma distribution.

The four parameter beta-pareto distribution was defined and studied by Akinsete et al (2008).

Akinsete and Lowe (2009) proposed the beta-Rayleigh distribution.

Amusan (2010) defined the beta-Maxwell distribution and studied some of its properties.

The figure 1.2 below illustrates how the generator approach has been applied in the two parameters classical beta distribution to derive other generalized beta distributions.

iii. Generalized beta generated

Barreto-Souza et al. (2010) proposed the generalized beta exponential distribution and discussed maximum likelihood estimation of its parameters.

iv. Beta-exponentiated distributions

Since 1995, the exponentiated distributions have been widely studied in statistics with the development of various classes of these distributions. Mudholkar et.al. (1995) proposed the exponentiated Weibull distribution.

Gupta et al. (1998) introduced the exponentiated Pareto distribution.

Gupta and Kundu (1999) introduced the exponentiated exponential distribution as a generalization of the standard exponential distribution.

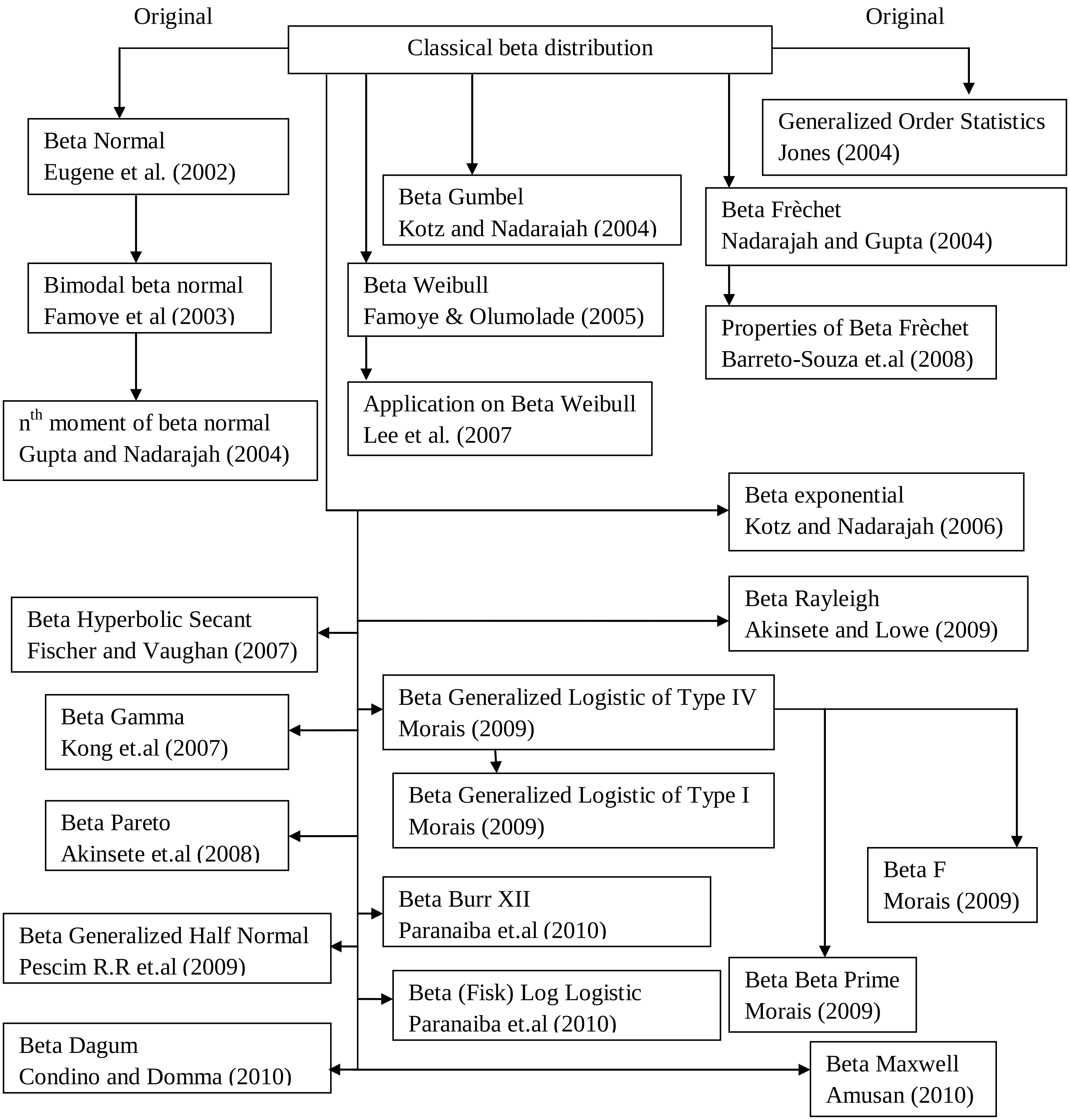


Fig.1. 2: Beta Generated Distributions

v. Based on special functions

Beta function is a special function. Other special functions have been used to generalize the beta distribution.

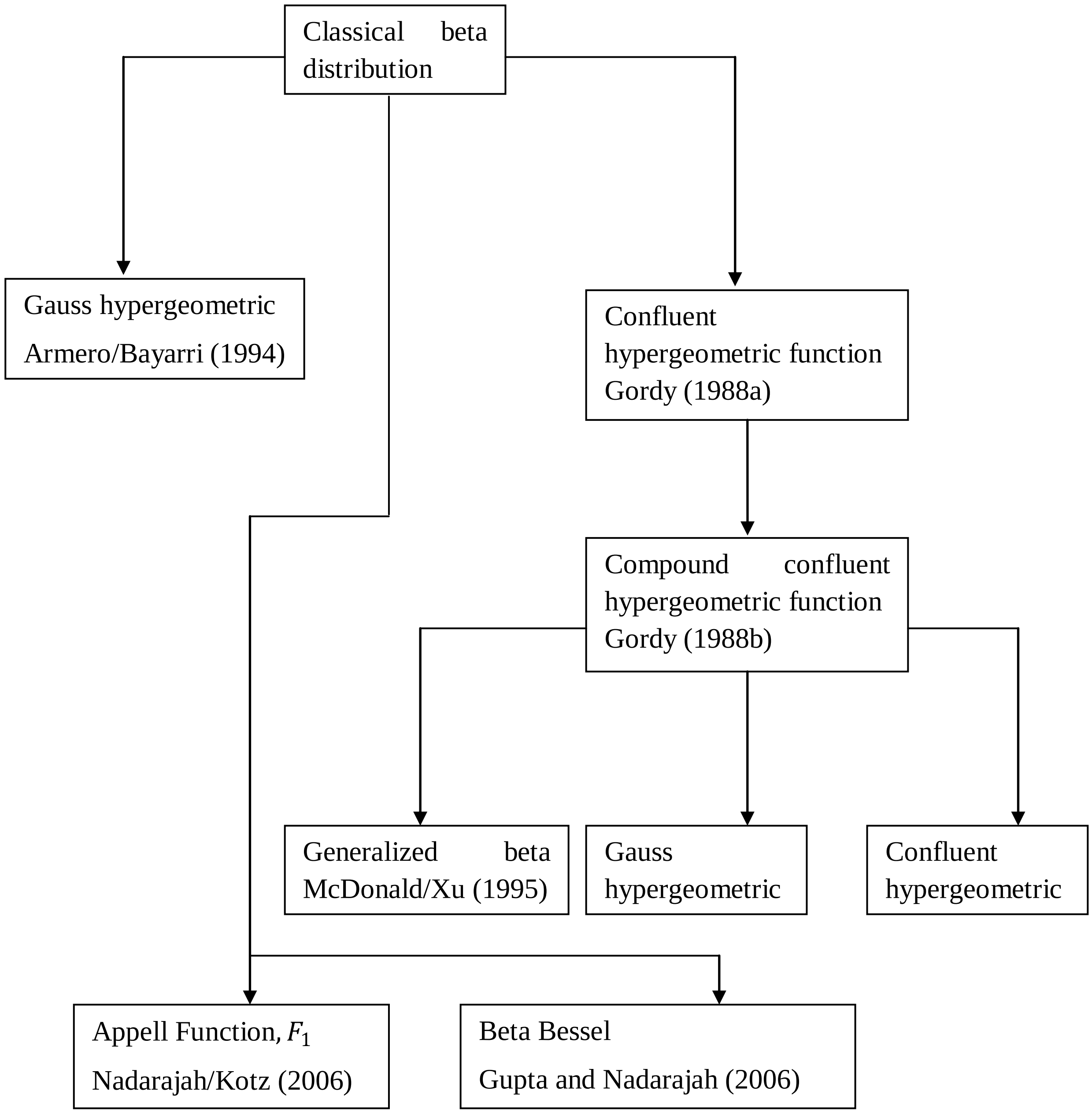
Armero and Bayarri (1994) suggested Gauss hypergeometric function distribution in connection with marginal prior/posterior distribution. They obtained the generalized beta distribution by dividing the classical beta distribution by certain algebraic function. Nadarajah and Kotz (2004) also studied the beta distribution based on the Gauss hypergeometric function.

Gordy (1988a) introduced confluent hypergeometric distribution and applied it to auction theory. He (Gordy (1988b)) also introduced the compound confluent hypergeometric distribution which contains McDonald and Xu's generalized beta, Gauss hypergeometric and confluent hypergeometric as special cases. Gordy generalized the classical beta distribution using the confluent hypergeometric function in the way Armero and Bayarri (1994) used Gauss hypergeometric function to generalize the beta to the Gauss hypergeometric distribution.

Nadarajah and Kotz (2006) introduced F_1 beta distribution based on Appell function of the first kind. The distribution is very flexible and contains several of the known generalization of the classical beta distribution as particular cases.

The figure 1.3 below illustrates the relationship between the two parameters classical beta distribution and some of the special functions.

Fig.1. 3: Generalized Beta Distributions Based on Special Functions



2 Chapter II: The Classical Beta Distribution and its Special Cases

2.1 Beta and Gamma Functions

Beta distributions are based on gamma functions. A brief description of these two functions is therefore in order.

i. Beta Function

The beta function is a special function denoted by $B(a,b)$ and defined as

$$B(a,b) = \int_{0}^{1} x^{a-1}(1-x)^{b-1}dx\,, \qquad a > 0, \qquad b > 0$$

ii. Gamma Function

The gamma function is a special function denoted by Γ and defined as

$$\Gamma(a) = \int_{0}^{\infty} t^{a-1}e^{-t}dt$$

iii. Relationship between the Beta function and the Gamma function

There are two different methods for deriving the relationship between the beta function and the gamma function:

i. The formula of the beta function can be derived as a definite integral whose integrand depends on two variables a and b by reversing the order of integration of a double integral as follows:

$$\Gamma(a)\,\Gamma(b) = \int_{0}^{\infty} t^{a-1}e^{-t}dt \int_{0}^{\infty} u^{b-1}e^{-u}du$$

$$= \int_0^\infty \int_0^\infty t^{a-1} u^{b-1} e^{-(t+u)} dt du$$

Using the transformation $u = tv$ we obtain

$$\Gamma(a)\Gamma(b) = \int_0^\infty \int_0^\infty t^{a-1} (tv)^{b-1} e^{-(t+tv)} t dt dv$$

$$= \int_0^\infty \int_0^\infty t^{a-1} t^b v^{b-1} e^{-(t+tv)} dt dv$$

$$= \int_0^\infty \int_0^\infty t^{a+b-1} v^{b-1} e^{-t(1+v)} dt dv$$

Using the transformation $w = t(v+1)$ again gives

$$= \int_0^\infty \int_0^\infty \left(\frac{w}{v+1}\right)^{a+b-1} v^{b-1} e^{-w} (v+1)^{-1} dw dv$$

$$= \int_0^\infty v^{b-1} dv \int_0^\infty \frac{w^{a+b-1} e^{-w}}{(v+1)^{a+b}} dw$$

$$= \int_0^\infty \frac{v^{b-1}}{(v+1)^{a+b}} dv \int_0^\infty w^{a+b-1} e^{-w} dw$$

$$= \int_0^\infty \frac{v^{b-1}}{(v+1)^{a+b}} dv \, \Gamma(\mathrm{a} + \mathrm{b})$$

From which it follows that

$$B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} = \int_0^\infty \frac{v^{b-1}}{(v+1)^{a+b}} dv$$

Finally, using the transformation $v = \dfrac{x}{1-x}$ we get the alternative formulation

$$B(a,b)=\int_{0}^{\infty}\frac{\left(\frac{x}{1-x}\right)^{b-1}}{\left(\frac{x}{1-x}+1\right)^{a+b}}\frac{1}{(1-x)^2}dx$$

$$=\int_{0}^{\infty}\frac{(\frac{x}{1-x})^{b-1}}{\left(\frac{1}{1-x}\right)^{a+b}}\frac{1}{(1-x)^2}dx$$

$$=\int_{0}^{\infty}x^{b-1}(1-x)^{-b+1+a+b-2}\,dx$$

$$=\int_{0}^{1}x^{b-1}(1-x)^{a-1}\,dx$$

which is the beta function.

ii. Another method for deriving the relationship between beta function and gamma function is by using the polar co-ordinates as shown below.

$$\Gamma(a)\,\Gamma(b)=\left(\int_{0}^{\infty}x^{a-1}e^{-x}dx\right)\left(\int_{0}^{\infty}y^{b-1}e^{-y}dy\right)$$

$$=\int_{0}^{\infty}\int_{0}^{\infty}e^{-(x+y)}x^{a-1}\,y^{b-1}dxdy$$

let $x=\mathrm{s}^2$ and $y=\mathrm{t}^2$

therefore $dx=2sds$ and $dy=2tdt$

now

$$\Gamma(\mathrm{a})\Gamma(b)=\int_{0}^{\infty}\int_{0}^{\infty}e^{-(s^2+\mathrm{t}^2)}\mathrm{s}^{2(\mathrm{a}-1)}\;t^{2(b-1)}.\,2sds2tdt$$

$$=4\int_{0}^{\infty}\int_{0}^{\infty}e^{-(s^2+\mathrm{t}^2)}\mathrm{s}^{2\mathrm{a}-1}\;t^{2b-1}ds\,dt$$

Next, let $s=rsin\theta\;and\;t=rcos\theta$

Therefore $ds=rcos\theta\;d\theta\;and\;dt=-rsin\theta d\theta$

$$J = \begin{vmatrix} \frac{ds}{d\theta} & \frac{dt}{d\theta} \\ \frac{ds}{dr} & \frac{dt}{dr} \end{vmatrix} = \begin{vmatrix} rCos\theta & -rSin\theta \\ Sin\theta & Cos\theta \end{vmatrix} = rCos^2\theta + rSin^2\theta = r$$

$$\therefore \Gamma(a)\Gamma(b) = 4\int_{\mathbf{0}}^{\infty}\int_{\mathbf{0}}^{\pi/2} e^{-r^2}\,(rSin\theta)^{2a-1}(rCos\theta)^{2b-1}\,|J|d\theta dr$$

$$= 4\int_{\mathbf{0}}^{\infty}\int_{\mathbf{0}}^{\pi/2} e^{-r^2}\,r^{2a-1+2b-1}(Sin\theta)^{2a-1}(Cos\theta)^{2b-1}\,rd\theta dr$$

$$= 2\int_{\mathbf{0}}^{\infty}\left[2\int_{\mathbf{0}}^{\pi/2} (Sin\theta)^{2a-1}\,(Cos\theta)^{2b-1}d\theta\right]\,e^{-r^2}r^{2(a+b-1)}\,rdr$$

Recall that $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$

Let $x = Sin^2\theta \implies dx = 2Sin\theta Cos\theta d\theta$

$$\therefore B(a,b) = \int_{\mathbf{0}}^{\frac{\pi}{2}} (Sin\theta)^{2a-2}\,(Cos\theta)^{2b-2}.\,2Sin\theta Cos\theta d\theta$$

$$= 2\int_{\mathbf{0}}^{\frac{\pi}{2}} (Sin\theta)^{2a-1}\,(Cos\theta)^{2b-1}d\theta$$

$$\therefore \Gamma(\mathrm{a})\Gamma(\mathrm{b}) = 2\int_{\mathbf{0}}^{\infty} B(a,b)\,e^{-r^2}r^{2(a+b-1)}\,rdr$$

$$= B(a,b)\int_{\mathbf{0}}^{\infty} e^{-r^2}r^{2(a+b-1)}2r\,dr$$

Let $u = r^2 \implies du = 2rdr$

$$\therefore \Gamma(\mathrm{a})\Gamma(\mathrm{b}) = B(a,b)\int_{\mathbf{0}}^{\infty} e^{-u}u^{a+b-1}\,du$$

$$= B(a,b)\Gamma(a+b)$$

$$\therefore B(a,b) = \frac{\Gamma(\mathrm{a})\Gamma(\mathrm{b})}{\Gamma(a+b)}$$

2.2 Construction of the Classical Beta Distribution

2.2.1 From the beta function

By definition

$$B(a,b) = \int_{0}^{1} x^{a-1}(1-x)^{b-1}dx\,, \qquad a > 0, \qquad b > 0$$

Normalizing the beta function gives

$$1 = \int_{0}^{1} \frac{x^{a-1}(1-x)^{b-1}dx}{B(a,b)}$$

Thus the probability density function of the beta distribution with shape parameters *a* and *b* is expressed as

$$f(x;a,b) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\,, \qquad 0 < x < 1, \qquad a > 0, \qquad b > 0 \text{\_\_\_\_\_\_(2.1)}$$

The cumulative distribution function of equation 2.1 is given by

$$F(x) = \frac{1}{B(a,b)}\int_{0}^{x} t^{a-1}(1-t)^{b-1}\,dt, \qquad 0 < t < 1, \qquad a > 0, \qquad b > 0\text{\_\_\_\_\_\_(2.2)}$$

Where the parameters a and b are positive real quantities and the variable x satisfies $0 \leq x \leq 1$.The quantity $B(a,b)$ is the beta function. Equation (2.1) is also known as the standard beta or classical beta distribution.

2.2.2 From stochastic processes

Consider a Poisson process with arrival rate of λ events per unit time. Let w_k denote the waiting time until the k^{th} arrival of an event and w_s denote the waiting time until the s^{th} arrival, $s > k$. then, w_k and $w_s - w_k$ are independent gamma random variables with $w_k \sim$gamma $(k, \frac{1}{\lambda})$ and $w_s - w_k \sim$ gamma $(\mathrm{s} - \mathrm{k}, 1/\lambda)$.

The proportion of the time taken by the first k arrivals in the time needed for the first arrival is

$$\frac{w_k}{w_s} = \frac{w_k}{w_{k+(w_s - w_k)}} \sim beta\ (k, s-k)$$

In simple terms, a continuous beta distribution $B(a, b)$ can be derived from a Poisson process such that if $a + b$ events occur in a time interval, then the fraction of that interval until the a^{th} event occurs has a $B(a, b)$ distribution.

2.2.3 From ratio transformation of two independent gamma variables

Let X_1 and X_2 be independent gamma variables with joint pdf

$$h(x_1, x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} {x_1}^{\alpha-1} {x_2}^{\beta-1} e^{-x_1 - x_2}, \qquad 0 < x_1 < \infty, \qquad 0 < x_2 < \infty$$

where $\alpha > 0,\ \beta > 0$

Let $Y_1 = X_1 + X_2$ *and* $Y_2 = \frac{X_1}{X_1 + X_2}$

$$y_1 = g_1\ (x_1, x_2) = x_1 + x_2$$
$$y_2 = g_2\ (x_1, x_2) = x_1/(x_1 + x_2)$$

$$x_1 = h_1(y_1, y_2) = y_1 y_2$$
$$x_2 = h_2(y_1, y_2) = y_1(1 - y_2)$$

$$J = \begin{vmatrix} y_1 & y_2 \\ (1 - y_2) & -y_1 \end{vmatrix} = -y_1$$

The transformation is one-to-one and maps $\mathcal{A}$, the first quadrant of the $x_1 x_2$ plane onto

$\mathcal{B} = \{(y_1, y_2)\text{: } 0 < y_1 < 1, 0 < y_2 < 1\}$.

The joint pdf of Y_1, Y_2 is

$$f(y_1, y_2) = y_1/\Gamma(\alpha)\Gamma(\beta)\ (y_1 y_2)^{\alpha-1}[y_1(1-y_2)]^{\beta-1}e^{-y_1}$$
$$= \frac{{y_2}^{\alpha-1}(1-y_2)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} {y_1}^{\alpha+\beta-1}e^{-y_1},\ (y_1, y_2) \in \mathcal{B}.$$

Because $\mathcal{B}$ is a rectangular region and because g (y_1, y_2) can be factored into a function of y_1 and a function of y_2, it follows that Y_1 and Y_2 are statistically independent.

The marginal pdf of Y_2 is

$$f_{Y_2}(y_2) = \frac{{y_2}^{\alpha-1}(1-y_2)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty {y_1}^{\alpha+\beta-1}\, e^{-y_1} dy_1$$
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} {y_2}^{\alpha-1}(1-y_2)^{\beta-1} \_\_\_\_\_\_(2.3)$$

for $0 < y_2 < 1$.

This is the pdf of a beta distribution with parameters α and β.

Also, since $f(y_1, y_2) = f_{Y_1}(y_1)\, f_{Y_2}(y_2)$ it implies that

$$f_{Y_1}(y_1) = \frac{1}{\Gamma(\alpha+\beta)} {y_1}^{\alpha+\beta-1}e^{-y_1}$$

for $0 < y_1 < \infty$.

Thus, Y_1 has a gamma distribution with parameter values $\alpha + \beta$ and 1.

2.2.4 From the r^th order statistic of the Uniform distribution

Let $X_1, \dots, X_n$ be independent and identically distributed random sample from the standard Uniform distribution $\mathcal{U}(0,1)$. Let X_r be the r^th order statistics from this sample. Then the probability distribution of X_r is a beta distribution with parameters r and n-r+1.

2.3 Properties of the Classical Beta Distribution

2.3.1 r^th^-order moments

To obtain the r^{th}-order moments of the classical beta distribution, we evaluate

$$\mathrm{E}(\mathrm{X}^{\mathrm{r}}) = \int_0^1 x^r f(x;a,b)\ dx$$

$$= \int_0^1 x^r\ \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\ dx$$

$$= \frac{1}{\mathrm{B}(p,q)} \int_0^1 x^{a+r-1}(1-x)^{b-1}\ \ dx$$

$$= \frac{1}{B(a,b)} \int_0^1 \frac{x^{(a+r)-1}(1-x)^{b-1}}{B(a+r,b)}\ .B(a+r,b)dx$$

$$= \frac{B(a+r,b)}{B(a,b)}$$

$$= \frac{\Gamma(\mathrm{a}+\mathrm{r})\ \Gamma(\mathrm{b})}{\Gamma(\mathrm{a}+\mathrm{b}+\mathrm{r})} \cdot \frac{\Gamma(\mathrm{a}+\mathrm{b})}{\Gamma(\mathrm{a})\Gamma(\mathrm{b})}$$

$$= \frac{\Gamma(\mathrm{a}+\mathrm{r})\ \Gamma(\mathrm{a}+\mathrm{b})}{\Gamma(\mathrm{a}+\mathrm{b}+\mathrm{r})\ \Gamma(\mathrm{a})} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_(2.4)}$$

2.3.2 The first four moments

The mean is found by substituting $\mathrm{r} = 1$ in equation (2.4)

$$\text{i.e.,} \qquad m_1 \equiv \mu = \mathrm{E}(X) = \frac{\Gamma(\mathrm{a}+1)\ \Gamma(\mathrm{a}+\mathrm{b})}{\Gamma(\mathrm{a}+\mathrm{b}+1)\Gamma(\mathrm{a})}$$

$$= \frac{\mathrm{a}\Gamma(\mathrm{a})\ \Gamma(\mathrm{a}+\mathrm{b})}{(\mathrm{a}+\mathrm{b})\Gamma(\mathrm{a}+\mathrm{b})\ \Gamma(\mathrm{a})}$$

$$= \frac{\mathrm{a}}{\mathrm{a}+\mathrm{b}}\ (Christian\ Walck, 1996) \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(2.5)}$$

$$m_2 \equiv E(X^2) = \frac{\Gamma(a+b)\,\Gamma(a+2)}{\Gamma(a+b+2)\Gamma(a)}$$

$$= \frac{\Gamma(a+b)\;\,(a+1)\Gamma(a+1)}{(a+b+1)\Gamma(a+b+1)\,\Gamma(a)}$$

$$= \frac{\Gamma(a+b)\,(a+1)\,a\Gamma(a}{(a+b+1)(a+b)\Gamma(a+b)\,\Gamma(a)}$$

$$= \frac{(a+1)\,a}{(a+b+1)(a+b)}$$

$$= \frac{a(a+1)}{(a+b)(a+b+1)}$$

$$= m_1 \frac{(a+1)}{(a+b+1)}\ (Catherine\ et.al., 2010)\_\_\_\_\_\_\_\_\_\_(2.6)$$

$$m_3 \equiv E(X^3) = \frac{\Gamma(a+b)\,\Gamma(a+3)}{\Gamma(a+b+3)\,\Gamma(a)}$$

$$= \frac{\Gamma(a+b)\,(a+2)(a+1)a\Gamma(a)}{(a+b)(a+b+2)(a+b+1)\Gamma(a+b)\,\Gamma(a)}$$

$$= \frac{(a+2)(a+1)a}{(a+b+2)(a+b+1)(a+b)}$$

$$= m_2 \frac{(a+2)}{(a+b+2)}\,(Catherine\ et\ al., 2010)\_\_\_\_\_\_\_\_\_\_\_\_\_(2.7)$$

$$m_4 \equiv E(X^4) = \frac{\Gamma(a+b)\,\Gamma(a+4)}{\Gamma(a+b+4)\,\Gamma(a)}$$

$$= \frac{\Gamma(a+b)\,(a+3)(a+2)(a+1)a\Gamma(a)}{(a+b)(a+b+1)(a+b+2)(a+b+3)\Gamma(a+b)\,\Gamma(a)}$$

$$= \frac{(a+3)(a+2)(a+1)a}{(a+b+3)(a+b+2)(a+b+1)(a+b)}$$

$$= \frac{\mathrm{a(a+1)\,(a+2)(a+3)}}{\mathrm{(a+b)(a+b+1)(a+b+2)(a+b+3)}}$$

$$= m_3 \frac{\mathrm{(a+3)}}{\mathrm{(a+b+3)}}$$

2.3.3 Mode

Mode is obtained by solving

$$\frac{\mathrm{d}}{\mathrm{dx}} f(x; a, b) = 0$$

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{y^{a-1}(1-x)^{b-1}}{B(a,b)}\right) = 0$$

$$\frac{(a-1)x^{a-2}(1-x)^{b-1} + (b-1)(-1)(1-x)^{b-2}x^{a-1}}{B(a,b)} = 0$$

$$(a-1)x^{a-2}(1-x)^{b-1} = (b-1)(1-x)^{b-2}x^{a-1}$$

$$(a-1)x^{a-1}x^{-1}(1-x)^{b-1} = (b-1)(1-x)^{b-1}(1-x)^{-1}x^{a-1}$$

$$(a-1)x^{-1} = (b-1)(1-x)^{-1}$$

$$\frac{a-1}{b-1} = \frac{x}{1-x}, \qquad a > 1, \qquad b > 1$$

$$(1-x)(a-1) = x(b-1)$$

$$(a - 1 - xa + x) = xb - x$$

$$x(-a + 1 - b + 1) = 1 - a$$

$$x = \frac{(a-1)}{(a+b-2)}, \qquad x = 0 \; if \; a = 1 \; and \; x = 1 \; if \; b = 1 \_\_\_\_\_\_\_\_\_\_ (2.8)$$

2.3.4 Variance

Since variance can be expressed as

$$\mathrm{Var(X)} = E(X^2) - \{E(X)\}^2$$

$$= \mathrm{m}_2 - m_1^2$$

We can now get the variance as

$$\mathrm{Var(X)} = \sigma^2 \ = \frac{\mathrm{a(a+1)}}{\mathrm{(a+b)(a+b+1)}} - \left\{\frac{\mathrm{a}}{\mathrm{a+b}}\right\}^2$$

$$= \frac{\mathrm{a(a+1)(a+b)^2 - a^2(a+b)(a+b+1)}}{\mathrm{(a+b)(a+b+1)(a+b)^2}}$$

$$= \frac{\mathrm{a(a+1)(a+b) - a^2(a+b)(a+b+1)}}{\mathrm{(a+b+1)(a+b)^2}}$$

$$= \frac{\mathrm{a^3 + a^2b + a^2 + ab - a^3 - a^2b + a^2}}{\mathrm{(a+b+1)(a+b)^2}}$$

$$= \frac{\mathrm{ab}}{\mathrm{(a+b+1)(a+b)^2}} \, (Christian\, Walck, 1996) \_\_\_\_\_\_\_\_\_\_ (2.9)$$

2.3.5 Skewness

Skewness (the third central moment) can be expressed as

$$\mathrm{Skewness} = u_3 = E(X - E(X))^3$$

$$= E(X^3) - 3E(X^2)E(X) + 2\{E(X)\}^3$$

$$= \mathrm{m}_3 - 3m_2m_1 + 2m_1^3,$$

This can be manipulated algebraically as follows:

$$u_3 \ = \frac{\mathrm{a(a+1)(a+2)}}{\mathrm{(a+b)(a+b+1)(a+b+2)}} - 3\frac{\mathrm{a^2(a+1)}}{\mathrm{(a+b)^2(a+b+1)}} + 2\left\{\frac{\mathrm{a}}{\mathrm{a+b}}\right\}^3$$

$$= \frac{a(a+1)(a+2) - a^2(a+1)(a+b+2)}{(a+b)^2(a+b+1)(a+b+2)} + \frac{2a^3}{(a+b)^3}$$

$$= \frac{(a^3+3a^2+2a)(a+b) - 3a^4 - 3a^3b - 9a^3 - 3a^2b - 6a^2}{(a+b)^2(a+b+1)(a+b+2)} + \frac{2a^3}{(a+b)^3}$$

$$= \frac{a^4+3a^3+2a^2+a^3b+3a^2b+2ab-3a^4-3a^3b-9a^3-3a^2b-6a^2}{(a+b)^2(a+b+1)(a+b+2)}$$

$$+ \frac{2a^3}{(a+b)^3}$$

$$= \frac{-2a^4-6a^3-4a^2-2a^3b+2ab}{(a+b)^2(a+b+1)(a+b+2)} + \frac{2a^3}{(a+b)^3}$$

$$= \frac{(a+b)(-2a^4-6a^3-4a^2-2a^3b+2ab) + (2a^4+2a^3b+2a^3)(a+b+2)}{(a+b)^3(a+b+1)(a+b+2)}$$

$$= \frac{-2a^5-6a^4-2a^4b-4a^3+2a^2b-2a^4b-6a^3b-2a^3b^2-4a^2b+2ab^2+2a^5+2a^4b}{(a+b)^3(a+b+1)(a+b+2)}$$

$$\frac{+4a^4+2a^4b+2a^3b^2+4a^3b+2a^4+2a^3b+4a^3}{(a+b)^3(a+b+1)(a+b+2)}$$

$$= \frac{2ab^2-2a^2b}{(a+b)^3(a+b+1)(a+b+2)}$$

$$= \frac{2ab(b-a)}{(a+b)^3(a+b+1)(a+b+2)} \,(Christian\ Walck, 1996)\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(2.10)$$

The standardized skewness can be expressed as

$$E\left(\frac{X-E(X)}{\sigma}\right)^3$$

$$= \frac{1}{\sigma^3}\left(\frac{2ab(b-a)}{(a+b)^3(a+b+1)(a+b+2)}\right)$$

$$= \frac{1}{\left(\sqrt{\frac{ab}{(a+b+1)(a+b)^2}}\right)^3}\left(\frac{2ab(b-a)}{(a+b)^3(a+b+1)(a+b+2)}\right)$$

$$= \frac{(a+b+1)(a+b)^2}{ab}.\frac{\sqrt{(a+b+1)}(a+b)}{\sqrt{ab}}\left(\frac{2ab(b-a)}{(a+b)^3(a+b+1)(a+b+2)}\right)$$

$$= \frac{\sqrt{(a+b+1)}}{\sqrt{ab}}\left(\frac{2(b-a)}{(a+b+2)}\right)$$

$$= \frac{2(b-a)}{a+b+2}\sqrt{\frac{a+b+1}{ab}} \text{\_\_\_\_\_\_\_\_\_\_\_\_}(2.11)$$

2.3.6 Kurtosis

Kurtosis can be expressed as

$$\text{Kurtosis} = u_4 = E(X^4) - 4E(X)E(X^3) + 6\{E(X)\}^2E(X^2) - 3\{E(X)\}^4$$

$$= m_4 - 4m_1m_3 + 6{m_1}^2m_2 - 3m_1^4,$$

$$= \frac{a(a+1)(a+2)(a+3)}{(a+b)(a+b+1)(a+b+2)(a+b+3)}$$
$$-4\frac{a \quad a(a+1)(a+2)}{(a+b)(a+b)(a+b+1)(a+b+2)}$$
$$+6\left\{\frac{a}{a+b}\right\}^2\frac{a(a+1)}{(a+b)(a+b+1)} - 3\left\{\frac{a}{a+b}\right\}^4$$

$$= \frac{(a^2+a)(a^2+5a+6)}{(a+b)(a+b+1)(a+b+2)(a+b+3)}$$
$$-4\frac{a^2(a^2+3a+2)}{(a+b)^2(a+b+1)(a+b+2)}$$

$$+\frac{6a^4+6a^3}{(a+b)^3(a+b+1)}-3\left\{\frac{a}{a+b}\right\}^4$$

$$=\frac{(a+b)(a^4+5a^3+6a^2+a^3+5a^2+6a)-(4a^4+12a^3+8a^2)(a+b+3)}{(a+b)^2(a+b+1)(a+b+2)(a+b+3)}$$

$$+\frac{6(a+b)(a^4+a^3)-3a^4(a+b+1)}{(a+b)^4(a+b+1)}$$

$$=\frac{\begin{matrix}(a^5+5a^4+6a^3+a^4+5a^3+6a^2+a^4b+5a^3b+6a^2b+a^3b+5a^2b+6ab)-\\(4a^5+4a^4b+12a^4+12a^4+12a^3b+36a^3+8a^3+8a^2b+24a^2)\end{matrix}}{(a+b)^2(a+b+1)(a+b+2)(a+b+3)}$$

$$+\frac{(6a^5+6a^4+6a^4+6a^3b)-3a^5-3a^4b-3a^4)}{(a+b)^4(a+b+1)}$$

$$=\frac{-3a^5-18a^4-33a^3-18a^2-3a^4b-6a^3b-6a^3b+3a^2b+6ab}{(a+b)^2(a+b+1)(a+b+2)(a+b+3)}$$

$$+\frac{3a^5+3a^4+3a^4q+6a^3b}{(a+b)^4(a+b+1)}$$

$$=\frac{(a^2+2ab+b^2)(-3a^5-18a^4-33a^3-18a^2-3a^4b-6a^3b-6a^3b+3a^2b+6ab)}{(a+b)^4(a+b+1)(a+b+2)(a+b+3)}$$

$$+\frac{(a^2+b^2+2ab+5a+5b+6)(3a^5+3a^4+3a^4b+6a^3b)}{(a+b)^4(a+b+1)(a+b+2)(a+b+3)}$$

$$=\frac{\begin{pmatrix}-3a^7-18a^6-33a^5-18a^4-3a^6b-6a^5b+3a^4b-6a^6b+36a^5b-66a^4b\\-36a^3b-6a^5b^2\end{pmatrix}}{(a+b)^4(a+b+1)(a+b+2)(a+b+3)}$$

$$+\frac{\begin{pmatrix}-12a^4b^2+6a^3b^2+12a^2b^2-3a^5b^2-18a^4b^2-33a^3a^2-18a^2b^2-3a^4b^3\\-6a^3b^3+3a^2b^3\end{pmatrix}}{(a+b)^4(a+b+1)(a+b+2)(a+b+3)}$$

$$+\frac{\begin{pmatrix}6ab^3+3a^7+3a^6+3a^6b+6a^5b+3a^5b^2+3a^4b^2+3a^4b^3+6a^3b^3+6a^6b\\+6a^5b+6a^5b^2\end{pmatrix}}{(a+b)^4(a+b+1)(a+b+2)(a+b+3)}$$

$$+\frac{\left(\begin{array}{c}12a^4b^2+15a^6+15a^5+15a^5b+30a^4b+15a^5b+15a^4b^2+30a^3b^2+18a^5\\+18a^4+18a^4b+36a^3b\end{array}\right)}{(a+b)^4(a+b+1)(a+b+2)(a+b+3)}$$

$$=\frac{6a^3b+3a^3b^2-6a^2b^2+3a^2b^3+6ab^3}{(a+b)^4(a+b+1)(a+b+2)(a+b+3)}$$

$$=\frac{3ab[2a^2+a^2b-2ab+ab^2+2b^2]}{(a+b)^4(a+b+1)(a+b+2)(a+b+3)}$$

$$=\frac{3ab[a^2(b+2)-2ab+b^2(a+2)]}{(a+b)^4(a+b+1)(a+b+2)(a+b+3)}(Catherine\ et\ al.,2010)\_\_\_\_\_\_\_\_\_\_(2.12)$$

The standardized kurtosis can be expressed as

$$E\left(\frac{X-E(X)}{\sigma}\right)^4$$

$$=\frac{1}{\sigma^4}\left(\frac{3ab[a^2(b+2)-2ab+b^2(a+2)]}{(a+b)^4(a+b+1)(a+b+2)(a+b+3)}\right)$$

$$=\frac{1}{\left(\frac{ab}{(a+b+1)(a+b)^2}\right)^2}\left(\frac{3ab[a^2(b+2)-2ab+b^2(a+2)]}{(a+b)^4(a+b+1)(a+b+2)(a+b+3)}\right)$$

$$=\frac{(a+b+1)^2(a+b)^4}{(ab)^2}\left(\frac{3ab[a^2(b+2)-2ab+b^2(a+2)]}{(a+b)^4(a+b+1)(a+b+2)(a+b+3)}\right)$$

$$=\frac{(a+b+1)}{ab}\left(\frac{3[a^2(b+2)-2ab+b^2(a+2)]}{(a+b+2)(a+b+3)}\right)\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(2.13)$$

The standardized skewness and kurtosis are often denoted by $\left(\sqrt{\beta_1},\beta_2\right)$ in literature.

2.4 Shapes of Classical Beta Distribution

The shape of the distribution is governed by the two parameters a and b, and can be summarized as follows;

i. When a=b, the distribution is symmetrical about ½, the case of arcsine distribution, uniform distribution and parabolic distribution, and the skewness is zero, as a and b become larger, the distribution becomes more peaked and the variance decreases

ii. When $a \neq b$, the distribution is skewed with the degree of skewness increasing as a and b become more unequal. In particular, if $b > a$, then the skewness, $u_3 > 0$ and the distribution is skewed to the right as in figure 2.2 and if $a > b$, then the distribution is skewed to the left as in figure 2.1.

iii. When a=b=1, the distribution becomes uniform distribution and the density function has a constant value 1 over the interval (0,1) for the random variable X.

iv. Figures 2.2 and 2.3 below shows two cases of the beta unimodal distributions with $a > 0$ and $b > 0$. If a and/or b is less than one i.e $(a-1)(b-1) < 1$ $f(0) \to \infty$ and /or $f(1) \to \infty$ and the distribution is J- shaped. Figure 2.4 and 2.5 below show the beta distribution of two cases a=10, b=1 and a=1, b=10 respectively.

v. The Classical beta distribution is U shaped if a $<$ 1, and b $<$ 1 as shown in figure 2.1 below.

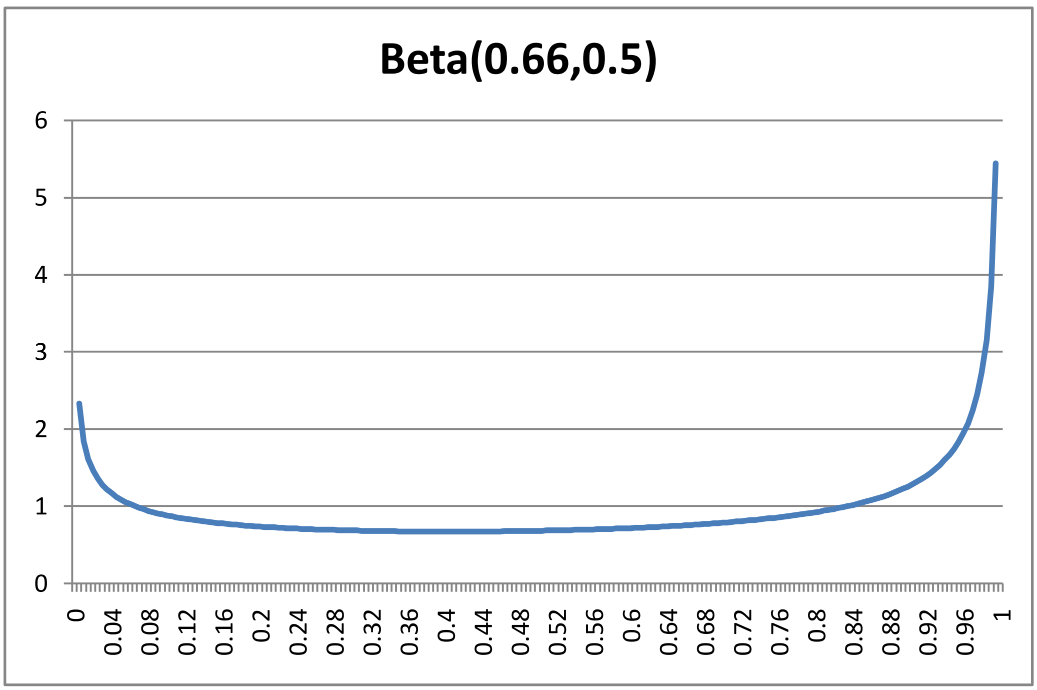


Fig.2. 1 : Left Skewedness

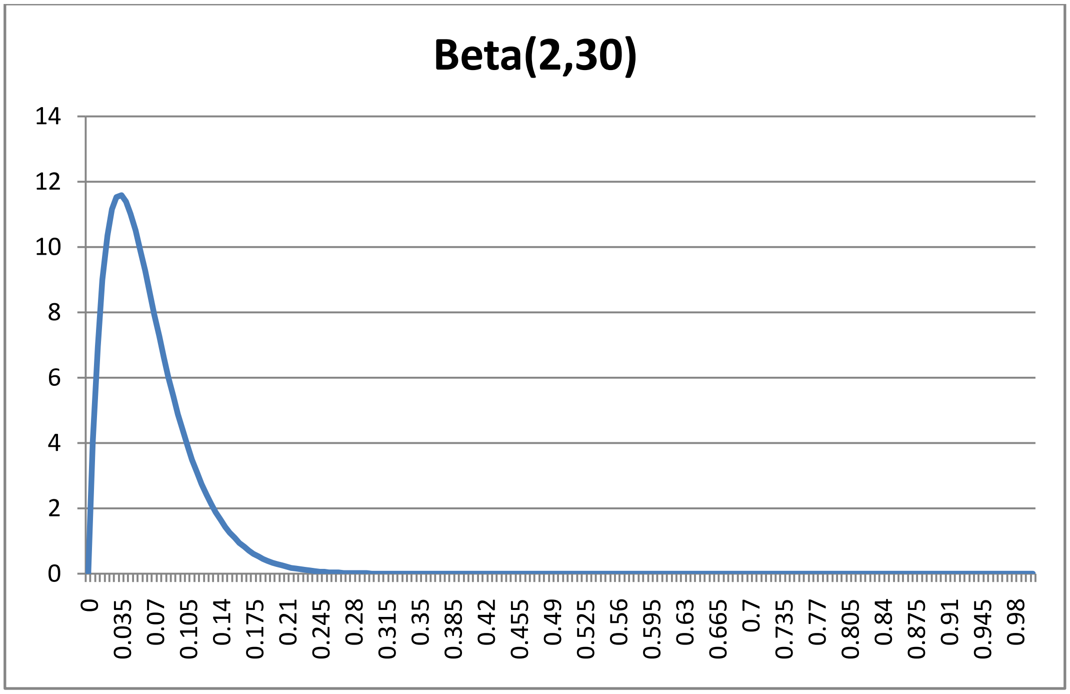


Fig.2. 2: Case of beta unimodal distribution with $a > 0$

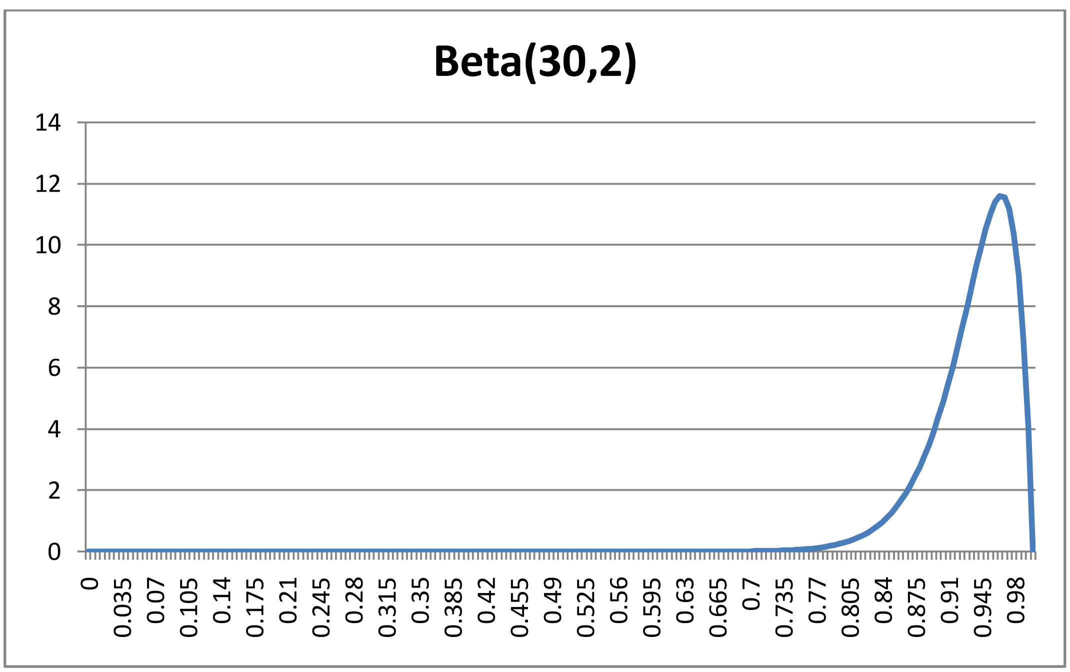


Fig.2. 3: Case of beta unimodal distribution with $b > 0$

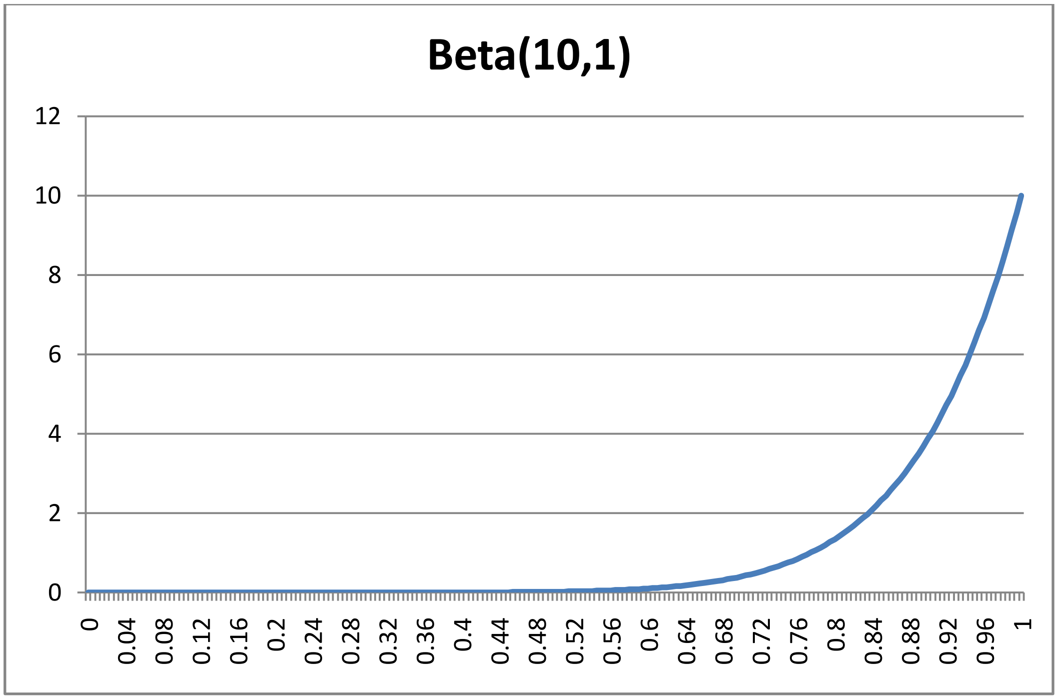


Fig.2. 4: *j*-Shaped beta distribution

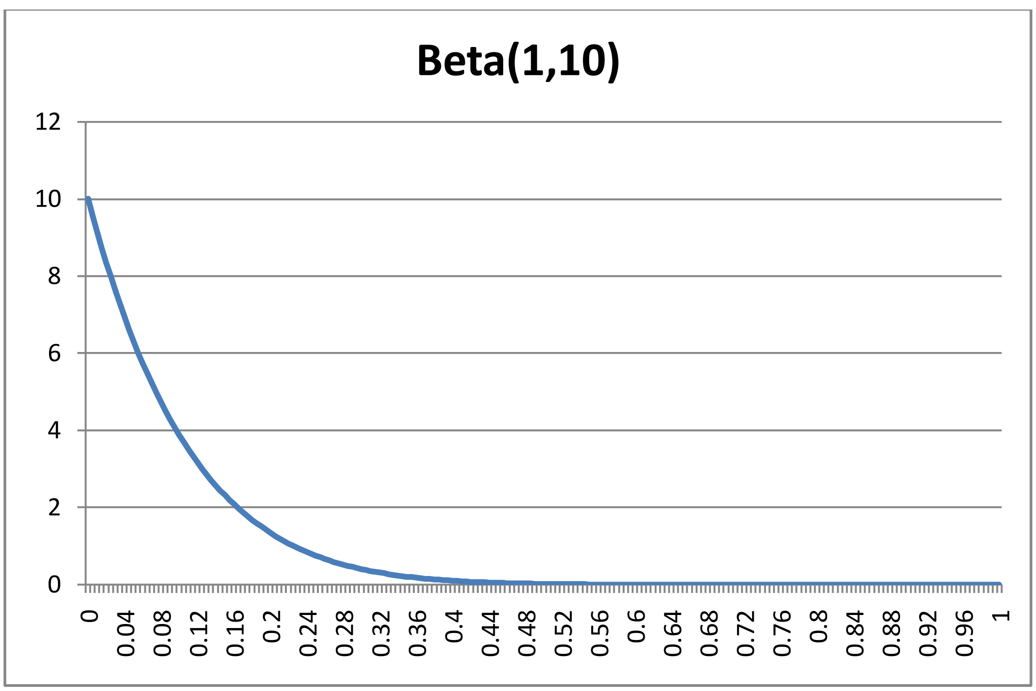


Fig.2. 5: inverted *j*-Shaped beta distribution

2.5 Applications of the Classical Beta Distribution

Income and Wealth

Thurow (1970) applied the Classical beta distribution to US Census Bureau data of income distribution for households (families and unrelated individuals) for every year from 1949 – 1966, stratified by race. He also studied the impact of various macroeconomic factors on the parameters of the distribution via regression techniques. McDonald and Ransom (1979a) employed the Classical beta distribution for approximating US family incomes for 1960 and 1969 through 1975. When utilizing three different estimators, it turns out that the distribution is preferable to the gamma and lognormal distributions but inferior to Singh Maddala. McDonald (1984) estimated beta distribution of the first kind for 1970, 1975 and 1980 US family incomes.

2.6 Special Cases of the Classical Beta Distribution

2.6.1 Power Distribution Function

This distribution is obtained when $b = 1$ in the formula of the classical beta distribution. Thus, from equation (2.1) the pdf becomes

$$f(x;a) = \frac{x^{a-1}}{B(a,1)}$$

$$= \frac{x^{a-1}}{\Gamma(a)\Gamma(1)}.\Gamma(a+1)$$

$$= \frac{x^{a-1}}{\Gamma(a)\Gamma(1)}.a\Gamma(a)$$

$$= ax^{a-1}, \qquad a > 0 \quad \_\_\_\_\_\_(2.14)$$

and the CDF is given by

$$F(x) = \int_0^{\infty} f(x)\,dx$$

$$= \int_0^\infty a x^{a-1} \ \ dx$$

$$= a \frac{x^{a-1+1}}{a}$$

$$= x^a \qquad , \qquad 0 < a < 1$$

Equation (2.14) is the power distribution.

Generally letting $x = y/\alpha$, with b=1 in equation (2.1) yield the general power distribution given by

$$f(y) = \frac{a y^{a-1}}{\alpha^a}, 0 < a < \alpha$$

(Leemis and McQueston, 2008)

2.6.2 Properties of power distribution

i. r^th-order moments

Putting b=1 in equation (2.4)

$$E(X^r) = \frac{\Gamma(a+r)\Gamma(a+1)}{\Gamma(\mathrm{a}+1+\mathrm{r})\Gamma(\mathrm{a})}$$

$$= \frac{\Gamma(a+r)a\Gamma(a)}{(\mathrm{a}+\mathrm{r})\Gamma(\mathrm{a}+\mathrm{r})\Gamma(\mathrm{a})}$$

$$= \frac{a}{\mathrm{a}+\mathrm{r}}$$

ii. Mean

$$E(X) = \frac{a}{\mathrm{a}+1}$$

iii. Mode

$$Mode = \frac{a-1}{\mathrm{a}+1-2}$$

$$= \frac{a-1}{\mathrm{a}-1} = 1$$

iv. **Variance**

$$Var(X) = \frac{a}{(\mathrm{a}+2)(\mathrm{a}+1)^2}$$

v. **Skewness**

$$Skewness = \frac{2a(1-a)}{(\mathrm{a}+1)^3(\mathrm{a}+2)(\mathrm{a}+3)}$$

While the standardized skewness is

$$\frac{2a(1-a)}{(\mathrm{a}+3)}\sqrt{\frac{(\mathrm{a}+2)}{\mathrm{a}}}$$

vi. **Kurtosis**

$$Kurtosis = \frac{3a[a^2(1+2)-2a+(a+2)]}{(\mathrm{a}+1)^4(\mathrm{a}+1+1)(\mathrm{a}+1+2)(\mathrm{a}+1+3)}$$

$$= \frac{3a[3a^2-a+2]}{(\mathrm{a}+1)^4(\mathrm{a}+2)(\mathrm{a}+3)(\mathrm{a}+4)}$$

and the standardized kurtosis is expressed as

$$\frac{a+2}{a}\left(\frac{3[3a^2-a+2]}{(\mathrm{a}+3)(\mathrm{a}+4)}\right)$$

2.6.3 Shape of Power distribution

The figure 2.6 below shows a graph of a power distribution with the parameter $a = 0.05$. This is an example of Power distribution graph, being used to demonstrate ranking of popularity. To the right is the long tail, and to the left are the few that dominate. Varying a gives different shapes of the power distribution. Special cases of $a = 1$ (Uniform distribution), a =2 (triangular distribution), $a < 1$ gives inverted J-shaped distributions and $a > 2$ (J-shaped distributions).

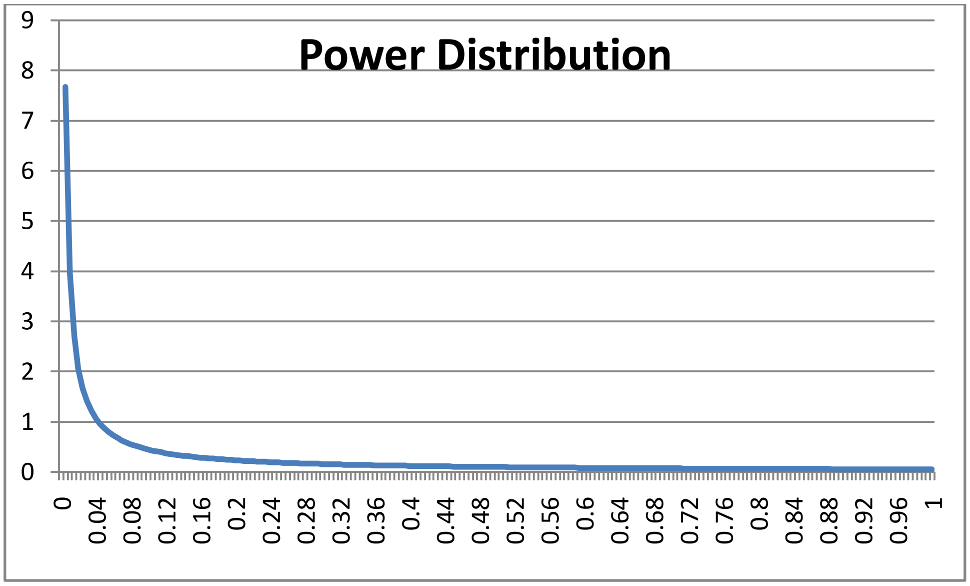


Fig.2. 6: Power distribution

2.6.4 Uniform Distribution Function

Special case when a=b=1 in equation (2.1)

$$f(x) = 1 \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (2.15)$$

Equation (2.15) is a Uniform distribution on the interval [0,1]. The Uniform distribution function is a special case of the power distribution function when a=1.

2.6.5 Properties of the Uniform distribution

i. r^{th}-order moments

Putting $a = 1$ and $b = 1$ in equation 2.4

$$E(x^r) = \frac{\Gamma(1+r)\Gamma(1+1)}{\Gamma(1+1+\mathrm{r})\Gamma(1)}$$

$$= \frac{\mathrm{r}\Gamma(r).1}{(1+\mathrm{r})(\mathrm{r})\Gamma(\mathrm{r}).1}$$

$$= \frac{1}{1+\mathrm{r}}$$

ii. **Mean**

$$\text{mean} = \frac{a}{a+b},$$

Letting a=b=1,

$$= \frac{1}{1+1} = 0.5$$

iii. **Mode**

Mode is any value in the interval [0,1]

iv. **Variance**

From the variance of the standard beta distribution given by

$$\text{Variance} = \frac{ab}{(a+b+1)(a+b)^2}$$

a=b=1,

$$= \frac{1}{(3)(2)^2}$$

$$= \frac{1}{12} = 0.0833$$

v. **Skewness**

$$\text{Skewness} = \frac{2ab(\mathrm{b}-\mathrm{a})}{(\mathrm{a}+\mathrm{b})^3(\mathrm{a}+\mathrm{b}+1)(\mathrm{a}+\mathrm{b}+2)} = 0$$

$$\text{Standardized skewness} = \frac{2(1-1)}{(1+3)}\sqrt{\frac{(1+2)}{1}} = 0$$

vi. **Kurtosis**

$$\text{kurtosis} = \frac{3ab[\mathrm{a}^2(\mathrm{b}+2) - 2\mathrm{ab} + \mathrm{b}^2(\mathrm{a}+2)]}{(\mathrm{a}+\mathrm{b})^4(\mathrm{a}+\mathrm{b}+1)(\mathrm{a}+\mathrm{b}+2)(\mathrm{a}+\mathrm{b}+3)}$$

$$= \frac{3[(3) - 2 + (3)]}{(2)^4(3)(4)(5)}$$

$$= \frac{1}{16(5)} = 0.0125$$

The standardized kurtosis is given by

$$\frac{a+2}{a}\left(\frac{3[3a^2 - a + 2]}{(a+3)(a+4)}\right)$$

$$= \frac{1+2}{1}\left(\frac{3[3 - 1 + 2]}{(1+3)(1+4)}\right)$$

$$= 3\left(\frac{3[4]}{(4)(5)}\right) = 1.8$$

2.6.6 Shape of Uniform distribution

Figure 2.7 below shows a graphical representation of the Uniform distribution on the interval (0,1)

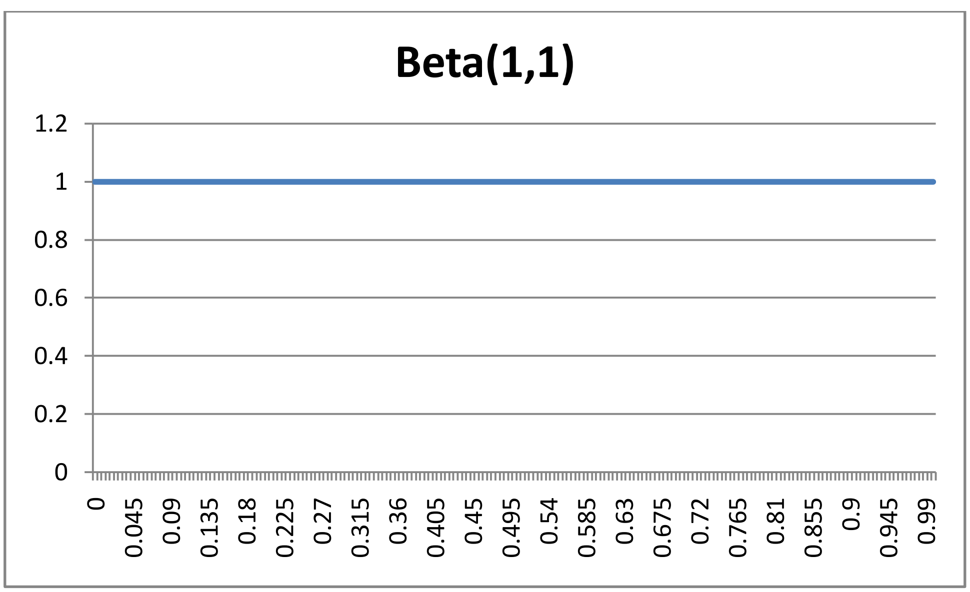


Fig.2. 7: Uniform distribution on the interval (0,1)

2.6.7 Arcsine Distribution Function

Special case of the standard beta distribution is the arcsine distribution, when a=b=1/2 in equation (2.1). In this special case, we have

$$f\left(x;\frac{1}{2},\frac{1}{2}\right)=\frac{x^{1/2-1}(1-\mathrm{x})^{1/2-1}}{B\left(\frac{1}{2},\frac{1}{2}\right)}, \qquad 0<x<1$$

$$f(x)=\frac{x^{-1/2}(1-\mathrm{x})^{-1/2}}{B\left(\frac{1}{2},\frac{1}{2}\right)}, \qquad 0<x<1$$

$$f(x)=\frac{1}{\pi\sqrt{x(1-x)}}, \qquad 0<x<1 \text{ \_\_\_\_\_\_(2.16)}$$

and the cumulative distribution function is given by

$$\mathrm{F(x)}=\frac{2}{\pi}\arcsin(\sqrt{x}).$$

Note: $\Gamma(1/2) = \sqrt{\pi}$

Equation (2.16) is an Arcsine distribution

2.6.8 Properties of the arcsine distribution

i. r^{th}-order moments

Putting $a=\frac{1}{2}$ and $b=\frac{1}{2}$ in equation 2.4

$$E(X^r)=\frac{\Gamma\left(\frac{1}{2}+r\right)\Gamma(1)}{\Gamma(1+\mathrm{r})\Gamma(1)}$$

$$=\frac{\Gamma\left(\frac{1}{2}+r\right)}{\mathrm{r}\Gamma(\mathrm{r})\Gamma\left(\frac{1}{2}\right)}$$

ii. Mean

From the mean of the beta distribution given by

$$\text{mean} = \frac{a}{a+b},$$

$\mathrm{a} = \mathrm{b} = \frac{1}{2}$

$$= \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}} = \frac{1}{2} = 0.5$$

iii. Mode

$$\text{mode} = \frac{a-1}{a+b-2} = \frac{\frac{1}{2}-1}{\frac{1}{2}+\frac{1}{2}-2}$$

$$= \frac{\frac{1}{2}-1}{\frac{1}{2}+\frac{1}{2}-2}$$

$$= \frac{1}{2}$$

iv. Variance

From the variance of the beta distribution given by

$$\text{Variance} = \frac{ab}{(a+b+1)(a+b)^2}$$

$$\mathrm{a} = \mathrm{b} = \frac{1}{2}$$

$$= \frac{\frac{1}{4}}{(2)(1)^2} = 0.125$$

v. Skewness

$$\text{Skewness} = \frac{2ab(\text{b} - \text{a})}{(\text{a} + \text{b})^3(\text{a} + \text{b} + 1)(\text{a} + \text{b} + 2)} = 0$$

$$\text{Standardized skewness} = \frac{2(\text{b} - \text{a})}{(\text{a} + \text{b} + 2)}\sqrt{\frac{a + b + 1}{ab}}$$

$$= \frac{2(\frac{1}{2} - \frac{1}{2})}{(1 + 2)}\sqrt{\frac{1 + 1}{\frac{1}{4}}} = 0$$

vi. Kurtosis

$$\text{kurtosis} = \frac{3ab[\text{a}^2(\text{b} + 2) - 2\text{ab} + \text{b}^2(\text{a} + 2)]}{(\text{a} + \text{b})^4(\text{a} + \text{b} + 1)(\text{a} + \text{b} + 2)(\text{a} + \text{b} + 3)}$$

$$= \frac{3\left(\frac{1}{4}\right)[\frac{1}{4}(\frac{1}{2} + 2) - 2(\frac{1}{4}) + \frac{1}{4}(\frac{1}{2} + 2)]}{(1)^4(1 + 1)(1 + 2)(1 + 3)}$$

$$= \frac{\left(\frac{3}{4}\right)[(\frac{5}{8}) - (\frac{4}{8}) + (\frac{5}{8})]}{(2)(3)(4)}$$

$$= \frac{\frac{18}{32}}{(24)} = 0.0234$$

$$\text{Standardized kurtosis} = \frac{(1 + 1)}{\frac{1}{4}}\left(\frac{3\left(\frac{1}{4}\right)\left[\left(\frac{1}{2} + 2\right) - 2\left(\frac{1}{4}\right) + \frac{1}{4}\left(\frac{1}{2} + 2\right)\right]}{(1 + 2)(1 + 3)}\right) = 1.5$$

2.6.9 Shape of Arcsine distribution

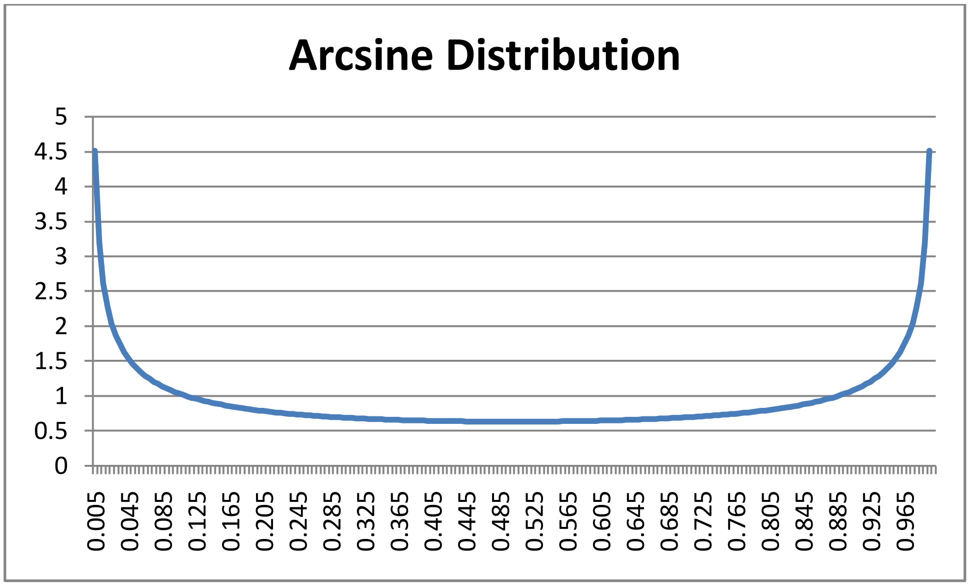


Fig.2. 8: Arcsine distribution

2.6.10 Triangular shaped distributions (a=1, b=2)

For $a=1$ and $b=2$ we get triangular shaped distribution as follows

$$f(x;1,2) = \frac{(1-x)}{B(1,2)}$$

$$= 2 - 2x \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (2.17)$$

With the cdf given as

$$F(x) = 2x - x^2$$

2.6.11 Properties of the Triangular shaped distribution (a=1, b=2)

i. r^{th}-order moments

Putting $a = 1$ and $b = 2$ in equation 2.4

$$E(X^r) = \frac{\Gamma(1+r)\Gamma(3)}{\Gamma(3+r)\Gamma(1)}$$

$$= \frac{2r\Gamma(r)}{\Gamma(r+3)}$$

ii. Mean

From the mean of the standard beta distribution given by

$$\text{mean} = \frac{a}{a+b},$$

$\mathrm{a} = 1, \mathrm{b} = 2$

$$= \frac{1}{3} = 0.3333$$

iii. Mode

$$\text{mode} = \frac{a-1}{a+b-2} = \frac{1-1}{1+2-2} = 0$$

iv. Variance

From the variance of the standard beta distribution given by

$$\text{Variance} = \frac{ab}{(a+b+1)(a+b)^2}$$

$\mathrm{a} = 1, \mathrm{b} = 2$

$$= \frac{2}{(4)(3)^2}$$

$$= \frac{1}{18} = 0.0556$$

Standard deviation, $\sigma = \sqrt{var} = 0.2358$

$$CV = \frac{\sigma}{\mu} = \frac{0.2358}{0.3333} = 0.7074$$

v. Skewness

$$\text{Skewness} = \frac{2ab(\mathrm{b}-\mathrm{a})}{(\mathrm{a}+\mathrm{b})^3(\mathrm{a}+\mathrm{b}+1)(\mathrm{a}+\mathrm{b}+2)}$$

$$= \frac{2(2)(1)}{(3)^3(4)(5)}$$

$$= \frac{1}{(3)^3(5)} = 0.0074$$

$$\text{Standardized skewness} = \frac{2(\mathrm{b}-\mathrm{a})}{(\mathrm{a}+\mathrm{b}+2)}\sqrt{\frac{a+b+1}{ab}}$$

$\mathrm{a} = 1, \mathrm{b} = 2$

$$= \frac{2}{5}\sqrt{2} = 0.5657$$

vi. Kurtosis

$$\text{kurtosis} = \frac{3ab[\mathrm{a}^2(\mathrm{b}+2) - 2\mathrm{ab} + \mathrm{b}^2(\mathrm{a}+2)]}{(\mathrm{a}+\mathrm{b})^4(\mathrm{a}+\mathrm{b}+1)(\mathrm{a}+\mathrm{b}+2)(\mathrm{a}+\mathrm{b}+3)}$$

$$= \frac{3(2)[1(4) - 4 + 4(3)]}{(3)^4(4)(5)(6)}$$

$$= \frac{1}{(3)^3(5)} = 0.0074$$

$$\text{Standardized kurtosis} = \frac{(3+1)}{2}\left(\frac{3(1)[(2+2) - 2(1)(2) + 4(1+2)]}{(3+2)(3+3)}\right) = 2.4$$

2.6.12 Shape of triangular distribution (a=1, b=2)

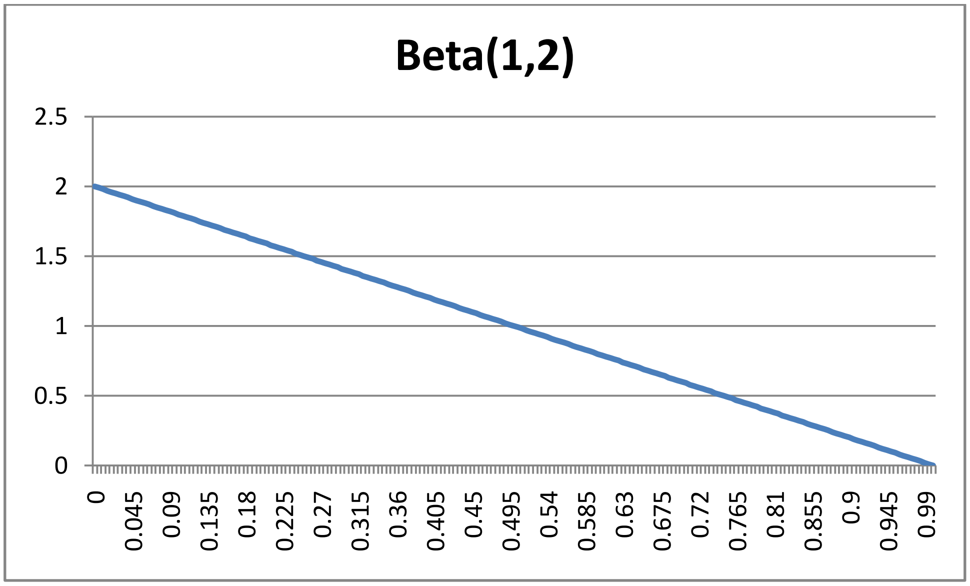


Fig.2. 9: Triangular shaped distribution

2.6.13 Triangular shaped distributions (a=2, b=1)

For *a=2* and *b=1* we get triangular shaped distribution as follows

$$f(x;2{,}1) = \frac{x}{B(2{,}1)}$$

$$= 2x \qquad (2.18)$$

The CDF of equation 2.18 is given by

$$F(x) = x^2$$

2.6.14 Properties of the Triangular shaped distribution (a=2, b=1)

i. r^{th} - order moments

Putting $a = 2$ and $b = 1$ in equation 2.4

$$E(X^r) = \frac{\Gamma(2+r)\Gamma(3)}{\Gamma(3+r)\Gamma(2)}$$

$$= \frac{2\Gamma(2 + r)}{\Gamma(\mathrm{r} + 3)}$$

ii. Mean

From the mean of the beta distribution given by

$$\text{mean} = \frac{a}{a + b},$$

a=2, b=1,

$$= \frac{2}{3} = 0.6667$$

iii. Mode

$$\text{mode} = \frac{a - 1}{a + b - 2} = \frac{2 - 1}{2 + 1 - 2} = 1$$

iv. Variance

From the variance of the beta distribution given by

$$\text{Variance} = \frac{ab}{(a + b + 1)(a + b)^2}$$

a=2, b=1,

$$= \frac{2}{(4)(3)^2}$$

$$= \frac{1}{18} = 0.0556$$

v. Skewness

$$\text{Skewness} = \frac{2ab(\mathrm{b} - \mathrm{a})}{(\mathrm{a} + \mathrm{b})^3(\mathrm{a} + \mathrm{b} + 1)(\mathrm{a} + \mathrm{b} + 2)}$$

$$= \frac{2(2)(-1)}{(3)^3(4)(5)}$$

$$= \frac{-1}{(3)^3(5)} = -0.0074$$

$$\text{Standardized skewness} = \frac{2(\mathrm{b}-\mathrm{a})}{(\mathrm{a}+\mathrm{b}+2)}\sqrt{\frac{a+b+1}{ab}}$$

a=2, b=1

$$= \frac{-2}{5}\sqrt{2} = -0.5657$$

vi. Kurtosis

$$\text{kurtosis} = \frac{3ab[\mathrm{a}^2(\mathrm{b}+2)-2\mathrm{ab}+\mathrm{b}^2(\mathrm{a}+2)]}{(\mathrm{a}+\mathrm{b})^4(\mathrm{a}+\mathrm{b}+1)(\mathrm{a}+\mathrm{b}+2)(\mathrm{a}+\mathrm{b}+3)}$$

$$= \frac{3(2)[4(3)-4+1(4)]}{(3)^4(4)(5)(6)}$$

$$= \frac{1}{(3)^3(5)} = 0.0074$$

$$\text{Standardized kurtosis} = \frac{(3+1)}{2}\left(\frac{3[4(1+2)-2(2)(1)+1(2+2)]}{(3+2)(3+3)}\right) = 2.4$$

2.6.15 Shape of triangular distribution (a=2, b=1)

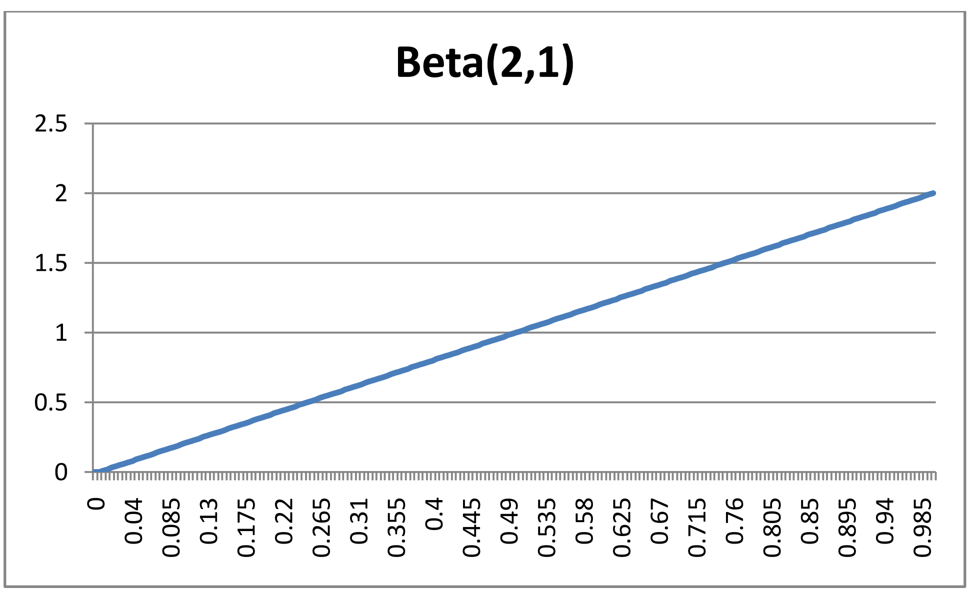


Fig.2. 10: Trangular shaped distribution

2.6.16 Parabolic shaped distribution

For a=b=2 we obtain a distribution of parabolic shape,

$$f(x;2,2) = \frac{x(1-x)}{B(2,2)}$$

$$= 6x(1-x)\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(2.19)$$

The cdf of equation 2.19 is given by

$$F(x) = 3x^2 - 2x^3$$

2.6.17 Properties of parabolic shaped distribution

i. rth-order moments

Putting $a = 2$ and $b = 2$ in equation 2.4

$$E(X^r) = \frac{\Gamma(2+r)\Gamma(4)}{\Gamma(4+\mathrm{r})\Gamma(2)}$$

$$= \frac{6}{(\mathrm{r}+3)(\mathrm{r}+2)}$$

ii. Mean

From the mean of the Classical beta distribution given by

$$\mathrm{E(X)} = \frac{a}{a+b},$$

$\mathrm{a = b = 2}$,

$$= \frac{2}{2+2} = 0.5$$

iii. Mode

From the mode of the Classical beta distribution given by

$$\mathrm{Mode} = \frac{a-1}{(a+b-2)}$$

$\mathrm{a = b = 2}$, $= 0.5$

iv. Variance

From the variance of Classical beta distribution given by

$$\text{Variance} = \frac{ab}{(a+b+1)(a+b)^2}$$

a = b = 2,

$$= \frac{4}{(5)(4)^2}$$

$$= \frac{1}{20} = 0.05$$

$$\text{standard deviation}, \sigma = \sqrt{0.05} = 0.2236$$

$$CV = \frac{0.2236}{0.5} = 0.4472$$

v. Skewness

From the skewness of the Classical beta distribution given by

$$\text{Skewness} = \frac{2ab(b-a)}{(a+b)(a+b+1)(a+b+2)}$$

a=b=2,

$$= 0$$

$$\text{Standardized skewness} = \frac{2(\mathrm{b}-\mathrm{a})}{(\mathrm{a}+\mathrm{b}+2)}\sqrt{\frac{a+b+1}{ab}} = 0$$

vi. Kurtosis

$$\text{kurtosis} = \frac{3ab[\mathrm{a}^2(\mathrm{b}+2) - 2\mathrm{ab} + \mathrm{b}^2(\mathrm{a}+2)]}{(\mathrm{a}+\mathrm{b})^4(\mathrm{a}+\mathrm{b}+1)(\mathrm{a}+\mathrm{b}+2)(\mathrm{a}+\mathrm{b}+3)}$$

$$= \frac{3(4)[4(4) - 8 + 4(4)]}{(4)^4(5)(6)(7)}$$

$$= \frac{3}{(4)^2(5)(7)} = 0.0054$$

$$\text{Standardized kurtosis} = \frac{(4+1)}{4}\left(\frac{3[4(2+2)-2(2)(2)+4(2+2)]}{(4+2)(4+3)}\right) = 2.1429$$

2.6.18 Shape of Parabolic distribution

The figure 2.11 below shows the shape of the parabolic distribution.

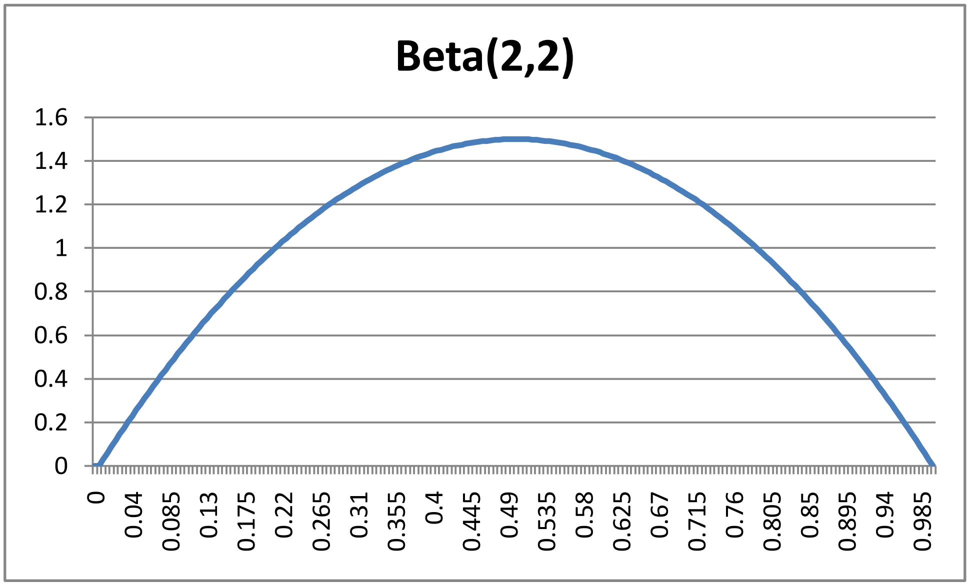


Fig.2. 11: Parabolic distribution

2.7 Transformation of Special Case of the Classical Beta Distribution

2.7.1 Wigner Semicircle Distribution Function

Letting a=b=3/2 in equation (2.1) we obtain the following distribution function

$$f\left(x;\frac{3}{2},\frac{3}{2}\right) = \frac{x^{3/2-1}(1-x)^{3/2-1}}{B\left(\frac{3}{2},\frac{3}{2}\right)}, \qquad 0 < x < 1$$

$$f(x) = \frac{2.x^{1/2}(1-x)^{1/2}}{\left(\left(\frac{1}{2}\right)\sqrt{\pi}\right)^2}, \qquad 0 < x < 1$$

$$f(x) = \frac{8\sqrt{x(1-x)}}{\pi}$$

Let

$$x = \frac{y+R}{2R}, \qquad |J| = \frac{1}{2R}$$

Limits changes as $x = 0 \ \rightarrow y > -R$ and $x = 1 \ \rightarrow y = R$

$$f(y) = \frac{8\sqrt{\frac{y+R}{2R}\left(1 - \frac{y+R}{2R}\right)}}{\pi} \cdot \frac{1}{2R}$$

$$= \frac{4\sqrt{\frac{y+R}{2R} - \left(\frac{y+R}{2R}\right)^2}}{R\pi}$$

$$= \frac{4\sqrt{\frac{2Ry + 2R^2 - y^2 - 2yR - R^2}{4R^2}}}{R\pi}$$

$$= \frac{2\sqrt{R^2 - y^2}}{R^2\pi}, \qquad -R < y < R \text{ \_\_\_\_\_\_(2.20)}$$

Equation (2.20) is the Wigner semicircle distribution.

2.7.2 Properties of Wigner Semicircle

i. rth-order moments

The rth-order moments for positive integers r, the 2rthmoments are given by

$$E(Y^{2r}) = \left(\frac{R}{2}\right)^{2r} C_r$$

Where C_r is the rth catalan number given by $C_r = \frac{1}{r+1}\binom{2r}{r}$ so that the moments are the catalan numbers if R=2. The odd numbers are zero.

ii. Mean

Mean of Wigner semi-circle, the first order moment, odd moment is zero.

iii. Mode

Mode of Wigner semi-circle is given by

$$\frac{d}{dy} f(y) \ = 0$$

$$\frac{2}{R^2\pi}\frac{d}{dy}\left(\sqrt{R^2 - y^2}\right) = 0$$

$$y = 0$$

iv. Variance

The variance of Wigner semi-circle is given by

$$\text{Var(Y)} = E(Y^2) - E(Y)$$

$$= \left(\frac{R}{2}\right)^2 \frac{1}{2}.2$$

$$= \frac{R^2}{4}$$

v. Skewness

The Skewness of Wigner semi-circle is zero, since it is the third order moment which is an odd moment.

vi. Kurtosis

The Kurtosis of Wigner semi-circle is given by

$$\text{Kurtosis} = E(Y^4) - 4E(Y)E(Y^3) + 6\{E(Y)\}^2 E(Y^2) - 3\{E(Y)\}^4$$

$$= E(Y^4)$$

$$= \left(\frac{R}{2}\right)^{2.2} \frac{1}{2+1} \binom{2.2}{2}$$

$$= \frac{R^4}{8}$$

$$\text{Standardized Kurtosis} = \frac{R^4}{8}.\left(\frac{4}{R^2}\right)^2 = 2$$

2.7.3 Shape of Wigner semicircle distribution

The figure 2.12 below shows the shape of the Wigner semicircle distribution.

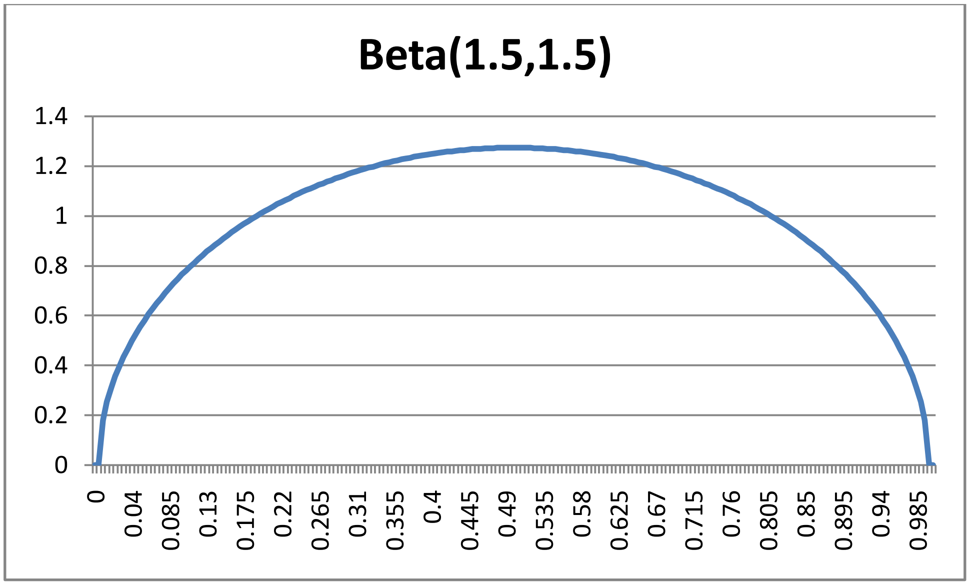


Fig.2. 12: Wigner semicircle distribution

3 Chapter III: Type II Two Parameter Inverted Beta Distribution

3.1 Construction of Beta Distribution of the Second Kind

The beta distribution of the second kind is derived by using the transformation

$$T_1: \qquad x = \frac{\mathrm{y}}{1+\mathrm{y}}$$

in equation (2.1)to give a new distribution as follows:

$$f(y;a,b) = f(x;a,b)|J|$$

The limits changes as follows $x + xy = y \implies y(1-x) = x, y = \frac{x}{1-x}$

when $\mathrm{x} = 0$, $\mathrm{y} = 0$ and when $\mathrm{x} = 1, \mathrm{y} = \infty \implies 0 < y < \infty$

Where

$$|J| = \frac{1}{(1+\mathrm{y})^2}$$

Therefore

$$f(y;a,b) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}.|J|$$

$$= \frac{(\frac{y}{1+y})^{a-1}\left(1-\frac{y}{1+y}\right)^{b-1}}{B(a,b)}.\frac{1}{(1+\mathrm{y})^2}$$

$$= \frac{y^{a-1}}{B(a,b)(1+\mathrm{y})^{\mathrm{a+b}}}, 0 < y < \infty, \qquad a,b > 0\_\_\_\_\_(3.1)$$

Equation (3.1) is a beta distribution of the second kind. It is also known as the standard form of a Pearson Type VI distribution, sometimes called beta prime distribution or Inverted beta distribution.

Another method of deriving beta distribution of the second kind is by using the transformation

$$x = \frac{1}{1+\mathrm{y}} \text{ in equation (2.1)}$$

$$\frac{dx}{dy} = \frac{-1}{(1+\mathrm{y})^2}$$

The limits changes as follows

$$x + xy = 1 \implies y = \frac{1-\mathrm{x}}{\mathrm{x}}, \text{when } \mathrm{x} = 0, \mathrm{y} = \infty \text{ and when } \mathrm{x} = 1, \mathrm{y} = 0 \implies 0 < y < \infty$$

$$f(y; a, b) = f(x; a, b)|J|$$

$$= \frac{\left(\frac{1}{1+y}\right)^{a-1}\left(1-\left(\frac{1}{1+y}\right)\right)^{b-1}}{B(a,b)} \cdot \left|\frac{1}{(1+\mathrm{y})^2}\right|$$

$$= \frac{(\frac{1}{1+y})^{a-1}\left(\frac{y}{1+y}\right)^{b-1}}{B(a,b)} \cdot \frac{1}{(1-\mathrm{y})^2}$$

$$= \frac{y^{b-1}}{B(a,b)(1+y)^{\mathrm{a+b}}}$$

3.2 Properties of Beta Distribution of the Second Kind

3.2.1 r^th-order moments

To obtain r^{th}-order moments of the beta distribution of the second kind, we evaluate

$$E(Y^r) = \int_0^\infty y^r f(y; a, b)\, dy$$

$$= \int_0^\infty y^r \frac{y^{a-1}}{B(a,b)(1+\mathrm{y})^{\mathrm{a+b}}}\, dy$$

$$= \frac{1}{B(a,b)} \int_0^\infty \frac{y^{r+a-1}}{(1+\mathrm{y})^{\mathrm{a+b}}}\, dy$$

To integrate the integral in above expression, substitute

$$y = \frac{x}{1-x}$$

$$\left|\frac{dy}{dx}\right| = \frac{1}{(1-x)^2}$$

$y(1-x) = x \qquad \Longrightarrow 1 + x = \frac{1}{(1-y)}$, and limits of integration become 0 to 1. Then substituting these terms in the above integral, we have

$$\int_0^\infty \frac{y^{r+a-1}}{(1+\mathrm{y})^{\mathrm{a+b}}}\, dy \quad = \int_0^1 \left(\frac{x}{1-x}\right)^{r+a-1} (1-x)^{a+b} (1-x)^{-2}\, dx$$

$$= \int_0^1 x^{r+a-1} (1-x)^{-r-a+1+a+b-2}\, dx$$

$$= \int_0^1 x^{(r+a)-1} (1-x)^{(b-r)-1}\, dx$$

$$= B(a+r, b-r), \qquad r < b$$

Consequently, the moments of beta distribution of the second kind are given by

$$E(Y^r) = \frac{B(a+r, b-r)}{B(a,b)}$$

$$= \frac{\Gamma(a+r)\Gamma(b-r)}{\Gamma(a)\Gamma(b)} \qquad r < b \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_(3.2)}$$

Using recursive relationship $\Gamma(\mathrm{a}+1) = \mathrm{a}\Gamma(\mathrm{a})$, the first four moments about zero can be written as follows

3.2.2 Mean

The mean can be obtained by substituting r=1 in equation 3.2

$$m_1 \equiv \mu = \mathrm{E(Y)} = \frac{B(a+r, b-r)}{B(a,b)}$$

$$= \frac{\Gamma(\mathrm{a}+1)\Gamma(\mathrm{b}-1)}{\Gamma(\mathrm{a}+\mathrm{b})} \cdot \frac{\Gamma(\mathrm{a}+\mathrm{b})}{\Gamma(\mathrm{a})\Gamma(\mathrm{b})}$$

$$= \frac{\mathrm{a}\Gamma(\mathrm{a})\Gamma(\mathrm{b}-1)}{\Gamma(\mathrm{a}+\mathrm{b})} \cdot \frac{\Gamma(\mathrm{a}+\mathrm{b})}{\Gamma(\mathrm{a})\Gamma(\mathrm{b})}$$

$$= \frac{\mathrm{a}\Gamma(\mathrm{b}-1)}{\Gamma(\mathrm{b})}$$

$$= \frac{\mathrm{a(b}-2)!}{(b-1)!}$$

$$= \frac{a(b-2)!}{(b-1)(b-2)!}$$

$$= \frac{a}{b-1}, \qquad b > 1 (Catherine\ et\ al., 2010) \_\_\_\_\_\_ (3.3)$$

If $\mathrm{b} \leq 1$, the mean is infinite i. e undefined

$$m_2 \equiv \mathrm{E(Y^2)} = \frac{B(a+2, b-2)}{B(a,b)}$$

$$= \frac{\Gamma(\mathrm{a}+2)\Gamma(\mathrm{b}-2)}{\Gamma(\mathrm{a}+\mathrm{b})} \cdot \frac{\Gamma(\mathrm{a}+\mathrm{b})}{\Gamma(\mathrm{a})\Gamma(\mathrm{b})}$$

$$= \frac{(\mathrm{a}+1)\mathrm{a}\Gamma(\mathrm{a})\Gamma(\mathrm{b}-2)}{\Gamma(\mathrm{a})\Gamma(\mathrm{b})}$$

$$= \frac{\mathrm{a(a}+1)(\mathrm{b}-3)!}{(\mathrm{b}-1)(\mathrm{b}-2)(\mathrm{b}-3)!}$$

$$= \frac{\mathrm{a(a}+1)}{(b-1)(b-2)}$$

$$= m_1 \frac{(\mathrm{a}+1)}{(b-2)}, \quad b > 2\ (Catherine\ et\ al., 2010)$$

$$m_3 \equiv \mathrm{E}(\mathrm{Y}^3) = \frac{B(a+3, b-3)}{B(a,b)}$$

$$= \frac{\Gamma(\mathrm{a}+3)\Gamma(\mathrm{b}-3)}{\Gamma(\mathrm{a}+\mathrm{b})} . \frac{\Gamma(\mathrm{a}+\mathrm{b})}{\Gamma(\mathrm{a})\Gamma(\mathrm{b})}$$

$$= \frac{(\mathrm{a}+2)(\mathrm{a}+1)\mathrm{a}\Gamma(\mathrm{a})\ (\mathrm{b}-4)!}{\Gamma(\mathrm{a})(\mathrm{b}-1)(\mathrm{b}-2)(\mathrm{b}-3)(\mathrm{b}-4)!}$$

$$= \frac{\mathrm{a}(\mathrm{a}+1)(\mathrm{a}+2)}{(\mathrm{b}-1)(\mathrm{b}-2)(\mathrm{b}-3)}$$

$$= \frac{m_2(\mathrm{a}+2)}{(\mathrm{b}-3)}\ , \quad b > 3\ (Catherine\ et\ al., 2010)$$

$$m_4 \equiv \mathrm{E}(\mathrm{Y}^4) = \frac{B(a+4, b-4)}{B(a,b)}$$

$$= \frac{\Gamma(\mathrm{a}+4)\Gamma(\mathrm{b}-4)}{\Gamma(\mathrm{a}+\mathrm{b})} . \frac{\Gamma(\mathrm{a}+\mathrm{b})}{\Gamma(\mathrm{a})\Gamma(\mathrm{b})}$$

$$= \frac{(\mathrm{a}+3)(\mathrm{a}+2)(\mathrm{a}+1)\mathrm{a}\Gamma(\mathrm{a})(\mathrm{b}-5)!}{\Gamma(\mathrm{a})\ (\mathrm{b}-1)(\mathrm{b}-2)(\mathrm{b}-3)(\mathrm{b}-4)(\mathrm{b}-5)!}$$

$$= \frac{\mathrm{a}(\mathrm{a}+1)(\mathrm{a}+2)(\mathrm{a}+3)}{(\mathrm{b}-1)(\mathrm{b}-2)(\mathrm{b}-3)(\mathrm{b}-4)}$$

$$= \frac{m_3(\mathrm{a}+3)}{(\mathrm{b}-4)}\ , \quad b > 4\ (Catherine\ et\ al., 2010)$$

3.2.3 Mode

Mode is obtained by solving

$$\frac{d}{dy} f(y) = 0$$

$$= \frac{d}{dy}\left(\frac{y^{b-1}}{B(a,b)(1+y)^{\mathrm{a}+\mathrm{b}}}\right) = 0$$

$$= \frac{1}{B(a,b)}\left(\frac{(b-1)y^{b-2}.(1+y)^{\mathrm{a+b}} - (\mathrm{a+b})(1+y)^{\mathrm{a+b-1}}.y^{b-1}}{(1+y)^{\mathrm{2a+2b}}}\right) = 0$$

$$= (b-1)y^{b-2}.(1+y)^{\mathrm{a+b}} - (\mathrm{a+b})(1+y)^{\mathrm{a+b-1}}.y^{b-1} = 0$$

$$(b-1)y^{b-2}.(1+y)^{\mathrm{a+b}} = (\mathrm{a+b})(1+y)^{\mathrm{a+b-1}}.y^{b-1}$$

$$(b-1)y^{b-1}.y^{-1}.(1+y)^{\mathrm{a+b}} = (\mathrm{a+b})(1+y)^{\mathrm{a+b}}(1+y)^{-1}.y^{b-1}$$

$$(b-1)y^{-1} = (\mathrm{a+b})(1+y)^{-1}$$

$$(1+y)(b-1) = y(\mathrm{a+b})$$

$$b - 1 + yb - y = ya + \mathrm{yb}$$

$$y(a+1) = b - 1$$

$$y = \frac{b-1}{a+1}, \text{if b} \geq 1, \qquad 0 \text{ otherwise} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(3.4)$$

3.2.4 Variance

Since variance is given by $E(Y^2) - \{E(Y)\}^2$, we have

$$Var(Y) = \frac{\mathrm{a(a+1)}}{(b-1)(b-2)} - \frac{a^2}{(b-1)^2}$$

$$= \frac{(b-1)(a^2+a) - a^2(b-2)}{(b-1)^2(b-2)}$$

$$= \frac{a^2b + ab - a^2 - a - a^2b + 2a^2}{(b-1)^2(b-2)}$$

$$= \frac{ab + a^2 - a}{(b-1)^2(b-2)}$$

$$= \frac{a(a+b-1)}{(b-1)^2(b-2)}, b > 2 \text{ } (Catherine\ et\ al., 2010) \_\_\_\_\_\_ (3.5)$$

3.2.5 Skewness

Skewness can be expressed as

$$\text{Skewness} = u_3 = E(Y^3) - 3E(Y^2)E(Y) + 2\{E(Y)\}^3$$

$$= \mathrm{m}_3 - 3m_2m_1 + 2m_1^3,$$

$$= \frac{a(a+1)(a+2)}{(b-1)(b-2)(b-3)} - 3\frac{\mathrm{a}(\mathrm{a}+1)}{(b-1)(b-2)}\frac{a}{(b-1)} + 2(\frac{a}{(b-1)})^3$$

$$= \frac{(a^2+a)(a+2)}{(b-1)(b-2)(b-3)} - \frac{3\mathrm{a}^3 + 3\mathrm{a}^2}{(b-1)^2(b-2)} + \frac{2\mathrm{a}^3}{(b-1)^3}$$

$$= \frac{(a^3+3a^2+2a)}{(b-1)(b-2)(b-3)} - \frac{3\mathrm{a}^3 + 3\mathrm{a}^2}{(b-1)^2(b-2)} + \frac{2\mathrm{a}^3}{(b-1)^3}$$

$$= \frac{(b-1)^2(a^3+3a^2+2a) - (b-1)(b-3)(3\mathrm{a}^3+3\mathrm{a}^2) + 2\mathrm{a}^3(b-2)(b-3)}{(b-1)^3(b-2)(b-3)}$$

$$= \frac{(b^2-2b+1)(a^3+3a^2+2a) - (b^2-4b+3)(3\mathrm{a}^3+3\mathrm{a}^2) + 2\mathrm{a}^3(b^2-5b+6)}{(b-1)^3(b-2)(b-3)}$$

$$= \frac{\begin{array}{c} a^3b^2 + 3a^2b^2 + 2ab^2 - 2a^3b - 6a^2b - 4ab + a^3 + 3a^2 + 2a - 3\mathrm{a}^3b^2 - 3\mathrm{a}^2b^2 + 12\mathrm{a}^3\mathrm{b} \\ +12p^2b - 9a^3 - 9a^2 + 2\mathrm{p}^3b^2 - 10\mathrm{a}^3b + 12\mathrm{a}^3 \end{array}}{(b-1)^3(b-2)(b-3)}$$

$$= \frac{2ab^2 + 6a^2b - 4ab + 4a^3 - 6a^2 + 2a}{(b-1)^3(b-2)(b-3)}$$

$$= \frac{2a[b^2 + 3ab - 2b + 2a^2 - 3a + 1]}{(b-1)^3(b-2)(b-3)}$$

$$= \frac{2a[2a^2 + 3ab - 3a + b^2 - 2b + 1]}{(b-1)^3(b-2)(b-3)}$$

$$= \frac{2a[2a^2 + 3a(b-1) + (b-1)^2]}{(b-1)^3(b-2)(b-3)}, \quad b > 3\_\_\_\_\_\_\_\_\_\_(3.6)$$

$(Catherine\ et\ al., 2010)$

Since skewness is positive, it can be concluded that the beta distribution of the second kind has usually along tail to the right.

3.2.6 Kurtosis

Kurtosis can be expressed as

$$\text{Kurtosis} = u_4 = E(Y^4) - 4E(Y)E(Y^3) + 6\{E(Y)\}^2E(Y^2) - 3\{E(Y)\}^4$$

$$= \mathrm{m}_4 - 4m_1m_3 + 6{m_1}^2m_2 - 3m_1^4,$$

$$= \frac{\mathrm{a(a+1)(a+2)(a+3)}}{\mathrm{(b-1)(b-2)(b-3)(b-4)}} - 4\frac{a}{(b-1)}\frac{\mathrm{a(a+1)(a+2)}}{\mathrm{(b-1)(b-2)(b-3)}} + 6\left(\frac{a}{(b-1)}\right)^2\frac{\mathrm{a(a+1)}}{(b-1)(b-2)} - 3\left(\frac{a}{(b-1)}\right)^4$$

$$= \frac{\mathrm{(a^2+a)(a^2+5a+6)}}{\mathrm{(b-1)(b-2)(b-3)(b-4)}} - \frac{\mathrm{4(a^3+a^2)(a+2)}}{\mathrm{(b-1)^2(b-2)(b-3)}} + \frac{\mathrm{6(a^4+a^3)}}{\mathrm{(b-1)^3}(b-2)} - \frac{\mathrm{3a^4}}{\mathrm{(b-1)^4}}$$

$$= \frac{\mathrm{(a^4+5a^3+6a^2+a^3+5a^2+6a)}}{\mathrm{(b-1)(b-2)(b-3)(b-4)}} - \frac{\mathrm{(4a^4+8a^3+4a^3+8a^2)}}{\mathrm{(b-1)^2(b-2)(b-3)}} + \frac{\mathrm{(6a^4+6a^3)}}{\mathrm{(b-1)^3}(b-2)} - \frac{\mathrm{3a^4}}{\mathrm{(b-1)^4}}$$

$$= \frac{\mathrm{(b-1)^3(a^4+5a^3+6a^2+a^3+5a^2+6a) - (b-1)^2(b-4)(4a^4+8a^3+4a^3+8a^2)}}{\mathrm{(b-1)^4(b-2)(b-3)(b-4)}}$$

$$+ \frac{\mathrm{(b-1)(b-3)(b-4)(6a^4+6a^3) - (b-1)(b-2)(b-3)(b-4)3a^4}}{\mathrm{(b-1)^4(b-2)(b-3)(b-4)}}$$

$$= \frac{(b-1)(b^2-2b+1)(a^4+6a^3+11a^2+6a)-(b^2-2b+1)(b-4)(4a^4+12a^3}{(b-1)^4(b-2)(b-3)(b-4)}$$

$$\frac{+8a^2)(b^2-4b+3)(b-4)(6a^4+6a^3)-(b-2)(b^2-7b+12)3a^4}{(b-1)^4(b-2)(b-3)(b-4)}$$

$$= \frac{(b^3-3b^2+3b-1)(a^4+6a^3+11a^2+6a)-(b^3-6b^2+9b-4)(4a^4+12a^3}{(b-1)^4(b-2)(b-3)(b-4)}$$

$$\frac{+8a^2)(b^3-8b^2+19b-12)(6a^4+6a^3)-(b-2)(3a^4b^2-21a^4b+36a^4)}{(b-1)^4(b-2)(b-3)(b-4)}$$

$= [(a^4b^3 + 6a^3b^3 + 11a^2b^3 + 6ab^3)+(-3a^4b^2-18a^3b^2-33a^2b^2 - 18ab^2)+(3a^4b + 18a^3b + 33a^2b + 18ab\ ) + (-a^4 - 6a^3 - 11a^2 - 6a)+(-4a^4b^3 - 12a^3b^3 - 8a^2b^3) + (24a^4b^2 + 72a^3b^2 + 48a^2b^2)+(-36a^4b - 108a^3b - 72a^2b) + (16a^4 + 48a^3 +$
32a2+6a4b3−48a4b2+114a4b−72a4+6a3b3−48a3b2+114a3b−72a3+−3a4b3+27
a4b2−78a4b+72a4]/b−14b−2b−3b−4

$$= \frac{3a^2b^3+6ab^3+6a^3b^2+15a^2b^2-18ab^2+3a^4b+24a^3b-39a^2b+18ab+15a^4-30a^3+21a^2-6a}{(b-1)^4(b-2)(b-3)(b-4)}$$

$$= \frac{3a(ab^3+2b^3+2a^2b^2+5ab^2-6b^2+a^3b+8a^2b-13ab+6b+5a^3-10a^2+7a-2)}{(b-1)^4(b-2)(b-3)(b-4)} \quad \_\_\_\_\_\_(3.7)$$

$(Catherine\ et\ al., 2010)$

3.2.7 Shapes of Beta distribution of the second kind

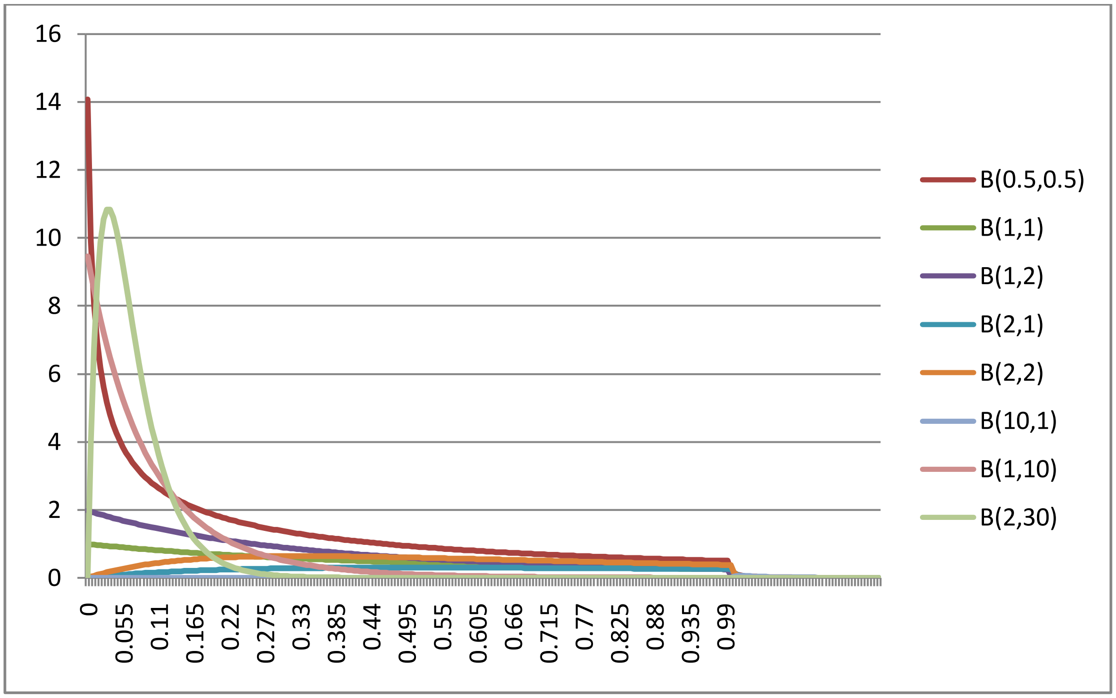


Fig.3. 1: Beta distribution of the second kind

3.3 Application of Beta Distribution of the Second Kind

Vartia and Vartia (1980) fitted a four parameter shifted beta distribution of the second kind under the name of scaled and shifted F distribution to the 1967 distribution of taxed income in Finland. Using maximum likelihood and method of moment's estimators, they found that their model fits systematically better than the two- and three-parameter lognormal distributions. McDonald (1984) estimated the beta distribution of the second kind and both beta of the first and second kind distributions for 1970, 1975 and 1980 US family incomes.

3.4 Special Cases of Beta Distribution of the Second Kind

3.4.1 Lomax (Pareto II) distribution function

Special case when a=1 in equation (3.1) gives the following density function

$$f(x;b) = \frac{1}{B(1,b)(1+\mathrm{x})^{1+\mathrm{b}}}\ ,0 < x < \infty, \qquad b > 0$$

$$B(1,b) = \frac{1}{b}$$

$$= \frac{b}{(1+\mathrm{x})^{1+\mathrm{b}}}, \quad x > 0 \quad , \qquad b > 0 \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(3.8)$$

Equation (3.8) is a Lomax distribution.

3.4.2 Properties of Lomax distribution

i. r^{th}-order moments

Putting $a = 1$ in equation 3.2

$$E(X^r) = \frac{\Gamma(1+r)\Gamma(b-r)}{\Gamma(1)\Gamma(b)}$$

$$= \frac{\Gamma(1+r)\Gamma(b-r)}{\Gamma(b)}$$

ii. Mean

$$\mathrm{E(X)} = \frac{\Gamma(1+1)\Gamma(b-1)}{\Gamma(b)}$$

$$= \frac{(\mathrm{b}-2)!}{(\mathrm{b}-1)(\mathrm{b}-2)!}$$

$$= \frac{1}{b-1}$$

iii. Mode

$$\mathrm{Mode} = \frac{b-1}{2}$$

iv. Variance

$$Var(X) = \frac{b}{(b-2)(b-1)^2}$$

v. **Skewness**

$$\text{Skewness} = \frac{2[2 + 3(b-1) + (b-1)^2]}{(b-1)^3(b-2)(b-3)}$$

$$= \frac{2b[b+1]}{(b-1)^3(b-2)(b-3)}$$

vi. **Kurtosis**

$$\text{kurtosis} = \frac{3(3\text{b}^3 + \text{b}^2 + 2\text{b})}{(\text{b}-1)^4(\text{b}-2)(\text{b}-3)(\text{b}-4)}$$

3.4.3 Shape of Lomax distribution

The figure 3.2 below shows the shape of the Lomax distribution with parameter b=10. The graph is equivalent to the graph of beta distribution of the second kind with parameters a=1 and b=10.

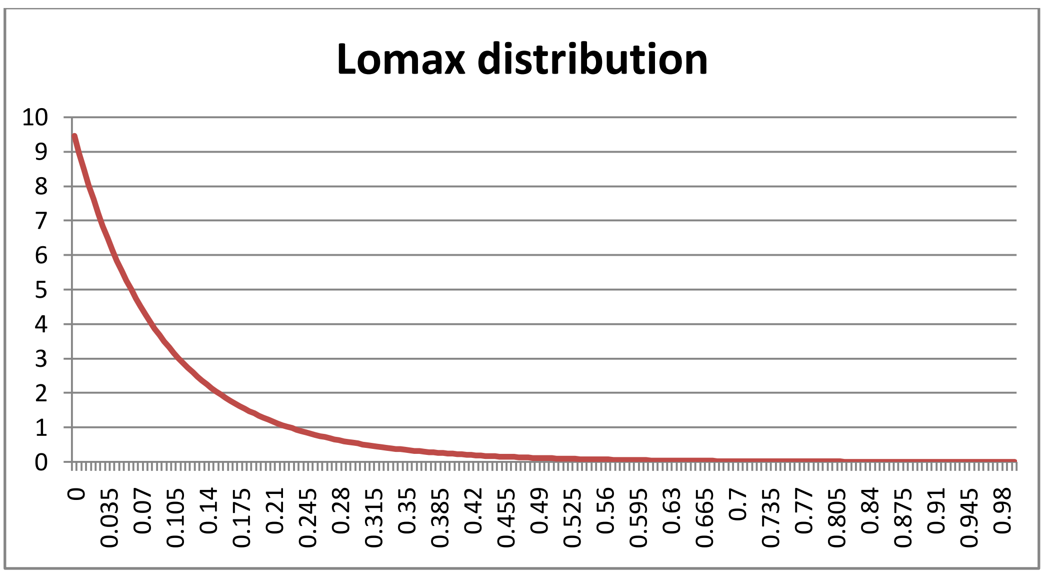


Fig.3. 2: Lomax distribution

3.4.4 Log-Logistic (Fisk) distribution function

Special case of beta distribution of the second kind when a=b=1 in equation (3.1)

$$f(x) = \frac{1}{(1+x)^2}, \qquad x > 0 \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ } (3.9)$$

Equation 3.9 is the log-logistic distribution, also known as Fisk distribution.

3.4.5 Properties of Log-logistic distribution

i. r^{th}-order moments

Putting $a = 1$ and $b = 1$ in equation 3.2

$$E(X^r) = \frac{\Gamma(1+r)\Gamma(1-r)}{\Gamma(1)\Gamma(1)}$$

$$= \Gamma(1+r)\Gamma(1-r)$$

$$= \frac{\text{r}\pi}{\sin \text{r}\pi}$$

ii. Mean

$$\text{E(X)} = \Gamma(1+r)\Gamma(1-r)$$

$$= \Gamma(1+1)\Gamma(-1+1)$$

$$= \frac{\pi}{\sin \pi}$$

iii. Mode

$$\text{Mode} = \frac{1-1}{1+1} = 0$$

iv. Variance

$$\text{Var(X)} = \frac{2\pi}{\sin 2\pi} - \left(\frac{\pi}{\sin \pi}\right)^2$$

v. Skewness

$$\text{Skewness} = E(X^3) - 3E(X^2)E(X) + 2\{E(X)\}^3$$

$$= \frac{3\pi}{\sin 3\pi} - 3\frac{2\pi}{\sin 2\pi}\left(\frac{\pi}{\sin \pi}\right) + 2\left(\frac{\pi}{\sin \pi}\right)^3$$

vi. **Kurtosis**

$$\text{kurtosis} = E(X^4) - 4E(X)E(X^3) + 6\{E(X)\}^2E(X^2) - 3\{E(X)\}^4$$

$$= \frac{4\pi}{\sin 4\pi} - 4\left(\frac{\pi}{\sin \pi}\right)\frac{3\pi}{\sin 3\pi} + 6\left(\frac{\pi}{\sin \pi}\right)^2\frac{2\pi}{\sin 2\pi} - 3\left(\frac{\pi}{\sin \pi}\right)^4$$

3.4.6 Shape of Log-logistic distribution

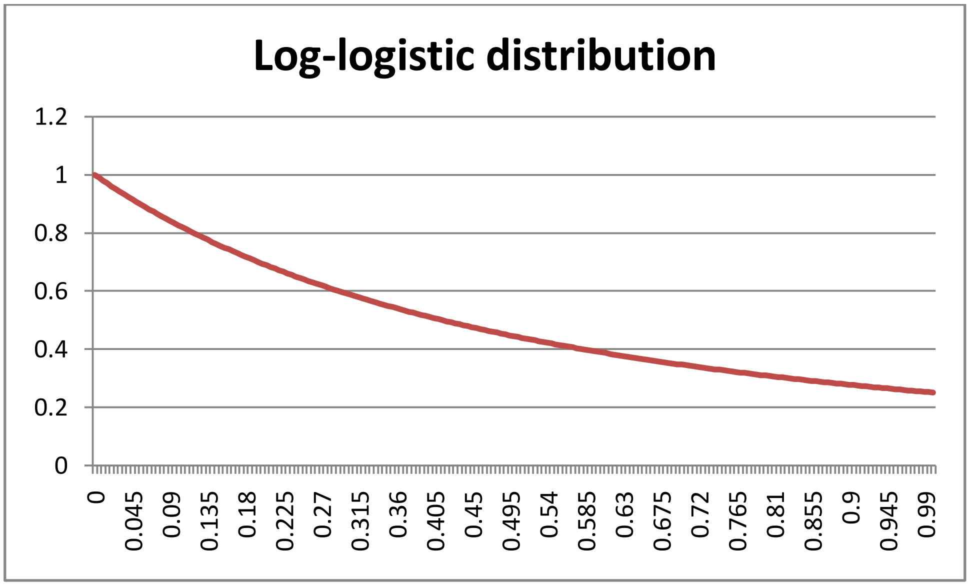


Fig.3. 3: Log-logistic distribution

3.5 Limiting Distribution

Gamma distribution

The two parameter beta distribution in equation 3.1 can be shown to approach the gamma distribution as b→∞ with scale parameter a changing with the parameter b as a= $b\beta$. The resultant probability density function is given by

$$f(\mathrm{x}; \mathrm{a}, \beta, \mathrm{b}) = \left(\frac{x^{b\beta-1}}{(a\beta)\Gamma(b\beta)}\right) e^{-\left(\frac{x}{b\beta}\right)} \quad 0 \leq x \text{ \_\_\_\_\_\_\_\_\_\_\_(3.10)}$$

Equation (3.10) is the gamma distribution function which is covered in details under the generalized beta distribution of the second kind in chapter VI.

4 Chapter IV: The Three Parameter Beta Distributions

4.1 Introduction

This chapter highlights some of the three parameter beta distribution functions used by various authors in literature. These three parameter beta distributions are special cases of the more generalized four parameter beta distributions.

4.2 Case I: Libby and Novick Three Parameter Beta Distribution

Libby and Novick (1982) used the following approach in deriving the three parameter beta distribution

Let $Y_0 = X_0$ and $Y_1 = \frac{X_1}{X_0+X_1}$

$\Longrightarrow X_0 = Y_0$ and $Y_1(X_0 + X_1) = X_1$

$$Y_1X_0 + Y_1X_1 = X_1$$

$$Y_1X_0 = (1 - Y_1)X_1$$

$$\therefore X_1 = \frac{Y_0Y_1}{1 - Y_1}$$

$$J = \begin{vmatrix} \frac{dx_0}{dy_0} & \frac{dx_0}{dy_1} \\ \frac{dx_1}{dy_0} & \frac{dx_1}{dy_1} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{Y_1}{1 - Y_1} & \frac{Y_0}{(1 - Y_1)^2} \end{vmatrix} = \frac{Y_0}{(1 - Y_1)^2}$$

$$g(y_{0,}y_1) = f(x_0, x_1)\,|J|$$

$$= f(x_0)\,f(x_1)\;\frac{Y_0}{(1 - Y_1)^2}$$

$$= \frac{u^a}{\Gamma(a)} e^{-ux_0} {x_0}^{a-1}\;\frac{v^b}{\Gamma(b)} e^{-vx_1} {x_1}^{b-1} \cdot \frac{Y_0}{(1 - Y_1)^2}$$

$$= \frac{u^a v^b}{\Gamma(a)\Gamma(b)} e^{-ux_0-vx_1} {x_0}^{a-1} {x_1}^{b-1}\;\frac{Y_0}{(1 - Y_1)^2}$$

$$g(y_{0,}y_1) = \frac{u^a v^b}{\Gamma(a)\Gamma(b)} e^{-ux_0-vx_1} {x_0}^{a-1} {x_1}^{b-1}\;\frac{Y_0}{(1 - Y_1)^2}$$

$$= \frac{u^a v^b}{\Gamma(a)\Gamma(b)} e^{-uy_0-v\left(\frac{Y_0Y_1}{1-Y_1}\right)} {y_0}^{a-1} \left(\frac{Y_0Y_1}{1 - Y_1}\right)^{b-1}\;\frac{Y_0}{(1 - Y_1)^2}$$

$$\therefore g(y_{0,}y_1) = \frac{u^a v^b}{\Gamma(\mathrm{a})\Gamma(\mathrm{b})} e^{-y_0\left(u+\left(\frac{vY_1}{1-Y_1}\right)\right)} \frac{{y_0}^{a+b-1}{y_1}^{b-1}}{(1-Y_1)^{b+1}}$$

$$= \frac{u^a v^b}{\Gamma(\mathrm{a})\Gamma(\mathrm{b})} e^{-uy_0\left(1+\left(\frac{\lambda_1 Y_1}{1-Y_1}\right)\right)} \frac{{y_0}^{a+b-1}{y_1}^{b-1}}{(1-Y_1)^{b+1}}$$

$$g(y_1) = \frac{u^a v^b}{\Gamma(\mathrm{a})\Gamma(\mathrm{b})} \frac{{y_1}^{b-1}}{(1-Y_1)^{b+1}} \int_0^\infty e^{-y_0\left(u+\left(\frac{vY_1}{1-Y_1}\right)\right)} {y_0}^{a+b-1}\, dy_0$$

$$x = cy_0 \Longrightarrow y_0 = \frac{x}{c}$$

$$dy_0 = \frac{dx}{c}$$

$$\int_0^\infty e^{-x} \left(\frac{x}{c}\right)^{a+b-1} \frac{dx}{c}$$

$$g(y_1) = \frac{u^a v^b}{\Gamma(\mathrm{a})\Gamma(\mathrm{b})} \frac{{y_1}^{b-1}}{(1-Y_1)^{b+1}} \int_0^\infty e^{-cy_0}\, {y_0}^{a+b-1}\, dy_0$$

$$\int_0^\infty e^{-x} \frac{x^{a+b-1}}{c^{a+b}} dx$$

$$\frac{\Gamma(\mathrm{a}+\mathrm{b})}{\Gamma(\mathrm{a})\Gamma(\mathrm{b})} \frac{u^a v^b}{\left(\mathrm{u}+\frac{\mathrm{v}y_1}{1-y_1}\right)^{\mathrm{a+b}}} \frac{{y_1}^{b-1}}{(1-Y_1)^{b+1}}$$

$$\frac{1}{\mathrm{B(a,b)}} \frac{u^a v^b}{\mathrm{u^{ab}}\left(1+\frac{\lambda_1 y_1}{1-y_1}\right)^{\mathrm{a+b}}} \frac{{y_1}^{b-1}}{(1-Y_1)^{b+1}}$$

$$\frac{\left(\frac{\mathrm{v}}{\mathrm{u}}\right)^{\mathrm{b}}}{\mathrm{B(a,b)}} \frac{\left(\frac{y_1}{1-y_1}\right)^{b-1}\left(\frac{1}{1-Y_1}\right)^2}{\left(1+\lambda_1\left(\frac{y_1}{1-y_1}\right)\right)^{\mathrm{a+b}}}, 0 < y_1 < 1$$

$$\frac{{\lambda_1}^{\mathrm{b}}}{\mathrm{B(a,b)}} \frac{\left(\frac{y_1}{1-y_1}\right)^{b-1}\left(\frac{1}{1-Y_1}\right)^2}{\left(1+\lambda_1\left(\frac{y_1}{1-y_1}\right)\right)^{\mathrm{a+b}}}$$

$$g(y_1) = \frac{{\lambda_1}^{\mathrm{b}}}{\mathrm{B(a,b)}} \frac{\left(\frac{y_1}{1-y_1}\right)^{b-1}\left(\frac{1}{1-Y_1}\right)^2}{\left(1+\lambda_1\left(\frac{y_1}{1-y_1}\right)\right)^{\mathrm{a+b}}}$$

$$= \frac{\lambda_1{}^{b}}{B(a,b)} \frac{y_1{}^{b-1}}{(1-Y_1)^{b+1}\left(1+\lambda_1\left(\frac{y_1}{1-y_1}\right)\right)^{a+b}}$$

$$= \frac{\lambda_1{}^{b}}{B(a,b)} \frac{y_1{}^{b-1}}{(1-Y_1)^{b+1}\left(\frac{1-y_1+\lambda_1 y_1}{1-y_1}\right)^{a+b}}$$

$$= \frac{\lambda_1{}^{b}}{B(a,b)} \frac{y_1{}^{b-1}(1-Y_1)^{a-1}}{(1-(1-\lambda_1)y_1)^{a+b}} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(4.1)$$

Equation (4.1) was first used by Libby and Novick (1982) for utility function fitting and by Chen and Novick (1984) as prior in some binomial sampling model. The equation becomes standard beta distribution when c=1.

An alternative method of deriving the three parameter beta distribution introduced by Libby and Novick (1982) is by using transformation as follows:

From the Classical beta distribution in equation (2.1), a third parameter c is introduced using the transformation

$$x = \frac{cy}{1-y+cy}$$

$$\left|\frac{dx}{dy}\right| = \frac{[c(1-y+cy]-[(-1+c)cy]}{(1-y+cy)^2}$$

$$= \frac{[c-cy+cy^2]-[-cy+cy^2]}{(1-y+cy)^2}$$

$$= \frac{c}{(1-y+cy)^2}$$

We get

$$f(y;c,a,b)=\frac{\left(\frac{cy}{1-y+cy}\right)^{a-1}\left(1-\frac{cy}{1-y+cy}\right)^{b-1}}{B(a,b)}\cdot\frac{c}{(1-y+cy)^2}$$

$$=\frac{1}{B(a,b)}\frac{c^{a-1}y^{a-1}}{(1-y+cy)^{a-1}}\frac{(1-y+cy-cy)^{b-1}}{(1-y+cy)^{b-1}}\cdot\frac{c}{(1-y+cy)^2}$$

$$=\frac{1}{B(a,b)}\frac{c^{a-1+1}y^{a-1}(1-y)^{b-1}}{(1-y+cy)^{a-1+b-1+2}}$$

$$=\frac{1}{B(a,b)}\frac{c^{a}y^{a-1}(1-y)^{b-1}}{(1-(1-c)y)^{a+b}}$$

$$\text{for } 0<y<1,\qquad c>0,\qquad a>0,\qquad b>0$$

4.3 Case II: McDonald's Three Parameter Beta I Distribution

The general three parameter beta (a,b,c) can be derived from the standard beta distribution with two parameters in equation (2.1) by using the transformation

$$x=\frac{y}{c}$$

$$\left|\frac{dx}{dy}\right|=\frac{1}{c}$$

$$f(y;a,b,c)=\frac{\left(\frac{y}{c}\right)^{a-1}\left(1-\frac{y}{c}\right)^{b-1}}{B(a,b)}\cdot\frac{1}{c}$$

$$=\frac{\left(\frac{y}{c}\right)^{a-1}\left(1-\left(\frac{y}{c}\right)\right)^{b-1}}{c\,B(a,b)}$$

$$= \frac{y^{a-1}\left(1-\left(\frac{y}{c}\right)\right)^{b-1}}{c^{a}\ B(a,b)}\ , \qquad 0 < y < c\ ,\ \ a > 0, b > 0, \qquad c > 0\_\_\_\_\_\_\_\_(4.2)$$

The three parameter beta distribution in equation 4.2 is referred to as the beta of the first kind in McDonald, 1995. The Classical beta distribution corresponds to equation 4.2 when c=1.

4.4 Gamma Distribution as a Limiting Distribution

We show that the three parameter generalized beta distribution in equation 4.2 approaches the gamma distribution as b→∞ as follows:

Let the scale factor b changes with b as c=$\beta(a+b)$. Substituting this in equation 4.2 gives

$$f(x; a, b, c) = \frac{x^{a-1}\left(1-\frac{x}{\beta(a+b)}\right)^{b-1}}{(\beta(a+b)\ )^{a} B(a,b)}$$

Collecting the terms yields

$$= \left(\frac{x^{a-1}}{\Gamma(a)\beta^{a}}\right)\left(\frac{\Gamma(a+b)}{\Gamma(b)(a+b)^{a}}\right)\left(1-\frac{x}{\beta(a+b)}\right)^{b-1}$$

For large values of b, the gamma function can be approximated by Stirling's formula

$$\Gamma(b) = e^{-b} b^{b-\frac{1}{2}\sqrt{2\pi}}$$

$$\Rightarrow \left(\frac{\Gamma(a+b)}{\Gamma(b)(a+b)^{a}}\right) = \frac{e^{-a-b}(a+b)^{a+b-\frac{1}{2}\sqrt{2\pi}}}{e^{-b} b^{b-\frac{1}{2}\sqrt{2\pi}}(a+b)^{a}}$$

$$\therefore \lim_{b\to\infty} \frac{e^{-a}(a+b)^{b-\frac{1}{2}}\sqrt{2\pi}}{b^{b-\frac{1}{2}}\sqrt{2\pi}} = 1$$

Similarly, $$\lim_{b\to\infty} \left(1 - \frac{x}{\beta(a+b)}\right)^{b-1} = e^{-\left(\frac{x}{\beta}\right)}$$

$$\therefore \quad f(x; a, \beta) = \frac{x^{a-1}}{\beta^{a}\Gamma(a)}\, e^{-\left(\frac{x}{\beta}\right)} \quad 0 \le x \text{ \_\_\_\_\_\_\_\_\_\_\_\_(4.3)}$$

Equation 4.3 is the gamma distribution function.

4.5 Case III

4.5.1 3 Parameter Beta Distribution Derived from the Beta Distribution of the Second Kind

From the Classical beta distribution in equation 2.1, let

$$x = \frac{cy}{1+y}$$

$$\left|\frac{dx}{dy}\right| = \frac{c}{(1+y)^2}$$

$$f(y; c, a, b) = f(x; a, b)|J|$$

$$= \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}.|J|$$

$$= \frac{\left(\frac{cy}{1+y}\right)^{a-1}\left(1-\frac{cy}{1+y}\right)^{b-1}}{B(a,b)}.\frac{c}{(1+y)^2}$$

$$= \frac{c^{a}y^{a-1}(1+(1-c)y)^{b-1}}{B(a,b)(1+y)^{a+b}}\ , 0 < y < 1 \text{ \_\_\_\_\_\_(4.4)}$$

The three parameter beta distribution in equation 4.4 reduces to the beta distribution of the second kind when $c = 1$. When $c = 2$ it reduces to beta type 3 given by

$$f(y;a,b)=\frac{2^a y^{a-1}(1-y)^{b-1}}{B(a,b)(1+\mathrm{y})^{\mathrm{a+b}}}\ ,0<y<1\_\_\_\_\_\_\_\_\_\_(4.5)$$

which is defined on a finite interval and may serve as an alternative to the Classical beta distribution for many practical applications. The beta type 3 was introduced and studied by Cardeńo et.al (2004)

4.5.2 Case IV

An alternative form of the probability density function of the three parameter beta distribution of the second kind can be obtained by substituting $y = \mathrm{x}/\mathrm{c}\ ,\ c > 0$ in equation 3.1,

Then we obtain

$$\mathrm{f}(x;c,a,b) = f(y;a,b)|J|$$

$$=\frac{\left(\frac{x}{c}\right)^{a-1}}{cB(a,b)\left(1+\left(\frac{x}{c}\right)\right)^{\mathrm{a+b}}}$$

$$=\frac{x^{a-1}}{c^a B(a,b)\left(1+\left(\frac{x}{c}\right)\right)^{\mathrm{a+b}}}\ ,\qquad 0\le x<\infty,\qquad a>0,\qquad b>0\_\_\_\_\_\_(4.6)$$

The three parameter beta distribution in equation (4.6) was used as the beta of the second kind by Chotikapanich et.al (2010). It is referred to as beta of the second kind in McDonald and Xu, (1995). Equation 4.6 becomes the beta distribution of the second kind when c=1.

4.5.3 Case V

A three parameter probability density function of beta distribution of the second kind can be obtained by using the transformation $y = cx\ ,\ c > 0$ in equation 3.1,

Then we obtain

$$\mathrm{f}(x;c,a,b) = f(y;a,b)|J|$$

$$= \frac{(cx)^{a-1}}{B(a,b)\left(1+(cx)\right)^{\mathrm{a+b}}}.c$$

$$= \frac{c^a x^{a-1}}{B(a,b)\left(1+(cx)\right)^{\mathrm{a+b}}}, 0 \leq x < \infty, \_\_\_\_\_\_\_\_\_\_\_\_(4.7)$$

$$a > 0, \quad b > 0, \quad c > 0$$

Equation 4.7 becomes the beta distribution of the second kind when c=1.

4.5.4 Case VI

A three parameter probability density function of the Classical beta distribution can be obtained by using the transformation $x = y^c$ $\left|\frac{dx}{dy}\right| = cy^{c-1}$in equation 2.1,

$$f(y; a, b, c) = \frac{y^{ca-c}(1-y^c)^{b-1}}{B(a,b)}.cy^{c-1}$$

$$= \frac{cy^{ca-1}(1-y^c)^{b-1}}{B(a,b)}, \quad 0 < y < 1, \quad a > 0, \quad b > 0, \quad c > 0\_\_\_(4.8)$$

Special case when c=1 in equation 4.8 gives the Classical beta distribution given in equation 2.1.

Special case when a=1 in equation 4.8 gives

$$f(y; b, c) = \frac{cy^{c-1}(1-y^c)^{b-1}}{B(1,b)}$$

$$= cby^{c-1}(1-y^c)^{b-1} \quad 0 < y < 1, \quad a > 0, \quad b > 0, \quad c > 0\_\_\_\_(4.9)$$

Equation 4.9 is a Kumaraswamy distribution.

Special case when c=1 and b=1 in equation 4.8 gives

$$f(y; a, c) = ax^{a-1} \_\_\_\_\_\_\_\_\_\_\_\_(4.10)$$

Equation 4.10 is a power distribution function. When a=1 in equation 4.10 we get a Uniform distribution function.

5 Chapter V: The Four Parameter Beta Distributions

5.1 Introduction

This chapter provides for the construction, properties and special cases of four parameter generalized beta distribution of the first kind and second kind introduced by McDonald (1984) as an income distribution, whereas the beta distribution of the first kind was used for the same purpose more than a decade earlier by Thurow (1970). It also provides for the construction and properties of the five parameter generalized beta distributions which includes both generalized beta distribution of the first kind and second kind and many other distributions.

5.2 Generalized Four Parameter Beta Distribution of the First Kind

The generalized beta distribution of the first kind was introduced by McDonald (1984). It can be obtained from the Classical beta distribution in equation 2.1 by using the transformation

$$x = \left(\frac{y}{q}\right)^p \text{ in equation (2.1)}$$

$$\left|\frac{dx}{dy}\right| = \left(\frac{p}{q^p}\right) y^{p-1}$$

The limits changes as follows $xq^p = y^p$, when $x = 0$, $y^p = 0$ and when

$$y = 1, y^p = q^p$$

The new distribution is

$$f(y; p, q, a, b) = f(x; a, b)|J|$$

$$= \frac{\left(\frac{y}{q}\right)^{pa-p} \left(1 - \left(\frac{y}{q}\right)^p\right)^{b-1}}{B(a, b)} . \left|\frac{p}{q^p} y^{p-1}\right|$$

$$= \frac{|p|\left\{\left(\frac{y}{q}\right)^p\right\}^{a-1}\left(1-\left(\frac{y}{q}\right)^p\right)^{b-1}}{B(a,b)} \cdot \frac{1}{q^p} y^{p-1}$$

$$= \frac{|p| y^{pa-1}\left(1-\left(\frac{y}{q}\right)^p\right)^{b-1}}{q^{ap} B(a,b)}, \qquad 0 \le y^p \le q^p, \qquad \_\_\_\_(5.1)$$

The CDF of 5.1 is given by

$$F(y) = \frac{B\left(a, b, \left(\frac{y}{q}\right)^p\right)}{B(a,b)}, \qquad 0 < y < q$$

where all the four parameters a, b, c and q are positive. q is a scale parameter and a, b, p are shape parameters.

The probability distribution function in equation 5.1 is a four parameter generalized beta distribution of the first kind. If p=q=1, the generalized beta distribution of the first kind in equation 5.1 becomes the two parameter Classical beta distribution in equation 2.1 over the interval (0,1). If p=1, in equation 4.1, we obtain a three parameter beta distribution in equation 4.2. When p=-1 and b=1 in equation 5.1, one obtain the inverse beta distribution of the first kind (Pareto type 1) with density function given by

$$f(y; q, a) = \frac{y^{-a-1}}{q^{-a} B(a,1)}$$

$$= \frac{aq^a}{y^{1+a}}, \text{for } q < y$$

The generalized beta distribution of the first kind is supported on a bounded domain and is useful in the modeling of size phenomena, particularly the distribution of income.

5.2.1 Properties of generalized beta distribution of the first kind

i. r^{th}-order moments

To obtain the r^{th}-order moments of generalized beta distribution of the first kind, we evaluate

$$E(Y^r) = \int_0^q y^r f(y; a, b)\ dy$$

$$= \int_0^q \frac{|p| y^{r+pa-1} \left(1 - \left(\frac{y}{q}\right)^p\right)^{b-1}}{q^{pa} B(a,b)} dy$$

$$= \frac{1}{q^{pa} B(a,b)} \int_0^q \frac{|p| y^{p\left(\frac{r}{p}+a\right)-1} \left(1 - \left(\frac{y}{q}\right)^p\right)^{b-1}}{q^{p\left(\frac{r}{p}+a\right)} B\left(\frac{r}{p}+a, b\right)}\ q^{r+pa} B\left(\frac{r}{p}+a, b\right) dy$$

$$= \frac{q^{r+pa} B\left(\frac{r}{p}+a, b\right)}{q^{pa} B(a,b)}$$

$$= \frac{q^r B\left(\frac{r}{p}+a, b\right)}{B(a,b)}$$

$$= \frac{q^r \Gamma\left(\frac{r}{p}+a\right) \Gamma(b) \Gamma(\mathrm{a}+\mathrm{b})}{\Gamma\left(\frac{r}{p}+a+b\right) \Gamma(\mathrm{a}) \Gamma(\mathrm{b})}$$

$$= \frac{q^r \Gamma(\mathrm{a}+\mathrm{b}) \Gamma\left(a+\frac{r}{p}\right)}{\Gamma\left(a+b+\frac{r}{p}\right) \Gamma(\mathrm{a})}, \qquad a + \frac{r}{p} > 0 \_\_\_\_\_\_\_\_\_\_\_\_\_(5.2)$$

ii. Mean

The mean is given by substituting r=1 in equation 5.2 as

$$E(Y) = \frac{qB\left(a+\frac{1}{p}, b\right)}{B(a,b)}$$

$$= \frac{q\Gamma(\mathrm{a}+\mathrm{b})\Gamma\left(\mathrm{a}+\frac{1}{p}\right)}{\Gamma\left(\mathrm{a}+\mathrm{b}+\frac{1}{p}\right)\Gamma(\mathrm{a})} \_\_\_\_\_\_\_\_\_(5.3)$$

iii. Mode

The mode of generalized beta distribution of the first kind is given by

$$Mode = \frac{d}{dy} f(y) = 0$$

$$\frac{|p|}{q^{pa} B(a,b)} \frac{d}{dy}\left(y^{pa-1}\left(1-\left(\frac{y}{q}\right)^{p}\right)^{b-1}\right) = 0$$

$$\left((pa-1)y^{pa-2}\left(1-\left(\frac{y}{q}\right)^{p}\right)^{b-1}\right) + y^{pa-1}(b-1)\left(1-\left(\frac{y}{q}\right)^{p}\right)^{b-2} \frac{-p}{q^{p}} y^{p-1} = 0$$

$$(pa-1) - \frac{py^{p}(b-1)}{q^{p}\left(1-\left(\frac{y}{q}\right)^{p}\right)} = 0$$

$$pa-1 = \frac{py^{p}(b-1)}{q^{p}-y^{p}}$$

$$(pa-1)(q^{p}-y^{p}) = py^{p}(b-1)$$

$$y = \left(\frac{q^{p}(pa-1)}{pa+pb-1-p}\right)^{\frac{1}{p}}$$

iv. Variance

Variance of generalized beta distribution of the first kind can be obtained from the r^{th}-order moments as follows

$$\text{Variance} = E(Y^2) - \{E(Y)\}^2$$

$$= \frac{q^2\Gamma\left(\mathrm{a}+\frac{2}{\mathrm{p}}\right)\Gamma(\mathrm{b})\Gamma(\mathrm{a}+\mathrm{b})}{\Gamma\left(\mathrm{a}+\mathrm{b}+\frac{2}{\mathrm{p}}\right)} - \left(\frac{q\Gamma\left(\mathrm{a}+\frac{1}{\mathrm{p}}\right)\Gamma(\mathrm{b})\Gamma(\mathrm{a}+\mathrm{b})}{\Gamma\left(\mathrm{a}+\mathrm{b}+\frac{1}{\mathrm{p}}\right)}\right)^2$$

$$= q^2\left[\frac{\Gamma\left(a+\frac{2}{p}\right)\Gamma(b)\Gamma(a+b)}{\Gamma\left(a+b+\frac{2}{p}\right)} - \left(\frac{\Gamma\left(a+\frac{1}{p}\right)\Gamma(b)\Gamma(a+b)}{\Gamma\left(a+b+\frac{1}{p}\right)}\right)^2\right]$$

$$= q^2\left[\frac{\Gamma\left(a+\frac{2}{p}\right)\Gamma(b)\Gamma(a+b)}{\Gamma\left(a+b+\frac{2}{p}\right)} - \frac{\Gamma^2\left(a+\frac{1}{p}\right)\Gamma^2(b)\Gamma^2(a+b)}{\Gamma^2\left(a+b+\frac{1}{p}\right)}\right]$$

v. Skewness

Skewness can be given by

$$Skewness = E(Y^3) - 3E(Y)E(Y^2) + 2\{E(Y)\}^3$$

$$= \frac{q^3\Gamma(a+b)\Gamma\left(a+\frac{3}{p}\right)}{\Gamma\left(a+b+\frac{3}{p}\right)\Gamma(a)} - 3\frac{q\Gamma(a+b)\Gamma\left(a+\frac{1}{p}\right)}{\Gamma\left(a+b+\frac{1}{p}\right)\Gamma(a)}\frac{q^2\Gamma(a+b)\Gamma\left(a+\frac{2}{p}\right)}{\Gamma\left(a+b+\frac{2}{p}\right)\Gamma(a)} + 2\left(\frac{q\Gamma(a+b)\Gamma\left(a+\frac{1}{p}\right)}{\Gamma\left(a+b+\frac{1}{p}\right)\Gamma(a)}\right)^3$$

$$= q^3\left(\frac{\Gamma(a+b)\Gamma\left(a+\frac{3}{p}\right)}{\Gamma\left(a+b+\frac{3}{p}\right)\Gamma(a)} - 3\frac{\Gamma(a+b)\Gamma\left(a+\frac{1}{p}\right)}{\Gamma\left(a+b+\frac{1}{p}\right)\Gamma(a)}\frac{\Gamma(a+b)\Gamma\left(a+\frac{2}{p}\right)}{\Gamma\left(a+b+\frac{2}{p}\right)\Gamma(a)} + 2\left(\frac{\Gamma(a+b)\Gamma\left(a+\frac{1}{p}\right)}{\Gamma\left(a+b+\frac{1}{p}\right)\Gamma(a)}\right)^3\right)$$

vi. Kurtosis

Kurtosis can be given by

$$\text{Kurtosis} = E(Y^4) - 4E(Y)E(Y^3) + 6\{E(Y)\}^2E(Y^2) - 3\{E(Y)\}^4$$

$$= \frac{q^4\Gamma(a+b)\Gamma\left(a+\frac{4}{p}\right)}{\Gamma\left(a+b+\frac{4}{p}\right)\Gamma(a)} - 4\frac{q\Gamma(a+b)\Gamma\left(a+\frac{1}{p}\right)}{\Gamma\left(a+b+\frac{1}{p}\right)\Gamma(a)}\frac{q^3\Gamma(a+b)\Gamma\left(a+\frac{3}{p}\right)}{\Gamma\left(a+b+\frac{3}{p}\right)\Gamma(a)}$$

$$+ 6\left(\frac{q\Gamma(a+b)\Gamma\left(q+\frac{1}{p}\right)}{\Gamma\left(a+b+\frac{1}{p}\right)\Gamma(a)}\right)^2 \frac{q^2\Gamma(a+b)\Gamma\left(a+\frac{2}{p}\right)}{\Gamma\left(q+b+\frac{2}{p}\right)\Gamma(a)}$$

$$- 3\left(\frac{q\Gamma(a+b)\Gamma\left(a+\frac{1}{p}\right)}{\Gamma\left(a+b+\frac{1}{p}\right)\Gamma(a)}\right)^4$$

$$= q^4\left(\frac{\Gamma(a+b)\Gamma\left(a+\frac{4}{p}\right)}{\Gamma\left(a+b+\frac{4}{p}\right)\Gamma(a)} - 4\frac{\Gamma(a+b)\Gamma\left(a+\frac{1}{p}\right)}{\Gamma\left(a+b+\frac{1}{p}\right)\Gamma(a)}\frac{\Gamma(a+b)\Gamma\left(a+\frac{3}{p}\right)}{\Gamma\left(a+b+\frac{3}{p}\right)\Gamma(a)}\right.$$

$$+ 6\left(\frac{\Gamma(a+b)\Gamma\left(a+\frac{1}{p}\right)}{\Gamma\left(a+b+\frac{1}{p}\right)\Gamma(a)}\right)^2 \frac{\Gamma(a+b)\Gamma\left(a+\frac{2}{p}\right)}{\Gamma\left(a+b+\frac{2}{p}\right)\Gamma(a)}$$

$$\left. - 3\left(\frac{\Gamma(a+b)\Gamma\left(a+\frac{1}{p}\right)}{\Gamma\left(a+b+\frac{1}{p}\right)\Gamma(a)}\right)^4\right)$$

5.3 Applications of Generalized Beta Distribution of the First Kind

McDonald (1984) estimated both the generalized beta of the first and second kind and both beta of the first and second kind distributions for 1970, 1975 and 1980 US family incomes and found out that the performance of the generalized beta of the first kind is comparable to the generalized gamma and beta of the second kind distribution.

5.4 Special Cases of Generalized Beta Distribution of the First Kind

The four parameter probability density function of the generalized beta distribution of the first kind is very flexible and includes the following distributions:

5.4.1 Pareto type 1 distribution function

Special case of the generalized beta distribution of the first kind when *p=-1* and *b=1* in equation 5.1 gives

$$f(y;q,a) = \frac{y^{-a-1}}{q^{-a}B(a,1)}(1-(\frac{y}{q})^{-1})^{1-1}$$

$$= \frac{q^a}{B(a,1)y^{a+1}}$$

$$= \frac{aq^a}{y^{a+1}}, \qquad 0 < q \leq y \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (5.4)$$

The cdf of equation 5.4 is given by

$$F(y) = 1 - \left(\frac{q}{y}\right)^a, \qquad y \geq q$$

The probability distribution function in equation 5.4 is the Pareto type 1 distribution also known as classical Pareto distribution. This special case also include inverse beta of the first kind (IB1), implying Pareto (y;q,a) =IB1 (y;q,a,b=1) (McDonald 1995).

5.4.2 Properties of Pareto type 1 distribution

i. r^th^- order moments

Putting p=-1 and b=1 in equation 5.2

$$E(Y^r) = \frac{q^r\Gamma(\mathrm{a}+1)\Gamma(a-r)}{\Gamma(a+1-r)\Gamma(\mathrm{a})}$$

$$= \frac{q^r \mathrm{a}}{a-r}, \qquad a-r > 0$$

ii. Mean

$$E(Y) = \frac{q\Gamma(\mathrm{a}+1)\Gamma(a-1)}{\Gamma(a+1-1)\Gamma(\mathrm{a})}$$

$$= \frac{qa\Gamma(a-1)}{\Gamma(a)}$$

$$= \frac{qa(a-2)!}{(a-1)!}$$

$$= \frac{qa(a-2)!}{(a-1)(a-2)!}$$

$$= \frac{qa}{a-1}, a > 1$$

iii. Mode

Mode of Pareto type 1 distribution can be obtained by substituting b=1 and $\mathrm{p} = -1$ in the mode of generalized beta distribution of the first kind

$$y = \left(\frac{q^{-1}(-1a-1)}{-1a-1.1-1-1}\right)^{-\frac{1}{1}}$$

$$= q\left(\frac{(-a-1+1-1)}{(-a-1)}\right)$$

$$= q$$

iv. Variance

Substituting $p = -1$ and $\mathrm{b} = 1$ in the expression of variance for the generalized beta distribution of the second kind we obtain

$$\mathrm{Var(Y)} = \mathrm{m}_2 - \mathrm{m}_1{}^2$$

$$= \frac{q^2\mathrm{a}}{a-2} - \left(\frac{q\mathrm{a}}{a-1}\right)^2 q^2$$

$$= \frac{q^2\mathrm{a}(\mathrm{a}^2 - 2\mathrm{a} + 1) - q^2a^2(a-2)}{(\mathrm{a}-2)(\mathrm{a}-1)^2}$$

$$= \frac{q^2\mathrm{a}}{(\mathrm{a}-2)(\mathrm{a}-1)^2}$$

v. Skewness

Skewness of Pareto type 1 distribution can be obtained by substituting b=1 and $p = -1$ in the skewness of generalized beta distribution of the first kind

$$Skewness = q^3 \left(\frac{\Gamma(a+1)\Gamma(a-3)}{\Gamma(a+1-3)\Gamma(a)} - 3\frac{\Gamma(a+1)\Gamma(a-1)}{\Gamma(a+1-1)\Gamma(a)}\frac{\Gamma(a+1)\Gamma(a-2)}{\Gamma(a+1-2)\Gamma(a)} + 2\left(\frac{\Gamma(a+1)\Gamma(a-1)}{\Gamma(a+1-1)\Gamma(a)}\right)^3 \right)$$

$$Skewness = q^3 \left(\frac{a}{(a-3)} - \frac{3a^2}{(a-1)(a-2)} + 2\left(\frac{a}{(a-1)}\right)^3 \right)$$

$$= q^3 \left(\frac{a(a-1)^3(a-2) - 3a^2(a-1)^2(a-3) + 2a^3(a-2)(a-3)}{(a-1)^3(a-2)(a-3)} \right)$$

$$= q^3 \left(\frac{(a^5 - 5a^4 + 9a^3 - 7a^2 + 2a) + (-3a^5 + 15a^4 - 21a^3 + 9a^2) + (2a^5 - 10a^4 + 12a^3)}{(a-1)^3(a-2)(a-3)} \right)$$

$$= 2aq^3 \left(\frac{1+a}{(a-1)^3(a-2)(a-3)} \right)$$

$$standardized\ skewness = 2aq^3 \left(\frac{1+a}{(a-1)^3(a-2)(a-3)} \right) * \left(\frac{(a-2)(a-1)^2}{q^2 a} \right)^{\frac{3}{2}}$$

$$= 2aq^3 \left(\frac{1+a}{(a-1)^3(a-2)(a-3)} \right) * \left(\frac{(a-2)\sqrt{(a-2)}(a-1)^3}{q^3 a\sqrt{a}} \right)$$

$$= \frac{2(1+a)}{a-3}\left(\sqrt{\frac{(a-2)}{a}} \right), \qquad a > 3$$

vi. Kurtosis

Kurtosis of Pareto type 1 distribution can be obtained by substituting b=1 and $p = -1$ in the Kurtosis of generalized beta distribution of the first kind

$Kurtosis$

$$= q^4\left(\frac{\Gamma(a+1)\Gamma(a-4)}{\Gamma(a+1-4)\Gamma(a)} - 4\frac{\Gamma(a+1)\Gamma(a-1)}{\Gamma(a+1-1)\Gamma(a)}\frac{\Gamma(a+1)\Gamma(a-3)}{\Gamma(a+1-3)\Gamma(a)} + 6\left(\frac{\Gamma(a+1)\Gamma(a-1)}{\Gamma(a+1-1)\Gamma(a)}\right)^2\frac{\Gamma(a+1)\Gamma(a-2)}{\Gamma(a+1-2)\Gamma(a)} - 3\left(\frac{\Gamma(a+1)\Gamma(a-1)}{\Gamma(a+1-1)\Gamma(a)}\right)^4\right)$$

$$= q^4\left(\frac{a}{(a-4)} - \frac{4a^2}{(a-1)(a-3)} + \frac{6a^3}{(a-1)^2(a-2)} - \frac{3a^4}{(a-1)^4}\right)$$

$$= q^4\left(\frac{a(a-1)^4(a-2)(a-3) - 4a^2(a-1)^3(a-2)(a-4) + 6a^3(a-1)^2(a-3)(a-4) - 3a^4(a-2)(a-3)(a-4)}{(a-1)^4(a-2)(a-3)(a-4)}\right)$$

$$= q^4\left(\frac{(a^7-9a^6+32a^5-58a^4+57a^3-29a^2+6a) + (-4a^7+36a^6-116a^5+172a^4-120a^3+32a^2) + (6a^7-54a^6+162a^5-186a^4+72a^3) + (-3a^7+27a^6-78a^5+72a^4)}{(a-1)^4(a-2)(a-3)(a-4)}\right)$$

$$= 3aq^4\left(\frac{3a^2+a+2}{(a-1)^4(a-2)(a-3)(a-4)}\right)$$

$standardized\ Kurtosis$

$$= 3aq^4\left(\frac{3a^2+a+2}{(a-1)^4(a-2)(a-3)(a-4)}\right) * \left(\frac{(a-2)(a-1)^2}{q^2 a}\right)^2$$

$$= 3\left(\frac{3a^2+a+2}{(a-3)(a-4)}\right) * \frac{a-2}{a}$$

$$= \frac{3(3a^3-5a^2-4)}{a(a-3)(a-4)}$$

5.4.3 Stoppa (generalized Pareto type 1) distribution function

Special case when a=1 and p=-p in equation 5.1 gives the following density function

$$f(y;p,q,b) = pq^{p}by^{-p-1}\left(1-\left(\frac{y}{q}\right)^{-p}\right)^{b-1}, \qquad y \geq q > 0 \_\_\_\_\_\_\_\_\_\_(5.5)$$

The CDF of equation 5.5 is given by

$$F(y) = \left(1-\left(\frac{y}{q}\right)^{-p}\right)^{b}, \qquad y \geq q > 0$$

Equation 5.5 is a Stoppa distribution also known as the generalized Pareto type 1 distribution.

5.4.4 Properties of Stoppa distribution

i. r^th^- order moments

Putting a=1 and $p = -p$ in equation 5.2

$$E(Y^{r}) = \frac{q^{r}\Gamma(1+b)\Gamma\left(1-\frac{r}{p}\right)}{\Gamma\left(1+b-\frac{r}{p}\right)\Gamma(1)}$$

$$= \frac{q^{r}b\Gamma(b)\Gamma\left(1-\frac{r}{p}\right)}{\Gamma\left(1+b-\frac{r}{p}\right)}$$

ii. Mean

The mean is given by substituting r=1 in the expression for the r^{th}-order moments

$$E(Y) = \frac{qb\Gamma(b)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+b-\frac{1}{p}\right)}$$

iii. Mode

Putting a=1 and $p = -p$ in the mode of generalized beta distribution of the first kind

$$y = \left(\frac{q^{-p}(-p-1)}{-p-pb-1+p}\right)^{-\frac{1}{p}}$$

$$= q\left(\frac{1+pb}{(1+p)}\right)^{\frac{1}{p}}, \qquad b > 1$$

Compared to the classical Pareto distribution, the Stoppa distribution is more flexible since it has an additional shape parameter b that allows for unimodal (for $b > 1$) and zero-modal (for $b \leq 1$) densities.

iv. Variance

Putting a=1 and p=-p in the variance of the generalized beta distribution of the first kind

$$\mathrm{Var}\,(\mathrm{Y}) = \mathrm{q}^2 \left[\frac{\Gamma\left(1-\frac{2}{\mathrm{p}}\right)\Gamma(\mathrm{b})\Gamma(1+\mathrm{b})}{\Gamma\left(1+\mathrm{b}-\frac{2}{\mathrm{p}}\right)} - \frac{\Gamma^2\left(1-\frac{1}{\mathrm{p}}\right)\Gamma^2(\mathrm{b})\Gamma^2(1+\mathrm{b})}{\Gamma^2\left(1+\mathrm{b}-\frac{1}{\mathrm{p}}\right)} \right]$$

v. Skewness

Putting a=1 and $\mathrm{p} = -\mathrm{p}$ in the skewness of generalized beta distribution of the first kind

$$\text{Skewness} = q^3 \left(\frac{\Gamma(1+\mathrm{b})\Gamma\left(1-\frac{3}{p}\right)}{\Gamma\left(1+\mathrm{b}-\frac{3}{p}\right)\Gamma(1)} - 3\,\frac{\Gamma(1+\mathrm{b})\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+\mathrm{b}-\frac{1}{p}\right)\Gamma(1)}\,\frac{\Gamma(1+\mathrm{b})\Gamma\left(1-\frac{2}{p}\right)}{\Gamma\left(1+\mathrm{b}-\frac{2}{p}\right)\Gamma(1)} + 2\left(\frac{\Gamma(1+\mathrm{b})\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+\mathrm{b}-\frac{1}{p}\right)\Gamma(1)}\right)^3 \right)$$

$$\text{Skewness} = q^3 \left(\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1-\frac{3}{p}\right)}{\left(\mathrm{b}-\frac{3}{p}\right)\Gamma\left(\mathrm{b}-\frac{3}{p}\right)} - 3\,\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1-\frac{1}{p}\right)}{\left(\mathrm{b}-\frac{1}{p}\right)\Gamma\left(\mathrm{b}-\frac{1}{p}\right)}\,\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1-\frac{2}{p}\right)}{\left(\mathrm{b}-\frac{2}{p}\right)\Gamma\left(\mathrm{b}-\frac{2}{p}\right)} + 2\left(\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1-\frac{1}{p}\right)}{\left(\mathrm{b}-\frac{1}{p}\right)\Gamma\left(\mathrm{b}-\frac{1}{p}\right)}\right)^3 \right)$$

vi. Kurtosis

Putting a=1 and $\mathrm{p} = -\mathrm{p}$ in the Kurtosis of generalized beta distribution of the first kind

Kurtosis

$$= q^4 \left(\frac{\Gamma(1+b)\Gamma\left(1-\frac{4}{p}\right)}{\Gamma\left(1+b-\frac{4}{p}\right)\Gamma(1)} - 4\frac{\Gamma(1+b)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+b-\frac{1}{p}\right)\Gamma(1)} \frac{\Gamma(1+b)\Gamma\left(1-\frac{3}{p}\right)}{\Gamma\left(1+b-\frac{3}{p}\right)\Gamma(1)} \right.$$

$$+ 6\left(\frac{\Gamma(1+b)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+b-\frac{1}{p}\right)\Gamma(1)}\right)^2 \frac{\Gamma(1+b)\Gamma\left(1-\frac{2}{p}\right)}{\Gamma\left(1+b-\frac{2}{p}\right)\Gamma(1)}$$

$$\left. - 3\left(\frac{\Gamma(1+b)\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+b+\frac{1}{p}\right)\Gamma(1)}\right)^4 \right)$$

$$= q^4 \left(\frac{b\Gamma(b)\Gamma\left(1-\frac{4}{p}\right)}{\Gamma\left(1+b-\frac{4}{p}\right)} - 4\frac{b\Gamma(b)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+b-\frac{1}{p}\right)\Gamma(1)} \frac{b\Gamma(b)\Gamma\left(1-\frac{3}{p}\right)}{\Gamma\left(1+b-\frac{3}{p}\right)} \right.$$

$$+ 6\left(\frac{\Gamma(1+b)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+b-\frac{1}{a}\right)}\right)^2 \frac{\Gamma(1+b)\Gamma\left(1-\frac{2}{p}\right)}{\Gamma\left(1+b-\frac{2}{p}\right)}$$

$$\left. - 3\left(\frac{\Gamma(1+b)\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+b+\frac{1}{p}\right)}\right)^4 \right)$$

5.4.5 Kumaraswamy distributions

Special case when a=1 and q=1 in equation 5.1 gives the following density function

$$f(y; p, b) = \frac{p y^{p-1}(1-y^p)^{b-1}}{B(1,b)}$$

$$= pby^{p-1}(1-y^p)^{b-1}, \qquad b > 0 \_\_\_\_\_\_\_\_\_ (5.6)$$

Equation 5.6 is a Kumaraswamy distribution. On the other hand, if b=1 in equation 5.6 then we get the power distribution $f(y; p) = pz^{p-1}$ as in equation 2.12 or equation 4.10.

This implies that the power distribution is a special case of Kumaraswamy distribution with parameters (a,1). If p=b=1 in equation 5.6, then the Kumaraswamy distribution becomes a uniform distribution.

5.4.6 Properties of Kumaraswamy distribution

i. r^{th}-order moments

The r^{th}-order moments are obtained by substituting a=1 and q=1 in equation 5.2

$$E(Y^r) = \frac{B\left(\frac{r}{p}+1, b\right)}{B(1,b)}$$

$$= \frac{\Gamma\left(\frac{r}{p}+1\right) \mathrm{b}\Gamma(b)}{\Gamma\left(\frac{r}{p}+1+b\right)}$$

$$= \frac{\mathrm{b}\Gamma\left(\frac{r}{p}+1\right) \Gamma(b)}{\Gamma(1+\frac{r}{p}+b)}$$

ii. Mean

The mean is given by substituting r=1 in the expression for the r^{th}-order moments

$$\mathrm{E(Y)} = \frac{\mathrm{b}\Gamma\left(\frac{1}{p}+1\right) \Gamma(b)}{\Gamma(1+\frac{1}{p}+b)}$$

iii. Mode

The mode of the Kumaraswamy distribution occurs at

$$\frac{\mathrm{d}}{\mathrm{dy}}\mathrm{f(y)} = 0$$

$$\frac{\mathrm{d}}{\mathrm{dy}}\left(\mathrm{pby^{p-1}(1-y^p)^{b-1}}\right) = 0$$

$$\mathrm{pb}\left[(\mathrm{p}-1)\mathrm{y^{p-2}}(1-\mathrm{y^p})^{\mathrm{b}-1} + (\mathrm{b}-1)(1-\mathrm{y^p})^{\mathrm{b}-2}.-\mathrm{py^{p-1}}.\mathrm{y^{p-1}}\right] = 0$$

$$\left[(\mathrm{p}-1)\mathrm{y^{p-1}}.\mathrm{y^{-1}}(1-\mathrm{y^p})^{\mathrm{b}-1} - (\mathrm{b}-1)(1-\mathrm{y^p})^{\mathrm{b}-1}(1-\mathrm{y^p})^{-1}\mathrm{py^{p-1}y^p y^{-1}}\right] = 0$$

$$(\mathrm{p}-1) - (\mathrm{b}-1)(1-\mathrm{y^p})^{-1}\mathrm{py^p} = 0$$

$$(\mathrm{p}-1)=(\mathrm{b}-1)(1-\mathrm{y^p})^{-1}\mathrm{py^p}=\frac{(\mathrm{b}-1)\mathrm{py^p}}{(1-\mathrm{y^p})}$$

$$(\mathrm{p}-1)(1-\mathrm{y^p})=(\mathrm{b}-1)\mathrm{py^p}$$

$$(\mathrm{p}-\mathrm{py^p}-1+\mathrm{y^p})=(\mathrm{pby^p}-\mathrm{py^p})$$

$$\mathrm{y^p}(-\mathrm{p}+1-\mathrm{pb}+\mathrm{p})=1-\mathrm{p}$$

$$\mathrm{y^p}=\frac{1-p}{1-pb}$$

$$y=\left(\frac{1-p}{1-pb}\right)^{\frac{1}{p}}\ \text{for p}\geq 1,\qquad \mathrm{b}\geq 1,\qquad (\mathrm{p},\mathrm{b})\neq(1,1)$$

iv. Variance

Substituting a=1 and q=1in the variance of the generalized beta distribution of the first kind we get

$$\mathrm{Var\,(Y)}=\frac{\Gamma\left(1+\frac{2}{\mathrm{p}}\right)\Gamma(\mathrm{b})\Gamma(1+\mathrm{b})}{\Gamma\left(1+\mathrm{b}+\frac{2}{\mathrm{p}}\right)}-\frac{\Gamma^2\left(1+\frac{1}{\mathrm{p}}\right)\Gamma^2(\mathrm{b})\Gamma^2(1+\mathrm{b})}{\Gamma^2\left(1+\mathrm{b}+\frac{1}{\mathrm{p}}\right)}$$

$$=\mathrm{b}!\,(\mathrm{b}-1)!\left[\frac{\left(\frac{2}{\mathrm{p}}\right)!}{\left(\frac{2}{\mathrm{p}}+\mathrm{b}\right)!}-\mathrm{b}!\,(\mathrm{b}-1)!\left(\frac{\left(\frac{1}{\mathrm{p}}\right)!}{\left(\frac{1}{\mathrm{p}}+\mathrm{b}\right)!}\right)^2\right]$$

v. Skewness

Putting a=1 and q = 1 in the skewness of generalized beta distribution of the first kind and is given by

$$Skewness$$

$$=\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(\mathrm{a}+\frac{3}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{3}{p}\right)}-3\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{1}{p}\right)}\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{2}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{2}{p}\right)}$$

$$+2\left(\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{1}{p}\right)}\right)^3$$

vi. Kurtosis

Putting a=1 and $q = 1$ in the Kurtosis of generalized beta distribution of the first kind and is given by

$$Kurtosis$$

$$= \frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{4}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{4}{p}\right)} - 4\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{1}{p}\right)}\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{3}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{3}{p}\right)}$$

$$+ 6\left(\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{1}{p}\right)}\right)^2 \frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{2}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{2}{p}\right)} - 3\left(\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{1}{p}\right)}\right)^4$$

5.4.7 Generalized Power distribution Function

Special case when p=1 and b=1 in equation (5.1) gives the following density

$$f(y;q,a) = \frac{y^{a-1}}{q^a B(a,1)}$$

$$= \frac{a y^{a-1}}{q^a}, \qquad 0 < y < q \quad \_\_\_\_\_\_(5.7)$$

Equation (5.7) is a generalized power distribution function.

5.4.8 Properties of power distribution

i. r^th^-order moments

Putting $p = 1$ and $b = 1$ in equation 5.2

$$E(Y^r) = \frac{q^r \Gamma(\mathrm{a}+1)\Gamma(a+r)}{\Gamma(a+1+r)\Gamma(\mathrm{a})}$$

$$= \frac{q^r \mathrm{a}\Gamma(\mathrm{a})\Gamma(a+r)}{(\mathrm{a}+\mathrm{r})\Gamma(a+r)\Gamma(\mathrm{a})}$$

$$= \frac{q^r \mathrm{a}}{(\mathrm{a}+\mathrm{r})}, \qquad a + r > 0$$

ii. **Mean**

$$E(Y) = \frac{qa}{\mathrm{a}+1}$$

iii. **Variance**

Putting $p = 1$ and $\mathrm{b} = 1$ in the expression of variance for the generalized beta distribution of the second kind we obtain

$$\mathrm{Var(Y)} = q^2\left[\frac{\Gamma(\mathrm{a}+2)\Gamma(1)\Gamma(\mathrm{a}+1)}{\Gamma(\mathrm{a}+1+2)} - \frac{\Gamma^2(\mathrm{a}+1)\Gamma^2(1)\Gamma^2(\mathrm{a}+1)}{\Gamma^2(\mathrm{a}+1+2)}\right]$$

$$= q^2\left[\frac{\mathrm{a}\Gamma(\mathrm{a})}{(\mathrm{a}+2)} - \frac{\mathrm{a}\Gamma(\mathrm{a})\mathrm{a}\Gamma(\mathrm{a})}{(\mathrm{a}+1)^2(\mathrm{a}+2)^2}\right]$$

$$= q^2\left[\frac{(\mathrm{a}+1)^2(\mathrm{a}+2)\mathrm{a}\Gamma(\mathrm{a}) - \mathrm{a}^2\Gamma^2(\mathrm{a})}{(\mathrm{a}+1)^2(\mathrm{a}+2)^2}\right]$$

iv. **Skewness**

Putting p=1 and b = 1 in the skewness of generalized beta distribution of the first kind and is given by

Skewness

$$= q^3\left(\frac{\Gamma(\mathrm{a}+1)\Gamma(\mathrm{a}+3)}{\Gamma(\mathrm{a}+1+3)\Gamma(\mathrm{a})} - 3\frac{\Gamma(\mathrm{a}+1)\Gamma(\mathrm{a}+1)}{\Gamma(\mathrm{a}+1+1)\Gamma(\mathrm{a})}\frac{\Gamma(\mathrm{a}+1)\Gamma(\mathrm{a}+2)}{\Gamma(\mathrm{a}+1+2)\Gamma(\mathrm{a})}\right.$$

$$\left.+ 2\left(\frac{\Gamma(\mathrm{a}+1)\Gamma(\mathrm{a}+1)}{\Gamma(\mathrm{a}+1+1)\Gamma(\mathrm{a})}\right)^3\right)$$

$$= q^3\left(\frac{\mathrm{a}}{\mathrm{a}+3} - \frac{3\mathrm{a}^2}{(\mathrm{a}+1)(\mathrm{a}+2)} + \frac{2\mathrm{a}^3}{(\mathrm{a}+1)^3}\right)$$

$$= q^3\left(\frac{\mathrm{a}(\mathrm{a}+1)^3(\mathrm{a}+2) - 3\mathrm{a}^2(\mathrm{a}+1)^2(\mathrm{a}+3) + 2\mathrm{a}^3(\mathrm{a}+2)(\mathrm{a}+3)}{(\mathrm{a}+1)^3(\mathrm{a}+2)(\mathrm{a}+3)}\right)$$

$$= q^3\left(\frac{(\mathrm{a}^5+5\mathrm{a}^4+9\mathrm{a}^3+7\mathrm{a}^2+2\mathrm{a}) + (-3\mathrm{a}^5 - 15\mathrm{a}^4 - 21\mathrm{a}^3 - 9\mathrm{a}^2) + 2\mathrm{a}^5 + 10\mathrm{a}^4 + 12\mathrm{a}^3}{(\mathrm{a}+1)^3(\mathrm{a}+2)(\mathrm{a}+3)}\right)$$

$$= \frac{2aq^3(1-\mathrm{a})}{(\mathrm{a}+1)^3(\mathrm{a}+2)(\mathrm{a}+3)}$$

Standardized Skewness

$$= \frac{2aq^3(1-\mathrm{a})}{(\mathrm{a}+1)^3(\mathrm{a}+2)(\mathrm{a}+3)}\cdot\left[\frac{(\mathrm{a}+1)^2(\mathrm{a}+2)^2}{q^2(\mathrm{a}+1)^2(\mathrm{a}+2)\mathrm{a}\Gamma(\mathrm{a}) - \mathrm{a}^2\Gamma^2(\mathrm{a})}\right]^{\frac{3}{2}}$$

$$= \frac{2a(-a^3 - 3a^2 + 4)}{(a+3)} \left[\frac{1}{(a+1)^2(a+2)a\Gamma(a) - a^2\Gamma^2(a)}\right]^{\frac{3}{2}}$$

v. Kurtosis

Putting p=1 and b = 1 in the Kurtosis of generalized beta distribution of the first kind and is given by

$$Kurtosis$$

$$= q^4\left(\frac{\Gamma(a+1)\Gamma(a+4)}{\Gamma(a+1+4)\Gamma(a)} - 4\frac{\Gamma(a+1)\Gamma(a+1)}{\Gamma(a+1+1)\Gamma(a)}\frac{\Gamma(a+1)\Gamma(a+3)}{\Gamma(a+1+3)\Gamma(a)}\right.$$

$$\left. + 6\left(\frac{\Gamma(a+1)\Gamma(a+1)}{\Gamma(a+1+1)\Gamma(a)}\right)^2 \frac{\Gamma(a+1)\Gamma(a+2)}{\Gamma(a+1+2)\Gamma(a)} - 3\left(\frac{\Gamma(a+1)\Gamma(a+1)}{\Gamma(a+1+1)\Gamma(a)}\right)^4\right)$$

$$= q^4\left(\frac{a}{a+4} - \frac{4a^2}{(a+1)(a+3)} + \frac{6a^3}{(a+1)^2(a+2)} - \frac{3a^4}{(a+1)^4}\right)$$

$$Standardized\ Kurtosis$$

$$= \left[\frac{(a+1)^2(a+2)^2}{q^2(a+1)^2(a+2)a\Gamma(a) - a^2\Gamma^2(a)}\right]^2 . q^4\left(\frac{a}{a+4} - \frac{4a^2}{(a+1)(a+3)}\right.$$

$$\left. + \frac{6a^3}{(a+1)^2(a+2)} - \frac{3a^4}{(a+1)^4}\right)$$

5.4.9 Generalized Gamma Distribution Function

We show that the generalized beta distribution of the first kind approaches the generalized gamma distribution as b→∞ as follows:

Let the scale factor q changes with b as q=$\beta(a+b)^{\frac{1}{p}}$. Substituting this in equation 5.1 we get

$$f(y; p, \beta, a) = \frac{|p|y^{pa-1}\left(1 - \left(\frac{y}{\beta(a+b)^{\frac{1}{p}}}\right)^p\right)^{b-1}}{(\beta(a+b)^{\frac{1}{p}})^{pa} B(a,b)}$$

Collecting the terms yields

$$= \left(\frac{|p|y^{pa-1}}{\Gamma(a)\beta^{pa}}\right)\left(\frac{\Gamma(a+b)}{\Gamma(b)(a+b)^{a}}\right)\left(1-\frac{y^{p}}{\beta^{p}(a+b)}\right)^{b-1}$$

For large values of b, the gamma function can be approximated by Stirling's formula

$$\Gamma(b) = e^{-b}b^{b-\frac{1}{2}\sqrt{2\pi}}$$

$$\Rightarrow \left(\frac{\Gamma(a+b)}{\Gamma(b)(a+b)^{a}}\right) = \frac{e^{-a-b}(a+b)^{a+b-\frac{1}{2}\sqrt{2\pi}}}{e^{-b}b^{b-\frac{1}{2}\sqrt{2\pi}}(a+b)^{a}}$$

$$\therefore \lim_{b\to\infty} \frac{e^{-a}(a+b)^{b-\frac{1}{2}\sqrt{2\pi}}}{b^{b-\frac{1}{2}\sqrt{2\pi}}} = 1$$

Similarly, $\lim_{b\to\infty}\left(1-\frac{y^{p}}{\beta^{p}(a+b)}\right)^{b-1} = e^{-\left(\frac{y}{\beta}\right)^{p}}$

$$\therefore \quad f(y;p,\beta,a) = \frac{|p|y^{pa-1}}{\beta^{pa}\Gamma(a)}\, e^{-\left(\frac{y}{\beta}\right)^{p}} \qquad 0 \le y \text{ \_\_\_\_\_\_\_\_\_\_\_(5.8)}$$

Equation 5.8 is the generalized Gamma (GG) distribution function. See the relationship of generalized gamma to other distribution under the GG derived from generalized beta of the second kind where it includes lognormal, Weibull, Gamma, Exponential, Normal and Pareto distributions as special or limiting cases.

5.4.10 Unit Gamma Distribution Function

Special case of generalized beta distribution of the second kind when $a = \delta/p$ and getting the limit as $p \to 0$ in equation 5.1 gives

$$f(\mathrm{y};\mathrm{q},\delta,\mathrm{b}) = \lim_{p\to 0}\left(\frac{|p|y^{p\left(\frac{\delta}{p}\right)-1}(1-(\frac{y}{q})^p)^{b-1}}{(\mathrm{q^p})^{\frac{\delta}{p}}\,\mathrm{B}(\frac{\delta}{p},\mathrm{b})}\right)$$

$$= \frac{y^{\delta-1}}{\mathrm{q}^{\delta}} \lim_{p\to 0}\left(\frac{|p|(1-(\frac{y}{q})^p)^{b-1}}{\mathrm{B}(\frac{\delta}{p},\mathrm{b})}\right)$$

$$= \frac{y^{\delta-1}}{\mathrm{q}^{\delta}} \lim_{p\to 0}\left(\frac{|p|(1-(\frac{y}{q})^p)^{b-1}}{\Gamma\left(\frac{\delta}{p}\right)\Gamma(\mathrm{b})}.\Gamma(\frac{\delta}{p}+\mathrm{b})\right)$$

$$= \frac{y^{\delta-1}}{\mathrm{q}^{\delta}\Gamma(\mathrm{b})} \lim_{p\to 0}\left(\frac{|p|(1-(\frac{y}{q})^p)^{b-1}}{\Gamma\left(\frac{\delta}{p}\right)}.\Gamma\left(\frac{\delta}{p}+\mathrm{b}\right)\right)$$

$$= \frac{y^{\delta-1}}{\mathrm{q}^{\delta}\Gamma(\mathrm{b})}\left[\delta^{a}\left(\ln\left(\frac{y}{q}\right)\right)^{b-1}\right], \qquad 0 < y < q \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_(5.9)}$$

Equation 5.9 is the Unit gamma distribution. The unit gamma with q=1 is mentioned in Patil et al (1984) and has been used as a mixing distribution for the parameter p in the binomial, Grassia (1977).

5.5 Generalized Four Parameter Beta Distribution of the Second Kind

The generalized beta distribution of the second kind can be derived from the beta distribution of the second kind in equation 3.1 by using the transformation

$$y = \left(\frac{x}{q}\right)^{p}$$

$$\left|\frac{dy}{dx}\right| = \left(\frac{p}{q^{p}}\right)x^{p-1}$$

The new limits are as follows $(yq^{p})^{\frac{1}{p}} = x$

When $y = 0$, $x = 0$ and when $y = \infty, x = \infty$

The new distribution is

$$f(x; p, q, a, b) = f(y; a, b)|J|$$

$$= \frac{\left((\frac{x}{q})^p\right)^{a-1}}{B(a,b)\left(1 + (\frac{x}{q})^p\right)^{\mathrm{a+b}}} \left|\left(\frac{p}{q^p}\right) x^{p-1}\right|$$

$$= \frac{|p| x^{pa-1}}{q^{pa} B(a,b)(1 + (\frac{x}{q})^p)^{\mathrm{a+b}}} \quad , \quad 0 < x < \infty \, , a, b, p, q > 0 \qquad \_\_\_\_(5.10)$$

The probability density function in equation (5.10) is a four parameter generalized beta distribution of the second kind. It is a result of the pioneering work of McDonald (1984) and Venter (1983) which led to this generalized beta of the second kind (or transformed beta) distribution. It is also known in the statistical literature as the generalized F (see, e.g., Kalbfleisch and Prentice, 1980). If p=q=1, then the generalized beta distribution of the second kind in equation 5.10 reduces to beta distribution of the second kind in equation 3.1. If p=1 in equation 5.10, one obtain the three parameter beta distribution in equation 4.6.

5.5.1 Properties of generalized beta distribution of the second kind

i. r^{th}-order moments

To obtain the r^{th}-order moments of generalized beta distribution of the second kind, we evaluate

$$E(Y^r) = \int_0^\infty y^r f(y; a, b) \; dy$$

$$= \int_0^\infty \frac{|p| y^{r+pa-1}}{q^{pa} B(a,b)\left(1 + (\frac{y}{q})^p\right)^{a+b}} dy$$

$$= \frac{1}{B(a,b)} \int_0^{\infty} \frac{|p| y^{p\left(a+\frac{r}{p}\right)-1}}{q^{pa}\left(1+\left(\frac{y}{q}\right)^p\right)^{a+b}} dy$$

$$= \frac{1}{q^{pa} B(a,b)} \int_0^{\infty} \frac{|p| y^{p\left(a+\frac{r}{p}\right)-1} \left(1-(\frac{y}{q})^p\right)^{b-1}}{q^{p\left(a+\frac{r}{p}\right)}\left(1+(\frac{y}{q})^p\right)^{\left(a+\frac{r}{p}\right)+b-\frac{r}{p}}} \; q^{p\left(a+\frac{r}{p}\right)} dy$$

$$= \frac{q^{p\left(a+\frac{r}{p}\right)}}{q^{pa} B(a,b)} B\left(a+\frac{r}{p}, b-\frac{r}{p}\right)$$

$$= \frac{q^{pa+r-pa}}{B(a,b)} \; B\left(a+\frac{r}{p}, b-\frac{r}{p}\right)$$

$$= \frac{q^r B\left(a+\frac{r}{p}, b-\frac{r}{p}\right)}{B(a,b)}$$

$$= \frac{q^r \Gamma\left(a+\frac{r}{p}\right)\Gamma\left(b-\frac{r}{p}\right)}{\Gamma(a)\Gamma(b)}, \qquad -pa < r < pb \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (5.11)$$

ii. Mean

The mean of the generalized beta distribution of the second kind is found by substituting r=1 in equation 5.11

$$m_1 \equiv E(Y) = \frac{qB\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)}$$

$$= \frac{q\Gamma\left(a+\frac{1}{p}\right)\Gamma\left(b-\frac{1}{p}\right)}{\Gamma(a+b)} . \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

$$= \frac{q\Gamma\left(a+\frac{1}{p}\right)\Gamma\left(b-\frac{1}{p}\right)}{\Gamma(a)\Gamma(b)}, \qquad b, p > 1$$

$$m_2 \equiv E(Y^2) = \frac{q^r B\left(a+\frac{r}{p}, b-\frac{r}{p}\right)}{B(a,b)}$$

$$= \frac{q^2 B\left(a+\frac{2}{p}, b-\frac{2}{p}\right)}{B(a,b)}$$

iii. Mode

The mode of the generalized beta distribution of the second kind occurs at

$$\frac{d}{dy} f(y; p, q, a, b) = 0$$

$$\frac{d}{dy}\left(\frac{|p| y^{pa-1}}{q^{pa} B(a,b)(1+(\frac{y}{q})^p)^{\mathrm{a+b}}}\right) = 0$$

$$\frac{|p|}{q^{pa} B(a,b)} \frac{d}{dy}\left(\frac{y^{pa-1}}{(1+(\frac{y}{q})^p)^{\mathrm{a+b}}}\right) = 0$$

$$\left(\frac{(pa-1)y^{pa-1-1}\left((1+(\frac{y}{q})^p)^{\mathrm{a+b}}\right) - (\mathrm{a+b})\left((1+(\frac{y}{q})^p)^{\mathrm{a+b-1}}\left(\frac{\mathrm{py^{p-1}}}{\mathrm{q^p}}\right)\right) y^{pa-1}}{\left(\left(1+(\frac{y}{q})^p\right)^{\mathrm{a+b}}\right)^2}\right)$$

$$= 0$$

$$\left((pa-1)y^{pa-1-1}\left((1+(\frac{y}{q})^p)^{\mathrm{a+b}}\right) - (\mathrm{a+b})\left((1+(\frac{y}{q})^p)^{\mathrm{a+b-1}}\left(\frac{\mathrm{py^{p-1}}}{\mathrm{q^p}}\right)\right) y^{pa-1}\right)$$

$$= 0$$

$$\left((pa-1)y^{pa-1}y^{-1}\left((1+(\frac{y}{q})^p)^{\mathrm{a+b}}\right)\right.$$

$$\left. - (\mathrm{a+b})\left((1+(\frac{y}{q})^p)^{\mathrm{a+b}}(1+(\frac{y}{q})^p)^{-1}\left(\frac{\mathrm{py^p y^{-1}}}{\mathrm{q^p}}\right)\right) y^{pa-1}\right) = 0$$

$$(pa-1)=(\mathrm{a}+\mathrm{b})\left(\frac{1}{(1+(\frac{y}{q})^{p})}\left(\frac{\mathrm{py^p}}{\mathrm{q^p}}\right)\right)$$

$$(\mathrm{q^p}+\mathrm{q^p}(\frac{y}{q})^{p})(pa-1)=(\mathrm{apy^p}+\mathrm{bpy^p})$$

$$\mathrm{q^p}pa-\mathrm{q^p}+y^{p}pa-y^{p}=\mathrm{apy^p}+\mathrm{bpy^p}$$

$$y^{p}(pa+pb-pa+1)=\mathrm{q^p}(pa-1)$$

$$y^{p}=\frac{q^{p}(pa-1)}{(pb+1)}$$

$$y=q\left(\frac{pa-1}{pb+1}\right)^{\frac{1}{p}}\ \text{if}\ pa\geq 1$$

iv. Variance

The variance of the generalized beta distribution of the second kind is given by

$$\mathrm{Var}(Y)=E(Y^{2})-\{E(Y)\}^{2}$$

$$=q^{2}\left[\frac{B\left(a+\frac{2}{p},b-\frac{2}{p}\right)}{B(a,b)}-\left\{\frac{B\left(a+\frac{1}{p},b-\frac{1}{p}\right)}{B(a,b)}\right\}^{2}\right]$$

Coefficient of Variation

$$CV=\frac{\sqrt{q^{2}\left[\frac{B\left(a+\frac{2}{p},b-\frac{2}{p}\right)}{B(a,b)}-\left\{\frac{B\left(a+\frac{1}{p},b-\frac{1}{p}\right)}{B(a,b)}\right\}^{2}\right]}}{\frac{qB\left(a+\frac{1}{p},b-\frac{1}{p}\right)}{B(a,b)}}$$

$$=\sqrt{\left[\frac{B\left(a+\frac{2}{p},a-\frac{2}{p}\right)B^{2}(a,b)}{B(a,b)B^{2}\left(a+\frac{1}{p},b-\frac{1}{p}\right)}-\left\{\frac{B\left(a+\frac{1}{p},b-\frac{1}{p}\right)}{B(a,b)}\right\}^{2}\frac{B^{2}(a,b)}{B^{2}\left(a+\frac{1}{p},b-\frac{1}{p}\right)}\right]}$$

$$= \sqrt{\frac{B\left(a+\frac{2}{p}, b-\frac{2}{p}\right) B(a,b)}{B^2\left(a+\frac{1}{p}, b-\frac{1}{p}\right)} - 1}$$

v. Skewness

Skewness of the generalized beta distribution of the second kind can be given by

$$Skewness = E(Y^3) - 3E(Y)E(Y^2) + 2\{E(Y)\}^3$$

$$= \frac{q^3 B\left(a+\frac{3}{p}, b-\frac{3}{p}\right)}{B(a,b)} - 3\frac{qB\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)} \frac{q^2 B\left(a+\frac{2}{p}, b-\frac{2}{p}\right)}{B(a,b)} + 2\left(\frac{qB\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)}\right)^3$$

$$= q^3 \left(\frac{B\left(a+\frac{3}{p}, b-\frac{3}{p}\right)}{B(a,b)} - 3\frac{B\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)} \frac{B\left(a+\frac{2}{p}, b-\frac{2}{p}\right)}{B(a,b)} + 2\left(\frac{B\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)}\right)^3\right)$$

$$\text{Standardized Skewness} = \frac{E(Y^3) - 3E(Y)E(Y^2) + 2\{E(Y)\}^3}{\sigma^3}$$

vi. Kurtosis

Kurtosis of generalized beta distribution of the second kind can be given by

$$Kurtosis = E(Y^4) - 4E(Y)E(Y^3) + 6E\{(Y)\}^2 E(Y^2) - 3\{E(Y)\}^4$$

$$= \frac{q^4 B\left(a+\frac{4}{p}, b-\frac{4}{p}\right)}{B(a,b)} - 4\frac{qB\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)}\frac{q^3 B\left(a+\frac{3}{p}, b-\frac{3}{p}\right)}{B(a,b)} + 6\left(\frac{qB\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)}\right)^2 \frac{q^2 B\left(a+\frac{2}{p}, b-\frac{2}{p}\right)}{B(a,b)} - 3\left(\frac{qB\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)}\right)^4$$

$$= q^4\left(\frac{B\left(a+\frac{4}{p}, b-\frac{4}{p}\right)}{B(a,b)} - 4\frac{B\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)}\frac{B\left(a+\frac{3}{p}, b-\frac{3}{p}\right)}{B(a,b)} + 6\left(\frac{B\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)}\right)^2 \frac{B\left(a+\frac{2}{p}, b-\frac{2}{p}\right)}{B(a,b)} - 3\left(\frac{B\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)}\right)^4\right)$$

$$Standardized\ Kurtosis = \frac{E(Y^4) - 4E(Y)E(Y^3) + 6E\{(Y)\}^2 E(Y^2) - 3\,\{E(Y)\}^4}{\sigma^4}$$

5.6 Special Case of the Generalized Beta Distribution of the Second Kind

The generalized beta distribution of the second kind nests many distributions as special or limiting cases, below is a consideration of some of the distributions and their properties

5.6.1 Singh-Maddala (Burr 12) Distribution

Special case of the generalized beta distribution of the second kind when a=1 in equation 5.10 gives the following density function

$$f(y; p, q, b) = \frac{|p| y^{p-1}}{q^p B(1,b)(1 + (\frac{y}{q})^p)^{1+\mathrm{b}}}$$

$$= \frac{|p|by^{p-1}}{q^p \left(1 + (\frac{y}{q})^p\right)^{1+b}}, \qquad y > 0 \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (5.12)$$

The CDF of equation 2.1 is given by

$$\mathrm{F(y)} = 1 - \left(1 + (\frac{y}{q})^p\right)^{-\mathrm{b}}$$

Where all three parameters p, q, b are positive. q is a scale parameter and p, b are shape parameters; b only affects the right tail, whereas p affects both tails (Kleiber and Kotz, 2003). Singh Maddala distribution is known under a variety of names: Usually called Burr Type 12 distribution, or simply the Burr distribution. The Singh-Maddala distribution includes the Weibull and Fisk distributions as special cases.

5.6.2 Properties of Singh-Maddala distribution

i. r^th^-order moments

The r^{th}-order moments of the Singh-Maddala distribution can be obtained from the expression of the r^{th} –order moments of the generalized beta distribution of the second kind by substituting a=1 in equation 5.11. Thus

$$E(Y^r) = \frac{q^r B\left(1 + \frac{r}{p}, b - \frac{r}{p}\right)}{B(1, b)}$$

$$= \frac{q^r \Gamma\left(1 + \frac{r}{p}\right) \Gamma\left(b - \frac{r}{p}\right)}{\Gamma(\mathrm{b})}, \qquad -p < r < pb$$

ii. Mean

The mean is given by

$$E(Y) = \frac{q\Gamma\left(1 + \frac{1}{p}\right) \Gamma\left(b - \frac{1}{p}\right)}{\Gamma(\mathrm{b})}$$

iii. Mode

The mode of the Singh-Maddala distribution is given by

$$\text{Mode} = q\left(\frac{p-1}{pb+1}\right)^{\frac{1}{p}} \text{ if } p \geq 1$$

and at zero elsewhere. Thus the mode is decreasing with b, reflecting the fact that the right tail becomes lighter as b increases

iv. Variance

$$\text{Var(Y)} = \sigma^2 = \frac{q^2 B\left(1+\frac{2}{p}, b-\frac{2}{p}\right)}{B(1,q)} - \left\{\frac{qB\left(1+\frac{1}{p}, b-\frac{1}{p}\right)}{B(1,b)}\right\}^2$$

$$= \frac{q^2 \text{b}\Gamma\left(1+\frac{2}{p}\right)\Gamma\left(\text{b}-\frac{2}{\text{p}}\right)}{\text{b}\Gamma(b)} - \left\{\frac{qb\Gamma(1+\frac{1}{p})\Gamma\left(b-\frac{1}{p}\right)}{\text{b}\Gamma(b)}\right\}^2$$

$$= q^2\left\{\frac{\Gamma(\text{b})\Gamma\left(1+\frac{2}{p}\right)\Gamma\left(b-\frac{2}{p}\right) - \Gamma^2(1+\frac{1}{p})\Gamma^2\left(b-\frac{1}{p}\right)}{\Gamma^2(b)}\right\}$$

Hence the coefficient of variation is given by

$$CV = \frac{\sigma}{\mu}$$

$$CV = \frac{\sqrt{q^2\left\{\frac{\Gamma(\text{b})\Gamma\left(1+\frac{2}{p}\right)\Gamma\left(b-\frac{2}{p}\right) - \Gamma^2\left(1+\frac{1}{p}\right)\Gamma^2\left(b-\frac{1}{p}\right)}{\Gamma^2(b)}\right\}}}{\frac{q\Gamma(1+\frac{1}{p})\Gamma(b-\frac{1}{p})}{\Gamma(\text{b})}}$$

$$= \frac{\frac{q}{\Gamma(b)}\sqrt{\left\{\Gamma(\text{b})\Gamma\left(1+\frac{2}{p}\right)\Gamma\left(b-\frac{2}{p}\right) - \Gamma^2\left(1+\frac{1}{p}\right)\Gamma^2\left(b-\frac{1}{p}\right)\right\}}}{\frac{q\Gamma(1+\frac{1}{p})\Gamma(b-\frac{1}{p})}{\Gamma(\text{b})}}$$

$$= \frac{\sqrt{\left\{\Gamma(\text{b})\Gamma\left(1+\frac{2}{p}\right)\Gamma\left(b-\frac{2}{p}\right) - \Gamma^2\left(1+\frac{1}{p}\right)\Gamma^2\left(b-\frac{1}{p}\right)\right\}}}{\Gamma(1+\frac{1}{p})\Gamma(b-\frac{1}{p})}$$

$$= \sqrt{\left\{\frac{\Gamma(\mathrm{b})\Gamma\left(1+\frac{2}{p}\right)\Gamma\left(b-\frac{2}{p}\right)}{\Gamma^2\left(1+\frac{1}{p}\right)\Gamma^2\left(b-\frac{1}{p}\right)} - \frac{\Gamma^2\left(1+\frac{1}{p}\right)\Gamma^2\left(b-\frac{1}{p}\right)}{\Gamma^2\left(1+\frac{1}{p}\right)\Gamma^2\left(b-\frac{1}{p}\right)}\right\}}$$

$$\sqrt{\left(\frac{\Gamma(\mathrm{b})\Gamma\left(1+\frac{2}{p}\right)\Gamma\left(b-\frac{2}{p}\right)}{\Gamma^2\left(1+\frac{1}{p}\right)\Gamma^2\left(b-\frac{1}{p}\right)} - 1\right)}$$

v. Skewness

Putting a=1in the Skewness of generalized beta distribution of the second kind

$$Skewness = q^3\left(\frac{B\left(1+\frac{3}{p}, b-\frac{3}{p}\right)}{B(1,b)} - 3\frac{B\left(1+\frac{1}{p}, b-\frac{1}{p}\right)}{B(1,b)}\frac{B\left(1+\frac{2}{p}, b-\frac{2}{p}\right)}{B(1,b)} + 2\left(\frac{B\left(1+\frac{1}{p}, b-\frac{1}{p}\right)}{B(1,bb)}\right)^3\right)$$

$$= q^3\left(bB\left(1+\frac{3}{p}, b-\frac{3}{p}\right) - 3bB\left(1+\frac{1}{p}, b-\frac{1}{p}\right)bB\left(1+\frac{2}{p}, b-\frac{2}{p}\right) + 2\left(bB\left(1+\frac{1}{p}, b-\frac{1}{p}\right)\right)^3\right)$$

vi. Kurtosis

Kurtosis of Singh-Maddala distribution can be obtained from the Kurtosis of generalized beta distribution of the second kind by letting a=1

$$Kurtosis = E(Y^4) - 4E(Y)E(Y^3) + 6E\{(Y)\}^2E(Y^2) - 3\{E(Y)\}^4$$

$$= q^4 \left(\frac{B\left(1+\frac{4}{p}, b-\frac{4}{p}\right)}{B(1,b)} - 4\frac{B\left(1+\frac{1}{p}, b-\frac{1}{p}\right)}{B(1,b)} \frac{B\left(1+\frac{3}{p}, b-\frac{3}{p}\right)}{B(1,b)} \right.$$

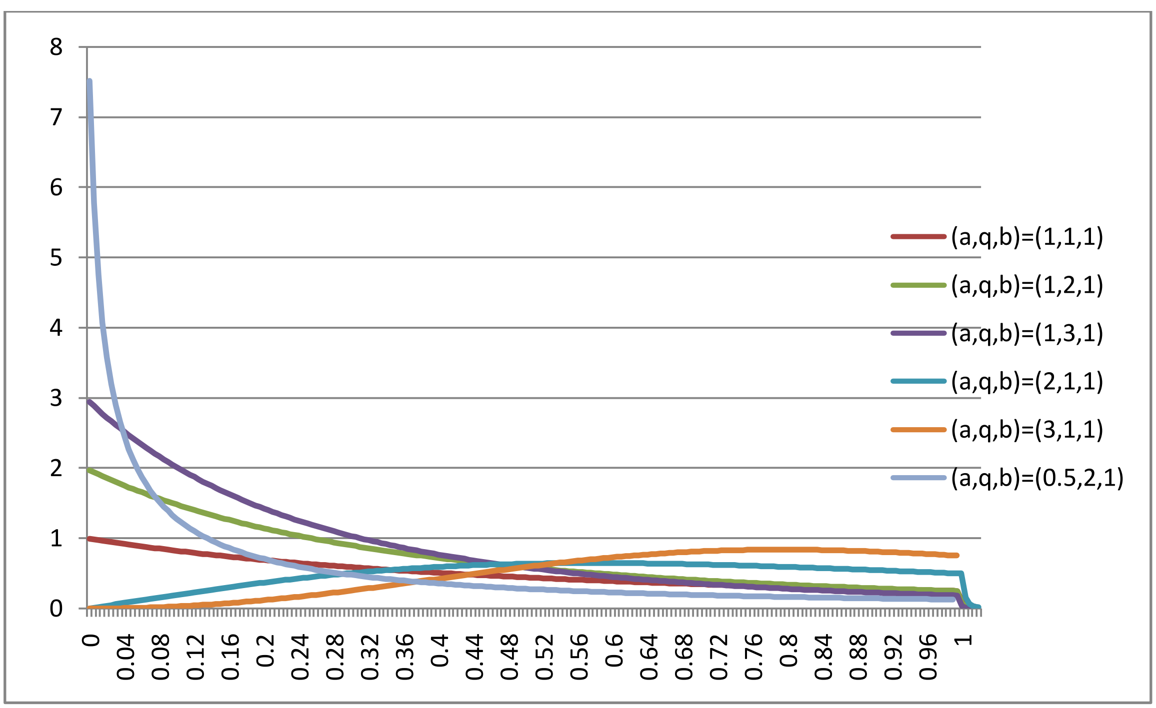
$$\left. + 6\left(\frac{B\left(1+\frac{1}{p}, b-\frac{1}{p}\right)}{B(1,b)}\right)^2 \frac{B\left(1+\frac{2}{p}, b-\frac{2}{p}\right)}{B(1,b)} - 3\left(\frac{B\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(1,b)}\right)^4 \right)$$

$$= q^4 \left(bB\left(1+\frac{4}{p}, b-\frac{4}{p}\right) - 4b^2 B\left(1+\frac{1}{p}, b-\frac{1}{p}\right) B\left(1+\frac{3}{p}, b-\frac{3}{p}\right) \right.$$

$$+ 6\left(bB\left(1+\frac{1}{p}, b-\frac{1}{p}\right)\right)^2 bB\left(1+\frac{2}{p}, b-\frac{2}{p}\right)$$

$$\left. - 3\left(bB\left(a+\frac{1}{p}, b-\frac{1}{p}\right)\right)^4 \right)$$

5.6.3 Shapes of Singh-Maddala distribution



5.6.4 Generalized Lomax (Pareto type II) Distribution

Special case of the generalized beta distribution of the second kind when p=1, $q = \frac{1}{\lambda}$ and $a = 1$ in equation 5.10 gives

$$f(\mathrm{y}; \lambda, \mathrm{b}) = \frac{|1| y^{1(1)-1}}{(\frac{1}{\lambda})^{1.1} B(1, b)(1 + (\frac{y}{\frac{1}{\lambda}})^{1})^{1+\mathrm{b}}}$$

$$= \frac{\lambda b}{(1 + \lambda y)^{1+\mathrm{b}}}, \qquad y > 0, \quad \_\_\_\_\_\_\_\_(5.13)$$

The CDF is given by

$$\mathrm{F(y)} = 1 - (1 + \lambda y)^{-\mathrm{b}}$$

Equation 5.13 is the general expression of Lomax distribution which becomes equation 3.8 when $\lambda = 1$.

5.6.5 Properties of Generalized Lomax distribution

i. r^{th}-order moments

The r^{th}-order moments of the generalized Lomax distribution can be obtained from the expression of the r^{th}–order moments of the generalized beta distribution of the second kind by substituting $p = 1,\ q = \frac{1}{\lambda}, a = 1$ in equation 5.11.

$$E(Y^r) = \frac{\left(\frac{1}{\lambda}\right)^r \Gamma(1 + r)\Gamma(b - r)}{\Gamma(\mathrm{b})}$$

ii. Mean

The mean is given by

$$E(Y) = \frac{\frac{1}{\lambda}\Gamma(1 + 1)\Gamma(b - 1)}{\Gamma(\mathrm{b})}$$

$$= \frac{\frac{1}{\lambda}(b-2)!}{(\mathrm{b}-1)(\mathrm{b}-2)!} = \frac{1}{\lambda(\mathrm{b}-1)}, \qquad b > 1$$

iii. Mode

The mode of generalized Lomax distribution is given by

$$\text{Mode} = \frac{1}{\lambda}\left(\frac{1-1}{b+1}\right) = 0$$

iv. Variance

The variance of generalized Lomax distribution can be obtained from the variance of generalized beta distribution of the second kind by substituting $p = 1, q = \frac{1}{\lambda}$ and $a = 1$

$$\text{Var(Y)} = \frac{\left(\frac{1}{\lambda}\right)^2 B(1+2, b-2)}{B(1,b)} - \left\{\frac{qB(1+1, b-1)}{B(1,b)}\right\}^2$$

$$= \frac{2\left(\frac{1}{\lambda}\right)^2 (\mathrm{b}-3)!}{(\mathrm{b}-1)(\mathrm{b}-2)(\mathrm{b}-3)!} - \left\{\frac{\left(\frac{1}{\lambda}\right)(b-2)!}{(\mathrm{b}-1)(\mathrm{b}-2)!}\right\}^2$$

$$= \left(\frac{1}{\lambda}\right)^2 \frac{1}{(\mathrm{b}-1)}\left(\frac{2}{(\mathrm{b}-2)} - \frac{1}{(\mathrm{b}-1)}\right)$$

$$= \left(\frac{1}{\lambda}\right)^2 \frac{1}{(\mathrm{b}-1)}\left(\frac{2(b-1)-(b-2)}{(\mathrm{b}-2)(\mathrm{b}-1)}\right)$$

$$= \frac{\mathrm{b}}{\lambda^2(\mathrm{b}-1)^2(\mathrm{b}-2)}, \qquad b > 2$$

Hence the coefficient of variation is given by

$$CV = \frac{\sigma}{\mu}$$

$$CV = \frac{\sqrt{\frac{\mathrm{b}}{\lambda^2(\mathrm{b}-1)^2(\mathrm{b}-2)}}}{\frac{1}{\lambda(\mathrm{b}-1)}}$$

$$= \sqrt{\frac{\lambda^2(\mathrm{b}-1)^2\mathrm{b}}{\lambda^2(\mathrm{b}-1)^2(\mathrm{b}-2)}}$$

$$= \sqrt{\frac{\mathrm{b}}{(\mathrm{b}-2)}}$$

v. Skewness

Skewness of generalized Lomax distribution is given by

$$skewness = E(Y^3) - 3E(Y^2)E(Y) + 2\big(E(Y)\big)^3$$

$$= \frac{\left(\frac{1}{\lambda}\right)^3 \Gamma(4)\Gamma(b-3)}{\Gamma(\mathrm{b})} - 3\frac{\left(\frac{1}{\lambda}\right)^2 \Gamma(3)\Gamma(b-2)}{\Gamma(\mathrm{b})}\frac{1}{\lambda(\mathrm{b}-1)} + 2\frac{1}{\lambda^3(\mathrm{b}-1)^3}$$

$$= \frac{1}{\lambda^3}\left(\frac{6}{(\mathrm{b}-1)(\mathrm{b}-2)(\mathrm{b}-3)} - \frac{6}{(\mathrm{b}-1)^2(\mathrm{b}-2)} + \frac{2}{(\mathrm{b}-1)^3}\right)$$

$$= \frac{1}{\lambda^3}\left(\frac{6(\mathrm{b}^2-2\mathrm{b}+1)+6(-\mathrm{b}^2+4\mathrm{b}-3)+2(\mathrm{b}^2-5\mathrm{b}+6)}{(\mathrm{b}-1)^3(\mathrm{b}-2)(\mathrm{b}-3)}\right)$$

$$= \frac{1}{\lambda^3}\left(\frac{2\mathrm{b}(\mathrm{b}+1)}{(\mathrm{b}-1)^3(\mathrm{b}-2)(\mathrm{b}-3)}\right)$$

$$Standardized\ skewness = \frac{2\mathrm{b}(\mathrm{b}+1)}{\lambda^3(\mathrm{b}-1)^3(\mathrm{b}-2)(\mathrm{b}-3)}.\frac{1}{\left(\sqrt{\frac{\mathrm{b}}{\lambda^2(\mathrm{b}-1)^2(\mathrm{b}-2)}}\right)^3}$$

$$= \frac{2\mathrm{b}(\mathrm{b}+1)}{\lambda^3(\mathrm{b}-1)^3(\mathrm{b}-2)(\mathrm{b}-3)}.\lambda^3(\mathrm{b}-1)^3\left(\sqrt{\frac{(\mathrm{b}-2)}{\mathrm{b}}}\right)^3$$

$$= \frac{2(\mathrm{b}+1)}{(\mathrm{b}-3)}\left(\sqrt{\frac{(\mathrm{b}-2)}{\mathrm{b}}}\right)$$

vi. Kurtosis

Kurtosis of generalized Lomax distribution can be obtained from the kurtosis of generalized beta distribution of the second kind by substituting $p = 1, q = \frac{1}{\lambda}$ and $a = 1$

Kurtosis

$$= \left(\frac{1}{\lambda}\right)^4 \left(\frac{B(1+4,b-4)}{B(1,b)} - 4\frac{B(1+1,b-1)}{B(1,b)}\frac{B(1+3,b-3)}{B(1,b)} + 6\left(\frac{B(1+1,b-1)}{B(1,b)}\right)^2 \frac{B(1+2,b-2)}{B(1,b)} - 3\left(\frac{B(1+1,b-1)}{B(1,b)}\right)^4\right)$$

$$= \left(\frac{1}{\lambda}\right)^4 \left(bB(5,b-4) - 4bB(2,b-1)bB(4,b-3) + 6\big(bB(2,b-1)\big)^2 bB(3,b-2) - 3\big(bB(2,b-1)\big)^4\right)$$

$$= \left(\frac{1}{\lambda}\right)^4 \left(\frac{24}{(\mathrm{b}-1)(\mathrm{b}-2)(\mathrm{b}-3)(\mathrm{b}-4)} - \frac{4}{(\mathrm{b}-1)}\ \frac{6}{(\mathrm{b}-1)(\mathrm{b}-2)(\mathrm{b}-3)} + 6\left(\frac{1}{(\mathrm{b}-1)}\right)^2 \frac{2}{(\mathrm{b}-1)(\mathrm{b}-2)} - 3\left(\frac{1}{(\mathrm{b}-1)}\right)^4\right)$$

$$= \left(\frac{1}{\lambda}\right)^4 \left(\frac{24}{(\mathrm{b}-1)(\mathrm{b}-2)(\mathrm{b}-3)(\mathrm{b}-4)} - \frac{24}{(\mathrm{b}-1)^2(\mathrm{b}-2)(\mathrm{b}-3)} + \frac{12}{(\mathrm{b}-1)^3(\mathrm{b}-2)} - \frac{3}{(\mathrm{b}-1)^4}\right)$$

5.6.6 Inverse Lomax Distribution

Special case of the generalized beta distribution of the second kind when b=1and p=1 in equation 5.10 gives

$$f(\mathrm{y;q,a}) = \frac{ay^{a-1}}{q^a(1+\frac{y}{q})^{1+\mathrm{a}}}, \qquad y > 0, \quad q > 0, \qquad a > 0 \_\_\_\_\_\_\_\_\_\_\_\_\_\_ (5.14)$$

Equation 5.14 is the inverse Lomax distribution

5.6.7 Properties of Inverse Lomax distribution

i. r^{th}-order moments

The r^{th}-order moments of Inverse Lomax distribution can be obtained from the expression of the r^{th}–order moments of the generalized beta distribution of the second kind by substituting $p = 1,\ \ b = 1$ in equation 5.11.

$$E(Y^r) = \frac{(q)^r\Gamma(a+r)\Gamma(1-r)}{\Gamma(a)\Gamma(1)}$$

ii. Mean

The mean of inverse Lomax distribution is given by

$$E(Y) = \frac{(q)\Gamma(a+1)\Gamma(1-1)}{\Gamma(a)\Gamma(1)}$$

iii. Mode

The mode of inverse Lomax distribution is given by

$$Mode = q\left(\frac{a-1}{2}\right)$$

iv. Variance

The variance of inverse Lomax distribution can be obtained from the variance of generalized beta distribution of the second kind by substituting $p = 1,$ and $b = 1$

$$\text{Var(Y)} \ = q^2[bB(3, b-2) - \{bB(2, b-1)\}^2]$$

$$\text{Var(Y)} = q^2[aB(a+2,1-2) - \{aB(a+1,1-1)\}^2]$$

$$= q^2[aB(a+2,-1) - \{aB(a+1,0)\}^2]$$

v. Skewness

The skewness of inverse Lomax distribution can be obtained from the skewness of generalized beta distribution of the second kind by substituting $p = 1,$ and $b = 1$

$$= q^3\left(\frac{B(a+3,1-3)}{B(a,1)} - 3\frac{B(a+1,1-1)}{B(a,1)}\frac{B(a+2,1-2)}{B(a,1)} + 2\left(\frac{B(a+1,1-1)}{B(a,1)}\right)^3\right)$$

vi. Kurtosis

The Kurtosis of inverse Lomax distribution can be obtained from the kurtosis of generalized beta distribution of the second kind by substituting $p = 1,$ and $b = 1$

$$Kurtosis$$

$$= q^4 \left(\frac{B(a+4,1-4)}{B(a,1)} - 4 \frac{B(a+1,1-1)}{B(a,1)} \frac{B(a+3,1-3)}{B(a,1)} + 6 \left(\frac{B(a+1,1-1)}{B(a,1)} \right)^2 \frac{B(a+2,1-2)}{B(a,1)} - 3 \left(\frac{B(a+1,1-1)}{B(a,1)} \right)^4 \right)$$

5.6.8 Dagum (Burr type 3) Distribution

Special case of the generalized beta distribution of the second kind with b=1 in equation 5.10 gives the following density function

$$f(\mathrm{y;p,q,a}) = \frac{|p| a y^{pa-1}}{q^{pa} (1 + (\frac{y}{q})^p)^{1+\mathrm{a}}}, \qquad y > 0, \qquad p > 0, \qquad q > 0,$$

$$a > 0 \_\_\_\_\_\_\_\_\_ (5.15)$$

Equation 5.15 is the Dagum distribution. The Dagum distribution is also known as Burr Type 3 distribution in the statistic literature (Kleiber and Kotz, 2003)

5.6.9 Properties of Dagum distribution

i. r^{th}-order moments

The r^{th}-order moments of the Dagum distribution can be obtained from the expression of the generalized beta distribution of the second kind by substituting b=1 in equation 5.11. Thus

$$E(Y^r) = \frac{q^r B\left(a + \frac{r}{p}, 1 - \frac{r}{p}\right)}{B(a,1)}$$

$$= \frac{q^r \Gamma(a + \frac{r}{p}) \Gamma(1 - \frac{r}{p})}{\Gamma(\mathrm{a})}, \qquad -pa < r < p$$

ii. Mean

The mean is given by

$$E(Y)=\frac{q\Gamma(a+\frac{1}{p})\Gamma(1-\frac{1}{p})}{\Gamma(\mathrm{a})}$$

iii. Mode

The mode of the Dagum distribution is given by

$$\text{Mode}=q\left(\frac{pa-1}{p+1}\right)^{\frac{1}{p}}\ \text{if}\ pa\geq 1$$

and at zero elsewhere.

iv. Variance

$$\text{Var(Y)}=\sigma^2\ =\frac{q^2B\left(a+\frac{2}{p},1-\frac{2}{p}\right)}{B(a,1)}-\left\{\frac{qB\left(a+\frac{1}{p},1-\frac{1}{p}\right)}{B(a,1)}\right\}^2$$

$$=\frac{q^2\mathrm{a}\Gamma\left(a+\frac{2}{p}\right)\Gamma\left(1-\frac{2}{\mathrm{p}}\right)}{\mathrm{a}\Gamma(a)}-\left\{\frac{qa\Gamma(a+\frac{1}{p})\Gamma\left(1-\frac{1}{p}\right)}{\mathrm{a}\Gamma(a)}\right\}^2$$

$$=q^2\left\{\frac{\Gamma(\mathrm{a})\Gamma\left(a+\frac{2}{p}\right)\Gamma\left(1-\frac{2}{p}\right)-\Gamma^2(a+\frac{1}{p})\Gamma^2\left(1-\frac{1}{p}\right)}{\Gamma^2(a)}\right\}$$

Hence the coefficient of variation is

$$CV=\sqrt{\left(\frac{\Gamma(\mathrm{a})\Gamma\left(a+\frac{2}{p}\right)\Gamma\left(1-\frac{2}{p}\right)}{\Gamma^2(a+\frac{1}{p})\Gamma^2\left(1-\frac{1}{p}\right)}-1\right)}$$

5.6.10 Para-logistic distribution

Special case of the generalized beta distribution of the second kind with a=1 and b=p in equation 5.10 (Klugman, Panjer and Willmot, 1998)

$$f(\mathrm{y};\mathrm{p},\mathrm{q}) = \frac{|p|y^{p-1}}{q^p B(1,p)(1+(\frac{y}{q})^p)^{1+\mathrm{p}}}$$

$$= \frac{p^2 y^{p-1}}{q^p(1+(\frac{y}{q})^p)^{1+\mathrm{p}}}\,, \qquad y > 0 \_\_\_\_\_\_\_\_\_\_\_\_\_\_(5.16)$$

5.6.11 General Fisk (Log-Logistic) Distribution Function

Special case of the generalized beta distribution of the second kind when a=b=1 and $q = \frac{1}{\lambda}$ in equation 5.10

$$f(\mathrm{y};\mathrm{p},\lambda) = \frac{|p|z^{p.1-1}}{\left(\frac{1}{\lambda}\right)^{p.1} B(1,1)(1+(\frac{y}{\frac{1}{\lambda}})^p)^{1+1}}$$

$$= \frac{\lambda|p|(\lambda y)^{p-1}}{(1+(y\lambda)^p)^2}\,, y > 0 \_\_\_\_\_\_\_\_\_\_\_\_\_\_(5.17)$$

Equation 5.17 is the general form of the log logistic distribution. The Fisk (Log-logistic) distribution can also be obtained from Singh-Maddala distribution by letting b=1 and q= $\frac{1}{\lambda}$ or from Dagum distribution by letting a=1 and $q = \frac{1}{\lambda}$

5.6.12 Properties of Fisk (log-logistic) distribution

i. r^{th}-order moments

The r^{th}-order moments is obtained from the r^{th}–order moments of the generalized beta distribution of the second kind by substituting a=b=1 in equation 5.11. Thus we get

$$E(Y^r) = q^r B\left(1+\frac{r}{p}, 1-\frac{r}{p}\right)$$

$$= q^r \Gamma\left(1+\frac{r}{p}\right)\Gamma\left(1-\frac{r}{p}\right), \qquad -p < r < p$$

ii. Mean

The mean is given by

$$E(Y) = q\Gamma\left(1+\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)$$

$$= q\frac{1}{p}\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)$$

iii. Mode

The mode of the Fisk distribution is given by

$$\text{Mode} = q\left(\frac{p-1}{p+1}\right)^{\frac{1}{p}} \text{ if } p > 1$$

and at zero for $p \leq 1$.

iv. Variance

$$\text{Var(Y)} = \sigma^2 = q^2 B\left(1+\frac{2}{p}, 1-\frac{2}{p}\right) - \left\{qB\left(1+\frac{1}{p}, 1-\frac{1}{p}\right)\right\}^2$$

$$= q^2\Gamma\left(1+\frac{2}{p}\right)\Gamma\left(1-\frac{2}{\mathrm{p}}\right)-\left\{q\Gamma(1+\frac{1}{p})\Gamma\left(1-\frac{1}{p}\right)\right\}^2$$

$$= q^2\left\{\Gamma\left(1+\frac{2}{p}\right)\Gamma\left(1-\frac{2}{p}\right)-\Gamma^2(1+\frac{1}{p})\Gamma^2\left(1-\frac{1}{p}\right)\right\}$$

5.6.13 Fisher (F) Distribution Function

Special case of the generalized beta distribution of the second kind when $a=\frac{u}{2}, b=\frac{v}{2}, p=1$ and $q=\frac{v}{u}$ in equation 5.10

$$f(\mathrm{y};u,v)=\frac{|1|y^{1.\frac{u}{2}-1}}{\left(\frac{v}{u}\right)^{1.\frac{u}{2}}B\left(\frac{u}{2},\frac{v}{2}\right)\left(1+\left(\frac{y}{\frac{v}{u}}\right)^{1}\right)^{\frac{u}{2}+\frac{v}{2}}}$$

$$=\frac{\Gamma\left(\frac{u+v}{2}\right)\left(\frac{u}{v}\right)^{\frac{u}{2}}y^{\frac{u}{2}-1}}{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{v}{2}\right)\left(1+\left(\frac{yu}{v}\right)\right)^{\frac{u+v}{2}}} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(5.18)$$

Equation 5.18 is the Fisher (F) distribution.

5.6.14 Properties of F distribution

i. r^{th}-order moments

The r^{th}-order moments is obtained from the r^{th}–order moments of the generalized beta distribution of the second kind by substituting $a=\frac{u}{2}, b=\frac{v}{2}, p=1$ and $q=\frac{v}{u}$ in equation 5.11. Thus we get

$$E(Y^r)=\frac{\left(\frac{v}{u}\right)^r B\left(\frac{u}{2}+r,\frac{v}{2}-r\right)}{B\left(\frac{u}{2},\frac{v}{2}\right)}$$

$$=\frac{\left(\frac{v}{u}\right)^r\Gamma\left(\frac{u}{2}+r\right)\Gamma\left(\frac{v}{2}-r\right)}{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{v}{2}\right)}, \qquad -\frac{u}{2}<r<1$$

ii. Mean

The mean of F distribution is given by

$$E(Y) = \frac{\left(\frac{v}{u}\right)\Gamma\left(\frac{u}{2}+1\right)\Gamma\left(\frac{v}{2}-1\right)}{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{v}{2}\right)}$$

$$= \frac{\left(\frac{v}{u}\right)\frac{u}{2}\Gamma\left(\frac{u}{2}\right)\left(\frac{v}{2}-2\right)!}{\Gamma\left(\frac{u}{2}\right)\left(\frac{v}{2}-1\right)\left(\frac{v}{2}-2\right)!}$$

$$= \frac{\left(\frac{v}{2}\right)}{\left(\frac{v}{2}-1\right)}$$

$$= \frac{v}{v-2}, \qquad v > 2$$

iii. Mode

The mode of F distribution is given by

$$\text{Mode} = \frac{v}{u}\left(\frac{\frac{u}{2}-1}{\frac{v}{2}+1}\right)$$

$$= \frac{v}{u}\left(\frac{u-2}{v+2}\right)$$

$$= \left(\frac{u-2}{u}\right)\left(\frac{v}{v+2}\right), u > 2$$

iv. Variance

The variance for F distribution can be given by

$$\text{Var(Y)} = \frac{(\frac{v}{u})^2 B\left(\frac{u}{2}+2, \frac{v}{2}-2\right)}{B(\frac{u}{2}, \frac{v}{2})} - \left\{\frac{v}{v-2}\right\}^2$$

$$= \frac{(\frac{v}{u})^2 \Gamma\left(\frac{u}{2}+2\right)\Gamma\left(\frac{v}{2}-2\right)\Gamma\left(\frac{\mathrm{u+v}}{2}\right)}{\Gamma\left(\frac{\mathrm{u+v}}{2}\right)\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{v}{2}\right)} - \left\{\frac{v}{v-2}\right\}^2$$

$$= \frac{(\frac{v}{u})^2 \left(\frac{u}{2}+1\right)\frac{u}{2}\Gamma\left(\frac{u}{2}\right)\left(\frac{v}{2}-3\right)!}{\Gamma\left(\frac{u}{2}\right)\left(\frac{v}{2}-1\right)\left(\frac{v}{2}-2\right)\left(\frac{v}{2}-3\right)!} - \left\{\frac{v}{v-2}\right\}^2$$

$$= \frac{(\frac{v}{u})^2 \left(\frac{u}{2}+1\right)\frac{u}{2}}{\left(\frac{v}{2}-1\right)\left(\frac{v}{2}-2\right)} - \left\{\frac{v}{v-2}\right\}^2$$

$$= v^2\left(\frac{\frac{1}{u}\left(\frac{u+2}{2}\right)\frac{1}{2}}{\left(\frac{v-2}{2}\right)\left(\frac{v-4}{2}\right)} - \left\{\frac{1}{v-2}\right\}^2\right)$$

$$= v^2\left(\frac{u+2}{u(v-2)(v-4)} - \left\{\frac{1}{v-2}\right\}^2\right)$$

$$= v^2\left(\frac{(vu+2v-2u-4)-uv+4u}{u(v-2)^2(v-4)}\right)$$

$$= 2v^2\left(\frac{u+v-2}{u(v-2)^2(v-4)}\right), v > 4$$

5.6.15 Half Student's t distribution

Special case of the generalized beta distribution of the second kind when $a = \frac{1}{2}, b = \frac{r}{2}, p = 2$ and $q = \sqrt{r}$ in equation 5.10

$$f(\mathrm{y}; r) = \frac{|2| y^{2.\frac{1}{2}-1}}{(\sqrt{r})^{2.\frac{1}{2}} B\left(\frac{1}{2}, \frac{r}{2}\right)\left(1+\left(\frac{z}{\sqrt{r}}\right)^2\right)^{\frac{1}{2}+\frac{r}{2}}}$$

$$= \frac{2\Gamma\left(\frac{1+r}{2}\right)}{\sqrt{r\pi}\,\Gamma\left(\frac{r}{2}\right)\left(1+\frac{y^2}{r}\right)^{\frac{1+r}{2}}}, \qquad -\infty < y < \infty \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_\_(5.19)}$$

Equation 5.19 is the Half student's t distribution.

5.6.16 Logistic Distribution Function

Letting $y = e^x$ and $\left|\frac{dy}{dx}\right| = e^x$

In equation 5.10 for the generalized beta distribution of the second kind, we get the following density function

$$f(\mathrm{x}; \mathrm{p}, \lambda) = \frac{|p| e^{pax-x}}{q^{pa} B(a,b)(1 + (\frac{e^x}{q})^p)^{\mathrm{a+b}}} \cdot e^x$$

$$= \frac{|p| e^{pax}}{e^{palog\,(q)} B(a,b)(1 + (e^x e^{-\log(q)})^p)^{\mathrm{a+b}}}$$

$$= \frac{|p| e^{pa(x-log\,(q))}}{B(a,b)(1 + e^{p(x-\log\,\mathrm{q})})^{\mathrm{a+b}}}, \qquad -\infty < x < \infty \quad \_\_\_\_\_\_\_\_\_\_\_\_(5.20)$$

Equation 5.20 is the generalized logistic distribution. McDonald and Xu (1995) referred to equation 5.20 as the density of an exponential generalized distribution. The standard logistic distribution can be obtained by setting $\mathrm{p = q = a = b = 1}$ as

$$f(x) = \frac{e^x}{(1 + e^x)^2}$$

5.6.17 Generalized Gamma Distribution Function

We show that the generalized beta distribution of the second kind approaches the generalized gamma distribution as b→∞ as follows:

Let $\mathrm{q} = \beta \mathrm{b}^{\frac{1}{p}}$. Substituting this expression in equation 5.10 yields

$$f(y; p, \beta, a, b) = \frac{|p| y^{pa-1}}{(\beta \mathrm{b}^{\frac{1}{p}})^{pa} B(a,b)(1 + (\frac{y}{\beta \mathrm{b}^{\frac{1}{p}}})^p)^{a+b}}$$

$$= \frac{|p| y^{pa-1}}{b^a \beta^{pa} B(a,b)(1 + \frac{1}{b}(\frac{y}{\beta})^p)^{a+b}}$$

Grouping the terms gives

$$f(y;p,\beta,a,b) = \left(\frac{|p|y^{pa-1}}{\Gamma(a)\beta^{pa}}\right)\left(\frac{\Gamma(a+b)}{\Gamma(b)\mathrm{b}^{a}}\right)\left(\frac{1}{(1+\frac{1}{b}(\frac{y}{\beta})^{p})}\right)^{a+b}$$

For large values of b, the gamma function can be approximated by Stirling's formula

thus

$$\left(\frac{\Gamma(a+b)}{\Gamma(b)\mathrm{b}^{a}}\right) = \frac{e^{-a-b}(a+b)^{a+b-\frac{1}{2}}\sqrt{2\pi}}{e^{-b}\,b^{b-\frac{1}{2}}\sqrt{2\pi}\,b^{a}}$$

$$\lim_{\mathrm{b}\to\infty}\left(\frac{\Gamma(a+b)}{\Gamma(b)\mathrm{b}^{a}}\right) = \lim_{\mathrm{b}\to\infty}\frac{e^{-a}(a+b)^{b-\frac{1}{2}}\sqrt{2\pi}}{b^{b-\frac{1}{2}}\sqrt{2\pi}} = 1$$

Similarly, $$\lim_{\mathrm{b}\to\infty}\left(\frac{1}{(1+\frac{1}{b}(\frac{y}{\beta})^{p})^{a+b}}\right) = e^{-\left(\frac{y}{\beta}\right)^{p}}$$

taking the limit of $f(y;p,\beta,a,b)$ as $b\to\infty$ yields

$$f(\mathrm{y};\mathrm{p},\beta,\mathrm{a}) = \frac{|p|y^{pa-1}}{\beta^{pa}\Gamma(a)}\,e^{-\left(\frac{y}{\beta}\right)^{p}}\quad,\quad 0\le y \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(5.21)$$

Equation 5.21 is the generalized gamma (GG) distribution function equivalent to the one derived in equation 5.8.

5.6.18 Properties of Generalized Gamma distribution

i. r^{th}-order moments

To obtain the r^{th}-order moments of the generalized gamma distribution, we evaluate

$$E(Y^r) = \int_0^\infty y^r f(y; p, \beta, a) dy$$

$$= \int_0^\infty \frac{y^r |p| z^{pa-1}}{\beta^{pa}\Gamma(a)} \; e^{-\left(\frac{y}{\beta}\right)^p} dy$$

$$= \int_0^\infty \frac{|p| y^{(pa+r)-1}}{\beta^{pa}\Gamma(a)} \; e^{-\left(\frac{y}{\beta}\right)^p} dy$$

$$= \int_0^\infty \frac{|p| y^{p(a+\frac{r}{p})-1} . \Gamma\left(a+\frac{r}{p}\right)}{\beta^{p(a+\frac{r}{p})} . \beta^{-r} . \Gamma(a)\Gamma\left(a+\frac{r}{p}\right)} \; e^{-\left(\frac{z}{\beta}\right)^a} dy$$

$$= \frac{\beta^r \Gamma\left(a+\frac{r}{p}\right)}{\Gamma(a)} \int_0^\infty \frac{|p| y^{p(a+\frac{r}{p})-1}}{\beta^{p(a+\frac{r}{p})}\Gamma\left(a+\frac{r}{p}\right)} \; e^{-\left(\frac{y}{\beta}\right)^p} dy$$

The expression under the integral sign integrates to 1

$$\therefore E(Y^r) = \frac{\beta^r \Gamma\left(a+\frac{r}{p}\right)}{\Gamma(a)}, \qquad -\text{pa} < r < \infty \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (5.22)$$

ii. Mean

The mean is given by

$$E(Y) = \frac{\beta\Gamma\left(a+\frac{1}{p}\right)}{\Gamma(a)}$$

iii. Mode

The mode of the generalized gamma distribution occurs at

$$\frac{\text{d}}{\text{dy}} f(y) = 0$$

$$\frac{\text{d}}{\text{dy}}\left(\frac{|p| y^{pa-1}}{\beta^{pa}\Gamma(a)} \; e^{-\left(\frac{y}{\beta}\right)^p}\right) = 0$$

$$(pa-1)y^{pa-1} . y^{-1} e^{-\left(\frac{y}{\beta}\right)^p} - \frac{p y^{p-1}}{\beta^p} . y^{pa-1} e^{-\left(\frac{y}{\beta}\right)^p} = 0$$

$$(pa-1) = \frac{p y^p}{\beta^p}$$

$$y = \beta\left(a - \frac{1}{p}\right)^{\frac{1}{p}}$$

and at zero for $p \leq 1$.

iv. Variance

$$\mathrm{Var(Y)} = \frac{\beta^2\Gamma\left(a+\frac{2}{p}\right)}{\Gamma(\mathrm{a})} - \left\{\frac{\beta\Gamma\left(a+\frac{1}{p}\right)}{\Gamma(\mathrm{a})}\right\}^2$$

$$= \frac{\beta^2\Gamma(\mathrm{a})\Gamma\left(a+\frac{2}{p}\right) - \beta^2\Gamma^2\left(a+\frac{1}{p}\right)}{\Gamma^2(\mathrm{a})}$$

$$= \frac{\beta^2}{\Gamma^2(\mathrm{a})}\left(\Gamma(\mathrm{a})\Gamma\left(a+\frac{2}{p}\right) - \Gamma^2\left(a+\frac{1}{p}\right)\right)$$

v. Skewness

Skewness of the generalized gamma distribution can be expressed as follows

$$\text{Skewness} = E(Y^3) - 3E(Y^2)E(Y) + 2\{E(Y)\}^3$$

$$= \frac{\beta^3\Gamma\left(a+\frac{3}{p}\right)}{\Gamma(a)} - \frac{3\beta^3\Gamma\left(a+\frac{2}{p}\right)\Gamma\left(a+\frac{1}{p}\right)}{\Gamma^2(a)} + \frac{2\beta^3\Gamma^3\left(a+\frac{1}{p}\right)}{\Gamma^3(a)}$$

$$= \frac{\beta^3\Gamma^2(a)\Gamma\left(a+\frac{3}{p}\right) - 3\beta^3\Gamma(\mathrm{a})\Gamma\left(a+\frac{2}{p}\right)\Gamma\left(a+\frac{1}{p}\right) + 2\beta^3\Gamma^3\left(a+\frac{1}{p}\right)}{\Gamma^3(a)}$$

vi. Kurtosis

Kurtosis of the generalized gamma distribution can be expressed as follows

$$\text{Kurtosis} = E(Y^4) - 4E(Y^3)E(Y) + 6E(Y^2)\{E(Y)\}^2 - 3\{E(Y)\}^4$$

$$= \frac{\beta^4\Gamma\left(a+\frac{4}{p}\right)}{\Gamma(a)} - \frac{4\beta^3\Gamma\left(a+\frac{3}{p}\right)}{\Gamma(a)}\frac{\beta\Gamma\left(a+\frac{1}{p}\right)}{\Gamma(\mathrm{a})} + \frac{6\beta^2\Gamma\left(a+\frac{2}{p}\right)}{\Gamma(a)}\frac{\beta^2\Gamma^2\left(a+\frac{1}{p}\right)}{\Gamma^2(a)}$$

$$- 3\frac{\beta^4\Gamma^4\left(a+\frac{1}{p}\right)}{\Gamma^4(a)}$$

$$= \beta^4 \left(\frac{\Gamma\left(a+\frac{4}{p}\right)}{\Gamma(a)} - \frac{4\Gamma\left(a+\frac{3}{p}\right)\Gamma\left(a+\frac{1}{p}\right)}{\Gamma^2(a)} + \frac{6\Gamma\left(a+\frac{2}{p}\right)\Gamma^2\left(a+\frac{1}{p}\right)}{\Gamma^3(a)} - 3\frac{\Gamma^4\left(a+\frac{1}{p}\right)}{\Gamma^4(a)} \right)$$

5.6.19 Gamma Distribution Function

Special case of the generalized gamma distribution derived in equation 5.21 when p=1 gives

$$f(\mathrm{y};\beta,\mathrm{a}) = \frac{y^{a-1}}{\beta^a\Gamma(a)}\, e^{-\left(\frac{y}{\beta}\right)}, \quad 0 \le y \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_\_(5.23)}$$

Equation 5.23 is the gamma distribution.

5.6.20 Properties of Gamma distribution

i. r^{th}-order moments

The r^{th}-order moments of the Gamma distribution can be obtained from the expression of the moments of the generalized Gamma distribution by substituting p=1 in equation 5.22. Thus

$$E(Y^r) = \frac{\beta^r\Gamma\left(a+\frac{r}{p}\right)}{\Gamma(a)}$$

$$= \frac{\beta^r\Gamma(a+r)}{\Gamma(a)}$$

ii. Mean

The mean is given by

$$E(Y) = \frac{\beta\Gamma(a+1)}{\Gamma(a)} = \beta\mathrm{a}$$

iii. Mode

The mode of the Gamma distribution occurs at

$$\text{Mode} = \beta\left(a - \frac{1}{p}\right)^{\frac{1}{p}} = \beta(\mathrm{a}-1), \qquad \mathrm{a} > 1$$

iv. Variance

The variance of gamma distribution can be obtained from the variance of generalized gamma distribution by substituting p=1 as follows:

$$\mathrm{Var(Y)} = \frac{\beta^2}{\Gamma^2(\mathrm{a})}\left(\Gamma(\mathrm{a})\Gamma(a+2) - \Gamma^2(a+1)\right)$$

$$= \frac{\beta^2}{\Gamma^2(\mathrm{a})}\left(\Gamma(\mathrm{a})(\mathrm{a}+1)(\mathrm{a})\Gamma(a) - (\mathrm{a})\Gamma(a)(\mathrm{a})\Gamma(a)\right)$$

$$= \frac{\beta^2}{\Gamma^2(\mathrm{a})}(\mathrm{a})\Gamma^2(\mathrm{a})\left((\mathrm{a}+1) - (\mathrm{a})\right)$$

$$= \mathrm{a}\beta^2$$

v. Skewness

$$Skewness = E(Y^3) - 3E(Y^2)E(Y) + 2\{E(Y)\}^3$$

$$= \frac{\beta^3\Gamma(a+3)}{\Gamma(a)} - 3\left(\frac{\beta^2\Gamma(a+2)}{\Gamma(a)}\right).a\beta + 2(a\beta)^3$$

$$= \frac{\beta^3(\mathrm{a}+2)(\mathrm{a}+1)(\mathrm{a})\Gamma(a)}{\Gamma(a)} - \left(\frac{3a\beta^3(\mathrm{a}+1)(\mathrm{a})\Gamma(a)}{\Gamma(a)}\right) + 2(a\beta)^3$$

$$= \beta^3(\mathrm{a}+2)(\mathrm{a}+1)(\mathrm{a}) - 3a\beta^3(\mathrm{a}+1)(\mathrm{a}) + 2(a\beta)^3$$

$$= \mathrm{a}^3\beta^3 + 3\mathrm{a}^2\beta^3 + 2a\beta^3 - 3\mathrm{a}^3\beta^3 - 3\mathrm{a}^2\beta^3 + 2\mathrm{a}^3\beta^3$$

$$= 2a\beta^3$$

The standardized skewness is given by

$$Standardized\ skewness = \frac{2a\beta^3}{(a\beta^2)^{\frac{3}{2}}}$$

$$= \frac{2}{\sqrt{a}}$$

vi. Kurtosis

Kurtosis is given by

$$Kurtosis = E(Y^4) - 4E(Y^3)E(Y) + 6E(Y^2)\{E(Y)\}^2 - 3\{E(Y)\}^4$$

$$= \left(\frac{\beta^4\Gamma(a+4)}{\Gamma(a)}\right) - 4\left(\frac{\beta^3\Gamma(a+3)}{\Gamma(a)}\right).a\beta + 6\left(\frac{\beta^2\Gamma(a+2)}{\Gamma(a)}\right).(a\beta)^2 - 3(a\beta)^4$$

$$= \left(\beta^4(a+3)(a+2)(a+1)(a)\right) - 4a\beta^4(a+2)(a+1)(a) + 6a^2\beta^4(a+1)(a) - 3a^4\beta^4$$

$$= \left(a^4\beta^4 + 6a^3\beta^4 + 11a^2\beta^4 + 6a\beta^4 - 4a^4\beta^4 - 12a^3\beta^4 - 8a^2\beta^4 + 6a^4\beta^4 + 6a^3\beta^4 - 3a^4\beta^4\right)$$

$$= 3a^2\beta^4 + 6a\beta^4$$

The standardized kurtosis is given by

$$Standardized\ kutosis = \frac{3a^2\beta^4 + 6a\beta^4}{a^2\beta^4}$$

$$= 3 + \frac{6}{a}$$

5.6.21 Inverse Gamma distribution

Special case of the generalized gamma distribution derived in equation 5.21 when $p = -1$ gives

$$f(y;\beta,a) = \frac{y^{-a-1}}{\beta^{-a}\Gamma(a)}\ e^{-\left(\frac{y}{\beta}\right)^{-1}}, \quad 0 \le y$$

$$= \frac{\beta^a}{y^{1+a}\Gamma(a)}\ e^{-\left(\frac{\beta}{y}\right)}, \quad 0 \le y \text{\_\_\_\_\_\_\_\_\_\_\_\_}(5.24)$$

Equation 5.24 is the inverse gamma distribution

5.6.22 Properties of Inverse Gamma distribution

i. r^{th}-order moments

The r^{th}-order moments of the Inverse Gamma distribution can be obtained from the expression of the moments of the generalized Gamma distribution by substituting $p = -1$ in equation 5.22.

$$E(Y^r) = \frac{\beta^r \Gamma(a-r)}{\Gamma(a)}$$

ii. Mean

The mean is given by

$$E(Y) = \frac{\beta\Gamma(a-1)}{\Gamma(a)}$$

$$= \frac{\beta(a-2)!}{(\mathrm{a}-1)(\mathrm{a}-2)!}$$

$$= \frac{\beta}{\mathrm{a}-1}$$

iii. Mode

The mode of the inverse Gamma distribution occurs at

$$\text{Mode} = \beta(a+1)^{-1}$$

$$= \frac{\beta}{(\mathrm{a}+1)}, \qquad \mathrm{a} > 1$$

iv. Variance

The variance of inverse gamma distribution can be obtained from the variance of generalized gamma distribution by substituting $p = -1$ as follows:

$$\text{Var}(Y) = \frac{\beta^2}{\Gamma^2(\mathrm{a})}\left(\Gamma(\mathrm{a})\Gamma(a-2) - \Gamma^2(a-1)\right)$$

$$= \frac{\beta^2((a-1)!\,(a-3)! - (a-2)!\,(a-2)!}{(\mathrm{a}-1)!\,(\mathrm{a}-1)!}$$

$$= \beta^2 \left(\frac{1}{(\mathrm{a}-1)(\mathrm{a}-2)} - \frac{1}{(\mathrm{a}-1)(\mathrm{a}-1)}\right)$$

$$= \frac{\beta^2}{(a-1)^2(\mathrm{a}-2)}, \qquad a > 2$$

v. Skewness

Skewness of inverse gamma distribution can be as follows

$$Skewness = E(Y^3) - 3E(Y^2)E(Y) + 2\{E(Y)\}^3$$

$$= \frac{\beta^3\Gamma(a-3)}{\Gamma(a)} - 3\left(\frac{\beta^2\Gamma(a-2)}{\Gamma(a)}\right).\frac{\beta}{\mathrm{a}-1} + 2\left(\frac{\beta}{\mathrm{a}-1}\right)^3$$

$$= \frac{\beta^3(\mathrm{a}-4)!}{(\mathrm{a}-1)(\mathrm{a}-2)(\mathrm{a}-3)(\mathrm{a}-4)!} - \left(\frac{3\beta^3(\mathrm{a}-3)!}{(\mathrm{a}-1)^2(\mathrm{a}-2)(\mathrm{a}-3)!}\right) + \frac{2\beta^3}{(a-1)^3}$$

$$= \frac{\beta^3}{(\mathrm{a}-1)(\mathrm{a}-2)(\mathrm{a}-3)} - \left(\frac{3\beta^3}{(\mathrm{a}-1)^2(\mathrm{a}-2)}\right) + \frac{2\beta^3}{(a-1)^3}$$

$$= \beta^3\left(\frac{(a^2-2a+1)-(3a^2-12a+9)+2(a^2-5a+6)}{(\mathrm{a}-1)^3(\mathrm{a}-2)(\mathrm{a}-3)}\right)$$

$$= \beta^3\left(\frac{4}{(\mathrm{a}-1)^3(\mathrm{a}-2)(\mathrm{a}-3)}\right)$$

$$= \frac{4\beta^3}{(\mathrm{a}-1)^3(\mathrm{a}-2)(\mathrm{a}-3)}$$

The standardized skewness is given by

$$Standardized\ skewness = \frac{\left(\frac{4\beta^3}{(\mathrm{a}-1)^3(\mathrm{a}-2)(\mathrm{a}-3)}\right)}{\left(\frac{\beta^2}{(a-1)^2(\mathrm{a}-2)}\right)^{\frac{3}{2}}}$$

$$= \frac{4\beta^3}{(\mathrm{a}-1)^3(\mathrm{a}-2)(\mathrm{a}-3)}\frac{(\mathrm{a}-1)^3(\mathrm{a}-2)\sqrt{(\mathrm{a}-2)}}{\beta^3}$$

$$= \frac{4\sqrt{(a-2)}}{(a-3)}$$

vi. Kurtosis

Kurtosis is given by

$$Kurtosis = E(Y^4) - 4E(Y^3)E(Y) + 6E(Y^2)\{E(Y)\}^2 - 3\{E(Y)\}^4$$

$$= \left(\frac{\beta^4\Gamma(a-4)}{\Gamma(a)}\right) - 4\left(\frac{\beta^3\Gamma(a-3)}{\Gamma(a)}\right).\frac{\beta}{a-1} + 6\left(\frac{\beta^2\Gamma(a-2)}{\Gamma(a)}\right).\left(\frac{\beta}{a-1}\right)^2 - 3\left(\frac{\beta}{a-1}\right)^4$$

$$= \frac{\beta^4}{(a-1)(a-2)(a-3)(a-4)} - \frac{4\beta^4}{(a-1)^2(a-2)(a-3)} + \frac{6\beta^4}{(a-1)^3(a-2)} - 3\left(\frac{\beta}{a-1}\right)^4$$

$$= \beta^4\left(\frac{(a-1)^3 - 4(a-1)^2(a-4) + 6(a-1)(a-3)(a-4) - 3(a-2)(a-3)(a-4)}{(a-1)^4(a-2)(a-3)(a-4)}\right)$$

$$= \beta^4\left(\frac{a^3 - 3a^2 + 3a - 1 - 4a^3 + 24a^2 - 36a + 16 + 6a^3 - 48a^2 + 114a - 72 - 3a^3 + 27a^2 - 78a + 72}{(a-1)^4(a-2)(a-3)(a-4)}\right)$$

$$= \beta^4\left(\frac{3a+15}{(a-1)^4(a-2)(a-3)(a-4)}\right)$$

The standardized kurtosis is given by

$$Standardized\ kurtosis = \beta^4\left(\frac{3a+15}{(a-1)^4(a-2)(a-3)(a-4)}\right).\frac{(a-1)^4(a-2)^2}{\beta^4}$$

$$= \frac{3(a^2+3a-10)}{(p-3)(a-4)}$$

$$= \frac{3(a-2)(a+5)}{(a-3)(a-4)}, \qquad a > 4$$

5.6.23 Weibull Distribution Function

Special case of the generalized gamma distribution derived in equation 5.21 when a=1

$$f(\mathrm{y};\mathrm{p},\beta) = \frac{|p|y^{p-1}}{\beta^p}\ e^{-\left(\frac{y}{\beta}\right)^p} \quad , \quad 0 \le y$$

$$= \left(\frac{|p|}{\beta^p}\right) y^{p-1}\ e^{-\left(\frac{y}{\beta}\right)^p} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_(5.25)}$$

The cdf of the distribution in equation 5.25 is given by

$$F(y) \ = 1 - \ e^{-\left(\frac{y}{\beta}\right)^p}$$

Equation 5.25 is the Weibull distribution with $p > 0$ as the shape parameter and $\beta > 0$ as the scale parameter.

5.6.24 Properties of Weibull distribution

i. r^{th}-order moments

The r^{th}-order moments of the Weibull distribution can be obtained from the expression of the moments of the generalized Gamma distribution by substituting a=1 in equation 5.22. Thus

$$E(Y) = \frac{\beta^r \Gamma\left(1 + \frac{r}{p}\right)}{\Gamma(1)}$$

$$= \beta^r \Gamma\left(1 + \frac{r}{p}\right)$$

ii. Mean

The mean is given by

$$E(Y) = \beta\Gamma\left(1 + \frac{1}{p}\right) = \frac{\beta}{\mathrm{p}}\Gamma\left(\frac{1}{\mathrm{p}}\right)$$

iii. Mode

The mode occurs at

$$\frac{d}{dy}f(y) = 0$$

$$\frac{d}{dz}\left(\frac{|p|}{\beta^p}\right)y^{p-1}\ e^{-\left(\frac{y}{\beta}\right)^p} = 0$$

$$\left(\frac{|p|}{\beta^p}\right)\left((p-1)y^{p-2}\ e^{-\left(\frac{y}{\beta}\right)^p} - \frac{py^{p-1}}{\beta^p}e^{-\left(\frac{y}{\beta}\right)^p}.\,y^{p-1}\right) = 0$$

$$(p-1)y^{-1} = \frac{p}{\beta^p}.\,y^{p-1}$$

$$\beta^p(p-1) = py^p$$

$$y = \beta\left(\frac{p-1}{p}\right)^{\frac{1}{p}}, \qquad p > 0$$

iv. Variance

$$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2$$

$$= \beta^2\Gamma\left(1+\frac{2}{p}\right) - \beta^2\Gamma^2\left(1+\frac{1}{p}\right)$$

$$= \beta^2\Gamma\left(1+\frac{2}{p}\right) - \mu^2$$

5.6.25 Exponential distribution Function

Special case of the generalized gamma distribution in equation 5.21 with p=a=1 gives

$$f(\mathrm{y;p}=1,\beta,\mathrm{a}=1) = \frac{|1|z^{1-1}}{\beta^1\Gamma(1)}\ e^{-(\frac{y}{\beta})^1} \qquad 0 \le y$$

$$= \frac{1}{\beta}\ e^{-(\frac{y}{\beta})}\ , \quad 0 \le y \_\_\_\_\_\_\_\_\_\_\_\_\_ (5.26)$$

and the cdf is given by

$$F(Y) = 1 - e^{-(\frac{y}{\beta})}$$

Equation 5.26 is the exponential distribution. This exponential distribution can be expressed as a special case of gamma distribution in equation 5.23 with a=1 and also a special case of Weibull distribution in equation 5.25 with p=1.

5.6.26 Properties of exponential distribution

i. r^{th}-order moments

The r^{th}-order moments for the exponential distribution can be obtained from the expression of moments of the generalized gamma distribution in equation 5.22 by substituting p = a =1

$$E(Y^r) = \frac{\beta^r \Gamma\left(1 + \frac{r}{1}\right)}{\Gamma(1)}$$

$$= \beta^r \Gamma(1 + r)$$

$$= \beta^r r\Gamma(r)$$

ii. Mean

The mean is given by

$$E(Y) = \beta^1(1)$$

$$= \beta$$

iii. Mode

The mode of the exponential distribution occurs at

$$\frac{d}{dy} f(y) = 0$$

$$= \frac{d}{dy}\left(\frac{1}{\beta} e^{-\frac{y}{\beta}}\right) = 0$$

$$= \frac{1}{\beta}\left(-\frac{1}{\beta} e^{-\frac{y}{\beta}}\right) = 0$$

$$-\frac{y}{\beta} = 0$$

$$y = 0$$

iv. Variance

From the r^th^-order moments given by

$$\mathrm{E}(\mathrm{Y}^{\mathrm{r}}) = \beta^{r}\,\mathrm{r}$$

the variance is given by $\sigma^2 = 2\beta^2 - (\beta)^2$

$$= \beta^2$$

5.6.27 Chi-Squared distribution function

Special case of the generalized gamma distribution in equation 5.21 when p=1, $\mathrm{a} = \frac{\mathrm{n}}{2}$ and $\beta = 2$ gives

$$f(\mathrm{y};\ \mathrm{n}) = \frac{|1|y^{\frac{\mathrm{n}}{2}-1}}{2^{\frac{\mathrm{n}}{2}}\Gamma\left(\frac{\mathrm{n}}{2}\right)}\ e^{-\left(\frac{y}{2}\right)^{1}}\ ,\quad 0 \leq y$$

$$= \frac{1}{2^{\frac{\mathrm{n}}{2}}\Gamma\left(\frac{\mathrm{n}}{2}\right)} y^{\frac{\mathrm{n}}{2}-1} e^{-\frac{y}{2}}\ ,\quad 0 \leq y \_\_\_\_\_\_\_\_\_\_\_\_\_(5.27)$$

Equation 5.27 is the chi-squared distribution. Chi-squared distribution is a special case of the gamma distribution in equation 5.24 when $a = \frac{n}{2}$ and $\beta = 2$

5.6.28 Properties of Chi-squared distribution

i. r^{th}-order moments

The r^{th}-order moments for the Chi-squared distribution can be obtained from the expression of moments of the generalized gamma distribution in equation 5.22 by substituting $p = 1, \mathrm{a} = \frac{\mathrm{n}}{2}$ and $\beta = 2$

$$E(Y^r) = \frac{2^r\Gamma\left(\frac{\mathrm{n}}{2} + \frac{r}{1}\right)}{\Gamma\left(\frac{\mathrm{n}}{2}\right)}$$

$$= \frac{2^r \Gamma\left(\frac{n}{2} + r\right)}{\Gamma\left(\frac{n}{2}\right)}$$

ii. Mean

The mean is given by

$$E(Y) = \frac{2\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n}{2}\right)}$$

$$= \frac{2(\frac{n}{2})\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

$$= n$$

iii. Mode

The mode of the Chi-squared distribution can be obtained from the mode of generalized gamma distribution by substituting $p = 1, a = \frac{n}{2}$ and $\beta = 2$ and is given by

$$y = 2\left(\frac{n}{2} - 1\right)$$

$$= n - 2$$

iv. Variance

The variance of Chi-squared distribution can be obtained from the variance of generalized gamma distribution by substituting $p = 1, a = \frac{n}{2}$ and $\beta = 2$

$$\mathrm{Var}(Y) = \frac{4}{\Gamma^2\left(\frac{n}{2}\right)}\left(\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2} + 2\right) - \Gamma^2\left(\frac{n}{2} + 1\right)\right)$$

$$= 4\left(\frac{n}{2}\left(\frac{n}{2} + 1\right) - \left(\frac{n}{2}\right)^2\right)$$

$$= 2n$$

5.6.29 Rayleigh distribution function

Special case of the generalized gamma distribution in equation 5.21 when p=2, $a = 1$ and $\beta = \alpha\sqrt{2}$ gives

$$f(\mathrm{y}; \alpha) = \frac{|2| y^{2-1}}{(\alpha\sqrt{2})^2 \Gamma(1)} e^{-\left(\frac{y}{\alpha\sqrt{2}}\right)^2}, \quad 0 \le y$$

$$= \frac{y}{\alpha^2} e^{-\left(\frac{y}{\alpha\sqrt{2}}\right)^2}, \quad 0 \le y \_\_\_\_\_\_\_\_\_\_\_\_ (5.28)$$

The cdf is given by

$$F(\mathrm{y}) = 1 - e^{-\frac{y^2}{2\alpha^2}}, \quad 0 \le y$$

Equation 5.28 is the Rayleigh distribution. Rayleigh distribution is a special case of Weibull distribution with p=2 and $\beta = \alpha\sqrt{2}$ in equation 5.25.

5.6.30 Properties of Rayleigh distribution

i. r^{th}-order moments

The r^{th}-order moments for the Rayleigh distribution can be obtained from the expression of moments of the generalized gamma distribution in equation 5.22 by substituting p=2, a=1 and $\beta = \alpha\sqrt{2}$

$$E(Y^r) = \frac{\left(\alpha\sqrt{2}\right)^r \Gamma\left(1 + \frac{r}{2}\right)}{\Gamma(1)}$$

$$= \left(\alpha\sqrt{2}\right)^r \Gamma\left(1 + \frac{r}{2}\right)$$

ii. Mean

The mean is given by

$$E(Y) = (\alpha\sqrt{2})\Gamma\left(1 + \frac{1}{2}\right)$$

$$= \frac{\alpha}{2}\sqrt{2\pi} = \alpha\sqrt{\frac{\pi}{2}}$$

iii. Mode

The mode of the Rayleigh distribution occurs at

$$\frac{d}{dy}f(y) = 0$$

$$= \frac{d}{dy}\left(\frac{y}{\alpha^2}e^{-\frac{y^2}{2\alpha^2}}\right) = 0$$

$$= \frac{1}{\alpha^2}\left(e^{-\frac{y^2}{2\alpha^2}} - y\frac{2y}{2\alpha^2}e^{-\frac{y^2}{2\alpha^2}}\right) = 0$$

$$1 - \frac{y^2}{\alpha^2} = 0$$

$$y = \alpha$$

iv. Variance

From the r^{th}-order moments given by

$$E(Y^r) = (\alpha\sqrt{2})^r\Gamma\left(1 + \frac{r}{2}\right)$$

the variance is given by $\sigma^2 = (\alpha\sqrt{2})^2\Gamma\left(1 + \frac{2}{2}\right) - \left((\alpha\sqrt{2})\Gamma\left(1 + \frac{1}{2}\right)\right)^2$

$$= \frac{4\alpha^2 - \alpha^2\pi}{2}$$

$$= \alpha^2\left(\frac{4 - \pi}{2}\right)$$

5.6.31 Maxwell distribution function

Special case of the generalized gamma distribution in equation 5.21 when p=2, $a = \frac{3}{2}$ and $\beta = \alpha\sqrt{2}$ gives

$$f(\mathrm{y};\alpha) = \frac{|2| y^{2\left(\frac{3}{2}\right)-1}}{(\alpha\sqrt{2})^{2\left(\frac{3}{2}\right)}\Gamma\left(\frac{3}{2}\right)} \; e^{-\left(\frac{y}{\alpha\sqrt{2}}\right)^2} \;, \quad 0 \le y$$

$$= \frac{2y^2}{\left(\alpha\sqrt{2}\right)^3\Gamma\left(\frac{1}{2}+1\right)} \; e^{-y^2/(\alpha\sqrt{2})^2}$$

$$= \frac{2y^2}{2\sqrt{2}\,\alpha^3\,\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \; e^{-\left(\frac{y^2}{2\alpha^2}\right)}$$

$$= \frac{2y^2}{\sqrt{2}\,\alpha^3\sqrt{\pi}} \; e^{-\left(\frac{y^2}{2\alpha^2}\right)}$$

$$= \frac{y^2}{\sqrt{\frac{\pi}{2}}\;\alpha^3} \; e^{-\left(\frac{y^2}{2\alpha^2}\right)}$$

$$= \sqrt{\frac{2}{\pi}}\frac{y^2 e^{-\left(\frac{y^2}{2\alpha^2}\right)}}{\alpha^3} \;, \quad 0 \le y \_\_\_\_\_\_\_\_\_\_\_\_\_\_ (5.29)$$

Equation 5.29 is the Maxwell distribution.

5.6.32 Properties of Maxwell distribution

i. r^{th}-order moments

The r^{th}-order moments for the Maxwell distribution can be obtained from the expression of moments of the generalized gamma distribution in equation 5.22 by substituting p=2, a=3/2 and $\beta = \alpha\sqrt{2}$

$$E(Y^r) = \frac{\left(\alpha\sqrt{2}\right)^r\Gamma\left(\frac{3}{2}+\frac{r}{2}\right)}{\frac{1}{2}\sqrt{\pi}}$$

ii. Mean

The mean is given by

$$E(Y)=\frac{(\alpha\sqrt{2})\Gamma\left(\frac{3}{2}+\frac{1}{2}\right)}{\frac{1}{2}\sqrt{\pi}}$$

$$=\frac{2\alpha\sqrt{2}}{\sqrt{\pi}}$$

iii. Mode

The mode of the Maxwell distribution occurs at

$$\frac{d}{dy}f(y)=0$$

$$=\frac{d}{dy}\left(y^2e^{-\frac{y^2}{2\alpha^2}}\right)=0$$

$$\left(2ye^{-\frac{y^2}{2\alpha^2}}-y^2\frac{2y}{2\alpha^2}e^{-\frac{y^2}{2\alpha^2}}\right)=0$$

$$2y-\frac{y^3}{\alpha^2}=0$$

$$y=\sqrt{2}\alpha$$

iv. Variance

From the rth-order moments given by

$$\mathrm{E}(\mathrm{Y}^{\mathrm{r}})=(\alpha\sqrt{2})^r\Gamma\left(\frac{1}{2}+\frac{r}{2}\right)$$

the variance is given by

$$Var(Y)=\frac{(\alpha\sqrt{2})^2\Gamma\left(\frac{3}{2}+\frac{2}{2}\right)}{\frac{1}{2}\sqrt{\pi}}-\left(\frac{2\alpha\sqrt{2}}{\sqrt{\pi}}\right)^2$$

$$= 3\alpha^2 - \frac{8\alpha^2}{\pi}$$

$$= \alpha^2\left(\frac{3\pi - 8}{\pi}\right)$$

5.6.33 Half Standard normal function

Special case of the generalized gamma distribution in equation 5.21 when p=2, $a = \frac{1}{2}$ and $\beta = \sqrt{2}$

$$f(\mathrm{y}) = \frac{\|2\| y^{2\left(\frac{1}{2}\right)-1}}{\left(\sqrt{2}\right)^{2\left(\frac{1}{2}\right)}\Gamma\left(\frac{1}{2}\right)}\ e^{-\left(\frac{y}{\sqrt{2}}\right)^2}$$

$$= \frac{2}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}}\ , \quad 0 \le y \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(5.30)$$

5.6.34 Properties of Half Standard normal distribution

i. r^th^-order moments

The r^{th}-order moments of the Half-standard normal distribution can be obtained from the expression of the moments of the generalized gamma distribution by substituting p=2, $a = \frac{1}{2}$ and $\beta = \sqrt{2}$ in equation 5.22 as follows

$$E(Y^r) = \frac{\left(\sqrt{2}\right)^r \Gamma\left(\frac{1}{2} + \frac{r}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

ii. Mean

The mean is given by

$$E(Y) = \frac{\left(\sqrt{2}\right)^1 \Gamma\left(\frac{1}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

$$= \sqrt{\frac{2}{\pi}}$$

iii. Mode

The mode is given by

$$y = \sqrt{2}\left(\frac{1}{2} - \frac{1}{2}\right)^{\frac{1}{2}} = 0$$

iv. Variance

Variance is given by

$$\text{Var(Y)} = \frac{\left(\sqrt{2}\right)^2 \Gamma\left(\frac{1}{2} + \frac{2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} - \frac{2}{\pi}$$

$$= 1$$

5.6.35 Half-normal distribution function

Special case of the generalized gamma distribution in equation 5.21 when $p = 2,\ a = \frac{1}{2}$ and $\beta = \sigma\sqrt{2}$ gives

$$f(\mathrm{y}; \sigma) = \frac{|2| y^{2\left(\frac{1}{2}\right) - 1}}{(\sigma\sqrt{2})^{2\left(\frac{1}{2}\right)} \Gamma\left(\frac{1}{2}\right)}\ e^{-\left(\frac{y}{\sigma\sqrt{2}}\right)^2} \quad , \quad 0 \leq y$$

$$= \frac{2}{\left(\sigma\sqrt{2}\right)\Gamma\left(\frac{1}{2}\right)}\ e^{-y^2/(\sigma\sqrt{2})^2}$$

$$= \frac{2}{\sigma\sqrt{2}\sqrt{\pi}}\ e^{-\left(\frac{y^2}{2\sigma^2}\right)}$$

$$= \frac{\sqrt{2}}{\sigma\sqrt{\pi}}\ e^{-\left(\frac{y^2}{2\sigma^2}\right)}$$

$$= \sqrt{\frac{2}{\pi}} \frac{e^{-\left(\frac{y^2}{2\sigma^2}\right)}}{\sigma} \quad , \quad 0 \leq y \_\_\_\_\_\_\_\_\_\_\_\_ (5.31)$$

Equation 5.31 is the Half-normal distribution.

5.6.36 Properties of Half-normal distribution

i. r^{th}-order moments

The r^{th}-order moments of the Half-normal distribution can be obtained from the expression of the moments of the generalized gamma distribution by substituting p=2, $a = \frac{1}{2}$ and $\beta = \sigma\sqrt{2}$ in equation 5.22 as follows

$$E(Y^r) = \frac{(\sigma\sqrt{2})^r \Gamma\left(\frac{1}{2} + \frac{r}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

ii. Mean

The mean is given by

$$E(Y) = \frac{(\sigma\sqrt{2})\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)}{\sqrt{\pi}}$$
$$= \frac{\sigma\sqrt{2}}{\sqrt{\pi}}$$

iii. Mode

The mode of the Half-normal distribution is given by

$$y = \sigma\sqrt{2}\left(\frac{1}{2} - \frac{1}{2}\right)^{\frac{1}{2}}$$
$$= 0$$

iv. Variance

Variance is given by

$$\text{Var(Y)} = \frac{(\sigma\sqrt{2})^2 \Gamma\left(\frac{1}{2} + \frac{2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} - \frac{2\sigma^2}{\pi}$$
$$= \sigma^2\left(1 - \frac{2}{\pi}\right)$$

5.6.37 Half Log-normal distribution function

Letting

$$y = (\ln(\text{x}) - \mu) \text{ and } \left|\frac{dy}{dx}\right| = \frac{1}{\text{x}} \text{ in equation 5.21 we get}$$

$$f(\text{x}; \mu, \sigma) = \frac{|p|(\ln(x) - \mu)^{\text{pa}-1}}{\beta^{\text{pa}}\Gamma(a)} e^{-\left(\frac{\ln(x)-\mu}{\beta}\right)^p} \cdot \frac{1}{x}$$

Substituting p=2, $\text{a} = \frac{1}{2}$ and $\beta = \sigma\sqrt{2}$ gives

$$f(\text{x}; \mu, \sigma) = \frac{2}{\sigma x\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}$$

$$f(\text{x}; \mu, \sigma) = \frac{2}{\sigma x\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}, \qquad x > 0\_\_\_\_\_\_\_\_\_\_\_\_(5.32)$$

Equation 5.32 is the Half Log-normal distribution. The log-normal distribution can be obtained as a limiting case of the generalized gamma distribution by setting $\beta = (\sigma^2\text{p}^2)^{\frac{1}{\text{p}}}$ and $\text{a} = \frac{\text{p}\mu+1}{\sigma^2\text{p}^2}$ as $\text{p} \to 0$.

5.6.38 Properties of Half Log-normal distribution

i. r^{th}-order moments

The r^{th}-order moments of the Half Log-normal distribution is given by

$$E(Y^r) = \text{e}^{r\mu + \frac{1}{2}r^2\sigma^2}$$

ii. Mean

The mean is given by

$$E(Y) = \text{e}^{\mu + \frac{1}{2}\sigma^2}$$

iii. Mode

The mode of the Half Log-normal distribution is given by

$$y = e^{\mu+\sigma^2}$$

iv. Variance

$$\text{Var(Y)} = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$$

$$= e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$$

$$CV = \frac{e^{\mu+\frac{1}{2}\sigma^2}\left(e^{\sigma^2} - 1\right)^{\frac{1}{2}}}{e^{\mu+\frac{1}{2}\sigma^2}}$$

$$= \left(e^{\sigma^2} - 1\right)^{\frac{1}{2}}$$

5.7 Four Parameter Generalized Beta Distribution Function

From the beta distribution with two shape parameters a and b in equation (2.1), $f(x; a, b) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b))}$, supported on the range 0 – 1. The location and scale parameters of the distribution can be altered by introducing two further parameters p and q representing the minimum and maximum values of the distribution respectively using the following transformation:-

$$x = \frac{y-p}{q-p}$$

$$\left|\frac{dx}{dy}\right| = \frac{1}{q-p}$$

$$\therefore f(y; p, q, a, b) = \frac{1}{B(a,b)} \frac{(y-p)^{a-1}}{(q-p)^{a-1}} \left(1 - \frac{y-p}{q-p}\right)^{b-1} \frac{1}{q-p}$$

$$= \frac{1}{B(a,b)} \frac{(y-p)^{a-1}}{(q-p)^{a-1}} \left(\frac{q-p-y+p}{q-p}\right)^{b-1} \frac{1}{q-p}$$

$$= \frac{1}{B(a,b)}(y-p)^{a-1}(q-y)^{b-1}\frac{1}{(q-p)^{a-1+b-1+1}}$$

$$= \frac{1}{B(a,b)}(y-p)^{a-1}(q-y)^{b-1}\frac{1}{(q-p)^{a-1+b-1+1}}$$

$$= \frac{(y-p)^{a-1}(q-y)^{b-1}}{B(a,b)(q-p)^{a+b-1}} \text{ ______________ } (5.33)$$

The probability distribution function in equation 5.33 is a four parameter beta distribution function.

5.7.1 Properties of the four parameter generalized beta distribution

i. **Mean**

$$E(Y) = \int_0^{\infty} yf(y;p,q,a,b)dy = \int_0^{\infty} \frac{y(y-p)^{a-1}(q-y)^{b-1}}{B(a,b)(q-p)^{a+b-1}}dy$$

$$= \frac{1}{B(a,b)(q-p)^{a+b-1}}\int_0^{\infty} y(y-p)^{a-1}(q-y)^{b-1}\,dy$$

$$= \frac{aq+bp}{a+b}$$

ii. **Mode**

$$= \frac{(a-1)q+(b-1)p}{a+b-2} \; for\; a>1, b>1$$

iii. **Variance**

$$= \frac{ab(q-p)^2}{(a+b)^2(a+b+1)}$$

6 Chapter VI: The Generalized Beta Distribution Based on Transformation Technique

6.1 Introduction

This chapter provides the construction, properties and special cases of the five parameter generalized beta distributions due to McDonald's. It gives an extension of the McDonald's five parameter to a five parameter weighted generalized beta distribution. It further considers a five parameter exponential generalized beta distribution function. The five parameter generalized beta distribution nests both the generalized beta distribution of the first kind and the generalized beta distribution of the second kind. It includes many distributions as special or limiting cases.

6.2 Five Parameter Generalized Beta Distribution Function Due to McDonald

McDonald and Xu (1995) introduced the five parameter generalized beta distribution and showed that it contains the distributions previously mentioned under 4-3-2 parameters, as special or limiting cases. The probability density function of the five parameter generalized beta distribution can be obtained from the classical beta distribution with two shape parameters a and b in equation (2.1), using the following transformation

$$x = \frac{y^p}{q^p + cy^p}$$

$$\left|\frac{dx}{dy}\right| = \frac{py^{p-1}[q^p + cy^p] - [pcy^{p-1}(y^p)]}{(q^p + cy^p)^2}$$

$$= \frac{pq^p y^{p-1}}{(q^p + cy^p)^2}$$

$$\therefore f(y; p, q, c, a, b) = \frac{y^{pa-p}}{(q^p + cy^p)^{a-1}}\left(1 - \frac{y^p}{q^p + cy^p}\right)^{b-1} \frac{1}{B(a, b)} \frac{pq^p y^{p-1}}{(q^p + cy^p)^2}$$

$$= \frac{y^{pa-p}}{(q^p + cy^p)^{a-1}} \left(\frac{q^p + cy^p - y^p}{q^p + cy^p} \right)^{b-1} \frac{1}{B(a,b)} \frac{pq^p y^{p-1}}{(q^p + cy^p)^2}$$

$$= \frac{pq^p y^{pa-p+p-1} (q^p + cy^p - y^p)^{b-1}}{B(a,b)(q^p + cy^p)^{a-1+b-1+2}}$$

$$= \frac{pq^p y^{pa-1} (q^p)^{b-1} \left(1 + c \frac{y^p}{q^p} - \frac{y^p}{q^p} \right)^{b-1}}{B(a,b)(q^p)^{a+b} \left(1 + c \frac{y^p}{q^p} \right)^{a+b}}$$

$$= \frac{pq^{p+pb-p} y^{pa-1} \left(1 + (c-1)(\frac{y}{q})^p \right)^{b-1}}{B(a,b) q^{pa+pb} \left(1 + c(\frac{y}{q})^p \right)^{a+b}}$$

$$= \frac{pq^{pb} y^{pa-1} \left(1 - (1-c)\left(\frac{y}{q}\right)^p \right)^{b-1}}{q^{pa+pb} B(a,b) \left(1 + c(\frac{y}{q})^p \right)^{a+b}}$$

$$= \frac{py^{pa-1} \left(1 - (1-c)\left(\frac{y}{q}\right)^p \right)^{b-1}}{q^{pa} B(a,b) \left(1 + c\left(\frac{y}{q}\right)^p \right)^{a+b}} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (6.1)$$

for $0 < y^p < \frac{q^p}{1-c}$, $0 \leq c \leq 1$ and $p > 0,\ q > 0,\ a > 0,\ b > 0$

Equation 6.1 is a five parameter generalized beta distribution. The five parameter generalized beta distribution encompasses both the four parameter generalized beta distribution in equation 5.1 and 5.10 as follows:-

i. Letting c=0 in the five parameter generalized beta distribution in equation 6.1 gives a four parameter generalized beta distribution in equation 5.1
ii. Letting c=1 in the five parameter generalized beta distribution in equation 6.1 gives the four parameter generalized beta distribution in equation 5.10.

However, when fitting this five parameter generalized beta distribution model to 1985 family income, McDonald and Xu (1995) found that the four parameter generalized beta of the second kind subfamily is selected (in terms of likelihood and several other criteria). Thus, it appears that the five parameter generalized beta distribution does not provide additional flexibility.

6.3 Properties of the Five Parameter Generalized Beta Distribution

6.3.1 r^th^–order moments

To obtain the r^{th}-order moments for the five parameter generalized beta distribution, we evaluate

$$E(Y^r) = \int_{-\infty}^{\infty} y^r f(y; p, q, c, a, b) dy$$

$$= \int_{-\infty}^{\infty} \frac{y^r p y^{pa-1} (1 - (1-c)(\frac{y}{q})^p)^{b-1}}{q^{pa} B(a,b)(1 + c(\frac{y}{q})^p)^{a+b}} dy$$

replacing $\left(1 - (1-c)(\frac{y}{q})^p\right)^{b-1}$ by its binomial expansion, letting

$u = (1-c)(\frac{y}{q})^p$ and collecting like terms yield

$$E(Y^r) = q^r \sum_{i=0}^{\infty} \frac{(a+b)_i \left(\frac{-c}{1-c}\right)^i}{(1-c)^{a+\frac{r}{p}+i}} \int_0^1 u^{a+\frac{r}{p}+i} (1-u)^{b-1}\, du$$

$$= q^r \sum_{i=0}^{\infty} \frac{(a+b)_i \left(\frac{-c}{1-c}\right)^i}{(1-c)^{a+\frac{r}{p}+i}} B(a + \frac{r}{p} + i, b)$$

$$= q^r \sum_{i=0}^{\infty} \frac{(a+b)_i \left(\frac{-c}{1-c}\right)^i \Gamma\left(a + \frac{r}{p} + i\right) \Gamma(\mathrm{b})}{(1-c)^{a+\frac{r}{p}+i} \Gamma\left(a + \frac{r}{p} + i + b\right)}$$

$$= \frac{q^r B(a + \frac{r}{p}, b)}{(1-c)^{a+\frac{r}{p}} B(a,b)} \, {}_2F_1 \begin{bmatrix} a+b, & b + \frac{r}{p}; & \frac{-c}{1-c} \\ a + b + \frac{r}{p} & & \end{bmatrix}$$

$$= \frac{q^r B(a+\frac{r}{p},b)}{B(a,b)} \; {}_2F_1\begin{bmatrix} a+\frac{r}{p}, & \frac{r}{p}; & c \\ a+b+\frac{r}{p} & & \end{bmatrix} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(6.2)$$

Equation 6.2 is defined for all r if $c < 1$ or for $-a < \frac{r}{p} < b$ if $c = 1$

6.3.2 Mean

The mean is given by

$$E(Y) = \frac{qB(a+\frac{1}{p},b)}{(1-c)^{a+\frac{1}{p}}B(a,b)} \; {}_2F_1\begin{bmatrix} a+\frac{1}{p}, & \frac{1}{p}; & c \\ a+b+\frac{1}{p} & & \end{bmatrix}$$

6.3.3 Variance

$$\mathrm{Var(Y)} = \frac{\mathrm{q^2B(a+\frac{2}{p},b)}}{\mathrm{(1-c)^{a+\frac{2}{p}}B(a,b)}} \; {}_2\mathrm{F}_1\begin{bmatrix} \mathrm{a}+\frac{2}{\mathrm{p}}, & \frac{2}{\mathrm{p}}; & \mathrm{c} \\ \mathrm{a+b}+\frac{2}{\mathrm{p}} & & \end{bmatrix} - \left(\frac{\mathrm{qB(a+\frac{1}{p},b)}}{\mathrm{(1-c)^{a+\frac{1}{p}}B(a,b)}} \; {}_2\mathrm{F}_1\begin{bmatrix} \mathrm{a}+\frac{1}{\mathrm{p}}, & \frac{1}{\mathrm{p}}; & \mathrm{c} \\ \mathrm{a+b}+\frac{1}{\mathrm{p}} & & \end{bmatrix}\right)^2$$

6.4 Special Cases of the Five Parameter Generalized Beta Distribution Function

6.4.1 The four parameter generalized beta distribution of the first kind

This is a special case when $c = 0$ in equation 6.1

$$f(y;a,b,p,q) = \frac{py^{ap-1}\left(1-(1-0)\left(\frac{y}{q}\right)^p\right)^{b-1}}{q^{ap}B(a,b)\left(1+0\left(\frac{y}{q}\right)^p\right)^{a+b}}$$

$$= \frac{py^{ap-1}\left(1-\left(\frac{y}{q}\right)^p\right)^{b-1}}{q^{ap}B(a,b)}$$

for $0 < y^p < \ q^p$, and $p > 0,\ q > 0,\ a > 0,\ b > 0$

6.4.2 The four parameter generalized beta distribution of the second kind

This is a special case when $c = 1$ in equation 6.1

$$f(y;a,b,p,q) = \frac{py^{pa-1}\left(1-(1-1)\left(\frac{y}{q}\right)^p\right)^{b-1}}{q^{pa}B(a,b)\left(1+1\left(\frac{y}{q}\right)^p\right)^{a+b}}$$

$$= \frac{py^{pa-1}}{q^{pa}B(a,b)\left(1+\left(\frac{y}{q}\right)^p\right)^{a+b}}$$

for $0 < y^p < \ \infty$, $p > 0,\ q > 0,\ a > 0,\ b > 0$

6.4.3 The four parameter generalized beta distribution

This is a special case when a $= 1$ in equation 6.1

$$f(y;b,c,p,q) = \frac{bpy^{p-1}\left(1-(1-c)\left(\frac{y}{q}\right)^p\right)^{b-1}}{q^p\left(1+c\left(\frac{y}{q}\right)^p\right)^{1+b}}$$

for $0 < y^p < \ \infty$, $p > 0,\ q > 0,\ a > 0,\ b > 0$

6.4.4 The four parameter generalized Logistic distribution function

This is a special case when $c = 1$, y $= e^x$, and $|J| = e^x$ in equation 6.1

$$f(e^x;a,b,p,q) = \frac{p(e^x)^{pa-1}\left(1-(1-1)\left(\frac{(e^x)}{q}\right)^p\right)^{b-1}.(e^x)}{q^{pa}B(a,b)\left(1+1\left(\frac{(e^x)}{q}\right)^p\right)^{a+b}}$$

$$= \frac{pe^{ap(x-logp)}}{q^{pa}B(a,b)(1+e^{p(x-logq)})^{a+b}}$$

for $-\infty < x < \infty$, $a > 0,\ b > 0,\ p > 0,\ q > 0$

6.4.5 The three parameter Inverse Beta distribution of the first kind

This is a special case when $c = 0$, and $\mathrm{p} = -1$ in equation 6.1

$$f(y; a, b, q) = \frac{(-1)y^{a(-1)-1}\left(1-(1-0)\left(\frac{y}{q}\right)^{(-1)}\right)^{b-1}}{q^{a(-1)}B(a,b)\left(1+0\left(\frac{y}{q}\right)^{(-1)}\right)^{a+b}}$$

$$= \frac{q^{a}\left(1-\frac{q}{y}\right)^{b-1}}{B(a,b)y^{a+1}}$$

for $0 < y < 1$, and $a > 0,\ b > 0,\ q > 0$

6.4.6 The three parameter Stoppa (generalized pareto type I) distribution function

This is a special case when $c = 0$, $\mathrm{a} = 1$ and $\mathrm{p} = -\mathrm{p}$ in equation 6.1

$$f(y; b, p, q) = \frac{|-p|y^{(1)(-p)-1}\left(1-(1-0)\left(\frac{y}{q}\right)^{-p}\right)^{b-1}}{q^{(1)(-p)}B(1,b)\left(1+0\left(\frac{y}{q}\right)^{-p}\right)^{1+b}}$$

$$= pq^{p}by^{-p-1}\left(1-\left(\frac{y}{q}\right)^{-p}\right)^{b-1}$$

for $y \geq q > 0$, and $p > 0,\ b > 0$

6.4.7 The three parameter Singh-Maddala (Burr12) distribution function

This is a special case when $c = 1$, and $\mathrm{a} = 1$ in equation 6.1

$$f(y; b, p, q) = \frac{py^{p(1)-1}\left(1-(1-1)\left(\frac{y}{q}\right)^{p}\right)^{b-1}}{q^{p(1)}B(1,b)\left(1+1\left(\frac{y}{q}\right)^{p}\right)^{(1)+b}}$$

$$= \frac{bpy^{p-1}}{q^{p}\left(1+\left(\frac{y}{q}\right)^{p}\right)^{1+b}}$$

for $0 < y^p < \infty$, $b > 0$, $p > 0$, $q > 0$

6.4.8 The three parameter Dagum (Burr3) distribution function

This is a special case when $c = 1$, and b $= 1$ in equation 6.1

$$f(y; a, p, q) \ = \frac{py^{pa-1}\left(1-(1-1)\left(\frac{y}{q}\right)^p\right)^{(1)-1}}{q^{pa}B(a,1)\left(1+1\left(\frac{y}{q}\right)^p\right)^{a+(1)}}$$

$$= \frac{apy^{pa-1}}{q^{pa}\left(1+\left(\frac{y}{q}\right)^p\right)^{a+1}}$$

for $0 < y^p < \infty$, $a > 0$, $p > 0$, $q > 0$

6.4.9 The two parameter Classical Beta distribution

This is a special case when $c = 0$, p $= 1$, and q $= 1$ in equation 6.1

$$f(y; a, b) = \frac{(1)y^{a(1)-1}\left(1-(1-0)\left(\frac{y}{(1)}\right)^{(1)}\right)^{b-1}}{(1)^{a(1)}B(a,b)\left(1+0\left(\frac{y}{(1)}\right)^{(1)}\right)^{a+b}}$$

$$= \frac{y^{a-1}(1-y)^{b-1}}{B(a,b)}$$

for $0 < y < 1$, and $a > 0$, $b > 0$

6.4.10 The two parameter Pareto type I distribution function

This is a special case when $c = 0$, p $= -1$, and b $= 1$ in equation 6.1

$$f(y; a, q) = \frac{(-1)y^{a(-1)-1}\left(1-(1-0)\left(\frac{y}{q}\right)^{(-1)}\right)^{1-1}}{q^{a(-1)}B(a,1)\left(1+0\left(\frac{y}{q}\right)^{(-1)}\right)^{a+1}}$$

$$= \frac{aq^{a}}{y^{a+1}}$$

for $0 < y < \ 1$, and $a > 0,\ q > 0,$

6.4.11 The two parameter Kumaraswamy distribution function

This is a special case when $c = 0$, a $= 1$ and q $= 1$ in equation 6.1

$$f(y; b, p) = \frac{py^{(1)p-1}\left(1-(1-0)\left(\frac{y}{1}\right)^{p}\right)^{b-1}}{1^{(1)p}B(1,b)\left(1+0\left(\frac{y}{1}\right)^{p}\right)^{1+b}}$$

$$= pby^{p-1}(1-(y)^{p})^{b-1}$$

for $0 < y^{p} < \ 1$, and $p > 0,\ b > 0$

6.4.12 The two parameter generalized Power distribution function

This is a special case when $c = 0$, b $= 1$, and p $= 1$ in equation 6.1

$$f(y; a, q) = \frac{(1)y^{a(1)-1}\left(1-(1-0)\left(\frac{y}{q}\right)^{(1)}\right)^{1-1}}{q^{a(1)}B(a,1)\left(1+0\left(\frac{y}{q}\right)^{(1)}\right)^{a+1}}$$

$$= \frac{ay^{a-1}}{q^{a}}$$

for $0 < y < 1$, and $a > 0,\ q > 0,$

6.4.13 The two parameter Beta distribution of the second kind

This is a special case when $c = 1$, p $= 1$, and q $= 1$ in equation 6.1

$$f(y; a, b) = \frac{(1)y^{(1)a-1}\left(1-(1-1)\left(\frac{y}{(1)}\right)^{(1)}\right)^{b-1}}{(1)^{(1)a}B(a,b)\left(1+1\left(\frac{y}{(1)}\right)^{(1)}\right)^{a+b}}$$

$$= \frac{y^{a-1}}{B(a,b)(1+y)^{a+b}}$$

for $0 < y < \ \infty$, $a > 0,\ b > 0$

6.4.14 The two parameter Inverse Lomax distribution function

This is a special case when $c = 1, \mathrm{b} = 1,$ and $\mathrm{p} = 1$ in equation 6.1

$$f(y; a, q) = \frac{(1)y^{(1)a-1}\left(1-(1-1)\left(\frac{y}{q}\right)^{(1)}\right)^{1-1}}{q^{(1)a}B(a,1)\left(1+1\left(\frac{y}{q}\right)^{(1)}\right)^{a+1}}$$

$$= \frac{ay^{a-1}}{q^{a}\left(1+\left(\frac{y}{q}\right)\right)^{a+1}}$$

for $0 < y < \infty$, $a > 0,\ q > 0$

6.4.15 The two parameter General Fisk (Log-Logistic) distribution function

This is a special case when $c = 1, \mathrm{a} = 1, \mathrm{b} = 1$ and $\mathrm{q} = \frac{1}{\lambda}$ in equation 6.1

$$f(y; \lambda, p) = \frac{py^{p(1)-1}\left(1-(1-1)\left(\frac{y}{\frac{1}{\lambda}}\right)^{p}\right)^{(1)-1}}{\left(\frac{1}{\lambda}\right)^{p(1)}B(1,1)\left(1+1\left(\frac{y}{\frac{1}{\lambda}}\right)^{p}\right)^{(1)+(1)}}$$

$$= \frac{\lambda p(\lambda y)^{p-1}}{(1+(\lambda y)^{p})^{2}}$$

for $0 < y^{p} < \infty$, $\lambda > 0,\ p > 0,\ q > 0$

6.4.16 The two parameter Fisher (F) distribution function

This is a special case when $c = 1, \mathrm{a} = \frac{\mathrm{u}}{2}, \mathrm{b} = \frac{\mathrm{v}}{2}, \mathrm{p} = 1$ and $\mathrm{q} = \frac{\mathrm{v}}{\mathrm{u}}$ in equation 6.1

$$f(y;\ u, v) = \frac{(1)y^{(1)\left(\frac{u}{2}\right)-1}\left(1-(1-1)\left(\frac{y}{\frac{\mathrm{v}}{\mathrm{u}}}\right)^{(1)}\right)^{\left(\frac{u}{2}\right)-1}}{\left(\frac{\mathrm{v}}{\mathrm{u}}\right)^{(1)\left(\frac{u}{2}\right)}B\left(\frac{u}{2},\frac{v}{2}\right)\left(1+1\left(\frac{y}{\frac{\mathrm{v}}{\mathrm{u}}}\right)^{(1)}\right)^{\left(\frac{u}{2}\right)+\left(\frac{v}{2}\right)}}$$

$$= \frac{\Gamma\left(\frac{u+v}{2}\right)\left(\frac{\mathrm{u}}{\mathrm{v}}\right)^{\left(\frac{u}{2}\right)} y^{\left(\frac{u}{2}\right)-1}}{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{v}{2}\right)\left(1+\left(\frac{y\mathrm{u}}{\mathrm{v}}\right)\right)^{\frac{u+v}{2}}}$$

for $0 < y < \infty$, $\mathrm{u} > 0$, $v > 0$

6.4.17 The one parameter Power distribution function

This is a special case when $c = 0, \mathrm{a} = 1, \mathrm{b} = 1$ and $\mathrm{q} = 1$ in equation 6.1

$$f(y;p) = \frac{p y^{(1)p-1}\left(1-(1-0)\left(\frac{y}{1}\right)^{p}\right)^{1-1}}{1^{(1)p} B(1{,}1)\left(1+0\left(\frac{y}{1}\right)^{p}\right)^{1+1}}$$

$$= p y^{p-1}$$

for $0 < y < 1$, and $p > 0$,

6.4.18 The one parameter Wigner semicircle distribution function

This is a special case when $c = 0, \mathrm{a} = \frac{3}{2}, \mathrm{b} = \frac{3}{2}, \mathrm{p} = 1$ and $\mathrm{q} = 1$ in equation 6.1

$$f(y) = \frac{(1) y^{\left(\frac{3}{2}\right)(1)-1}\left(1-(1-0)\left(\frac{y}{1}\right)^{(1)}\right)^{\frac{3}{2}-1}}{1^{\left(\frac{3}{2}\right)(1)} B\left(\frac{3}{2},\frac{3}{2}\right)\left(1+0\left(\frac{y}{1}\right)^{(1)}\right)^{\frac{3}{2}+\frac{3}{2}}}$$

$$= \frac{8\sqrt{y(1-y)}}{\pi}, \qquad 0 < y < 1$$

Letting

$$y = \frac{x+R}{2R} \ and \ \ |J| = \frac{1}{2R}$$

Limits changes as $y = 0 \implies x > -R$ and $y = 1 \implies x = R$

$$f(x) = \frac{8\sqrt{\frac{y+R}{2R}\left(1-\frac{y+R}{2R}\right)}}{\pi} \cdot \frac{1}{2R}$$

$$= \frac{2\sqrt{R^2-y^2}}{R^2\pi}, \qquad -R < y < R$$

6.4.19 The one parameter Lomax (pareto type II) distribution function

This is a special case when $c = 1, \mathrm{a} = 1, \mathrm{p} = 1,$ and $\mathrm{q} = 1$ in equation 6.1

$$f(y;b) = \frac{(1)y^{(1)(1)-1}\left(1-(1-1)\left(\frac{y}{(1)}\right)^{(1)}\right)^{b-1}}{(1)^{(1)(1)}B(1,b)\left(1+1\left(\frac{y}{(1)}\right)^{(1)}\right)^{(1)+b}}$$

$$= \frac{b}{(1+y)^{\mathbf{1}+b}}$$

for $0 < y < \infty$, $b > 0$

6.4.20 The one parameter half student's (t) distribution function

This is a special case when $c = 1, \mathrm{a} = \frac{1}{2}, \mathrm{b} = \frac{\mathrm{r}}{2}, \mathrm{p} = 2$ and $\mathrm{q} = \sqrt{\mathrm{r}}$ in equation 6.1

$$f(y;\ r) = \frac{(2)y^{(2)\left(\frac{1}{2}\right)-1}\left(1-(1-1)\left(\frac{y}{\sqrt{\mathrm{r}}}\right)^{(2)}\right)^{\left(\frac{r}{2}\right)-1}}{\left(\sqrt{\mathrm{r}}\right)^{(2)\left(\frac{1}{2}\right)}B\left(\frac{1}{2},\frac{r}{2}\right)\left(1+1\left(\frac{y}{\sqrt{\mathrm{r}}}\right)^{(2)}\right)^{\left(\frac{1}{2}\right)+\left(\frac{r}{2}\right)}}$$

$$= \frac{2\Gamma\left(\frac{1+r}{2}\right)}{\sqrt{\mathrm{r}\pi}\,\Gamma\left(\frac{r}{2}\right)\left(1+\frac{y^{\mathbf{2}}}{r}\right)^{\left(\frac{\mathbf{1}+\boldsymbol{r}}{\mathbf{2}}\right)}}$$

for $0 < y < \infty$, $r > 0$

6.4.21 Uniform distribution function

This is a special case when $c = 0, \mathrm{a} = 1, \mathrm{b} = 1, \mathrm{p} = 1$ and $\mathrm{q} = 1$ in equation 6.1

$$f(y) = \frac{(1)y^{(1)(1)-1}\left(1-(1-0)\left(\frac{y}{1}\right)^{(1)}\right)^{1-1}}{1^{(1)(1)}B(1{,}1)\left(1+0\left(\frac{y}{1}\right)^{(1)}\right)^{1+1}}$$

$$= 1$$

6.4.22 Arcsine distribution function

This is a special case when $c = 0, \mathrm{a} = \frac{1}{2}, \mathrm{b} = \frac{1}{2}, \mathrm{p} = 1$ and $\mathrm{q} = 1$ in equation 6.1

$$f(y) = \frac{(1)y^{\left(\frac{1}{2}\right)(1)-1}\left(1-(1-0)\left(\frac{y}{1}\right)^{(1)}\right)^{\frac{1}{2}-1}}{1^{\left(\frac{1}{2}\right)(1)}B\left(\frac{1}{2},\frac{1}{2}\right)\left(1+0\left(\frac{y}{1}\right)^{(1)}\right)^{\frac{1}{2}+\frac{1}{2}}}$$

$$= \frac{1}{\pi\sqrt{y(1-y)}}, \qquad 0 < y < 1$$

6.4.23 Triangular shaped distribution functions

i. This is a special case when $c = 0, \mathrm{a} = 1, \mathrm{b} = 2, \mathrm{p} = 1$ and $\mathrm{q} = 1$ in equation 6.1

$$f(y) = \frac{(1)y^{(1)(1)-1}\left(1-(1-0)\left(\frac{y}{1}\right)^{(1)}\right)^{2-1}}{1^{(1)(1)}B(1,2)\left(1+0\left(\frac{y}{1}\right)^{(1)}\right)^{1+2}}$$

$$= 2(1-y), \qquad 0 < y < 1$$

ii. This is a special case when $c = 0, \mathrm{a} = 2, \mathrm{b} = 1, \mathrm{p} = 1$ and $\mathrm{q} = 1$ in equation 6.1

$$f(y) = \frac{(1)y^{(2)(1)-1}\left(1-(1-0)\left(\frac{y}{1}\right)^{(1)}\right)^{1-1}}{1^{(2)(1)}B(2,1)\left(1+0\left(\frac{y}{1}\right)^{(1)}\right)^{2+1}}$$

$$= 2y, \qquad 0 < y < 1$$

6.4.24 Parabolic shaped distribution functions

This is a special case when $c = 0, \mathrm{a} = 2, \mathrm{b} = 2, \mathrm{p} = 1$ and $\mathrm{q} = 1$ in equation 6.1

$$f(y) = \frac{(1)y^{(2)(1)-1}\left(1-(1-0)\left(\frac{y}{1}\right)^{(1)}\right)^{2-1}}{1^{(2)(1)}B(2,2)\left(1+0\left(\frac{y}{1}\right)^{(1)}\right)^{2+2}}$$

$$= 6y(1-y), \qquad 0 < y < 1$$

6.4.25 Log-Logistic (Fisk) distribution function

This is a special case when $c = 1$, a $= 1$, b $= 1$, p $= 1$, and q $= 1$ in equation 6.1

$$f(y) = \frac{(1)y^{(1)(1)-1}\left(1-(1-1)\left(\frac{y}{(1)}\right)^{(1)}\right)^{(1)-1}}{(1)^{(1)(1)}B(1,1)\left(1+1\left(\frac{y}{(1)}\right)^{(1)}\right)^{(1)+(1)}}$$

$$= \frac{1}{(1+y)^{\mathbf{2}}}$$

for $0 < y < \infty$

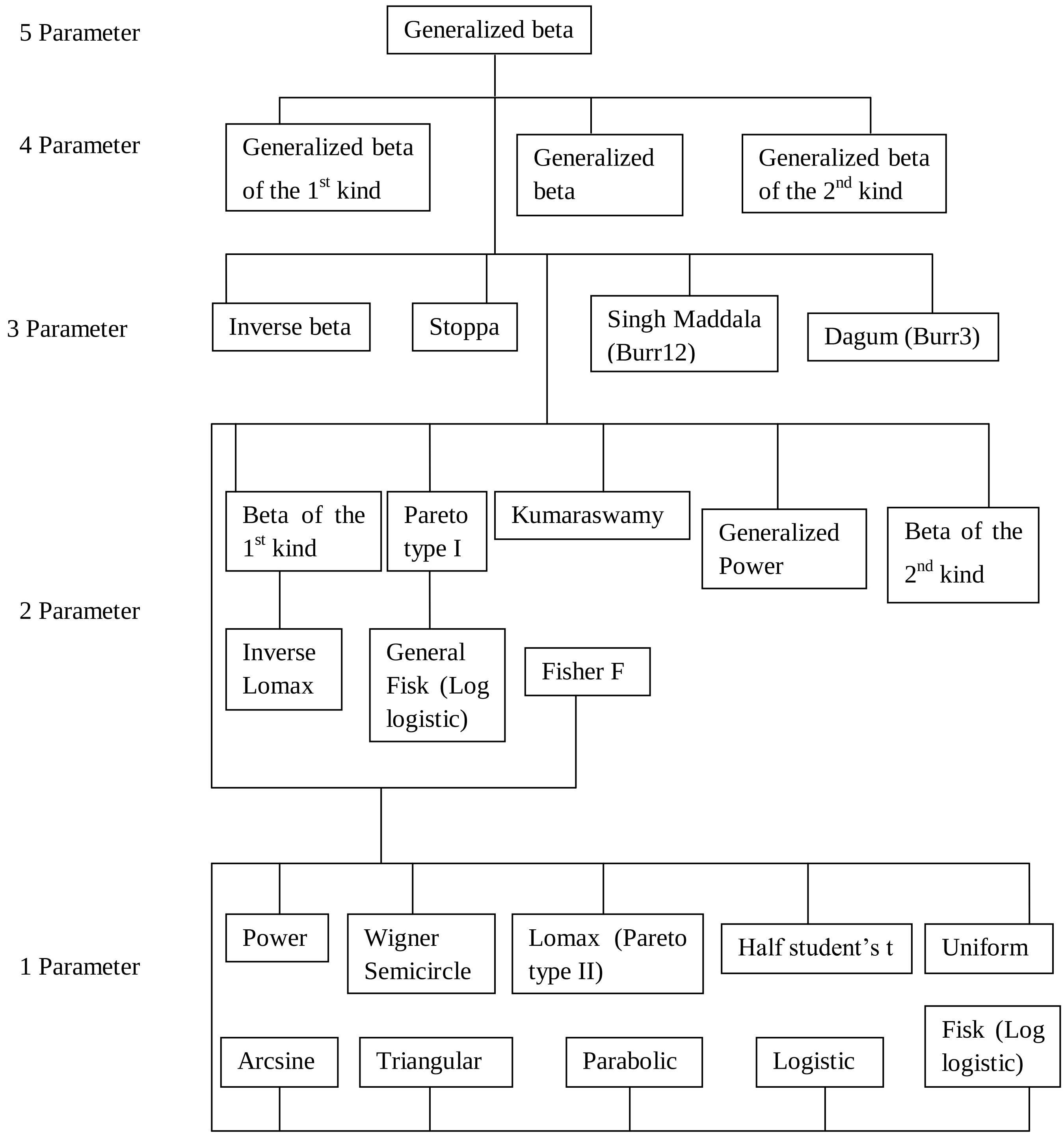
6.4.26 Logistic distribution function

This is a special case when $c = 1$, a $= 1$, b $= 1$, p $= 1$, q $= 1$, y $= e^x$, and $|J| = e^x$ in equation 6.1

$$f(x) = \frac{(1)(e^x)^{(1)(1)-1}\left(1-(1-1)\left(\frac{(e^x)}{(1)}\right)^{(1)}\right)^{(1)-1} .\, (e^x)}{(1)^{(1)(1)}B(1,1)\left(1+1\left(\frac{(e^x)}{(1)}\right)^{(1)}\right)^{(1)+(1)}}$$

$$= \frac{e^x}{(1+e^x)^2}, \qquad \text{for} -\infty < x < \infty$$

The Five Parameter Beta Distribution and its Special Cases Diagrammatically



6.5 Construction of the Five Parameter Weighted Generalized Beta Distribution Function

Extending the McDonald's five parameter model by introducing a sixth parameter, k gives a six parameter weighted generalized beta distribution as follows.

Let

$$f(y;a,b,c,k,p,q)=\frac{py^{ap+k-1}\left(1-(1-c)\left(\frac{y}{q}\right)^{p}\right)^{b-1}}{q^{ap+k}B\left(a+\frac{k}{p},b-\frac{k}{p}\right)\left(1+c\left(\frac{y}{q}\right)^{p}\right)^{a+b}} \quad \text{———————(6.3)}$$

for $0<y^{p}<\frac{q^{p}}{1-c}$, $0\leq c\leq 1$ and $p>0,\ q>0,\ a+\frac{k}{p}>0,\ b-\frac{k}{p}>0$

The Weighted generalized beta distribution of the second kind with polynomial weight function $w(y)=y^{k}$ is a special case of equation 6.3 when c =1. Alternatively, it can be obtained from the generalized beta distribution of the second kind by adding some weight to the parameters a and b.

Let $a=a+\frac{k}{p}$ and $b=b-\frac{k}{q}$ in equation 4.1 of generalized beta distribution of the first kind.

$$f(y;p,q,a,b,k)=\frac{|p|y^{p\left(a+\frac{k}{p}\right)-1}}{q^{p\left(a+\frac{k}{p}\right)}B\left(a+\frac{k}{p},\ b-\frac{k}{p}\right)\left(1+\left(\frac{y}{q}\right)^{p}\right)^{a+b}}$$

$$=\frac{|p|y^{ap+k-1}}{q^{pa+k}B\left(a+\frac{k}{p},\ b-\frac{k}{p}\right)\left(1+\left(\frac{y}{q}\right)^{p}\right)^{a+b}} \quad \text{——————————(6.4)}$$

$$y>0,\qquad p,q,a,b>0, \text{and} -pa<k<pb$$

Equation 6.4 is a very flexible five parameter weighted generalized beta distribution of the second kind presented by Yuan et al. (2012).

6.6 Properties of Five Parameter Weighted Generalized Beta Distribution

6.6.1 r^{th}-order moments

To obtain the r^{th}-order moments for the weighted generalized beta distribution, we evaluate

$$E(Y^r) = \int_{-\infty}^{\infty} y^r f(y;p,q,c,a,b)dy$$

$$= \int_{-\infty}^{\infty} \frac{y^r p y^{pa+k-1}}{q^{pa+k} B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)\left(1+c(\frac{y}{q})^p\right)^{a+b}} dy$$

$$= \frac{q^r B\left(a+\frac{k}{p}+\frac{r}{p}, b-\frac{k}{p}-\frac{r}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)} \qquad (6.5)$$

6.6.2 Mean

The mean is given by

$$E(Y) = \frac{qB\left(a+\frac{k}{p}+\frac{1}{p}, b-\frac{k}{p}-\frac{1}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)} \qquad (6.6)$$

6.6.3 Variance

$$\mathrm{Var(Y)} = q^2\left[\frac{B\left(a+\frac{k+2}{p}, b-\frac{k+2}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)} - \left(\frac{a+\frac{k+1}{p}, b-\frac{k-1}{p}}{\left(a+\frac{k}{p}, b-\frac{k}{p}\right)}\right)^2\right]$$

The coefficient of variation is given by

$$CV = \sqrt{\frac{B\left(a+\frac{k+2}{p}, b-\frac{k+2}{p}\right)B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)}{B^2\left(a+\frac{k+1}{p}, b-\frac{k+1}{p}\right)} - 1}$$

6.6.4 Skewness

The skewness is given by

$$Skewness = E(Y^3) - 3E(Y^2)E(Y) + 2(E(Y))^3$$

$$= \frac{q^3 B\left(a+\frac{k+3}{p}, b-\frac{k+3}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)} - 3\frac{q^2 B\left(a+\frac{k+2}{p}, b-\frac{k+2}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)}.$$

$$\frac{qB\left(a+\frac{k}{p}+\frac{1}{p}, b-\frac{k}{p}-\frac{1}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)} + 2\left(\frac{qB\left(a+\frac{k}{p}+\frac{1}{p}, b-\frac{k}{p}-\frac{1}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)}\right)^2$$

6.6.5 Kurtosis

The kurtosis is given by

$$kurtosis = E(Y^4) - 4E(Y^3)E(Y) + 6E(Y^2)\big(E(Y)\big)^2 - 3\big(E(Y)\big)^4$$

$$Kurtosis$$

$$= \frac{q^4 B\left(a+\frac{k+4}{p}, b-\frac{k+4}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)}$$

$$- 4\frac{q^3 B\left(a+\frac{k+3}{p}, b-\frac{k+3}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)}.\frac{qB\left(a+\frac{k}{p}+\frac{1}{p}, b-\frac{k}{p}-\frac{1}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)}$$

$$+ 6\frac{q^2 B\left(a+\frac{k+2}{p}, b-\frac{k+2}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)}.\left(\frac{qB\left(a+\frac{k}{p}+\frac{1}{p}, b-\frac{k}{p}-\frac{1}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)}\right)^2$$

$$- 3\left(\frac{qB\left(a+\frac{k}{p}+\frac{1}{p}, b-\frac{k}{p}-\frac{1}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{pp}\right)}\right)^4$$

6.7 Special Cases of the Five Parameter Weighted Generalized Beta Distribution Function

The five parameter weighted generalized beta distribution includes both the generalized beta distribution of the first and second kinds as special cases, it also includes several other weighted distributions as special or limiting cases: weighted generalized gamma, weighted beta of the second kind, weighted Singh-Maddala, weighted Dagum, weighted gamma, weighted Weibull, and weighted exponential distributions among others as shown below.

6.7.1 Weighted beta of the second distribution function

This is a special case when p = 1 and q = 1 in equation 6.4

$$f(y;a,b,k) = \frac{y^{a+k-1}}{B(a+k,\ \ b-k)(1+y)^{a+b}} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (6.7)$$

$$y > 0, \qquad a,b,k > 0, \text{and} -1 < k < b$$

6.7.2 Weighted Singh-Maddala (Burr12) distribution function

This is a special case when a = 1 in equation 6.4

$$f(y;b,k,p,q) = \frac{|p|y^{p+k-1}}{q^{p+k}B\left(1+\frac{k}{p},\ \ b-\frac{k}{p}\right)\left(1+\left(\frac{y}{q}\right)^{p}\right)^{1+b}} \text{\_\_\_\_\_\_\_\_\_\_} (6.8)$$

$$y > 0, \qquad p,q,b > 0, \text{and} -p < k < pb$$

6.7.3 Weighted Dagum (Burr3) distribution function

This is a special case when b = 1 in equation 6.4

$$f(y;p,q,a,k) = \frac{|p|y^{ap+k-1}}{q^{pa+k}B\left(a+\frac{k}{p},\ \ 1-\frac{k}{p}\right)\left(1+\left(\frac{y}{q}\right)^{p}\right)^{a+1}} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (6.9)$$

$$y > 0, \qquad p,q,a > 0, \text{and} -pa < k < p$$

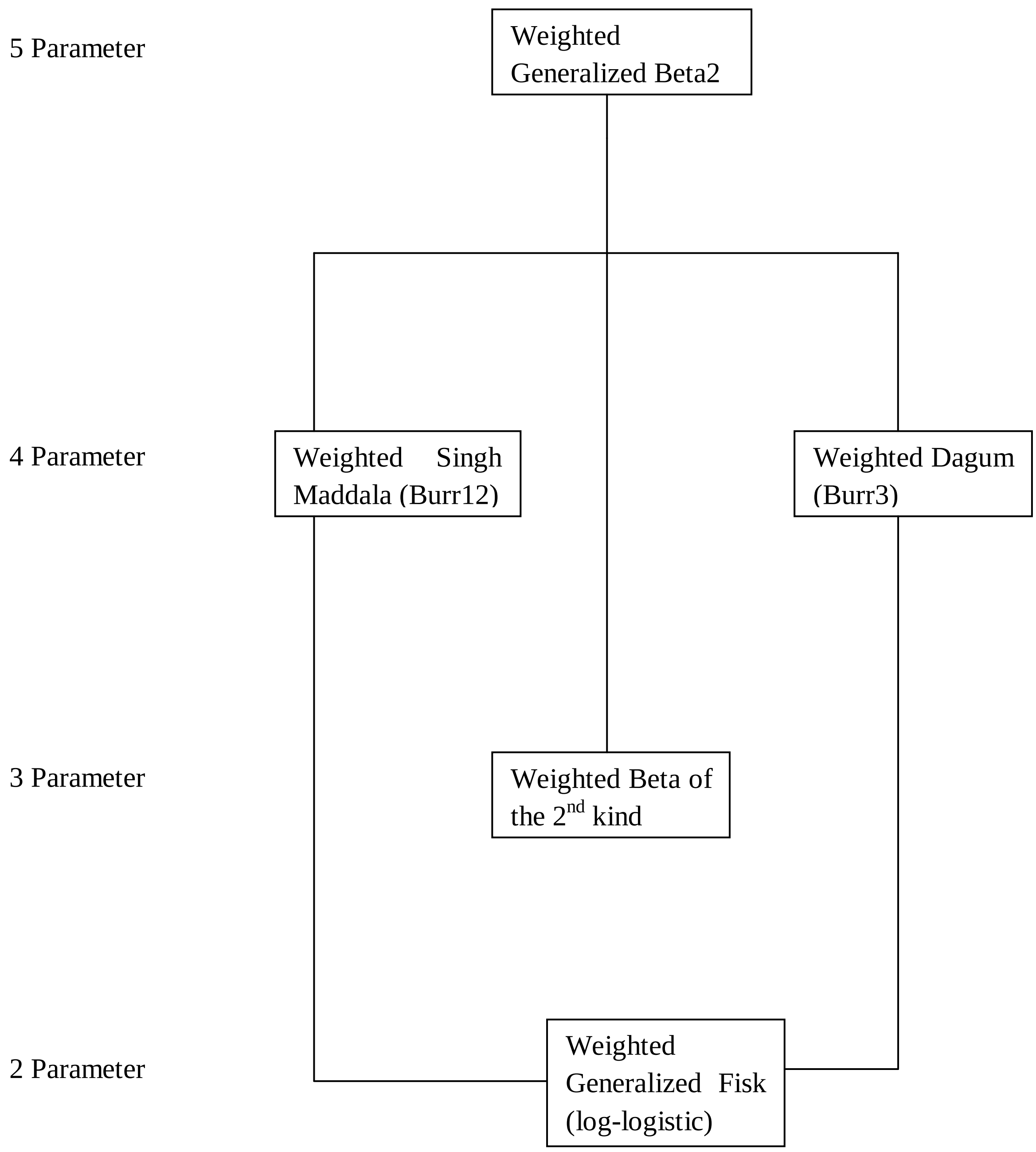
6.7.4 Weighted Generalized Fisk (Log-Logistic) distribution function

This is a special case when , $a = 1, b = 1$ and $q = \frac{1}{\lambda}$ in equation 6.4

$$f(y; p, \lambda, k) = \frac{p y^{p+k-1}}{\left(\frac{1}{\lambda}\right)^{p+k} B\left(a + \frac{k}{p},\ 1 - \frac{k}{p}\right)(1 + \lambda y^p)^2} \qquad (6.10)$$

$$y > 0, \qquad p, \lambda > 0, \text{and} - p < k < p$$

The Five Parameter Weighted Beta Distribution and its Special Cases Diagrammatically



6.8 Construction of the Five Parameter Exponential Generalized Beta Distribution Function

This section considers a five parameter exponential generalized beta distribution function, its construction, properties and special cases. If a random variable Y follows the generalized beta distribution given in equation 6.1 with parameters (a, b, c, p, q), then the random variable $Z = In(Y)$ is said to be distributed as an exponential generalized beta, with the pdf given by

Let

$$y = e^z$$

and

$$\left|\frac{dy}{dz}\right| = e^z$$

$$f(z; a, b, c, p, q) = \frac{p(e^z)^{pa-1}\left(1-(1-c)\left(\frac{e^z}{q}\right)^p\right)^{b-1} . e^z}{q^{pa}B(a,b)\left(1+c\left(\frac{e^z}{q}\right)^p\right)^{a+b}}$$

$$= \frac{p(e^z)^{pa}\left(1-(1-c)\left(\frac{e^z}{q}\right)^p\right)^{b-1}}{q^{pa}B(a,b)\left(1+c\left(\frac{e^z}{q}\right)^p\right)^{a+b}} \qquad (6.11)$$

Let $p = \frac{1}{\sigma}$ and $q = e^\delta$ in equation 6.11

$$f(z; a, b, c, \sigma, \delta) = \frac{\frac{1}{\sigma}(e^z)^{\frac{1}{\sigma}a}\left(1-(1-c)\left(\frac{e^z}{e^\delta}\right)^{\frac{1}{\sigma}}\right)^{b-1}}{(e^\delta)^{\frac{1}{\sigma}a}B(a,b)\left(1+c\left(\frac{e^z}{e^\delta}\right)^{\frac{1}{\sigma}}\right)^{a+b}}$$

$$= \frac{e^{\frac{a(z-\delta)}{\sigma}}\left(1-(1-c)e^{\frac{z-\delta}{\sigma}}\right)^{b-1}}{\sigma B(a,b)\left(1+ce^{\frac{z-\delta}{\sigma}}\right)^{a+b}} \qquad (6.12)$$

for $-\infty < \frac{z-\delta}{\sigma} < In\left(\frac{1}{1-c}\right)$

Equation 6.12 is the five parameter exponential generalized beta distribution function.

6.9 Properties of Five Parameter Exponential Generalized Beta Distribution

6.9.1 Moment-generating function

The moment-generating function of the exponential generalized beta distribution variates can be obtained by using the substitution $u = e^{(z-\delta)/\sigma}$ in $E(e^{tz}\ )$ and combining terms:

$$M(t) = E(tZ)$$

$$= e^{\delta t}\ E(Y^{\sigma t}; p = 1, q = 1, c, a, b)$$

$$= \frac{e^{\delta t}\ B(a+\sigma t, b)}{B(a,b)}\ {}_2F_1\left[\begin{matrix} a+\sigma t, & \sigma t; & c \\ a+b+\sigma t & & \end{matrix}\right], \quad (6.13)$$

Equation 6.13 is the moment-generating function of the five parameter exponential generalized beta distribution. The other moment generating functions related to special cases can be obtained from equation 6.13 as necessary; for instance, the moment generating function for the exponential generalized beta distribution of the first kind can be obtained as a special case when $c = 0$ and is given by

$$M(t) = \frac{e^{\delta t}\ B(a+\sigma t, b)}{B(a,b)}$$

While for the exponential generalized beta distribution of the second kind can be obtained as a special case when $c = 1$ and is given by

$$M(t) = \frac{e^{\delta t}\ B(a+\sigma t, b-\sigma t)}{B(a,b)}.$$

6.9.2 Mean of the exponential generalized beta distribution

The mean of the exponential generalized beta distribution can be obtained by differentiating the natural logarithm of equation 6.13, the cumulant-generating function, and substituting $t = 0$ to yield

$$E(Z) = \delta + \sigma[\psi(a) - \psi(b)] \frac{c\sigma a}{a+b} \; {}_3F_1 \begin{bmatrix} 1, & 1, & a+1; & c \\ 2, & a+b+1; & & \end{bmatrix}$$

The means for the exponential generalized beta of the first kind, exponential generalized beta of the second kind, and exponential generalized gamma, respectively, can be given by

$$E_1(Z) = \delta + \sigma[\psi(a) - \psi(a+b)],$$

$$E_2(Z) = \delta + \sigma[\psi(a) - \psi(b)],$$

$$E_3(Z) = \delta + \sigma\psi(b),$$

Where $\psi()$ denotes the digamma function. The higher-order moments for the exponential generated beta distribution are quite involving, however, relatively simple expressions for the variance, skewness and Kurtosis for the exponential generated beta distribution of the first kind, the exponential generated beta distribution of the second kind and the exponential generated gamma distribution can be easily obtained by differentiating the desired cumulant-generating functions.

6.10 Special Cases of Exponential Generalized Beta Distribution

Since the five parameter exponential generalized beta and the five parameter generalized beta distributions are related by the logarithmic transformation, similar ‘exponential’ distributions can be obtained by transforming each of the distributions mentioned under the five parameter generalized beta distribution in section 6.2.3 which corresponds to the special cases given below:

6.10.1 Exponential generalized beta of the first kind

This is a special case of the exponential generalized beta distribution in equation 6.12 when $c = 0$

$$f(z; a, b, \sigma, \delta) = \frac{e^{\frac{a(z-\delta)}{\sigma}}\left(1-e^{\frac{z-\delta}{\sigma}}\right)^{b-1}}{\sigma B(a, b)} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(6.14)}$$

for $-\infty < \frac{z-\delta}{\sigma} < 0$

Equation 6.14 is just an alternative representation of the generalized exponential distribution.

6.10.2 Exponential generalized beta of the second kind

This is a special case of the exponential generalized beta distribution in equation 6.12 when $c = 1$

$$f(z; a, b, \sigma, \delta) = \frac{e^{\frac{a(z-\delta)}{\sigma}}}{\sigma B(a, b)\left(1 + e^{\frac{z-\delta}{\sigma}}\right)^{a+b}} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(6.15)}$$

for $-\infty < \frac{z-\delta}{\sigma} < \infty$

Equation 6.15 is just an alternative representation of the generalized logistic distribution.

6.10.3 Exponential generalized gamma

The exponential generalized gamma distribution can be derived as a limiting case $(b \rightarrow \infty)$and $\delta * = \sigma \text{In} q + \delta$ using the same method given in section 5.6.17. It is defined by the following probability density function:

$$f(z; a, \sigma, \delta) = \frac{e^{\frac{a(z-\delta)}{\sigma}} e^{-e^{\frac{z-\delta}{\sigma}}}}{\sigma \Gamma \Xi(p)} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(6.16)$$

$$\text{for } -\infty < \frac{z-\delta}{\sigma} < \infty$$

Equation 6.16 is an alternative representation of the generalized Gompertz distribution.

6.10.4 Generalized Gumbell distribution

This is a special case of the exponential generalized beta distribution in equation 6.12 when $c = 1$ and $\text{a} = \text{b}$

$$f(z; a, \sigma, \delta) = \frac{e^{\frac{a(z-\delta)}{\sigma}}}{\sigma B(a, a) \left(1 + e^{\frac{z-\delta}{\sigma}}\right)^{2a}} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(6.17)$$

$$\text{for } -\infty < \frac{z-\delta}{\sigma} < \infty$$

6.10.5 Exponential Burr 12

This is a special case of the exponential generalized beta distribution in equation 6.12 when $c = 1$ and $\text{b} = 1$

$$f(z; a, \sigma, \delta) = \frac{a e^{\frac{a(z-\delta)}{\sigma}}}{\sigma \left(1 + e^{\frac{z-\delta}{\sigma}}\right)^{a+1}} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(6.18)$$

$$\text{for } -\infty < \frac{z-\delta}{\sigma} < \infty$$

6.10.6 Exponential power distribution function

The 3 parameter exponential power distribution function is a special case of the exponential generalized beta distribution in equation 6.12 when $c = 0$ and $b = 1$

$$f(z; a, \sigma, \delta) = \frac{ae^{\frac{a(z-\delta)}{\sigma}}}{\sigma} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (6.19)$$

$$\text{for } -\infty < \frac{z-\delta}{\sigma} < 0$$

6.10.7 Two parameter exponential distribution function

The 2 parameter exponential distribution function is a special case of the exponential generalized beta distribution in equation 6.12 when $c = 0, a = 1$ and $b = 1$

$$f(z; \sigma, \delta) = \frac{e^{\frac{z-\delta}{\sigma}}}{\sigma} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (6.20)$$

$$\text{for } -\infty < \frac{z-\delta}{\sigma} < 0$$

6.10.8 One parameter exponential distribution function

The 1 parameter exponential distribution function is a special case of the exponential generalized beta distribution in equation 6.12 when $c = 0, a = 1, \ b = 1$ and $\delta = 0$

$$f(z; \sigma) = \frac{e^{\frac{z}{\sigma}}}{\sigma} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (6.21)$$

$$\text{for } -\infty < \frac{z}{\sigma} < 0$$

6.10.9 Exponential Fisk (logistic) distribution

The exponential fisk distribution is a special case of the exponential generalized beta distribution in equation 6.12 when $c = 1,\ \mathrm{a} = 1$ and $\mathrm{b} = 1$

$$f(z;\sigma,\delta) = \frac{e^{\frac{z-\delta}{\sigma}}}{\sigma\left(1+e^{\frac{z-\delta}{\sigma}}\right)^2} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (6.22)$$

for $-\infty < \frac{z-\delta}{\sigma} < \infty$

Equation 6.22 of the exponential fisk distribution is also known as the logistic distribution.

6.10.10 Standard logistic distribution

The standard logistic distribution is a special case of the exponential generalized beta distribution in equation 6.12 when $c = 1,\ \mathrm{a} = 1,\ \mathrm{b} = 1$ and $\delta = 0$

$$f(z;\sigma) = \frac{e^{\frac{z}{\sigma}}}{\sigma\left(1+e^{\frac{z}{\sigma}}\right)^2} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (6.23)$$

for $-\infty < \frac{z}{\sigma} < \infty$

6.10.11 Exponential weibull distribution

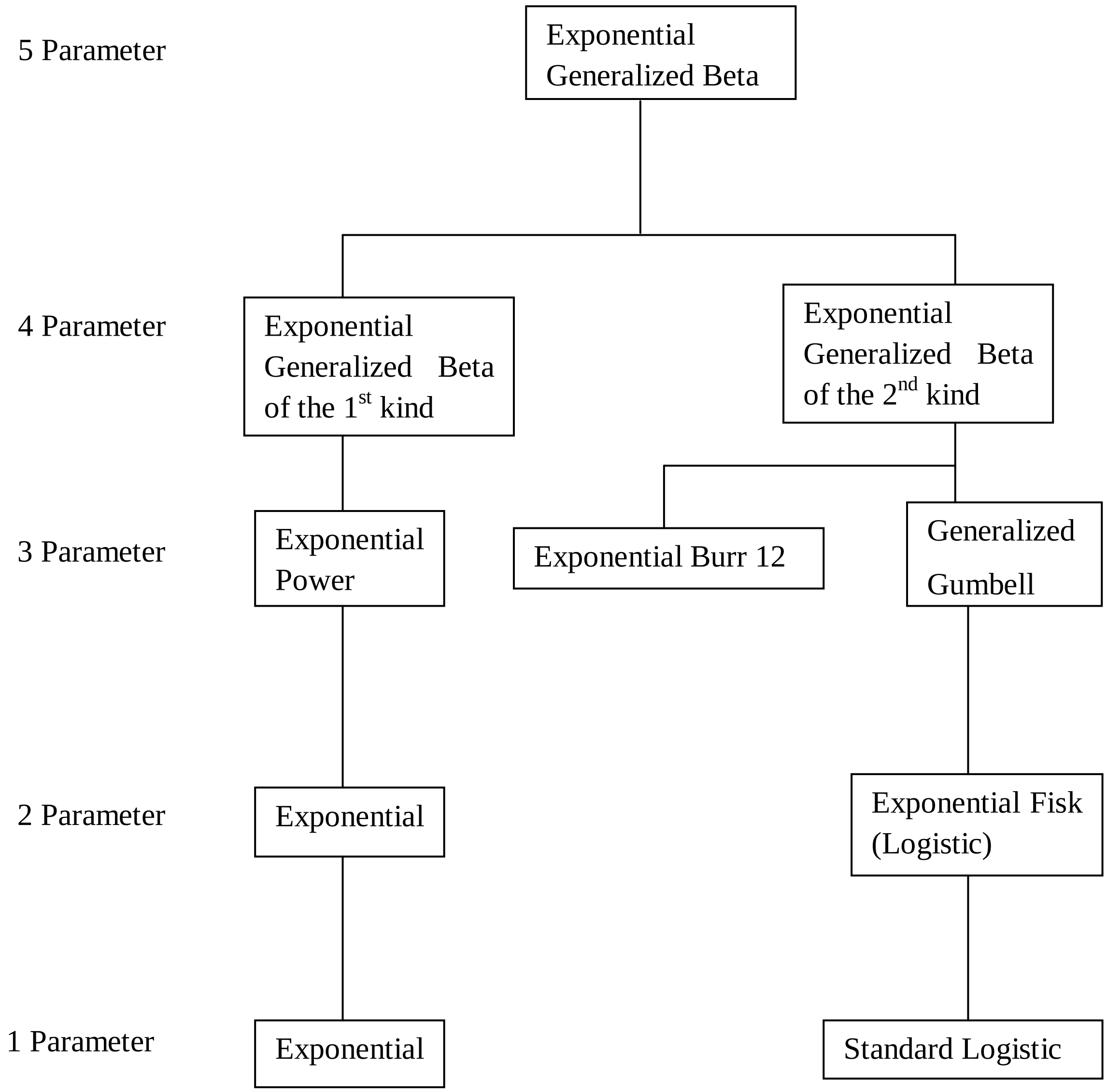
The exponential weibull distribution is a special case of the exponential generalized gamma distribution in equation 6.16 when $\mathrm{a} = 1$

$$f(z;b,\sigma,\delta) = \frac{e^{\frac{z-\delta}{\sigma}}e^{-e^{\frac{z-\delta}{\sigma}}}}{\sigma\Gamma\Xi(p)} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (6.24)$$

for $-\infty < \frac{z-\delta}{\sigma} < \infty$

The exponential weibull distribution is the extreme value type 1 distribution or the Gompertz distribution. Equation 6.24 of the exponential weibull distribution can also be derived as a limiting case ($b \rightarrow \infty$ with $\delta = \delta^{*} + \sigma \text{In} q$) from the exponential burr 12 distribution.

The Five Parameter exponential Beta Distribution and its Special Cases Diagrammatically



7 Chapter VII: Generalized Beta Distributions Based on Generated Distributions

7.1 Introduction

This chapter looks at the new family of generalized beta distributions. The new family of generalized beta distributions is based on beta generators and can be classified as follows:

- Beta generated distributions
- Exponentiated generated distributions
- Generalized beta generated distributions

7.2 Beta Generator Approach

The beta generated distribution was first introduced by Eugene et. al (2002) through its cumulative distribution function (cdf). The cdf of a beta distribution is defined by

$$W(t) = \int_0^{y} \frac{t^{a-1}(1-\mathrm{t})^{b-1}}{B(a,b)} dt, \qquad a > 0, \qquad b > 0$$

Note that $0 < y < 1$ since $0 < t < 1$

Replace y by a cdf say $\mathrm{G}(x)$, of any distribution, since $0 < \mathrm{G}(x) < 1$,

for $-\infty < x < \infty$

$$\therefore W(\mathrm{G}(x)) = \frac{1}{B(a,b)} \int_0^{\mathrm{G}(x)} t^{a-1}(1-\mathrm{t})^{b-1} dt, \qquad a > 0, \qquad b > 0$$

$$W(\mathrm{G}(x)) = A\ cdf\ of\ a\ cdf$$

$$= A\ function\ of\ x$$

$$= F(x)$$

$$F(x) = F(\mathrm{G}(x))$$

$$\therefore \frac{d}{dx}(F(x) = \frac{d}{dx} F(\mathrm{G}(x))$$

$$f(x) = \{F'(\mathrm{G}(x))\}\, \mathrm{G}'(x)$$

$$= \{f(\mathrm{G}(x))\}\, \mathrm{g}(x)$$

$$\Rightarrow f(x) = \frac{g(x)[G(x)]^{a-1}[1-G(x)]^{b-1}}{B(a,b)} \qquad (7.1)$$

for $0 < G(x) < 1$, $-\infty < x < \infty$ where $a > 0$, $b > 0$, $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function.

Equation 7.1 is the beta generator or beta generated distribution. It is also referred to as the generalized beta-F distribution (Sepanski and Kong, 2007). The equation can be used to generate a new family of beta distributions usually referred to as beta generated distributions. Jones (2004) called it generalized ith order statistic because the order statistic distribution is a special case when $a = i, b = n - i + 1$ which gives the following density function

$$f\big(G(y;\, a,b)\big) = \frac{g(y)[G(y)]^{i-1}[1-G(y)]^{(n-i+1)-1}}{B(i, n-i+1)}$$

$$= \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n-i+1)} g(y)[G(y)]^{i-1}[1-G(y)]^{n-i}$$

$$= \frac{n!}{(i-1)!\,(n-i)!} g(y)[G(y)]^{i-1}[1-G(y)]^{n-i} \qquad (7.2)$$

Equation 7.2 is the ith-order statistics generated from the beta distribution.

7.3 Various Beta Generated Distributions

7.3.1 Beta-Normal Distribution - (Eugene et. al (2002))

Eugene et. al (2002) introduced the beta-normal distribution, based on a composition of the classical beta distribution and the normal distribution. Its importance is more than just generalize the normal distribution. The beta-normal distribution generalizes the normal distribution and has flexible shapes, giving it greater applicability. Since then, many authors generalized other distributions similar to the beta-normal distribution.

The beta-normal distribution is obtained as follows:

Let

$$G(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

be the cumulative density function of normal distribution with parameters μ and $\sigma,$ and

$$g(x) = \phi\left(\frac{x-\mu}{\sigma}\right)$$

be the probability density function of the normal distribution.

By using equation 7.1, the density function of beta-normal distribution is given by

$$f(x;a,b,\mu,\sigma) = \frac{\sigma^{-1}\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{a-1}\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{b-1}}{B(a,b)}\phi\left(\frac{(x-\mu)}{\sigma}\right) \_\_(7.3)$$

Where $a > 0, b > 0, \sigma > 0,\ \ \mu \in \mathbb{R}\ and\ x \in \mathbb{R}$

The parameters a and b are the shape parameters characterizing the skewness, kurtosis and bimodality of the beta normal distribution. The parameters μ and σ have the same role as in normal distribution where, μ is a location parameter and σ is a scale parameter that stretches out or shrinks the distribution.

Special case of equation 7.3 when $\mu = 0$ and $\sigma = 1$ gives the standard beta-normal distribution given by

$$f(x;a,b,) = \frac{[\Phi(x)]^{a-1}[1-\Phi(x)]^{b-1}}{B(a,b)}\phi(x) \_\_\_\_\_\_\_\_\_\_\_\_(7.4)$$

Where $a > 0, b > 0,\ \ x \in \mathbb{R}$

Equation 7.3 is the beta-normal distribution introduced by Eugene et.al who also discussed some of its properties.

7.3.2 Beta-Exponential distribution – (Nadarajah and Kotz (2006))

Nadarajah and Kotz (2006) introduced the beta-exponential distribution, based on a composition of the classical beta distribution and the exponential distribution. They discussed in their paper the expression for the rth moment, properties of the hazard function, results for the distribution of the sum of beta-exponential random variables, maximum likelihood estimation and some asymptotic results.

The beta-exponential distribution is obtained as follows:

Let

$$G(x) = 1 - \exp(-\lambda x)$$

be the cdf of exponential distribution with parameter λ, and

$$g(x) = \lambda \exp(-\lambda x)$$

be the pdf of exponential distribution

By using equation 7.1, the density of the beta-exponential distribution is given by

$$f(x; a, b, \lambda) = \frac{\lambda \exp(-\lambda x)[1 - \exp(-\lambda x)\,]^{a-1}[1 - (1 - \exp(-\lambda x))]^{b-1}}{B(a, b)}$$

$$= \frac{\lambda \exp(-b\lambda x)[1 - \exp(-\lambda x)\,]^{a-1}}{B(a, b)} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(7.5)}$$

$a > 0, b > 0, \lambda > 0, x > 0$

Equation 7.5 is the beta-exponential distribution introduced by Nadarajah and Kotz (2006). They also discussed some of its properties. This beta-exponential distribution contains the exponentiated-exponential distribution (Gupta and Kundu (1999)) as a special case for b=1. i.e

$$f(x; a, \lambda) = a\lambda \exp(-\lambda x)[1 - \exp(-\lambda x)\,]^{a-1}$$

$a > 0,\ \lambda > 0, x > 0$

When $a = 1$, the beta-exponential distribution coincides with the exponential distribution with parameters $b\lambda$ i.e

$$f(x; b, \lambda) = \mathrm{b}\lambda \exp(-b\lambda x)$$

$b > 0,\ \lambda > 0, x > 0$

7.3.3 Beta-Weibull Distribution – (Famoye et al. (2005))

Famoye et.al (2005) introduced the beta-weibull distribution, based on a composition of the classical beta distribution and the weibull distribution. Lee et.al (2007) gave some properties of the hazard function, entropies and an application to censored data. Cordeiro et.al (2005) derived additional mathematical properties of beta-weibull such as expression for moments.

The beta-weibull distribution is obtained as follows:

Let

$$G(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{c}}$$

be the cdf of Weibull distribution and

$$g(x) = \left(\frac{|c|}{\beta^{c}}\right) x^{c-1}\, e^{-\left(\frac{x}{\beta}\right)^{c}}$$

be the pdf of Weibull distribution

By using equation 7.1, the density function of the beta-weibull distribution is given by

$$f(x; a, b, c, \beta) = \frac{\left(\frac{|c|}{\beta^{c}}\right) x^{c-1}\, e^{-\left(\frac{x}{\beta}\right)^{c}} \left[1 - e^{-\left(\frac{x}{\beta}\right)^{c}}\right]^{a-1} \left[1 - \left(1 - e^{-\left(\frac{x}{\beta}\right)^{c}}\right)\right]^{b-1}}{B(a,b)}$$

$$= \frac{\left(\frac{|c|}{\beta^{c}}\right) x^{c-1}\, e^{-\left(\frac{x}{\beta}\right)^{c}} \left[1 - e^{-\left(\frac{x}{\beta}\right)^{c}}\right]^{a-1} \left[e^{-\left(\frac{x}{\beta}\right)^{c}}\right]^{b-1}}{B(a,b)}$$

$$= \frac{\left(\frac{|c|}{\beta^{c}}\right) x^{c-1}\, e^{-b\left(\frac{x}{\beta}\right)^{c}} \left[1 - e^{-\left(\frac{x}{\beta}\right)^{c}}\right]^{a-1}}{B(a,b)}, \qquad a, b, c, \beta > 0 \_\_\_\_\_ (7.6)$$

Equation 7.6 is the four parameter beta-Weibull distribution introduced by Famoye et.al (2005) and studied by Lee et.al (2007).

7.3.4 Beta-Hyperbolic Secant Distribution – (Fischer and Vaughan. (2007))

Fisher and Vaughan (2007) introduced the beta-hyperbolic secant distribution, based on a composition of the classical beta distribution and the hyperbolic secant distribution. They gave some of its properties like the moments, special and limiting cases and compare it with other distributions using a real data set.

The beta-hyperbolic secant distribution is obtained as follows:

Let

$$G(x) = \frac{2}{\pi}\arctan(e^{x})$$

be the cdf of the hyperbolic secant distribution and

$$g(x) = \frac{1}{\pi\mathrm{Cosh(x)}}$$

be the pdf of the hyperbolic secant distribution

By using equation 7.1, the density function of the beta-hyperbolic secant is given by

$$f(x; a, b, \pi) = \frac{\left[\frac{2}{\pi}\arctan(e^{x})\right]^{a-1}\left[1-\frac{2}{\pi}\arctan(e^{x})\right]^{b-1}}{B(a,b)\pi\cosh(\mathrm{x})} \text{\_\_\_\_\_\_\_\_\_\_\_\_}(7.7)$$

$$a > 0, b > 0 \text{ and } x \in \mathbb{R}$$

Equation 7.7 is the beta-hyperbolic secant distribution introduced and studied by. Fischer and Vaughan (2007).

7.3.5 Beta-Gamma – (Kong et al. (2007))

Kong et.al (2007) introduced the beta-gamma distribution, based on a composition of the classical beta distribution and the gamma distribution. They derived in their paper some properties of the limit of the density function and of the hazard function. They gave an expression for the moments when the shape parameter a is an integer and made an application of the beta-gamma distribution.

The beta-gamma distribution is obtained as follows:

Let

$$G(x) = \frac{\Gamma_{\frac{x}{\lambda}}(\rho)}{\Gamma(\rho)}$$

be the cdf of gamma distribution where

$$\Gamma_x(\rho) = \int_0^x y^{\rho-1} e^{-y}\, dy$$

is the incomplete gamma function and

$$g(x) = \left(\frac{x}{\lambda}\right)^{\rho-1} e^{-\frac{x}{\lambda}}$$

is the pdf of the gamma function.

By using equation 7.1, the density function of the beta-gamma distribution is given by

$$f(x; a, b, \rho, \lambda) = \frac{x^{\rho-1} e^{-\frac{x}{\lambda}} \Gamma_{\frac{x}{\lambda}}(\rho)^{a-1} \left[1 - \frac{\Gamma_{\frac{x}{\lambda}}(\rho)}{\Gamma(\rho)}\right]^{b-1}}{B(a,b)\Gamma(\rho)^a \lambda^\rho} \qquad (7.8)$$

$$a, b, \rho, \lambda, x > 0$$

Equation 7.8 is the beta-gamma distribution introduced by Kong et al. (2007). They also derived some properties of the limit of the density function and hazard function.

7.3.6 Beta-Gumbel Distribution – (Nadarajah and Kotz (2004))

Nadarajah and Kotz (2004) introduced the beta-Gumbel distribution, based on a composition of the classical beta distribution and the Gumbel distribution. They calculated expressions for the r^{th} moment, gave some particular cases, and studied the density function.

The beta-Gumbel distribution is obtained as follows:

Let

$$G(x) = \exp\left\{-\exp\left\{-\frac{x-\mu}{\sigma}\right\}\right\}$$

be the cdf of Gumbel distribution and

$$g(x) = \frac{1}{\sigma}\exp\left\{-\frac{x-\mu}{\sigma}\right\}\exp\left\{-\exp\left\{-\frac{x-\mu}{\sigma}\right\}\right\}$$

$$= \frac{u}{\sigma}\exp\{-u\}$$

is the pdf of the gumbel distribution. Where

$$u = \exp\left\{-\frac{x-\mu}{\sigma}\right\}$$

By using equation 7.1, the density function of the beta-gumbel distribution is given by

$$f(x;a,b,u,\sigma) = \frac{1}{B(a,b)}[\exp\{-u\}]^{a-1}\,[1-\exp\{-u\}]^{b-1}.\frac{u}{\sigma}\exp\{-u\}$$

$$= \frac{u}{\sigma B(a,b)}e^{-au}(1-e^{-u})^{b-1} \qquad (7.9)$$

$$a,b,u,\sigma,x > 0$$

Equation 7.9 is the beta-gumbel distribution introduced by Nadarajah and Kotz (2004). They calculated expressions for the r^{th} moments, gave some particular cases and studied the density function.

7.3.7 Beta-Frèchet Distribution (Nadarajah and Gupta (2004))

Nadarajah and Gupta (2004) introduced the beta- Frèchet distribution, based on a composition of the classical beta distribution and the Frèchet distribution. They derived in their paper some of its properties and hazard function and of the hazard functions. They also gave an expression for the r^{th} moment. Barreto-Souza et.al (2008) gave another expression for the moments and derived some additional mathematical properties.

The beta- Frèchet distribution is obtained as follows:

Let

$$G(x) = \exp\left\{-\left(\frac{\mathrm{x}}{\sigma}\right)^{-\lambda}\right\}$$

be the cdf of standard Frèchet distribution and

$$g(x) = \frac{\lambda\sigma^{\lambda}}{x^{1+\lambda}}\exp\left\{-\left(\frac{\mathrm{x}}{\sigma}\right)^{-\lambda}\right\}$$

be the pdf of the Frèchet distribution.

By using equation 7.1, the density function of the beta-Frèchet distribution is given by

$$f(x; a, b, \mu, \sigma) = \frac{\lambda\sigma^{\lambda}\exp\left\{-\mathrm{a}\left(\frac{\mathrm{x}}{\sigma}\right)^{-\lambda}\right\}\left(1-\exp\left\{-\left(\frac{\mathrm{x}}{\sigma}\right)^{-\lambda}\right\}\right)^{\mathrm{b}-1}}{x^{1+\lambda}B(a,b)} \quad \_\_\_\_(7.10)$$

$$a, b, \lambda, \sigma, \mathrm{x} > 0$$

Equation 7.10 is the beta-Frèchet distribution introduced by Nadarajah and Gupta (2004). They derived some of its properties and gave an expression for the r^{th} moment. Barreto-Souza et.al (2008) gave another expression for the moments and derived some additional mathematical properties.

7.3.8 Beta-Maxwell Distribution (Amusan (2010))

Amusan (2010) introduced the beta-Maxwell distribution, based on a composition of the classical beta distribution and the Maxwell distribution. She defined and studied the three-parameter Maxwell distribution. She also discussed various properties of the distribution.

The beta-maxwell distribution is obtained as follows:

Let

$$G(x) = \frac{2\gamma\left(\frac{3}{2}, \frac{x^2}{2}\right)}{\sqrt{\pi}}$$

be the cdf of maxwell distribution and

$$g(x) = \sqrt{\frac{2}{\pi}} \frac{x^2 e^{-\left(\frac{x^2}{2\alpha^2}\right)}}{\alpha^3}$$

be the pdf of the maxwell distribution.

By using equation 7.1, the density function of the beta-maxwell distribution is given by

$$f(x; a, b, \alpha) = \frac{\left(\frac{2\gamma\left(\frac{3}{2}, \frac{x^2}{2\alpha}\right)}{\sqrt{\pi}}\right)^{a-1} \left(1 - \frac{2\gamma\left(\frac{3}{2}, \frac{x^2}{2\alpha}\right)}{\sqrt{\pi}}\right)^{b-1} \sqrt{\frac{2}{\pi}} \frac{x^2 e^{-\left(\frac{x^2}{2\alpha^2}\right)}}{\alpha^3}}{B(a, b)}$$

$$= \frac{1}{B(a,b)} \left(\frac{2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{x^2}{2\alpha}\right)\right)^{a-1} \left(1 - \frac{2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{x^2}{2\alpha}\right)\right)^{b-1} \sqrt{\frac{2}{\pi}} \frac{x^2 e^{-\left(\frac{x^2}{2\alpha^2}\right)}}{\alpha^3} \_\_\_\_(7.11)$$

$for\ a, b, \alpha, x > 0\ and\ \gamma(a, b)$ is an incomplete gamma function.

Equation 7.11 is the beta-Maxwell distribution introduced by Amusan (2010).

7.3.9 Beta-Pareto Distribution (Akinsete et.al (2008))

Akinsete et.al (2008) introduced the beta-Pareto distribution, based on a composition of the classical beta distribution and the Pareto distribution. They defined and studied properties of the four-parameter beta-pareto distribution.

The beta-pareto distribution is obtained as follows:

Let

$$G(x) = 1 - \left(\frac{x}{\theta}\right)^{-k}$$

be the cdf of pareto distribution and

$$g(x) = \frac{k\theta^k}{x^{k+1}}$$

be the pdf of the pareto distribution.

By using equation 7.1, the density function of the beta-pareto distribution is given by

$$f(x;a,b,k,\theta)=\frac{\left(1-\left(\frac{x}{\theta}\right)^{-\mathrm{k}}\right)^{\mathrm{a}-1}\left(1-\left(1-\left(\frac{x}{\theta}\right)^{-\mathrm{k}}\right)\right)^{\mathrm{b}-1}\frac{k\theta^{k}}{x^{k+1}}}{B(a,b)}$$

$$=\frac{k}{\theta B(a,b)}\left(1-\left(\frac{x}{\theta}\right)^{-\mathrm{k}}\right)^{\mathrm{a}-1}\left(\frac{x}{\theta}\right)^{-kb-1}\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(7.12)$$

$$a,b,k,\theta,x>0$$

Equation 7.12 is the beta-pareto distribution defined by Akinsete et.al.

7.3.10 Beta-Rayleigh Distribution (Akinsete and Lowe (2009))

Akinsete and Lowe (2009) introduced the beta-Rayleigh distribution, based on a composition of the classical beta distribution and the Rayleigh distribution.

The beta-Rayleigh distribution is obtained as follows:

Let

$$G(x)=1-e^{-\frac{x^{2}}{2\alpha^{2}}}$$

be the cdf of Rayleigh distribution and

$$g(x)=\frac{x}{\alpha^{2}}\ e^{-\left(\frac{x}{\alpha\sqrt{2}}\right)^{2}}$$

be the pdf of the Rayleigh distribution.

By using equation 7.1, the density function of the beta-Rayleigh distribution is given by

$$f(x;a,b,\alpha)=\frac{\left(1-e^{-\frac{x^{2}}{2\alpha^{2}}}\right)^{\mathrm{a}-1}\left(1-\left(1-e^{-\frac{x^{2}}{2\alpha^{2}}}\right)\right)^{\mathrm{b}-1}\frac{x}{\alpha^{2}}\ e^{-\left(\frac{x}{\alpha\sqrt{2}}\right)^{2}}}{B(a,b)}$$

$$=\frac{x}{\alpha^{2}B(a,b)}\left(1-e^{-\frac{x^{2}}{2\alpha^{2}}}\right)^{\mathrm{a}-1}e^{-b\left(\frac{x}{\alpha\sqrt{2}}\right)^{2}}\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(7.13)$$

$$a,b,k,\alpha>0$$

Equation 7.13 is the beta-Rayleigh distribution. If $a = b = 1$, equation 7.13 reduces to Rayleigh distribution with parameter α. When $a = 1$, the beta-rayleigh distribution reduces to the Rayleigh distribution with parameter $k = \frac{\alpha}{\sqrt{b}}$.

7.3.11 Beta-generalized-logistic of type IV distribution (Morais (2009))

Morais (2009) introduced the beta-generalized logistic of type IV distribution, based on a composition of the classical beta distribution and the generalized logistic of type IV distribution. She discussed some of its special cases that belong to the beta-G such as the beta –generalized logistic of types I, II, and III and some related distributions like the beta-beta prime and the beta-F. The generalized logistic of type IV distribution was proposed by Prentice (1976) as an alternative to modeling binary response data with the usual symmetric logistic distribution.

The beta-generalized logistic of type IV distribution is obtained as follows:

Let

$$G(x) = \frac{\mathrm{B}_{\frac{1}{1+\mathrm{e}^{-\mathrm{x}}}}(\mathrm{p}, \mathrm{q})}{\mathrm{B}(\mathrm{p}, \mathrm{q})}$$

be the cdf of the generalized logistic of type IV distribution given by a normalized incomplete beta function and

$$g(x) = \frac{\mathrm{e}^{-\mathrm{qx}}}{\mathrm{B}(\mathrm{p}, \mathrm{q})(1 + \mathrm{e}^{-\mathrm{x}})^{\mathrm{p}+\mathrm{q}}}$$

be the pdf of the generalized logistic of type IV distribution.

By using equation 7.1, the density function of the beta-generalized logistic of type IV distribution is given by

$$f(x; a, b, \mathrm{p}, \mathrm{q}) = \frac{\mathrm{B}(\mathrm{a}, \mathrm{b})^{1-\mathrm{a}-\mathrm{b}}}{\mathrm{B}(\mathrm{a}, \mathrm{b})} \frac{\mathrm{e}^{\mathrm{qx}}}{(1 - \mathrm{e}^{-\mathrm{x}})^{\mathrm{p}+\mathrm{q}}} [\mathrm{B}_{\frac{1}{1+\mathrm{e}^{-\mathrm{x}}}}(\mathrm{p}, \mathrm{q})]^{\mathrm{a}-1} [\mathrm{B}_{\frac{\mathrm{e}^{-\mathrm{x}}}{1+\mathrm{e}^{-\mathrm{x}}}}(\mathrm{q}, \mathrm{p})]^{\mathrm{b}-1} \text{\_\_\_\_\_}(7.14)$$

$$a, b, \mathrm{p}, \mathrm{q}, > 0, \qquad x > R$$

Equation 7.14 is the beta-generalized logistic of type IV distribution introduced by Morais (2009).

7.3.12 Beta-generalized-logistic of type I distribution (Morais (2009))

Morais (2009) introduced the beta-generalized logistic of type I distribution which is a special case of the beta-generalized logistic of type IV distribution with $\mathrm{q} = 1$.

The density function of the beta-generalized logistic of type I distribution is given by

$$f(x; a, b, \mathrm{p}) = \frac{\mathrm{pe^{-x}[(1 + e^{-ax})^p - 1]^{b-1}}}{\mathrm{B(a, b)(1 + e^{-x})^{a+pb}}}, \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (7.15)$$

$$a, b, \mathrm{p} > 0, \qquad x > R$$

Equation 7.15 is the beta-generalized logistic of type I distribution introduced by Morais (2009).

7.3.13 Beta-generalized-logistic of type II distribution (Morais (2009))

Morais (2009) introduced the beta-generalized logistic of type II distribution which is a special case of the beta-generalized logistic of type IV distribution with $\mathrm{p} = 1$. This distribution can also be obtained through the transformation $X = -Y$ where Y follows the beta-generalized logistic of type I distribution.

The density function of the beta-generalized logistic of type II distribution is given by

$$f(x; a, b, \mathrm{q}) = \frac{\mathrm{qe^{-bqx}}}{\mathrm{B(a, b)(1 + e^{-x})^{qb+1}}} \left[1 - \frac{\mathrm{e^{-qx}}}{\mathrm{B(a, b)(1 + e^{-x})^q}}\right] \_\_\_\_\_\_ (7.16)$$

$$a, b, \mathrm{q}, > 0, \qquad x > R$$

Equation 7.16 is the beta-generalized logistic of type II distribution introduced by Morais (2009).

7.3.14 Beta-generalized-logistic of type III distribution (Morais (2009))

Morais (2009) introduced the beta-generalized logistic of type III distribution which is a special case of the beta-generalized logistic of type IV distribution with $\mathrm{p} = \mathrm{q}$. This distribution is symmetric when a=b.

The density function of the beta-generalized logistic of type III distribution is given by

$$f(x; a, b, \mathrm{q}) = \frac{\mathrm{B(q,q)}^{1-\mathrm{a}-\mathrm{b}}}{\mathrm{B(a,b)}} \frac{\mathrm{e}^{-\mathrm{qx}}}{(1+\mathrm{e}^{-\mathrm{x}})^{2\mathrm{q}}} [\mathrm{B}_{\frac{1}{1+\mathrm{e}^{-\mathrm{x}}}}(\mathrm{q},\mathrm{q})]^{\mathrm{a}-1} [\mathrm{B}_{\frac{\mathrm{e}^{-\mathrm{x}}}{1+\mathrm{e}^{-\mathrm{x}}}}(\mathrm{q},\mathrm{q})]^{\mathrm{b}-1} \_\_\_(7.17)$$

$$a, b, \mathrm{q}, > 0, \qquad x > R$$

Equation 7.17 is the beta-generalized logistic of type III distribution introduced by Morais (2009).

7.3.15 Beta-beta prime distribution (Morais (2009))

The beta-beta prime distribution can be obtained from the beta-generalized logistic of type IV distribution using the transformation

$$x = \mathrm{e}^{-\mathrm{y}}$$

where Y is a random variable following the beta-generalized logistic of type IV distribution.

The density function of the beta-beta prime distribution is given by

$$f(x; a, b, p, \mathrm{q}) = \frac{B(p,q)^{1-a-b}}{B(a,b)} \frac{x^{q-1}}{(1+x)^{p+q}} [B_{\frac{x}{1+x}}(q,p)]^{a-1} [B_{\frac{1}{1+x}}(p,q)]^{b-1} \_\_\_(7.18)$$

$$a, b, q, > 0, \qquad x > R$$

Equation 7.18 is the beta-beta prime distribution introduced by Morais (2009).

7.3.16 Beta-F distribution (Morais (2009))

The beta-F distribution can be obtained as follows:

Let

$$G(x) = \frac{1}{\mathrm{B}\left(\frac{u}{2},\frac{v}{2}\right)} \mathrm{B}_{\frac{\left(\frac{v}{u}\right)x}{1+\left(\frac{v}{u}\right)x}}\left(\frac{v}{2},\frac{u}{2}\right)$$

be the cdf of the F distribution and

$$g(x) = \frac{\left(\frac{v}{u}\right)^{\frac{q}{2}}}{\mathrm{B}\left(\frac{u}{2},\frac{v}{2}\right)} \frac{x^{\frac{v}{2}-1}}{\left(1+\left(\frac{v}{u}\right)x\right)^{(u+v)/2}}$$

be the pdf of the F distribution.

By using equation 7.1, the density function of the beta-F distribution is given by

$$f(x;a,b,\mathrm{u},\mathrm{v}) = \frac{\mathrm{B}(\mathrm{a},\mathrm{b})^{-1}}{\mathrm{B}\left(\frac{u}{2},\frac{v}{2}\right)} \frac{\left(\frac{v}{u}\right)^{\frac{v}{2}} x^{\frac{v}{2}-1}}{\left(1+\left(\frac{v}{u}\right)x\right)^{(u+v)/2}} \left[\mathrm{I}_{\frac{\left(\frac{v}{u}\right)x}{1+\left(\frac{v}{u}\right)x}}\left(\frac{v}{2},\frac{u}{2}\right)\right]^{a-1} \left[\mathrm{I}_{\frac{1}{1+\left(\frac{v}{u}\right)x}}\left(\frac{v}{2},\frac{u}{2}\right)\right]^{b-1} \quad \text{\_\_(7.19)}$$

$$a, b, \mathrm{u}, \mathrm{v}, \mathrm{x} > 0$$

Equation 7.19 is the beta-F distribution. It can easily be verified that if Y follows the beta generalized logistic of type IV distribution with parameters $a, b, \mathrm{u}, \mathrm{v}$ then $\mathrm{F} = \frac{\mathrm{v}}{\mathrm{u}} e^{-y}$ follows the beta F distribution with parameters $a, b, 2\mathrm{u}, 2\mathrm{v}$,.

7.3.17 Beta-Burr XII distribution (Paranaíba et.al (2010))

Paranaíba et.al (2010) introduced the beta-burr XII distribution, based on a composition of the classical beta distribution and the burr XII distribution. They defined and studied the five parameter beta-burr XII distribution. They also derived some of its properties like moment generating function, mean and proposed methods of estimating its parameters.

The beta- burr XII distribution is obtained as follows:

Let

$$G(x) = 1 - \left(1 + \left(\frac{x}{\mathrm{q}}\right)^{\mathrm{p}}\right)^{-k}$$

be the cdf of burr XII distribution and

$$g(x) = pkq^{-p}\left(1 + \left(\frac{x}{\mathrm{q}}\right)^{\mathrm{p}}\right)^{-k-1} x^{p-1}$$

be the pdf of the burr XII distribution.

By using equation 7.1, the density function of the beta- burr XII distribution is given by

$$f(x; a, b, p, q, k) = \frac{\left(1 - \left(1 + \left(\frac{x}{\mathrm{q}}\right)^{\mathrm{p}}\right)^{-k}\right)^{\mathrm{a}-1} \left(\left(1 + \left(\frac{x}{\mathrm{q}}\right)^{\mathrm{p}}\right)^{-k}\right)^{\mathrm{b}-1} pkq^{-p}\left(1 + \left(\frac{x}{\mathrm{q}}\right)^{\mathrm{p}}\right)^{-k-1} x^{p-1}}{B(a,b)}$$

$$= \frac{pkq^{-p}x^{p-1}}{B(a,b)}\left(1 - \left(1 + \left(\frac{x}{\mathrm{q}}\right)^{\mathrm{p}}\right)^{-k}\right)^{\mathrm{a}-1} \left(1 + \left(\frac{x}{\mathrm{q}}\right)^{\mathrm{p}}\right)^{-kb-1} \quad \_\_\_\_\_\_(7.20)$$

for $a, b, p, q, k, x > 0$.

Equation 7.20 is the beta-burr XII distribution introduced by Paranaíba et.al (2010). The beta-burr XII distribution contains well-known distributions as special sub-models.

For $a = b = 1$, equation 7.20 gives the burr XII distribution

$$f(x; p, q, k) = pkq^{-k}x^{p-1}\left(1 + \left(\frac{x}{\mathrm{q}}\right)^{\mathrm{p}}\right)^{-k-1}$$

For $b = 1$, equation 7.20 gives the exponentiated burr XII distribution

$$f(x; a, p, q, k) = apkq^{-k}x^{p-1}\left(1 - \left(1 + \left(\frac{x}{\mathrm{q}}\right)^{\mathrm{p}}\right)^{-k}\right)^{\mathrm{a}-1} \left(1 + \left(\frac{x}{\mathrm{q}}\right)^{\mathrm{p}}\right)^{-k-1}$$

For $a = b = 1, q = \lambda^{-1}$ and $k = 1$ equation 7.20 gives the log logistic distribution

$$f(x;\ p, q,) = p\lambda^{p}x^{p-1}\ (1 + (x\lambda)^{\mathrm{p}})^{-2}$$

For $a = p = 1$ equation 7.20 gives the beta-pareto type II

$$f(x; b, k, q,) = \frac{bk}{q}\left(1 + \left(\frac{x}{\mathrm{q}}\right)\right)^{-kb-1}$$

For $a = b = p = 1$ equation 7.20 gives the beta-pareto type II

$$f(x; k, q,) = \frac{k}{q}\left(1 + \left(\frac{x}{\mathrm{q}}\right)\right)^{-k-1}$$

7.3.18 Beta-Dagum distribution (Condino and Domma (2010))

Condino and Domma (2010) introduced the beta-dagum distribution, based on a composition of the classical beta distribution and the dagum distribution. They defined and studied its properties.

The beta- dagum distribution is obtained as follows:

Let

$$G(x) = (1 + qx^{-p})^{-a}$$

be the cdf of dagum distribution and

$$g(x) = \frac{|p|qax^{-p-1}}{(1 + qx^{-p})^{1+\mathrm{a}}}$$

be the pdf of the dagum distribution.

By using equation 7.1, the density function of the beta- dagum distribution is given by

$$f(x) = \frac{1}{B(a,b)}((1 + qx^{-p})^{-a})^{\mathrm{a}-1}\,(1 - (1 + qx^{-p})^{-a}\,)^{\mathrm{b}-1}\,\frac{|p|qax^{-p-1}}{(1 + qx^{-p})^{1+\mathrm{a}}} \text{ ____(7.21)}$$

for $a, b, p, q, x > 0$.

Equation 7.21 is the beta-dagum distribution introduced by Condino and Domma (2010). The beta-dagum distribution contains the beta-fisk (log-logistic) distribution when $a = 1$ and $q = \frac{1}{\lambda}$ and the dagum distribution when $a = b = 1$.

7.3.19 Beta-(fisk) log logistic distribution

This is a new distribution referred to by Paranaíba et.al (2010) as a special sub-model of beta-burr XII distribution. It can be constructed based on a composition of the classical beta distribution and the log logistic (fisk) distribution as follows.

Let

$$G(x) = \frac{(\lambda x)^{\mathrm{p}}}{1 + (\lambda x)^{\mathrm{p}}}$$

be the cdf of log logistic distribution and

$$g(x) = \frac{\lambda p (\lambda x)^{\mathrm{p}-1}}{(1 + (\lambda x)^{\mathrm{p}})^2}$$

be the pdf of the log logistic distribution.

By using equation 7.1, the density function of the beta- log logistic distribution is given by

$$f(x; a, b, p, \lambda) = \frac{\left(\frac{(\lambda x)^{\mathrm{p}}}{1 + (\lambda x)^{\mathrm{p}}}\right)^{\mathrm{a}-1} \left(1 - \frac{(\lambda x)^{\mathrm{p}}}{1 + (\lambda x)^{\mathrm{p}}}\right)^{\mathrm{b}-1} \frac{\lambda p (\lambda x)^{\mathrm{p}-1}}{(1 + (\lambda x)^{\mathrm{p}})^2}}{B(a, b)}$$

$$= \frac{\lambda p (\lambda x)^{\mathrm{ap}-1}}{B(a, b)(1 + (\lambda x)^{\mathrm{p}})^{\mathrm{a}+\mathrm{b}}} \text{———————————————}(7.22)$$

for $a, b, p, \lambda, x > 0$.

Equation 7.22 is the beta-log logistic distribution referred to by Paranaíba et.al (2010).

7.3.20 Beta-generalized half normal Distribution (Pescim, R.R, et.al (2009))

Pescim, R.R, et.al (2009) introduced the beta-generalized half normal distribution, based on a composition of the classical beta distribution and the generalized half normal distribution. They derived expansions for the cumulative distribution and density functions. They also obtained formal expressions for the moments of the four-parameter beta- generalized half normal distribution.

The beta- generalized half normal distribution is obtained as follows:

Let

$$G(x) = 2\phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right] - 1 = \operatorname{erf}\left(\frac{\left(\frac{x}{\sigma}\right)^{\alpha}}{\sqrt{2}}\right)$$

Where

$$\phi(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right] \text{ and } \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x exp(-t^2)\,dt$$

be the cdf of generalized half normal distribution and

$$g(x) = \sqrt{\frac{2}{\pi}}\left(\frac{\alpha}{x}\right)\left(\frac{x}{\sigma}\right)^{\alpha} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^{2\alpha}}, \quad 0 \le x$$

be the pdf of the generalized half normal distribution.

By using equation 7.1, the density function of the beta-generalized half normal distribution is given by

$$f(x; a, b, \alpha, \sigma) = \frac{\left(2\phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right] - 1\right)^{a-1}\left(2 - 2\phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right]\right)^{b-1}\sqrt{\frac{2}{\pi}}\left(\frac{\alpha}{x}\right)\left(\frac{x}{\sigma}\right)^{\alpha} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^{2\alpha}}}{B(a,b)}$$

$a, b, \alpha, \sigma > 0$ ____(7.23)

Equation 7.23 is the beta- generalized half normal distribution defined by Pescim, R.R, et.al. (2009) It contains as special sub-models well known distributions.

for $\alpha = 1$, it reduces to beta half normal distribution given as

$$f(x; a, b, \sigma) = \frac{\left(2\phi\left(\frac{x}{\sigma}\right) - 1\right)^{a-1}\left(2 - 2\phi\left(\frac{x}{\sigma}\right)\right)^{b-1}\sqrt{\frac{2}{\pi}}\frac{e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^{2}}}{\sigma}}{B(a,b)}$$

$$a, b, \sigma > 0$$

for $a = b = 1$, it reduces to generalized half normal distribution given by

$$f(x;\alpha,\sigma) = \sqrt{\frac{2}{\pi}}\left(\frac{\alpha}{x}\right)\left(\frac{x}{\sigma}\right)^{\alpha} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^{2\alpha}}$$

$$\alpha,\sigma > 0$$

further if $\alpha = 1$, the generalized half normal distribution gives the half normal distribution

$$f(x;\alpha,\sigma) = \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^{2}}}{\sigma}$$

$$\alpha,\sigma > 0$$

for $b = 1$, it leads to the exponentiated generalized half normal distribution given as

$$f(x;a,\alpha,\sigma) = a\sqrt{\frac{2}{\pi}}\left(\frac{\alpha}{x}\right)\left(\frac{x}{\sigma}\right)^{\alpha} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^{2\alpha}}\left(2\phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right]-1\right)^{a-1}$$

7.4 Beta Exponentiated Generated Distributions

Beta exponentiated generated distributions are based on the power of the cdfs.

7.4.1 Exponentiated generated distribution

Let

$$F(x) = [G(x)]^{\alpha}$$

Where $G(x)$ is the old or parent cdf and $F(x)$ is the new cdf

The new cdf in terms of the old is therefore given by

$$f(x) = \alpha[G(x)]^{\alpha-1}g(x)$$

The following are some of the examples of the exponentiated exponential distributions

7.4.2 Exponentiated exponential distribution (Gupta and Kundu (1999))

Let

$$g(x) = \frac{1}{\beta}\, e^{-\left(\frac{x}{\beta}\right)}, \;\; \beta > 0, \qquad 0 < x < \infty$$

$$G(x) = 1 - e^{-(\frac{x}{\beta})}$$

$$F(x) = \left(1 - \, e^{-\left(\frac{x}{\beta}\right)}\right)^{\alpha} \text{ and}$$

$$f(x) = \frac{\alpha}{\beta}\left(1 - \, e^{-\left(\frac{x}{\beta}\right)}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(7.24)$$

Equation 7.24 is an example of the exponentiated generated distribution known as the exponentiated exponential distribution which has been extensively covered by Gupta and Kundu (1999).

7.4.3 Exponentiated Weibull distribution (Mudholkar et.al. (1995))

Let

$$g(x) = \left(\frac{p}{\beta^{p}}\right) x^{p-1} e^{-\left(\frac{x}{\beta}\right)^{p}}$$

$$G(x) = 1 - \, e^{-\left(\frac{x}{\beta}\right)^{p}}$$

$$F(x) = \left(1 - \, e^{-\left(\frac{x}{\beta}\right)^{p}}\right)^{\alpha}$$

$$f(x) = \frac{\alpha}{\beta^{p}}\left(1 - \, e^{-\left(\frac{x}{\beta}\right)^{p}}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{p}} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(7.25)$$

Equation 7.25 is the exponentiated Weibull distribution proposed by Mudholkar et.al. (1995).

7.4.4 Exponentiated Pareto distribution (Gupta et.al (1998))

Let

$$g(x) = pq^{\mathrm{p}} b x^{-p-1}\left(1-\left(\frac{x}{q}\right)^{-p}\right)^{b-1}$$

$$G(x) = \left(1-\left(\frac{x}{q}\right)^{-p}\right)^{b}$$

$$F(x) = \left(\left(1-\left(\frac{x}{q}\right)^{-p}\right)^{b}\right)^{\alpha}$$

$$f(x) = \alpha\left(\left(1-\left(\frac{x}{q}\right)^{-p}\right)^{b}\right)^{\alpha-1}\frac{pb}{q}x^{-pb-1} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(7.26)$$

Equation 7.26 is the exponentiated Pareto distribution.

7.4.5 Beta exponentiated generated distribution

$$f(x) = \frac{g(x)\alpha[G(x)]^{\alpha-1}([G(x)]^{\alpha})^{a-1}(1-[G(x)]^{\alpha})^{b-1}}{B(a,b)}$$

$$= \frac{g(x)\alpha[G(x)]^{a\alpha-1}(1-[G(x)]^{\alpha})^{b-1}}{B(a,b)}, \quad a>0, \qquad \alpha>0, \qquad b>0,$$

$$-\infty<x<\infty, \; 0<[G(x)]^{\alpha}<1$$

is the exponentiated generator distribution or exponentiated generated distribution. Below are some of the examples on exponentiated generated distributions.

7.4.6 Generalized beta generated distributions

To obtain generalized beta distributions, we simply replace $G(x)$ by $[G(x)]^{p}$ in the beta generated distribution. Some of the many distributions which can arise within the generalized beta generated class of distributions can be derived from an arbitrary parent cumulative distribution function (cdf) $G(x)$, and the probability density function (pdf) $f(x)$ of the new class of generalized beta distributions is defined as

$$f(G(x);\ a,b)) = \frac{|p|g(x)[G(x)]^{ap-1}\left[1-\left(G(x)\right)^p\right]^{b-1}}{B(a,b)} \qquad (7.27)$$

where $a > 0,\ b > 0$ and $p > 0$ are additional shape parameters which aim to introduce skewness and to vary tail weight and $g(x) = dG(x)/dx$. the new class of generalized beta distributions generated by equation 7.27 includes as special sub-models the beta generated distributions discussed in section 7.2 above. Eugene et.al (2002) introduced the beta generated distributions for $p = 1$ i.e

$$f(x) = \frac{g(x)[G(x)]^{a-1}\left[1-\left(G(x)\right)\right]^{b-1}}{B(a,b)}$$

and Cordeiro and Castro (2010) introduced Kumaraswamy (Kw) generated distributions for $a = 1$ i. e

$$f(x) = b|p|g(x)[G(x)]^{p-1}\left[1-\left(G(x)\right)^p\right]^{b-1}$$

Alternatively, the generalized beta generated distribution in equation 7.27 can be derived from the generalized beta distribution of the first kind in equation 5.1 as follows:

Letting $q = 1$

$$g(y) = \frac{|p|y^{pa-1}\left(1-\left(\frac{y}{1}\right)^p\right)^{b-1}.g(x)}{1^{ap}B(a,b)} \quad ,0 < y^p < 1, \qquad a,b,p > 0$$

Now consider

$$F(x) = \int_0^{G(x)} \frac{|p|y^{pa-1}(1-y^p)^{b-1}}{B(a,b)}dy$$

$$\therefore f(x) = \frac{|p|g(x)\left(G(x)\right)^{ap-1}\left(1-\left(G(x)\right)^p\right)^{b-1}}{B(a,b)}, \qquad -\infty < x < \infty, \qquad a,b,p > 0$$

7.4.7 Generalized beta exponential distribution (Barreto-Souza et al. (2010))

Let

$$G(x) = 1 - \exp(-\lambda x)$$

be the cdf of exponential distribution with parameter λ, and

$$g(x) = \lambda \exp(-\lambda x)$$

be the pdf of exponential distribution.

By using equation 7.27, the density function of the generalized beta-exponential distribution is given by

$$f(x; a, b, p, \lambda) = \frac{|p|\lambda \exp(-\lambda x)[1 - \exp(-\lambda x)\,]^{pa-1}[1 - (1 - \exp(-\lambda x))^{p}]^{b-1}}{B(a, b)} \quad \text{\_\_\_\_\_\_\_(7.28)}$$

Equation 7.28 is the generalized beta-exponential distribution.

7.4.8 Generalized beta-Weibull distribution

Let

$$G(x) = 1 - \mathrm{e}^{-\left(\frac{x}{\beta}\right)^{c}}$$

be the cdf of Weibull distribution and

$$g(x) = \left(\frac{|c|}{\beta^{c}}\right) x^{c-1}\, e^{-\left(\frac{x}{\beta}\right)^{c}}$$

be the pdf of Weibull distribution.

By using equation 7.27, the density function of the generalized beta-weibull distribution is given by

$$f(x; p, c, a, b, \beta) = \frac{|p| \left(\frac{|c|}{\beta^{c}}\right) x^{c-1}\, e^{-\left(\frac{x}{\beta}\right)^{c}} [1 - \mathrm{e}^{-\left(\frac{x}{\beta}\right)^{c}}\,]^{pa-1} \left[1 - \left(1 - \mathrm{e}^{-\left(\frac{x}{\beta}\right)^{c}}\right)^{p}\right]^{b-1}}{B(a, b)} \quad \text{\_\_(7.29)}$$

Equation 7.29 is the generalized beta-Weibull distribution.

7.4.9 Generalized beta-hyperbolic secant distribution

Let

$$G(x) = \frac{2}{\pi}\arctan\left(e^{\frac{x-\mu}{\sigma}}\right)$$

be the cdf of the hyperbolic secant distribution and

$$g(x) = \frac{1}{\sigma\pi\text{Cosh}\left(\frac{x-\mu}{\sigma}\right)}$$

be the pdf of the hyperbolic secant distribution

By using equation 7.27, the density function of the generalized beta-hyperbolic secant is given by

$$f(x; a, b, \mu, \sigma, \pi) = \frac{p\left[\frac{2}{\pi}\arctan\left(e^{\frac{x-\mu}{\sigma}}\right)\right]^{ap-1}\left[1-\left(\frac{2}{\pi}\arctan\left(e^{\frac{x-\mu}{\sigma}}\right)\right)^{p}\right]^{b-1}}{B(a,b)\sigma\pi\cosh\left(\frac{x-\mu}{\sigma}\right)} \quad \text{\_\_\_(7.30)}$$

$$a > 0, b > 0, \mu > 0, \sigma > 0 \text{ and } x \in \mathbb{R}$$

Equation 7.30 is the generalized beta-hyperbolic secant distribution. If $\mu = 0$ and $\sigma = 1$, the generalized beta-hyperbolic secant distribution reduces to the standard beta-hyperbolic distribution in equation 7.7.

7.4.10 Generalized beta-normal distribution

The generalized beta-normal distribution is obtained as follows:

Let

$$G(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

be the cumulative density function of normal distribution with parameters μ and σ, and

$$g(x) = \phi\left(\frac{x-\mu}{\sigma}\right)$$

be the probability density function of the normal distribution.

By using equation 7.27, the density function of the generalized beta-normal distribution is given by

$$f(x; a, b, \mu, \sigma, p) = \frac{p\sigma^{-1}[\Phi\left(\frac{x-\mu}{\sigma}\right)]^{ap-1}\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)^{\mathrm{p}}\right]^{b-1}}{B(a,b)}\phi\left(\frac{(x-\mu)}{\sigma}\right) \text{\_\_(7.31)}$$

Where $a > 0, b > 0, \sigma > 0,\ p > 0\ \mu \in \mathbb{R}\ and\ x \in \mathbb{R}$

Equation 7.31 is the generalized beta-normal distribution. If $\mu = 0$, $\sigma = 1$ and $p = 1$, the generalized beta-normal distribution reduces to the standard beta-normal distribution proposed by Eugene et.al (2002). If $\mu = 0, \sigma = 1, b = 1$ and $ac = 2$ the generalized beta-normal distribution coincides with the skew-normal distribution.

7.4.11 Generalized beta-log normal distribution

The generalized beta- log normal distribution is obtained as follows:

Let

$$G(x) = \Phi(\log x)$$

be the cumulative density function of standard log normal distribution, and

$$g(x) = \frac{\phi(\log x)}{x}$$

be the probability density function of the standard log normal distribution.

By using equation 7.27, the density function of the generalized beta-log normal distribution is given by

$$f(x; a, b, p) = \frac{p\phi(\log x)[\Phi(\log x)]^{ap-1}[1-\Phi(\log x)^{p}]^{b-1}}{xB(a,b)} \text{\_\_\_\_\_(7.32)}$$

Where $a > 0, b > 0, p > 0$ and $x \in \mathbb{R}$

Equation 7.32 is the generalized beta-log normal distribution.

7.4.12 Generalized beta-Gamma distribution

The generalized beta-gamma distribution is obtained as follows:

Let

$$G(x) = \frac{\Gamma_{\frac{x}{\lambda}}(\rho)}{\Gamma(\rho)}$$

be the cumulative density function of gamma distribution, and

$$g(x) = \left(\frac{x}{\lambda}\right)^{\rho-1} e^{-\frac{x}{\lambda}}$$

be the probability density function of the gamma distribution.

By using equation 7.27, the density function of the generalized beta-gamma distribution is given by

$$f(x; a, b, p, \rho, \lambda) = \frac{\left(\frac{x}{\lambda}\right)^{\rho-1} e^{-\frac{x}{\lambda}} \left[\frac{\Gamma_{\frac{x}{\lambda}}(\rho)}{\Gamma(\rho)}\right]^{ap-1} \left[1 - \left(\frac{\Gamma_{\frac{x}{\lambda}}(\rho)}{\Gamma(\rho)}\right)^{p}\right]^{b-1}}{B(a, b)} \text{\_\_\_}(7.33)$$

Where $a > 0, b > 0, p > 0, \rho > 0, \lambda > 0$ and $x \in \mathbb{R}$

Equation 7.33 is the generalized beta-gamma distribution.

7.4.13 Generalized beta-Frèchet distribution

The generalized beta- Frèchet distribution is obtained as follows:

Let

$$G(x) = \exp\left\{-\left(\frac{x}{\sigma}\right)^{-\lambda}\right\}$$

be the cumulative density function of Frèchet distribution, and

$$g(x) = \frac{\lambda}{\sigma} \exp\left\{-\left(\frac{x}{\sigma}\right)^{-\lambda}\right\} x^{\lambda-1}$$

be the probability density function of the Frèchet distribution.

By using equation 7.27, the density function of the generalized beta-Frèchet distribution is given by

$$f(x; a, b, p, \rho, \lambda) = \frac{p\frac{\lambda}{\sigma}\exp\left\{-\left(\frac{x}{\sigma}\right)^{-\lambda}\right\}x^{\lambda-1}\left[\exp\left\{-\left(\frac{x}{\sigma}\right)^{-\lambda}\right\}\right]^{ap-1}\left[1-\left(\exp\left\{-\left(\frac{x}{\sigma}\right)^{-\lambda}\right\}\right)^{p}\right]^{b-1}}{B(a,b)} \quad \text{\_\_(7.34)}$$

Where $a > 0, b > 0, p > 0, \rho > 0, \lambda > 0$ and $x \in \mathbb{R}$

Equation 7.34 is the generalized beta-Frèchet distribution.

8 Chapter VIII: Generalized Beta Distributions Based on Special Functions

8.1 Introduction

Beta, F, Gamma functions are among the many special functions. Most of these special functions can be expressed as hypergeometric function. This chapter looks at the generalized beta distributions based on special functions such as the hypergeometric distribution function, the confluent hypergeometric function, gauss Hypergeometric function among others.

8.2 Confluent Hypergeometric Function

The confluent hypergeometric (Kummer's) function is denoted by the symbol ${}_1F_1[a;c;x]$ and represent the series

$$ {}_1F_1[a;c,x] = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{x^3}{3!} + \cdots $$

$$ = \sum_{\mathrm{n}=0}^{\infty} \frac{a(a+1)(a+2)\ldots(a+n-1)}{c(c+1)(c+2)\ldots(c+n-1)}\frac{x^n}{n!} $$

for $c \neq 0, -1, -2, \ldots$ It can simply be expressed in the following form

$$ {}_1F_1[a;c,x] = \sum_{\mathrm{n}=0}^{\infty} \frac{(a)_{\mathrm{n}}\, x^n}{(c)_n\, n!}, \qquad \mathrm{c} \neq 0, -1, -2, \ldots $$

where $(a)_{\mathrm{n}}$ is the Pochhammer's symbol defined by

$$ (a)_{\mathrm{n}} = \mathrm{a(a+1)\ldots(a+n-1)} $$

$$ = \frac{\Gamma(\mathrm{a}+1)}{\Gamma(\mathrm{a})};\ \text{n is a positive integer} $$

$$ (a)_0 = 1 $$

If a is a non-positive integer, the series terminates.

The integral representation is given as

$$ {}_1F_1[a;c,x] = \frac{\Gamma(\mathrm{c})}{\Gamma(\mathrm{a})\Gamma(\mathrm{c}-\mathrm{a})}\int_0^1 u^{a-1}(1-u)^{c-a-1}e^{xu}\,du \qquad (8.1) $$

where $c > a > 0$

We verify that equation 8.1 is a distribution

$$\int_0^1 u^{a-1}(1-u)^{c-a-1}e^{xu}\,du = \int_0^1 u^{a-1}(1-u)^{c-a-1}\sum_{n=0}^{\infty}\frac{(xu)^n}{n!}\,du$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 x^n u^{n+a-1}\,(1-u)^{c-a-1}\,du$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!}\int_0^1 u^{n+a-1}\,(1-u)^{c-a-1}\,du$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!}\mathrm{B}(\mathrm{n+a,c-a})$$

$$\Rightarrow \int_0^1 u^{a-1}(1-u)^{c-a-1}e^{xu}\,du = \sum_{n=0}^{\infty}\frac{x^n}{n!}\frac{\Gamma(\mathrm{n+a})\Gamma(\mathrm{c-a})}{\Gamma(\mathrm{n+c})}$$

$$\text{but}\quad \Gamma(\mathrm{n+a}) = (\mathrm{n+a-1})\Gamma(\mathrm{n+a-1})$$

$$= \mathrm{a(a+1)(a+2)}\ldots(\mathrm{n+a-1})\Gamma(\mathrm{a})$$

$$\Rightarrow \int_0^1 u^{a-1}(1-u)^{c-a-1}e^{xu}\,du = \frac{\mathrm{a(a+1)(a+2)}\ldots(\mathrm{n+a-1})\Gamma(\mathrm{a})\Gamma(\mathrm{c-a})}{\mathrm{c(c+1)(c+2)}\ldots(\mathrm{n+c-1})\Gamma(\mathrm{c})}\frac{x^n}{n!}$$

$$= {}_1F_1[a;c,x]\mathrm{B(a,c-a)}$$

Thus

$$ {}_1F_1[a;c,x] = \frac{1}{B(a,c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}e^{xu}\,du$$

8.3 The Differential Equations

The first and second order differential equations can be expressed as follows;

$$\frac{d}{dx}\,{}_1F_1[a;c;x] = 0 + \frac{a}{c}\frac{1}{1!} + \frac{a(a+1)}{c(c+1)}\frac{2x}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{3x}{3!} + \cdots$$

$$= \frac{a}{c} + \frac{a}{c}\frac{(a+1)}{(c+1)}\frac{x}{1!} + \frac{a}{c}\frac{(a+1)(a+2)}{(c+1)(c+2)}\frac{x^2}{2!} + \cdots$$

$$= \frac{a}{c}\left\{1 + \frac{(a+1)}{(c+1)}\frac{x}{1!} + \frac{(a+1)(a+2)}{(c+1)(c+2)}\frac{x^2}{2!} + \cdots\right\}$$

$$= \frac{a}{c}\ {}_1F_1[a+1; c+1; x]$$

$$\frac{d^2}{dx^2}\ {}_1F_1[a; c; x] = \frac{a}{c}\left\{0 + \frac{(a+1)}{(c+1)} + \frac{(a+1)(a+2)}{(c+1)(c+2)}\frac{x}{1!} + \cdots\right\}$$

$$= \frac{a}{c}\frac{(a+1)}{(c+1)}\left\{1 + \frac{(a+2)}{(c+2)}\frac{x}{1!} + \cdots\right\}$$

$$= \frac{a(a+1)}{c(c+1)}\ {}_1F_1[a+2; c+2; x]$$

In applied mathematics, special functions are used in solving differential equation. Thus ${}_1F_1[a; c; x]$is a solution to a certain differential equation which is derived as follows:-

Using the operator

$$\delta = x\frac{d}{dx}$$

Therefore

$$\delta x^n = x\frac{d}{dx}x^n = xnx^{n-1} = nx^n$$

$$(\delta + c - 1)x^n = \delta x^n + (c-1)x^n$$

$$\therefore (\delta + c - 1)x^n = (n + c - 1)x^n$$

$$\delta(\delta + c - 1)x^n = \delta(n + c - 1)x^n$$

$$= \delta x^n(n + c - 1)$$

$$= nx^n(n + c - 1)$$

$$\therefore \delta(\delta + c - 1)x^n = n(n + c - 1)x^n$$

$$(\delta + a)(\delta + b)x^n = (\delta + a)(\delta x^n + bx^n)$$

$$= (\delta + a)(nx^n + bx^n)$$

$$= (\delta + a)nx^n + (\delta + a)bx^n$$

$$= \delta nx^n + anx^n + \delta bx^n + abx^n$$

$$= n^2x^n + anx^n + bnx^n + abx^n$$

$$= (n^2 + an + bn + ab)x^n$$

$$= (n(n+a) + b(n+a))x^n$$

$$\therefore (\delta + a)(\delta + b)x^n = (n+a)(n+b)\ x^n$$

$$\delta(\delta + c - 1)\,{}_1F_1[a;c;x] = \delta(\delta + c - 1)\sum_{\mathrm{n}=0}^{\infty}\frac{a(a+1)(a+2)\dots(a+n-1)}{c(c+1)(c+2)\dots(c+n-1)}\frac{x^n}{n!}$$

$$= \sum_{\mathrm{n}=0}^{\infty}\frac{a(a+1)(a+2)\dots(a+n-1)}{c(c+1)(c+2)\dots(c+n-1)}\delta(\delta + c - 1)\frac{x^n}{n!}$$

$$= \sum_{\mathrm{n}=0}^{\infty}\frac{a(a+1)(a+2)\dots(a+n-1)}{c(c+1)(c+2)\dots(c+n-1)}n(n + c - 1)\frac{x^n}{n!}$$

$$= \sum_{\mathrm{n}=0}^{\infty}\frac{a(a+1)(a+2)\dots(a+n-1)}{c(c+1)(c+2)\dots(c+n-2)}\frac{x^n}{(n-1)!}$$

$$= x\sum_{\mathrm{n}=1}^{\infty}\frac{a(a+1)(a+2)\dots(a+n-1)}{c(c+1)(c+2)\dots(c+n-2)}\frac{x^{n-1}}{(n-1)!}$$

$$= x\sum_{\mathrm{n}=1}^{\infty}\frac{a(a+1)(a+2)\dots(a+n)}{c(c+1)(c+2)\dots(c+n-1)}\frac{x^n}{n!}$$

by replacing n with n+1

$$= x\sum_{\mathrm{n}=0}^{\infty}\frac{a(a+1)(a+2)\dots(a+n-1)}{c(c+1)(c+2)\dots(c+n-1)}(a+n)\frac{x^n}{n!}$$

$$= x\sum_{\mathrm{n}=0}^{\infty}\frac{a(a+1)(a+2)\dots(a+n-1)}{c(c+1)(c+2)\dots(c+n-1)}(a+\delta)\frac{x^n}{n!}$$

$$= x(a+\delta)\sum_{\mathrm{n}=0}^{\infty}\frac{a(a+1)(a+2)\dots(a+n-1)}{c(c+1)(c+2)\dots(c+n-1)}\frac{x^n}{n!}$$

$$\therefore\ \delta(\delta + c - 1)\,{}_1F_1[a;c;x] = x(a+\delta)\ {}_1F_1[a;c;x]$$

i.e

$$\delta(\delta + c - 1)y = x(a+\delta)y$$

where

$$y = {}_1F_1[a;c;x]$$

$$\therefore [\delta^2 + \delta(c-1)]y = x(a+\delta)y$$

$$\Longrightarrow \qquad \delta^2 y + [\delta(c-1) - x(a+\delta)]y = 0$$

$$\therefore\quad \delta^2 y + (c-1)\delta y - xay - x\delta y = 0$$

$$\therefore \quad \delta^2 y + (c-1-x)\delta y - xay = 0$$

$$\delta(\delta y) + (c-1-x)\delta y - xay = 0$$

$$\delta\left[x\frac{d}{dx}y\right] + (c-1-x)x\frac{d}{dx}y - xay = 0$$

$$x\frac{d}{dx}\left[x\frac{d}{dx}y\right] + (c-1-x)x\frac{dy}{dx} - xay = 0$$

$$\therefore \frac{d}{dx}\left[x\frac{d}{dx}y\right] + (c-1-x)\frac{dy}{dx} - ay = 0$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + (c-1-x)\frac{dy}{dx} - ay = 0$$

$$x\frac{d^2y}{dx^2} + (c-x)\frac{dy}{dx} - ay = 0$$

Which is the Kummer's differential equation. The confluent hypergeometric function satisfies this equation.

The confluent hypergeometric distribution due to Gordy (1988a) is a special case of the confluent hypergeometric distribution given in equation 8.1 where

$c = a + b, u = x$ and $x = -\gamma$ such that

$${}_1F_1[a; a+b, -\gamma] = \frac{\Gamma(\mathrm{a+b})}{\Gamma(\mathrm{a})\Gamma(\mathrm{a+b-a})}\int_0^1 x^{a-1}(1-x)^{a+b-a-1}e^{-\gamma x}\,dx$$

$${}_1F_1[a; a+b, x] = \frac{\Gamma(\mathrm{a+b})}{\Gamma(\mathrm{a})\Gamma(\mathrm{b})}\int_0^1 x^{a-1}(1-x)^{b-1}e^{-\gamma x}\,dx$$

$$1 = \int_0^1 \frac{x^{a-1}(1-x)^{b-1}e^{-\gamma x}\,dx}{{}_1F_1[a; a+b, -\gamma]\mathrm{B(a,b)}}$$

$$f(x) = \frac{x^{a-1}[1-x]^{b-1}e^{(-\gamma x)}}{B(a,b){}_1F_1(a; a+b; -\gamma)}, \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(8.2)}$$

for $0 < x < 1$, a > 0, b > 0, $-\infty < \gamma < \infty$ and ${}_1F_1$ is the confluent hypergeometric function.

If $\gamma = 0$, then the moment generating function denoted by

$$M(t) = \frac{B(a-t,b)\,{}_1F_1(a-t; a+b-t; -\gamma)}{B(a,b){}_1F_1(a; a+b; -\gamma)} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(8.3)}$$

reduces to the standard beta pdf given by equation 2.1

8.4 Gauss Hypergeometric Function.

The Gaussian hypergeometric function denoted by the symbol ${}_2F_1[a,b;c;x]$ represents the series

$${}_2F_1[a,b;c;x] = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots$$

where $c \neq 0, -1, -2, \ldots$

It can simply be expressed in the following form;

$${}_2F_1[a,b;c;x] = \sum_{\mathrm{n}=0}^{\infty} \frac{(a)_\mathrm{n}\,(b)_\mathrm{n} x^n}{(c)_n\, n!}, \qquad \mathrm{c} \neq 0, -1, -2, \ldots \text{ \_\_\_\_\_\_\_\_\_\_(8.4)}$$

where $(a)_\mathrm{n}$ is the Pochhammer's symbol defined by

$$(a)_\mathrm{n} = \mathrm{a(a+1)} \ldots \mathrm{(a+r-1)}$$

$$= \frac{\Gamma(\mathrm{a}+1)}{\Gamma(\mathrm{a})}; \text{(n is a positive integer)}$$

$$(a)_0 = 1$$

If a is a non-positive integer, then $(a)_\mathrm{n}$ is zero for $\mathrm{n} > -a$, and the series terminates.

The Euler's integral representation is given as

$${}_2F_1[a,b;c;x] = \frac{\Gamma(\mathrm{c})}{\Gamma(\mathrm{a})\Gamma(\mathrm{c}-\mathrm{a})}\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}\,du \text{ \_\_\_\_\_\_\_\_\_\_(8.5)}$$

where $c > a > 0$

8.5 The Differential Equations

The first and second order differentials can be expressed as follows;

$$\frac{d}{dx}\,{}_2F_1[a,b;c;x] = \frac{ab}{c} + \frac{a(a+1)b(b+1)}{c(c+1)}x + \cdots$$

$$= \frac{ab}{c}\left[1 + \frac{(a+1)(b+1)}{(c+1)}x + \frac{(a+1)(a+2)(b+1)(b+2)}{(c+1)(c+2)}\frac{x^2}{2!} \ldots\right]$$

$$= \frac{ab}{c}\;{}_2F_1[a+1,b+1;c+1;x]$$

$$\frac{d^2}{dx^2}\,{}_2F_1[a,b;c;x] = \frac{ab}{c}\left[\frac{(a+1)(b+1)}{(c+1)} + \frac{(a+1)(a+2)(b+1)(b+2)}{(c+1)(c+2)}x + \cdots\right]$$

$$= \frac{ab}{c}\frac{(a+1)(b+1)}{(c+1)}\left[1 + \frac{(a+2)(b+2)}{(c+2)}x + \cdots\right]$$

$$= \frac{a(a+1)b(b+1)}{c(c+1)}\ {}_2F_1[a+2,b+2;c+2;x]$$

We now wish to derive a differential equation whose solution is ${}_2F_1[a,b;c;x]$

Let

$$\delta = x\frac{d}{dx}$$

then

$$\delta(\delta+c-1)\ {}_2F_1[a,b;c;x]$$

$$= \sum_{n=0}^{\infty}\frac{a(a+1)\dots(a+n-1)b(b+1)\dots(b+n-1)}{c(c+1)\dots(c+n-1)}\delta(\delta+c-1)\frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{a(a+1)\dots(a+n-1)b(b+1)\dots(b+n-1)}{c(c+1)\dots(c+n-2)}\frac{x^n}{(n-1)!}$$

$$= x\sum_{n=1}^{\infty}\frac{a(a+1)\dots(a+n-1)b(b+1)\dots(b+n-1)}{c(c+1)\dots(c+n-2)}\frac{x^{n-1}}{(n-1)!}$$

replace n by n+1. Then

$$\delta(\delta+c-1)\ {}_2F_1[a,b;c;x]$$

$$= x\sum_{n=1}^{\infty}\frac{a(a+1)\dots(a+n-1)b(b+1)\dots(b+n-1)(a+n)(b+n)}{c(c+1)\dots(c+n-1)}\frac{x^n}{n!}$$

$$= x\sum_{n=0}^{\infty}\frac{a(a+1)\dots(a+n-1)b(b+1)\dots(b+n-1)(a+\delta)(b+\delta)}{c(c+1)\dots(c+n-1)}\frac{x^n}{n!}$$

$$= x(a+\delta)(b+\delta)\sum_{n=0}^{\infty}\frac{a(a+1)\dots(a+n-1)b(b+1)\dots(b+n-1)}{c(c+1)\dots(c+n-1)}\frac{x^n}{n!}$$

$$\therefore\ \delta(\delta+c-1)\ {}_2F_1[a,b;c;x] = x(a+\delta)(b+\delta)\ {}_2F_1[a,b;c;x]$$

$$\delta(\delta+c-1)F = x(a+\delta)(b+\delta)\ F$$

where

$$F = {}_2F_1[a,b;c;x] \quad \text{and} \quad \delta = x\frac{d}{dx}$$

$$\therefore\ \delta[\delta F + (c-1)F] = x(a+\delta)(bF + \delta F)$$

$$\Rightarrow \delta\left[x\frac{dF}{dx} + (c-1)F\right] = x(a+\delta)\left[bF + x\frac{d}{dx}F\right]$$

$$\Rightarrow \delta\left[x\frac{dF}{dx}\right] + (c-1)\delta F = x(a+\delta)\left[bF + x\frac{dF}{dx}\right]$$

$$\Rightarrow x\frac{d}{dx}\left[x\frac{dF}{dx}\right] + (c-1)x\frac{dF}{dx} = x(a+\delta)\left[bF + x\frac{dF}{dx}\right]$$

$$\Rightarrow \frac{d}{dx}\left[x\frac{dF}{dx}\right] + (c-1)\frac{dF}{dx} = (a+\delta)\left[bF + x\frac{dF}{dx}\right]$$

$$x\frac{d^2F}{dx^2} + \cancel{\frac{dF}{dx}} + c\frac{dF}{dx} - \cancel{\frac{dF}{dx}} = abF + ax\frac{dF}{dx} + b\delta F + \delta x\frac{dF}{dx}$$

$$\therefore\ x\frac{d^2F}{dx^2} + c\frac{dF}{dx} = abF + ax\frac{dF}{dx} + bx\frac{dF}{dx} + x\frac{dF}{dx}\left(x\frac{dF}{dx}\right)$$

$$= abF + (a+b)x\frac{dF}{dx} + x\left[x\frac{d^2F}{dx^2} + \frac{dF}{dx}\right]$$

$$\therefore\ x(1-x)\frac{d^2F}{dx^2} + [c - ax - bx - x]\frac{dF}{dx} - abF = 0$$

$$\therefore\ x(1-x)\frac{d^2F}{dx^2} + [c - (a+b+1)x]\frac{dF}{dx} - abF = 0$$

This is the differential equation whose solution is ${}_2F_1[a,b;c;x]$ i.e the Gaussian hypergeometric function satisfies this second order linear differential equation.

The Gauss hypergeometric distribution due to Armero and Bayarri (1994) is a special case of the Gaussian hypergeometric distribution given in equation 8.5 where

$c = a+b, x = -z\ , u = x,$ and $-b = \gamma$ such that

$${}_2F_1[\gamma,b;a+b;-z] = \frac{\Gamma(\mathrm{a+b})}{\Gamma(\mathrm{a})\Gamma(\mathrm{a+b-a})}\int_0^1 x^{a-1}(1-x)^{a+b-a-1}(1+zx)^{-b}dx$$

$$1 = \int_0^1 \frac{x^{a-1}(1-x)^{b-1}(1+zx)^{\gamma}dx}{{}_2F_1[\gamma,b;a+b;-z]\mathrm{B(a,b)}}$$

$$f(x) = \frac{x^{a-1}(1- \ x)^{b-1}/(1+zx)^{\gamma}}{B(a,b)_2F_1(\gamma,a;a+b;-z)}$$ ____________(8.6)

for 0 < x < 1, a > 0, b > 0, and $-\infty < \gamma < \infty$. If $\gamma = 0$, then equation 8.6 reduces to beta of the first kind given by equation 2.1

Nadarajah and Kotz (2004) also studied the beta distribution based on the Gauss hypergeometric function. They defined the distribution as follows:

$$f(x) = \frac{bB(a,b)}{B(a,b+\gamma)} x^{a+b-1} \ {}_2F_1(1-\gamma,a;a+b;x),$$ _____________(8.7)

When $b = 0$ in equation 8.7, then it reduces to the Classical beta distribution with parameters a and γ.

8.6 Appell Function

Nadarajah and Kotz (2006) introduced a beta function based on the Appell function. The distribution is given by

$$f(x) = \frac{Cx^{\alpha-1}(1-x)^{\beta-1}}{(1-ux)^{\rho}(1-vx)^{\lambda}},$$ ____________________________(8.8)

for $0 < x < 1,\ \alpha > 0,\ \beta > 0,\ \rho > 0,\ \lambda > 0, -1 < u < 1, -1 < v < 1$

where C denotes the normalizing constant given by

$$\frac{1}{C} = B(\alpha,\beta)F_1(\alpha,\rho,\lambda,\alpha+\beta;u,v)$$

and F_1 denotes the Appell function of the first kind. This distribution is flexible and contains several of the known generalizations of the Classical beta distribution as particular cases. The Classical beta distribution is the particular case for either $\rho = 0$ and $\lambda = 0$ or $\rho = 0$ and $v = 0$ or $u = 0$ and $\lambda = 0$ or $u = 0$ and $v = 0$.Libby and Novick's beta distribution in equation 4.1 is the particular case for either $\rho = \alpha + \beta$ and $\lambda = 0$ or $u = v$ and $\rho + \lambda = \alpha + \beta$. Armero and Bayarri's Gauss hypergeometric distribution in equation 8.6 is the particular case for either $\rho = 0$ or $\lambda = 0$

8.7 Nadarajah and Kotz's Power Beta Distribution

The generalization due to Nadarajah and Kotz (2004) is based on the basic characterization that if X and Y are independent gamma distributed random variables with common scale parameter then the ratio *X/(X+Y)* is a beta distribution of the first kind. If one consider the distribution of

$W = X^c / (X^c + Y^c)\ C > 0$, referred to as *power beta*, then the pdf and cdf of W can be expressed as

$$f(w) = \frac{w^{-(1+b/c)}(1-w)^{b/c-1}}{bcB(a,b)}\left\{ b\,{}_2F_1\left(b, a+b; b+1; -\left(\frac{1-w}{w}\right)^{\frac{1}{c}}\right) - \left(\frac{1-w}{w}\right)^{\frac{1}{c}}\ {}_2F_1\left(b+1, a+b+1; b+2; -\left(\frac{1-w}{w}\right)^{\frac{1}{c}}\right)\right\}$$

and

$$F(w) = 1 - \frac{1}{bB(a,b)}\left(\frac{1-w}{w}\right)^{b/c}\ {}_2F_1\left(b, a+b; b+1; -\left(\frac{1-w}{w}\right)^{1/c}\right).$$

Simpler expressions can be obtained for integer values of a, b and c.

8.8 Compound Confluent Hypergeometric Function

The compound confluent hypergeometric distribution due to Gordy (1988b) is given by

$$f(x) = \frac{x^{a-1}[1-\ vx]^{b-1}\{\theta + (1-\theta)vx\}^{-r} e(-sx)}{B(a,b)v^{-a}\exp\left(-\frac{s}{v}\right)\Phi_1(b, r, a+b, \frac{s}{v}, 1-\theta)}, \qquad (8.9)$$

for $0 < a < 1$, $b > 0$, $-\infty < r < \infty$, $-\infty < s < \infty, 0 \le v \le 1$, $\theta > 0$ and Φ_1 is the confluent hypergeometric function of two variables defined by

$$\Phi_1(\alpha, \beta, \gamma, x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_n}{(\gamma)_{m+n} m!\, n!} x^m y^n$$

When $s = 0,\ r = a + b, v = \left(\frac{1-c}{q}\right)$ and $\theta = 1 - c$ in Equation 8.9, then it reduces to McDonald and Xu's five parameter generalized beta distribution in equation 6.1. When $s = 0,$ and $v = 1$ in Equation 8.9, then it reduces to gauss hypergeometric distribution due to Armero and Bayarri (1994) in equation 8.6. When $v = 1$, and $\theta = 1$ in Equation 8.9, then it reduces to confluent hypergeometric distribution due to Gordy (1988a) in equation 8.2. When $v = 0$, and > 0 , Equation 8.9 becomes the gamma distribution.

8.9 Beta-Bessel Distribution (A.K Gupta and S.Nadarajah (2006))

A.K Gupta and S. Nadarajah (2006) introduced three types of beta-bessel distributions, based on a composition of the classical beta distribution and the Bessel function. They derived expressions for their shapes, particular cases and the n^{th} moments. They also discussed estimation by the method of maximum likelihood and Bayes estimation.

The beta- bessel distribution is obtained as follows:

The first generalization of equation 2.1 is given by the pdf

$$f(x) = \mathrm{C}x^{\alpha-1}(1-x)^{\beta-1}I_v(cx)\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(8.10)$$

For $0 < x < 1, v > 0, \alpha > 0, \beta > 0\ and\ c \geq 0$, where C denotes the normalizing constant and can be determined as

$$\frac{1}{C} = \frac{c^v \Gamma(\alpha+v)\Gamma(\beta)}{2^v \Gamma(\alpha+\beta+v)\Gamma(v+1)} \quad {}_2F_3\left(\frac{\alpha+v}{2}, \frac{\alpha+v+1}{2}; v + 1, \frac{\alpha+v+\beta}{2}, \frac{\alpha+v+\beta+1}{2}; \frac{c^2}{4}\right)$$

$I_v(x)$ is the modified Bessel function defined by

$$I_v(x) = \frac{x^v}{\sqrt{\pi}\, 2^v (v+1/2)} \int_{-1}^{1} (1-t^2)^{(v+1/2)} \exp(xt)\, dt$$

The classical beta pdf (2.1) arises as particular case of 8.10 for $v = 0$ and $c = 0$.

The second generalization of equation 2.1 is given by the pdf

$$f(x) = \mathrm{C}x^{\boldsymbol{\alpha}-1}(1-x)^{\boldsymbol{\beta}-1}\exp(cx)I_v(cx) \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (8.11)$$

For $0 < x < 1, v > 0, \alpha > 0, \beta > 0 \; and \; c \geq 0$, where C denotes the normalizing constant and can be determined as

$$\frac{1}{C} = \frac{c^v \Gamma(\alpha+v)\Gamma(\beta)}{2^v \Gamma(\alpha+\beta+v)\Gamma(v+1)} \quad {}_2F_2\left(v+\frac{1}{2}, \alpha+v;\; 2v+1, \alpha+\beta+v; 2c\right)$$

The classical beta pdf (2.1) arises as the particular case of 8.11 for $v = 0$ and $c = 0$.

The third and final generalization of equation 2.1 is given by the pdf

$$f(x) = \mathrm{C}x^{\boldsymbol{\alpha}-1}(1-x)^{\boldsymbol{\beta}-1}\exp(-cx)I_v(cx) \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (8.12)$$

For $0 < x < 1, v > 0, \alpha > 0, \beta > 0 \; and \; c \geq 0$, where C denotes the normalizing constant and can be determined as

$$\frac{1}{C} = \frac{c^v \Gamma(\alpha+v)\Gamma(\beta)}{2^v \Gamma(\alpha+\beta+v)\Gamma(v+1)} \quad {}_2F_2\left(v+\frac{1}{2}, \alpha+v;\; 2v+1, \alpha+\beta+v; 2c\right)$$

The classical beta pdf (2.1) arises as the particular case of 8.12 for $v = 0$ and $c = 0$

9 Chapter IX: Summary and Conclusion

9.1 Summary

This project has looked at three methods of generalizing the classical beta distribution. The work could be summarized as follows:

In Chapter I, a general introduction covering the work on statistical distributions in general is given followed by a brief background on classical beta distribution. The limitation of the classical beta distribution is noted leading to the main objective of adding more parameters to make the distribution more flexible for analyzing empirical data. The recent works in literature is also highlighted.

In Chapter II, the classical beta distribution is constructed from the beta function and the relationship between the beta function and the Gamma function is shown. Its construction from ratio transformation of two independent gamma variables is also done. Other constructions from the r^{th} order statistic of the Uniform distribution and from the stochastic processes are also looked at. Some properties, shapes, and special cases of the classical beta are also covered under this chapter.

In Chapter III, looks at the transformation of the classical beta distribution on the domain (0,1) to the extended beta distribution of the second kind on the domain $(0, \infty)$. It also looked at some properties, shapes, and special cases of the beta distribution of the second kind.

In Chapter IV, highlights some of the three parameter beta distributions used by various authors in literature. In particular, the well known Libby and Novick (1982) three parameter beta distribution and the McDonald (1995) three parameter beta distribution.

In Chapter V, the four parameter beta distributions are constructed. Their properties and special cases are also given.

In Chapter VI, covers the construction of the five parameter beta distributions due to McDonald (1995) giving some of its properties and special cases. It further extends the five parameter beta distribution due to McDonald (1995) by introducing a sixth parameter to give a five parameter weighted beta distribution. Some of the properties and special cases of this five parameter weighted beta distribution have also been looked at. The chapter also looked at the construction of the five parameter exponential generalized beta distribution and its special cases.

In Chapter VII, gives a new family of generalized beta distributions based on the work of Eugene et al. (2002) using the generator approach. It covers the beta generated distributions, exponentiated generated distributions and the generalized beta generated distributions.

In Chapter VIII, the generalized beta distributions based on some special functions are covered. In particular, the Confluent Hypergeometric function, the Gauss Hypergeometric function, the Bessel and the Appell function.

In Chapter IX, a summary and concluding remark of the work is given followed by references and an appendix.

9.2 Conclusion

This project has looked at both the two kinds of two parameter beta distributions. It has reviewed various methods of constructing the beta distributions, their properties which include expressions for the r^{th} moments, first four moments, mode, coefficient of variation, coefficient of skewness, coefficient of kurtosis and their shapes. It has also presented the special or limiting cases. However, there are other methods which can be explored in extending this work e.g. using mixtures of distributions. Other areas of research which need to be looked at are: further properties of the beta distributions, estimation and applications.

10 Appendix

A.1 Classical Beta Distribution

Pdf	$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}, a > 0, \quad b > 0,$ $0 < x < 1$
Cumulative distribution function	$F(x) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1}\, dt$
r^{th} order moment	$\frac{\Gamma(a+r)\,\Gamma(a+b)}{\Gamma(a+b+r)\,\Gamma(a)}$
Mean	$\frac{a}{a+b}$
Mode	$\frac{(a-1)}{(a+b-2)}$
Variance	$\frac{ab}{(a+b+1)(a+b)^2}$
Skewness	$\frac{2(b-a)}{a+b+2}\sqrt{\frac{a+b+1}{ab}}$
Kurtosis	$\frac{(a+b+1)}{ab}\left(\frac{3[a^2(b+2)-2ab+b^2(a+2)]}{(a+b+2)(a+b+3)}\right)$

A.2 Power Distribution (b=1 in table A.1)

Pdf	$f(x) = ax^{a-1}, \quad a > 0, \quad 0 < x < 1$
Cumulative distribution function	$F(x) = x^a$
r^{th} order moment	$\frac{a}{a+r}$
Mean	$\frac{a}{a+1}$
Mode	1
Variance	$\frac{a}{(a+2)(a+1)^2}$
Skewness	$\frac{2a(1-a)}{(a+3)}\sqrt{\frac{(a+2)}{a}}$
Kurtosis	$\frac{a+2}{a}\left(\frac{3[3a^2-a+2]}{(a+3)(a+4)}\right)$

A.3 Uniform Distribution (a=b=1 in table A.1)

Pdf	$f(x) = 1, \qquad 0 < x < 1$
r^{th} order moment	$\frac{1}{1 + \mathrm{r}}$
Mean	$\frac{1}{2}$
Mode	Any value in the interval [0,1]
Variance	$\frac{1}{12}$
Skewness	0
Kurtosis	1.8

A.4 Arcsine Distribution (a=b= $\frac{1}{2}$ in table A.1)

Pdf	$f(x) = \frac{1}{\pi\sqrt{x(1-x)}}, 0 < x < 1$
Cumulative distribution function	$\mathrm{F(x)} = \frac{2}{\pi}\mathrm{arcsin}(\sqrt{x})$
r^{th} order moment	$\frac{\Gamma\left(\frac{1}{2} + r\right)}{\mathrm{r}\Gamma(\mathrm{r})\Gamma\left(\frac{1}{2}\right)}$
Mean	$\frac{1}{2}$
Mode	$\frac{1}{2}$
Variance	0.125
Skewness	0
Kurtosis	1.5

A.5 Triangular Shaped (a=1, b=2 in table A.1)

Pdf	$f(x) = 2 - 2x, 0 < x < 1$
Cumulative distribution function	$F(x) = 2x - x^2$
r^{th} order moment	$\frac{2r\Gamma(r)}{\Gamma(r+3)}$
Mean	0.3333
Mode	0
Variance	0.0556
Skewness	0.5657
Kurtosis	2.4

A.6 Triangular Shaped (a=2, b=1 in table A.1)

Pdf	$f(x) = 2x, \quad 0 < x < 1$
Cumulative distribution function	$F(x) = x^2$
r^{th} order moment	$\frac{2\Gamma(2+r)}{\Gamma(r+3)}$
Mean	0.6667
Mode	1
Variance	0.0556
Skewness	-0.5657
Kurtosis	2.4

A.7 Parabolic Shaped (a=b=2 in table A.1)

Pdf	$f(x) = 6x(1-x), 0 < x < 1$
Cumulative distribution function	$F(x) = 3x^2 - 2x^3$
r^{th} order moment	$\frac{6}{(r+3)(r+2)}$
Mean	0.5
Mode	0.5
Variance	0.05
Skewness	0
Kurtosis	2.1429

A.8 Wigner Semicircle (a=b=$\frac{3}{2}$, and $x = \frac{y+R}{2R}$ in table A.1)

Pdf	$f(x) = \frac{2\sqrt{R^2 - x^2}}{R^2\pi}, -R < x < R$
r^{th} order moment	$\left(\frac{R}{2}\right)^{2r} \frac{1}{r+1}\binom{2r}{r}$
Mean	0
Mode	0
Variance	$\frac{R^2}{4}$
Skewness	0
Kurtosis	2

A.9 Beta Distribution of the Second Kind (using $x = \frac{y}{1+y}$ or $\frac{1}{1+y}$ in table A.1)

Pdf	$f(x) = \frac{x^{a-1}}{B(a,b)(1+x)^{a+b}}, a > 0, \quad b > 0, \quad 0 < x < \infty$
r^{th} order moment	$\frac{\Gamma(a+r)\Gamma(b-r)}{\Gamma(a)\Gamma(b)}$
Mean	$\frac{a}{b-1}$
Mode	$\frac{b-1}{a+1}$
Variance	$\frac{a(a+b-1)}{(b-1)^2(b-2)}$
Skewness	$\frac{2a[2a^2 + 3a(b-1) + (b-1)^2]}{(b-1)^3(b-2)(b-3)}$
Kurtosis	$\frac{3a(ab^3 + 2b^3 + 2a^2b^2 + 5ab^2 - 6b^2 + a^3b + 8a^2b - 13ab + 6b + 5a^3 - 10a^2 + 7a - 2)}{(b-1)^4(b-2)(b-3)(b-4)}$

A.10 Lomax (Pareto II) (a=1 in table A.9)

pdf	$f(x) = \frac{b}{(1+x)^{1+b}}, b > 0, \quad 0 < x < \infty$
r^{th} order moment	$\frac{\Gamma(1+r)\Gamma(b-r)}{\Gamma(b)}$
Mean	$\frac{1}{b-1}$
Mode	$\frac{b-1}{2}$
Variance	$\frac{b}{(b-2)(b-1)^2}$
Skewness	$\frac{2b[b+1]}{(b-1)^3(b-2)(b-3)}$
Kurtosis	$\frac{3(3b^3+b^2+2b)}{(b-1)^4(b-2)(b-3)(b-4)}$

A.11 Log Logistic (Fisk) (a=b=1 in table A.9)

pdf	$f(x) = \frac{1}{(1+x)^2}, \quad 0 < x < \infty$
r^{th} order moment	$\frac{r\pi}{\sin r\pi}$
Mean	$\frac{\pi}{\sin \pi}$
Mode	0
Variance	$\frac{2\pi}{\sin 2\pi} - \left(\frac{\pi}{\sin \pi}\right)^2$
Skewness	$\frac{3\pi}{\sin 3\pi} - 3\frac{2\pi}{\sin 2\pi}\left(\frac{\pi}{\sin \pi}\right) + 2\left(\frac{\pi}{\sin \pi}\right)^3$
Kurtosis	$\frac{4\pi}{\sin 4\pi} - 4\left(\frac{\pi}{\sin \pi}\right)\frac{3\pi}{\sin 3\pi} + 6\left(\frac{\pi}{\sin \pi}\right)^2\frac{2\pi}{\sin 2\pi} - 3\left(\frac{\pi}{\sin \pi}\right)^4$

A.12 Libby and Novick (1982) Beta Distribution ($x = \frac{yc}{1-y+yc}$ in table A.1)

pdf	$f(x) = \frac{1}{B(a,b)} \frac{c^a y^{a-1}(1-y)^{b-1}}{(1-(1-c)y)^{a+b}}$, $a > 0, \quad b > 0, \quad c > 0, \quad 0 < x < 1$

A.13 McDonald's (1995) beta distribution of the first kind ($x = \frac{y}{c}$ in table A.1)

Pdf	$f(x) = \frac{x^{a-1}\left(1-\left(\frac{x}{c}\right)\right)^{b-1}}{c^a B(a,b)}$, $a > 0, \quad b > 0, \quad c > 0, \quad 0 < x < c$

A.14 Three Parameter Beta Distribution ($x = \frac{cy}{1+y}$ in table A.1)

Pdf	$f(x) = \frac{c^a x^{a-1}(1+(1-c)x)^{b-1}}{B(a,b)(1+x)^{a+b}}$, $a > 0, \quad b > 0, \quad c > 0, \quad 0 < x < 1$

A.15 Chotikapanich et.al (2010) Beta of the Second Kind Distribution ($x = \frac{y}{c}$ in table A.9)

Pdf	$f(x) = \frac{x^{a-1}}{c^a B(a,b)\left(1+\left(\frac{x}{c}\right)\right)^{a+b}}$, $a > 0, \quad b > 0, \quad c > 0, \quad 0 < x < \infty$

A.16 General Three Parameter Beta Distribution ($x = cy$ in table A.9)

Pdf	$f(x) = \frac{c^a x^{a-1}}{B(a,b)(1+cx)^{a+b}}$, $a > 0, \quad b > 0, \quad c > 0, \quad 0 < x < \infty$

A.17 Three Parameter Beta Distribution ($x = y^c$ in table A.1)

Pdf	$f(x) = \frac{cx^{ac-1}(1-x^c)^{b-1}}{\mathrm{B(a,b)}}$, $a > 0, \quad b > 0, \quad c > 0, \quad 0 < x < 1$

A.18 Generalized Four Parameter Beta Distribution of the First Kind ($x = \left(\frac{y}{q}\right)^c$ in table A.1)

Pdf	$f(x) = \frac{\lvert p\rvert x^{pa-1}\left(1-\left(\frac{x}{q}\right)^p\right)^{b-1}}{q^{ap}B(a,b)}, a > 0, \quad b > 0, \quad p > 0,$ $q > 0, \quad 0 \le x^p \le q^p$
r^{th} order moment	$\frac{q^r\Gamma(\mathrm{a+b})\Gamma\left(a+\frac{r}{p}\right)}{\Gamma\left(a+b+\frac{r}{p}\right)\Gamma(\mathrm{a})}$
Mean	$\frac{q\Gamma(\mathrm{a+b})\Gamma\left(\mathrm{a}+\frac{1}{p}\right)}{\Gamma\left(\mathrm{a+b}+\frac{1}{p}\right)\Gamma(\mathrm{a})}$
Mode	$\left(\frac{q^p(pa-1)}{pa+pb-1-p}\right)^{\frac{1}{p}}$
Variance	$q^2\left[\frac{\Gamma\left(\mathrm{a}+\frac{2}{\mathrm{p}}\right)\Gamma(\mathrm{b})\Gamma(\mathrm{a+b})}{\Gamma\left(\mathrm{a+b}+\frac{2}{\mathrm{p}}\right)} - \frac{\Gamma^2\left(\mathrm{a}+\frac{1}{\mathrm{p}}\right)\Gamma^2(\mathrm{b})\Gamma^2(\mathrm{a+b})}{\Gamma^2\left(\mathrm{a+b}+\frac{1}{\mathrm{p}}\right)}\right]$
Skewness	$q^3\left(\frac{\Gamma(\mathrm{a+b})\Gamma\left(\mathrm{a}+\frac{3}{p}\right)}{\Gamma\left(\mathrm{a+b}+\frac{3}{p}\right)\Gamma(\mathrm{a})} - 3\frac{\Gamma(\mathrm{a+b})\Gamma\left(\mathrm{a}+\frac{1}{p}\right)}{\Gamma\left(\mathrm{a+b}+\frac{1}{p}\right)\Gamma(\mathrm{a})}\frac{\Gamma(\mathrm{a+b})\Gamma\left(\mathrm{a}+\frac{2}{p}\right)}{\Gamma\left(\mathrm{a+b}+\frac{2}{p}\right)\Gamma(\mathrm{a})} + 2\left(\frac{\Gamma(\mathrm{a+b})\Gamma\left(\mathrm{a}+\frac{1}{p}\right)}{\Gamma\left(\mathrm{a+b}+\frac{1}{p}\right)\Gamma(\mathrm{a})}\right)^3\right)$

Kurtosis	$$q^4\left(\frac{\Gamma(a+b)\Gamma\left(a+\frac{4}{p}\right)}{\Gamma\left(a+b+\frac{4}{p}\right)\Gamma(a)}-4\frac{\Gamma(a+b)\Gamma\left(a+\frac{1}{p}\right)}{\Gamma\left(a+b+\frac{1}{p}\right)\Gamma(a)}\frac{\Gamma(a+b)\Gamma\left(a+\frac{3}{p}\right)}{\Gamma\left(a+b+\frac{3}{p}\right)\Gamma(a)}+6\left(\frac{\Gamma(a+b)\Gamma\left(a+\frac{1}{p}\right)}{\Gamma\left(a+b+\frac{1}{p}\right)\Gamma(a)}\right)^2\frac{\Gamma(a+b)\Gamma\left(a+\frac{2}{p}\right)}{\Gamma\left(a+b+\frac{2}{p}\right)\Gamma(a)}-3\left(\frac{\Gamma(a+b)\Gamma\left(a+\frac{1}{p}\right)}{\Gamma\left(a+b+\frac{1}{p}\right)\Gamma(a)}\right)^4\right)$$

A.19 Pareto Type I (b=1, p=-1 in table A.1)

Pdf	$f(x)=\frac{aq^a}{x^{a+1}}, a>0, \quad q>0,$ $0<q\leq x$
Cumulative distribution function	$1-\left(\frac{q}{x}\right)^a$
r^{th} order moment	$\frac{q^r a}{a-r}$
Mean	$\frac{qa}{a-1}$
Mode	q
Variance	$\frac{q^2 a}{(a-2)(a-1)^2}$
Skewness	$\frac{2(1+a)}{a-3}\left(\sqrt{\frac{(a-2)}{a}}\right)$
Kurtosis	$\frac{3(3a^3-5a^2-4)}{a(a-3)(a-4)}$

A.20 Stoppa (Generalized Pareto Type I) (a=1, p=-p in table A.1)

Pdf	$f(x) = \mathrm{p}q^{p}bx^{-p-1}\left(1-\left(\frac{x}{q}\right)^{-p}\right)^{\mathrm{b}-1}, b>0, \quad p>0, \quad q>0,$ $0<q\leq x$
Cumulative distribution function	$\left(1-\left(\frac{x}{q}\right)^{-p}\right)^{\mathrm{b}}$
rth order moment	$\frac{\mathrm{q}^{\mathrm{r}}b\Gamma(b)\Gamma\left(1-\frac{r}{p}\right)}{\Gamma\left(1+b-\frac{r}{p}\right)}$
Mean	$\frac{q\mathrm{b}\Gamma(b)\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+b-\frac{1}{p}\right)}$
Mode	$q\left(\frac{1+pb}{(1+p)}\right)^{\frac{1}{p}}$
Variance	$\mathrm{q}^{2}\left[\frac{\Gamma\left(1-\frac{2}{\mathrm{p}}\right)\Gamma(\mathrm{b})\Gamma(1+\mathrm{b})}{\Gamma\left(1+\mathrm{b}-\frac{2}{\mathrm{p}}\right)}-\frac{\Gamma^{2}\left(1-\frac{1}{\mathrm{p}}\right)\Gamma^{2}(\mathrm{b})\Gamma^{2}(1+\mathrm{b})}{\Gamma^{2}\left(1+\mathrm{b}-\frac{1}{\mathrm{p}}\right)}\right]$
Skewness	$q^{3}\left(\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1-\frac{3}{p}\right)}{\left(\mathrm{b}-\frac{3}{p}\right)\Gamma\left(\mathrm{b}-\frac{3}{p}\right)}-3\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1-\frac{1}{p}\right)}{\left(\mathrm{b}-\frac{1}{p}\right)\Gamma\left(\mathrm{b}-\frac{1}{p}\right)}\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1-\frac{2}{p}\right)}{\left(\mathrm{b}-\frac{2}{p}\right)\Gamma\left(\mathrm{b}-\frac{2}{p}\right)} + 2\left(\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1-\frac{1}{p}\right)}{\left(\mathrm{b}-\frac{1}{p}\right)\Gamma\left(\mathrm{b}-\frac{1}{p}\right)}\right)^{3}\right)$
Kurtosis	$\mathrm{q}^{4}\left(\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1-\frac{4}{p}\right)}{\Gamma\left(1+\mathrm{b}-\frac{4}{p}\right)}-4\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+\mathrm{b}-\frac{1}{p}\right)\Gamma(1)}\frac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1-\frac{3}{p}\right)}{\Gamma\left(1+\mathrm{b}-\frac{3}{p}\right)} + 6\left(\frac{\Gamma(1+\mathrm{b})\Gamma\left(1-\frac{1}{p}\right)}{\Gamma\left(1+\mathrm{b}-\frac{1}{a}\right)}\right)^{2}\frac{\Gamma(1+\mathrm{b})\Gamma\left(1-\frac{2}{p}\right)}{\Gamma\left(1+\mathrm{b}-\frac{2}{p}\right)} - 3\left(\frac{\Gamma(1+\mathrm{b})\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{1}{p}\right)}\right)^{4}\right)$

A.21 Kumaraswamy (a=1, q=1 in table A.1)

Pdf	$f(x) = \mathrm{pb}x^{p-1}(1 - x^{p})^{\mathrm{b}-1}, \quad b > 0, \quad p > 0$
r^{th} order moment	$\dfrac{\mathrm{b}\Gamma\left(\frac{r}{p}+1\right)\Gamma(b)}{\Gamma(1+\frac{r}{p}+b)}$
Mean	$\dfrac{\mathrm{b}\Gamma\left(\frac{1}{p}+1\right)\Gamma(b)}{\Gamma(1+\frac{1}{p}+b)}$
Mode	$\left(\dfrac{1-p}{1-pb}\right)^{\frac{1}{p}}$
Variance	$\mathrm{b}!\,(\mathrm{b}-1)!\left[\dfrac{\left(\frac{2}{\mathrm{p}}\right)!}{\left(\frac{2}{\mathrm{p}}+\mathrm{b}\right)!} - \mathrm{b}!\,(\mathrm{b}-1)!\left(\dfrac{\left(\frac{1}{\mathrm{p}}\right)!}{\left(\frac{1}{\mathrm{p}}+\mathrm{b}\right)!}\right)^2\right]$
Skewness	$\dfrac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(\mathrm{a}+\frac{3}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{3}{p}\right)} - 3\dfrac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{1}{p}\right)}\dfrac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{2}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{2}{p}\right)} + 2\left(\dfrac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{1}{p}\right)}\right)^3$
Kurtosis	$\dfrac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{4}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{4}{p}\right)} - 4\dfrac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{1}{p}\right)}\dfrac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{3}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{3}{p}\right)} + 6\left(\dfrac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{1}{p}\right)}\right)^2\dfrac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{2}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{2}{p}\right)} - 3\left(\dfrac{\mathrm{b}\Gamma(\mathrm{b})\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\mathrm{b}+\frac{1}{p}\right)}\right)^4$

A.22 Generalized Power (b=1, p=1 in table A.1)

pdf	$f(x) = \frac{ax^{a-1}}{q^a}, \quad a > 0, \quad q > 0, \quad 0 < x < q$
r^{th} order moment	$\frac{q^r \mathrm{a}}{(\mathrm{a}+\mathrm{r})}$
Mean	$\frac{qa}{\mathrm{a}+1}$
Variance	$q^2 \left[\frac{(\mathrm{a}+1)^2(\mathrm{a}+2)\mathrm{a}\Gamma(\mathrm{a}) - \mathrm{a}^2\Gamma^2(\mathrm{a})}{(\mathrm{a}+1)^2(\mathrm{a}+2)^2}\right]$
Skewness	$\frac{2a(-\mathrm{a}^3 - 3\mathrm{a}^2 + 4)}{(\mathrm{a}+3)} \left[\frac{1}{(\mathrm{a}+1)^2(\mathrm{a}+2)\mathrm{a}\Gamma(\mathrm{a}) - \mathrm{a}^2\Gamma^2(\mathrm{a})}\right]^{\frac{3}{2}}$
Kurtosis	$\left[\frac{(\mathrm{a}+1)^2(\mathrm{a}+2)^2}{q^2(\mathrm{a}+1)^2(\mathrm{a}+2)\mathrm{a}\Gamma(\mathrm{a}) - \mathrm{a}^2\Gamma^2(\mathrm{a})}\right]^2 . \mathrm{q}^4 \left(\frac{\mathrm{a}}{\mathrm{a}+4} - \frac{4\mathrm{a}^2}{(\mathrm{a}+1)(\mathrm{a}+3)} + \frac{6\mathrm{a}^3}{(\mathrm{a}+1)^2(\mathrm{a}+2)} - \frac{3\mathrm{a}^4}{(\mathrm{a}+1)^4}\right)$

A.23 Generalized Four Parameter Beta Distribution of the Second Kind ($x = \left(\frac{x}{q}\right)^p$ in table A.9)

Pdf	$f(x) = \frac{\lvert p\rvert x^{pa-1}}{q^{pa} B(a,b)(1+(\frac{x}{q})^p)^{a+b}}, a > 0, \quad b > 0, \quad p > 0,$ $q > 0, 0 < x < \infty$
rth order moment	$\frac{q^r \Gamma\left(a+\frac{r}{p}\right)\Gamma\left(b-\frac{r}{p}\right)}{\Gamma(a)\Gamma(b)}$
Mean	$\frac{q\Gamma\left(a+\frac{1}{p}\right)\Gamma\left(b-\frac{1}{p}\right)}{\Gamma(a)\Gamma(b)}$
Mode	$q\left(\frac{pa-1}{pb+1}\right)^{\frac{1}{p}}$
Variance	$q^2\left[\frac{B\left(a+\frac{2}{p}, b-\frac{2}{p}\right)}{B(a,b)} - \left\{\frac{B\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)}\right\}^2\right]$
Skewness	$q^3\left(\frac{B\left(a+\frac{3}{p}, b-\frac{3}{p}\right)}{B(a,b)} - 3\frac{B\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)}\frac{B\left(a+\frac{2}{p}, b-\frac{2}{p}\right)}{B(a,b)} + 2\left(\frac{B\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)}\right)^3\right)$
Kurtosis	$q^4\left(\frac{B\left(a+\frac{4}{p}, b-\frac{4}{p}\right)}{B(a,b)} - 4\frac{B\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)}\frac{B\left(a+\frac{3}{p}, b-\frac{3}{p}\right)}{B(a,b)} + 6\left(\frac{B\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)}\right)^2\frac{B\left(a+\frac{2}{p}, b-\frac{2}{p}\right)}{B(a,b)} - 3\left(\frac{B\left(a+\frac{1}{p}, b-\frac{1}{p}\right)}{B(a,b)}\right)^4\right)$

A.24 Singh Maddala (Burr 12) ($a = 1$ in table A.23)

Pdf	$f(x) = \dfrac{\lvert p\rvert b x^{p-1}}{q^p\left(1+(\frac{x}{q})^p\right)^{1+\mathrm{b}}}, b > 0, \quad p > 0, \quad q > 0$ $0 < x < \infty$
Cumulative distribution function	$F(x) = 1 - \left(1+(\frac{x}{q})^p\right)^{-\mathrm{b}}$
r^{th} order moment	$\dfrac{q^r\Gamma\left(1+\frac{r}{p}\right)\Gamma\left(b-\frac{r}{p}\right)}{\Gamma(\mathrm{b})}$
Mean	$\dfrac{q\Gamma\left(1+\frac{1}{p}\right)\Gamma\left(b-\frac{1}{p}\right)}{\Gamma(\mathrm{b})}$
Mode	$q\left(\dfrac{p-1}{pb+1}\right)^{\frac{1}{p}}$
Variance	$q^2\left\{\dfrac{\Gamma(\mathrm{b})\Gamma\left(1+\frac{2}{p}\right)\Gamma\left(b-\frac{2}{p}\right)-\Gamma^2(1+\frac{1}{p})\Gamma^2\left(b-\frac{1}{p}\right)}{\Gamma^2(b)}\right\}$
Skewness	$q^3\left(bB\left(1+\frac{3}{p},b-\frac{3}{p}\right)-3bB\left(1+\frac{1}{p},b-\frac{1}{p}\right)bB\left(1+\frac{2}{p},b-\frac{2}{p}\right)+2\left(bB\left(1+\frac{1}{p},b-\frac{1}{p}\right)\right)^3\right)$
Kurtosis	$q^4\left(bB\left(1+\frac{4}{p},b-\frac{4}{p}\right)-4b^2B\left(1+\frac{1}{p},b-\frac{1}{p}\right)B\left(1+\frac{3}{p},b-\frac{3}{p}\right)+6\left(bB\left(1+\frac{1}{p},b-\frac{1}{p}\right)\right)^2bB\left(1+\frac{2}{p},b-\frac{2}{p}\right)-3\left(bB\left(a+\frac{1}{p},b-\frac{1}{p}\right)\right)^4\right)$

A.25 Generalized Lomax (Pareto II) (a=1, p=1, q = $\frac{1}{\lambda}$ in table A.23)

Pdf	$f(x) = \frac{\lambda b}{(1+\lambda x)^{1+b}}, \quad b > 0, \quad \lambda > 0, \quad 0 < x < \infty$
Cumulative distribution function	$F(x) = 1 - (1+\lambda x)^{-b}$
r^{th} order moment	$\frac{\left(\frac{1}{\lambda}\right)^r \Gamma(1+r)\Gamma(b-r)}{\Gamma(b)}$
Mean	$\frac{1}{\lambda(b-1)}$
Mode	0
Variance	$\frac{b}{\lambda^2(b-1)^2(b-2)}$
Skewness	$\frac{2(b+1)}{(b-3)}\left(\sqrt{\frac{(b-2)}{b}}\right)$
Kurtosis	$\left(\frac{1}{\lambda}\right)^4 \left(\frac{24}{(b-1)(b-2)(b-3)(b-4)} - \frac{24}{(b-1)^2(b-2)(b-3)} + \frac{12}{(b-1)^3(b-2)} - \frac{3}{(b-1)^4}\right)$

A.26 Inverse Lomax (b=1, p=1 in table A.23)

Pdf	$f(x) = \frac{ax^{a-1}}{q^a\left(1+\frac{x}{q}\right)^{1+a}}, \quad a > 0, \quad q > 0, \quad 0 < x < \infty$
rth order moment	$\frac{(q)^r\Gamma(a+r)\Gamma(1-r)}{\Gamma(a)\Gamma(1)}$
Mode	$q\left(\frac{a-1}{2}\right)$
Variance	$q^2[aB(a+2,-1) - \{aB(a+1,0)\}^2]$
Skewness	$q^3\left(\frac{B(a+3,1-3)}{B(a,1)} - 3\frac{B(a+1,1-1)}{B(a,1)}\frac{B(a+2,1-2)}{B(a,1)} + 2\left(\frac{B(a+1,1-1)}{B(a,1)}\right)^3\right)$
Kurtosis	$q^4\left(\frac{B(a+4,1-4)}{B(a,1)} - 4\frac{B(a+1,1-1)}{B(a,1)}\frac{B(a+3,1-3)}{B(a,1)} + 6\left(\frac{B(a+1,1-1)}{B(a,1)}\right)^2\frac{B(a+2,1-2)}{B(a,1)} - 3\left(\frac{B(a+1,1-1)}{B(a,1)}\right)^4\right)$

A.27 Dagum (Burr 3) (b=1 in table A.23)

Pdf	$f(x) = \frac{\lvert p\rvert ax^{pa-1}}{q^{pa}(1+(\frac{x}{q})^p)^{1+a}}, \quad a > 0, \quad p > 0, \quad q > 0$ $0 < x < \infty$
rth order moment	$\frac{q^r\Gamma(a+\frac{r}{p})\Gamma(1-\frac{r}{p})}{\Gamma(a)}$
Mean	$\frac{q\Gamma(a+\frac{1}{p})\Gamma(1-\frac{1}{p})}{\Gamma(a)}$
Mode	$q\left(\frac{pa-1}{p+1}\right)^{\frac{1}{p}}$
Variance	$q^2\left\{\frac{\Gamma(a)\Gamma\left(a+\frac{2}{p}\right)\Gamma\left(1-\frac{2}{p}\right) - \Gamma^2(a+\frac{1}{p})\Gamma^2\left(1-\frac{1}{p}\right)}{\Gamma^2(a)}\right\}$

A.28 Para-Logistic (a=1, b = p in table A.23)

Pdf	$f(x) = \frac{p^2 x^{p-1}}{q^p(1+(\frac{x}{q})^p)^{1+p}}, \quad p > 0, \quad q > 0$ $0 < x < \infty$

A.29 General Fisk (Log –Logistic) (a=b=1, q = $\frac{1}{\lambda}$ in table A.23)

pdf	$f(x) = \frac{\lambda\lvert p\rvert(\lambda x)^{p-1}}{(1+(\lambda x)^p)^2}, \quad p > 0, \quad \lambda > 0, \quad 0 < x < \infty$
rth order moment	$q^r\Gamma\left(1+\frac{r}{p}\right)\Gamma\left(1-\frac{r}{p}\right)$
Mean	$q\frac{1}{p}\Gamma\left(\frac{1}{p}\right)\Gamma\left(1-\frac{1}{p}\right)$
Mode	$q\left(\frac{p-1}{p+1}\right)^{\frac{1}{p}}$
Variance	$q^2\left\{\Gamma\left(1+\frac{2}{p}\right)\Gamma\left(1-\frac{2}{p}\right)-\Gamma^2(1+\frac{1}{p})\Gamma^2\left(1-\frac{1}{p}\right)\right\}$

A.30 Fisher (F) (a=$\frac{u}{2}$, b=$\frac{v}{2}$, q = $\frac{v}{u}$ in table A.23)

pdf	$f(x) = \frac{\Gamma\left(\frac{u+v}{2}\right)\left(\frac{u}{v}\right)^{\frac{u}{2}} x^{\frac{u}{2}-1}}{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{v}{2}\right)\left(1+\left(\frac{xu}{v}\right)\right)^{\frac{u+v}{2}}}, \quad u > 0, \quad v > 0$ $0 < x < \infty$
rth order moment	$\frac{\left(\frac{v}{u}\right)^r\Gamma\left(\frac{u}{2}+r\right)\Gamma\left(\frac{v}{2}-r\right)}{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{v}{2}\right)}$
Mean	$\frac{v}{v-2}$
Mode	$\left(\frac{u-2}{u}\right)\left(\frac{v}{v+2}\right)$
Variance	$2v^2\left(\frac{u+v-2}{u(v-2)^2(v-4)}\right)$

A.31 Half Student's t (a=$\frac{1}{2}$, b=$\frac{r}{2}$, q = $\sqrt{r}$ in table A.23)

pdf	$f(x) = \frac{2\Gamma\left(\frac{1+r}{2}\right)}{\sqrt{r\pi}\Gamma\left(\frac{r}{2}\right)\left(1+\frac{x^2}{r}\right)^{\frac{1+r}{2}}}, \quad -\infty < x < \infty$

A.32 Logistic (x = e^y in table A.23)

pdf	$f(x) = \frac{\lvert p\rvert e^{pa(x-log(q)}}{B(a,b)(1+e^{p(x-\log(q))})^{a+b}}, \quad -\infty < x < \infty$

A.33 Generalized Gamma (q $= \beta b^{\frac{1}{p}}$, $\lim\limits_{b\to\infty}$ in table A.23)

pdf	$f(x) = \frac{\lvert p\rvert x^{pa-1}}{\beta^{pa}\Gamma(a)} e^{-\left(\frac{x}{\beta}\right)^p}, \quad a > 0, \quad p > 0, \quad \beta > 0,$ $0 < x < \infty$
r[th] order moment	$\frac{\beta^r\Gamma\left(a+\frac{r}{p}\right)}{\Gamma(a)}$
Mean	$\frac{\beta\Gamma\left(a+\frac{1}{p}\right)}{\Gamma(a)}$
Mode	$\beta\left(a-\frac{1}{p}\right)^{\frac{1}{p}}$
Variance	$\frac{\beta^2}{\Gamma^2(a)}\left(\Gamma(a)\Gamma\left(a+\frac{2}{p}\right)-\Gamma^2\left(a+\frac{1}{p}\right)\right)$
Skewness	$\frac{\beta^3\Gamma^2(a)\Gamma\left(a+\frac{3}{p}\right)-3\beta^3\Gamma(a)\Gamma\left(a+\frac{2}{p}\right)\Gamma\left(a+\frac{1}{p}\right)+2\beta^3\Gamma^3\left(a+\frac{1}{p}\right)}{\Gamma^3(a)}$
Kurtosis	$\beta^4\left(\frac{\Gamma\left(a+\frac{4}{p}\right)}{\Gamma(a)}-\frac{4\Gamma\left(a+\frac{3}{p}\right)\Gamma\left(a+\frac{1}{p}\right)}{\Gamma^2(a)}+\frac{6\Gamma\left(a+\frac{2}{p}\right)\Gamma^2\left(a+\frac{1}{p}\right)}{\Gamma^3(a)} - 3\frac{\Gamma^4\left(a+\frac{1}{p}\right)}{\Gamma^4(a)}\right)$

A.34 Gamma (p = 1, in table A.33)

pdf	$f(x) = \frac{x^{a-1}}{\beta^a \Gamma(a)} e^{-\left(\frac{x}{\beta}\right)}, a > 0,\ \beta > 0,\ 0 < x < \infty$
rth order moment	$\frac{\beta^r \Gamma(a+r)}{\Gamma(a)}$
Mean	$\mathrm{B}a$
Mode	$\beta(\mathrm{a}-1)$
Variance	$\mathrm{a}\beta^2$
Skewness	$\frac{2}{\sqrt{a}}$
Kurtosis	$3 + \frac{6}{a}$

A.35 Inverse Gamma (p = −1, in table A.33)

pdf	$f(x) = \frac{\beta^a}{x^{1+a}\Gamma(a)} e^{-\left(\frac{\beta}{x}\right)}, \quad a > 0, \quad \beta > 0, \quad 0 < x < \infty$
rth order moment	$\frac{\beta^r \Gamma(a-r)}{\Gamma(a)}$
Mean	$\frac{\beta}{\mathrm{a}-1}$
Mode	$\frac{\beta}{(\mathrm{a}+1)}$
Variance	$\frac{\beta^2}{(a-1)^2(\mathrm{a}-2)}$
Skewness	$\frac{4\sqrt{(\mathrm{a}-2)}}{(\mathrm{a}-3)}$
Kurtosis	$\frac{3(a-2)(a+5)}{(a-3)(a-4)}$

A.36 Weibull ($a = 1$, in table A.33)

pdf	$f(x) = \left(\frac{\lvert p \rvert}{\beta^p}\right) x^{p-1}\, e^{-\left(\frac{x}{\beta}\right)^p}, \quad p > 0, \quad \beta > 0, \quad 0 < x < \infty$
Cumulative distribution function	$1 - e^{-\left(\frac{y}{\beta}\right)^p}$
r^{th} order moment	$\beta^r \Gamma\left(1 + \frac{r}{p}\right)$
Mean	$\frac{\beta}{p} \Gamma\left(\frac{1}{p}\right)$
Mode	$\beta \left(\frac{p-1}{p}\right)^{\frac{1}{p}}$
Variance	$\beta^2 \Gamma\left(1 + \frac{2}{p}\right) - \mu^2$

A.37 Exponential ($a = p = 1$, in table A.33)

pdf	$f(x) = \frac{1}{\beta}\, e^{-\left(\frac{x}{\beta}\right)}, \; \beta > 0, \quad 0 < x < \infty$
Cumulative distribution function	$F(x) = 1 - e^{-(\frac{x}{\beta})}$
r^{th} order moment	$\beta^r r\Gamma(r)$
Mean	β
Mode	0
Variance	β^2

A.38 Chi-Squared ($a = \frac{n}{2}, p = 1, \beta = 2$, in table A.33)

pdf	$f(x) = \frac{1}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, \quad 0 < x < \infty$
r^{th} order moment	$\frac{2^r \Gamma\left(\frac{n}{2} + r\right)}{\Gamma\left(\frac{n}{2}\right)}$
Mean	n
Mode	$n - 2$
Variance	$2n$

A.39 Rayleigh ($a = 1, p = 2, \beta = \alpha\sqrt{2}$, in table A.33)

pdf	$f(x) = \frac{x}{\alpha^2}\, e^{-\left(\frac{x}{\alpha\sqrt{2}}\right)^2}, \quad \alpha > 0, \quad 0 < x < \infty$
Cumulative distribution function	$F(x) = 1 - e^{-\frac{x^2}{2\alpha^2}}$
r^{th} order moment	$(\alpha\sqrt{2})^r\, \Gamma\left(1 + \frac{r}{2}\right)$
Mean	$\alpha\sqrt{\frac{\pi}{2}}$
Mode	α
Variance	$\alpha^2\left(\frac{4-\pi}{2}\right)$

A.40 Maxwell ($a = \frac{3}{2}, p = 2, \beta = \alpha\sqrt{2}$, in table A.33)

pdf	$f(x) = \sqrt{\frac{2}{\pi}}\, \frac{x^2 e^{-\left(\frac{x^2}{2\alpha^2}\right)}}{\alpha^3}, \quad \alpha > 0, \quad 0 < x < \infty$
r^{th} order moment	$\frac{(\alpha\sqrt{2})^r\, \Gamma\left(\frac{3}{2} + \frac{r}{2}\right)}{\frac{1}{2}\sqrt{\pi}}$
Mean	$\frac{2\alpha\sqrt{2}}{\sqrt{\pi}}$
Mode	$\sqrt{2}\alpha$
Variance	$\alpha^2\left(\frac{3\pi - 8}{\pi}\right)$

A.41 Half Standard Normal ($a = \frac{1}{2}, p = 2, \beta = \sqrt{2}$, in table A.33)

pdf	$f(x) = \frac{2}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}, \quad 0 < x < \infty$
r^{th} order moment	$\frac{(\sqrt{2})^r\, \Gamma\left(\frac{1}{2} + \frac{r}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$
Mean	$\sqrt{\frac{2}{\pi}}$
Mode	0
Variance	1

A.42 Half Normal ($\mathbf{a = \frac{1}{2}, p = 2, \beta = \sigma\sqrt{2}}$, in table A.33)

pdf	$f(x) = \sqrt{\frac{2}{\pi}} \frac{e^{-\left(\frac{x^2}{2\sigma^2}\right)}}{\sigma}, \qquad 0 < x < \infty$
r^{th} order moment	$\frac{\left(\sigma\sqrt{2}\right)^r \Gamma\left(\frac{1}{2} + \frac{r}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$
Mean	$\frac{\sigma\sqrt{2}}{\sqrt{\pi}}$
Mode	0
Variance	$\sigma^2 \left(1 - \frac{2}{\pi}\right)$

A.43 Half Log Normal ($\boldsymbol{x = (ln(y) - \mu, a = \frac{1}{2}, \beta = \sigma\sqrt{2}}$ in table A.33)

pdf	$f(x) = \frac{2}{\sigma x \sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}, \qquad 0 < x < \infty$
r^{th} order moment	$e^{r\mu + \frac{1}{2}r^2\sigma^2}$
Mean	$e^{\mu + \frac{1}{2}\sigma^2}$
Mode	$e^{\mu + \sigma^2}$
Variance	$e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right)$

A.44 Four Parameter Generalized Beta Distribution ($\boldsymbol{x = \frac{y-p}{q-p}}$ in table A.1)

pdf	$f(x) = \frac{(x-p)^{a-1}(q-x)^{b-1}}{B(a,b)(q-p)^{a+b-1}}, \qquad a, b, p, q > 0, 0 < x < \infty$
Mean	$\frac{aq + bp}{a + b}$
Mode	$\frac{(a-1)q + (b-1)p}{a + b - 2}$
Variance	$\frac{ab(q-p)^2}{(a+b)^2(a+b+1)}$

A.45 Five Parameter Generalized Beta Distribution ($x = \frac{y^p}{q^p+cy^p}$ in table A.1)

pdf	$f(x) = \dfrac{px^{ap-1}\left(1-(1-c)\left(\frac{x}{q}\right)^p\right)^{b-1}}{q^{ap}B(a,b)\left(1+c\left(\frac{x}{q}\right)^p\right)^{a+b}}, \qquad a,b,p,q > 0, 0 < c < 1$
r^{th} order moment	$\dfrac{q^r B(a+\frac{r}{p},b)}{B(a,b)}\ {}_2F_1\begin{bmatrix} a+\frac{r}{p}, & \frac{r}{p}; & c \\ a+b+\frac{r}{p} & & \end{bmatrix}$
Mean	$\dfrac{qB(a+\frac{1}{p},b)}{B(a,b)}\ {}_2F_1\begin{bmatrix} a+\frac{1}{p}, & \frac{1}{p}; & c \\ a+b+\frac{1}{p} & & \end{bmatrix}$
Mode	$\dfrac{(a-1)q+(b-1)p}{a+b-2}$
Variance	$\dfrac{q^2 B(a+\frac{2}{p},b)}{(1-c)^{a+\frac{2}{p}}B(a,b)}\ {}_2F_1\begin{bmatrix} a+\frac{2}{p}, & \frac{2}{p}; & c \\ a+b+\frac{2}{p} & & \end{bmatrix} - \left(\dfrac{qB(a+\frac{1}{p},b)}{(1-c)^{a+\frac{1}{p}}B(a,b)}\ {}_2F_1\begin{bmatrix} a+\frac{1}{p}, & \frac{1}{p}; & c \\ a+b+\frac{1}{p} & & \end{bmatrix}\right)^2$

A.46 Five Parameter Weighted Generalized Beta Distribution ($a = a + \frac{k}{p}$, $b = b - \frac{k}{q}$ in table A.18)

pdf	$f(x) = \dfrac{px^{ap+k-1}}{q^{ap+k} B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)\left(1+\left(\frac{x}{q}\right)^p\right)^{a+b}}, \quad a,b,p,q > 0,$ $0 < x < \infty, \quad -ap < k < bp$
r^{th} order moment	$\dfrac{q^r B\left(a+\frac{k}{p}+\frac{r}{p}, b-\frac{k}{p}-\frac{r}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)}$
Mean	$\dfrac{qB\left(a+\frac{k}{p}+\frac{1}{p}, b-\frac{k}{p}-\frac{1}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)}$
Variance	$q^r\left[\dfrac{B\left(a+\frac{k+2}{p}, b-\frac{k+2}{p}\right)}{B\left(a+\frac{k}{p}, b-\frac{k}{p}\right)} - \left(\dfrac{\left(a+\frac{k+1}{p}, b-\frac{k-1}{p}\right)}{\left(a+\frac{k}{p}, b-\frac{k}{p}\right)}\right)^2\right]$

A.47 Weighted Beta of the Second Kind ($p = q = 1$ in table A.46)

pdf	$f(x) = \dfrac{x^{a+k-1}}{q^{a+k} B(a+k, b-k)(1+x)^{a+b}}, \quad a,b,> 0,$ $0 < x < \infty, \quad -a < k < b$

A.48 Weighted Singh-Maddala (Burr12) ($a = 1$ in table A.46)

Pdf	$f(x) = \dfrac{px^{p+k-1}}{q^{p+k} B\left(1+\frac{k}{p}, b-\frac{k}{p}\right)\left(1+\left(\frac{x}{q}\right)^p\right)^{1+b}}, \quad b,p,q > 0,$ $0 < x < \infty, \quad -p < k < bp$

A.49 Weighted Dagum (Burr3) ($b = 1$ in table A.46)

Pdf	$f(x) = \dfrac{px^{ap+k-1}}{q^{ap+k} B\left(a+\frac{k}{p}, 1-\frac{k}{p}\right)\left(1+\left(\frac{x}{q}\right)^p\right)^{a+1}}, \quad a,p,q > 0,$ $0 < x < \infty, \quad -ap < k < p$

A.50 Weighted Generalized Fisk (Log-Logistic) ($a = b = 1,\ q = \frac{1}{\lambda}$ in table A.46)

Pdf	$f(x) = \frac{px^{p+k-1}}{\left(\frac{1}{\lambda}\right)^{p+k} B\left(1+\frac{k}{p}, 1-\frac{k}{p}\right)(1+(\lambda x)^p)^2}, \quad p, q, \lambda > 0,$ $0 < x < \infty, \quad -p < k < p$

A.51 Five Parameter Exponential Generalized Beta Distribution ($x = e^y,\ p = \frac{1}{\sigma}$ and $q = e^\delta$ in table A.18)

Pdf	$f(x) = \frac{e^{\frac{a(x-\delta)}{\sigma}}\left(1-(1-c)e^{\frac{(x-\delta)}{\sigma}}\right)^{b-1}}{\sigma B(a,b)\left(1+c\left(\frac{e^x}{e^\delta}\right)^{\frac{1}{\sigma}}\right)^{a+b}}, \quad a, b, c, \sigma, \delta > 0,$ $-\infty < \frac{x-\delta}{\sigma} < \ln\left(\frac{1}{1-c}\right)$

A.52 Exponential Generalized Beta Distribution of the First Kind ($c = 0$ in table A.51)

Pdf	$f(x) = \frac{e^{\frac{a(x-\delta)}{\sigma}}\left(1-e^{\frac{(x-\delta)}{\sigma}}\right)^{b-1}}{\sigma B(a,b)}, \quad a, b, \sigma, \delta > 0,$ $-\infty < \frac{x-\delta}{\sigma} < 0$

A.53 Exponential Generalized Beta Distribution of the Second Kind ($c = 1$ in table A.51)

Pdf	$f(x) = \frac{e^{\frac{a(x-\delta)}{\sigma}}}{\sigma B(a,b)\left(1+e^{\frac{x-\delta}{\sigma}}\right)^{a+b}}, \quad a, b, \sigma, \delta > 0,$ $-\infty < \frac{x-\delta}{\sigma} < \infty$

A.54 Exponential Generalized Gamma (Generalized Gompertz) ($b \to \infty$ and $\delta^* = \sigma lnq + \delta$ in table A.51)

Pdf	$f(x) = \frac{e^{\frac{a(x-\delta)}{\sigma}} e^{-e(x-\delta)/\sigma}}{\sigma\Gamma\Xi p}, \quad a, \sigma, \delta > 0,$ $-\infty < \frac{x-\delta}{\sigma} < \infty$

A.55 Generalized Gumbell ($c = 1$ and $a = b$ in table A.51)

Pdf	$f(x) = \frac{e^{a(x-\delta)/\sigma}}{\sigma B(a,a)\left(1 + e^{\frac{x-\delta}{\sigma}}\right)^{2a}}, \quad a, \sigma, \delta > 0,$ $-\infty < \frac{x-\delta}{\sigma} < \infty$

A.56 Exponential Power ($c = 0$ and $b = 1$ in table A.51)

Pdf	$f(x) = \frac{ae^{a(x-\delta)/\sigma}}{\sigma}, \quad a, \sigma, \delta > 0,$ $-\infty < \frac{x-\delta}{\sigma} < 0$

A.57 Exponential (Two Parameter) ($c = 0, a = 1$ and $b = 1$ in table A.51)

Pdf	$f(x) = \frac{e^{(x-\delta)/\sigma}}{\sigma}, \quad \sigma, \delta > 0,$ $-\infty < \frac{x-\delta}{\sigma} < 0$

A.58 Exponential (One Parameter) ($c = 0, a = 1, b = 1$ and $\delta = 0$ in table A.51)

Pdf	$$f(x) = \frac{e^{x/\sigma}}{\sigma}, \quad \sigma > 0,$$ $-\infty < \frac{x}{\sigma} < 0$

A.59 Exponential Fisk (Logistic) ($c = 1, a = 1, b = 1$ in table A.51)

Pdf	$$f(x) = \frac{e^{(x-\delta)/\sigma}}{\sigma(1 + e^{(x-\delta)/\sigma})^2}, \quad \sigma > 0,$$ $-\infty < \frac{x-\delta}{\sigma} < \infty$

A.60 Standard Logistic ($c = 1, a = 1, b = 1, \delta = 0$ in table A.51)

Pdf	$$f(x) = \frac{e^{x/\sigma}}{\sigma(1 + e^{x/\sigma})^2}, \quad \sigma > 0,$$ $-\infty < \frac{x}{\sigma} < \infty$

A.61 Exponential Weibull ($b \to \infty, \delta^* = \sigma lnq + \delta$ and $a = 1$ in table A.51)

Pdf	$$f(x) = \frac{e^{\frac{(x-\delta)}{\sigma}} e^{-e(x-\delta)/\sigma}}{\sigma \Gamma \Xi p}, \quad \sigma, \delta > 0,$$ $-\infty < \frac{x-\delta}{\sigma} < \infty$

Other Generalized Beta Distributions

A.62 Beta Generator ($x = G(y)$ in table A.1)

Pdf	$$f(x) = \frac{g(x)\big(G(x)\big)^{a-1}\big(1 - G(x)\big)^{b-1}}{B(a,b)}, \; a > 0, b > 0,$$ $-\infty < x < \infty, 0 < G(x) < 1$

A.63 i^th Order Statistics (*a=i, b=n-i+1* in table A.62)

Pdf	$f(x) = \frac{n!}{(i-1)!\,(n-i)!} g(x)[G(x)]^{i-1}[1-G(x)]^{n-i}, \quad i > 0,$ $n - i > 0, -\infty < x < \infty, 0 < G(x) < 1$

A.64 Beta-Normal (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$f(x) = \frac{\sigma^{-1}\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{a-1}\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{b-1}}{B(a,b)}\phi\left(\frac{(x-\mu)}{\sigma}\right)$ $a > 0, \quad b > 0, \quad \sigma > 0, \ \mu \in \mathbb{R}\ and\ x \in \mathbb{R}$
Parent cdf	$G(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
Parent pdf	$g(x) = \phi\left(\frac{x-\mu}{\sigma}\right)$

A.65 Beta-Exponential (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$f(x) = \frac{\lambda\exp(-b\lambda x)[1-\exp(-\lambda x)]^{a-1}}{B(a,b)}$ $a > 0, b > 0, \lambda > 0, x > 0$
Parent cdf	$G(x) = 1 - \exp(-\lambda x)$
Parent pdf	$g(x) = \lambda\exp(-\lambda x)$

A.66 Beta-Weibull (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$f(x) = \frac{\left(\frac{\lvert c\rvert}{\beta^c}\right)x^{c-1}\, e^{-b\left(\frac{x}{\beta}\right)^c}\left[1 - e^{-\left(\frac{x}{\beta}\right)^c}\right]^{a-1}}{B(a,b)}, \quad a, b, c, \beta > 0$
Parent cdf	$G(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^c}$
Parent pdf	$g(x) = \left(\frac{\lvert c\rvert}{\beta^c}\right)x^{c-1}e^{-\left(\frac{x}{\beta}\right)^c}$

A.67 Beta-Hyperbolic Secant (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$f(x) = \frac{\left[\frac{2}{\pi}\arctan(e^x)\right]^{a-1}\left[1-\frac{2}{\pi}\arctan(e^x)\right]^{b-1}}{B(a,b)\pi\cosh(x)}$ $a > 0, b > 0$ and $x \in \mathbb{R}$
Parent cdf	$G(x) = \frac{2}{\pi}\arctan(e^x)$
Parent pdf	$g(x) = \frac{1}{\pi\cosh(x)}$

A.68 Beta-Gamma (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$f(x) = \frac{x^{\rho-1}e^{-\frac{x}{\lambda}}\Gamma_{\frac{x}{\lambda}}(\rho)^{a-1}\left[1-\frac{\Gamma_{\frac{x}{\lambda}}(\rho)}{\Gamma(\rho)}\right]^{b-1}}{B(a,b)\Gamma(\rho)^{a}\lambda^{\rho}}$ $a, b, \rho, \lambda, x > 0$
Parent cdf	$G(x) = \frac{\Gamma_{\frac{x}{\lambda}}(\rho)}{\Gamma(\rho)}$
Parent pdf	$g(x) = \left(\frac{x}{\lambda}\right)^{\rho-1} e^{-\frac{x}{\lambda}}$

A.69 Beta-Gumbel (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$f(x) = \frac{\exp\left\{-\frac{x-\mu}{\sigma}\right\}}{\sigma B(a,b)} e^{-a\left(\exp\left\{-\frac{x-\mu}{\sigma}\right\}\right)}\left(1-e^{-\exp\left\{-\frac{x-\mu}{\sigma}\right\}}\right)^{b-1}$ $a, b, u, \sigma, x > 0$
Parent cdf	$G(x) = \exp\left\{-\exp\left\{-\frac{x-\mu}{\sigma}\right\}\right\}$
Parent pdf	$g(x) = \frac{\exp\left\{-\frac{x-\mu}{\sigma}\right\}}{\sigma}\exp\left\{-\left(\exp\left\{-\frac{x-\mu}{\sigma}\right\}\right)\right\}$

A.70 Beta-Frèchet (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$f(x) = \frac{\lambda\sigma^{\lambda} \exp\left\{-a\left(\frac{x}{\sigma}\right)^{-\lambda}\right\}\left(1-\exp\left\{-\left(\frac{x}{\sigma}\right)^{-\lambda}\right\}\right)^{b-1}}{x^{1+\lambda}B(a,b)}$ $a, b, \lambda, \sigma, x > 0$
Parent cdf	$G(x) = \exp\left\{-\left(\frac{x}{\sigma}\right)^{-\lambda}\right\}$
Parent pdf	$g(x) = \frac{\lambda\sigma^{\lambda}}{x^{1+\lambda}} \exp\left\{-\left(\frac{x}{\sigma}\right)^{-\lambda}\right\}$

A.71 Beta-Maxwell (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$f(x) = \frac{1}{B(a,b)}\left(\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{x^2}{2\alpha}\right)\right)^{a-1}\left(1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{x^2}{2\alpha}\right)\right)^{b-1}\sqrt{\frac{2}{\pi}}\frac{x^2e^{-\left(\frac{x^2}{2\alpha^2}\right)}}{\alpha^3}$ $a, b, \alpha, x > 0$ *and* $\gamma(a, b)$ is an incomplete gamma function
Parent cdf	$G(x) = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{x^2}{2\alpha}\right)$
Parent pdf	$g(x) = \sqrt{\frac{2}{\pi}}\frac{x^2e^{-\left(\frac{x^2}{2\alpha^2}\right)}}{\alpha^3}$

A.72 Beta-Pareto (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$f(x) = \frac{k}{\theta B(a,b)}\left(1-\left(\frac{x}{\theta}\right)^{-k}\right)^{a-1}\left(\frac{x}{\theta}\right)^{-kb-1}$ $a, b, k, \theta, x > 0$
Parent cdf	$G(x) = 1-\left(\frac{x}{\theta}\right)^{-k}$
Parent pdf	$g(x) = \frac{k\theta^k}{x^{k+1}}$

A.73 Beta-Rayleigh (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$$f(x) = \frac{x}{\alpha^2 B(a,b)} \left(1 - e^{-\frac{x^2}{2\alpha^2}}\right)^{a-1} e^{-b\left(\frac{x}{\alpha\sqrt{2}}\right)^2}$$ $a, b, k, \alpha > 0$
Parent cdf	$$G(x) = 1 - e^{-\frac{x^2}{2\alpha^2}}$$
Parent pdf	$$g(x) = \frac{x}{\alpha^2} e^{-\left(\frac{x}{\alpha\sqrt{2}}\right)^2}$$

A.74 Beta-Generalized-Logistic of Type IV (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$$f(x) = \frac{B(a,b)^{1-a-b}}{B(a,b)} \frac{e^{qx}}{(1 - e^{-x})^{p+q}} [B_{\frac{1}{1+e^{-x}}}(p,q)]^{a-1} [B_{\frac{e^{-x}}{1+e^{-x}}}(q,p)]^{b-1}$$ $a, b, p, q > 0, \quad x > R$
Parent cdf	$$G(x) = \frac{B_{\frac{1}{1+e^{-x}}}(p,q)}{B(p,q)}$$
Parent pdf	$$g(x) = \frac{e^{qx}}{(1 + e^{-x})^{p+q}}$$

A.75 Beta-Generalized-Logistic of Type I (Substituting *q=1* in table A.74)

Pdf	$$f(x) = \frac{pe^{-x}[(1 + e^{-ax})^p - 1]^{b-1}}{B(a,b)(1 + e^{-x})^{a+pb}}$$ $a, b, p > 0, \quad x > R$

A.76 Beta-Generalized-Logistic of Type II (Substituting *q=1* in table A.74)

Pdf	$$f(x) = \frac{qe^{-bqx}}{B(a,b)(1 + e^{-x})^{qb+1}} \left[\left(1 - \frac{e^{-qx}}{B(a,b)(1 + e^{-x})^q}\right)^p\right]$$ $a, b, q > 0, \quad x > R$

A.77 Beta-Generalized-Logistic of Type III (Substituting *q=1* in table A.74)

Pdf	$$f(x) = \frac{B(q,q)^{1-a-b}}{B(a,b)} \frac{e^{-qx}}{(1 + e^{-x})^{2q}} [B_{\frac{1}{1+e^{-x}}}(q,q)]^{a-1} [B_{\frac{e^{-x}}{1+e^{-x}}}(q,q)]^{b-1}$$ $a, b, q > 0, \quad x > R$

A.78 Beta-Beta Prime ($x = e^{-y}$ in table A. 74)

Pdf	$f(x) = \frac{B(p,q)^{1-a-b}}{B(a,b)} \frac{x^{q-1}}{(1+x)^{p+q}} [B_{\frac{x}{1+x}}(q,p)]^{a-1}[B_{\frac{1}{1+x}}(p,q)]^{b-1}$ $a, b, q, > 0, \quad x > R$

A.79 Beta-F (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$f(x) = \frac{\mathrm{B(a,b)^{-1}}}{\mathrm{B}\left(\frac{u}{2},\frac{v}{2}\right)} \frac{\left(\frac{v}{u}\right)^{\frac{v}{2}} x^{\frac{v}{2}-1}}{\left(1+\left(\frac{v}{u}\right)x\right)^{(u+v)/2}} \left[I_{\frac{\left(\frac{v}{u}\right)x}{1+\left(\frac{v}{u}\right)x}}\left(\frac{v}{2},\frac{u}{2}\right)\right]^{a-1} \left[I_{\frac{1}{1+\left(\frac{v}{u}\right)x}}\left(\frac{v}{2},\frac{u}{2}\right)\right]^{b-1}$ $a, b, \mathrm{u}, \mathrm{v}, \mathrm{x} > 0$
Parent cdf	$G(x) = \frac{1}{\mathrm{B}\left(\frac{u}{2},\frac{v}{2}\right)} \mathrm{B}_{\frac{\left(\frac{v}{u}\right)x}{1+\left(\frac{v}{u}\right)x}}\left(\frac{v}{2},\frac{u}{2}\right)$
Parent pdf	$g(x) = \frac{\left(\frac{v}{u}\right)^{\frac{q}{2}}}{\mathrm{B}\left(\frac{u}{2},\frac{v}{2}\right)} \frac{x^{\frac{v}{2}-1}}{\left(1+\left(\frac{v}{u}\right)x\right)^{(u+v)/2}}$

A.80 Beta-Burr XII (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$f(x) = \frac{pkq^{-p}x^{p-1}}{B(a,b)} \left(1-\left(1+\left(\frac{x}{q}\right)^{p}\right)^{-k}\right)^{a-1} \left(1+\left(\frac{x}{q}\right)^{p}\right)^{-kb-1}$ $a, b, p, q, k, x > 0.$
Parent cdf	$G(x) = 1-\left(1+\left(\frac{x}{q}\right)^{p}\right)^{-k}$
Parent pdf	$g(x) = pkq^{-p}\left(1+\left(\frac{x}{q}\right)^{p}\right)^{-k-1} x^{p-1}$

A.81 Beta-Dagum (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$f(x) = \frac{1}{B(a,b)}((1+qx^{-p})^{-a})^{a-1}(1-(1+qx^{-p})^{-a})^{b-1} \frac{\lvert p\rvert qax^{-p-1}}{(1+qx^{-p})^{1+a}}$ $a, b, p, q, x > 0$
Parent cdf	$G(x) = (1+qx^{-p})^{-a}$
Parent pdf	$g(x) = \frac{\lvert p\rvert qax^{-p-1}}{(1+qx^{-p})^{1+a}}$

A.82 Beta-Fisk (Log-Logistic) (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$f(x) = \frac{\lambda p(\lambda x)^{ap-1}}{B(a,b)(1+(\lambda x)^{p})^{a+b}}$ $a, b, p, \lambda, x > 0$
Parent cdf	$G(x) = \frac{(\lambda x)^{p}}{1+(\lambda x)^{p}}$
Parent pdf	$g(x) = \frac{\lambda p(\lambda x)^{p-1}}{(1+(\lambda x)^{p})^{2}}$

A.83 Beta-Generalized Half Normal (Substituting *G(X)* and *g(X)* in table A.62)

Pdf	$f(x) = \frac{\left(2\phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right]-1\right)^{a-1} \left(2-2\phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right]\right)^{b-1} \sqrt{\frac{2}{\pi}}\left(\frac{\alpha}{x}\right)\left(\frac{x}{\sigma}\right)^{\alpha} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^{2\alpha}}}{B(a,b)}$ $a, b, \alpha, \sigma > 0$
Parent cdf	$G(x) = 2\phi\left[\left(\frac{x}{\sigma}\right)^{\alpha}\right]-1$
Parent pdf	$g(x) = \sqrt{\frac{2}{\pi}}\left(\frac{\alpha}{x}\right)\left(\frac{x}{\sigma}\right)^{\alpha} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^{2\alpha}}$

A.84 Generalized Beta Generator ($\mathbf{x = [G(y)]^{\alpha}}$ in table A.1)

Pdf	$f(x) = \frac{g(x)\alpha[G(x)]^{\alpha-1}([G(x)]^{\alpha})^{a-1}(1-[G(x)]^{\alpha})^{b-1}}{B(a,b)}, \; a > 0,$ $\alpha > 0, b > 0,$ $-\infty < x < \infty, 0 < [G(x)]^{\alpha} < 1$

A.85 Exponentiated Exponential (Substituting *G(X)* and *g(X)* in table A.84)

Pdf	$f(x) = \frac{\left(1-e^{-\left(\frac{x}{\beta}\right)}\right)^{\alpha}\left[\left(1-e^{-\left(\frac{x}{\beta}\right)}\right)^{\alpha}\right]^{a-1}\left[1-\left(1-e^{-\left(\frac{x}{\beta}\right)}\right)^{\alpha}\right]^{b-1}}{B(a,b)}$
Parent cdf	$G(x) = \left(1-e^{-\left(\frac{x}{\beta}\right)}\right)^{\alpha}$
Parent pdf	$g(x) = \frac{\alpha}{\beta}\left(1-e^{-\left(\frac{x}{\beta}\right)}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)}$

A.86 Exponentiated Weibull (Substituting *G(X)* and *g(X)* in table A.84)

Pdf	$$f(x)=\frac{\left(\frac{p}{\beta^{p}}\right)y^{p-1}e^{-\left(\frac{y}{\beta}\right)^{p}}[1-\ e^{-\left(\frac{y}{\beta}\right)^{p}}]^{a-1}\left[1-\left(1-\ e^{-\left(\frac{y}{\beta}\right)^{p}}\right)\right]^{b-1}}{B(a,b)}$$
Parent cdf	$$G(x)=1-\ e^{-\left(\frac{y}{\beta}\right)^{p}}$$
Parent pdf	$$g(x)=\left(\frac{p}{\beta^{p}}\right)y^{p-1}e^{-\left(\frac{y}{\beta}\right)^{p}}$$

A.87 Exponentiated Pareto (Substituting *G(X)* and *g(X)* in table A.84)

Pdf	$$f(x)=\frac{p\mathrm{q}^{\mathrm{p}}by^{-p-1}\left(1-\left(\frac{y}{q}\right)^{-p}\right)^{b-1}[\left(1-\left(\frac{y}{q}\right)^{-p}\right)^{b}]^{a-1}\left[1-\left(1-\left(\frac{y}{q}\right)^{-p}\right)^{b}\right]^{b-1}}{B(a,b)}$$
Parent cdf	$$G(x)=\left(1-\left(\frac{y}{q}\right)^{-p}\right)^{b}$$
Parent pdf	$$g(x)=\ p\mathrm{q}^{\mathrm{p}}by^{-p-1}\left(1-\left(\frac{y}{q}\right)^{-\mathrm{p}}\right)^{b-1}$$

A.88 Generalized Beta Exponential (Substituting *G(X)* and *g(X)* in table A.84)

Pdf	$$f(x)=\frac{\lvert p\rvert\lambda\exp(-\lambda\mathrm{x})[1-\exp(-\lambda\mathrm{x})\ ]^{pa-1}[1-(1-\exp(-\lambda\mathrm{x}))^{p}]^{b-1}}{B(a,b)}$$
Parent cdf	$$G(x)=1-\exp(-\lambda\mathrm{x})$$
Parent pdf	$$g(x)=\ \lambda\exp(-\lambda\mathrm{x})$$

A.89 Generalized Beta Weibull (Substituting *G(X)* and *g(X)* in table A.84)

Pdf	$$f(x)=\frac{\lvert p\rvert\left(\frac{\lvert c\rvert}{\beta^{c}}\right)x^{c-1}\ e^{-\left(\frac{x}{\beta}\right)^{c}}[1-\mathrm{e}^{-\left(\frac{\mathrm{x}}{\beta}\right)^{\mathrm{c}}}]^{pa-1}\left[1-\left(1-\ \mathrm{e}^{-\left(\frac{\mathrm{x}}{\beta}\right)^{\mathrm{c}}}\right)^{p}\right]^{b-1}}{B(a,b)}$$
Parent cdf	$$G(x)=1-\ \mathrm{e}^{-\left(\frac{\mathrm{x}}{\beta}\right)^{\mathrm{c}}}$$
Parent pdf	$$g(x)=\ \left(\frac{\lvert c\rvert}{\beta^{c}}\right)x^{c-1}\ e^{-\left(\frac{x}{\beta}\right)^{c}}$$

A.90 Generalized Beta Hyperbolic Secant (Substituting *G(X)* and *g(X)* in table A.84)

Pdf	$f(x) = \dfrac{p\left[\frac{2}{\pi}\arctan\left(e^{\frac{x-\mu}{\sigma}}\right)\right]^{ap-1}\left[1-\left(\frac{2}{\pi}\arctan\left(e^{\frac{x-\mu}{\sigma}}\right)\right)^{p}\right]^{b-1}}{B(a,b)\sigma\pi\cosh\left(\frac{x-\mu}{\sigma}\right)}$
Parent cdf	$G(x) = \frac{2}{\pi}\arctan\left(e^{\frac{x-\mu}{\sigma}}\right)$
Parent pdf	$g(x) = \dfrac{1}{\sigma\pi\cosh\left(\frac{x-\mu}{\sigma}\right)}$

A.91 Generalized Beta Normal (Substituting *G(X)* and *g(X)* in table A.84)

Pdf	$f(x) = \dfrac{p\sigma^{-1}[\Phi\left(\frac{x-\mu}{\sigma}\right)]^{ap-1}\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)^{p}\right]^{b-1}}{B(a,b)}\phi\left(\frac{(x-\mu)}{\sigma}\right)$
Parent cdf	$G(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
Parent pdf	$g(x) = \phi\left(\frac{(x-\mu)}{\sigma}\right)$

A.92 Generalized Beta Log Normal (Substituting *G(X)* and *g(X)* in table A.84)

Pdf	$f(x) = \dfrac{p\phi(\log x)[\Phi(\log x)]^{ap-1}[1-\Phi(\log x)^{p}]^{b-1}}{xB(a,b)}$
Parent cdf	$G(x) = \Phi(\log x)$
Parent pdf	$g(x) = \dfrac{\phi(\log x)}{x}$

A.93 Generalized Beta Gamma (Substituting *G(X)* and *g(X)* in table A.84)

Pdf	$f(x) = \dfrac{\left(\frac{x}{\lambda}\right)^{\rho-1} e^{-\frac{x}{\lambda}}\left[\frac{\Gamma_{\frac{x}{\lambda}}(\rho)}{\Gamma(\rho)}\right]^{ap-1}\left[1-\left(\frac{\Gamma_{\frac{x}{\lambda}}(\rho)}{\Gamma(\rho)}\right)^{p}\right]^{b-1}}{B(a,b)}$
Parent cdf	$G(x) = \left(\frac{\Gamma_{\frac{x}{\lambda}}(\rho)}{\Gamma(\rho)}\right)$
Parent pdf	$g(x) = \left(\frac{x}{\lambda}\right)^{\rho-1} e^{-\frac{x}{\lambda}}$

A.94 Generalized Beta Frèchet (Substituting *G(X)* and *g(X)* in table A.84)

Pdf	$f(x) = \dfrac{\frac{p\lambda}{\sigma}\exp\left\{-\left(\frac{x}{\sigma}\right)^{-\lambda}\right\}x^{\lambda-1}\left[\exp\left\{-\left(\frac{x}{\sigma}\right)^{-\lambda}\right\}\right]^{ap-1}\left[1-\left(\exp\left\{-\left(\frac{x}{\sigma}\right)^{-\lambda}\right\}\right)^{p}\right]^{b-1}}{B(a,b)}$
Parent cdf	$G(x) = \left(\exp\left\{-\left(\frac{x}{\sigma}\right)^{-\lambda}\right\}\right)$
Parent pdf	$g(x) = \frac{\lambda}{\sigma}\exp\left\{-\left(\frac{x}{\sigma}\right)^{-\lambda}\right\}x^{\lambda-1}$

A.95 Confluent Hypergeometric

Pdf	$f(x) = \frac{1}{B(a,c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}e^{xu}du$

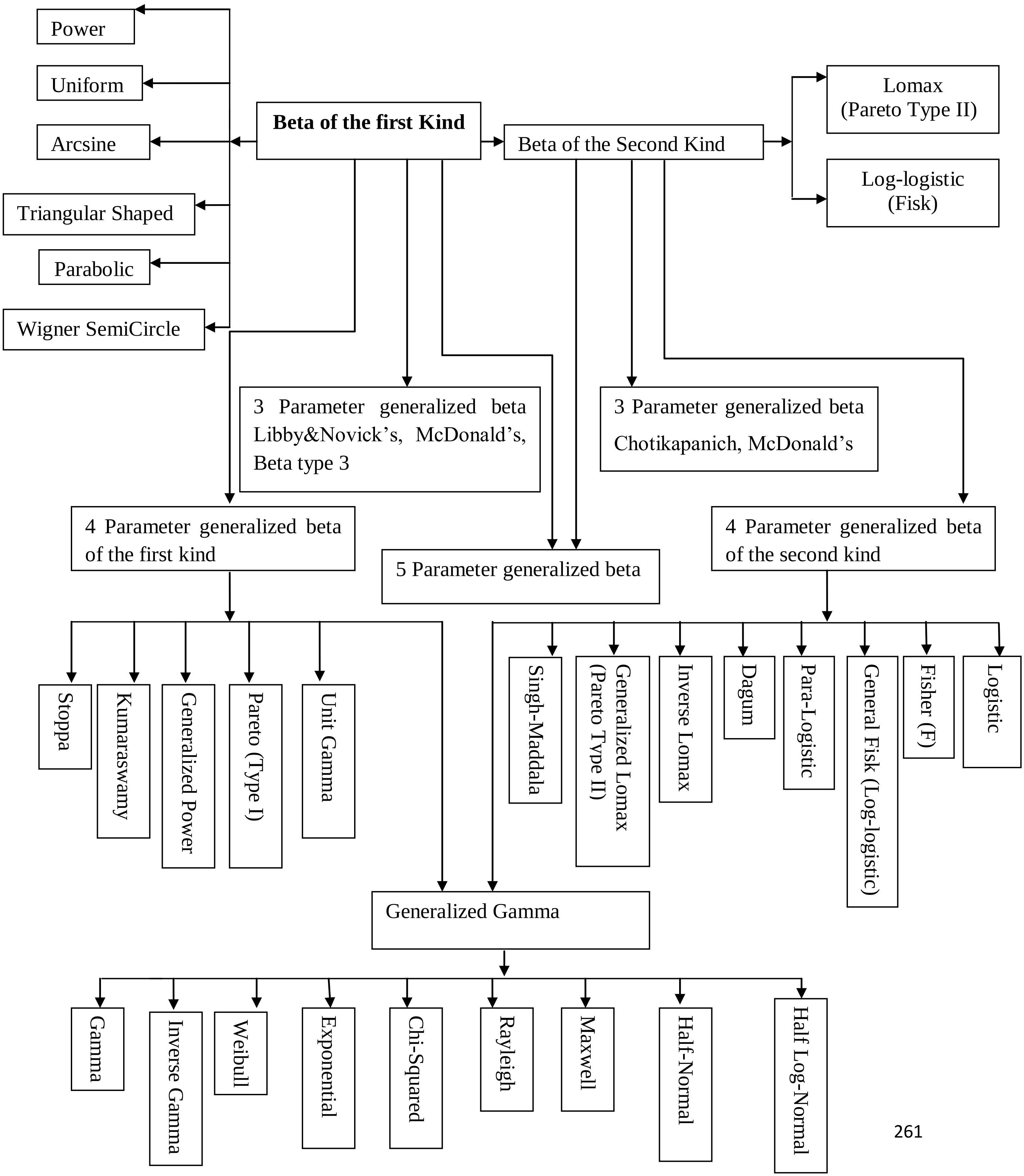
A.96 Gauss Hypergeometric

Pdf	$f(x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}du$

A.97 Appell

Pdf	$f(x) = \frac{Cx^{\alpha-1}(1-x)^{\beta-1}}{(1-ux)^{\rho}(1-vx)^{\lambda}},$ $0 < x < 1,\ \alpha > 0,\ \beta > 0,\ \rho > 0,\ \lambda > 0,\ -1 < u < 1,\ -1 < v < 1$ and C is the normalizing constant given by $\frac{1}{C} = B(\alpha,\beta)F_1(\alpha,\rho,\lambda,\alpha+\beta;u,v)$

SUMMARY: FRAMEWORK FOR CLASSICAL TO GENERALIZED BETA DISTRIBUTIONS



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