ON APARCH LÉVY FILTER OPTION PRICING FORMULA FOR DEVELOPED AND EMERGING MARKETS

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Declaration and Approval

Declaration

I, the undersigned, declare that this thesis contains my own work. To the best of my knowledge, no portion of this work has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

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Dedication

To

My parents, *My wife Lynette*, and *My son Gift*.

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Abstract

Popular models such as Black-Scholes-Merton (BSM73) lack most of empirically found stylized features of financial data, such as volatility clustering, leptokurtic nature of log returns, joint covariance structure and aggregational Gaussianity, hence it may not consistently price all the European and exotic options that are quoted in one specific market. Such simplifying assumptions in real financial markets, translate to the implied volatility curves typically skewed, with smiley shapes or even more complex structures.

In this study, we model components of return distribution, assumed to be directed by a latent news process, in developed and emerging economies. We endeavor to identify stochastic processes that govern equity market indices as the underlying process, bearing in mind arithmetic Brownian motion of Louis Bachelier, geometric Brownian motion of Samuelson, exponential Lévy, binomial trees, pentanomial lattices, Duan GARCH model and AR-PARCH-Levy model as some of the formulas which could be implemented for use universally.

Daily log returns assets from both markets (developed and emerging) are found to exhibit positive autocorrelation and changing variances. Possibility of option pricing under linear autoregressive power ARCH type dynamics and conditional leptokurtic residuals of the underlying process is investigated and AR APARCH Lévy filter model developed. Further, the unconditional variance of the AR APARCH Levy model under the local risk neutral valuation relationship equivalent measure is formulated.

Mean mixture distribution, here termed the generalized hyperbolic distribution and some of its subclasses, such as normal inverse Gaussian, hyperbolic and variance gamma distributions, are used to construct stochastic process for a strictly stationary filtered residuals. The need to study these distribution(s) is to accommodate certain frequently observed stylistic facts of daily log returns, ranging from temporal joint dependance structures to the presence of jumps.

While Brownian motion generates a normal innovations, a non-gaussian innovations can be generated by a pure jump Lévy process. To capture changing volatility, we apply ARCH type model. A time changed Lévy process as used in variance gamma process was used to draw comparison in both economies. Developed markets were observed to have a faster business clock, less or no autoregressive mean and pronounced changing correlated volatilities as opposed to emerging markets.

A closed-form option pricing model, APARCH Lévy filter, which nests BSM73 model, minimizes the "consistent volatility smiles" and incorporates most of the stylized features observed in developed and emerging economies is constructed and presented. An extensive empirical analysis based on S&P500 index options and Nairobi stock NSE20 index is used to compare performance of proposed model against BSM73 and GARCH option pricing model of Duan 1995. We explore the application of APARCH Levy filter model to other types of exotic options such as lookback options and arithmetic Asian type options.

Notations and Abbreviations

AA_t	The price of arithmetic average Asian option
GA_t	Geometric average Asian option price
BSM73	Black Scholes and Merton model of 1973
MJD76	Merton-Jump-diffusion model of 1976
Ω	the sample space
$\mathcal{F}$	the sigma algebra of events
$\mathbb{P}$	Physical probability measure i.e. in real world
$\mathbb{Q}$	Risk neutral probability measure
FFT	Fast Fourier Transform
VaR	Value at Risk
MEMM	Minimal Entropy Martingale Measure
i.i.d.	Identically, independently distributed
$\Phi(x)$	standard normal distribution function
$\phi_X(u)$	characteristic function of random variable X
E.M.	Expectation Maximization algorithm
M.C.	Monte Carlo simulation technique
NM	Normal distribution i.e. $N(\mu, \sigma^2)$
LRNVR	Local risk neutral valuation relationship
EWMA	Exponential weighted moving average
ARCH	Autoregressive conditional heteroscedasticity
GJR	Glosten Jaganathan Runkle model
GARCH	Generalized Autoregressive Conditional Heteroscedasticity
APARCH	Asymmetric Power Autoregressive Conditional Heteroscedasticity
IGARCH	Integrated Generalized ARCH
VG	Variance Gamma density function
NIG	Normal Inverse Gaussian density function
HY	Hyperbolic distribution
$N(\mu, \sigma^2)$	normal distribution
OTC	Over the counter
$\mathrm{Var}^{\mathbb{Q}}(X)$	Variance of X under $\mathbb{Q}$
$\mathbb{E}^{\mathbb{P}}(X)$	Expected (mean), expected value of X
$\mathcal{F}_t^B$	natural filtration
$(\mathcal{F}_t^X)_{t\in[0,T]}$	the σ-algebra generated by the past values of the X process
$(\Omega, \mathcal{F}, \mathbb{P})$	Probability space
$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$	A filtered probability space, $(\mathcal{F}_t)_{t\in[0,T]}$, $\mathcal{F}_t \subseteq \mathcal{F}$

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Chapter 1

Introduction

The whole thesis consists of six chapters apart from the introduction and conclusions chapter. Initially, each of these chapters were meant to be separate articles, as a result, there are many repetitions especially definitions, probability density functions and kernel densities which appear across the chapters.

Throughout this thesis we use the term asset to describe any financial object whose value is known at present but is liable to change in future. We further focus on shares or equities transacted in an emerging economy and developed stock markets respectively. At times the price of equities fluctuate, exposing the owner to a risk of loosing his or her financial position and vise versa. However such a negative drop in share prices could be minimized if *call options* were in place. Our main objective is to develop a valuation formulae for plain vanilla and exotic options. This will require knowledge of stochastic process such as Brownian motion and Lévy process.

Let Ω be a given set, $\mathcal{F}$ a σ-algebra on Ω, then the pair $(\Omega, \mathcal{F})$ is called a measurable space, and the triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

1.1 Stochastic process

A *stochastic process* is a parameterized collection of random variables $\{X_t\}_{t\in\mathbb{R}}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A stochastic process is said to be continuous in probability (stochastically continuous) if for every $t \geq 0$ and ε :

$$\lim_{s\to t} \mathbb{P}\left(|X_s - X_t| > \varepsilon\right) = 0.$$

1.1.1 Filtration

An increasing family of $\sigma-$fields $(\mathcal{F}_t)_{t\in[0,T]}$: $\forall t \geq s \geq 0, \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ is called a filtration or information flow on $\Omega, \mathcal{F}, \mathbb{P}$. We can interpret $\mathcal{F}_t$ as the information known at time t. $\mathcal{F}_t$ increases as time progresses.

1.1.2 Non-anticipating Process

A stochastic process $(X_t)_{t\in[0,T]}$ is said to be non-anticipating (adapted) with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ or $\mathcal{F}_t-$adapted if the value of X_t is revealed at time t for each $t \in [0,T]$.

1.1.3 Martingales

Consider a trend of a time series of a stochastic process. A stochastic process is said to be a martingale if its time series have no trend. A process with increasing trend is called a submartingale and a process with decreasing trend is called a supermartingale.

Definition 1.1.1. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration (information flow) $\mathcal{F}_t$. A cadlag process $(X_t)_{t\in[0,T]}$ is said to be a martingale with respect to its filtration $\mathcal{F}_t$ and the probability measure $\mathbb{P}$ if X is nonanticipating (adapted to $\mathcal{F}_t$), $\mathbb{E}[|X_t|]$ is finite for any $t \in [0,T]$ and

$$\forall s > t, \mathbb{E}[X_s|\mathcal{F}_t] = X_t.$$

The best prediction of a martingale's future value is its present value. The definition of martingale makes sense only when the underlying probability measure $\mathbb{P}$ and the information flow $(\mathcal{F}_t)_{t\in[0,T]}$ have been specified. Note that the stock price discounted by the risk free interest rate $e^{-r\Delta}S_{t+\Delta}$ is not a martingale under $\mathbb{P}$. But interestingly, non-martingales can be converted to martingales by changing the probability measure.

1.2 Brownian motion

Definition 1.2.1 (Brownian motion). A stochastic process $B = \{B_t, t \geq 0\}$ is a standard Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if

i) $B_0 = 0$

ii) B has independent and stationary increments

iii) $B_{t+s} - B_t$ is normally distributed with mean 0 and variance s.

Brownian motion is the basic example of a Lévy process, moreover we will always work with natural filtration $\mathcal{F}_t^B = \{\mathcal{F}_t, 0 \leq t \leq T\}$ of B. To this end, Brownian motion is adapted with respect to this filtration and that the increments $B_{t+s} - B_t$ are independent of $\mathcal{F}_t$

1.2.1 Lévy process

The term "Lévy process" honors the work of the French mathematician Paul Lévy who, although not alone in his contribution, played an instrumental role in bringing together an understanding and characterization of process with independent increments.

Lévy processes are defined as stochastically continuous processes with stationary and independent increments and can be viewed as analogues of random walks in continuous time. We state key notions and elementary properties of Lévy processes.

Definition 1.2.2. A function $f : [0, T] \to \mathbb{R}$ is said to be cadlag if it is right-continuous with left limits: for each $t \in [0, T]$ the

$$f(t-) = \lim_{s \to t, s < t} f(s) \qquad\qquad f(t+) = \lim_{s \to t, s > t} f(s)$$

exist and $f(t) = f(t+)$.

Remark 1.2.1. Any continuous function is cadlag but cadlag function can have discontinuities. A cadlag function f can have at most a countable number of discontinuities : $\{t \in [0, T], f(t) \neq f(t-)\}$ is finite or countable.

Definition 1.2.3. A cadlag stochastic process $(L_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}$ such that $L_0 = 0$ is called a Lévy process if it possesses the following properties.

i) independent increments: for every increasing sequence of times $t_0, ..., t_n$ the random variable $L_{t_0}, L_{t_1} - L_{t_0}, ..., L_{t_n} - L_{t_{n-1}}$ are independent,

ii) stationary increments: the law of $L_{t+h} - L_t$ does not depend on t.

iii) Stochastic continuity: $\forall \epsilon > 0, \lim_{h \to 0} \mathbb{P}(|L_{t+h} - L_t| \geq \epsilon) = 0$

In particular, Lévy processes include many important processes as special cases, e.g. Brownian motion, the Poisson process, stable and self-decomposable processes and subordinators. Therefore Lévy processes provide powerful modeling tools which are applied in various fields like econometrics, finance, telecommunications and physics. For more general treatment of Lévy processes see Sato (1999).

Definition 1.2.4. A stochastic process is said to be self-similar if there exists $H > 0$ such that for any scaling factor $c > 0$, the process $(X_{ct})_{t \geq 0}$ and $(c^H X_t)_{t \geq 0}$ have the same law

$$(X_{ct})_{t \geq 0} \stackrel{D}{=} (c^H X_t)_{t \geq 0} \tag{1.2.1}$$

H is called the self similar exponent of the process X. Browian motion(without drift) is an example of self similar process with self similarity exponent $H = 1/2$.

Choose $c = 1/t, \forall t > 0, X_t = t^H X_1$. So the distribution of X_t for any t, is completely determined by the distribution X_1

$$F_t(x) = \mathbb{P}(t^H X_1 \leq x) \tag{1.2.2}$$

$$= F_1\left(\frac{x}{t^H}\right) \tag{1.2.3}$$

Definition 1.2.5. A probability distribution function F on $\mathbb{R}$ is said to be infinite divisible if for any integer $n \geq 2$, there exist n i.i.d. random variables $Y_1, ..., Y_n$ such that $Y_1 + ... + Y_n$ has the distribution F.

Thus, if $X = \{X(t)\}_{t \geq 0}$ is a Lévy process, for any $t > 0$ the distribution of $X(t)$ is infinitely divisible. In view of the above definition, one may establish whether a given random variable has infinitely divisible distribution via its characteristic exponent and the exponent is best expressed by Lévy-Khintchine formula.

Theorem 1.2.1. *A probability law μ of a real valued random variable is infinitely divisible with characteristic exponent Ψ*

$$\int_{\mathbb{R}} e^{i\theta x} \mu(dx) = e^{-\Psi(\theta)} \quad for \quad \theta \in \mathbb{R} \tag{1.2.4}$$

if and only if there exist a triple (a, σ, Π) where $a \in \mathbb{R}, \sigma \geq 0$ and Π is a measure concentrated on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}} \min\{1, x^2\}\Pi(dx) < \infty$ such that

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbb{I}_{(|x|<1)})\Pi(dx). \tag{1.2.5}$$

Proof:
For an overview and further exposition of Levy processes we refer to Sato (1999). In conclusion, any Levy process has the property that for all $t \geq 0$

$$\mathbb{E}^{\mathbb{P}}(e^{i\theta X_t}) = e^{-t\Psi(\theta)},$$

where $\Psi(\theta) := \Psi_1(\theta)$ is the characteristic exponent of X_1 which has an infinitely divisible distribution.

In this study, the density of Levy increments were captured by fitting Generalized hyperbolic distribution and its subclasses like Variance Gamma (VG) distribution.

1.2.2 The Generalized Hyperbolic distribution

In this section we provide definition of the Generalized Hyperbolic distribution (GH) along with its classical representation as a variance mean mixture of the normal with Generalized Inverse Gaussian (GIG) distribution.

Definition 1.2.6. A random variable W is said to have a generalized inverse gaussian distribution if its probability density function is given by

$$f_{GIG}(w; \lambda, \gamma, \delta) = \frac{(\gamma/\delta)^\lambda}{2K_\lambda(\gamma\delta)} w^{\lambda-1} \exp\left(-\frac{1}{2}(\frac{\delta^2}{w} + \gamma^2 w)\right), w > 0$$

where K_λ is a modified Bessel function of the third kind with the index λ, i.e.,

$$K_\lambda(\omega) = \frac{1}{2}\int_0^\infty \exp\left[-\frac{\omega}{2}(v^{-1} + v)\right] v^{\lambda-1} dv \tag{1.2.6}$$

and the parameters $\lambda \in \mathbb{R}, \gamma \geq 0, \delta \geq 0$ such that $\gamma \neq \delta$ if either of them takes the value zero.

We note that if $\gamma > 0$ and $\delta > 0$ then

$$\mathbb{E}W^k = \left(\frac{\delta}{\gamma}\right)^k \frac{K_{\lambda+k}(\gamma\delta)}{K_\lambda(\gamma\delta)}, \quad k \in \mathbb{Z}$$

Definition 1.2.7. (Normal Mean-Variance Mixture)

A random variable Y is said to have a normal mean-variance distribution if

$$Y = \mu + \beta W + \sigma\sqrt{W}Z$$

where $Z \sim N(0,1)$, W is a positive random variable independent of Z; μ , β and σ > 0. From the definition, the conditional distribution of Y given W is normal with mean $\mu + \beta W$ and variance $\sigma^2 W$.

Note that, if the mixture variable W is $GIG(\lambda, \gamma, \delta)$ distributed, then Y is a Generalized Hyperbolic distribution with the $(\lambda, \alpha, \beta, \delta, \mu)$ parametrization, where $\alpha^2 = \gamma^2 + \beta^2$.

The probability density function of the one-dimensional Generalized Hyperbolic distribution is given by the following:

$$f_{GH}(y; \alpha, \beta, \delta, \mu, \lambda) = \frac{(\gamma/\delta)^{\lambda}}{\sqrt{2\pi}K_{\lambda}(\delta\gamma)} \frac{K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2 + (y-\mu)^2})}{(\sqrt{\delta^2 + (y-\mu)^2}/\alpha)^{\frac{1}{2}-\lambda}} e^{\beta(y-\mu)} \tag{1.2.7}$$

According to Barndorff-Nielsen (1977), the parameters domain is given by

$$\alpha > 0 \quad \alpha^2 > \beta^2 \quad \delta \geq 0 \quad \text{for} \quad \lambda > 0,$$

$$\alpha > 0 \quad \alpha^2 > \beta^2 \quad \delta > 0 \quad \text{for} \quad \lambda = 0,$$

$$\alpha > 0 \quad \alpha^2 \geq \beta^2 \quad \delta > 0 \quad \text{for} \quad \lambda < 0.$$

In all cases, μ is the location parameter and can take any real value, δ is a scale parameter; α and β determine the distribution shape and λ defines the subclasses of GH and is related to the tail flatness.
Characteristic function of the GH is given by

$$\phi_{GH}(u) = e^{iu\mu}\left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2}\right)^{\lambda/2} \frac{K_{\lambda}\left(\delta\sqrt{\alpha^2 - (\beta + iu)^2}\right)}{K_{\lambda}\left(\delta\sqrt{\alpha^2 - \beta^2}\right)}, \tag{1.2.8}$$

while mean and variance are given respectively by the followings

$$E(Y) = \mu + \frac{\beta\delta}{\sqrt{\alpha^2 - \beta^2}} \frac{K_{\lambda+1}(\zeta)}{K_{\lambda}(\zeta)} \tag{1.2.9}$$

and

$$Var(Y) = \delta^2\left(\frac{K_{\lambda+1}(\zeta)}{\zeta K_{\lambda}(\zeta)} + \frac{\beta^2}{\alpha^2 - \beta^2}\left[\frac{K_{\lambda+2}(\zeta)}{K_{\lambda}(\zeta)} - \left(\frac{K_{\lambda+1}(\zeta)}{K_{\lambda}(\zeta)}\right)^2\right]\right) \tag{1.2.10}$$

where $\zeta = \delta\sqrt{\alpha^2 - \beta^2}$, then

$$X \sim GH(-\frac{1}{2}, \alpha, \beta, \delta, \mu) \quad \text{Normal-Inverse Gaussian distribution}$$

$$X \sim GH(1, \alpha, \beta, \delta, \mu) \quad \text{Hyperbolic distribution} \tag{1.2.11}$$

$$X \sim GH(\lambda, \alpha, \beta, 0, \mu) \quad \text{Variance Gamma distribution.} \tag{1.2.12}$$

Parameterizations

Although the parametrization $(\alpha, \beta, \delta, \mu, \lambda)$ is mostly used in literature we have other parameterization like (χ, ξ, δ, μ) which is invariant under the transformation of the scale and location parameters with $\xi = (1 + \delta\sqrt{\alpha^2 - \beta^2})^{-1/2}$ and $\chi = \xi\beta/\alpha$. Hu (2005) and McNeil et al. (2005) used the following parameterizations $(\lambda, \chi, \psi, \mu, \sigma, \gamma)$ where

$$\lambda = \lambda,\ \beta = \frac{\gamma}{\sigma^2},\ \delta = \sigma\sqrt{\chi},\ \alpha = \sqrt{\frac{\psi}{\sigma^2} + \beta^2} \tag{1.2.13}$$

The parametrization $(\lambda, \bar{\alpha}, \mu, \sigma, \gamma)$, is derived if we set

$$\bar{\alpha} = \sqrt{\psi\chi},\ \text{and}\ \sqrt{\frac{\chi}{\psi}}\frac{K_{\lambda+1}(\sqrt{\psi\chi})}{K_\lambda(\sqrt{\psi\chi})} = 1,\quad \text{which implies},\quad \psi = \bar{\alpha}\frac{K_{\lambda+1}(\bar{\alpha})}{K_\lambda(\bar{\alpha})},\quad \chi = \bar{\alpha}\frac{K_\lambda(\bar{\alpha})}{K_{\lambda+1}(\bar{\alpha})},$$

similar parametrization is used in **ghyp** R package.

The Variance Gamma (VG) distribution

The Variance Gamma VG(C,G,M) distribution (see Carr and Madan (1998), Carr et al. (2002) for more details) on $\mathbb{R}$ can be constructed as the difference of two gamma random variables. Suppose that $X \sim \Gamma(C, M)$ and $Y \sim \Gamma(C, G)$ are random variables and that they are independent of each other. Then

$$X - Y \sim VG(C, G, M).$$

Note that the characteristic functions of X and Y are given by

$$\phi_X(u) = (1 - iu/M)^{-C}, \quad \phi_{-Y}(u) = (1 + iu/G)^{-C} \tag{1.2.14}$$

Summing the two independent random variables, the resulting characteristic function gives,

$$\begin{aligned} \phi_{X-Y}(u) &= (1 - iu/M)^{-C}(1 + iu/G)^{-C} && (1.2.15)\\ &= \left(\frac{GM}{GM + (M - G)iu + u^2}\right)^C && (1.2.16) \end{aligned}$$

similarly, VG distribution can be defined as mixture of normal variate with mean θ, and a positive number $\sigma > 0$ and a gamma variate with parameters $\alpha = 1/\nu$ and $\beta = 1/\nu$. Applying basic probability techniques, the characteristic function of $VG(\sigma, \nu, \theta)$ distribution is given by

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \exp[iuX] &= \phi_{VG}(u; \sigma, \nu, \theta) && (1.2.17)\\ &= (1 - iu\theta\nu + \sigma^2\nu u^2/2)^{-1/\nu} && (1.2.18) \end{aligned}$$

where

$$\begin{aligned} C &= 1/\nu,\\ G &= \left(\sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} + \frac{\theta\nu}{2}\right)^{-1} > 0,\\ M &= \left(\sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} - \frac{\theta\nu}{2}\right)^{-1} > 0. \end{aligned}$$

GARCH models

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $(\mathcal{F}_t)_{t\in[0,T]}$; be a filtered probability space. Assume that time series $(X_t)_{t\in\mathbb{Z}}$ is adapted to some filtration $\mathcal{F}_t = \sigma(\{X_s : s \leq t\})$. The process $(X_t)_{t\in\mathbb{Z}}$ is an ARCH(p) process if it is strictly stationary and if it satisfies the equations

$$\begin{aligned} X_t &= \sigma_t Z_t, \quad Z_t \sim i.i.d(0,1), \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2, \quad \alpha_0 > 0, \alpha_i \geq 0, i = 1, ..., p. \end{aligned} \tag{1.2.19}$$

for all $t \in \mathbb{Z}$ and some strictly positive -valued process $(\sigma_t)_{t\in\mathbb{Z}}$

The process $(X_t)_{t\in\mathbb{Z}}$ is a GARCH(p,q) process if it is strictly stationary and if it satisfies,

$$\begin{aligned} X_t &= \sigma_t Z_t, \quad Z_t \sim i.i.d(0,1) \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \quad \alpha_0 > 0, \alpha_i \geq 0, \beta_j \geq 0. \text{ and} \\ Var(X_t|\mathcal{F}_{t-1}) &= \sigma_t^2 \text{ and } Z_t = X_t/\sigma_t \text{ is } i.i.d. \ (Strong\ GARCH), \\ Var(X_t|\mathcal{F}_{t-1}) &= \sigma_t^2 \ (semi-Strong\ GARCH), \end{aligned} \tag{1.2.20}$$

for all $t \in \mathbb{Z}$ and some strictly positive valued process $(\sigma_t)_{t\in\mathbb{Z}}$ and $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$

AR Asymmetric Power ARCH

The (Autoregressive Asymmetric Power ARCH) AR(p)-APARCH(m,n) model of Ding et al. (1993) can be written as follows

$$\begin{aligned} X_t &= \sum_{k=1}^{p} \phi_k X_{t-j} + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad z_t \sim i.i.d(0,1) \\ \sigma_t^\delta &= \alpha_0 + \sum_{i=1}^{m} \alpha_i (|\varepsilon_{t-i}| - \gamma_i \varepsilon_{t-i})^\delta + \sum_{j=1}^{n} \beta_j \sigma_{t-j}^\delta \end{aligned} \tag{1.2.21}$$

subject to $\alpha_0 > 0, \delta \geq 0, \alpha_i \geq 0, -1 < \gamma < 1$, for $i = 1, ..., q$,$\beta_j \geq 0, for j = 1, ..., n.$ and

$$\sum_j k_j + \sum_i \beta_i < 1, \quad where \quad k_j = \alpha_j (|\varepsilon_{t-j}| - \gamma_j \varepsilon_{t-j})^\delta \tag{1.2.22}$$

The model introduces a Box-Cox power transformation on the conditional standard deviation process and on the asymmetric innovations, $\alpha_i(|\varepsilon_{t-i}| - \gamma_i \varepsilon_{t-i})^\delta$, adds flexibility of a varying exponent with an asymmetry coefficient to take the leverage effect into account. The properties of APARCH model have been studied (see for example He and Terasvirta (1999) and Sebastien (2004)). The model nests seven other ARCH extensions as special cases, of which we list six models.

i) ARCH model of Engle (1982) when $\delta = 2, \gamma_i$ and $\beta_j = 0 \ \forall \ \ i\&j$.

ii) Bollerslev (1986) GARCH model when $\delta = 2$, and $\gamma_i = 0$

iii) GJR-GARCH Model of Glosten et al. (1993) when $\delta = 2$

iv) TARCH Model of Zakoian (1994) when $\delta = 1$

v) The Log-ARCH Model of Geweke (1986) and Pentula (1986) when $\delta \to 0$

vi) The NARCH of Higgins and Bera (1992) when $\gamma_i = 0 (i = 1, ..., q)$ and $\beta_j = 0 (j = 1, ..., p)$

Note that $\mathbb{E}^{\mathbb{P}}(X_t|\mathcal{F}_{t-1}) = \mu_t$ denote the conditional mean given the information set $\mathcal{F}_{t-1}$ available at time $t-1$. The innovation process for the conditional mean is then given by $\varepsilon_t = X_t - \mu_t$ with corresponding unconditional variance σ^2 and zero unconditional mean. The conditional variance is defined as $\text{Var}(X_t|\mathcal{F}_{t-1}) = \text{Var}_{t-1}(X_t) = \sigma_t$.

1.3 Different Types of Options

Options are financial contracts that give the holder certain rights. As a holder you buy the rights stipulated in the contract. It can be the right to buy or sell financial contract, or else it can be the right to exchange one commodity for another. There are many different kinds of contracts on any underlying asset. Two of the simplest types of contracts that are traded are *European options* and *American options*. *European call option* gives the holder the right (but not the obligation) to purchase from the writer a prescribed asset for a prescribed price at a prescribed time in the future.

American options, gives the holder the right to buy/sell the underlying asset at any time before the time to maturity for a certain price.

There are numerous exotic options traded, like Asian, lookback, Parsian, Bermudan and so on. The buyer of the option pays the seller an amount $C(t, s)$ (the premium) at time $t = 0$, in return for the right, but not the obligation to buy the stock at time $t + T$ at a price K (the exercise price or strike) which is set when the contract is signed at time $t = 0$. The profit linked to a call is unlimited, and the losses are limited to $C(t, s)$ where $s = S(t)$ is the current stock price.

The valuation of an option, depends on determining the price $C(t, s)$. BSM73 formula and its binomial counterpart are the most used probability model/tool in every day use. Literally tens of thousands of people, including traders, market makers, and salespeople, use option formulas as documented in Haug (2007), several times a day (see for example Figure 1.1) as they trade in derivatives. Moreover, we develop stochastic volatility valuation model that captures important features of equity return process commonly overlooked by BSM73 model.

1.4 Black Scholes and Merton (BSM73) Model

In 1973, Fischer Black and Myron Scholes derived a partial differential equation for option prices, when asset prices behave according to the geometric Brownian motion. The standard BSM73 model of the financial market consists of two assets; a bond $\mathcal{B}$ and a stock S-with dynamics given by

$$\begin{aligned} d\mathcal{B}(t) &= r\mathcal{B}(t)dt \qquad (1.4.1)\\ dS(t) &= \alpha(t,S(t))S(t)dt + \sigma(t,S(t))S(t)dB(t). \qquad (1.4.2)\end{aligned}$$

Here B is a Brownian process, $r \in \mathbb{R}^+$ is the short rate of interest, $\alpha \in \mathbb{R}$ is the local mean of return of S and $\sigma \in \mathbb{R}$ is the volatility of S. The stock price in a risk neutral world moves according to a geometric Brownian motion, that is

$$dS(t) = S(t)rdt + S(t)\sigma dB(t), S(0) \neq 0. \qquad (1.4.3)$$

The stochastic process (1.4.3) is a specific case of a diffusive process. $B(t)$ is a (Wiener) standard Brownian process defined by $dB(t) \sim N(0,dt)$ and the terms $rS(t)$ and $\sigma S(t)$ are known as drift and diffusion of process. Technically, a stochastic process in continuous time $S(t), t \leq T$ is defined with respect to a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ where $\mathcal{F}_t = \sigma(S(u); u \leq t)$ is the smallest $\sigma-$field containing sets of the form $\{a \leq S(u) \leq b\}, 0 \leq u \leq t$: more intuitively $\mathcal{F}_t$ represents the amount of information available at time t.

The increasing $\sigma-$fields $\{\mathcal{F}_t\}$ form a so called filtration:

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}_t :$$

Not only is the filtration increasing, but also contains all the events with zero measure and these are typically referred to as "the usual assumptions". The increasing property corresponds to the fact that at least in financial applications the amount of information is continuously increasing as time elapses.

Clearly $S(T)$ can be written in a more explicit form

$$S(T) = S(0)\exp\left((r - \frac{\sigma^2}{2})T + \sigma B_T\right)$$

In order to calculate the value of the option, we calculate the expected value of the payoff and discount it using a risk-free rate.

1.5 Option Pricing

By the risk-neutral valuation, the price $C(K,t)$ at a time t, of a contingent claim with payoff $G(\{S_u, t \leq u \leq T\})$ is given by

$$C(K,t) = \exp\left(-(T-t)r\right)\mathbb{E}^{\mathbb{Q}}\left[G(\{S_t, 0 \leq t \leq T\})\right], \quad t \in [0,T] \qquad (1.5.1)$$

If the payoff function depends on time T, value of the stock, i.e. $G(\{S_t, 0 \leq t \leq T\}) = G(S_T)$, then

$$
\begin{aligned}
C(K,t) &= \exp(-(T-t)r)\mathbb{E}^{\mathbb{Q}}G(S_T), &(1.5.2)\\
&= \exp(-(T-t)r)\mathbb{E}^{\mathbb{Q}}G\left(S_0 \exp(r-\frac{\sigma^2}{2})(T-t)+\sigma B_t\right), &(1.5.3)\\
&= \exp(-(T-t)r)\int_{-\infty}^{+\infty} G\left(S_0 \exp\left[(r-\frac{\sigma^2}{2})(T-t)+\sigma B_t\right]\right) f_{B_t}(x;0,(T-t))dx
\end{aligned}
$$

A European call on a stock with strike K and maturity T

$$
\begin{aligned}
G(\{S_t, 0 \leq t \leq T\}) &= G(S_T) &(1.5.4)\\
&= (S_T - K)^+ &(1.5.5)\\
\therefore C(K,T) &= \exp(-(T-t)r)\int_{-\infty}^{+\infty}(S_T-K)^+ f_{B_t}(x)dx &(1.5.6)\\
&= S_0\Phi(d_1) - K\exp(-r(T-t))\Phi(d_2), &(1.5.7)\\
\text{where} \quad d_1 &= \frac{\log(S_0/K)+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} &(1.5.8)\\
\text{and} \quad d_2 &= d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

The pricing formulas obtained in Black and Scholes (1973) represented a major breakthrough in understanding financial derivatives, to such an extent that financial institutions and traders immediately adopted the new methodology.

We shall focus our attention on statistical modeling of data observed in stock markets (see for example Figure1.1), and analyze it with a view towards improving at-the-money bias of BSM73 model.

1.6 Shortfalls of BSM73 Model

The influence of the BSM73 model is best shown by traders quoting option contracts using their implied volatilities. Yet its influence extends beyond the traditional use of hedging, arbitrage and speculation.

But, despite its popularity, BSM73 has some serious systematic biases. For example, when one plots implied volatilities of exchange-traded options against their exercise prices for a fixed maturity, the curve is typically convex and is known as the "smile", see Eberlein and Keller (1995), Carr and Madan (1998), Barndorff-Nielsen (1998), Carr et al. (2002) and references therein. While the theoretical prediction of the model suggests a horizontal line, the reality is far from it. If BSM73 model is a good description of market-pricing behavior, the implied volatility for any exercise price, as a function of the option maturity, must also be a horizontal line.

While BSM73 model is the industry benchmark for options, empirical studies reveal that the normality of the log-returns, as assumed by BSM73, cannot capture features like heavy tails and

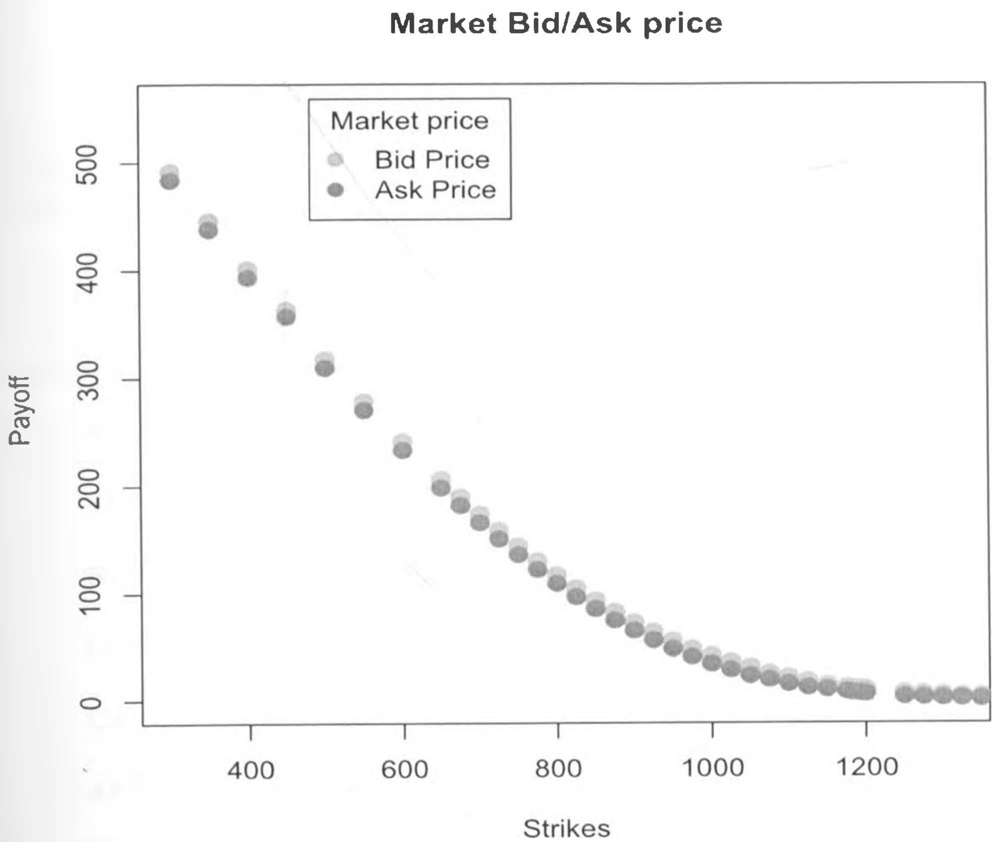

Figure 1.1: S&P500 European call option S_0=800.3, r=2.8%, T=81 days as on November 21,2008

asymmetries observed in market-data log return densities. In addition, a numerical inversion of the BSM73 formula based on data from different strikes and fixed maturity resembles a skew or a smile (Carr et al. (2003), Schoutens (2003)).

In line with the above stated drawbacks of the model, the critical task is fitting the smile so that a continuum of a call option prices in the exercise price and maturity dimension can be generated. Let us consider basic properties of log returns of any underlying asset.

1.6.1 Returns

Let the daily log returns X be defined as $X_t = \ln S(t) - \ln S(t-1)$. The skewness (measures the degree to which a distribution is asymmetric) is defined to be the ratio the third moment about the mean, to the third power of standard deviation

$$skewness = \frac{\mathbb{E}(X - \mu_X)^3}{Var(X)^{3/2}} \tag{1.6.1}$$

The behavior and peakedness of the distribution are measured by kurtosis, which is defined by

$$kurtosis = \frac{\mathbb{E}(X - \mu_X)^4}{[Var(X)]^2} \tag{1.6.2}$$

for the normal distribution (mesokurtic), the kurtosis is 3. If daily log returns are normally distributed, we expect skewness and kurtosis to be 0 and 3 respectively. However, empirical evidence as reported in table 1.1 does not concur with the model assumption(s).

Table 1.1: Basic statistics, i.e. mean, standard deviation, skewness and kurtosis of daily log returns major indices in developed and emerging market (Kenya-NSE20)

Index	Years	n	mean ($\bar{X}$)	st.dev (s)	skewness	kurtosis
S&P500	1990-2008	4763	0.0001	0.01112	−0.1928	12.7871
NDX100	1990-2008	1796	0.0003	0.01911	+0.0995	8.0454
DAX	1990-2008	4541	0.0002	0.01446	−0.1019	8.3366
CAC40	1990-2008	4729	0.0001	0.01392	−0.0658	7.9473
AEX	1990-2008	4101	0.0001	0.01418	−0.1767	9.9826
NSE20	1998-2007	2316	0.0001	0.00775	+0.0285	9.2279
RUT2000	1990-2008	4778	0.0002	0.01242	−0.5589	11.3834

1.6.2 Implied volatility

Let $C(t, s)$ be a European call option with strike K and time to maturity T. We calculate σ (volatility), the only free parameter $\sigma = \sigma(K, T)$ such that the theoretical price under BSM73 model match the empirical one, say C_{mkt}.

There is no closed formula to extract the implied volatility out of the call option price. One method to find numerically implied volatilities is the classical Newton-Rhapson iteration procedure. Let C_{mkt} be the market option price for a given strike K and time to maturity T. Let $\sigma_0 = 0.2$ be the initial value. If we denote σ_n the value obtained after n iteration steps, and the next value σ_{n+1} is given by

$$\begin{aligned}
\sigma_{n+1} &= \sigma_n - \frac{C(\sigma_n) - C_{mkt}}{C'(\sigma_n)}, \\
\text{where, } C'(\sigma_n) &= S_0\sqrt{T}\Phi(d_1) \\
\text{and } d_1 &= \left(\frac{\log(S_0/K) + (r + \frac{\sigma_n^2}{2})T}{\sigma_n\sqrt{T}}\right)
\end{aligned}$$

where S_0 is the current stock price, $\Phi(x)$ is the cumulative probability distribution of standard normal random variable. Under the Black-Scholes model, all $\sigma's$ should be the same. It is clearly observed that there is a huge variation in this volatility parameter both in strike and in time to maturity.

1.7 Objectives

The objectives of this study are:-

(i) To identify the "true" dynamics of the underlying asset process in developed and emerging economies.

(ii) To explore the dynamic response of volatilities of innovations.

(iii) To develop a pricing model that may be used to price financial derivatives in emerging and developed economies.

(iv) To minimize (or eliminate or offer an explanation) implied volatility smile or smirk.

1.8 Literature review

Option pricing theory has a long and illustrious history even before Bachelier (1900) publication. Thereafter, it underwent a revolutionary change in 1973. At that time, Black and Scholes (1973) presented the first complete satisfactory equilibrium option pricing model followed by Merton (1973) extending their model in several important ways. Later, Cox and Ross (1976) proposed jump process model as a special case of Black and Scholes model. Option valuation techniques have been extended to more realistic assumptions in a number of ways for the underlying stock processes (e.g. Rubinstein (1976), Cox et al. (1979), Carr and Wu (2004), Hull and White (1990), Derman and Kani (1994), Duan (1995), Eberlein and Keller (1995), Geman et al. (2001), Barndorff-Nielsen et al. (2002), Carr et al. (2003), Duan et al. (2006), Carr et al. (2007), Primbs et al. (2007) and many more).

A common assumption underlying most well known and widely used option pricing BSM73 model is that, the logarithm of stock price are normally distributed. An extensive empirical literature in finance has documented not only the presence of anomalies in BSM73 model, but also the term structure of these anomalies (see for instance, the behavior of the volatility smile, riding on smile, and pricing with a smile which can be found in Dupire (1994), Duan (1996), Das and Sundaram (1999), Bringo and Mercurio (2000), Meziou (2004)).

Distributional assumptions concerning risky asset log returns play a key role in option pricing. According to research finding of Mandelbrot (1963), evidence indicates that the empirical distributions of daily stock returns differ significantly from the traditional Gaussian model. In search of satisfactory descriptive models for financial data, large number of distributions have been tried, see for example, Fama (1965), Press (1967), Praetz (1972), Clark (1973), Blattberg and Gonedes (1974), Bates (1983), Madan and Seneta (1990), Eberlein and Keller (1995), Hurst et al. (1997), Schoutens (2003), Lindberg (2008)), Barndorff-Nielsen (1997).

The deviations from normality become more severe when more frequent data are used to calculate stock returns. Various studies have shown that the normal distribution does not accurately describe observed stock return data. Over the past several decades, some stylized facts have emerged about the statistical behavior of speculative market returns such as aggregational Gaussianity, volatility clustering, etc see Rydberg (2000), Cont (2001), Tsay (2002). On the same note, most of the literature for example Eberlein and Keller (1995), Carr and Madan (1998), Barndorff-Nielsen (1998), Carr et al. (2002) and references therein, make a simplifying assumptions, that daily log returns can be modeled by exponential Lévy processes, finding a number of explicit formulaes for pricing derivatives (see also Carr et al. (2003), Schoutens (2003), Carr and Wu (2004)) or modeling stock price process by a geometric Lévy process (Chan (1999)) in exact analogy with the ubiquitous geometric Brownian motion model.

The presence of a greater degree of excess kurtosis (and possibly skewness) in unconditional returns distributions, and the presence of implied volatility smile in options data, confirm the presence of leptokurtic distribution inconsistent with assumed normality. Theoretical efforts in the literature addressing these anomalies have largely focused on two extensions of the BSM73 model. Introducing jumps into the return process, and allowing volatility to be stochastic(see Merton (1976), Hull and White (1987), Stein and Stein (1991), Heston (1993), Bakshi et al. (1997), Duan et al. (2006)). The class of jump-diffusion models, augments BSM73 assumed risky asset with a Poisson-driven jump process.

There are two important directions in the literature regarding these type of stochastic volatility models. Continuous-time stochastic volatility process represented in general by a bivariate diffusion process, and the discrete time autoregressive conditionally heteroscedastic (ARCH) model of Engle (1982) or its generalization (GARCH) as first defined by Bollerslev (1986). In the last few years, much interest has been given to the discrete-time GARCH option pricing models. The most important papers which study the empirical fitting of these model include Pagan and Schwert (1990), Glosten et al. (1993), Bollerslev et al. (1994). Option pricing in GARCH models has been typically done using the Local Risk Neutral Valuation Relationship (LRNVR) pioneered by Duan (1995). The crucial assumptions in his construction are the conditional normality distribution of the asset returns under the underlying probability space and the invariance of the conditional volatility to the change of measure. The empirical performance of these normal option pricing models has been studied by many authors, for example Duan (1996), Hardle and Hafner (2000), Heston and Nandi (2000), Christoffersen and Jacobs (2004).

Lattices for option pricing were first introduced in 1979 in the pioneering work of Cox et al.

(1979). In particular, they used binomial lattice to model geometric Brownian motion and Rendleman and Bartter (1979) used binomial lattice to model exponential Poisson process. An attractive property of their model is that the binomial lattice for geometric Brownian motion is consistent with the standard Black and Scholes (1973) formula for European options. Due to simplicity and versatility of lattice models, a number of extensions to the basic model have been proposed, see Derman and Kani (1994), Ritchken and Trevor (1999), Yamada and Primbs (2001), Wu (2006) for example. Florescu and Viens (2008) use quadrinomial tree to model stochastic volatility in option pricing, while Primbs et al. (2007) price options with a pentanomial lattice. It is worthy noting that an efficient lattice method, may be significantly faster than a Monte Carlo method for valuing some types of path dependent options.

1.9 Problem formulation

The premise that price movement in mature economies and associated emerging markets do not lend themselves easily to explanations currently offered by conventional capital theory. Standard models fail to reproduce observed prices of vanilla options because implied volatilities exhibit a term structure of smiles.

The presence of implied volatility, which is a moderately downward-sloping or U-shaped function of the strike price, observed in international markets, suggests an inconsistency with Black and Scholes (1973) constant volatility assumption (see for example Constantinides et al. (2008)). According to BSM73 model, the behavior of a common stock is assumed to be described by the Geometric Brownian Motion. However in recent years, empirical evidence, has questioned the wisdom of such assumptions, particularly, in the context of emerging markets where stock returns are observed to exhibit non random walk behavior.

We seek to explore the dynamic response of volatilities of innovations, aimed at developing a model that may be used to construct pricing formulae for financial derivatives in emerging and developed economies. It is expected that the resulting model explains the significant deviation of observed European call values against BSM73 prices. See Figure 1.2 and Figure 1.4 for at-the-money problem and varying volatility respectively.

Moreover, our main focus will be on understanding the source of the "grimaced/smile of option" and endeavor to construct a model as an alternative for option pricing in any economy (be it emerging or developed).

1.10 Significance of study

The widely reported phenomena that the implied volatility is not constant as other parameters show that the BSM73 formulas fail to describe perfectly the option values that arise in the market place. Many attempts have been made to "fix" the nonconstant volatility discrepancy in the Black-Scholes theory. A few of these have been successful to some degree, but none lead to the simple formulas and clear interpretations of the original work.

This study examines the evolution dynamics in emerging economies and developed economies as captured by stock markets index namely, Kenya (NSE20), Morroco (MASI), United States (S&P 500), United Kingdom(FTSE100) and German (DAX 30).

For a newly established stock markets, the earning potential of a formerly state owned firm which has become listed as a result of privatization, cannot be measured or realized precisely. This is because the current price of the firm's shares is generally different from its potential equilibrium value, so that in practice individuals buy and sell shares on the basis of estimates of the later. At any point in time, the price of the firm's share represents the markets best guess of the equilibrium value in the new stock market. However, information on which to calculate estimates on new equilibrium value becomes more readily available as stock market matures.

To evaluate and understand non-random-walk behavior, we develop a stochastic price evolution model, namely APARCH-Levy Filter, which explains the randomness of the movements of stock prices slightly better. We further compare and contrast the prices of the proposed model and the Black and Scholes (1973) model and real market options data.

The role of a stock market in any economy may not be underestimated. Consequently, this study hopes to shed some light on the relationships between emerging and developed markets, especially to investors and policymakers. There has been a consistent flow of funds into these emerging financial markets as foreign participation in them continue to increase.

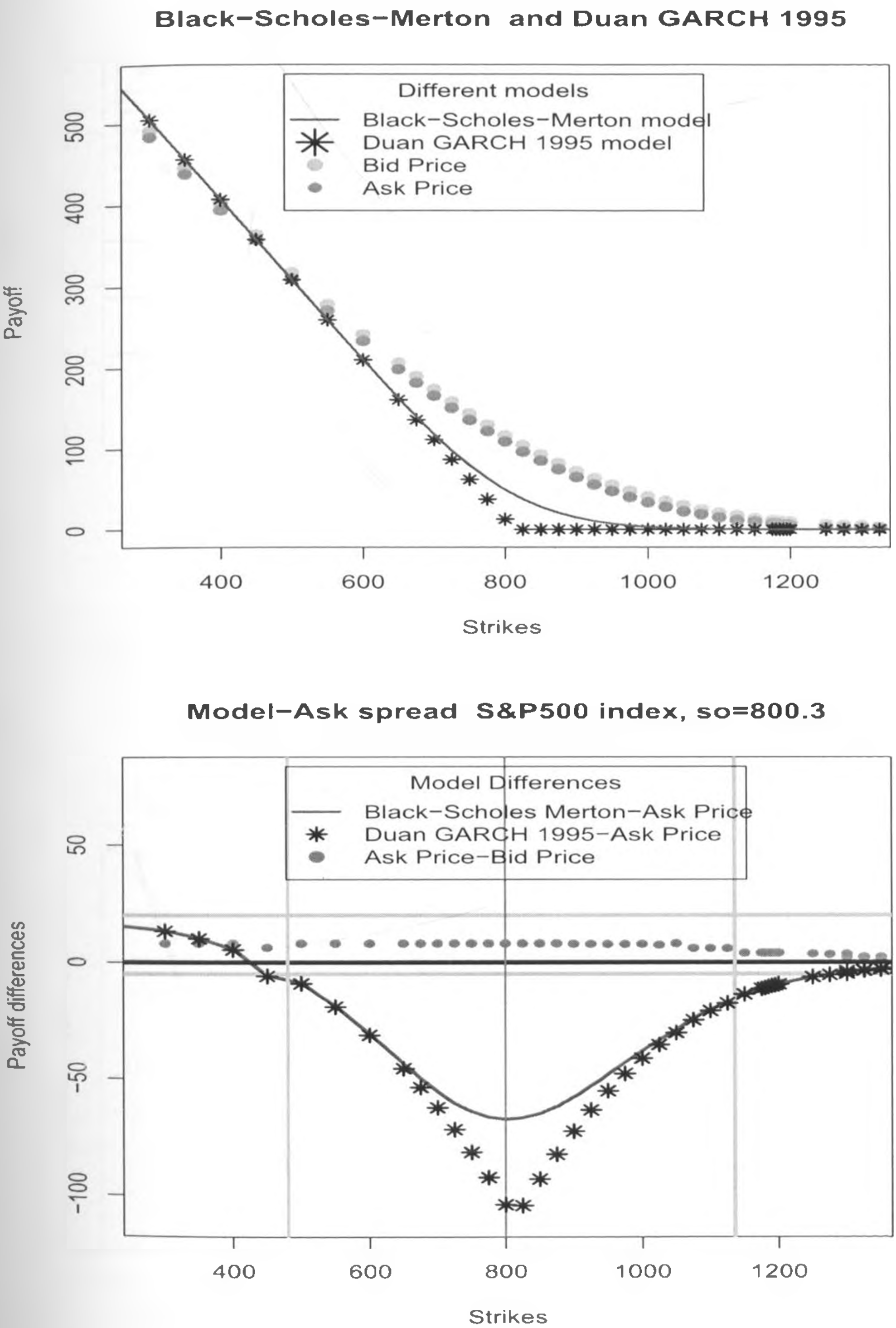

Figure 1.2: Black-Scholes-Merton model and GARCH model $T = 145$ days, $S_0 = 800, r = 2.83$

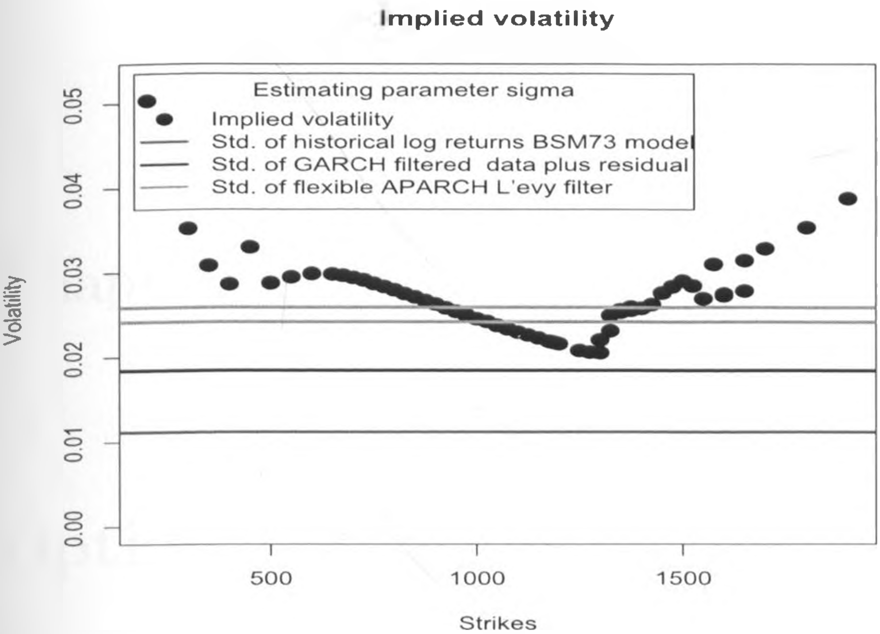

Figure 1.3: S&P500 Implied volatility $T = 145$ days, $S_0 = 800, r = 2.83$

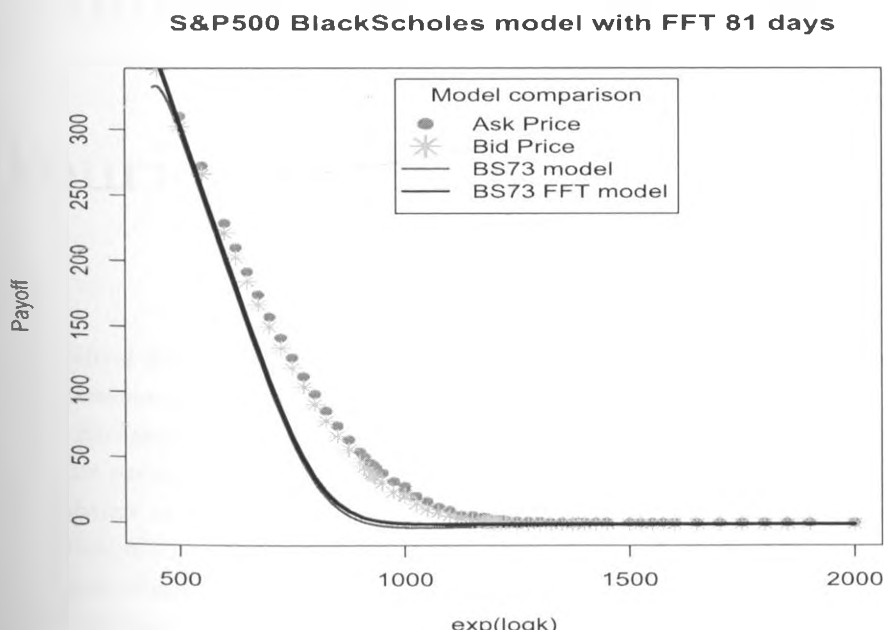

Figure 1.4: S&P500 Fast Fourier Transform BSM73 option pricing 81 days

Chapter 2

Option Valuation in Developed and Emerging Economies: Exponential Lévy model and Fast Fourier Transform

Generalized Hyperbolic distribution and some of it subclasses like normal, Hyperbolic and Variance Gamma distributions are used to fit daily log returns of eight listed companies in Nairobi Stock Exchange (NSE) and Montreal Exchange. EM-based ML estimation procedure is used to locate parameters of the model. Densities of simulated and empirical data and goodness of fit statistics of proposed distributions are compared to measure how well model fits the data. Empirical results show that Generalized Hyperbolic distribution seems to correct bias of BSM73 normality assumption both in developed and emerging market. Both markets seem to have different stochastic times. However, there is no significant difference between BSM73 model and exponential Levy model using Fast Fourier Transform.

2.1 Introduction

The Black and Scholes (1973) and Merton (1973) methodology has become the dominant paradigm for valuing options and other derivatives. This method uses a delta hedging argument to value options based on the absence of arbitrage strategies that profit instantaneously.

Despite the success of the Black Scholes model on Brownian motion, there is a non negligible discrepancy between the model and the real market data. The model fails to reflect the stylized facts of equity log returns. In Cont (2001), an extended list of stylized features of financial data is given. Among the facts are: the asymmetric leptokurtic feature of financial data, i.e the returns distribution is not only skewed but heavy tailed; aggregational Gaussianity as time scale increases over returns; evidence of volatility clustering as opposed to the assumed constant volatility in the BSM73 model; and the presence of large fluctuations in price jumps and market crashes.

To incorporate the asymmetric leptokurtic features in asset pricing, Madan and Seneta (1990) studied time changed Brownian motion and Eberlein and Keller (1995) introduced Hyperbolic distribution to finance. Most of the Levy processes studied in literature attempt to explain asset returns behavior and correct the bias of the celebrated Black-Scholes model (see Carr and Wu (2004)). Among these models, we consider the generalized hyperbolic Lévy motion and it subclasses; such as normal density, hyperbolic, variance gamma, skewed student t and normal inverse gaussian.

We focus our attention on the Generalized Hyperbolic (GH), Hyperbolic (HY) and Variance Gamma (VG) distributions. The Generalized Hyperbolic distribution was first introduced by Barndorff-Nielsen (1977) in the context of sand project, and later Eberlein and Keller (1995) applied the Hyperbolic distribution to price vanilla options based on German stocks using the Esscher transform. Asymmetric Variance Gamma process was introduced by Madan et al. (1998) as time-change Brownian motion.

In this chapter we fit generalized hyperbolic distributions and some of it subclasses like the hyperbolic and the variance gamma distributions of daily log returns of eight listed companies (*four from each exchange*) in **Montréal** and **Nairobi Stock Exchange**. We will use kernel densities versus maximum likelihood parameter estimates of hypothesized normal, GH, HY, VG and QQ plot, frequency distributions and Kolmogorov distance to compare goodness of fit of the selected distributions. We apply Mean-Correcting Martingale measure (see for example Schoutens (2003)) as a risk neutral measure to price a European type option in the Montreal market and Nairobi Stock Exchange.

We note that, to the best of our knowledge Futures and Options Market Segment (FORMS) *is not yet operational in NSE* and no research of this type has been conducted in Kenyan market, apart from testing presence of random walks in African stock markets by Smith et al. (2002).

The rest of this chapter is organized as follows. Section 2 presents empirical evidence of imperfections of Black Scholes model. In Section 3, the generalized hyperbolic distribution and some of its subclasses, Hyperbolic and variance gamma are defined. Data and details of estimation methodology are presented. In section 4, Kolmogorov distance and frequency distribution are used to test the normality assumption and proposed model fit. Section 5 connects Lévy processes and option pricing using numerical results of previous sections and a risk neutral measure to value European options.

2.2 Imperfection of BSM73 model

In Black-Scholes world, the financial asset is modeled by geometric Brownian motion. For stocks, the model assumes that the price process $S = \{S_t, t \geq 0\}$ of an asset is given by $S_t = S_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma B_t)$, where μ is the drift, σ is the assumed market constant volatility and B_t is standard Brownian motion. The simplifying assumptions of the classical Black-Scholes model about the dynamics of the underlying such as stock prices, are on some occasions dotted with several shortcomings as pointed out in Schoutens (2003). They include: normality assumption, continuous sample paths of Brownian motion and extreme events. Release of information, which is immediately absorbed by the market leads to jumps in the processes, hence prices are in reality driven by jumps. This is expected to be true in both markets.

As an illustration in Figure 2.1 and Figure 2.2, we use the kernel density versus maximum likelihood fit of the normal distribution of daily log returns for Barclays Bank of Kenya (**BBK**), Kenya Commercial Bank (**KCB**) listed in Nairobi Stock Market compared to Monterial listed Alcan Inc (**Al**) and Royal Bank of Canada (**RY**) Company's daily log returns. Clearly the normal distribution does not reflect the empirical distribution. Similar results can be obtained by plotting QQ plots which deviates from straight line.

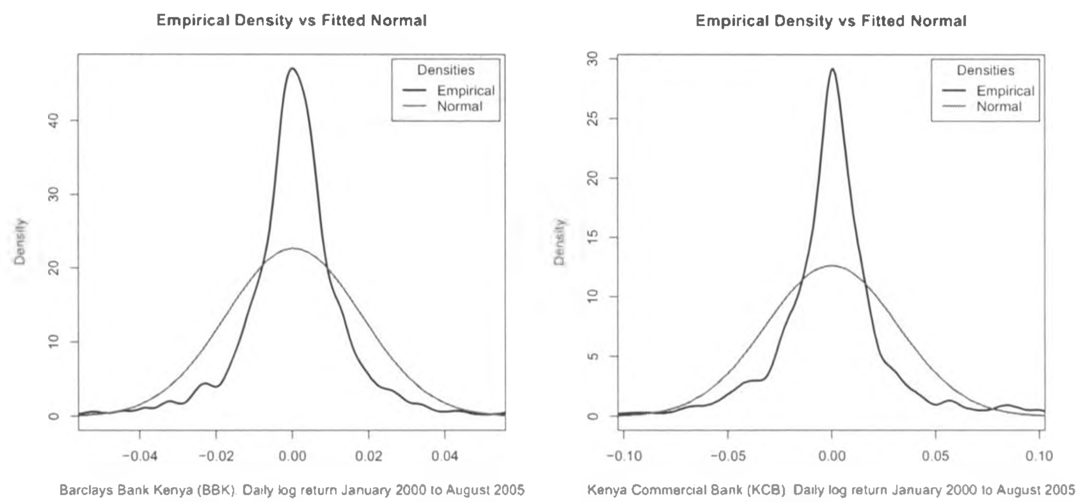

Figure 2.1: Maximum Likelihood Estimate of Normal distribution $\sim N(\mu - .5\sigma^2, \sigma)$ and Kernel densities for **BBK** and **KCB** from Nairobi Stock Exchange

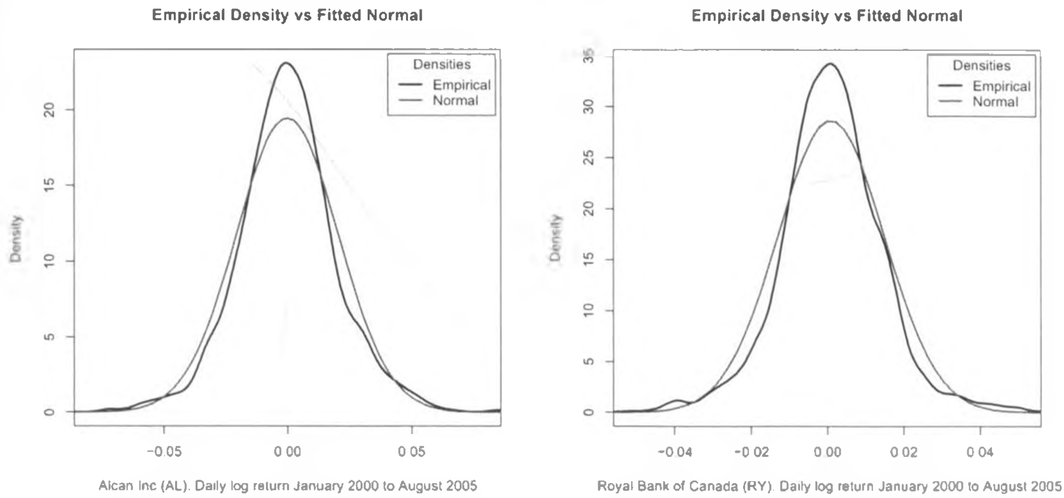

Figure 2.2: Maximum likelihood of Normal distribution $\sim N(\mu - .5\sigma^2, \sigma)$ and Kernel density for **AL** and **RY**. Monterial Exchange

Skewness and Kurtosis

Values of daily skewness and excess kurtosis are statistically far from that of the symmetric normal distribution. As an example, among the data we have studied, the daily skewness and excess kurtosis of **BBK** are -0.614181 and 13.327313 and for **RY** 0.187034 and 3.003643 respectively. Therefore, the tails of the empirical distribution are heavier than those of the normal distribution from both markets.

Many modifications of the model have been proposed, especially for fitting log returns, among them is the Generalized hyperbolic (GH) distribution. The GH distributions possess a lot of attractive properties, such as asymmetry, skewness and presence of semi-heavy tails. Therefore, the class of GH distributions appears to be a good candidate for modeling log returns.

2.3 The Generalized Hyperbolic distribution

In this section we discuss subclasses of the Generalized Hyperbolic distribution, parameter estimation and some empirical results.

2.3.1 Some Limiting Distributions and Subclasses

Many distributions are obtained as limiting distribution of the generalized hyperbolic distribution and by varying parameter λ to obtain subclasses.

Hyperbolic distributions

When $\lambda = 1$, using the fact that $K_{1/2}(x) = \sqrt{\frac{\pi}{2}}x^{-1/2}e^{-x}$, we obtain the subclass of hyperbolic distribution with probability density function and characteristic function given respectively by the followings

$$f_{HY}(x;\alpha,\beta,\delta,\mu) = \frac{\sqrt{\alpha^2-\beta^2}}{2\alpha\delta K_1\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\exp\left[-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)\right]. \tag{2.3.1}$$

$$\phi_{HY}(u) = e^{iu\mu}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)^{1/2}\frac{K_1\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)}{K_1\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \tag{2.3.2}$$

$$Var(X) = \delta^2\left(\frac{K_2(\zeta)}{\zeta K_1(\zeta)}+\frac{\beta^2}{\alpha^2-\beta^2}\left[\frac{K_3(\zeta)}{K_1(\zeta)}-\left(\frac{K_2(\zeta)}{K_1(\zeta)}\right)^2\right]\right) \tag{2.3.3}$$

$$\mathbb{E}(X) = \mu+\frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}}\frac{K_2(\zeta)}{K_1(\zeta)} \tag{2.3.4}$$

Variance Gamma distribution

Madan et al. (1998) introduced Variance Gamma process to model asset returns by time-changing a drifted Brownian motion using a gamma process with unit mean. We make a slight modification in the definition of the process by adding a drift term μ by considering the process $(Y(t))_{t\geq 0}$ defined for every nonnegative real number t by $Y(t) = \theta t + \sigma B(t)$, where θ is a constant, σ is positive real number and B a standard Brownian motion. Let $(\gamma(t,\nu))_{t\geq 0}$ be a unit mean gamma process independent of Y. The process $(X(t))_{t\geq 0}$ defined for every $t \geq 0$ by

$$X(t;\theta,\sigma,\nu,\mu) = \mu t + \theta\gamma(t,\nu) + \sigma W(\gamma(t,\nu)), \tag{2.3.5}$$

is the so-called the Variance Gamma process with drift.

Note that the parameters determining the Variance Gamma process are the drift term μ, volatility σ of the Brownian motion, θ and the variance ν of the gamma process. The variance gamma process does not have any continuous component, thus it is a pure jump process.

When $\delta = 0$, using the fact that $K_\lambda(x) \sim \Gamma(\lambda)2^{\lambda-1}x^{-\lambda}$, as $x \to 0$, we obtain

$$f_{GH}(x;\alpha,\beta,\mu,\lambda) = \frac{(\alpha^2-\beta^2)^{\lambda/2}|x-\mu|^{\lambda-1/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}2^{\lambda-1}}e^{\beta(x-\mu)}K_{\lambda-1/2}(\alpha|x-\mu|).$$

If we take $\beta = \frac{\theta}{\sigma^2}$, $\lambda = 1/\nu$ and $\alpha = \sqrt{\beta^2 + (2\lambda/\sigma^2)}$, we obtain the probability density function of the variance gamma process. The characteristic function of the drifted variance gampa distribution is given by

$$\phi_{VG}(u) = e^{iu\mu}\left(1 - i\theta\nu u + \frac{\sigma^2\nu}{2}u^2\right)^{-1/\nu}. \tag{2.3.6}$$

The mean and variance of the Variance Gamma distributed random variate X are given respectively by $\mathbb{E}(X) = \mu + \theta$ and $Var(X) = \sigma^2 + \nu\theta^2$.

2.3.2 Data Description

A sample of eight listed companies (*four from each exchange*) were collected from Nairobi Stock Exchange (NSE) and the rest was downloaded from yahoo finance for Monterial exchange,i.e. Barclay's Bank of Kenya (**BBK**), Kenya Commercial Bank (**KCB**), East Africa Breweries limited (**EABL**), Kenya Airways (**KQ**) for NSE and Angnico Eagle Mines limited (**AEM**), Alcan Inc (**AL**), Royal Bank of Cananda (**RY**)and Sun Life Financial (**SLF**) for Monterial.

Daily adjusted closing prices as from January 4, 2000 to August 30, 2005, were used to determine daily log returns. Let $P_j := P(t_j)$ be the price on day $t_j, j = 0, 1, 2, ..., n-1$. Sample increments of log returns is defined by $x_j = \log P_j - \log P_{j-1}, \quad j = 1, 2, ..., n-1$. Moreover, we use the historical daily record of S&P500 index over the period January 03,1990 to November 21,2008 to fit Variance Gamma model and there after use the estimated parameter to price European call option using Fast Fourier transform.

We make a simplifying assumption that the sequence $(x_j)_{0<j\leq n-1}$ is independent and identically distributed to model stock prices by continuous time Lévy process $S = (S_t)_{t\geq 0}$. The discrete financial time series data correspond to the value of the continuous time process S at equidistant integer points. Therefore, to obtain a discrete time series from our continuous model, we shall consider

$$S_n = S_0 \exp\left(\sum_{j=1}^{n} x_j\right).$$

2.3.3 Parameter Estimation and Empirical results

Assume that the daily log returns $x_1, x_2, ..., x_n$ are i.i.d.s and the parameters to estimate are denoted by $\Theta = (\lambda, \bar{\alpha}, \mu, \sigma, \gamma)$. We maximize

$$\ln L_X(\Theta; x_1, x_2, ..., x_n) = \sum_{i=1}^{n} \ln f_X(x_i; \Theta).$$

The parameters of the mixture are found by introducing EM algorithm concept. It is assumed that log returns are the observed data and the incomplete data is generated by latent mixing variable $w_1, w_2, ..., w_n$ which is GIG distributed. Thus the joint log likelihood function

$$\ln L_{XW}(\Theta; x_1, ..., x_n, w_1, ..., w_n) = \sum_{i=n}^{n} \ln f_{X|W}(X_i|w_i; \mu, \sigma, \gamma) + \sum_{i=1}^{n} \ln f_W(w_i; \lambda, \bar{\alpha}) \tag{2.3.7}$$

Calculate the conditional expectation (E-step) of the joint log likelihood given the data and the current estimates of the parameters followed by M-step Maximizing the objective function with respect to Θ to obtain an updated estimate. Repeating the E step and M step sequentially will obtain the maximum likelihood estimation of the parameter set Θ (see Madan and Seneta (1989) and Hu (2005)).

E-step: Get the objective function

$$Q(\Theta; \Theta^{[k]}) = \mathbb{E}(\log L_W(\Theta; x_1, ..., x_n, w_1, ..., w_n)|x_1, ..., x_n; \Theta^{[k]})$$

by conditioning (2.3.7) given the data $x_1, ..., x_n$.

M-step: Maximize the objective function with respect to Θ to obtain the next set of estimates $\Theta^{[k+1]}$. Repeated steps yields maximum likelihood estimation of the parameter set Θ (see McNeil et al. (2005) for elegant presentation of EM and MCECM estimation procedure.)

Empirical Results

Variance gamma subclass

Estimated parameters for the variance gamma distribution are set in Table 2.1. The variance gamma QQ-plots are in Figures 2.3 and 2.4. From Table 2.1, it is clear from both markets that the distributions are skewed. The value of θ determine the nature of skewness for the Variance Gamma distribution. Most important, though we notice that both markets have different stochastic time. The value of ν in developed market lies between $(0.52, 0.87)$ which implies higher frequency of business activity (time) compared to emerging market $(1.5, 2.6)$ relatively lower business activity in NSE in comparison to Monterial. Note that, $X_t = \mu + \theta T_t + \sigma W(T_t)$, where the activity time $\{T_t\}$, is a positive increasing random process with stationary differences. The parameter $1/\nu$ would measure the intensity of occurrence in business time scale.

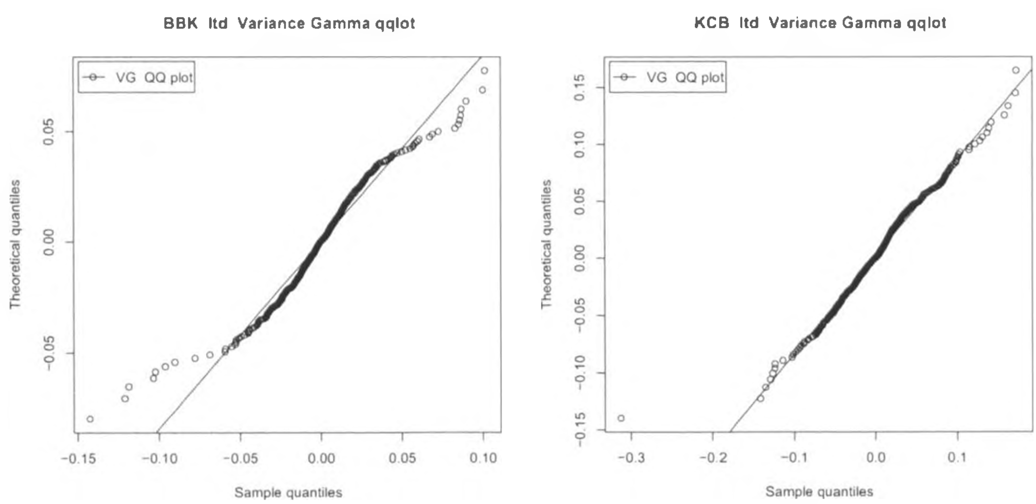

Figure 2.3: Variance Gamma Q-Q plots for **BBK** and **KCB**

Table 2.1: MLE parameters of Variance Gamma density fitted Monterial and NSE data

Co.	σ	ν	θ	μ	LLF
AEM	3.0731e-02	0.569706	4.0296e-03	−3.6020e-03	2960.673
AL	2.0313e-02	0.532999	1.2492e-03	−1.2469e-03	3547.072
RY	1.3709e-02	0.582838	7.0330e-04	2.9993e-04	4110.377
SLF	1.8965e-02	0.873982	1.1115e-03	−2.6059e-06	3543.324
BBK	1.5454e-02	1.543812	−4.1409e-04	1.00995e-03	4075.327
KCB	2.7678e-02	2.150426	2.0188e-03	4.11348e-10	3227.641
KQ	2.6047e-02	2.571561	6.9144e-04	3.8438e-10	3722.603
EABL	2.1660e-02	1.705037	1.68104e-03	1.01118e-03	3542.802

Hyperbolic subclass

From Table 2.2, we note that, the values of β are greater than zero from both markets. We observe that for all the samples from NSE, the value of the scale parameter δ is very close to zero.

Generalized Hyperbolic distribution

Table 2.3 gives the Generalized Hyperbolic parameters estimated. The QQ plots are given in Figure 2.7 and Figure 2.8. In most cases, the log likelihood value for the fitted Generalized Hyperbolic distribution is higher than those of Hyperbolic distribution and Variance Gamma. Therefore, the Generalized Hyperbolic distribution model which is a five parameters model seems to fit better the returns of asset prices from both markets than models with less parameters like Hyperbolic, Variance Gamma and gaussian, result in agreement with Figure 2.5.

2.4 Goodness of Fit Test and Frequency distribution

We analyze and compare the goodness of fit of the generalized hyperbolic distributions and some of their subclasses, using Kolmogorov distance and frequency distribution.

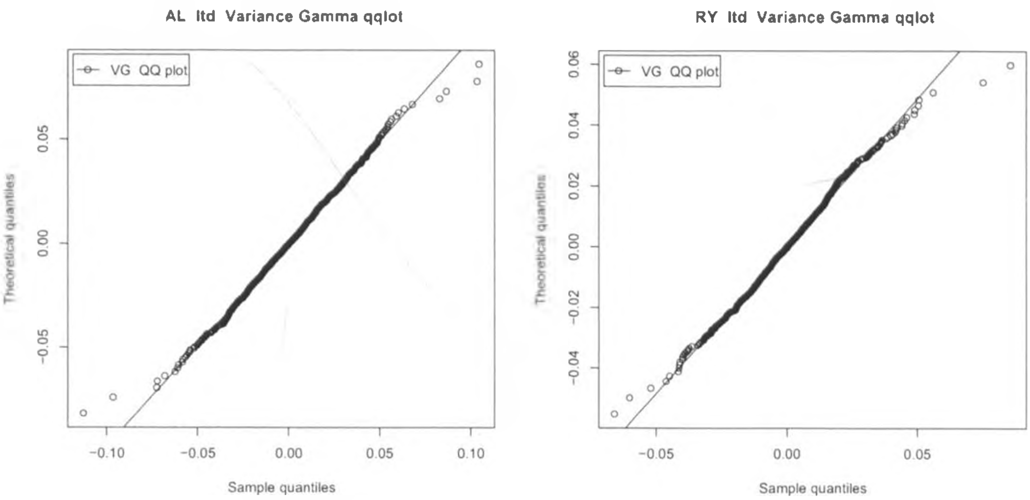

Figure 2.4: Variance Gamma Q-Q plots for **AL** and **RY**

Kolmogorov Distance

Kolmogorov distance is the supremum over the absolute differences between two density functions. It's expression is given by:

$$KS = \sup_{x \in \mathbf{R}} |F_{emp}(x) - F_{est}(x)| \tag{2.4.1}$$

where F_{emp} and F_{est} are the empirical and the estimated CDFs respectively. Kolmogorov distance is used because it pays more attention to the tails of distributions see Györfi et al. (1996) and Prause (1999). The Kolmogorov distances of the normal, the Hyperbolic, the Variance Gamma and the Generalized Hyperbolic distributions are presented in Table 2.4.
We make inference based on the p-value being the measure of how much evidence one can have against the null hypothesis. The general rule is that a small p-value is evidence against the null hypothesis, while a large p-value means no evidence against the null hypothesis. At 1% level of significance, we accept the null hypothesis that the data fits the three models for all the log returns from Montreal exchange. Generalized hyperbolic appears to fit well the data in both markets. **BBK** and **EABL** of the Nairobi stock Exchange fit VG at 1% level of confidence. Moreover, of all the four stocks from **NSE** considered Hyperbolic distribution is rejected .

Frequency distributions

Frequency distributions as in Eberlein and Keller (1995), in each column of the table, the relative frequencies of the returns in the intervals $(-k\sigma, k\sigma)$ i.e $P(|x| < k\sigma), k = 1, 2, 3, 4, 5$. are compared with the probabilities of the fitted distributions. We observe from table 2.5 and table 2.6 that among the fitted distributions, the probabilities of the generalized hyperbolic distribution are closer to the empirical probabilities in most cases.

Table 2.2: MLE parameters of Hyperbolic distribution fitted for Montréal and NSE data

Co.	α	β	δ	μ	LLF
AEM	53.4421	4.30726	1.837152e-02	-3.634096e-03	2963.292
AL	81.7215	3.00987	1.310892e-02	-1.241524e-03	3548.007
RY	118.0387	3.43599	7.72583e-03	3.571136e-04	4112.330
SLF	78.6708	3.53894	4.845935e-03	-1.421884e-04	3545.441
BBK	96.29101	-1.628226	9.984908e-08	9.5041e-08	4054.704
KCB	52.47160	0.457439	4.733096e-07	3.25479e-04	3171.505
KQ	74.35194	2.753337	2.399309e-07	5.68611e-4	3688.820
EABL	69.75716	2.761331	3.247389e-07	5.840876e-04	3509.165

2.5 Option pricing

Many of the option pricing models assume that a stock price process $\{S_t; 0 \leq t \leq T\}$ follows an exponential (geometric) Lévy process: $S_t = S_0 e^{L_t}$ where $\{L_t; 0 \leq t \leq T\}$ is a Levy process. In this section, we give the definition of Lévy processes and some related results which can be found in Applebaum (2004).

2.5.1 Lévy processes

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Classic Black and Scholes (1973) model chooses a Brownian motion with drift process which is the only continuous Levy process as their choice of a (risk-neutral) Levy process.

$$S_T = S_0 e^{L_t} \quad (2.5.1)$$

$$= S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma B_t} \quad (2.5.2)$$

where $\{B_t; 0 \leq t \leq T\}$ is a standard Brownian motion process. Moreover, the European call price can simply be calculated as the discounted value of the expected terminal payoff under

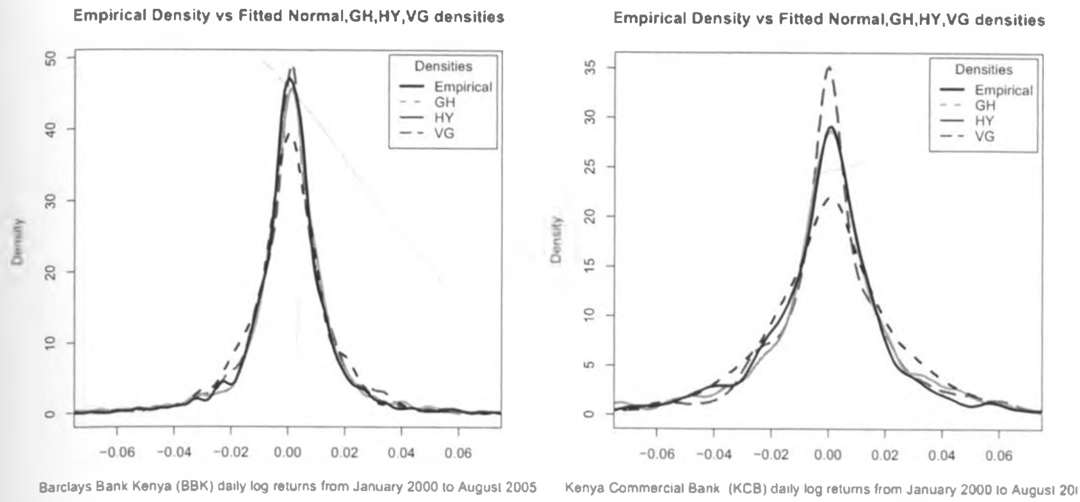

Figure 2.5: Kernel density vs Generalized Hyperbolic,Hyperbolic and Variance Gamma estimated maximum likelihood estimates for Barclays Bank Kenya(**BBK**) and Kenya Commercial Bank (**KCB**)

the risk-neutral measure $\mathbb{Q}$.

$$\begin{aligned}
C(S_0,T) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}(S_T-K)^+|\mathfrak{F}_0 && (2.5.3)\\
&= e^{-rT}\int_{-\infty}^{+\infty}(S_T-K)^+\mathbb{Q}(S_T|\mathfrak{F}_0)dS_T && (2.5.4)\\
&= e^{-rT}\int_{K}^{\infty}(S_T-K)\frac{1}{S_T\sqrt{2\pi\sigma^2T}}\exp\left(-\frac{\{\ln S_T-(\ln S_0+(r-\frac{1}{2}\sigma^2)T)\}}{2\sigma^2T}\right)dS_T
\end{aligned}$$

This implies that as far as a conditional risk-neutral density of terminal stock price is given, plain vanilla option pricing reduces to a closed form solution, however, for general exponential Lévy models $\mathbb{Q}(S_T|\mathfrak{F}_0)$ may not be expressed using special functions. Therefore to price plain vanilla options we use characteristic functions of general exponential Levy processes.

For simplicity, without loss of generality, from (2.5.3) we use change of variable technique from S_T to $\ln S_T$

$$\begin{aligned}
C(S_0,T) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}(S_T-K)^+|\mathfrak{F}_0 && (2.5.5)\\
&= e^{-rT}\int_{-\infty}^{+\infty}(S_T-K)^+\mathbb{Q}(\ln S_T|\mathfrak{F}_0)d\ln S_T && (2.5.6)\\
&= e^{-rT}\int_{\ln K}^{+\infty}(e^{\ln S_T}-e^{\ln K})^+\mathbb{Q}(\ln S_T|\mathfrak{F}_0)d\ln S_T
\end{aligned}$$

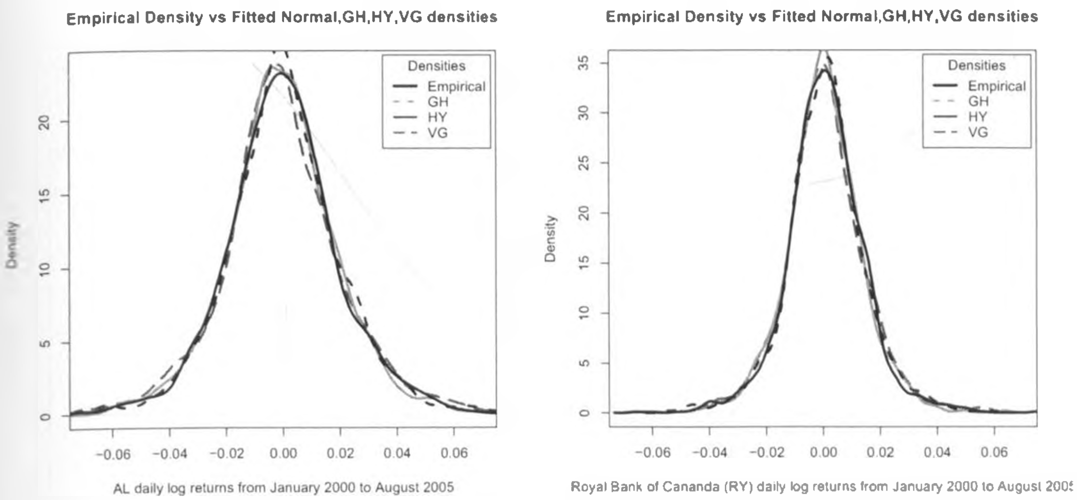

Figure 2.6: Kernel density vs Generalized Hyperbolic,Hyperbolic and Variance Gamma estimated maximum likelihood estimates for **AL** and **RY**

The characteristic function of BSM73 log terminal stock price i.e. $\ln S_T$ is easily obtained as

$$\begin{aligned}
\phi_{\ln S_T}(u) &= \int_{-\infty}^{\infty} e^{iu \ln S_T} \mathbb{Q}(\ln S_T) d \ln S_T &(2.5.7)\\
&= \exp\left(i \left\{ \ln S_0 + (r - \frac{1}{2}\sigma^2)T \right\} u - \frac{(\sigma^2 T u^2)}{2} \right) &(2.5.8)\\
\text{where } \mathbb{Q}(\ln S_T) &= \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left(-\frac{\{\ln S_T - (\ln S_0 + (r - \frac{1}{2}\sigma^2)T)\}^2}{2\sigma^2 T} \right) &(2.5.9)
\end{aligned}$$

2.5.2 Fast Fourier transform (FFT)method

The fast Fourier transform is a powerful computational method which was first introduced by Walker (1996). In this subsection, we follow the work of Carr and Madan (1998) on option pricing using the Fourier transform when the characteristic function is explicitly known under the risk-neutral probability measure. European call option with time to maturity T and strike price K is given by

$$C(K,T) = \frac{e^{-\alpha \ln K}}{\pi} \int_0^{+\infty} e^{-iu \ln K} \varrho(u) du \qquad (2.5.10)$$

where

$$\begin{aligned}
\varrho(u) &= e^{-rT} \frac{\mathbb{E}^{\mathbb{Q}} \exp(i[u - (\alpha+1)i \ln S_T])}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} &(2.5.11)\\
&= e^{-rT} \frac{\phi(u - (\alpha+1)i)}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} &(2.5.12)
\end{aligned}$$

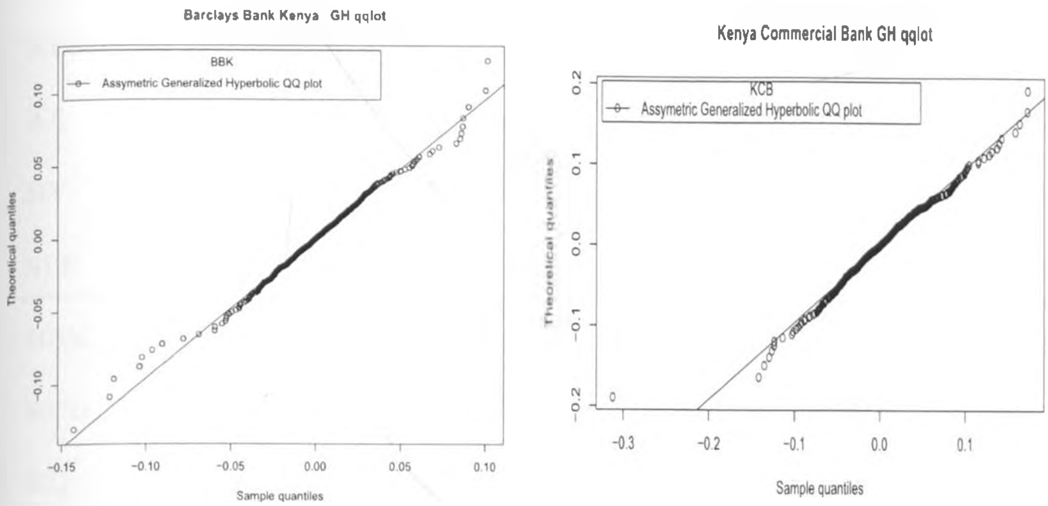

Figure 2.7: **BBK** and **KCB** Generalized Hyperbolic QQplot

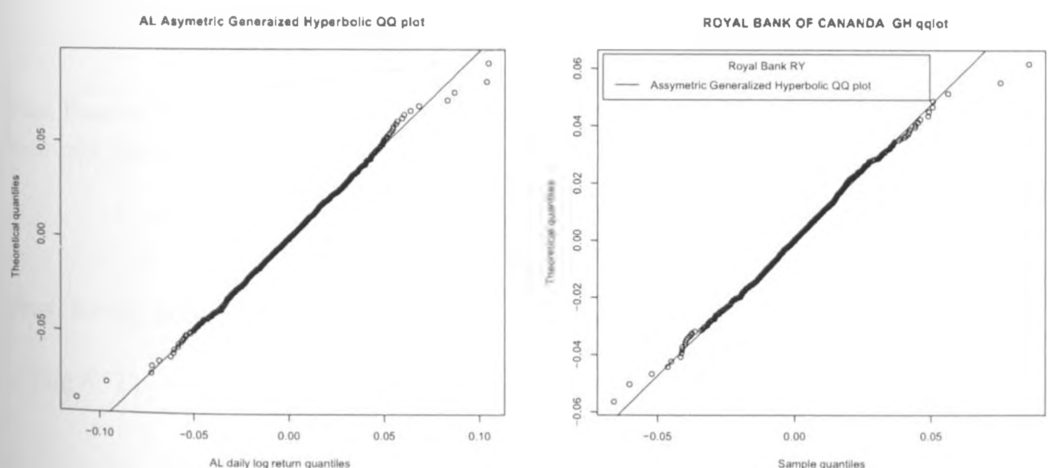

Figure 2.8: **RY** and **AL** Generalized Hyperbolic QQplot

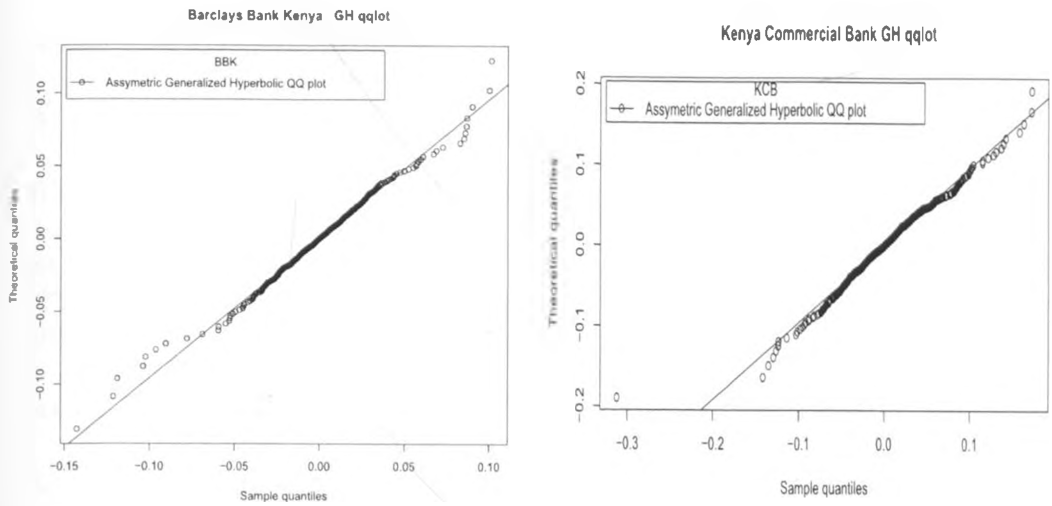

Figure 2.7: **BBK** and **KCB** Generalized Hyperbolic QQplot

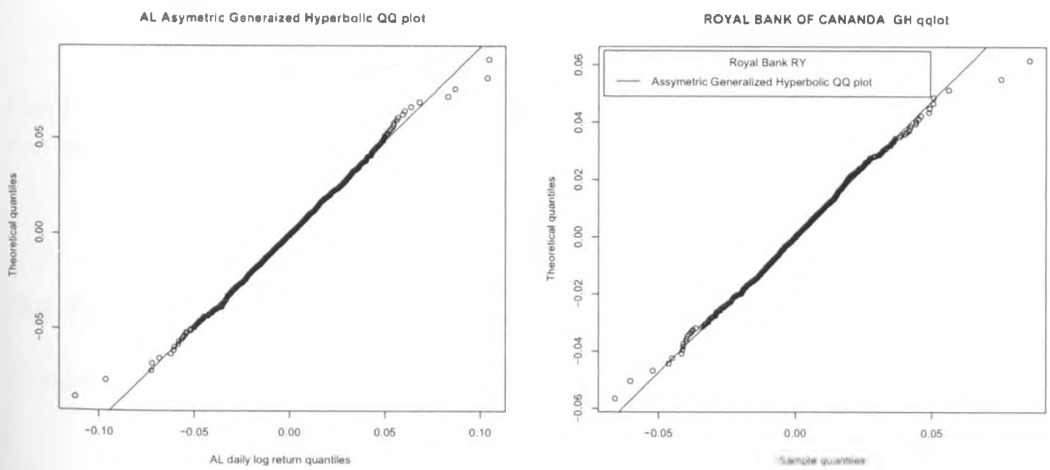

Figure 2.8: **RY** and **AL** Generalized Hyperbolic QQplot

Table 2.3: Generalized Hyperbolic parameter estimates for Montréal and NSE data

Co.	λ	α	β	δ	μ	LogLik
AEM	−1.43421e-00	22.45251	3.182777	0.04385096	−3.10649e-03	2968.365
AL	−4.47653e-02	66.30114	3.247621	0.0211068	−1.34398e-03	3548.421
RY	−3.90192e-01	90.85274	3.480525	0.01536024	4.77593e-04	4113.503
SLF	−9.69996e-01	24.72129	1.517417	0.01800045	3.79528e-04	3555.739
BBK	−5.81346e-01	20.63537	−1.00749	7.77159e-03	9.05012e-04	4109.781
KCB	3.75006e-02	22.39679	9.50887e-02	5.91669e-03	5.78596e-04	3222.247
KQ	−4.91616e-01	6.32854	1.66513	7.37481e-03	1.01943e-03	3764.946
EABL	−1.75863e-01	23.86991	4.13654e-01	6.52984e-03	1.01915e-03	3559.446

An approximation for the integral in the Carr-Madan formula

$$C = e^{-\alpha \ln K}\frac{1}{\pi}\int_0^\infty e^{-iu\ln K}\varrho(u)du \quad (2.5.13)$$

$$= e^{-\alpha \ln K}\frac{1}{\pi}\int_0^\infty e^{-iu\ln K}\frac{e^{-rT}\phi_T(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}du \quad (2.5.14)$$

$$\text{where, } \phi_T(\varpi) = \exp\left[i\left\{\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)T\right\}\varpi - \frac{\sigma^2 T}{2}\varpi^2\right]. \quad (2.5.15)$$

Fast Fourier Transform(FFT) is an efficient algorithm for computing the following transformation of a vector $a_n, n = 1, ..., N$ into a vector $b_n, n = 1, ..., N$:

$$b_n = \sum_{j=1}^{N}\exp\left(-\frac{2\pi(j-1)(n-1)}{N}\right)a_j \quad (2.5.16)$$

thus, for the integral in the Car-Madan formula Carr and Madan (1998)

$$C(\log K, T) = \exp(-\alpha\log K)\frac{1}{\pi}\int_0^\infty \exp(-iv\log K)\varrho(v)dv \quad (2.5.17)$$

$$\approx e^{-\alpha\log K_n}\frac{1}{\pi}\sum_{j=1}^{N}e^{-iv_j\log K}\frac{e^{-rT}\phi_T(v_j-(\alpha+1)i)}{\alpha^2+\alpha-v_j^2+i(2\alpha+1)v_j}\frac{\eta}{3}(3+(-1)^j) \quad (2.5.18)$$

$$= e^{-\alpha\log K_n}\frac{1}{\pi}\sum_{j=1}^{N}e^{-iv_j\log K_n}\frac{e^{-rT}\phi_T(v_j-(\alpha+1)i)}{\alpha^2+\alpha-v_j^2+i(2\alpha+1)v_j}\frac{\eta}{3}(3+(-1)^j) \quad (2.5.19)$$

$$= e^{-\alpha\log K_n}\frac{1}{\pi}\sum_{j=1}^{N}e^{-i\eta\lambda(n-1)(j-1)}e^{-ibv_j}\frac{e^{-rT}\phi_T(v_j-(\alpha+1)i)}{\alpha^2+\alpha-v_j^2+i(2\alpha+1)v_j}\frac{\eta}{3}(3+(-1)^j)$$

where, $\log K_n = -\log b + \lambda(n-1), \quad v_j = \eta(j-1), \quad \eta\lambda = \frac{2\pi}{N}, \quad n, j = 1, ..., N.$

Table 2.4: Kolmogorov distances

	Normal		Generalized H		VGamma		Hyperbolic	
	K_{Dist}	p-value	K_{Dist}	p-value	K_{Dist}	p-value	K_{Dist}	p-value
AEM	0.0706	1.483e-06	0.0423	1.243e-02	0.0311	1.301e-01	0.0289	1.863e-01
AL	0.0459	5.143e-03	0.0261	2.888e-01	0.0282	2.088e-01	0.0233	4.258e-01
RY	0.0543	4.641e-04	0.0212	5.491e-01	0.0219	5.064e-01	0.0191	6.823e-01
SLF	0.0845	7.252e-09	0.0309	1.496e-01	0.0265	2.968e-01	0.0198	6.558e-01
BBK	0.1318	2.2e-16	0.0298	1.640e-01	0.0333	8.733e-02	0.0376	3.373e-02
KCB	0.1387	2.2e-16	0.0407	1.922e-02	0.0536	6.437e-04	0.0508	1.438e-03
KQ	0.1595	2.2e-16	0.0262	2.864e-01	0.0532	6.892e-04	0.0631	2.661e-05
EABL	0.1577	2.2e-16	0.0247	3.708e-01	0.0298	1.736e-01	0.0509	1.614e-03

In VG world the characteristic function of the log stock price $\log S_t$.

$$S_T = S_0 \exp((r+\omega)T + X_T), \quad X \sim VG(C,G,M) \tag{2.5.20}$$

$$\log S_T = \log(S_0) + T(r+\omega) + X_T \tag{2.5.21}$$

$$\log S_T \sim \log(S_0) + T(r+\omega) + VG(CT,G,M) \tag{2.5.22}$$

$$\phi_{VG}(u;T) = \exp(iu(\log S_0 + (r+\omega)T))\left(\frac{GM}{GM + (M-G)iu + u^2}\right)^{CT} \tag{2.5.23}$$

Note that the result of figure 2.9, the following values were used

$$N = 4096, \quad \lambda := (1/N)\log(2000/400) = 3.696601 \times 10^{-4},$$
$$\eta = \frac{2\pi}{\lambda N} = 4.149706$$
$$v_j = \eta(0:(N-1))$$

2.6 Concluding Remarks

In summary, the role of building a Levy process amounts to measuring returns in relation to the level of activity and news instead of calendar time.

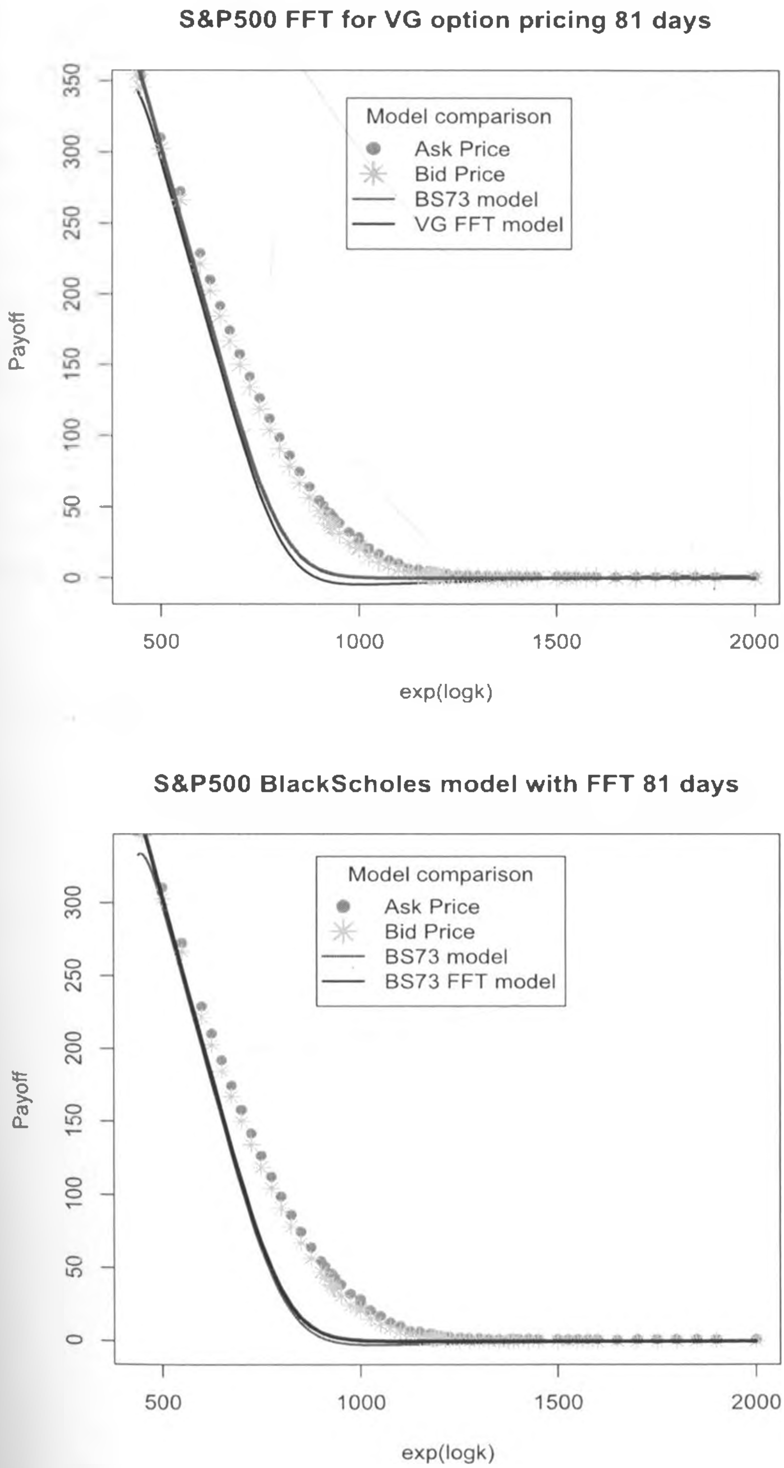

Figure 2.9: S&P500 Fast Fourier Transform Black Scholes and Variance Gamma option pricing 81 days.

In the preceeding sections we have tried to present, in some detail, Lévy process to model log returns of the terminal stock price S_T, which allows us to capture more stylized facts from real data. We fitted four asset prices coming from the Montréal exchange and four asset prices coming from the Nairobi Stock Exchange to three models based on generalized hyperbolic distribution and its subclasses. Empirical evidence suggest that the underlying distribution is heavy tailed in both marktets and not normal distribution. About 83-84% of empirical data from Nairobi Stock Exchange lies between the first standard deviation compared to 72-75 % from Monterial Exchange instead of 68% if log returns were to be normally distributed.

It is clear from our study that emerging market and developed market too are affected by shocks that produce diversity of jumps. Generalized Hyperbolic distribution, which is a five parameter model seems to explain better the behavior of empirical distribution of log returns from both markets.

One can argue that, from the estimated Kolmogorov distance in addition to calculated p-value, the Hyperbolic distribution is inferior when modeling returns from emerging market as opposed to developed market. The data used was not large enough to make a generalized observation. Empirically, both markets are not operating on the same business time.

It is worthy noting that implementing Fast Fourier Transform (FFT) method to price options, there is no significant change (say, interms of price improvement at the money for all options at different maturities) in comparison to Black-Scholes model.

Table 2.5: Frequency distribution of log returns from Monterial Exchange

Co.	Density	$\lvert x \rvert < \sigma$	$< 2\sigma$	$< 3\sigma$	$< 4\sigma$	$< 5\sigma$	$\lvert x \rvert > 5\sigma$
AEM	EMP	0.7558	0.9569	0.9894	0.9950	0.9971	1.07268e-01
	NM	0.6826	0.9544	0.9973	0.9999	0.9999	5.7334e-07
	HY	0.7493	0.9499	0.9902	0.9981	0.9996	3.7232e-04
	VG	0.7465	0.9497	0.9907	0.9983	0.9997	2.8439e-04
	gHY	0.7560	0.9468	0.9869	0.9962	0.9988	1.1885e-03
AL	EMP	0.7374	0.9484	0.9915	0.9957	0.9978	3.0345e-02
	NM	0.6826	0.9544	0.9972	0.9999	0.9999	5.7406e-07
	HY	0.7376	0.9462	0.9894	0.9979	0.9996	3.9088e-04
	VG	0.7348	0.9461	0.9901	0.9982	0.9997	2.8777e-04
	GH	0.7418	0.9479	0.9893	0.9977	0.9996	4.9973e-04
RY	EMP	0.7459	0.9470	0.9872	0.9964	0.9985	7.0571e-03
	NM	0.6816	0.9540	0.9972	0.9999	0.9999	6.0508e-07
	HY	0.7426	0.9464	0.9892	0.9978	0.9995	4.2351e-04
	VG	0.7395	0.9461	0.9897	0.9981	0.9996	3.2644e-04
	GH	0.7518	0.9516	0.9901	0.9978	0.9995	4.7676e-04
SLF	EMP	0.7972	0.9456	0.9779	0.9955	0.9977	3.6737e-02
	NM	0.6821	0.9542	0.9972	0.9999	0.9999	5.8953e-07
	HY	0.7647	0.9487	0.9888	0.9975	0.9994	5.3066e-04
	VG	0.7627	0.9469	0.9883	0.9974	0.9994	5.5580e-04
	GH	0.7793	0.9447	0.9820	0.9931	0.9971	2.8843e-03

Table 2.6: Frequency distribution of log returns from Nairobi Stock Exchange

Co.	Density	$\lvert x\rvert < \sigma$	$< 2\sigma$	$< 3\sigma$	$< 4\sigma$	$< 5\sigma$	$\lvert x\rvert > 5\sigma$
BBK	EMP	0.8454	0.9553	0.9787	0.9879	0.9929	2.5513e-02
	NM	0.6825	0.9544	0.9972	0.9999	0.9999	5.7815e-07
	HY	0.8153	0.9660	0.9937	0.9988	0.9997	2.1305e-04
	VG	0.8141	0.9565	0.9892	0.9972	0.9993	6.9737e-04
	GH	0.8410	0.9501	0.9790	0.9899	0.9947	5.2044e-03
EABL	EMP	0.8364	0.9534	0.9811	0.9920	0.9934	5.0145e-02
	NM	0.6818	0.9541	0.9972	0.9999	0.9999	5.9781e-07
	HY	0.8204	0.9676	0.9941	0.9989	0.9998	1.9446e-04
	VG	0.8169	0.9551	0.9882	0.9967	0.9991	8.9893e-04
	GH	0.8324	0.9465	0.9792	0.9912	0.9960	3.9322e-03
KCB	EMP	0.8255	0.9406	0.9771	0.9899	0.9964	9.5067e-02
	NM	0.6826	0.9544	0.9972	0.9999	0.9999	5.7349e-07
	HY	0.8094	0.9636	0.9930	0.9986	0.9997	2.5145e-04
	VG	0.8272	0.9543	0.9868	0.9960	0.9987	1.2368e-03
	GH	0.8191	0.9424	0.9787	0.9915	0.9965	3.4884e-03
KQ	EMP	0.8454	0.9376	0.9666	0.9900	0.9971	5.8114e-02
	NM	0.6819	0.9541	0.9972	0.9999	0.9999	5.9651e-07
	HY	0.8201	0.9675	0.9941	0.9989	0.9998	1.9516e-04
	VG	0.7759	0.9208	0.9695	0.9878	0.9949	5.0005e-03
	GH	0.8298	0.9270	0.9588	0.9739	0.9822	1.7734e-02

Table 2.7: Comparison of option prices on **RY** and **BBK** for 10 and 20 days to maturity under the mean-reverting martingale measure.

Days	RY					BBK			
T=10	K	BS	VG	HY	GH	BS	VG	HY	GH
	94	6.3277	7.1861	7.1852	7.2718	6.5227	6.9045	6.8616	7.0669
	97	3.7751	4.4870	4.4861	4.5461	4.1405	4.3255	4.2549	4.5159
	98.5	2.7204	3.3210	3.3201	3.3613	3.1503	3.2273	3.1469	3.4189
	100	1.8570	2.339	2.3329	2.3554	2.3165	2.3020	2.2172	2.4894
	101.5	1.1950	1.5508	1.5501	1.5571	1.6428	1.5674	1.4840	1.7495
	103	0.7222	0.9722	0.9720	0.9690	1.1219	1.0200	0.9431	1.1966
	106	0.2163	0.3216	0.3223	0.3144	0.4652	0.3828	0.3289	0.5380
T=20	K	BS	VG	HY	GH	BS	VG	HY	GH
	94	6.8246	8.4537	8.4514	8.6169	7.2296	7.9102	7.8297	8.1928
	97	4.5005	5.8783	5.8764	6.0037	5.0672	5.5029	5.3912	5.8273
	98.5	3.5226	4.74090	4.7393	4.8432	4.1436	4.4566	4.3333	4.7922
	100	2.6843	3.7302	3.7289	3.8087	3.3328	3.5336	3.4033	3.8733
	101.5	1.9889	2.8592	2.8582	2.9151	2.6353	2.7413	2.6092	3.0779
	103	1.4312	2.1327	2.1321	2.1690	2.0479	2.0800	1.9515	2.4074
	106	0.6772	1.0914	1.0916	1.1010	1.1726	1.1216	1.0132	1.4140

Chapter 3

On evolution Dynamics and Equity Market Risk in Developed and Emerging Markets

Evolution dynamics that govern developed and emerging stock markets daily index log returns are investigated in view of computing Value at Risk. AR-APARCH models conditioned on student t and Gaussian distribution, are used to filter first and second moment serial correlation of log returns. The i.i.d. white noise residuals are calibrated using Generalized Hyperbolic distribution. We identify appropriate models for estimating and forecasting daily volatility for four stock indices, ***SP500, DAX, MASI*** *and* ***NSE20****. Estimated parameters of the proposed densities of residuals, are used to calculate Value at Risk in all markets. Univariate daily log returns decompose into three components: (semi martingale) object analogous to drift (ARMA filter) time dependent, (GARCH filter) similar to Brownian part and jump density of Lévy increments).*

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3.1 Introduction

Recent financial disasters have emphasized the need for accurate risk measures for financial institutions. The Value at risk (VaR), has established itself as the most prominent measure

of market risk and an essential management tool in investment decisions. Owing to Basle Committee's (1995,1996) internal model approach, which allows banks to implement in-house VaR models for calculating capital requirements, the number of methods continues to increase (see Dowd and Blake (2006) and Hartz et al. (2006)).

Despite VaR's conceptual simplicity its measuring is a very challenging statistical problem and none of the methodologies developed so far gives a satisfying solution to the problem (see Sebastien (2004), Engle and Manganelli (2004) McMillan and Speight (2007) and references therein). The existing models for VaR estimation are classified into three broad categories namely Parametric (Riskmetrics, GARCH); Nonparametric (Historical simulation and the Hybrid Model) and semi parametric (Extreme Value Theory, Quasi-Maximum likelihood GARCH). We follow the parametric approach in developing our model. Before we embark on modeling the problem, we state some of the stylized facts of financial times series.

Stylized Facts

(i) The time series of log return of share price $S = (S_t)_{t\in\mathbb{Z}}$ process and other basic financial instruments are not stationary, but instead possess at least a local trend.

(ii) We define the returns of the index for given time interval Δt eg one day as $X_t = \log S_t - \log S_{t-\Delta t}$. It is well noted that as we progressively increase the interval of the returns by moving from daily to weekly, monthly, quarterly and yearly data, the volatility phenomena decreases and log returns tend to be i.i.d. and less heavy tailed. Without loss of generality we take $\Delta t = 1$ *day* in all our analysis.

(iii) That log returns X_t have a leptokurtic distribution, i.e. empirically estimated kurtosis is most cases grater than 3.

It is quite evident from Figure 3.1 that daily log returns and the square of filtered returns, are not i.i.d. they show strong correlation of second moment, see for example McNeil et al. (2005) when more than 5% of the estimated correlations lie outside the dashed blue lines. We observe that extreme returns seem to appear in clusters. A good candidate for modeling financial time series should therefore represent the properties of stochastic processes and stylized facts.

To capture these facts, ARCH models introduced by Engle (1982) and generalized as GARCH by Bollerslev (1986) are widely used in financial econometrics. It is worthy noting that volatility is modeled as the conditional standard deviation of returns given historical information, i.e natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$. The presence of volatility clustering suggests that the conditional expected returns are consistently changing partly due to predictable component and market excitement.

To explain our specific concerns and contributions, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space. Define a process $X = (X_t)_{t\in[0,T]}$ such that $\{X_t\}_{t=0}^{T}$ denote the time series of portfolio returns and T denote the sample size. We want to find VaR_t such that $Pr[X_t < -VaR_t|\mathcal{F}_t] = \alpha$ where $\mathcal{F}_t$ is natural filtration. It is interesting to extract and find an appropriate model for computing VaR. Let $X_1, X_2, ..., X_T$ denote the observed returns at times $1, 2, ..., T$. The GARCH

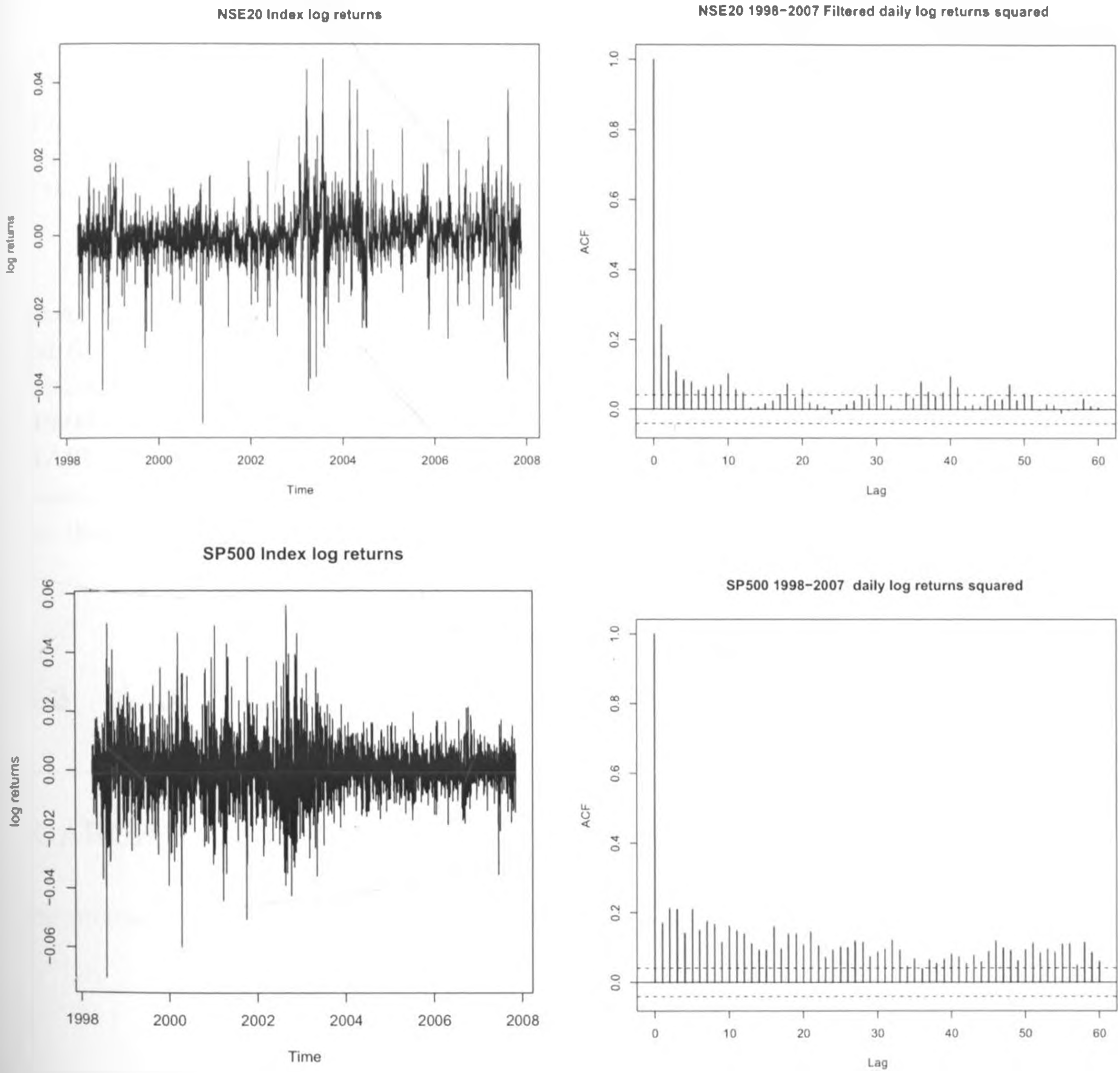


Figure 3.1: ACF of continuously compounded daily log returns and ACF of squared daily log returns respectively of NSE20 stock index and SP500 stock index as from 01 March 1998 to 10 July 2007 and 08 May1998 to 11 July 2007.

models currently in literature decompose X_t at time t into the form

$$X_t = \mu_t + \sigma_t Z_t, \quad Z_t \sim i.i.d.(0,1). \tag{3.1.1}$$

Here μ_t represents an expected (or structural) component and $\sigma_t Z_t$ represent an error (or random) component. It is assumed that μ_t and σ_t are at most functions of past returns $X_1, X_2, ..., X_{t-1}$ while $Z_1, Z_2, ..., Z_T$ are independent random variables with common cumulative distribution function say G. Given the observed returns $X_1, X_2, ..., X_T$ the expected components μ_{T+1} and the volatility σ_{T+1} of the next return are predictable but the innovation component Z_{T+1} is uncertain, which can be measured as risk due to innovations. As a result, the structure of the model simplifies to

$$X_t = \mu_t + \sigma_t Z_t + \xi_t \tag{3.1.2}$$

Where Z_t is the random component from a known or assumed cumulative distribution function and ξ_t is the random component related to Levy process. Maximum likelihood estimates of parameters of proposed densities ξ_t/σ_t are determined by fitting daily log return data from **SP500** index of **New York Stock Exchange**, **DAX** index of **Frankfurt Stock Exchange**, **MASI** of **Casablanca Exchange** and **NSE20** of **Nairobi Stock Exchange**. The ever overlooked and assumed nonsignificant non Gaussian residuals, are carefully studied and calibrated and thereafter used to compute VaR in comparison to classical Riskmetrics of JPMorgan.

3.2 ARCH type Models

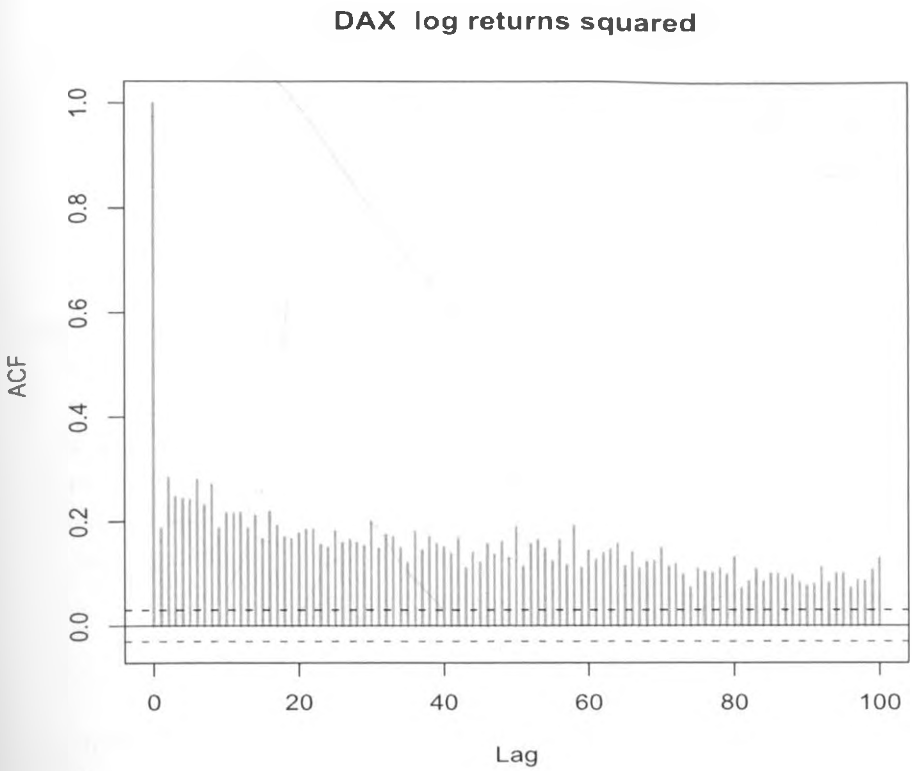
IGARCH

The process $X = (X_t)_{t\in[0,T]}$ is said to be IGARCH(1,1) process when

$$\begin{aligned} X_t &= \sigma_t z_t, \quad z_t \sim i.i.d.(0,1) \\ \sigma_t^2 &= \alpha(\sigma_{t-1}z_{t-1})^2 + \beta\sigma_{t-1}^2, \quad \alpha+\beta=1 \end{aligned} \tag{3.2.1}$$

The exponential Weighted Moving Average (EWMA) model used in $RiskMetrics^{TM}$(1995) in their VaR methodology for daily data is a special case of (3.2.1) with zero intercept and $\beta = 0.94$. Using $r_{t-1} = \sigma_{t-1}\varepsilon_{t-1}$ one can rewrite the volatility equation (3.2.1) as

$$\begin{aligned} \sigma_t^2 &= \alpha\sigma_{t-1}^2 + (1-\alpha)r_{t-1}^2 \quad 1 > \alpha > 0 \\ &= \alpha\sigma_{t-1}^2 + (1-\alpha)(\sigma_{t-1}^2\varepsilon_{t-1}^2) \\ &= \sigma_{t-1}^2 - \sigma_{t-1}^2 + (\alpha)\sigma_{t-1}^2 + (1-\alpha)\sigma_{t-1}^2\varepsilon_{t-1}^2 \\ &= \sigma_{t-1}^2 + (1-\alpha)\sigma_{t-1}^2(\varepsilon_{t-1}^2 - 1) \quad \forall t \\ \sigma_{t+i}^2 &= \sigma_{t+i-1}^2 + (1-\alpha)\sigma_{t+i-1}^2(\varepsilon_{t+i-1}^2 - 1) \quad \forall i = 1,2,...,k. \end{aligned} \tag{3.2.2}$$



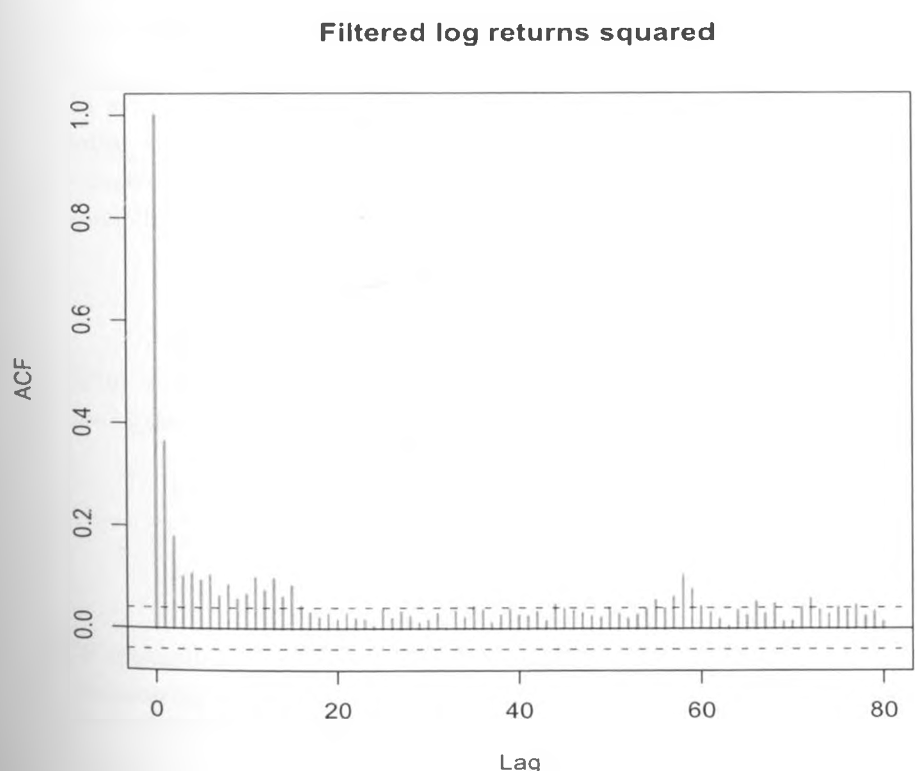


Figure 3.2: A comparison of inherent second order serial correlation in DAX index compared to MASI index of Casablanca stock exchange in Morocco. The intensity and nature of correlation are visibly different

3.2.1 The ARCH Framework of Estimating Volatility

The common methodology used for ARCH estimation is the maximum Likelihood. Under the assumption of i.i.d. innovations and for $D(z_t; v)$ denoting their density function, the Log-likelihood function of $\{z_t(\theta)\}$ for a sample of T observations is given by

$$\log L_T(\{z_t\}; \theta) = \sum_{t=1}^{T} \left[\ln(D(z_t(\theta); \nu)) - \frac{1}{2} \ln(\sigma_t^2(\theta)) \right] \tag{3.2.3}$$

where θ is the vector of the parameters that have to be estimated for the conditional mean and conditional variance and the density function, while $z_t(\theta) = \frac{\varepsilon_t(\theta)}{\sigma_t(\theta)}$. The maximum estimator θ maximizes (3.2.3). Under normality assumption the likelihood GARCH(p,q) process is conditionally Gaussian, in the sense that for given values of $\{z_s, s = t-1, t-2, ..., t-p\}$, z_t is Gaussian with known distribution. The likelihood of $z_{p+1}, ..., z_T$ conditional on $\{z_1, ..., z_p\}$ is formulated and maximized numerically to obtain maximum likelihood estimates. Let $\Theta = (\alpha_0, ..., \alpha_p, \beta_1, ..., \beta_q)$,

$$L_T(z_t, \Theta) = \prod_{t=p+1}^{T} \frac{1}{\sqrt{2\pi(\alpha_0 + \sum_{i=1}^{q} \sigma_{t-i}^2 + \sum_{j=1}^{p} \alpha_j z_{t-j})}} \exp\left\{ -\frac{z_t^2}{2\sigma_t^2} \right\} \tag{3.2.4}$$

where $\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \beta_i \sigma_{t-i}^2 + \sum_{j=1}^{p} \alpha_j (\sigma_{t-j} z_{t-j})^2$

$$\log L_T(\{z_t\}; \Theta) = -\frac{1}{2} \left[T \ln(2\pi) + \sum_{t=p+1}^{T} z_t^2 + \sum_{t=p+1}^{T} \ln(\sigma_t^2) \right] \tag{3.2.5}$$

while in some applications it is more appropriate to assume that z_t follows a heavy tailed distribution such as a standardized student t distribution. Let x_ν be a student t distribution with ν degrees of freedom. Then $Var(x_\nu) = \nu/(\nu-2)$ for $\nu > 2$ and use $z_t = x_\nu/\sqrt{\nu/\nu - 2}$. The probability density function of z_t is

$$f(z_t; v) = \frac{\Gamma(\nu+1)/2}{\Gamma(\nu/2)\sqrt{(\nu-2)\pi}} \left(1 + \frac{z_t^2}{\nu - 2} \right)^{-(\nu+1)/2}, \nu > 2 \tag{3.2.6}$$

where $\Gamma(v) = \int_0^\infty e^x x^{v-1} dx$ the gamma function and ν is the parameter which describes the thickness of the distribution tails. The maximum likelihood estimation for t distribution is

$$\begin{aligned} \log l_T(\{z_t\}; \theta) &= T \left[\ln \Gamma\left(\frac{\nu+1}{2}\right) - \ln \Gamma\left(\frac{\nu}{2}\right) - \frac{1}{2} \ln(\pi(\nu-2)) \right] \\ &-\frac{1}{2} \sum_{t=1}^{T} \left[\ln(\sigma_t^2) + (1+\nu) \ln\left(1 + \frac{z_t^2}{\nu-2}\right) \right] \end{aligned} \tag{3.2.7}$$

GARCH maximum optimization of mean corrected observations $\{z_t\}$, are estimated by numerically maximizing the likelihood of $\tilde{z}_t = z_t - \mu$ $z_{p+1}, ..., \tilde{z}_n$ conditional on the known values $z_1, ..., z_p$ and with assumed values 0 for each $\tilde{z}_t, t \leq 0$, and $\hat{\sigma}^2$ for each $\sigma_t, t \leq 0$ where $\hat{\sigma}^2$ is the sample variance of $\bar{z}_1, ..., \bar{z}_n$. We maximize

$$\begin{aligned} l(z_t, \theta) &= \prod_{t=p+1}^{T} \frac{1}{\sigma_t} g(\frac{\bar{z}_t}{\sigma_t}) \\ &= \prod_{t=p+1}^{T} \frac{\sqrt{\nu}}{\sigma_t \sqrt{\nu - 2}} t_\nu \left(\frac{\bar{z}_t \sqrt{\nu}}{\sigma_t \sqrt{\nu - 2}} \right) \\ &= \prod_{t=p+1}^{T} \frac{\sqrt{\nu} \left(\alpha_0 + \sum_{j=1}^{m} k_j + \sum_{j=1}^{n} \beta_j \sigma_{t-j}^{\delta} \right)^{-\delta}}{\sqrt{\nu - 2}} \\ &\times \quad t_\nu \left(\frac{\bar{z}_t \sqrt{\nu}}{\left(\alpha_0 + \sum_{j=1}^{m} k_j + \sum_{j=1}^{n} \beta_j \sigma_{t-j}^{\delta} \right)^{\delta} \sqrt{\nu - 2}} \right) \end{aligned} \tag{3.2.8}$$

where $k_j = \alpha_j(|\varepsilon_{t-j}| - \gamma_j \varepsilon_{t-j})^{\delta}$, $\delta > 0$ and $-1 < \gamma_i < 1$ which adds flexibility of a varying exponent with an asymmetry coefficient to take the leverage effect into account. The objective function is maximized subject to following constraints $\alpha_0 > 0, \hat{\alpha_1} + ... + \hat{\alpha_p} + \beta_1 + ... + \beta_q < 1$.
Note: If $\alpha_0 = 0$ and $\sum \alpha_i + \sum \beta_j > 1$, then the residues are not covariance stationary.

3.3 Log Returns Model

Under Black and Scholes (1973) model,

$$dS_t = S_t(\mu dt + \sigma dB_t), S_0 > 0, \mu \in \mathbb{R}, \quad \sigma > 0 \tag{3.3.1}$$

the stochastic differential equation for the log price $d \log S_t = (\mu - \frac{\sigma^2}{2})dt + \sigma dB_t$. To account for the fact that volatility change over time, we look for a stochastic process $\sigma^2 = \{\sigma_t^2, t \geq 0\}$ describing the nervousness of market through time. Of particular interest is the model for which σ^2 is an Ornstein-Uhlenbeck (OU) processes in the context of Barndorff-Nielsen et. al. (2002) models. In this case the $d\sigma_t^2 = -\lambda \sigma_t^2 dt + dz_{\lambda t}$ where $z = \{z_t, t \geq 0\}$ is a Lévy process with positive increments(a subordinator). The resulting log-stock price process follows the dynamics

$$dX_t = (\mu - \frac{1}{2}\sigma_t^2)dt + \sigma_t d\bar{B}_t + \rho dz_{\lambda t}, X_0 = x \tag{3.3.2}$$

where ρ is a non-positive real parameter which accounts for the positive leverage effect. The Brownian motion and the Background Driving Levy Process are independent, and filtration $(\mathcal{F}_t)$ is generated by the pair $(\bar{B}, z)$, see Schoutens (2003).

In the same way, we model most of the stylized facts of volatility embedded in financial data (discrete case), by AR filtering log returns and using appropriate ARCH type model to remove volatility, then calibrate the residual which is heavy tailed and leptokurtic. In literature, the residuals are assumed to be non-significant. In view of this, it seems therefore that there is no obvious reason why one should ignore the AR-APARCH Lévy filtered residuals while calibrating stock market log returns data. A more general class of model which explains dynamics of daily returns in developed and emerging markets is hereby proposed.

The general structure is as follows, Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a stochastic basis and $X = (X_t)_{t\in[0,T]}$ be the stochastic process (daily log returns) where $a(t, \omega) = \mu_t, b(t, \omega) = \sigma_t$ is adapted to the natural filtration $\mathcal{F}_t$ driven by Brownian motion $Z = (B_t)_{t\in[0,T]}$ and Lévy process $Y = (Y)_{t\in[0,T]}$. In continuous time economy it seems that the process $(S_t)_{t\geq 0}$ can be broken into three components; namely, drift, Brownian part and Lévy part ie,

$$dS_t = a(t, S_t)dt + \sigma(t, S_t)dZ_t + \sigma_2(t, S_t)dY_t \tag{3.3.3}$$

$a(t, S_t), \sigma(t, S_t), \sigma_2(t, S_t)$ are adapted to $\mathcal{F}_t$. As a simplifying assumption let $\sigma(t, S_t) = \sigma_2(t, S_t) = \sigma_t$ and $a(t, S_t) = \mu_t$ and the structure of the model reduces to

$$dS_t = S_t\mu_t dt + S_t\sigma_t(dZ_t + dY_t)$$

Without loss of generality we assume the discrete version of the solution is of the form

$$\begin{aligned} \log S_t - \log S_{t-1} &= AR(s) + APARCH(p,q) + (Y_t - Y_{t-1}) \\ & AR(s), \quad t, s \in \mathbb{Z} \\ & APARCH(p,q) \quad p, q \in \mathbb{Z} \end{aligned} \tag{3.3.4}$$

where $Y_t - Y_{t-1} \sim GH$. This leads to a brief review of Lévy processes and assumed densities of proposed models for residual calibration as outlined in chapter 1.

3.3.1 Lévy Processes

A cadlag adapted real valued stochastic process $L = (L_t)_{t\geq 0}$ with $L_0 = 0$ a.s. is called a Lévy process if the following conditions are satisfied.

(L1) L has independent increments, i.e. $L_t - L_s$ is independent of $\mathcal{F}_s$ for any $0 \leq s < t \leq T$

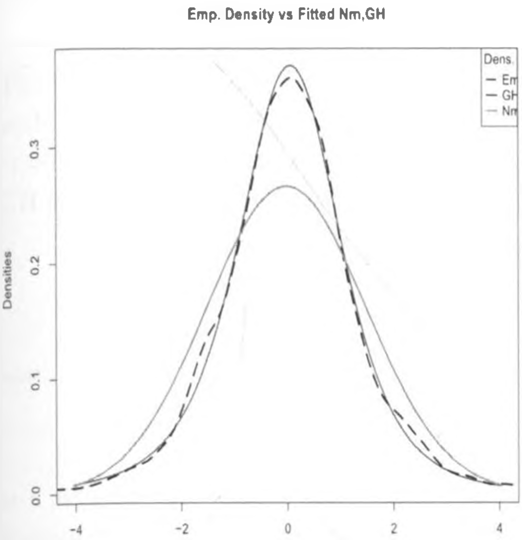
(L2) L has stationary increments ie for any $L_t \geq 0$ the distribution of $L_{t+s} - L_t$ does not depend on t.

(L3) L is stochastically continuous ie for every $t \geq 0$ and $\epsilon : \lim_{s\to t} \mathbb{P}(|L_t - L_s| > \epsilon) = 0$.

The class of infinitely divisible distributions and the class of Lévy processes are in a one-to-one relationship. Therefore if a specific infinitely divisible distribution is characterized by a few parameters the same holds for the corresponding Lévy process. We use this fact while estimating parameters of the residuals which are assumed to be driven by Lévy process. As an example Brownian motion is characterized by the parameters of Normal distribution μ and σ^2. We fit residuals of the APARCH type model to five different infinitely divisible distributions, Generalized Hyperbolic distribution, Hyperbolic distribution, Normal Inverse Gaussian, Generalized Skew t distribution and Variance Gamma.

3.3.2 Hyperbolic and NIG Lévy process

Hyperbolic distributions which generate hyperbolic Lévy process $X = (X_t)_{t\geq 0}$ constitute a four parameter class of distributions. Their Lebesgue density is given by taking $\lambda = 1$ in



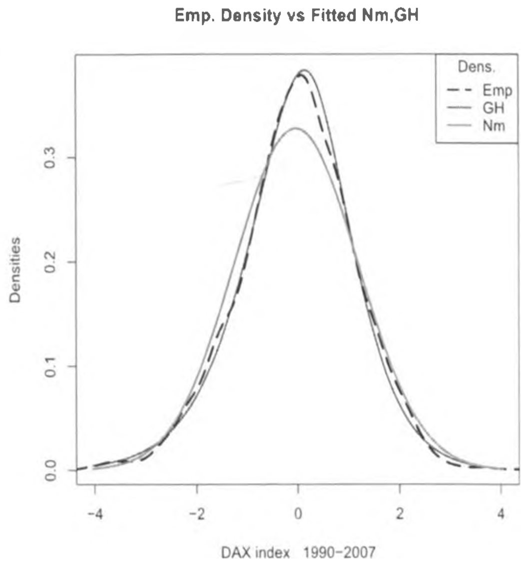


Figure 3.3: Kernel densities of empirical data **Left:** NSE20 index AR-GARCH filtered Lévy increments calibrated, **Right:** DAX index APARCH Lévy increments calibrated

Generalized hyperbolic density. Similarly, for $\lambda = -1/2$ we get a special case of the GH called normal inverse Gaussian NIG distribution. It was introduced to finance by Barndorff-Nielsen (1997). Note that $K_{-1/2}(\omega) = K_{1/2}(\omega) = \sqrt{\pi/(2\omega)e^{-\omega}}$. The density is

$$f_{NIG}(x) = \frac{\alpha}{\pi}\exp(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu))\frac{K_1(\alpha\delta\sqrt{1+x})}{\sqrt{1+x^2}} \tag{3.3.5}$$

The characteristic function of NIG is

$$\Phi_{NIG}(u) = \exp(iu\mu)\exp(\delta\sqrt{\alpha^2-\beta^2})\exp(-\delta\sqrt{\alpha-(\beta+iu)^2}) \tag{3.3.6}$$

$$\mathbb{E}X = \mu + \frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}} \quad and \quad Var(X) = \frac{\delta\alpha^2}{(\alpha^2-\beta^2)^{3/2}} \tag{3.3.7}$$

The skewness and kurtosis are given by

$$s = 3\frac{\beta}{\alpha}\frac{1}{\delta^{1/2}(\alpha^2-\beta^2)^{1/4}}, \qquad k = 3\left(1+4(\frac{\beta}{\alpha})^2\right)\frac{1}{\delta(\alpha^2-\beta^2)^{1/2}} \tag{3.3.8}$$

It follows that the kurtosis-skewness pairs must satisfy $|s| \le \sqrt{\frac{3}{5}k}$

The four parameter of the NIG distribution have natural interpretations relating to the overall shape of the density. The α parameter controls the steepness of the density, in the sense that the steepness of the density increases monotonically with increasing α. Skewness is represented by the parameter β, while μ is a centrality or translation parameter. δ is a scale parameter. This subclass is closed under convolution for fixed parameters α and β,i.e. by expanding (3.3.6) to power t, one gets the same form of equation with parameters $t\delta$ and $t\mu$.

3.3.3 GH skew T

The GH skew student's t-distribution may be represented as a normal variance-mean mixture with the generalized inverse mean mixture GIG distribution as a mixing distribution see Blæsild (1981) and Barndorff-Nielsen (1977). Letting $\lambda = -v/2$ and $\alpha \rightarrow |\beta|$ in (7.3.4) we obtain the GH skew Student's t-distribution. Its density is given by

$$f_{Gst}(x) = \frac{2^{\frac{1+\nu}{2}} \delta^{\nu} |\beta|^{\frac{\nu+1}{2}} K_{\frac{\nu+1}{2}} \left(\sqrt{(\beta^2(\delta^2 + (x-\mu))^2)} \right) \exp(\beta(x-\mu))}{\Gamma(\frac{\nu}{2}) \sqrt{\pi} \left(\sqrt{\delta^2 + (x-\mu)^2} \right)^{\frac{\nu+1}{2}}}, \beta \neq 0, \qquad (3.3.9)$$

and

$$f_{st}(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{2\pi}\Gamma(\nu/2)} \left(1 + \frac{(x-\mu)}{\delta^2} \right)^{(\nu+1)/2}, \quad \beta = 0 \qquad (3.3.10)$$

is a limiting case of the GH distribution. The mean and variance of GH skew student t distributed random variable are

$$\mathbb{E}(X) = \mu + \frac{\beta\gamma^2}{\nu - 2}, \quad Var(X) = \frac{2\beta^2\sigma^4}{(\nu-2)^2(\nu-4)} + \frac{\delta^2}{\nu-2}, \quad \nu > 4 \qquad (3.3.11)$$

3.3.4 Variance Gamma distribution

Carr and Madan (1998) introduced the Variance Gamma process to model asset returns by time-changing a drifted Brownian motion using a gamma process with unit mean. We define this model by considering the process $(Y(t))_{t \geq 0}$ defined for every nonnegative real number t by $Y(t) = \theta t + \sigma W(t)$, where θ is a constant, σ is positive real number and W a Wiener process. Let $(\gamma(t,\nu))_{t\geq 0}$ be a unit mean gamma process independent of Y. The process $(Y(t))_{t\geq 0}$ defined for every $t \geq 0$ by

$$Y(t) = Y\left(t, \theta, \sigma, \nu)\right) = \mu t + \theta\gamma(t,\nu) + \sqrt{\gamma(t,\nu)}\sigma W(1), \quad W(1) \sim N(0,1). \qquad (3.3.12)$$

is the Variance Gamma process
The PDF of Y_t is given by

$$f_{VG}(y) = \frac{2\exp(\gamma(y-\mu)/\sigma^2)}{\sigma\sqrt{2\pi}\lambda^{-\lambda}\Gamma(\lambda)} \left(\frac{|y-\mu|}{\sqrt{2\gamma^2\lambda + \sigma^2}} \right)^{\lambda - 0.5} K_{\lambda-1/2} \left(\frac{|x-\mu|\sqrt{2\sigma^2\lambda + \theta^2}}{\sigma^2} \right) \qquad (3.3.13)$$

The characteristic function of Y_t is

$$\phi_{VG}(u) = e^{iu\mu} \left(1 - i\frac{\gamma}{\lambda}u + \frac{\sigma^2}{2\lambda}u^2 \right)^{-\lambda} \qquad (3.3.14)$$

The mean and variance of the Variance Gamma distributed random variate Y are given respectively by $\mathbb{E}(Y) = \mu + \theta$ and $Var(Y) = \sigma^2 + \nu\theta^2$.

3.4 Risk Management

Value at risk is a statistical estimate of the worst expected loss over a given horizon at a given confidence level. Value at Risk (VaR) has become the standard measure that financial analysts use to quantify market risk. It is defined as the potential change in value of a portfolio of financial instruments with a given probability over a certain horizon. We restrict our attention to univariate methodologies. From a statistical point of view VaR entails the estimation of a quantile for a given (assumed) distribution of returns.

Riskmetrics

In (3.2.1), we show that Riskmetrics model is equivalent to normal integrated GARCH (IGARCH) model, where the autoregressive parameter is a set of pre-specified value λ and the coefficient of ε_{t-1}^2 is equal to $1-\lambda$. In the specification of daily data $\lambda = .94$ and $\varepsilon = \sigma_t z_t$ where z_t is i.i.d N(0,1) and σ_t^2 is defined as $\sigma_t^2 = (1-\lambda)\varepsilon_{t-1}^2 + \lambda\sigma_{t-1}^2$. The one step ahead VaR computed in $t-1$ for long trading position is given by $\mu_t + z_\alpha\sigma_t$ while for short trading positions is given by $\mu_t + z_{1-\alpha}\sigma_t$ with z_α being the left quantile at α% for the normal distribution and $z_{1-\alpha}$ is the right quantile at α%. This approach underestimates the value at risk especially in emerging economies.

Proposed VaR Model

In line with the AR-APARCH model, let $S = (S_t)_{t\in[0,T]}$ be observed financial time series, then

$$\begin{aligned}
\log\left(\frac{S_t}{S_{t-1}}\right) &= X_t + \mu, \quad \mu = \frac{1}{T}\sum_{t=1}^{T}(\log S_t - \log S_{t-1}) \\
X_t &= \mu_t + \sigma_t(Z_t + Y_t), \quad Z_t \sim i.i.d.(0,1), \\
\sigma_t^\delta &= (\alpha_0 + \alpha_1(|\varepsilon_{t-1}| - \gamma_1\varepsilon_{t-1})^\delta + \beta\sigma_{t-1}^\delta). \qquad (3.4.1)
\end{aligned}$$

Note that Y_t is the non normal residual term calibrated as increments of Levy process. VaR at confidence level $\alpha \in (0,1)$ for loss L of a security or portfolio is defined to be

$$\begin{aligned}
VaR_\alpha(X) &= inf\{x \in \mathbb{R} : F_L(x) \geq \alpha\} \\
&= \hat{\mu} + \hat{\mu}_t + \hat{\sigma}_t\left(\int_0^1 dF_z + \int_\alpha^1 dF_y\right) \\
&= \hat{\mu} + \hat{\mu}_t + \hat{\sigma}_t\left(F_z^{-1}(\alpha) + F_y^{-1}(\alpha)\right) \qquad (3.4.2)
\end{aligned}$$

where F_z and F_y are cumulative density function of the variable Z and Y respectively. As an Example, if L is normal distribution $N(\mu, \sigma^2)$ then $VaR_\alpha = \mu + \sigma\Phi^{-1}(\alpha)$ where $\Phi^{-1}(\alpha)$ is the α quantile of standard normal. While If L is a student t distribution $t(\nu, \mu, \sigma^2)$, then $VaR_\alpha = \mu + \sigma t^{-1}(\alpha)$ where $t_\nu^{-1}(\alpha)$ is the α quintile of standard t with degree of freedom ν.

Student and Normal APARCH

The normal APARCH one-step-ahead VaR is computed as the conditional standard deviation σ_t evaluated at its MLE, while the Student APARCH the VaR for long and short positions is given by $\mu_t + st_{\alpha,\nu}\sigma_t$ and $\mu_t + st_{1-\alpha}\sigma_t$ with $st_{\alpha,\nu}$ being the left quantile at $\alpha\%$ for the student t distribution with (estimated degrees) of freedom ν. Note that because $z_\alpha = z_{1-\alpha}$ for the normal distribution and $st_{\alpha,\nu} = -st_{1-\alpha,\nu}$ for the forecasted long and short VaR will be equal in both cases.

3.5 Data sets

The indices are based on the most liquid shares of blue chip companies traded on stock market.

- SP500 index NewYork Stock Exchange May 8,1998 to July 11,2007; USA
  $n = 2307$ observations.
- DAX index Frankfurt Stock Exchange November 26.1990 to October 20,2007; Germany.
  $n = 4265$ observations.
- NSE20 index Nairobi Stock Exchange March 2,1998 to July 11,2007; Kenya
  $n = 2328$ observations.
- MASI index Casablanca Stock Exchange. February 4,1998 to August 28,2007; Morocco
  $n = 2383$ observations.

3.5.1 Parameter Estimations and kernel densities

The Ljung-Box Q statistics of order 10 on the squared series indicate a high serial correlation in the second moment or variance. All returns distributions exhibits fat tails. See Figures (3.5), (3.8), (3.7) and (3.6).

Portmanteau Test

Financial applications often require that joint tests about several autocorrelations of $\log S_t - \log S_{t-1}, \;\; \forall \;\; t \in \{1, ..., T\}$ are zero. Box and Pierce (1970) propose the Portmanteau statistic $Q^*(m) = T\sum_{l=1}^{m} \rho_l^2$ as a test statistic for null hypothesis $H_0 : \rho_1 = ... = \rho_m = 0$ against the alternative hypothesis $H_A : \rho \neq 0$ for some $i \in \{1, ..., m\}$ against the alternative hypothesis $H_A : \rho_i \neq 0$ under the assumption that $\{\log S_t - \log S_{t-1}\}$ is an i.i.d. sequence with certain moment conditions, $Q^*(m)$ is asymptotically a chi-squared random variable with m degrees of freedom. Ljung and Box (1978) modified the $Q^*(m)$ statistic to increase the power of the test in finite samples

$$Q(m) = T(T+2)\sum_{l=1}^{m} \frac{\hat{\rho}_l^2}{T-l} \tag{3.5.1}$$

A time series X_t is called a white noise if $\{X_t\}$ is a sequence of independent and identically distributed variable with finite mean and variance. Empirical evidence show that the filtered

GARCH residues are not serially correlated in second moment. See results in table(3.1) which implies no serial correlation to the centered and normalized residuals. Moreover, One can argue that they fit increments of Lévy process as driven by Generalized hyperbolic distribution and some of its subclasses such as NIG, HY, VG and GH-student densities.

Table 3.1: Maximum likelihood parameter estimates of the AR(p)APARCH(1,1) model for four indices, using Gaussian and student t distributions appropriately.

	NSE20	DAX	SP500	MASI
n	2317	4265	2306	2382
ϕ_1	0.2740(.0210)	0	0	0.3435(.0192)
ϕ_2	0.1634(.0211)	0	0	0
ϕ_3	0.0440(.0208)	0	0	0
$\alpha_0 \times 10^4$.1036(.02476)	0.0313(.0053)	0.0313(.0053)	0.0101(.0031)	1.0993(.0314)
α_1	0.3175(.0525)	0.07808(.0089)	0.0659(.0100)	0.2754(.0223)
β	0.5352(.0725)	0.9029(.01064)	0.9265(.0109)	0.7308(.0183)
δ	2	2	2	1.2986(.0199)
γ	0	0	0	0.0114(.0372)
$Q^2(10)$	6.943(.539)	1.248(.996)	9.534(.299)	6.856(.552)
$\alpha_1(\lvert\varepsilon\rvert - \gamma\varepsilon)^{\delta} + \beta$	0.8437	0.9875	0.9923	0.9584
$f_Z(z)$	$z \sim t_{3.97(.38)}$	$z \sim N(0,1)$	$z \sim N(0,1)$	$z \sim N(0,1)$
$\log L$	8528	12881	7345	8833

3.5.2 Model(s) residual calibration

NSE20 Index student AR(3)GARCH(1,1) model

Let X_t be the mean centered process. Within a class of heteroscedastic variances we try to find reasonable models of $\{X_t\}$. From Table 3.1 the process $\{X_t\}$ is represented as AR-GARCH

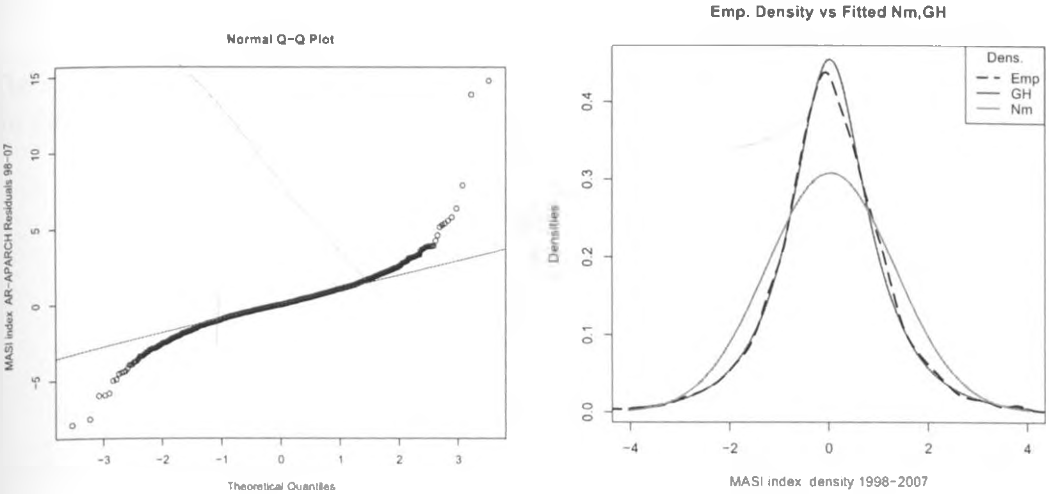


Figure 3.4: Casablanca, MASI share index daily log return GARCH Lévy filtered residuals calibrated.

noise of order (1, 1) for $t = 0, \pm1, \pm2, \pm3, ...$

$$\begin{aligned} X_t &= \mu + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + \varepsilon_t + \xi_t \\ \varepsilon_t &= \sigma_t z_t, \quad z_t \sim i.i.d.(0,1) \\ \sigma_t^2 &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \\ \xi_t &= \sigma_t Y_t, \quad Y_t - Y_{t-1} \sim GH \end{aligned} \tag{3.5.2}$$

The Ljung-Box statistics of the standardized residuals $\{\hat{z_t}^2\}$ gives $Q(10) = 6.97429$ and p-value 0.5394087. Thus the fitted GARCH(1,1) model with t-distribution is adequate. The unconditional variance is 6.74683×10^{-5}.

MASI Index

The AR(1)APARCH(1,1) model can be written as

$$\begin{aligned} X_t &= \phi X_{t-1} + \varepsilon_t + \xi_t \\ \varepsilon_t &= \sigma_t z_t \\ \sigma_t^\delta &= \alpha_0 + \alpha_1(|\varepsilon_{t-1}| - \gamma_1\varepsilon_{t-1})^\delta + \beta_1\sigma_{t-1}^\delta \end{aligned}$$

where $\delta > 0$ and $-1 < \gamma < 1$. This model was introduced by Ding et al. (1993) which was meant to add flexibility of varying exponent and asymmetry leverage effect into account. A stationary solution exists if $\alpha_0 > 0$ and $\alpha_1 k_j + \beta_1 < 1$ where $k_1 = (|z| + \gamma_1 z)^\delta$.

Similarly, other models can be extracted from table 3.1.

3.5.3 Residue calibration

The standard procedure E.M. algorithm was used to find maximum likelihood parameters. See Hu (2005), McNeil et al. (2005) and references there in, for detailed exposition of the model.

Table 3.2: Generalized Hyperbolic Distribution Maximum likelihood estimation.

GH	NSE20	DAX	SP500	MASI
λ	−1.846700	3.14556	3.052334	−1.793393
α	0.070542	2.15818	2.279172	0.013714
β	−0.057946	−0.26470	−0.224353	0.012858
δ	1.935255	0.016210	0.215735	1.628145
μ	0.074937	0.33469	0.259252	0.018427
$\log l$	−3967.591	−6723.702	−3469.757	−3721.059

Table 3.3: Maximum likelihood estimates of Hyperbolic distribution.

HY	NSE20	DAX	SP500	MASI
λ	1.000000	1.000000	1.000000	1.000000
α	1.068512	1.79988	1.938602	1.211495
β	−0.080233	−0.2838095	−0.241376	0.033726
δ	0.462694	1.520427	1.462422	0.323587
μ	0.109201	0.358756	0.278058	−0.010359
$\log l$	−3986.997	−6718.59	−3469.808	−3738.791

3.5.4 Testing goodness of fit

Once the maximum likelihood estimators have been calculated, the next step is to judge the quality of the fit obtained. One approach is to plot the fitted density $g(z, \Theta)$ and empirical or non parametric density estimate on the same graph, or Q-Q plot in addition to calculation of the Kolmogorov distance which is the supremum over the absolute differences between two

Table 3.4: Normal Inverse Gaussian.

NIG	NSE20	DAX	SP500	MASI
λ	−0.500000	−0.500000	−0.500000	−0.500000
α	0.596943	1.466975	1.650328	0.666512
β	−0.065622	−0.2744828	−0.247784	0.019438
δ	1.241263	1.968419	1.943713	1.042988
μ	0.085814	0.346604	0.285681	0.009591
$\log l$	−3974.308	−6715.163	−3469.501	−3726.955

Table 3.5: Variance Gamma Maximum likelihood of parameters.

VG	NSE20	DAX	SP500	MASI
$\lambda = 1/\nu$	1.272183	3.14992	3.118843	1.185112
σ	1.420518	1.17045	1.094472	1.229365
γ	−0.154469	−0.3630732	−0.267364	0.058057
μ	0.103105	0.191216	0.3347879	−0.018108
$\log l$	−3991.336	−6623.703	−3469.728	−3741.387

density functions whose expression is given by:

$$D_n = \sup_{x \in \mathbb{R}} |F_{emp}(x) - F_{est}(x)| \tag{3.5.3}$$

where n is the sample size, F_{emp} and F_{est} are the empirical and the estimated CDFs respectively. Kolmogorov distance is used because it pays more attention to the tails of distributions see Gyorfi et al. (1996) and Prause (1999). The Kolmogorov distances of the normal, the hyperbolic, the variance gamma, normal inverse Gaussian, generalized skew t and the generalized hyperbolic distributions are presented in Table 3.8

3.6 Conclusions

In this chapter we have proposed a formal procedure of estimating VaR in emerging and developed markets. This was made possible by fitting an AR-APARCH model first by maximum

Table 3.6: t distribution maximum likelihood estimates.

T-uv	NSE20	DAX	SP500	MASI
λ	−1.853741	−4.589006	−5.051762	−1.793984
α	0.882379	2.318619	2.628921	0.975141
β	−0.057144	−0.2751691	−0.267519	0.012596
δ	1.939159	3.117849	3.098538	1.628585
μ	0.073945	0.3463958	0.307584	0.018830
$\log l$	−3967.600	−6702.904	−3467.620	−3721.059

likelihood, and estimating drift term μ_t and the varying volatility σ_t appropriately. In developed market, μ_t was found to be non significant unlike the case of emerging market. The residuals are then calibrated using distributions of Lévy increments and the parameters tabulated. Graphical procedures and Kolmogorov Smirnoff distance was used to determine adequacy of fit.

The actual return distributions appear fat tailed and skewed compared to the assumed normal. In both markets volatility appears time varying and clustered where else returns are serially uncorrelated in fully developed market but correlated in emerging markets. VaR answers the basic concerns of risk management about the potential loss of the portfolio value.

There is need for further investigations about the exact solution of the proposed dynamics (AR APARCH Lévy filter) of the underlying asset, which seems to capture most of the stylized features of financial time series data.

In future it will be interesting to decompose covariance non-stationary residuals observed in emerging markets into poisson part and pure white noise part (semi-martingale).

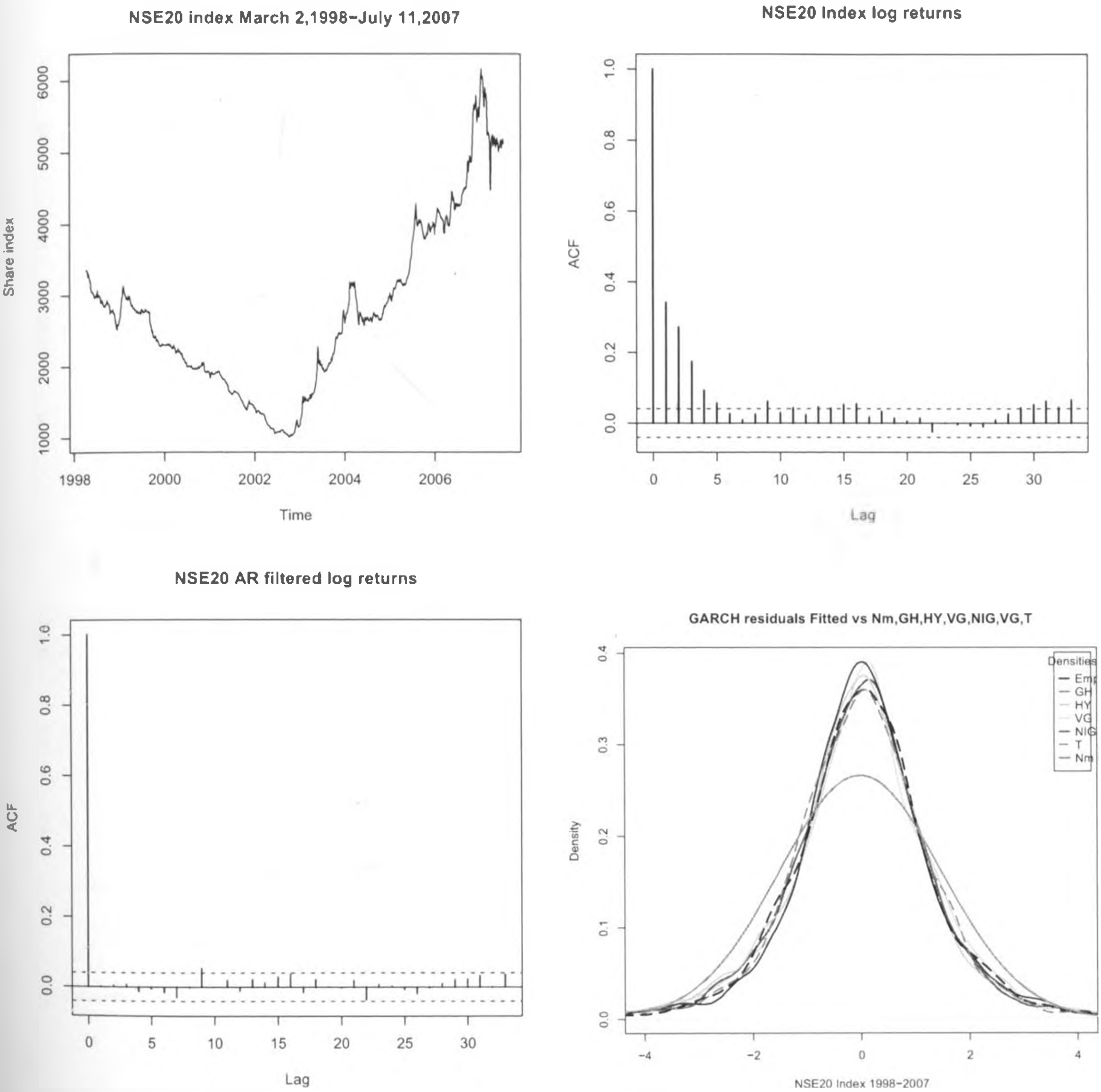


Figure 3.5: Nairobi Stock Exchange NSE20 share index daily time series plot and its market risk analysis using AR(3)-GARCH(1,1) conditioned on scaled t-distribution and residuals calibrated using Generalized hyperbolic distribution as from March 2,1998 to July 11, 2007.

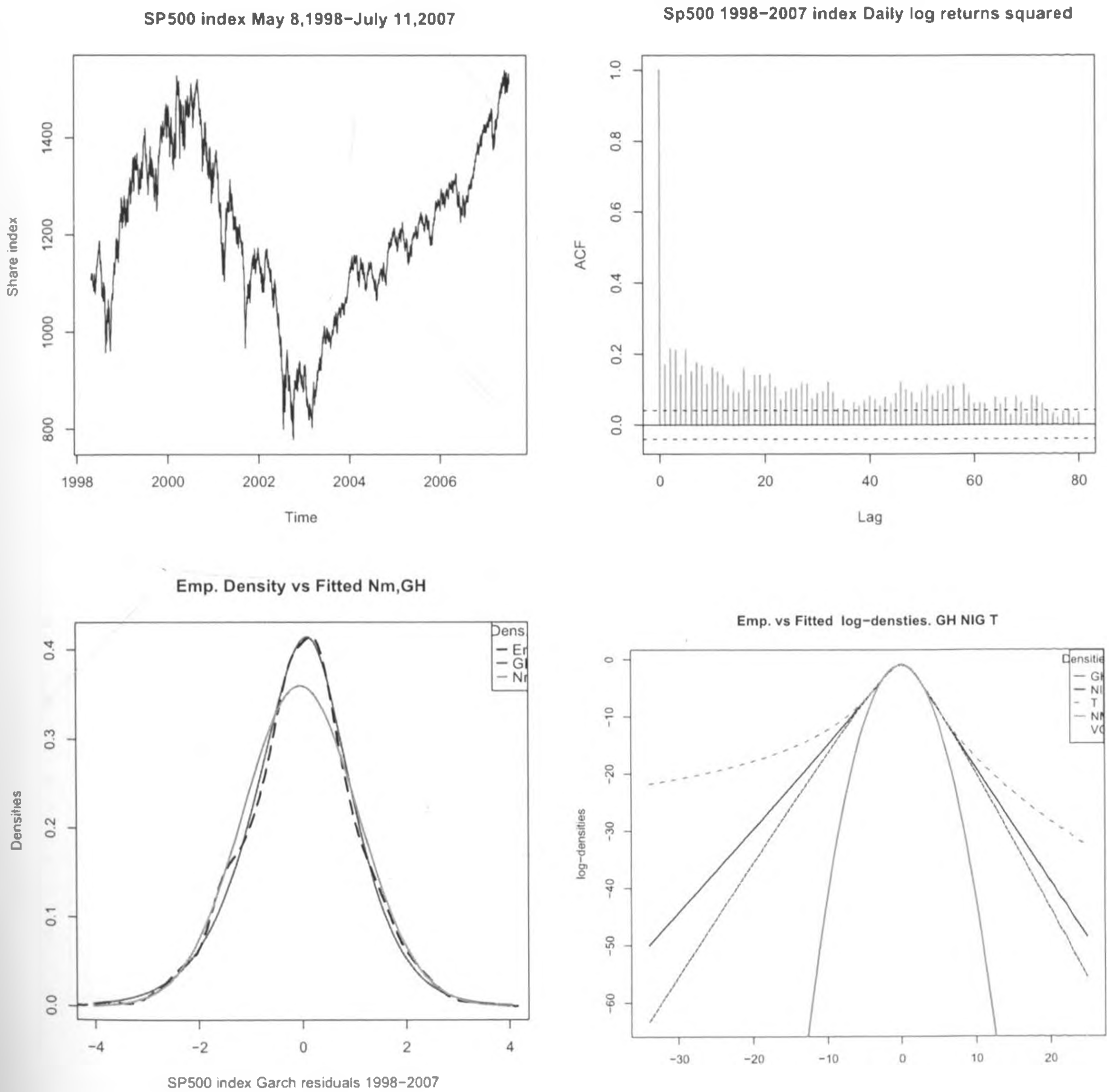


Figure 3.6: SP500 share index daily time series plot and its market risk analysis using GARCH(1,1) conditioned on Gaussian distribution and residuals calibrated using Generalized Hyperbolic and its subclasses as from May 8,1998 to July 11, 2007.

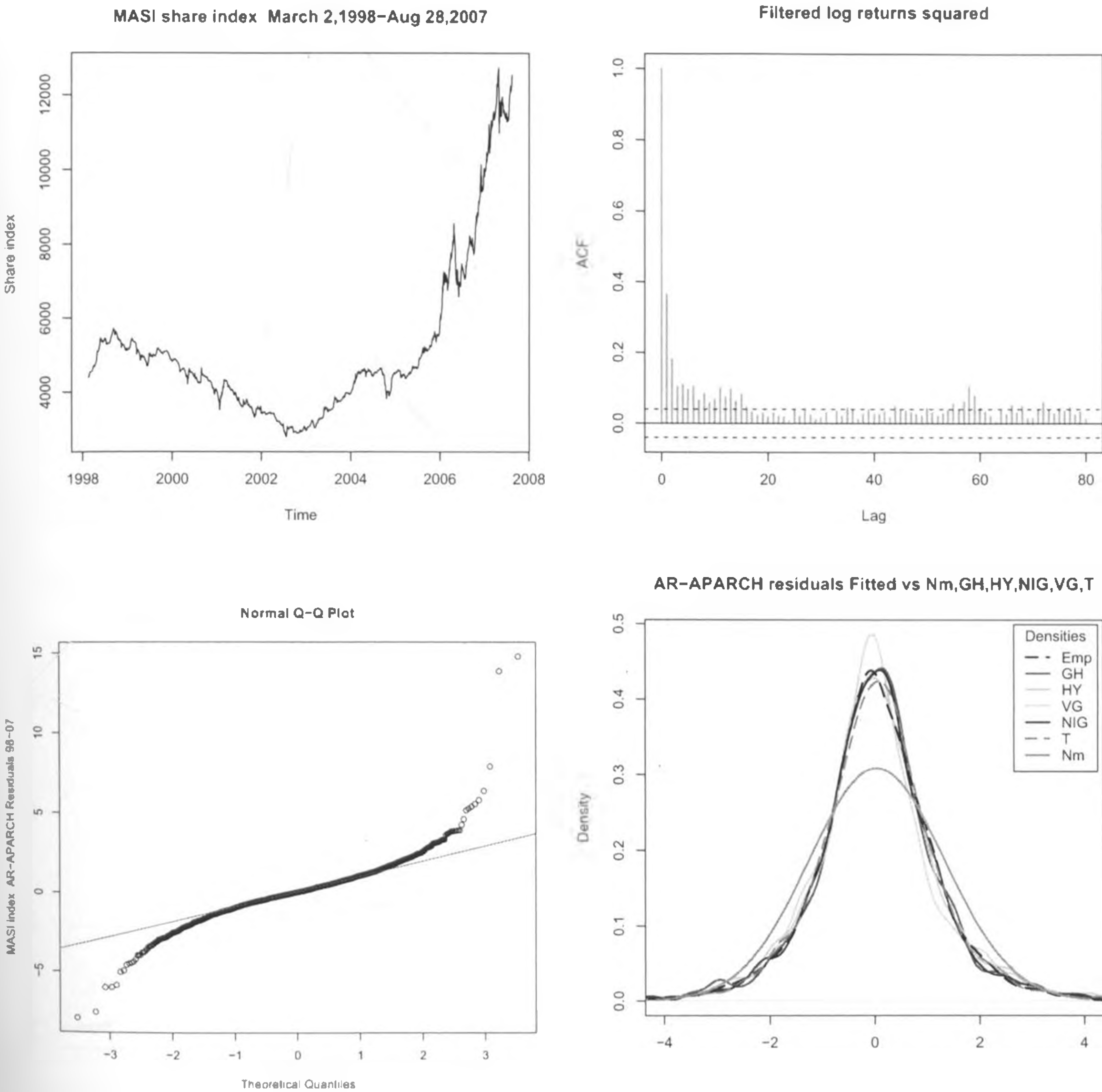


Figure 3.7: MASI share index of Casablanca Stock exchange Morocco daily time series plot analysis using AR(2)-APARCH(1,1) conditioned on Gaussian distribution and residuals calibrated using a Leptokurtic distribution as from February 04,1998 to August 28, 2007.

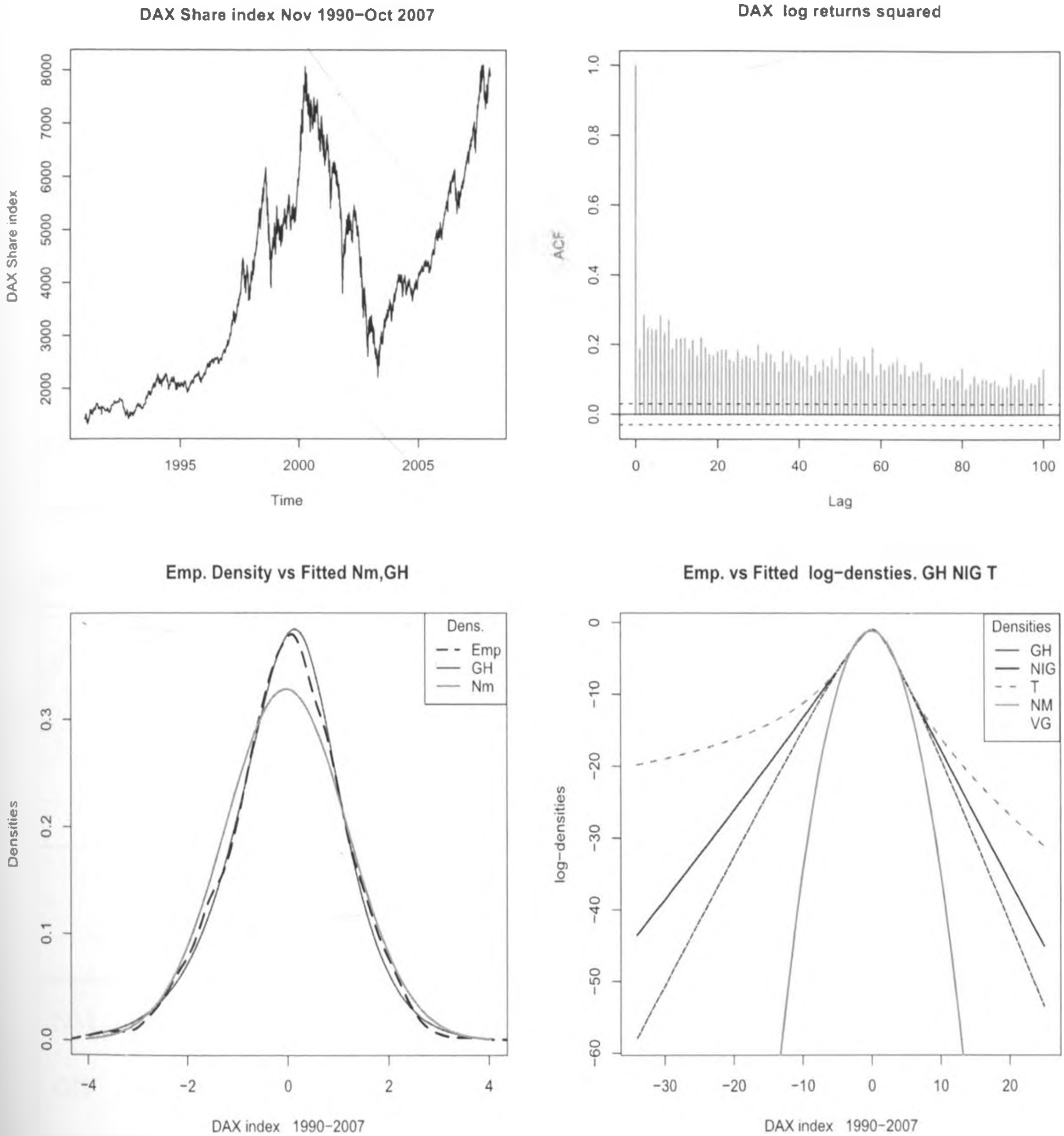


Figure 3.8: DAX share index of Frankfurt Stock exchange German daily time series plot and its market risk analysis GARCH(1,1) conditioned on Gaussian distribution and residuals calibrated using a Levy process as from November 26,1990 to October 20, 2007.

Table 3.7: One day VaR results for NSE20 DAX SP500 and MASI (in-sample).

Index Long position	5%	2.5%	1%	0.5%	0.25%
NSE20(AR(3)-GARCH(1,1))	**0.02998**	**0.03769**	**0.04832**	**0.05699**	**0.06639**
Riskmetrics	0.01293	0.01537	0.01821	0.02015	0.02194
DAX(GARCH(1,1))	**0.044878**	**0.054152**	**0.06541**	**0.07338**	**0.0810**
Riskmetrics	0.027108	0.02275	0.03217	0.03562	0.03882
SP500(GARCH(1,1))	**0.03870**	**0.04684**	**0.05672**	**0.06371**	**0.07040**
Riskmetrics	0.01871	0.02229	0.02646	0.02931	0.03193
MASI(AR(3)APARCH(1,1))	**0.06176**	**0.07629**	**0.09498**	**0.10881**	**0.1226**
Riskmetrics	0.01294	0.01533	0.01812	0.02001	0.02177

Table 3.8: Kolmogorov distances.

	NSE20		SP500		MASI		DAX	
	K_{Dist}	p-value	K_{Dist}	p-value	K_{Dist}	p-value	K_{Dist}	p-value
NM	0.355	$2.2e-06$	0.261	2.2e−06	0.3212	2.2e−16	0.2746	2.2e−16
GH	0.0241	0.135	0.0230	0.1749	0.0185	0.3906	0.0138	0.3879
HY	0.0176	0.4654	0.0178	0.4596	0.0181	0.1307	0.0098	0.8025
NIG	0.0275	0.0592	0.0165	0.5582	0.0252	0.4194	0.0185	0.1071
VG	0.0215	0.2329	0.0178	0.4596	0.0239	0.1307	0.0192	0.8537
ST	0.0228	0.1787	0.0204	0.2935	0.0235	0.1437	0.0117	0.6008

Chapter 4

European Call Option Pricing under AR-APARCH Lévy Filter

In order to minimize mispricing due to heteroscedastic nature of the underlying, flexible volatility models are required. In this chapter we develop option pricing model when the dynamics of the underlying process is driven by AR-APARCH Levy process. Arguably, empirical evidence of daily log returns of financial assets in emerging and parts of developed economies, can be characterized by the proposed underlying process. Conditional variance in risk neutral world of different conditional heteroscedastic models are derived. The proposed model is used to price European call options in developed and emerging economies. Pricing such quantities require knowledge of risk neutral cumulative return distribution which is generally unknown. Numerical analysis suggest that AR-APARCH-Levy model may be able to explain some well documented systematic biases associated with BSM73 model.

4.1 Introduction

In recent years more attention has been given to stochastic models of financial markets which depart from the traditional Black and Scholes (1973) model. Some of the most popular and still tractable models are the Lévy models. For an introduction to these models applied to finance we refer to Eberlein and Keller (1995), Prause (1999), Sato (1999), Raible (2000b), Barndorff-Nielsen et al. (2002), Schoutens (2003), Eberlein and Ozkan (2003), Shoutens (2006), Shoutens (2006) and references therein.

It is well known that the stock prices do not follow a pure random walk as documented by Lo and Mackinlay (1988). Price changes are neither independent nor identically distributed. There are linear and nonlinear dependencies between successive price changes. Maheu and Mccurdy (2004) for example, study news arrival and jump dynamics of asset returns as components of returns distributions. In view of this we focus on the variation of higher order moments as used in econometrics and time series techniques.

Following the work of Engle (1982) and Bollerslev (1986), a voluminous econometric literature has been developed on volatility estimation and forecasting. There is a general consensus that asset returns exhibit variances that change through time. GARCH models are a popular choice to model these changing variances.

Motivated by the successes of GARCH models in fitting asset returns; researchers have extended the GARCH model in the domain of option valuation. Duan (1995) characterizes the transition between the physical and risk neutral probability distributions if the underlying security is given by GARCH processes, and thus establishes the foundation for the valuation of European options. A common feature of all the tests to date is the assumption that the volatility of asset return is equal to volatility of the pricing process. That is, a risk neutral investor prices the option as if the distribution of its return had a different drift but unchanged volatility. This is certainly a tribute to the pervasive intellectual influence of the Black and Scholes (1973) model on option pricing. However, Black and Scholes derived the above property under very special assumptions, (perfect complete markets, continuous time and price processes and constant volatility). Changing volatility in real markets makes perfect replication argument of Black-Scholes invalid. Markets are then incomplete in the sense that perfect replication of contingent claims using only the underlying assets and riskless bond is impossible.

Several papers have investigated certain aspects of the empirical performance of GARCH option pricing models heteroscedastic nature of volatility see for example Bauwens and Lubrano (2002) and Christoffersen and Jacobs (2004). In most studies, volatility of the pricing process is considered. It is well known fact that volatility of pricing process is different from volatility of asset process. This will occur because investors will set state prices to reflect their aggregate preferences. Pricing distribution will then be different from the return distribution see Härdle and Hafner (2000), Hafner and Herwartz (2001), Barone-Adesi et al. (2007).

In this chapter we develop a pricing model for options on asset whose continuously compounded returns follow linear autoregressive asymmetric power autoregressive conditional heteroscedastic Lévy (AR-APARCH-Lévy) filter, which nests GARCH process of Bollerslev (1986) and its variants. We link the robust model to the contingent pricing literature in developed and emerging economies. In option pricing literature, the underlying process of asset value is assumed to follow a diffusion process, which leads to a well documented biases associated to Black-Scholes model. Moreover, concerning AR-APARCH-Lévy filter, it can be argued empirically that it explains most of the BSM73 model biases (underpricing of short maturity in the money options as well as long maturity at and in the money in relation to strike price).

The remainder of this chapter is organized as follows. The next section, outlines the estimation methodology and option valuation using Monte Carlo simulation. Numerical examples from developed and emerging economies are presented in Section 5.

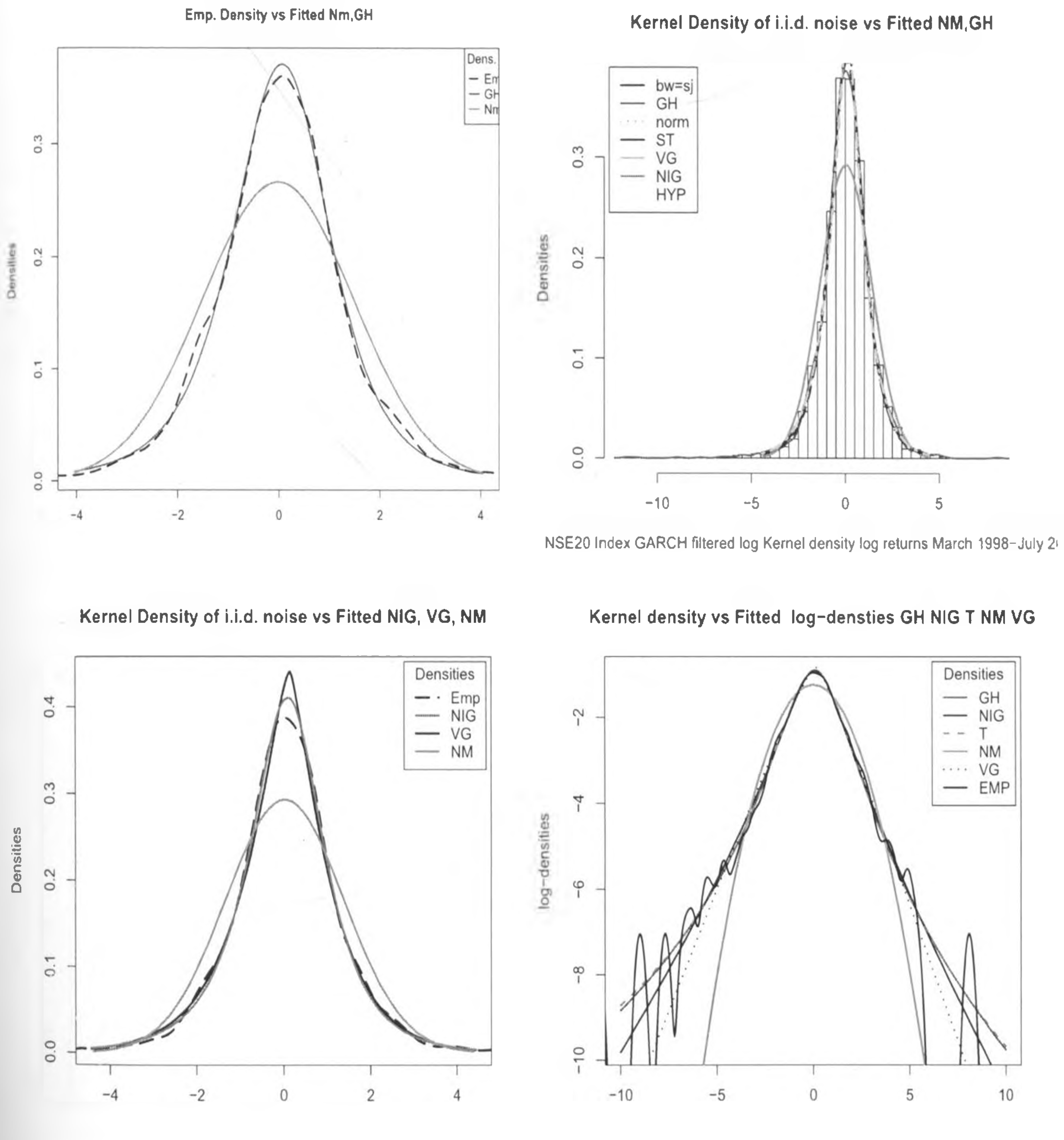


Figure 4.1: Empirical density of daily NSE20 AR-GARCH filtered Levy increments calibrated vs.density of fitted infinitely divisible distributions and normal distributions. The ordinate axis is on a log scale in order to exhibit the tails clearly.

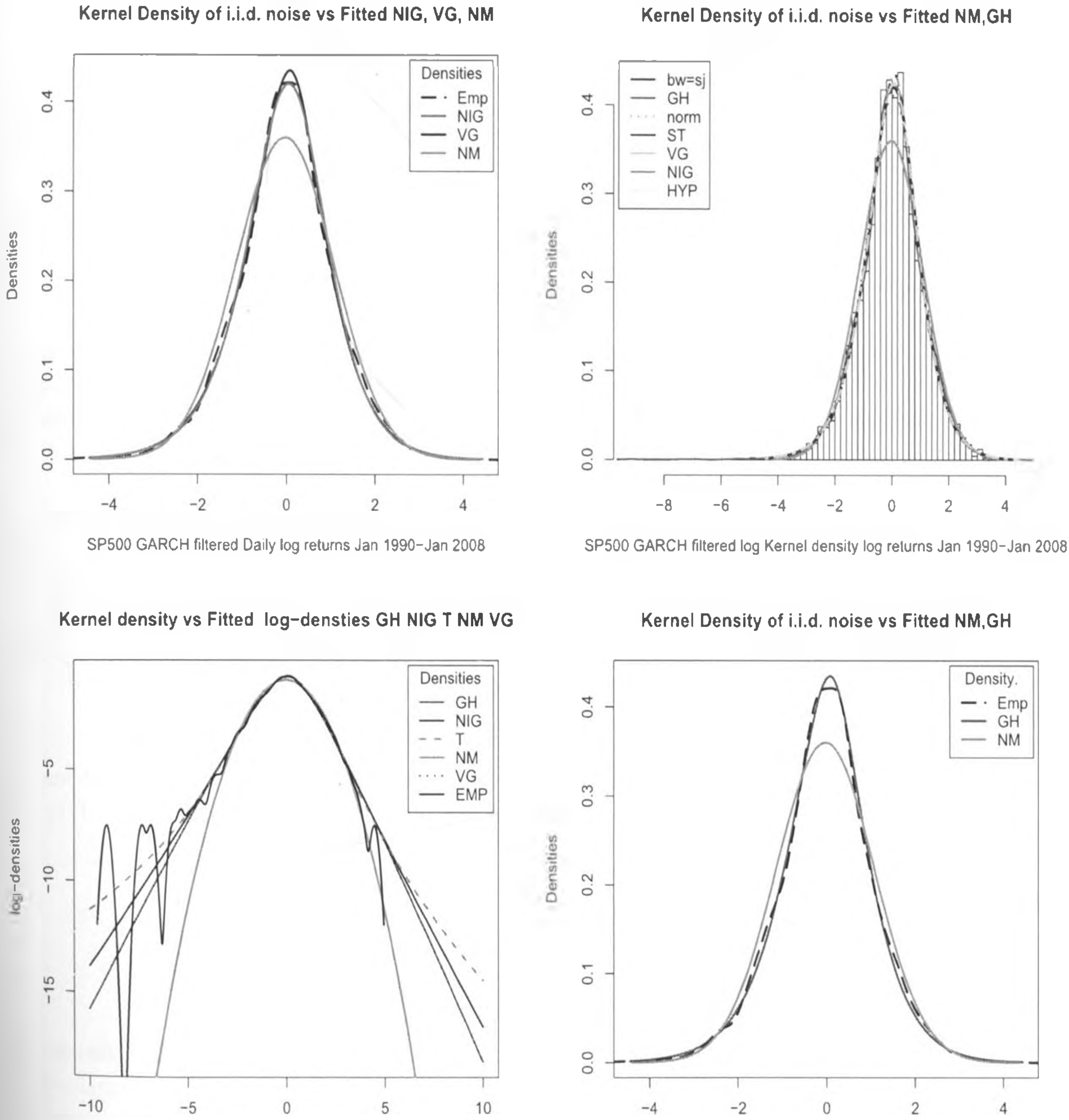


Figure 4.2: Empirical density of daily SP500 GARCH filtered Lévy increments calibrated vs. density of fitted infinitely divisible distributions and normal distributions.

4.2 The AR-APARCH-Lévy option pricing model

4.2.1 ARCH type models

ARCH type models are discrete-time models used to estimate volatility of financial time series such as stock returns, interest rates and foreign exchange rates see Bollerslev et al. (1992) and Bollerslev et al. (1994) and references therein for overview and empirical evidence.

Consider a discrete-time economy and let S_t be the asset price at time t. A general time series model for financial returns would be

$$\log(S_t/S_{t-1}) := X_t = \mu_t + \sigma_t(Z_t + L_t), \quad Z_t \sim N(0,1), \tag{4.2.1}$$

with $\mu_t = v + \phi X_{t-1}$, where ϕ is the coefficient of the conditional autoregressive part, σ_t standard deviation conditional on the past and L_t Lévy process. σ_t can be stochastic itself or determined by past history of the time series. Its one period rate of return is assumed to be conditionally distributed under physical probability measure $\mathbb{P}$.

We note general specifications of σ_t^2 in (4.2.1) common in most of ARCH type models. Ding et al. (1993) and Hentschel (1995) provide a general specifications of volatility dynamic that nest most existing work. In this connection volatility dynamics can be written as

$$\begin{aligned}
\sigma_t^2 &= \omega + \alpha Z_{t-1}\sigma_{t-1}^2 + \beta\sigma_{t-1}^2 \\
&= \omega + \beta\sigma_{t-1}^2 + \alpha\sigma_{t-1}^2(Z_{t-1}^2), \qquad \text{w.l.o.g.}, \\
&= \omega + \beta\sigma_{t-1}^2 + \alpha\sigma_{t-1}^2 f(Z_{t-1})
\end{aligned}$$

Different GARCH models are mainly characterized by differences in the innovation function $f(Z_{t-1})$. Consider the following specifications of f

$$f(Z_{t-1}) = \begin{cases}
Z_{t-1}^2, & \text{Simple ;} \\
(Z_{t-1} - \theta)^2, & \text{Leverage;} \\
\{|Z_{t-1} - \theta| - k(Z_{t-1} - \theta)^2\}, & \text{News;} \\
(Z_{t-1} - \theta)^{2\gamma}, & \text{Power;} \\
\{|Z_{t-1} - \theta| - \kappa(Z_{t-1} - \theta)^{2\gamma}\}, & \text{News and power;}
\end{cases} \tag{4.2.2}$$

These models can be generalized to allow nonlinearity of volatility dynamics by using Box-Cox transformation as follows

$$\sigma_t^{\psi} = \omega + \beta\sigma_{t-1}^{\psi} + \alpha\sigma_{t-1}^{\psi} f(Z_{t-1}), \quad \text{with} \quad f(Z_{t-1}) = (Z_{t-1} - \theta)^{2\psi} \tag{4.2.3}$$

which implies modeling news and power, will nest most of the proposed GARCH models in literature. Note that the leverage parameter θ shifts the innovation function, the news parameter κ tilts the innovation, and the power parameters γ and ψ flatten or steepen the innovation function. Such a model (4.2.3) is the Asymmetric Power Autoregressive Conditional Heteroscedastic model i.e. APARCH model defined in (1.2.21). We estimate risk neutral probability measure in the manner of Duan (1995) and Hafner and Herwartz (2001).

4.2.2 Risk neutralization incomplete market

First we estimate the parameters of (4.2.1) under the physical probability measure $\mathbb{P}$ from asset returns. Then, the parameters are converted to conform to a risk neutral measure $\mathbb{Q}$.

In order to develop the option pricing model, the conventional risk neutral valuation has to be generalized to accommodate heteroscedasticity of the asset return process. Duan (1995) introduced the GARCH option pricing model by generalizing the traditional risk neutral valuation methodology to the case of conditional heteroscedasticity, the so called Local Risk Neutral Valuation Relationship (LRNVR).

Definition 4.2.1. A pricing measure $\mathbb{Q}$ is said to satisfy the locally risk-neutral valuation relationship (LRNVR) if measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$, and

$$\mathbb{E}^{\mathbb{Q}}[X_t|\mathcal{F}_{t-1}] = r \quad (4.2.4)$$

$$\mathrm{Var}^{\mathbb{Q}}(X_t|\mathcal{F}_{t-1}) = \mathrm{Var}^{\mathbb{P}}(X_t|\mathcal{F}_{t-1}) \quad (4.2.5)$$

almost surely with respect to measure $\mathbb{P}$.

For some commonly used assumptions concerning utility functions and distributions of change of consumption, Duan (1995) shows that a representative agent maximizes his expected utility using the LRNVR measure $\mathbb{Q}$. Risk neutralization should leave the variance unchanged and should transform the conditional expectation so that the discounted expected price of the underlying asset becomes a martingale. It is worth noting that in the case of homoscedasticity process, $(p = 0, \ q = 0)$, the conditional variances become the same constant and the LRNVR reduces to conventional risk neutral valuation relationship.

Consider the general model of daily log returns under the data generating probability measure $\mathbb{P}$ as

$$X_t = \ln(S_t/S_{t-1}), \quad \text{where}, \quad \begin{cases} X_t = \mu_t + \varepsilon_t + \xi_t; \\ \varepsilon_t|\mathcal{F}_{t-1} \sim N(0, \sigma_t); \\ \sigma_t^2 = \omega + \alpha\varepsilon_t + \beta\sigma_{t-1}^2 \end{cases} \quad (4.2.6)$$

where the parameters $\omega > 0, \alpha > 0$ and $\beta > 0$ and $1 - \beta - \alpha > 0$ and given σ_0. The sequence $\{\varepsilon_t\}$ and $\{\xi_t\}$ are conditionally independent, while $\mathcal{F}_{t-1}$ is the past information set. μ_t represents the conditional expectation of returns.

The pricing measure $\mathbb{Q}$ shifts the error term ε_t by some measurable function λ_t, so that the conditional expectation of X_t becomes equal to r. In the case of AR(1)APARCH(1,1)- Levy filter, we follow the Hafner and Herwartz (2001) argument. Therefore under the equivalent martingale measure $\mathbb{Q}$ the model (4.2.6) translates to

$$X_t = \mu_t + \varepsilon_{1t} + \varepsilon_{2t}; \tag{4.2.7}$$

$$= \mu_t + \sigma_t(Z_t - \lambda_{1t}) + \sigma_t(L_t - \lambda_{2t}), \quad \begin{cases} \mu_t = \nu + \phi X_{t-1}; \\ \lambda_{1t} = (\mu_t - r)/\sigma_t; \\ \lambda_{2t} = \mathbb{E}L_t; \\ \sigma_t^2 = f(\sigma_s^2, Z_s, \lambda_{1s}; -\infty < s < t); \end{cases} \tag{4.2.8}$$

The LRNVR implies that under the risk neutral measure $\mathbb{Q}$ the return process evolves as

$$\begin{aligned} X_t &= r + \sigma_t(Z_t + L_t - \mathbb{E}L_t), \quad \begin{cases} Z_t \sim N(0,1), L_t \sim NIG(\Theta); \\ \Theta = (\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG}); \end{cases} && (4.2.9) \\ \sigma_t^2 &= \omega + \alpha(Z_{t-1} - \lambda_{t-1})^2\sigma_{t-1}^2 + \beta\sigma_{t-1}^2, && (4.2.10) \\ \lambda_{t-1} &= (\mu_{t-1} - r)/\sigma_{t-1}, && (4.2.11) \\ \mu_{t-1} &= v + \phi X_{t-2}, && (4.2.12) \end{aligned}$$

It follows quite easily that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[X_t|\mathcal{F}_{t-1}] = r \ \text{ and } \ \mathrm{Var}^{\mathbb{P}}(X_t|\mathcal{F}_{t-1}) &= \mathrm{Var}^{\mathbb{Q}}(X_t|\mathcal{F}_{t-1}) \\ &= \sigma_t^2(1 + \mathrm{Var}^{\mathbb{Q}}L_t) \end{aligned} \tag{4.2.13}$$

The following propositions provide the unconditional variance for the process X_t under $\mathbb{Q}$

Proposition 4.2.1. *Consider AR(3) APARCH(1,1) Levy filter, with $\delta = 2$ and $k = 0$ which implies AR(3)GARCH(1,1) Lévy model, the unconditional variance of X_t under the LRNVR equivalent measure $\mathbb{Q}$ is*

$$\mathrm{Var}^{\mathbb{Q}}X_t = \frac{(1 + \mathrm{Var}L_t)\left(\omega + \alpha\left[v - r(1 - \sum_{j=1}^{3}\phi_j)\right]^2 + 2r\sum_{i\neq j}^{3}\phi_j\phi_i\right)}{1 - \alpha[1 + (1 + \mathrm{Var}L_t)(\sum_{j=1}^{3}\phi_j^2)] - \beta}$$

Proof of proposition 4.2.1
Given $X_t = r + \sigma_t(Z_t + L_t - \mathbb{E}L_t); \lambda_t = (\mu_t - r)/\sigma_t; \quad \mu_t = v + \sum_{j=1}^{3}\phi_j y_{t-j}$ We note that $\mathbb{E}^{\mathbb{Q}}X_t = r$ and

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[X_t^2] &= \mathbb{E}^{\mathbb{Q}}(r^2 + 2r\sigma_t Z_t(L_t - \mathbb{E}L_t) + \sigma_t^2 Z_t^2(L - \mathbb{E}L_t)^2) \\ &= r^2 + \mathbb{E}^{\mathbb{Q}}\sigma_t^2(1 + \mathrm{Var}L_t) \\ \mathbb{E}^{\mathbb{Q}}[\sigma_t^2] &= \omega + \alpha\mathbb{E}^{\mathbb{Q}}(Z_{t-1} - \lambda_{t-1})^2\sigma_{t-1}^2 + \beta\mathbb{E}^{\mathbb{Q}}\sigma_{t-1}^2 \\ &= \omega + \alpha(\mathbb{E}^{\mathbb{Q}}[\sigma_{t-1}^2] + \mathbb{E}^{\mathbb{Q}}(\mu_t - r)^2) + \beta\mathbb{E}^{\mathbb{Q}}\sigma_{t-1}^2 \end{aligned}$$

after rearranging and simple algebra

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[\mu_{t-1} - r]^2 &= v^2 + (r^2 + \mathbb{E}^{\mathbb{Q}}\sigma_{t-1}^2(1 + \mathrm{Var}L_t))(\sum_{j=1}^{3}\phi_j^2) - 2vr(1 - \sum_{j=1}^{3}\phi_j) + r^2(1 - \sum_{j=1}^{3}\phi_j) + \dots \\ &= v^2 + \left(r^2 + \mathbb{E}^{\mathbb{Q}}\sigma_{t-1}^2(1 + \mathrm{Var}L_t)\right)(\sum_{j=1}^{3}\phi_j) + r(1 - 2v)(1 - \sum_{j=1}^{3}\phi_j) + 2r^2\sum_{j\neq k}^{3}\phi_j\phi_k \end{aligned}$$

Thus under stationarity, the unconditional expectations are independent of t

$$\mathbb{E}^{\mathbb{Q}}[\sigma_t^2] = \frac{\omega + r^2(\sum_{j=1}^3 \phi_j) + r(1-2v)(1-\sum_{j=1}^3 \phi_j) + 2r^2 \sum_{j\neq k}^3 \phi_j\phi_k}{1-\alpha[1+(1+\text{Var}L_t)\sum_{j=1}^3 \phi_j^2] - \beta}$$

Therefore the unconditional variance of AR(3)GARCH(1,1)Levy filter model under LRNVR equivalent martingale measure is

$$\begin{aligned}\text{Var}^{\mathbb{Q}} X_t &= \frac{(1+\text{Var}L_t)\left(\omega + \alpha\left[v^2 - 2vr(1-\sum_{j=1}^3 \phi_j) + r^2(1-2\sum_{j=1}^3 \phi_j + \sum \phi_j^2) + 2r\alpha \sum_{j\neq k}^3 \phi_j\phi_k\right]\right.}{1-\alpha[1+(1+\text{Var}L_t)(\sum_{j=1}^3 \phi_j)] - \beta} \\ &= \frac{(1+\text{Var}L_t)\left(\omega + \alpha\left[v - r(1-\sum_{j=1}^3 \phi_j)\right]^2 + 2\alpha r \sum_{i\neq j}^3 \phi_j\phi_i\right)}{1-\alpha[1+(1+\text{Var}L_t)(\sum_{j=1}^3 \phi_j^2)] - \beta}\end{aligned}$$

. □

Proposition 4.2.2. *A special case of AR(1)GARCH(1,1)Lévy filter the unconditional variance under the LRNVR equivalent measure $\mathbb{Q}$ is given by*

$$\text{Var}^{\mathbb{Q}} X_t = \frac{(1+\text{Var}L_t)[\omega + \alpha(v - r(1-\phi))^2]}{1-\alpha(1+\phi^2(1+\text{Var}L_t)) - \beta}$$

Proof of proposition 4.2.2

This is a special case of (4.2.1) with $\phi_1 = \phi$ and $\phi_2 = \phi_3 = 0$.□

Example 4.2.1. *In case of Hyperbolic distribution we substitute mean and variance respectively into (4.2.14). Where the parameters used maximize the likelihood function of Hyperbolic distribution. i.e. Let $\zeta_{HP} = \delta_{HP}\sqrt{\alpha_{HP}^2 - \beta_{HP}^2}$ then,*

$$\mathbb{E}L_t = \mu_{HP} + \frac{\beta_{HP}\delta_{HP}}{\sqrt{\alpha_{HP}^2 - \beta_{HP}^2}}\frac{K_2(\zeta_{HP})}{K_1(\zeta_{HP})}, \quad and \tag{4.2.14}$$

$$= 0.0073397 \tag{4.2.15}$$

$$\text{Var}L_t = \delta_{HP}^2\left(\frac{K_2(\zeta_{HP})}{\zeta_{HP}K_1(\zeta_{HP})} + \frac{\beta_{HP}^2}{\alpha_{HP}^2 - \beta_{HP}^2}\left[\frac{K_3(\zeta_{HP})}{K_1(\zeta_{HP})} - \left(\frac{K_2(\zeta_{HP})}{K_1(\zeta_{HP})}\right)^2\right]\right) \tag{4.2.16}$$

$$= 1.713026 \tag{4.2.17}$$

Consider a discrete time economy, where interest rates and returns are paid after each time interval of equispaced length. Suppose there is a price for risk, measured in terms of a risk premium that is added to the risk free interest rate r to build the expected next period return. As in Duan (1995), we adopt and extend the ARCH-M model of Engle et al. (1987) with the risk premium being linear functional of the conditional standard deviation, hence the following model under $\mathbb{P}$,

$$X_t = r + \lambda\sigma_t + \varepsilon_t \quad \text{where} \begin{cases} \varepsilon_t|\mathcal{F}_{t-1} = \sigma_t(Z_t + L_t), & L_t \text{ infinitely divisible density:} \\ Z_t \sim N(0,1), & Z_t \text{ Standard normal;} \\ \sigma_t^2 = \omega + (\alpha\sigma_{t-1}Z_{t-1})^2 + \beta\sigma_{t-1}^2, & \text{GARCH(1,1);} \end{cases} \tag{4.2.18}$$

The parameters ω, α, and β are constant parameters satisfying stationarity and positivity conditions, while the constant parameter λ may be interpreted as the unit price for risk. If we change the function σ_t^2 in (4.2.18) to model news impact, we get threshold GARCH model of Glosten et al. (1993) where

$$g(x) = \omega + \alpha_1 x^2 \mathbb{I}_{x<0} + \alpha_2 x^2 \mathbb{I}_{x>0} \tag{4.2.19}$$

hence the resulting TGARCH Levy filter model

$$X_t = r + \lambda\sigma_t + \varepsilon_t \quad \text{where} \begin{cases} \varepsilon_t|\mathcal{F}_{t-1} = \sigma_t(Z_t + L_t), & L_t \text{ infinitely divisible density:} \\ Z_t \sim N(0,1), & Z_t \text{ Standard normal;} \\ \sigma_t^2 = g(\sigma_{t-1}Z_{t-1}) + \beta\sigma_{t-1}^2, & \text{TGARCH(1,1);} \end{cases} \tag{4.2.20}$$

Proposition 4.2.3. *The unconditional variance of the TGARCH-M Lévy filter model under equivalent martingale measure $\mathbb{Q}$ is*

$$\text{Var}^{\mathbb{Q}} X_t = \frac{\omega(1 + \text{Var}L_t)}{1 - \alpha_1\psi(\lambda) - \alpha_2(1 + \lambda^2 - \psi(\lambda)) - \beta} \tag{4.2.21}$$

where

$$\psi(u) = \frac{u}{\sqrt{2\pi}}\exp(-\frac{1}{2}u^2) + (1 + u^2)\Phi(u) \tag{4.2.22}$$

and $\Phi(u)$ denoting the cumulative standard normal distribution function.

Proof:

Under measure $\mathbb{Q}$

$$\begin{aligned} X_t &= r + \varepsilon_t \\ &= r + \sigma_t(\lambda + Z_t + L_t - \mathbb{E}L_t) \end{aligned}$$

where λ is the risk premium and

$$\sigma_t^2 = \omega + \alpha_1\sigma_{t-1}^2(Z_{t-1} - \lambda)^2_{\mathbb{I}_{Z'_{t-1}<0}} + \alpha_2\sigma_{t-1}^2(Z_{t-1} - \lambda)^2_{\mathbb{I}_{Z'_{t-1}<0}}$$

$$\begin{aligned}
\mathrm{Var}^{\mathbb{Q}} X_t &= \mathbb{E}^{\mathbb{Q}} y_t^2 - r^2 \\
&= r^2 + \mathbb{E}\sigma_{t-1}^2(1 + \mathrm{Var}L_t) - r^2 \\
\mathbb{E}^{\mathbb{Q}}\sigma_t^2 &= \omega + \alpha_1\psi(\lambda)\mathbb{E}^{\mathbb{Q}}\sigma_{t-1}^2 + \alpha_2[1 + \lambda^2 - \psi(\lambda)]\mathbb{E}^{\mathbb{Q}}\sigma_{t-1}^2 + \beta\mathbb{E}^{\mathbb{Q}}\sigma_{t-1}^2 \\
&= \frac{\omega}{1 - \alpha_1\psi(\lambda) - \alpha_2(1 + \lambda^2 - \psi(\lambda)) - \beta} \\
\text{thus } \mathrm{Var}^{\mathbb{Q}}(X_t) &= \frac{\omega(1 + \mathrm{Var}L_t)}{1 - \alpha_1\psi(\lambda) - \alpha_2(1 + \lambda^2 - \psi(\lambda)) - \beta} \\
\text{where, } \psi(u) &= \frac{u}{\sqrt{2\pi}}\exp(-\frac{1}{2}u^2) + (1 + u^2)\Phi(u)
\end{aligned}$$

and $\phi(u)$ denoting the cumulative standard normal distribution. Note that $Z'_{t-1} \sim N(-\lambda, 1)$ and

$$\begin{aligned}
\mathbb{E}[Z'^2_t \mathbb{I}_{Z'_t<0}|\mathcal{F}_{t-1}] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} Z^2\exp(-(Z+\lambda)^2/2)dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} (u-\lambda)^2\exp(-u^2/2)du \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} u^2\exp(-u^2/2)du - \frac{2\lambda}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} u\exp(-u^2/2)du + \frac{\lambda^2}{\sqrt{2\pi}}\int_{-\infty}^{\lambda}\exp(-u^2/2 \\
&= \frac{-\lambda}{\sqrt{2\pi}}\exp(-\lambda^2/2) + \Phi(\lambda) + \frac{2\lambda}{\sqrt{2\pi}}\exp(-\lambda^2/2) + \lambda^2\Phi(\lambda) \\
&= \frac{\lambda}{\sqrt{2\pi}}\exp(-\frac{\lambda^2}{2}) + (1+\lambda^2)\phi(\lambda) \\
&=: \psi(\lambda)
\end{aligned}$$

Proposition 4.2.4. *The unconditional variance of the GARCH-M Levy filter model under the LRNVR equivalent martingale measure* $\mathbb{Q}$ *is*

$$\mathrm{Var}^{\mathbb{Q}} X_t = \frac{\omega(1 + \mathrm{Var}L_t)}{1 - \alpha(1+\lambda^2) - \beta} \tag{4.2.23}$$

Proof of proposition 4.2.4
It is a special case of proposition 4.2.3 when we take $\alpha_1 = \alpha$ and $\alpha_2 = 0$.□

4.3 Application to option valuation

4.3.1 The Black-Scholes and Merton (BSM73) model

In its simplest form, the Black and Scholes (1973) and Merton (1973) model involves only two underlying assets, a riskless asset cash bond and a risky asset stock. It is assumed that a cash bond appreciates at a short rate r, while the share price S_t of a risky asset at time t is assumed to follow a geometric differential equation of the form $dS_t = \mu S_t dt + \sigma S_t dW_t$ where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion, while μ and σ are constants. The price of a European call option at time t for a give strike price K is given by

$$C^{BS}(S_t, T-t, \sigma, K) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\max(S_T - K, 0)\right] \quad (4.3.1)$$
$$= S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \quad (4.3.2)$$
$$d_1 = \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \quad (4.3.3)$$
$$d_2 = d_1 - \sigma\sqrt{T-t} \quad (4.3.4)$$

Note: As stated in Duan (1995), the Black-Scholes option price in the GARCH framework and for our case AR-GARCH-Lévy process respectively should be interpreted as an incorrect homoscedasticity and hence an incorrect unconditional standard deviation for the risk neutralized asset return process. Specifically we substitute σ in (4.3.3) with

$$\sigma_g^2 = \omega(1-\alpha-\beta)^{-1}, \text{ and } \sigma_l^2 = \omega(1 + \mathrm{Var}L_t)(1-\alpha-\beta)^{-1}$$
$$\sigma^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \bar{X})^2 \quad (4.3.5)$$

To compute respective differences in option prices under BS73 model and observed market option prices as strike price k varies, $C^{mrkt}(S_t, T-t, k)$, we define and compute the following functions

$$f_{BM}(k) = C^{BS}(S_t, T-t, \sigma, k) - C^{mrkt}(S_t, T-t, k), \quad (4.3.6)$$
$$f_{GM}(k) = C^{BS}(S_t, T-t, \sigma_g, k) - C^{mrkt}(S_t, T-t, k), \quad (4.3.7)$$
$$f_{LM}(k) = C^{BS}(S_t, T-t, \sigma_l, k) - C^{mrkt}(S_t, T-t, k). \quad (4.3.8)$$

where k represents strike price $k \in (S_t - M, S_t + M)$, $M \in \mathbb{R}$. The corresponding results are presented graphically in Figures(4.3) and (4.4).

4.3.2 The GARCH option pricing model (Duan95)

According to Duan (1995), pricing contingent payoffs requires temporally aggregating one period asset returns to arrive at a random terminal asset price at some future point in time T.

$$S_T = S_t \exp\left[(T-t)r - \frac{1}{2}\sum_{s=t+1}^{T}\sigma_s^2 + \sum_{s=t+1}^{T}\xi_s\right] \quad (4.3.9)$$
$$\xi_t|\mathcal{F}_{t-1} \sim N(0, \sigma_t^2)$$
$$\sigma_t^2 = \alpha_0 + \alpha(\xi_{t-1} - \lambda\sigma_{t-1})^2 + \beta\sigma_{t-1}^2$$
$$C^{D95}(S_t, T-t, \sigma_t, K) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0)|\mathcal{F}_t] \quad (4.3.10)$$

Because there is no analytic expression for the expectation in (7.3.12), we use numerical techniques to simulate the option price. That is the distribution of the payoff function $\max(S_T - K, 0)$ at the terminal date is simulated by generating m stock price processes and taking their averages.

$$C^{D95}(S_t, T-t, \sigma_t, K) = e^{-r(T-t)}\frac{1}{m}\sum_{j=1}^{m}\max(S_T^{(j)} - K, 0) \quad (4.3.11)$$

We define and study the following differences

$$\begin{aligned} g_{DM}(S_t, T-t, \sigma_t, k) &= C^{D95}(S_t, T-t, \sigma_t, k) - C^{mrkt}(S_t, T-t, k) \\ g_{DB}(S_t, T-t, \sigma_t, k) &= C^{D95}(S_t, T-t, \sigma_t, k) - C^{BS}(S_t, T-t, \sigma, k) \end{aligned} \quad (4.3.12)$$

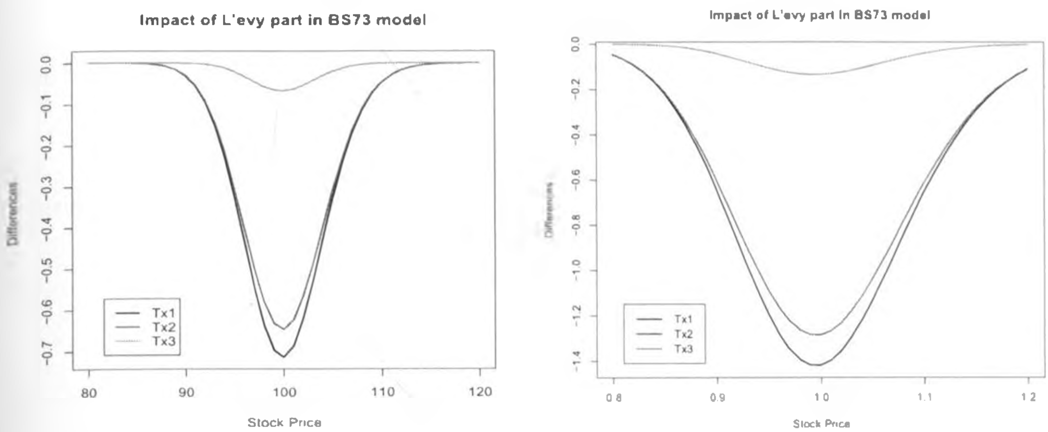


Figure 4.3: Differences between Black and Scholes prices, BS73 unconditional GARCH(1,1) variance model and unconditional GARCH-Lévy variance model for S&P500 index log returns compared for 10 days and 40 days respectively to maturity. The X-axis represent the moneyness (S_t/K) of the option.

4.3.3 AR-APARCH-Lévy filter model

The price of a European call option with strike price K and time to maturity T days, is computed by simulating (4.2.9)-(4.2.12) under risk neutral measure $\mathbb{Q}$. We apply Monte Carlo simulation which was introduced in finance several decades ago, (see Boyle (1977), Boyle et al. (1997), McLeish (2005)) to price options using AR-APARCH-Lévy filter model. We draw $(T+1)$ independent standard normal random variables and Normal Inverse Gaussian $NIG(\Theta)$ or any other infinitely divisible distribution such as Generalized hyperbolic and Variance gamma densities which fit the standardized residuals as shown in figure 5.1 and figure 4.2.

We simulate $(X_j, \sigma_j)_{j=0,\ldots,T}$ and compute

$$S_T^{(n)} = S_0 \exp(Tr + \sum_{j=0}^{T} X_j), n = 1, \ldots, N$$

where

$$X_j = \sigma_j(z_j + L_j - \mathbb{E}L_j), \quad \sigma_j^2 = \omega + \alpha(\sigma_j^2 z_0^2 - 2z_0(v-r) + (v-r)^2) + \beta\sigma_{j-1}^2,$$
$$\sigma_1^2 = \omega + \alpha(\sigma_0^2 z_0^2 - 2z_0(v-r) + (v-r)^2) + \beta\sigma_0^2$$
$$\sigma_0^2 = \mathrm{Var}^{\mathbb{Q}}(X_t),$$

Then we compute the discounted call option payoff

$$C^{(n)} = \exp(-rT)\max(0, S_T^{(n)} - K).$$

Iterating the procedure N times gives the Monte Carlo estimate for a call option price

$$C_{mc}(K,T) := N^{-1}\sum_{n=1}^{N} C^{(n)}.$$

To reduce the variance of the Monte Carlo estimate for the call option price, we calibrate the mean as in the empirical martingale simulation proposed by Duan and Simonato (1998). Scaling the simulated values $S_T^{(n)}$, $n = 1, ..., N$ by a multiplicative factor, the method ensures that the risk neutral expectation of the underlying asset is equal to the forward price.

$$N^{-1}\sum_{n=1}^{N} S_T^{(n)} = S_0\exp(rT) \quad \text{where} \tag{4.3.13}$$

$$\bar{S}_T^{(n)} := S_T^{(n)} S_0 \exp(rT)\left(N^{-1}\sum_{n=1}^{N} S_T^{(n)}\right)^{-1}. \tag{4.3.14}$$

We compute option prices using $\bar{S}_T^{(n)}$. i.e.

$$C^{APL}(S_t, T, \sigma_t, k) = \exp(-rT)\max(0, \bar{S}_T^{(n)} - K). \tag{4.3.15}$$

4.4 Empirical data

For simplicity we focus on daily closing indices $\{S_t\}$ as reported in Nairobi stock exchange for NSE20 share index and S&P500 index in New-York Stock Exchange. Daily log-returns X_t of S&P500 index are computed from January 03,1990 to January 18, 2008 for a total of 4550 daily observations. While for NSE20 share index are computed from March 02,1998 to July 11, 2007 for a total of 2317 daily observations.

All return series exhibit strong conditional heteroscedasticity. The Ljung and Box test rejects the hypothesis of homoscedasticity at all common levels both for returns in S&P500 index and AR(3)residuals of linear regression in NSE20 share index. We estimate GARCH type models assuming conditional normality. With respect to the absolute value of parameter estimates, we find that $(0 < \alpha + \beta < 1)$ but different for both indices (NSE20 $(0 < \alpha + \beta = 0.924238 < 1)$, S&P500 $(0 < \alpha + \beta = 0.994097 < 1)$), indicating the typical higher persistence of shocks in volatility in New York Stock exchange compared to Nairobi Stock exchange.

where

$$X_j = \sigma_j(z_j + L_j - \mathbb{E}L_j),\quad \begin{aligned} \sigma_j^2 &= \omega + \alpha(\sigma_j^2 z_0^2 - 2z_0(v-r) + (v-r)^2) + \beta\sigma_{j-1}^2, \\ \sigma_1^2 &= \omega + \alpha(\sigma_0^2 z_0^2 - 2z_0(v-r) + (v-r)^2) + \beta\sigma_0^2 \\ \sigma_0^2 &= \mathrm{Var}^{\mathbb{Q}}(X_t), \end{aligned}$$

Then we compute the discounted call option payoff

$$C^{(n)} = \exp(-rT)\max(0, S_T^{(n)} - K).$$

Iterating the procedure N times gives the Monte Carlo estimate for a call option price

$$C_{mc}(K,T) := N^{-1}\sum_{n=1}^{N} C^{(n)}.$$

To reduce the variance of the Monte Carlo estimate for the call option price, we calibrate the mean as in the empirical martingale simulation proposed by Duan and Simonato (1998). Scaling the simulated values $S_T^{(n)}$, $n = 1, \ldots, N$ by a multiplicative factor, the method ensures that the risk neutral expectation of the underlying asset is equal to the forward price.

$$N^{-1}\sum_{n=1}^{N} S_T^{(n)} = S_0\exp(rT) \quad \text{where} \tag{4.3.13}$$

$$\tilde{S}_T^{(n)} := S_T^{(n)} S_0 \exp(rT)\left(N^{-1}\sum_{n=1}^{N} S_T^{(n)}\right)^{-1} \tag{4.3.14}$$

We compute option prices using $\tilde{S}_T^{(n)}$. i.e.

$$C^{APL}(S_t, T, \sigma_t, k) = \exp(-rT)\max(0, \tilde{S}_T^{(n)} - K). \tag{4.3.15}$$

4.4 Empirical data

For simplicity we focus on daily closing indices $\{S_t\}$ as reported in Nairobi stock exchange for NSE20 share index and S&P500 index in New-York Stock Exchange. Daily log-returns X_t of S&P500 index are computed from January 03,1990 to January 18, 2008 for a total of 4550 daily observations. While for NSE20 share index are computed from March 02,1998 to July 11, 2007 for a total of 2317 daily observations.

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Model (5.3.14) is estimated using Pseudo Maximum Likelihood estimator based on the assumption of conditional normal innovations. The parameter estimates are reported in Table 5.4. For more on derivations of conditional likelihoods and GARCH models see Hamilton (1994), Tsay (2002).

Since the standard benchmark model in option pricing is Black and Scholes (1973), we compare simulated GARCH model (7.3.13) and AR-APARCH Levy model. We therefore estimate two models and compare the resulting option price differences graphically as days to maturity vary. The results are shown in figure(4.5) and figure(4.4)

Table 4.1: GARCH and GJR model estimates for the indices.

$$\begin{aligned} X_t &= \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + \varepsilon_t, \ \varepsilon_t = \sigma_t Z_t, \ Z_t \sim N(0,1), \\ \sigma_t^\delta &= \alpha_0 + \sum_{i=1}^{m} \alpha_i (|\varepsilon_{t-i}| - \gamma_i \varepsilon_{t-i})^\delta + \sum_{j=1}^{n} \beta_j \sigma_{t-j}^\delta \end{aligned}$$

	NSE20		S&P500	
Parameter	GARCH	GJR ($\delta = 2$)	GARCH	GJR($\delta = 2$)
ϕ_1	0.18915(.024496)	0.18136(0.02424)		
ϕ_2	0.16451(.023785)	0.16245(0.02352)		
ϕ_3	0.11388(.023413)	0.11516(0.02308)		
$\omega * 10^4$	0.03549(.006902)	0.03458(0.00647)	0.006577(.001645)	0.01088(0.00204)
α	0.15023(.017978)	0.18578(0.02528)	0.056461(.0067528)	0.00322(0.00512)
β	0.78763(.024753)	0.79045(0.02373)	0.937566(.0074845)	0.93202(0.0079)
GJR(γ)		-0.07332(0.02592)		0.10558(0.0123)
$Q(10)$	9.3468(0.2287)	8.8337(0.2648)	16.5309(0.08541)	15.2862(0.1220)
$Q^2(10)$	7.1689(0.5739)	8.46159(0.38973)	6.8918(0.54835)	5.9298(0.6551)
lgl	-8363.5	-8367.7	-15090.9	-15090.9
n	2316	2316	4549	4549

Notes: Standard errors are in parenthesis. lgl is the log likelihood.

Table 4.2: Calibration of AR-GARCH(1,1) residuals to GH and its subclasses.

$$
\begin{aligned}
f_{GH}(x;\alpha,\beta,\delta,\mu,\lambda) &= \frac{(\gamma/\delta)^{\lambda}}{\sqrt{2\pi}K_{\lambda}(\delta\gamma)}\frac{K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2+(x-\mu)^2})}{(\sqrt{\delta^2+(x-\mu)^2}/\alpha)^{\frac{1}{2}-\lambda}}e^{\beta(x-\mu)} \\
f_{NIG}(x;\alpha,\beta,\delta,\mu) &= \frac{\alpha}{\pi}\exp(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu))\frac{K_1(\alpha\delta\sqrt{1+x^2})}{\sqrt{1+x^2}}
\end{aligned}
$$

NSE20	GH	HY	NIG	S&P500	GH	HY	NIG
λ	-1.79233	1.0000	-0.5000	λ	2.38336	1.0000	-0.500
α	0.98225	1.15813	0.66862	α	0.14671	1.68640	1.33977
β	-0.05226	-0.06604	-0.05864	β	-0.14279	-0.14976	-0.15755
δ	1.79373	0.45207	1.18530	δ	0.04052	1.04004	1.59588
μ	0.12296	0.13923	0.13014	μ	0.14292	0.15130	0.16032

4.5 Conclusions

This chapter develops an AR APARCH Lévy filter option pricing model using a local risk neutralization. The proposed pricing model may be computationally demanding, but the rewards are enormous.

Our method delivers predictive distribution of the payoff function for a given econometric model. This probability distribution could be useful to market participants who wish to compare the model predictions to potential prices on the market or other participants who wish to compare the model predictions in developed and emerging markets.

The paper shows that we can price options from developed and emerging economies using AR-APARCH-Lévy filter. Analytically approximating the distribution function of the terminal asset price under locally neutralized pricing measure can be done by calculating AR-APARCH Lévy parameters from market data and applying Monte Carlo simulation.

Option prices crucially depends on volatility estimates. The presence of autoregressive dynamics affects prices directly, especially in developing economies as evidenced by NSE20 share index. To account for stylized facts of stock returns in both economies, we evaluate option prices under risk neutral pricing measure using Monte Carlo simulation. Our results may be summarized as follows:

a) Developed market and emerging economies may not have the same underlying dynamics as universally assumed in literature, AR-GARCH effects imply U-shaped differences of

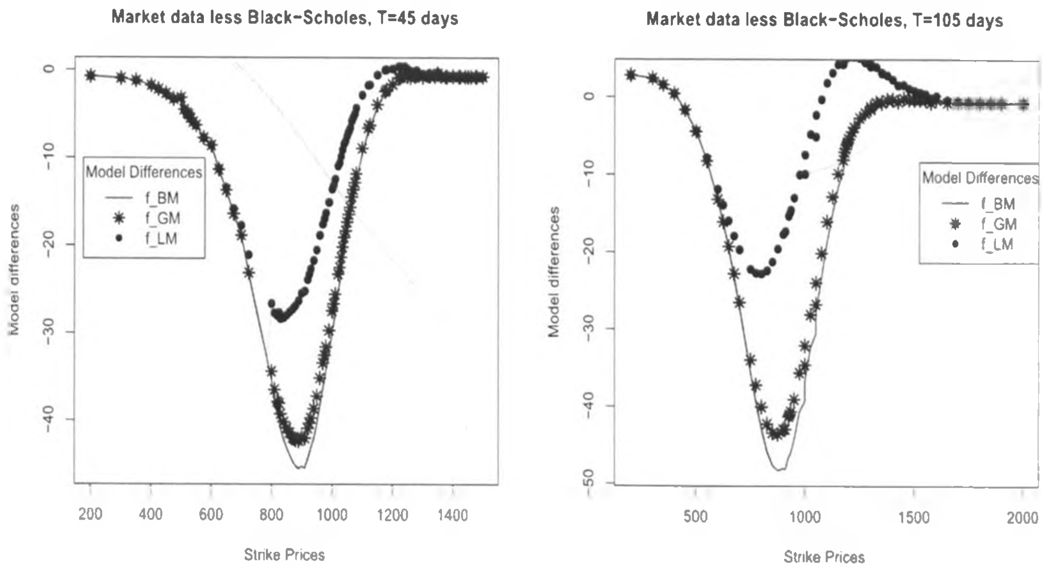


Figure 4.4: Investigating the difference between BSM73 model when σ is replaced with different unconditional risk-neutral variance from AR APARCH Levy Filter.

options to BSM73 price viewed as function of moneyness.

b) The presence of linear autoregressive dynamics AR(3) effects in NSE20 index affects the unconditional variance in risk neutral world and the computed option prices. The U-shaped differences are relatively smaller in magnitude compared to the S&P500 case in all maturities viewed as function of moneyness.

c) The option prices differences by both models (BSM73 and AR-APARCH Levy filter) from both markets for longer maturities say 120 days, is highest for all values of moneyness. One can conjecture that the proposed model may have superior predictive power as opposed to BSM73 model.

d) Both models, sparingly agree on short term maturities say 10 days, far out of money and in the money but not at the money viewed as function of moneyness.

Overall, we find that AR-APARCH-Levy filter deliver promising empirical performance and we hope to provide additional empirical support in future work. Casting the option pricing problem in incomplete market allows for more flexibility in calibration of market prices. It may be interesting to study continuous autoregressive moving average (CARMA) and COGARCH model driven by any class of Levy process. Further refinements and extensions are left for future research.

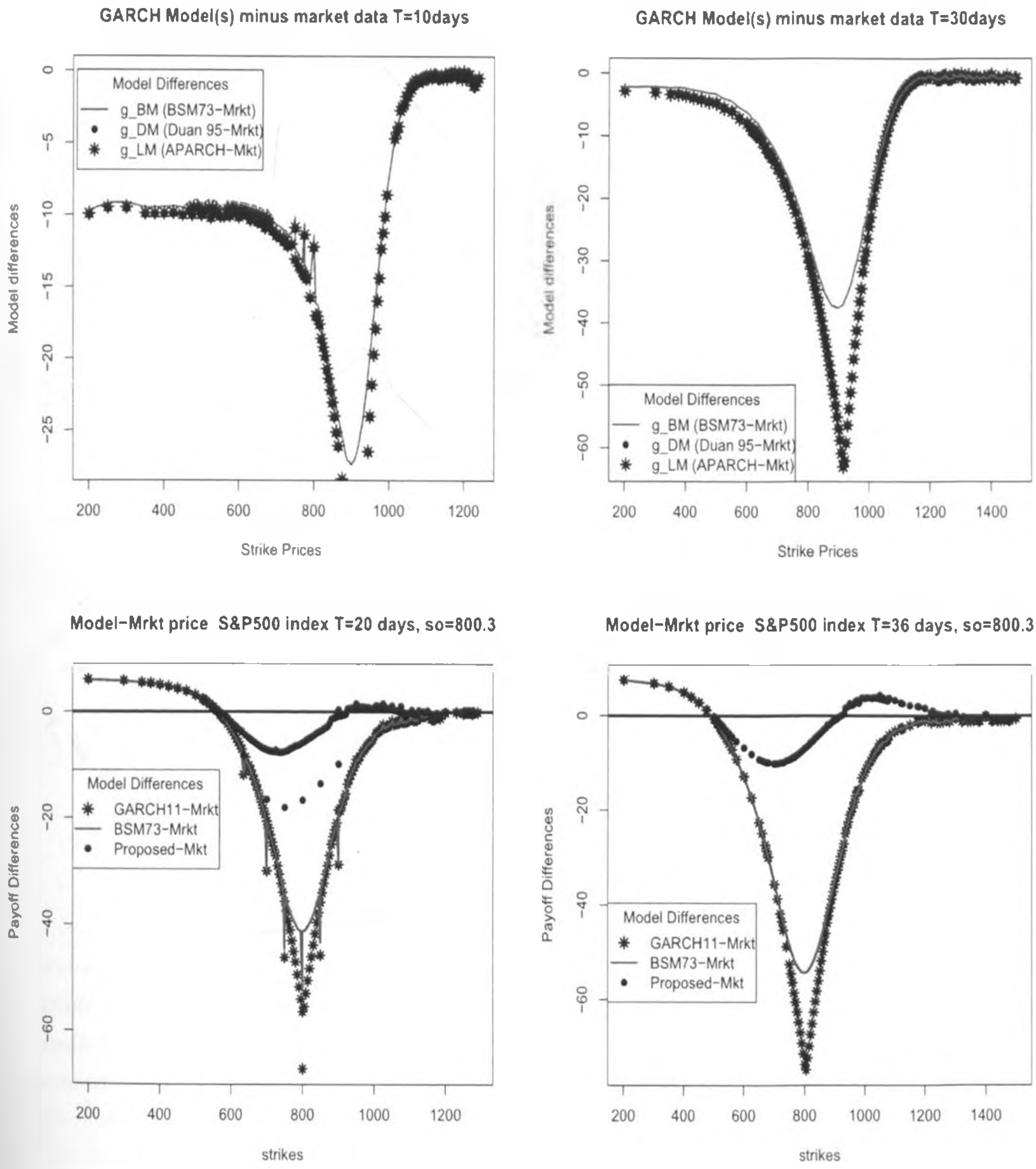


Figure 4.5: Investigating the difference between BSM73 model and simulated AR APARCH Levy filter option prices model when applied to NSE20 data and SP500 index data. The x-axis represent the Strike price K of the option.

Chapter 5

Asian and Lookback Option Pricing under AR-APARCH Lévy Filter

This chapter examines the pricing of Asian and lookback options, when the underlying asset is assumed to follow an AR-APARCH-Lévy process in any economy (developed or emerging). We show how one can price such exotic options whose value depends on the price path of the underlying asset. Empirical results, reveal that decomposition of log returns into three components, improves pricing performance compared to Black Scholes and Merton (BSM73) model. Numerical illustrations and comparisons are presented.

5.1 Introduction

European and American calls and puts are by far the most popular financial options. However, the development of financial markets has spawned many other types of options, eg exotic options: marketed as part of financial package. In the last two decades, interest in exotic options have been growing, especially in the over the counter (OTC) market. Most of these contingent claims are path dependent.

An option is called path-dependent if its price depends on the path followed by the underlying asset since the inception of the contract, in addition to the underlying asset price and the remaining time to maturity. Usually there exists one sufficient statistic describing the past. For example for an average rate (Asian) currency option, which is popular with the firms that have a streams of cashflow in a foreign currency, this statistic is the average of past values. We confine our study to Asian and lookback options.

Popular models such as the Black and Scholes (1973) model, based on the geometric Brownian motion, have very nice mathematical properties which have been extensively used to price options. The value of a European call option for example, is derived as a result of calculating the expectation

$$C = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+],$$

with respect to a risk neutral measure. Most of the published research on path dependent options assumes that the underlying asset is driven by geometric Brownian motion (GBM),- see Vecer and Xu (2004), Albrecher et al. (2008). These models, however, lack most of the empirically found features of financial data (see Cont (2001), Maheu and Mccurdy (2004)). More elaborate models can therefore be found in the literature which try to encompass these empirical findings.

Empirical studies of late have shown that the daily increments of asset prices are generally unlikely to be lognormally distributed. In this connection, more attention has been given to stochastic models which depart from the inspiring classical Black and Scholes (1973) model. Some of the most popular and still tractable models are the exponential Lévy models, advanced stochastic Lévy models etc. For an introduction on these models applied to finance we refer to Eberlein and Keller (1995), Prause (1999), Sato (1999), Raible (2000b), Barndorff-Nielsen et al. (2002), Schoutens (2003), Eberlein and Ozkan (2003), Shoutens (2006) and references therein.

It is widely accepted that financial time series of different assets share a common set of well established stylized features. Models with jump processes are thought to be more representative of actual market behavior (see Cont and Tankov (2004)). Daily log returns are known to display heavy tailed distributions, aggregational gaussianity, quasi long-range dependency in higher order central moments as documented by Duan (1995) and Rydberg (2000). To explain time series data and variation in option prices across strike price and maturity date, advanced stochastic volatility models of Ornstein-Uhlenbeck type proposed in literature employ six to ten parameters (Nicolato and Venardos (2003), Carr et al. (2003), Carr et al. (2007)), which is a far cry from a single parameter model originally proposed by Black and Scholes (1973) and Merton (1973) model.

In recent years it has been observed that distributions of log returns can be fitted to a class of Levy motions like normal inverse Gaussian, hyperbolic and Variance Gamma distributions among others. These distributions are infinitely divisible and generates a Lévy process which gives rise to exponential Lévy model proposed in literature to describe the dynamics of stock price. The exponential Lévy model $S_t = S_0 e^{X_t}$, where $\{X_t\}$ is a Lévy process, assumes that log returns $\log(S_{t+s}/S_t)$ increments of length s generate a stochastic process which is stationary and independent in all its central moments, which is not empirically observed. This approach fails to consider dependencies inherent in second and higher central moments. AR-APARCH-Lévy filter has been proposed in literature to improve on this assumption.

The outline of this chapter is as follows. In Section 2 AR-APARCH-Levy filter and infinitely divisible distributions are reviewed. Option pricing of Asian and lookback options using proposed model are documented in Section 3. Numerical demonstration is carried out in Section 4. Data sets from Nairobi stock exchange and Paris exchange are used to price exotic options, using Monte Carlo simulations. Discussions and concluding remarks are presented in Section 5.

5.2 Review of APARCH-Lévy filter

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a stochastic basis and $S = (S_t)_{t\in[0,T]}$ be the stochastic process. In continuous time economy, we assume that the price process of the stock $S = (S(t))$ is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ and is given by the exponential $\exp(X_t)$ with $X = (X_t)$ satisfying

$$dX_t = \mu_t + \sigma_B dB_t + \sigma_Y dY_t + \sigma_J dJ_t \tag{5.2.1}$$

where $\sigma_B(t, S_t), \sigma_Y(t, S_t)$ and $\sigma_J(t, S_t)$ are adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Note that $\{B_t\}_{\in[0,T]}$ denotes Brownian motion, while $\{Y_t\}_{t\in[0,T]}$ is a Lévy process and $\{J_t\}_{t\in[0,T]}$ is a jump innovation. Note that B_t, Y_t and J_t are assumed to be contemporaneously independent of each other. Analogously, in discrete time, the model is described by

$$\ln\left(\frac{S_t}{S_{t-1}}\right) := X_t \quad = \quad \mu_t + \sigma_t(Z_t + Y_t)$$

where $\sigma_B(t, S_t)$ is taken to be equal to $\sigma_Y(t, S_t) = \sigma_J(t, S_t)$. Without loss of generality, in discrete time economy, (5.2.1) is expressed in the form

$$\begin{aligned}
X_t &= \mu_t + \sigma_t(Z_t + Y_t), Z_t \sim i.i.d.(0,1), Z_0 = 0, Y_0 = 0, \\
\mu_t &= AR(s) := \sum_{j=1}^{s} \phi_j X_{t-j}, \quad t, s \in \mathbb{Z}, \phi_j \in \mathbb{R} \\
\sigma_t &= APARCH(p,q) \quad p, q \in \mathbb{Z}.
\end{aligned} \tag{5.2.2}$$

See Glosten et al. (1993), Seneta (2004), for more details about APARCH model.

5.2.1 Infinitely divisible distributions

Suppose $\phi(u)$ is the characteristic function of a distribution. If for every positive integer $n, \phi(u)$ is the n^{th} power of a characteristic function, we say that the distribution is infinitely divisible. One can define for every such an infinitely divisible distribution a stochastic process $Y = \{Y_t, t \geq 0\}$ called a Lévy process, which starts at zero, with independent and stationary increments. The distribution of an increment over $[s, s+t], s, t \geq 0$ has $(\phi(u))^t$ as its characteristic function. We consider Hyperbolic (HY), Variance Gamma (VG) and Normal Inverse Gaussian (NIG) distribution(s) which are special cases of Generalized Hyperbolic distribution(GH) as earlier defined in chapter one.

Variance Gamma distribution

The VG process is defined by time changing the arithmetic Brownian motion with drift γ and volatility σ by an independent gamma process with unit mean rate and variance rate $1/\lambda$. Let $G(t;\lambda)$ be the gamma process, then the Variance Gamma process may be written as $Y_{VG}(t;\sigma,\lambda,\gamma) = \gamma G(t;\lambda) + \sigma B(G(t;\lambda))$ where $B(t)$ is an independent standard Brownian motion. Carr and Madan (1998) show that the VG process can be expressed as the difference of two independent gamma process. The PDF of Y_t is given by

$$f_{VG}(y) = \frac{2\exp(\gamma(y-\mu)/\sigma^2)}{\sigma\sqrt{2\pi}\lambda^{-\lambda}\Gamma(\lambda)}\left(\frac{|y-\mu|}{\sqrt{2\gamma^2\lambda+\sigma^2}}\right)^{\lambda-0.5} K_{\lambda-1/2}\left(\frac{|y-\mu|\sqrt{2\sigma^2\lambda+\theta^2}}{\sigma^2}\right) \quad (5.2.3)$$

The characteristic function of Y_t is

$$\phi_{VG}(u) = e^{iu\mu}\left(1 - i\frac{\gamma}{\lambda}u + \frac{\sigma^2}{2\lambda}u^2\right)^{-\lambda}. \quad (5.2.4)$$

The mean and variance of the Variance Gamma distributed random variate Y are given respectively by $\mathbb{E}(Y) = \mu + \gamma$ and $Var(Y) = \sigma^2 + \lambda^{-1}\gamma^2$.

As stated earlier, a VG process is interpreted as the difference of two independent gamma processes, due to possible factorization of characteristic function. i.e.

$$\frac{1}{(1 - i\frac{\gamma}{\lambda} + \frac{\sigma^2}{2\lambda}u^2)} = \left(\frac{1}{1-i\eta_p u}\right)\left(\frac{1}{1-i\eta_n u}\right)$$

where

$$\eta_p - \eta_n = \gamma/\lambda, \quad \eta_p\eta_n = \frac{\sigma^2/\lambda}{2} \quad (5.2.5)$$

$$\eta_p = \sqrt{\frac{\gamma^2}{4\lambda^2} + \frac{\sigma^2}{2\lambda}} + \frac{\gamma}{2\lambda} \quad (5.2.6)$$

$$\eta_n = \sqrt{\frac{\gamma^2}{4\lambda^2} + \frac{\sigma^2}{2\lambda}} - \frac{\gamma}{2\lambda} \quad (5.2.7)$$

$$(5.2.8)$$

This leads to parametrization of CGMY process where VG process is a special case, with parameters C, G, M as expressed in (5.2.9)

$$C = \lambda = 1/\nu, \quad G = (\sqrt{\gamma^2\nu^2/4 + \sigma^2\nu/2} - \gamma\nu/2)^{-1} \quad (5.2.9)$$
$$M = (\sqrt{\gamma^2\nu^2/4 + \sigma^2\nu/2} + \gamma\nu/2)^{-1}$$

for more information, see Carr et al. (2002)

5.3 Option pricing

In this section, we value Asian and lookback options using APARCH Lévy filter model. Consider options on oil, which commonly tie the exercise price of the option to the average price of the barrel of oil, in the month before the exercise date. Such an option is useful to a company, which buys oil on monthly basis and wants to protect itself from loosing money during periods of high price volatility. An average is less volatile that the underlying asset itself. Such contingent claims, should be of interest for thinly traded assets in any market.

5.3.1 Asian options

Asian call option gives the owner the right to buy (or sell, if it is an Asian put) a share of stock for an average price using some period between the beginning of the contract and the exercise date of the option. We consider pricing of a European style arithmetic average call option with strike price K, maturity T and t_n averaging days where $0 = t_0 \leq t_1 < ... < t_n = T$.

Let A_{t_n} be the arithmetic average over the interval $[t_0, t_n]$. We express the payoff of the option as $(A_{t_n} - K)^+$. A_T is assumed to be

$$A_T = \left(\frac{1}{n+1}\sum_{i=0}^{n} S_{t_i}\right), \quad (5.3.1)$$

The price of arithmetic mean Asian option (AA_t), under risk neutral pricing measure $\mathbb{Q}$ at time t is given by

$$AA_t(K,T) = \frac{\exp\left(-r(T-t)\right)}{n+1}\mathbb{E}^{\mathbb{Q}}\left[\left(\sum_{j=0}^{n} S_{t_j} - (n+1)K\right)^+ \mid \mathcal{F}_t\right] \quad (5.3.2)$$

where $\{\mathcal{F}_t,\ 0 \leq t \leq T\}$ denotes the natural filtration of S.

The geometric average option price GA_t is given by

$$GA_t = e^{-(T-t)}\mathbb{E}^{\mathbb{Q}}[(G_T - K)^+ | \mathcal{F}_t] \quad (5.3.3)$$

where $\{G_T = (\prod_{k=1}^{n} S_{t_k})^{1/n}\}$, K denotes the strike price. Taking the logarithm of G_T, we get

$$\begin{aligned} \ln G_T &= \frac{1}{n}\sum_{k=1}^{n}\ln S_k = \ln S_0 + \frac{1}{n}\sum_{k=1}^{n}\ln\frac{S_k}{S_0} && (5.3.4)\\ &= \ln S_0 + \frac{1}{n}\left(n\ln\frac{S_1}{S_0} + (n-1)\ln\frac{S_2}{S_1} + ... + \ln\frac{S_n}{S_{n-1}}\right) && (5.3.5)\\ &= \ln S_0 + X_1 + \frac{n-1}{n}X_2 + ... + \frac{1}{n}X_n && (5.3.6) \end{aligned}$$

Note that if the input process $(X_t)_{t\in\mathbb{Z}^+}$ is AR-APARCH filtered, the resulting process,say $(Y_t)_{t\in\mathbb{Z}^+}$ is a Lévy process. The distribution of stationary daily increments is estimated and the resulting parameters are used to price exotic options. However, no analytical expression for (5.3.2) is available. Monte Carlo simulation technique is used to obtain numerical estimates in BSM73 world and APARCH-Lévy world.

5.3.2 Lookback options

Lookback options are contracts which at delivery time T allow the holder to take advantage of the realized maximum or minimum of the underlying price process over the entire contract period, see Boyle and Tian (1999), Bjork (2005), Musiela, M. et al (2004) for more contracts priced in Black-Scholes world. In this article we price the following European type lookback options numerically assuming APARCH Lévy model.

$$C_{lkbcall} = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[S(T) - \min_{t\leq T}(S(t))\right], \quad \text{lookback call,} \tag{5.3.7}$$

$$C_{lkbput} = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\max_{t\leq T}(S(t)) - S(T)\right], \quad \text{lookback put,} \tag{5.3.8}$$

$$C_{flkbcall} = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\max\left((\max_{t\leq T}(S(t)) - K), 0\right)\right], \quad \text{forward lookback call,} \tag{5.3.9}$$

$$C_{flkbput} = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\max\left((K - \max_{t\leq T}(S(t))), 0\right)\right], \quad \text{forward lookback put,} \tag{5.3.10}$$

It can be shown (see Bjork (2003) *Proposition 18.28*) in Black Scholes world that the price at $t = 0$ of the lookback put is given by

$$C_{BS_{lkbput}}(0) = -s\Phi(-d) + se^{-rT}\Phi(-d+\sigma\sqrt{T}) + s\frac{\sigma^2}{2r}\Phi(d) - se^{-rT}\frac{\sigma^2}{2r}\Phi(-d+\sigma\sqrt{T})$$

where $d = \dfrac{rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$ and Φ is the cumulative standard normal distribution

Numerical comparison of European lookback option prices using BSM73 model and APARCH Levy model respectively are presented in Table 5.3.

5.3.3 Risk neutralization in incomplete market

Risk neutralization should leave the variance unchanged while transforming the conditional expectation so that the discounted expected price of the underlying asset follows a martingale. We follow Duan (1995) general method for option pricing approach while assuming returns are predictable. He introduced the local risk neutral valuation relationship (LRNVR) which leaves the marginal variance unchanged.

Let

$$(S_t - S_{t-1})/S_{t-1} \approx \ln(S_t/S_{t-1}) := X_t$$

$$X_t = \mu_t + \varepsilon_t, \begin{cases} \varepsilon_t|\mathcal{F}_{t-1} \sim N(0, \sigma_t); \\ \sigma_t^2 = \omega + \alpha\varepsilon_t^2 + \beta\sigma_{t-1}^2; \quad \alpha,\ \beta,\ \omega > 0. \end{cases} \tag{5.3.11}$$

The starting value σ_0^2 is treated as a known constant, and the $\{\varepsilon_t\}$ sequence is conditionally independent. and μ_t represents the conditional expectation of returns.

The Pricing measure $\mathbb{Q}$ shifts the error term ε_t so that the conditional expectation of y_t becomes equal to r. The new error term is $\nu_t = \mu_t + \varepsilon_t - r$

$$X_t = \mu_t + \sigma_t(Z_t - \lambda_t), \begin{cases} Z_t \sim \ i.i.d.\ N(0,1); \\ \sigma_t^2 = f(\sigma_s^2, \varepsilon_s; -\infty < s < t; \theta); \\ \varepsilon_t = Z_t - \lambda_t; \\ \lambda_t = \frac{\mu_t - r}{\sigma_t}; \end{cases} \tag{5.3.12}$$

Similarly in the case of APARCH Lévy filter we follow the Hafner and Herwartz (2001) argument. Under the equivalent martingale measure $\mathbb{Q}$ the model translates to

$$X_t = \mu_t + \varepsilon_{1t} + \varepsilon_{2t} := \mu_t + \sigma_t(Z_t - \lambda_{1t}) + \sigma_t(Y_t - \lambda_{2t}), \begin{cases} \mu_t = \nu + \phi_1 y_{t-1}; \\ \lambda_{1t} = (\mu_t - r)/\sigma_t; \\ \lambda_{2t} = \mathbb{E}Y_t; \\ \sigma_t^2 = f(\sigma_s^2; Z_t, \lambda_{1t}, -\infty < s < t; \theta). \end{cases} \quad (5.3.13)$$

The LRNVR implies that under the risk neutral measure $\mathbb{Q}$ the return process evolves as

$$X_t = r + \sigma_t(Z_t + Y_t - \mathbb{E}Y_t), \begin{cases} Z_t \sim N(0,1), Y_t \sim NIG(\Theta); \\ \Theta = (\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG}); \\ \sigma_t^2 = \omega + \alpha(Z_{t-1} - \lambda_{t-1})^2\sigma_{t-1}^2 + \beta\sigma_{t-1}^2; \\ \lambda_{t-1} = (\mu_{t-1} - r)/\sigma_{t-1}, \quad \mu_{t-1} = v + \phi x_{t-2} \end{cases} \quad (5.3.14)$$

5.3.4 Monte Carlo simulations

In this subsection we use Monte Carlo simulations (see Boyle (1977), Glasserman et al. (1998), McLeish (2005) for more information), to compute the required arithmetic Asian option price.

Asian option based on BSM73 model

It is well known that if log returns are assumed to be normally distributed, then

$$\begin{aligned} S_t &= S_0 \exp(X_t), t = 1, ..., T \quad \text{where} \\ X_t &= \mu t + \sigma W_t, \quad \text{with} \quad \mu = r - \sigma^2/2, \quad W_t \sim N(0,t) \end{aligned} \quad (5.3.15)$$

The following steps are implemented. First we create M price paths using the above equation (5.3.15) appropriately, and for each path we compute the value

$$C_j = e^{-rT}(A_T - K)^+ \quad j = 1, ..., M.$$

where C_j is a payoff function and A_T is as defined in (5.3.1). After which we calculate the average Asian option value

$$\tilde{Asian}_{mc} = \frac{1}{M}\sum_{j=1}^{M} C_j$$

Asian option APARCH-Lévy model

We implement the following steps. Draw $(T+1)$ random variables from standard normal distribution, and normal inverse Gaussian $NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG})$ (or from hyperbolic or Variance Gamma) densities. In particular, if the market follows AR(1)-GARCH(1,1) process then, simulate (x_t, σ_t) $t = 0, \ldots T$ to compute $S_t^{(n)}, n = 1, \ldots, N$, for all t, $t = 1, \ldots, T$. i.e.

$$\begin{aligned}
S_T^{(n)} &= S_0 \exp\left[(\sum_{t=0}^{T} X_t) + Tr\right], \quad n = 1, \ldots, N \quad \text{where} \\
X_t = \sigma_t(Z_t + Y_t - \mathbb{E}Y_t), \quad \sigma_t^2 &= \omega + \alpha(\sigma_{t-1}^2 z_{t-1}^2 - 2z_{t-1}\sigma_{t-1}(v - r) + (v - r)^2 + \beta\sigma_{t-1}^2) \\
\sigma_1^2 &= \omega + \alpha(\sigma_0^2 z_0^2 - 2z_0\sigma_0(v - r) + (v - r)^2 + \beta\sigma_0^2) \\
\sigma_0^2 &= Var^{\mathbb{Q}}(X_t) \\
Var^{\mathbb{Q}} X_t &= \frac{(1 + VarY_t)[\omega + \alpha(v - r(1 - \phi))^2]}{1 - \alpha(1 + \phi^2(1 + VarY_t)) - \beta}
\end{aligned}$$

and $\{Y_t\}_{t \in \mathbb{Z}^+}$ is a Lévy process. Discounted call option payoff is computed by simulating $A_T^{(n)}$(defined in (5.3.1)) N times, which gives the Monte Carlo estimate average Asian call option price.

$$A_T^{(n)} = \frac{1}{T}\sum_{t=1}^{T} S_t^{(n)}, n = 1, \ldots, N \quad \text{say} \quad N := 10000 \text{ simulations} \tag{5.3.16}$$

$$AV_T^{(n)} = \exp(-rT)\max(0, A_T^{(n)} - K) \tag{5.3.17}$$

$$Asian_{mc} := \frac{1}{N}\sum_{n=1}^{N} AV_T^{(n)} \tag{5.3.18}$$

In particular, if the process follow AR(3)-GARCH(1,1) Lévy process. The unconditional variance under stationarity conditions (See Mwaniki (2007)) is finite if the denominator is greater than zero is given as

$$Var^{\mathbb{Q}} X_t = \frac{(1 + VarY_t)\left(\omega + \alpha\left[v - r(1 - \sum_{j=1}^{3}\phi_j)\right]^2 + 2r\sum_{i \neq j}^{3}\phi_j\phi_i\right)}{1 - \alpha[1 + (1 + VarY_t)(\sum_{j=1}^{3}\phi_j^2)] - \beta} \tag{5.3.19}$$

We price Asian option when the underlying follow AR(3)-GARCH(1,1) Lévy process. Simulate $(X_t, \sigma_t)_{j=0,\ldots,T}$ and compute

$$S_T^n = S_0 \exp\left[(\sum_{t=0}^{T} X_t) + Tr\right], \quad n = 1, \ldots, N \quad \text{where} \quad X_t = \sigma(Z_t + Y_t - \mathbb{E}Y_t) \tag{5.3.20}$$

and

$$\sigma_0^2 = Var^{\mathbb{Q}} X_t \tag{5.3.21}$$

$$\begin{aligned}
\sigma_1^2 &= \omega + \alpha(\sigma_0^2 z_0^2 - 2z_0(\mu_0 - r)\sigma_0 + (\mu_0 - r)^2) + \beta\sigma_0^2, \quad \text{where } \mu_0 = \phi_1 x_{-1} + \phi_2 x_{-2} + \phi_3 x_{-3} \\
\sigma_2^2 &= \omega + \alpha(\sigma_1^2 z_1^2 - 2z_1(\mu_1 - r)\sigma_1 + (\mu_1 - r)^2) + \beta\sigma_0^2, \quad \mu_1 = \phi_1 x_0 + \phi_2 x_{-1} + \phi_3 x_{-2} \\
\sigma_3^2 &= \omega + \alpha(\sigma_0^2 z_0^2 - 2z_0(\mu_0 - r)\sigma_2 + (\mu_2 - r)^2) + \beta\sigma_0^2, \quad \mu_2 = \phi_1 x_1 + \phi_2 x_0 + \phi_3 x_{-1}, \qquad (5.3.22) \\
\sigma_t^2 &= \omega + \alpha(\sigma_{t-1}^2 z_{t-1}^2 - 2z_{t-1}(\mu_{t-1} - r)\sigma_{t-1} + (\mu_{t-1} - r)^2) + \beta\sigma_{t-1}^2
\end{aligned}$$

$$\mu_{t-1} = \phi_1 x_{t-2} + \phi_2 x_{t-3} + \phi_3 x_{t-4} \tag{5.3.23}$$

In general it can be shown that

$$\mu_t = \phi_1\mu_{t-1} + \phi_2\mu_{t-2} + \phi_3\mu_{t-3}, \quad t = 4, \ldots, T \tag{5.3.24}$$

$$\mu_1 = \phi_1 x_0 + \phi_2 x_{-1} + \phi_3 x_{-2} \tag{5.3.25}$$

$$\mu_2 = \phi_1\mu_1 + \phi_2 x_0 + \phi_3 x_{-1} \tag{5.3.26}$$

$$\mu_3 = \phi_1\mu_2 + \phi_2\mu_1 + \phi_3 x_0 \tag{5.3.27}$$

in summary, the model becomes

$$
\begin{aligned}
S_T^{(n)} &= S_0 \exp\left[(\sum_{t=0}^{T} X_t) + Tr\right], \quad n = 1, ..., N \quad \text{where,} \quad X_t = \sigma_t(z_t + Y_t - \mathbb{E}Y_t) \\
\sigma_t^2 &= \omega + \alpha\left(\sigma_{t-1}^2 z_{t-1}^2 - 2z_{t-1}(\mu_{t-1} - r)\sigma_{t-1} + (\mu_{t-1} - r)^2\right) + \beta\sigma_{t-1}^2, \quad t = 1, ..., T. \\
\mu_t &= \phi_1\mu_{t-1} + \phi_2\mu_{t-2} + \phi_3\mu_{t-3}, \quad t = 4, ..., T, \quad \sigma_0^2 = Var^{\mathbb{Q}} X_t. \qquad (5.3.28)
\end{aligned}
$$

5.4 Data and Estimation methodology

The data used to facilitate the discussion in this chapter are daily indices, namely NSE20 share index of Nairobi stock exchange and CAC40 index of Paris stock exchange. NSE20 share index run from March 2, 1998 to July 11, 2007, while CAC40 share index runs from March 1,1990 to September 1,2008. As stated earlier, we let $\{S_t\}_{t\geq 0}$ denote the stock price process and $X_t = \log S_t - \log S_{t-1}$ denote the logarithmic increase (returns) of S over the interval $(t-1, t]$. Autocorrelation function (ACF) of squared log returns and APARCH-Levy filtered residuals in addition to fitting kernel densities (see Sheather and Jones (1991)) of NSE20 and CAC40 share indices are shown in Figure 5.1 and Figure 5.2 respectively.

Parameters of Variance Gamma (VG) distribution, Normal Inverse Gaussian (NIG) and Hyperbolic Distributions (HY) are estimated using maximum likelihood methods, see Seneta (2004), Blaesild and Sorensen (1992), and Hu (2005). Once the parameters have been estimated, the next step is to judge the quality of fit obtained. Our approach is to plot the fitted density and empirical or non-parametric density estimate on the same graph and same density estimation with log-vertical scale to compare their closeness (see Figure 5.1 and Figure 5.2). Computed prices are as shown in Figure 5.3 and Table 5.3.

Notes: Standard errors are in parentheses. lgd is the loglikelihood. n is the number of observations

5.5 Conclusions

In this chapter, we have described most of the features that characterize financial data, such as fat tail problem, volatility clustering, aggregational Gaussianity and quasi long range dependence in second central moment. In addition, the pricing of exotic path dependent options when the underlying can be characterized by APARCH-Levy filter is examined. In connection to that, we have proposed two formal procedures for pricing Asian option and lookback type options using filter model. As claimed in literature, about exponential Lévy model, we have argued that NIG, VG and HY Lévy increments fit filtered daily log return residuals of financial time series. This implies that the proposed model improves in characterizing distribution of daily log returns.

It was argued that the filtered stochastic process of daily log returns, in addition to calibrated residuals under Monte Carlo simulation, generates improved future asset path dynamics. Therefore, path depended options can be priced appropriately.

It is widely recognized today that there is a non-negligible discrepancy between the Black-Scholes model and real market behavior, which appears as the "smile effect" or "implied volatility smile" in options markets. With empirical evidence that implied volatility increases for in-the-money or out-of-the money options, our results seem to confirm the effect while the proposed model improve on the smile effect as shown in Figure 5.3.

Empirical evidence demonstrate that the prices of the exotic options (Asian and lookback) significantly deviates from those of the lognormal processes. One can infer that the accuracy of volatility estimation is most critical as it applies to pricing standard options with path dependent options. For contingent claims that depend on the extremum of the process, the prices are quite sensitive to the specification of the process.

Black-Schole's prices differ significantly from APARCH-Levy model in that they tend to be lower if the option is in and out of the money. These differences indicate that an appropriate choice of the model is of great importance for the issue of option pricing.

We cannot underestimate the repercussions of misspecified models, which may lead to mispricing of derivatives. This gives an idea why constructing a realistic model which explains real time stochastic phenomena will always be a challenging project, however we believe APARCH-Levy model takes care of observed stylistic features of stock market indices from developed and emerging economies.

It seems interesting to extend present investigation to pricing American type exotic options using multinomial lattices.

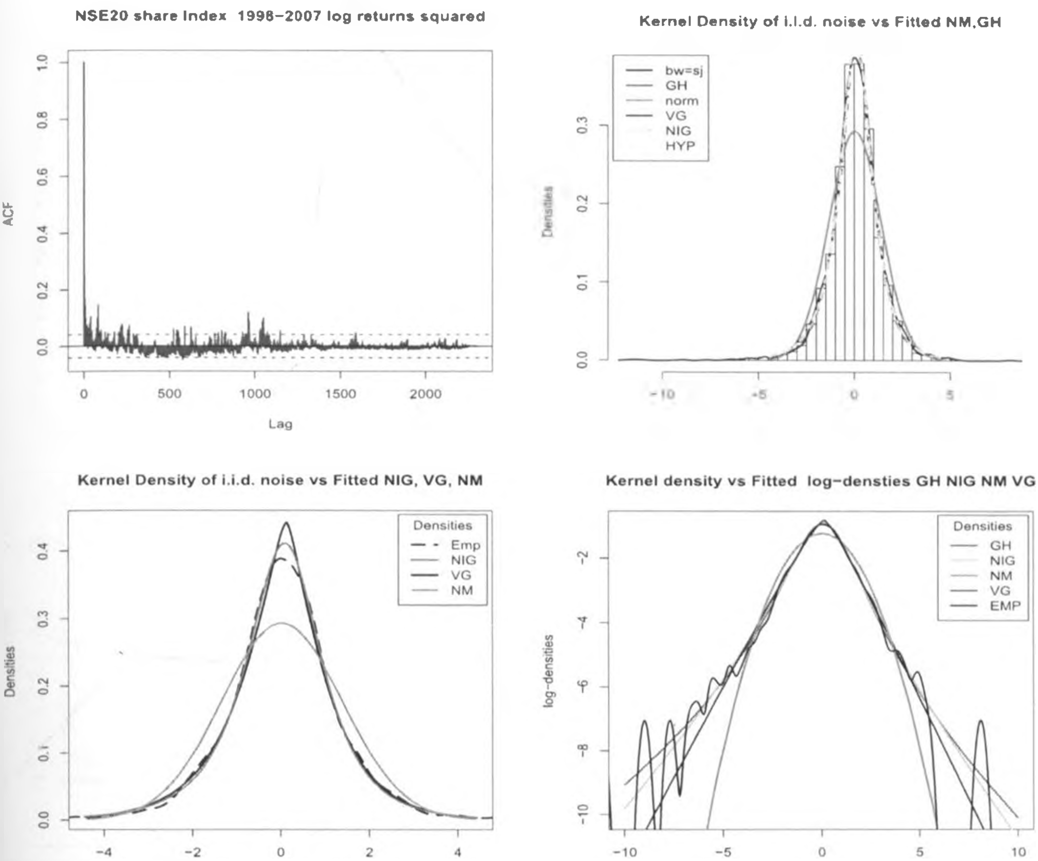


Figure 5.1: Plot of the autocorrelation function (ACF) of the squared NSE20 share index daily log returns in the period March 2, 1998 to July 11, 2007. The dashed lines indicate $\pm 1.96/\sqrt{n}$. 2500 lags are shown and each observation correspond to a business day. Long range dependence in second central moment is AR-GARCH filtered. The resulting residuals are calibrated to Normal Inverse Gaussian (NIG), Variance Gamma (VG) and Hyperbolic (HY) densities. Plot of the empirical densities(EMP) (rescaled histograms) and the estimated normal density(NM) are plotted. Note the ordinate axis is on log scale in order to exhibit tail decay.

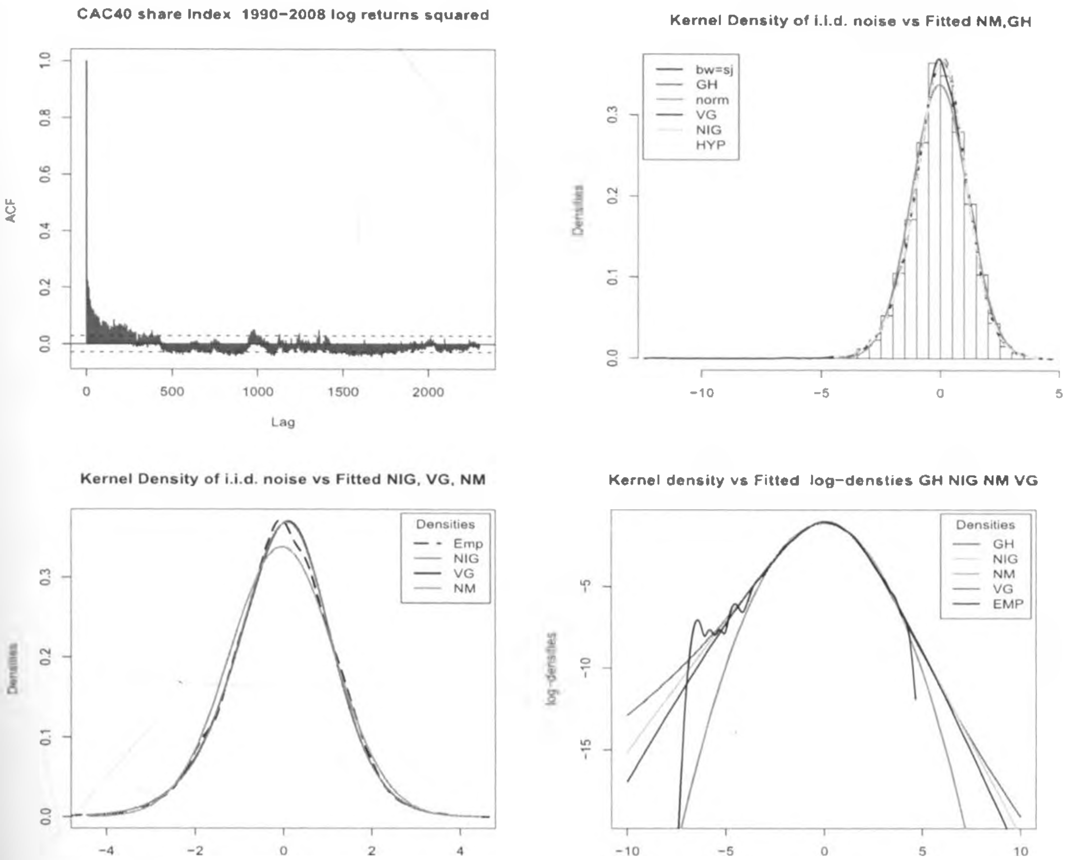


Figure 5.2: Plot of the autocorrelation function (ACF)of the squared CAC40 share index log returns in the period March 1, 1990 to September 1, 2008. The dashed lines indicate $\pm 1.96/\sqrt{n}$. 2500 lags are shown and each observation correspond to a business day. Long range dependence in second central moment is AR-GARCH filtered. The resulting residuals are calibrated to normal inverse gaussian (NIG), Variance Gamma (VG) and hyperbolic (HY) densities. Plot of the empirical densities(EMP) (see Silverman (1986) and Sheather and Jones (1991) for detailed information about kernel densities) (rescaled histograms) and the estimated normal density(NM) are plotted. Note the ordinate axis is on log scale in order to exhibit tail decay.

Table 5.1: Parameter estimates of AR(p)-GARCH(1,1) model for two indices, NSE20 ($p = 3$) and CAC40 ($p = 0$) respectively assuming normal innovations. $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + \sigma_t(Z_t + Y_t)$, $\sigma_t^2 = \omega + \sigma_{t-1}^2(\alpha Z_t^2 + \beta)$, $Z_t \sim N(0,1)$. $Q^2(10)$ are the modified Ljung-Box portmanteau test, for serial correlation in the squared standardized residuals with 10 lags while $Q(10)$ are the same test for serial correlation in the log returns. p-values are in the square brackets.

	NSE	CAC40
Parameters	GARCH	GARCH
ϕ_1	0.18915(0.024496)	
ϕ_2	0.16451(0.023785)	
ϕ_3	0.11388(0.023413)	
$\omega * 10^4$	0.03549(.006902)	0.031451(.00624)
α	0.15023(0.017978)	0.08249 (0.00907)
β	0.78763(0.024753)	0.899377(.010933)
$Q(10)$	9.3468[0.2287]	18.789[0.04302]
$Q^2(10)$	7.1689[0.5185]	10.041[0.26215]
lgd	-8363	-14082.8
n	2316	4669

Table 5.2: Calibration of AR-GARCH(1,1) residuals to a class of infinitely divisible distributions. We present maximum likelihood estimates of subclasses of Generalized Hyperbolic distribution (7.3.4), i.e. Hyperbolic distribution (HY) when $\lambda = 1$, Normal Inverse Gaussian (NIG) $\lambda = 0.5$ (1.2.7) and Variance Gamma (VG) (5.2.3). NSE20 share index of Nairobi stock exchange cover a period March 2, 1998 to July 11, 2007, while CAC40 share index of Paris exchange March 1,1990 to September 1,2008.

NSE20	HY	NIG	VG	CAC40	HY	NIG	VG
λ	1.0000	-0.5000	C=1.344094	λ	1.0000	-0.500	C=4.78872
α	1.15813	0.66862	G=1.030136	α	2.13751	1.8523	G=1.89652
β	-0.06604	-0.05864	M=1.130045	β	-0.33904	-0.15755	M=2.30262
δ	0.45207	1.18530		δ	2.05553	2.42002	
μ	0.13923	0.13014		μ	0.42889	0.41876	
lgd	-3809.9	-3801.8	-3814.0	lgd	-7323.6	-7322.4	-7326.6

Table 5.3: Comparison of simulated prices of exotic options (Arithmetic Asian call, lookback put, lookback call forward lookback put and call respectively when the underlying price process is assumed to follow AR-GARCH(1,1) NIG process) against Black Scholes and Merton lookback put option (lastcolumn). We used $r = 5\%p.a., S_0 = 80 : 120, K$ (*Strike price*) $= 100$, and $T = 50$ days to maturity. Daily CAC40 share index of Paris exchange in the period March 1, 1990 to September 1, 2008 was used as the underlying process

	Moneyness	AsianCall	lkbPut	lkbCall	FlkbCall	FlkbPut	BSM73lkbPut
1	0.80000	0.25736	9.60941	12.64613	2.52036	30.33528	5.70686
3	0.82000	0.37196	9.84964	12.96228	3.16402	28.61854	5.84953
5	0.84000	0.53006	10.08988	13.27843	3.91713	26.90180	5.99221
7	0.86000	0.74455	10.33011	13.59459	4.79401	25.18505	6.13488
9	0.88000	1.02855	10.57035	13.91074	5.80243	23.46831	6.27755
10	0.89000	1.20440	10.69047	14.06882	6.35833	22.60994	6.34888
12	0.91000	1.62502	10.93070	14.38497	7.57338	20.89329	6.49156
15	0.94000	2.46925	11.29105	14.85920	9.67640	18.32075	6.70556
18	0.97000	3.58286	11.65141	15.33343	12.15860	15.77296	6.91957
40	1.19000	19.22031	14.29400	18.81112	36.24631	3.49536	8.48896

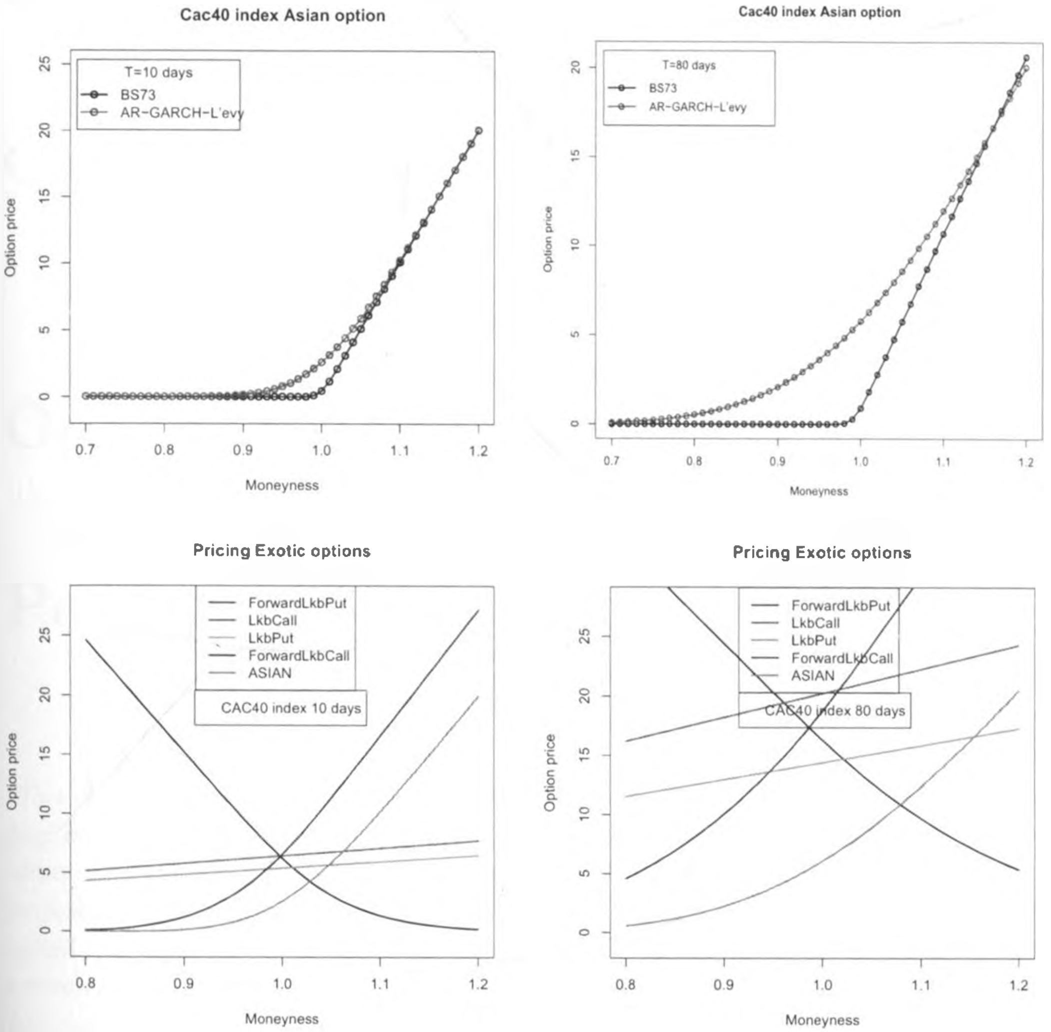


Figure 5.3: Arithmetic Asian option when the underlying asset process is assumed to be CAC40 share index covering the period March 1, 1990 to September 1, 2008. Note the significant difference between the proposed model and Geometric Brownian motion assumed in BSM73 model as we vary days to maturity. This might alluded to long range dependence in second central moment. Similarly, lookback options are priced.

Chapter 6

Option Pricing with AR GARCH Pentanomial Lattice

This chapter develops a GARCH pentanomial option pricing model that incorporates the first four moments of daily log returns AR-GARCH filtered residual distribution. The lattice is constructed using a moment matching procedure. We discuss the statistical properties of the proposed model, apply minimal entropy martingale measure (MEMM) to value European call option. An example illustrates some of the features of this model incorporating conditional heteroscedasticity of the underlying. One can observe that for long term options,i.e above say 300 days to maturity, Black-Scholes models seem to be close to the market data as opposed to proposed models, while for short term options, the reverse is quite evident.

6.1 Introduction

A common assumption underlying most option pricing models are that the logarithm of stock price is normally distributed. The well known Black and Scholes (1973) option pricing model was derived under such assumption. Various studies have shown that the normal distribution does not accurately describe observed stock return data see for example (Cont (2001), Barndorff-Nielsen (1998), Carr and Madan (1998), Eberlein and Keller (1995), Tsay (2002), Carr et al. (2002)). The deviations from normality become more severe when more frequent data are used to calculate stock returns.

Evidence from the financial markets suggest that the empirical returns distribution, both historical and implied do not arise from a diffusion process. On the same note, most of the literature for example Carr and Madan (1998), Rydberg (2000) and references therein, assume that daily log returns can be modeled by exponential Lévy processes, finding a number of explicit formulaes for pricing derivatives (see Carr et al. (2003), Schoutens (2003), Carr and Wu (2004)).

Modeling the underlying

Option pricing theory has a long and illustrious history even before Bachelier (1900) publication, but it also underwent a revolutionary change in 1973. At that time, Black and Scholes (1973) presented the first complete satisfactory equilibrium option pricing model followed by Merton (1973) extending their model in several important ways. Later, Cox and Ross (1976) proposed jump process model as a special case of BSM73 model. Option valuation techniques have been extended to more realistic assumptions in number of ways for the underlying stock processes (e.g. Rubinstein (1976), Cox et al. (1979), Barndorff-Nielsen et al. (2002), Carr and Wu (2004), Hull and White (1990), Derman and Kani (1994), Duan (1995), Eberlein and Keller (1995), Geman et al. (2001), Carr et al. (2003), Duan et al. (2006), Carr et al. (2007), Primbs et al. (2007) and many more).

Most empirical studies on equities, foreign exchange rate log returns exhibit leptokurtic behavior and clusters of high and low volatility, but not significant serial correlation especially in developed economies as opposed to emerging markets. These stylized facts can be reproduced by means of autoregressive conditional heteroscedasticity (ARCH) introduced by Engle (1982), later his student Bollerslev (1986) extended it to generalized autoregressive heteroscedasticity (GARCH). During the past decade, researchers have began to study generalized autoregressive (GARCH) models for option pricing because of their superior performance in describing asset returns. Duan (1995) developed a theory with which options can be priced when the evolution of asset returns follow a GARCH process. Heston and Nandi (2000), Christoffersen and Jacobs (2004), Härdle and Hafner (2000) show how GARCH models can be used to capture the pricing behavior of exchange traded options. Analytically, pricing European option requires the knowledge of the risk neutral distribution of the cumulative return with respect to a given model.

Lattices for option pricing were first introduced in 1979 in the pioneering work of Cox et al. (1979). In particular, they used binomial lattice to model geometric Brownian motion and Rendleman and Bartter (1979) used binomial lattice to model exponential Poisson process. An attractive property of their model is that the binomial lattice for geometric Brownian motion is consistent with the standard Black and Scholes (1973) formula for European options. Due to simplicity and versatility of lattice models, a number of extensions to the basic model have been proposed, see Derman and Kani (1994), Ritchken and Trevor (1999), Yamada and Primbs (2001), Wu (2006) for example. Florescu and Viens (2008) use quadrinomial tree to model stochastic volatility in option pricing, while Primbs et al. (2007) price options with a pentanomial lattice. It is worthy noting that an efficient lattice method, may be significantly faster than a Monte Carlo method for valuing some types of path dependent options.

Consider the stochastic distribution of the price of non-divided paying stock in a risk-neutral economy. Let the stock price be $S(t)$ at time t in a period $[t, T]$. An option pricing model is generally based on assumed process of the stock price or return. The Black and Scholes (1973), for example assume that the stock price movement is governed by the following process

$$dS(t) = rS(t)dt + \sigma S(t)dB_t \tag{6.1.1}$$

where r is the risk free rate and σ is the instantaneous volatility rate of the stock return distribution. This is equivalent to assuming daily log returns are normally distributed with mean $(r - \sigma^2/2)(T - t)$ and variance $\sigma^2(T - t)$. However such a process cannot incorporate nonlinear dependence in second moments and leptokurtic fat tails of ARCH type filtered standardized residuals, which are typically associated with empirical stock returns.

The objective of this chapter is to develop an option pricing model which combine the leptokurtic and heteroscedastic nature of daily log returns under an alternative distributional assumption, that is consistent with empirical stock returns. Minimal entropy martingale measure (MEMM) is used to change probability measure $\mathbb{P}$ to risk neutral economy, ARCH type pentanomial lattice is developed which incorporates the first four moments of the GARCH filtered residuals. Parameters of the model can be chosen to match the first four central moments of the residuals. Such a model thus has the potential of producing option prices that are more consistent with empirically observed stock return distributions.

The chapter proceeds as follows. In the second section, we establish the dynamics of the asset price over a time interval Δt. In section 3, a brief review of binomial, trinomial and pentanomial lattice is outlined. In section 4, option pricing formulaes are derived in pentanomial framework, considering three cases as a simplifying assumptions and minimal entropy martingale measure is applied to change measure $\mathbb{P}$ to risk neutral world $\mathbb{Q}$. Section 5 introduces numerical procedures in relation to derived formulaes using real market data. European call option is priced and numerical results compared. Section 6 draws conclusions showing its essential role in valuation by arbitrage methods.

6.2 Basic model setup

We consider a discrete time economy for a period of $[0, T]$ where the trading takes place at any of the $n + 1$ trading nodes $0, \Delta t, 2\Delta t, ..., n\Delta t$ where $\Delta t = \frac{T}{n}$. Suppose $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a given probability space, where $\mathbb{P}$ is the statistical or data generating probability measure. Here, the sample space $\mathbb{F}$ represents the uncertainty in our financial model. Let $\mathfrak{T}$ be the time index set $\{0, \Delta t, 2\Delta t, ..., n\Delta t\}$ of our financial model such that all economic activities take place at each time point $t \in \mathfrak{T}$. We equip our probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the information structure $\mathcal{F} := \{\mathcal{F}_\tau\}_{\tau \in \mathfrak{T}}$. That is, for each $\tau \in \mathfrak{T}$, $\mathcal{F}_\tau$ represents the information set of all market information up to and including time τ, where $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$. The asset price $S_{i\Delta t}$ is assumed to follow the process

$$\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}} = \exp\left(\mu + \sum_{k=1}^{d} \phi_k \ln\left(\frac{S_{(i-k)\Delta t}}{S_{(i-k-1)\Delta t}}\right) + \sigma_{i\Delta t}\left[Z_i + \sum_{j=1}^{N_i(\Delta t)} W_i^{(j)}(\Delta t)\right]\right) \tag{6.2.1}$$

where $Z_i \sim N(0,1)$ and $W_i^{(j)}(\Delta t) \sim N(\mu(\Delta t), \gamma^2(\Delta t))$, for $j = 1, 2, ...,$ and $N_i(\Delta t), i = 1, 2, ..., n$ are sequences of independent Poisson random variables with parameter $\lambda \Delta t$. The random variables $X_i^{(j)}$ are independent for $j = 0, 1, 2, ...,$ and $i = 1, 2, ..., n$. For $i = 1, 2, ..., n$,

$$Corr(X_i^{(j)}, \bar{X}_{i'}^{(j')}) = \begin{cases} \rho, & \text{if } i = i' \text{ and } j = j'; \\ 0, & \text{otherwise.} \end{cases} \tag{6.2.2}$$

Changes of daily log returns are known to be leptokurtic laced with Poisson mixture of normal distributions as it is shown in Figure 6.1. For more detailed exposition on modeling the dynamics of the underlying risky asset see Hsieh (1989), Nieuwland et al. (1994), Chan and Maheu (2002), Duan et al. (2006) etc.

6.2.1 GARCH processes

Let $(Z_t)_{t\in\mathbb{Z}}$ be a $N(0,1)$. The process $(X_t)_{t\in\mathbb{Z}}$ is a GARCH(p,q) process if it is strictly stationary and if it satisfies, for all $t \in \mathbb{Z}$, and some strict positive valued process $(\sigma_t)_{t\in\mathbb{Z}}$, the equations

$$X_t = \sigma_t Z_t, \quad \sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \tag{6.2.3}$$

where $\omega > 0$, $\alpha_i \geq 0$, $i = 1, ..., p$, and $\beta_j \geq 0$, $j = 1, ..., q$. It follows from (6.2.3) that

$$\sigma_t^2 = \omega + (\alpha Z_{t-1}^2 + \beta)\sigma_{t-1}^2 \tag{6.2.4}$$

$$= \omega + \alpha \sum_{i=1}^{\infty} \prod_{j=1}^{i} (\alpha Z_{t-j}^2 + \beta) \tag{6.2.5}$$

Numerous extensions and refinements of the GARCH model have been proposed to mimic additional stylized facts observed in financial markets. These extensions recognize that there may be important nonlinearity, asymmetry and long memory properties in the volatility process. Many of these are surveyed in Bollerslev et al. (1992), Bollerslev et al. (1994) and Engle (2004). In practice, low order GARCH processes are most widely used, however, we restrict ourselves to the GARCH(1,1) and GARCH(2,1) model(s).

6.2.2 Spectral density

Suppose that $\{X_k\}$ is a zero mean stationary time series with auto covariance function $\gamma(.)$ satisfying $\sum_{k=-\infty}^{\infty} |\gamma(k)| < \infty$. The spectral density of $\{X_k\}$ is the function $f(.)$ defined by

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k) e^{-ik\lambda}, \quad -\infty < \lambda < \infty, \tag{6.2.6}$$

Note that, If $\{X_k\} \sim WN(0, \sigma^2)$ then $\gamma(0) = \sigma^2$ and $\gamma(k) = 0$ for all $|k| > 0$. (WN stands for white noise). The process has a flat spectral density (see Figure 6.1)

$$f(\lambda) = \frac{\sigma^2}{2\pi}, \quad -\pi \leq \lambda \leq \pi \tag{6.2.7}$$

A process with this spectral density is called a white noise, since each frequency in the spectrum contributes equally to the variance of the process. In case of $X_k = \phi X_{k-1} + Z_k$ where $\{Z_k\} \sim WN(0, \sigma^2)$, then $\{X_k\}$ has a spectral density

$$f(\lambda) = \frac{\sigma^2}{2\pi}(1 - 2\phi\lambda + \phi^2)^{-1} \tag{6.2.8}$$

For more information about spectral analysis see Brockwell and Davis (2002).

In order to motivate the empirical relevance of these ideas, Figure 6.1 plots the lag 1 through 500 sample autocorrelations of the daily GARCH(1,1) filtered residuals, on CAC40 index (Paris Stock Exchange) from March 1,1990 through September 1,2008. Quick check of its spectral density differs significantly from similar check form daily covariance stationary AR(3)GARCH(1,1) filtered residuals on NSE20 share index (Nairobi Stock Exchange) from March 2,1998 to July 11,2007.

Our main focus is on developing lattice methods for the underlying process

$$X_i = \mu + \sigma_{i\Delta t}(Z_{i\Delta t} + Y_i). \tag{6.2.9}$$

where μ mean of daily log returns and $X_i := \log(S_{i\Delta t}/S_{(i-1)\Delta t})$, $i = 1, \ldots, n$, $\sigma_{i\Delta t}$ removes heteroscedastic nature of the data. $Z_i \sim N(0,1)$ $\forall i$. Y_i are assumed to be i.i.d random variable whose first four moments are known from the market data. The main challenge is to construct branching probabilities in the lattice. Our approach would be using moment matching technique.

BSM73 model

In BSM73 world for example, the value of a stock is a function of the values of a standard Brownian motion Z_j,

$$S_{j\Delta t} = S_0 \exp\left([rj\Delta t + \sigma Z_j] - \frac{1}{2}\sigma^2 j\Delta t \right) \tag{6.2.10}$$

where $\frac{1}{2}\sigma^2$ is the compensator. Given a lattice for Z_j, payoff options on S can be computed and discounted back. In this line of approach one does not factor the impact of skewness and kurtosis of the underlying, see Cox and Ross (1976), Cox et al. (1979) for example. To incorporate most of these factors appropriately, is a subject of ongoing research e.g. Kellezi and Webber (2004) values bermudan option using Lévy lattice when the underlying processes is driven by Lévy processes.

In the subsequent section we consider the following proposed dynamics

$$S_{j\Delta t} = S_{(j-1)\Delta t} \exp(\mu + \sigma_{j\Delta t}(Z_j + Y_j)) \tag{6.2.11}$$

$$S_{n\Delta t} = S_0 \exp\left(n\Delta t\mu + \sum_{j=1}^{n}(Z_j + Y_j)\sigma_{j\Delta t} \right) \tag{6.2.12}$$

where $Z_j \sim N(0,1)$ $\forall j$

$$\sigma^2_{j\Delta t} = \omega + \left(\alpha Z^2_{(j-1)} + \beta\right)\sigma^2_{(j-1)\Delta t} \tag{6.2.13}$$

6.2.3 Multinomial Lattices

One of the most important joint distributions is the multinomial distribution, which arises when a sequence of n independent and identical experiments are performed. Suppose that each experiment can result in any one of m possible outcomes, with respective probabilities $p_1, p_2, ..., p_m$, $\sum_{j=1}^{m} p_j = 1$. If we let X_j denote the number of the n experiments that result in outcome number j, then

$$P(X_1 = n_1, X_2 = n_2, ..., X_m = n_m) = \frac{n!}{n_1!n_2!...n_m!} p_1^{n_1} p_2^{n_2} ... p_m^{n_m} \tag{6.2.14}$$

whenever $\sum_{j=1}^{m} n_j = n$.

In multinomial lattice model, we need to determine the up and down rates u and d, and the probabilities $p_1, ..., p_L$ to fit the actual market data as closely as possible. This can be done by moment matching or directly from density function, see Kellezi and Webber (2004) for different ways of constructing branching probabilities in the lattice. Note that u and d may be thought of up and down factors at each step. Also it can be shown that the multinomial lattice still recombines even if u and d are time dependent when $u_n/d_n = c$ is satisfied for some constant $c > 1$ where u_n and d_n $n = 0, 1, ..., N-1$ are up and down factors in each time step, see Yamada and Primbs (2001), Yamada and Primps (2006) for more details. Let the up and down rates, u and d, be given as

$$u := \exp\left(\frac{\mathbb{E}(X_j)}{m-1} + \alpha\right), \quad d := \exp\left(\frac{\mathbb{E}(X_j)}{m-1} - \alpha\right) \tag{6.2.15}$$

where m is the number of branches, and $X_j = \log(S_{j\Delta t}/S_{(j-1)\Delta t})$ and $\alpha > 0$ are real numbers. Let the mean, variance, skewness and excess kurtosis of X_j be given as μ_x, σ_x^2, s_x, κ_x.

We first develop the basic theoretical set up to model the dynamics of the underlying with an objective to value options, in consideration to heteroscedastic nature of the process in discrete time. It is assumed that, trades occur only at discrete dates indexed by $\{0 < 1\Delta t, ..., < n\Delta t\}$, and the stock price at date $j\Delta t$ can take on values only in a discrete set specified exogenously by

$$\tilde{S}_{(k\Delta t, l)}, \quad l = 1, ..., (m-1)k + 1, \quad k = 0, ..., n.$$

The variables $k\Delta t$ and l index time and state respectively, while m is the possible number of future states for $\tilde{S}_{(k+1)\Delta t}$ from $\tilde{S}_{k\Delta t}$, i.e.

$$\tilde{S}_{((k+1)\Delta t, l)} = u^{m-l} d^{l-1} \tilde{S}_{k\Delta t}, \quad l = 1, ..., m. \tag{6.2.16}$$

with probabilities p_l, $l = 1, ..., m$, satisfying $p_1 + ... + p_m = 1$. In this case, the stock may achieve $k(m-1) + 1$ possible prices at time $t = k\Delta t$, $k = 0, ..., n$ given by

$$\tilde{S}_{(k\Delta t, k)} = u^{k(m-1)+1-k} d^{k-1} S_0, \quad k = 1, ..., n(m-1) + 1 \tag{6.2.17}$$

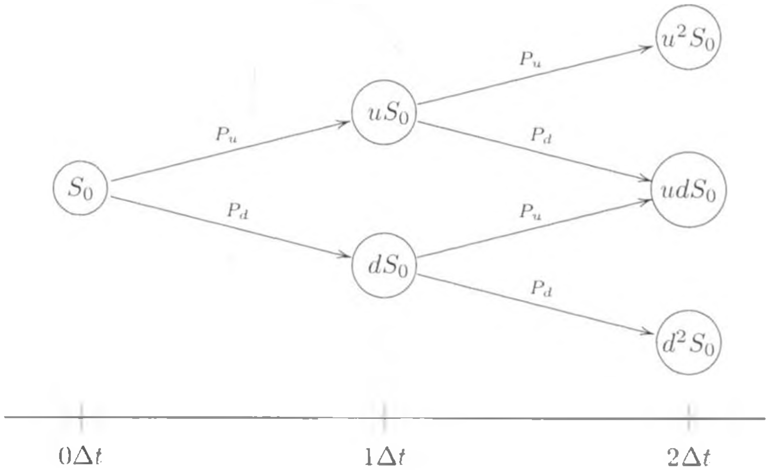
Recall $X_k = \log(S_{k\Delta t}/S_{(k-1)\Delta t})$, then its j^{th} central moment,

$$\mu_j = \mathbb{E}\left[(X_k - \mathbb{E}X_k)^j\right] = \alpha^j \sum_{l=1}^{m} p_l (m - 2l + 1)^j, \quad j \geq 2. \tag{6.2.18}$$

We briefly illustrate moment matching methodology, by considering the binomial, trinomial and pentanomial models for a two time steps in the following subsection.

Binomial Lattices

There are several approaches to the problem of option pricing based on different assumptions about the market, the dynamics of stock price behavior and individual preferences. We focus on no arbitrage theory which can be applied when the dynamics of the underlying stock takes certain forms as shown by the figure below.



(6.2.19)

The binomial option pricing model is an iterative solution that models the price evolution over the whole option validity period $[0, \Delta t]$. It represents the price evolution of the options underlying as the binomial lattices of all possible prices at equally spaced time steps from today under the assumption that at each step, the price can move up and down at fixed rates and with respective pseudo-probabilities p_u and p_d.. A standard Cox et al. (1979) binomial tree, consists of a set of nodes, representing possible future stock prices, with a constant logarithmic spacing between these nodes.

Consider the assumed dynamics of the underlying

$$X_j = \mu + \sigma_{j\Delta t}(Z_j + L_j) = \mu + \sigma_{j\Delta t} Y_j \tag{6.2.20}$$

Where $Y_j := Z_j + L_j$ is an i.i.d. random variable. The random variable Y_j has the following distribution

$$Y_j = \begin{cases} U = \ln u, & \text{with probability } p_u; \\ D = \ln d, & \text{With probability } p_d. \end{cases} \tag{6.2.21}$$

where u and d are two parameters chosen in such a way that $u > 1 + r > d$ to avoid arbitrage (if in risk neutral world). on assumption that the interest rate is constant. Individuals may borrow or lend as much as they wish at this rate and that there are no taxes, transaction costs, or margin requirements.

The necessary equations for the binomial lattice are $p_u + p_d = 1$,

$$\overset{2}{\underset{l\in\{u,d\}}{}} \qquad p_u - p_d = 0. \tag{6.2.22}$$

From these two equations, we obtain several possibilities of solutions e.g. $p_u = p_d = \frac{1}{2}$. or

$$p_u = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\Delta t}, \quad p_d = \frac{1}{2} - \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\Delta t}$$

In addition, suppose that the variance of X_n is given by ${\sigma_n}^2 = \sigma^2, \ \forall n$. This condition implies that $\alpha = \sigma$. Therefore,

$$u = e^{\mathbb{E}X_j + \sigma} \text{ and } d = e^{\mathbb{E}X_j - \sigma}. \tag{6.2.23}$$

Once the relevant parameters are estimated in risk neutral world, we get the binomial model (even though not the same as), see Cox et al. (1979) model. It can be deduced from the trinomial model of Kamrad and Ritchken (1991) by setting the probability of the middle jump equal to zero. A European call option with exercise price K and date n will have payoff in state $[n, j]$ given by

$$C(S_n, K) = \sum_{j=0}^{n} p_u^j (1 - p_u)^{n-j} \frac{n!}{j!(n-j)!} \left(\max\left[S_0 u^j d^{n-j} - K, 0\right]\right), \quad p_d, p_u \in \mathbb{Q} \tag{6.2.24}$$

See Hulle (1988), Boyle (1988), for numerical examples, eg Benninga and Wiener (1997) prices European option in binomial lattice model using mathematica, when the underlying is log-normally distributed.

Trinomial Lattices

Trinomial lattices provide another discrete representation of stock price movement, analogous to binomial lattices. It is characterized by the following five parameters.

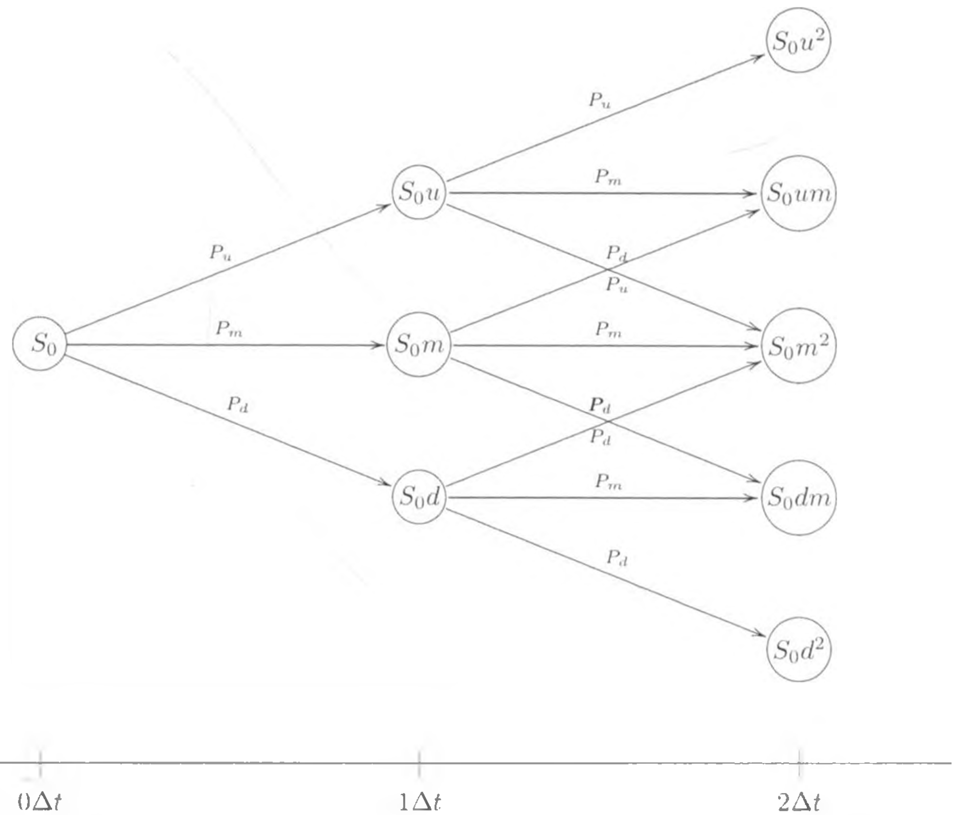
a) the probability of an up move p_u

b) the probability of down move p_d

c) the multiplier on the stock price for an up move u

d) the multiplier on the stock price for an middle move m

e) the multiplier on the stock price for a down move d



A recombining tree is computationally more efficient so we require $ud = m$. Figure above illustrates a two step in a trinomial lattice. The stock price at the beginning of the time step is S_0. During this time step the stock price can move to one of three nodes: with probability p_u to the up node,value uS_0; with probability p_d to the down node, value dS_0; and with probability p_m to the middle node, value dS_0. At the end of the time step, there are five unknown parameters: the probabilities p_u, p_m, p_d, u and d constants.

Boyle (1988) for example, used a multiperiod trinomial procedure to approximate a risk neutralized Geometric Wiener process. In Boyle's model the lattice jump probabilities were obtained by equating the first two moments of the underlying normal distributions of those approximating distributions. To ensure all jump probabilities were non negative, he introduced a stretch parameter, that had to be constrained. In a trinomial lattice, for any random variable Y assumed to be distributed as

$$Y = \begin{cases} \mathbb{E}Y + 2\alpha, & \text{with probability } p_u := p_1; \\ \mathbb{E}Y & \text{with probability } p_m := p_2; \\ \mathbb{E}Y - 2\alpha, & \text{with probability } p_d := p_3; \end{cases} \tag{6.2.25}$$

One can form a system of equations like binomial case and solve for parameters p_u, p_m, p_d. i.e. using the relation

$$\sum_{l=1}^{3} ([2l - 4]\alpha)^j \, p_l = \mu_j,$$

$\mu_j, j = 1, 2, 3.$ is the central moment for the random variable Y

$$\begin{cases} p_u + p_m + p_d = 1 \\ -2p_u + 0p_m + 2p_d = 0 \\ 4p_u + 0p_m + 4p_3 = \frac{\sigma_y^2}{\alpha^2} \end{cases}$$

On solving these three equations we get

$$(p_1, p_2, p_3) = \left(\frac{\sigma^2}{8\alpha^2}, 1 - \frac{\sigma^2}{4\alpha^2}, \frac{\sigma^2}{8\alpha^2}\right)$$

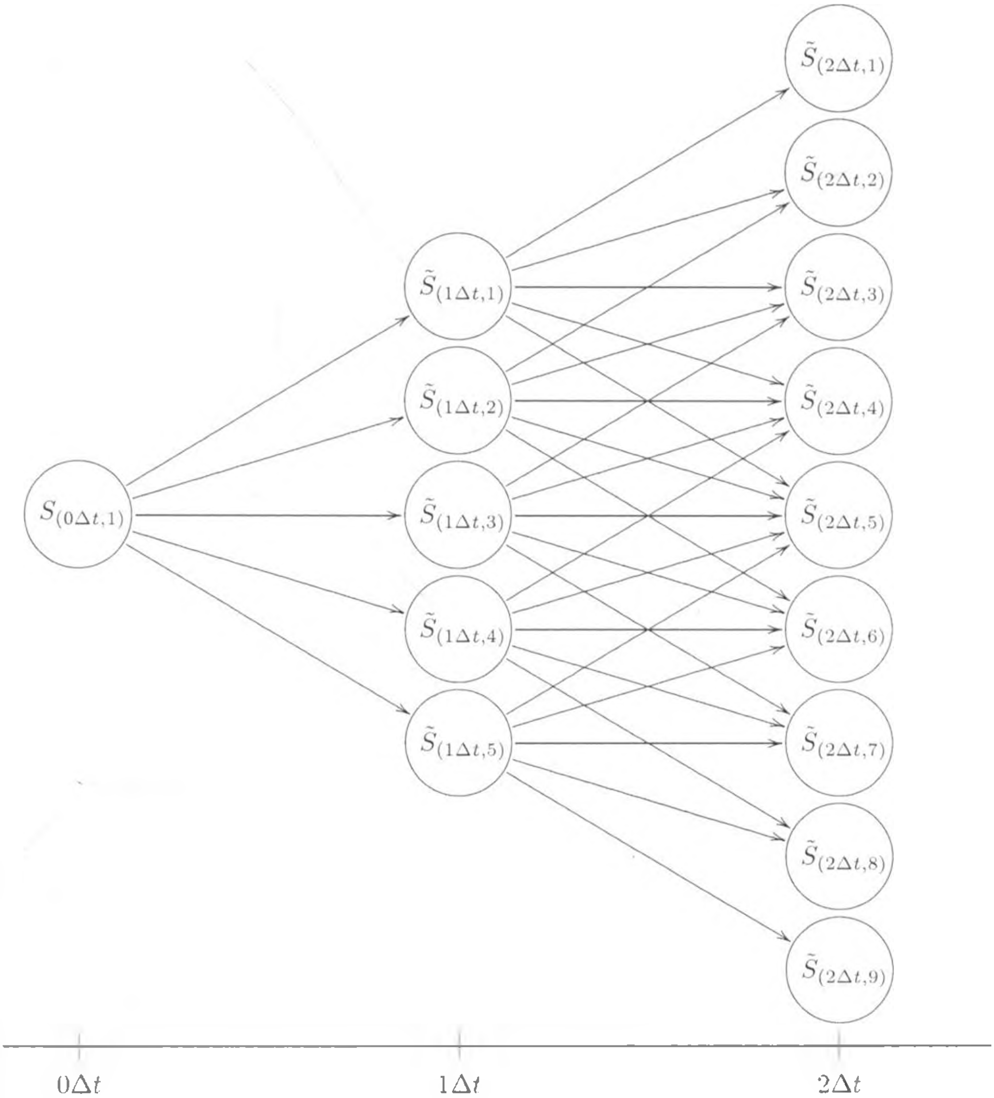
where α must be greater than $\frac{\sigma}{2}$ in order that p_u, p_m and p_d be strictly positive. e.g. Let $\alpha = \sigma$, then $(p_1, p_2, p_3) = (1/8, 3/4, 1/8)$. The corresponding jump amplitudes are given as follows:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} e^{\mu_Y + 2\alpha} \\ e^{\mu_Y} \\ e^{\mu_Y - 2\alpha} \end{pmatrix}.$$

where $\mathbb{E}Y = \mu_Y$. For more information about multinomial approximating models see Kamrad and Ritchken (1991), Kargin (2005) e.t.c.

6.3 Pentanomial lattice

We take state space for risky stock price dynamics over two periods. The figure below, defines the state space for the stock price distribution over the first two trading dates. At each date $k\Delta t$, the stock price can take on values in an exogenously specified discrete set indexed by j. The price $\tilde{S}(k\Delta t, j)$ denotes the stock price in state j at date $k\Delta t$. Denoting by $S_{k\Delta t}, k = 1, ..., N$ the original (known) underlying prices, and $\tilde{S}_{(k\Delta t, j)}$, $j = 1, ..., 4k + 1$.



6.3.1 Pentanomial option pricing

Consider the dynamics of the underlying

$$\log\left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}\right) := X_i = \mu + \phi X_{(i-1)\Delta t} + \sigma_{\Delta t}(Z_i + L_i) \tag{6.3.1}$$

then the first four moments of X_i are estimated, let $\phi = 0$.

$$X_i = \mu + \sigma_{i\Delta t}(Z_i + L_i) \tag{6.3.2}$$
$$\mathbb{E}X_i = \mu + \sigma_{i\Delta t}(\mathbb{E}L_i) = \mu_x \tag{6.3.3}$$

6.3.2 Different Cases:

Characteristic exponent

A characteristic exponent of a random variable X is defined as a log of a characteristic function $\phi_X(\omega)$

$$\Psi_X(\omega) = \ln \phi_X(\omega) \tag{6.3.4}$$

The nth cumulant is defined as

$$cumulant_n = \frac{1}{i^n}\frac{\partial^n \Psi_X(\omega)}{\partial \omega^n}|_{\omega=0} \tag{6.3.5}$$

Mean, variance, skewness and excess kurtosis of a random variable X can be obtained from cumulant as follows

$$\mathbb{E}(X) = cumulant_1 \tag{6.3.6}$$

$$\mathbb{E}(X - \mathbb{E}(X))^2 = cumulant_2 \tag{6.3.7}$$

$$\frac{\mathbb{E}(X - \mathbb{E}(X))^3}{\left[\sqrt{\mathbb{E}(X - \mathbb{E}(X))^2}\right]^3} = \frac{cumulant_3}{cumulant_2^{3/2}} = skewness \tag{6.3.8}$$

case I-$\sigma_{i\Delta t} \approx \sigma \quad \forall\ i$

Let $\sigma^2 = \omega/(1 - \alpha - \beta)$ while

$$\begin{aligned} Var(X_i) = \mathbb{E}(X_i - \mathbb{E}X_i)^2 &= \sigma^2(1 + VarL_i), \\ &= \sigma^2(1 + \sigma_L^2) = v_x \end{aligned} \tag{6.3.9}$$

$$\begin{aligned} Skew(X_i) = \frac{\mathbb{E}(X_i - \mathbb{E}X_i)^3}{Var(X_i)^{3/2}} &= \frac{\sigma^3\mathbb{E}(Z + L_i - \mathbb{E}L_i)^3}{(\sigma^2)^{3/2}(1 + \sigma_L^2)^{3/2}} \\ &= \frac{\mathbb{E}L_i^3}{(1 + \sigma_L^2)^{3/2}} = \frac{(\sigma_L^2)^{3/2}S_L}{(1 + \sigma_L^2)^{3/2}} = s_x \end{aligned} \tag{6.3.10}$$

$$Kurt(X_i) = \frac{\mathbb{E}(Z_i + \hat{L}_i)^4}{(1 + \sigma_L^2)^{4/2}} = \frac{\mathbb{E}Z_i^4 + 6VarL_i + \mathbb{E}\hat{L}_i^4}{(1 + \sigma_L^2)^{4/2}}, \quad \hat{L}_i = L_i - \mathbb{E}L_i \tag{6.3.11}$$

$$= \frac{3 + 6\sigma_L^2 + \sigma_L^4(3 + KurtL_i)}{(1 + \sigma_L^2)^2} = k_x \tag{6.3.12}$$

To construct a pentanomial model of stock prices, we need to examine the behavior of the stock price in a small interval $[0, \Delta t]$. As assumed earlier, log returns $X_i := \log(S(i\Delta t)/S(i-1)\Delta t)$ can be modeled as

$$\begin{aligned} X_i &= \mu + \sigma_{i\Delta t}(Z_i + \sum_{j=1}^{N_i(\Delta t)} W_i^{(j)}) \\ &= \mu + \sigma_{i\Delta t}(Z_i + L_i) \end{aligned} \tag{6.3.13}$$

where $Z_i \sim N(0, 1)$ for all $t \in [0, T]$ and $L_i's$ are assumed to be identically and independently distributed random variable. The discrete distribution of $Z_i + L_i$ over the interval $[0, \Delta t]$ can be approximated by pentanomial lattice. To model the stock price movement as deterministic part in addition to pentanomial process, the interval $[0, T]$ is divided into n equal subintervals of length $\Delta t = T/n$, where T is the maturity date of an option. The relation (6.3.14) is used to form system of linear equations.

$$\sum_{l=1}^{5} \left([2l-6]\alpha\right)^k p_l = \mu_k, \quad , k = 1, 2, 3, 4. \tag{6.3.14}$$

where

$$\begin{aligned} \mu_k &= \mathbb{E}(X_j - \mathbb{E}X_j)^k \\ &= \sum_{j=1}^{k} \binom{k-1}{j-1} c_j \mu_{k-j}. \end{aligned} \tag{6.3.15}$$

To calibrate the pentanomial lattice, we need to solve the following five equations

$$\begin{aligned} &p_1 + p_2 + p_3 + p_4 + p_5 = 1, \\ &-2p_1 - p_2 + p_4 + 2p_5 = 0, \\ &16p_1 + 4p_2 + 4p_4 + 16p_5 = v_x/\alpha^2, \\ &-64p_1 - 8p_2 + 8p_4 + 64p_5 = s_x v_x^{3/2}/\alpha^3, \\ &256p_1 + 16p_2 + 16p_4 + 256p_5 = k_x v_x^2/\alpha^4, \end{aligned} \tag{6.3.16}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & -1 & -2 \\ 16 & 4 & 0 & 4 & 16 \\ 64 & 8 & 0 & -8 & 64 \\ 256 & 16 & 0 & 16 & 256 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \frac{v_x}{\alpha^2} \\ \frac{s_x v_x^{3/2}}{\alpha^3} \\ \frac{k_x v_x^2}{\alpha^4} \end{bmatrix} \tag{6.3.17}$$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} = \frac{1}{384} \begin{bmatrix} 0 & 32 & -4 & -4 & 1 \\ 0 & -256 & 64 & 8 & -4 \\ 384 & 0 & -120 & 0 & 6 \\ 0 & 256 & 64 & -8 & -4 \\ 0 & -32 & -4 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \frac{v_x}{\alpha^2} \\ \frac{s_x v_x^{3/2}}{\alpha^3} \\ \frac{k_x v_x^2}{\alpha^4} \end{bmatrix} \tag{6.3.18}$$

The third and fourth equations arise from matching the third and fourth central moments of the approximating distribution to the third and fourth central moments respectively of the empirical distribution. The relationship of these central moments are related to skewness and excess kurtosis as described in equation (6.3.14). On solving these five equations, we get equation 6.3.20. Define

$$A = v_x/\alpha^2, \quad B = s_x v_x^{3/2}/\alpha^3, \quad C = k_x v_x^2/\alpha^4 \tag{6.3.19}$$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} = \frac{1}{384} \begin{bmatrix} -4A - 4B + C \\ 64A + 8B - 4C \\ 384 - 120A + 6C \\ -64A - 8B - 4C \\ -4A + 4B + C \end{bmatrix} \tag{6.3.20}$$

where α must be chosen in order to ensure positivity of probabilities p_1, p_2, p_3, p_4 and p_5. It so happens that if $\kappa_x \geq 3s_x^2 - 3$ and $\kappa_x \geq \frac{-23}{16}$ then, there exists a range of values of α (which includes $\alpha = \sqrt{\frac{(k_x)(1+VarL)}{12}}$) which will ensure that all the probabilities are strictly positive (see Yamada and Primbs (2001, 2004); Yamada and Primps (2006).) For this choice of α, we have the following jump amplitudes for the pentanomial lattice.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{cases} \exp\left(\mu_x - 4\sqrt{\frac{\kappa_x(1+VarL)}{12}}\right), & \text{with probability } p_1; \\ \exp\left(\mu_x - 2\sqrt{\frac{\kappa_x(1+VarL)}{12}}\right), & \text{with probability } p_2; \\ \exp(\mu_x), & \text{with probability } p_3; \\ \exp\left(\mu_x + 2\sqrt{\frac{\kappa_x(1+VarL)}{12}}\right), & \text{with probability } p_4; \\ \exp\left(\mu_x + 4\sqrt{\frac{\kappa_x(1+VarL)}{12}}\right), & \text{with probability } p_5. \end{cases} \tag{6.3.21}$$

case II-$\sigma_{j\Delta t} = f(\sigma_{(j-1)\Delta t}), \quad \forall \quad j = 1, ..., n$

In this case we assume given

$$X_j = \mu + \sigma_{j\Delta t}(Z_j + L_j) \tag{6.3.22}$$

$$\sigma^2_{j\Delta t} = \omega + \alpha(W^2_{j-1} + \beta)\sigma^2_{j-1\Delta t} \tag{6.3.23}$$

where W^2_{j-1} is a $\chi^2(1)$ and $Z_j \sim N(0,1)$ for all $j = 1, ..., n$ and L_j is assumed to be identically and independently distributed random variable for all j, while the updating process varies as we vary j. Let μ_i be the ith central moment of discrete random variable X_j. For each single step, we calculate $\mathbf{p}_j \in \mathbb{P}$ and corresponding risk neutral probability $\mathbf{q}_j \in \mathbb{Q}$. In this case the first four moment can be computed for every step j as follows;

$$\mathbb{E}X_{j|j-1} = \mu + \sigma_{j\Delta t|j-1}(\mathbb{E}L_j) := \mu_j x \tag{6.3.24}$$

$$VarX_{j|j-1} = \sigma^2_{j\Delta t}(1 + V_L) := \sigma^2_j x, \quad V_L = Var(L_j), \quad \forall \quad j \tag{6.3.25}$$

$$SkewX_{j|j-1} = \frac{\mathbb{E}[X_j - \mu_j x]^3}{V_j x^{3/2}} = \frac{\sigma^3_{j\Delta t|j-1}\mathbb{E}(Z_j + L_j - \mu_L)^3}{\sigma^3_{j\Delta t|j-1}(1 + V_L)^{3/2}} = \frac{S_L(V_L)^{3/2}}{(1 + V_L)^{3/2}} \tag{6.3.26}$$

$$KurtX_{j|j-1} = \frac{3 + 6V_L + V_L^2(3 + \kappa_L)}{1 + \sigma_L^2} := k_x \tag{6.3.27}$$

Note that $\kappa_L := KurtL - 3$ excess kurtosis of random variable L_j. To calibrate the pentanomial lattice, we need to solve the following four equations in addition to having $p_{1,j} + p_{2,j} + p_{3,j} + p_{4,j} + p_{5,j} = 1 \ \forall \ j = 1, ..., n$. as well as four other equations as a result of the relation (6.3.28) to form system of linear equations.

$$\sum_{l=1}^{5} ([2l - 6]\alpha)^k \, p_{(l,j)} = \mu_{k,j}, \quad \mu_k, k = 1, 2, 3, 4., \quad j = 1, ..., n. \tag{6.3.28}$$

$$\begin{aligned} &-4p_{1,j} - p_{2,j} + p_{4,j} + 2p_{5,j} = 0 \\ &16p_{1,j} + 4p_{2,j} + 4p_{4,j} + 16p_{5,j} = \sigma_j^2 x/\alpha_j^2, \quad \sigma_j x := \sqrt{1 + \sigma_L^2} \\ &-64p_{1,j} - 8p_{2,j} + 8p_{4,j} + 64p_{5,j} = s_j x \sigma_j^{1.5} x/\alpha_j^3 \\ &256p_{1,j} + 16p_{2,j} + 16p_{4,j} + 256p_{5,j} = k_x \sigma_j^2 x/\alpha_j^4, \end{aligned} \tag{6.3.29}$$

The third and fourth equations arise from matching the third and fourth central moments of the approximating distribution to the third and fourth central moments of the empirical distribution for each step j respectively. The relationships of these central moments are related to skewness and excess kurtosis as described in equation (6.3.29). On solving these five equations, we get equation 6.3.31. Define

$$A_j = v_j x/\alpha_j^2, \quad B_j = s_x v_j^{3/2} x/\alpha_j^3, \quad C_j = k_x v_j^2 x/\alpha_j^4 \tag{6.3.30}$$

$$\begin{bmatrix} p_{1,j} \\ p_{2,j} \\ p_{3,j} \\ p_{4,j} \\ p_{5,j} \end{bmatrix} = \frac{1}{384} \begin{bmatrix} -4A_j - 4B_j + C_j \\ 64A_j + 8B_j - 4C_j \\ 384 - 120A_j + 6C_j \\ -64A_j - 8B_j - 4C_j \\ -4A_j + 4B_j + C_j \end{bmatrix} \tag{6.3.31}$$

where $\alpha_j = \sigma_{j\Delta t|j-1}\sqrt{\frac{(k_x)(1+VarL)}{12}}$ must be chosen in order to ensure positivity of probabilities $p_{1,j}, p_{2,j}, p_{3,j}, p_{4,j}$ and $p_{5,j}$. (see Yamada and Primbs (2001, 2004); Yamada and Primps (2006)) which will ensure that all the probabilities are strictly positive for each j. For this choice of α_j, we have the following jump amplitudes

$$\begin{bmatrix} a_{1,j} \\ a_{2,j} \\ a_{3,j} \\ a_{4,j} \\ a_{5,j} \end{bmatrix} = \begin{cases} \exp\left(\mu_j x - 4\sigma_{j\Delta t|j-1}\sqrt{\frac{\kappa_x(1+VarL)}{12}}\right), & \text{with probability } p_{1,j}; \\ \exp\left(\mu_j x - 2\sigma_{j\Delta t|j-1}\sqrt{\frac{\kappa_x(1+VarL)}{12}}\right), & \text{with probability } p_{2,j}; \\ \exp(\mu_j x), & \text{with probability } p_{3,j}; \\ \exp\left(\mu_j x + 2\sigma_{j\Delta t|j-1}\sqrt{\frac{\kappa_x(1+VarL)}{12}}\right), & \text{with probability } p_{4,j}; \\ \exp\left(\mu_j x + 4\sigma_{j\Delta t|j-1}\sqrt{\frac{\kappa_x(1+VarL)}{12}}\right), & \text{with probability } p_{5,j}. \end{cases} \tag{6.3.32}$$

case III-$\sigma_{j\Delta t} = f(\sigma_{(j-1)\Delta t})$, $\forall \ j = 1, ..., n$ AR(1)-GARCH(1,1)

Consider the following model, given X_0,

$$\begin{aligned} X_j &= \phi X_{j-1} + \sigma_{j\Delta t}(Z_j + Y_j) && (6.3.33) \\ &= \phi^k X_{i-k} + \sum_{i=0}^{k-1} \phi^i (Z_{(j-i)} + Y_{(j-i)})\sigma_{(j-1)\Delta t} \\ &= \phi^j X_0 + \sum_{i=0}^{j-1} \phi^i (Z_{(j-i)} + Y_{(j-i)})\sigma_{(j-1)\Delta t} && (6.3.34) \\ \therefore \ X_n &= \phi^n X_0 + \sum_{i=0}^{n-1} \phi^i (Z_{(n-i)} + Y_{(n-i)})\sigma_{(n-1)\Delta t} && (6.3.35) \end{aligned}$$

We follow the same procedure as in **case II** and match the moments to find the corresponding probabilities in $\mathbb{P}$. With correct selection of α_j it will be possible to find jump amplitudes for each step j. It is assumed that at time step j, we know the previous estimates of $\sigma_{(j-m)\Delta t}$, $m = 1, ..., j-1$.

The first two moment of the assumed process are

$$\begin{aligned} \mathbb{E}X_j &= \phi^j X_0 + \sum_{i=0}^{j-i} \phi^i \mathbb{E}L_{(j-i)}\sigma_{(j-i)\Delta t} \\ &= \phi^j X_0 + \mu_L \sum_{i=0}^{j-i} \phi^i \sigma_{(j-1)\Delta t}, \quad \mu_L := \mathbb{E}L_{(j-i)} \end{aligned} \tag{6.3.36}$$

$$\begin{aligned} Var X_j &= \sum_{i=0}^{j-1} Var\left[\phi^i (Z_{(j-i)} + Y_{(j-i)})\sigma_{(j-i)\Delta t}\right] \\ &= \sum_{i=0}^{j-1} \phi^{2i}\sigma^2_{(j-i)\Delta t}\left[1 + Var(L_{j-i})\right] \\ &= (1+\sigma_L^2)\sum_{i=1}^{j-1} \phi^{2i}\sigma^2_{(j-i)\Delta t} \end{aligned} \tag{6.3.37}$$

6.3.3 Minimal entropy martingale measure

One of the important economic insight underlying the preference free option pricing result is the concept of perfect replication of contingent claims. This is by continuously adjusting a self-financing portfolio under the no-arbitrage principle. Cox et al. (1979) provided further insight in this concept by introducing the notion of risk-neutral valuation and establishing its relationship with no-arbitrage principle in a transparent way under a discrete-time binomial setting.

Harrison and Kreps (1979) and Harrison and Pliska (1981) established a solid mathematical foundation for the relationship between no-arbitrage principal and the notion of risk-neutral valuation using the modern language of probability theory. They proposed the "Fundamental theorem for asset pricing" which states that the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure. If the securities market is complete, there is a unique martingale measure and hence the unique price of any contingent claim is given by its discounted payoff at expiry. However, the assumption of market completeness is questionable in the real world securities market. Under an incomplete market, there is more than one equivalent martingale measure and hence a range of no-arbitrage prices for a contingent claim. One crucial issue is to identify an equivalent martingale measure which gives an economically consistent and justifiable price for the contingent claim.

Let $n \geq 2$ be the cardinality of Ω, $R = 1 + r$ (where r denotes single period interest rate) and $S = (S_{0\Delta t}, S_{1\Delta t})$ be the price process of the risky asset. Assuming that $S_{0\Delta t} = 1$ and the random variable $S_{1\Delta t}$ takes n different positive values $a_1, ..., a_n$ with the probability $\mathbb{P}(S_{1\Delta t} = a_i) = p_i > 0, \forall i = 1, ..., n, \quad p_1 + ... + p_n = 1$, then the minimal entropy martingale measure

$\mathbb{Q}_0 = (q_1, ..., q_n)$ is the solution to the problem

$$\mathbb{Q}_0 = \min_{q \in \mathbb{R}^n, q>0} \left(\sum_{i=1}^{n} q_i \ln \left(\frac{q_i}{p_i} \right) \right) \tag{6.3.38}$$

$$s.t. \sum_{j=1}^{n} q_j = 1, \quad \sum_{j=1}^{n} q_j a_j = R \tag{6.3.39}$$

It is evident that $\mathbb{Q}_0$ is given by

$$q_i = \frac{p_i e^{\gamma a_i}}{\sum_{j=1}^{n} p_j e^{\gamma a_j}}, \quad i = 1, ..., n, \tag{6.3.40}$$

where $\gamma \in \mathbb{R}$ is the unique real solution that always exists under the assumption of no-arbitrage opportunities of the following equation

$$\sum_{i=1}^{n} p_i(a_i - R)e^{\gamma a_i} = 0 \tag{6.3.41}$$

This part of lemma is due to Frittelli (2000) in which, he links the existence and uniqueness of γ to no arbitrage assumption. The minimal entropy martingale measure for the pentanomial lattice is given by $\widehat{\mathbb{Q}_0} = (\widehat{q}_1, \widehat{q}_2, \widehat{q}_3, \widehat{q}_4, \widehat{q}_5)$ where

$$\widehat{q}_j = \frac{p_j e^{\gamma a_j}}{\sum_{i=1}^{5} p_i e^{\gamma a_i}}, \; j = 1, \ldots, 5 \tag{6.3.42}$$

and γ is the unique solution to the equation $\sum_{i=1}^{5} \widehat{q}_i a_i = 1 + r$. See Ssebungenyi (2008) for more application(s) of minimal entropy martingale measure and Miyahara (2001), Fujiwara and Miyahara (2003), Esche and Schweizer (2005), Choulli and Striker (2006) for more literature about minimal entropy martingale measure.

6.4 Application to Stock market data

6.4.1 Empirical results

Our data set covers eight-year period, running from January 2000, to October 2008 on daily basis. These sets consist of daily adjusted closing index of Nairobi Stock exchange NSE20 (March 03,1998 to July 11,2007), Paris Stock exchange index CAC40, SP500 (Jan 04,2000 to Oct 03, 2008) index and IBM Inc (Jan 03,2000 to Sept 26,2008) company daily adjusted stock price. Over the entire period, we have the daily closing (adjusted) values of the indices which we use in our estimation of volatility process.

Option data of IBM Inc. as of October 31,2008 were used with expiry dates, November 21,2008, December 19.2008, January 16,2009, April 17,2009, January 15,2010. Due to economy of space, numerical results of CAC40 index are not tabulated, but referred to in second section. As pertains option data we present only shortest period before expiry and the longest duration before expiry.

In this subsection, we present empirical results from the estimation procedures outlined in the previous section especially case one model. Maximum likelihood estimates of the parameters and their heteroscedasticity consistent asymptotic error were obtained and reported in table 6.1.

The Ljung -Box test statistic $Q(.)$, $Q^2(.)$ for the standardized residuals and the standardized squared residuals respectively, from AR(3)-GARCH(1,1), GARCH(1,2) and AR(1)-GARCH(1,1) models take different values (see Table 6.1) and do not indicate any further first or second order serial dependence. The estimates of the conditional kurtosis(not tabulated) differs significantly from the normal value of three. Note that the estimated values for $\alpha + \beta_1 + \beta_2$ are close to one for the three indices. Moreover, daily log returns of NSE20 index is AR(3) correlated with GARCH(1,1) noise and leptokurtic standardized residual, similar to SP500 adjusted daily index.

As discussed earlier, specifications of binomial and pentanomial lattices are developed using the numerical procedure outlined in the previous section. Table 6.2 gives risk neutral probabilities for IBM Inc. company case. Of the proposed model, we took case I to demonstrate our results. The rest which can be done in a similar way was left for future research.

Once the parameters of discrete distributions are specified, pentanomial lattice building procedure is analogous to that of binomial and trinomial lattices. Option values are obtained through a recursive procedure, see Figure 6.2.

6.4.2 European call option prices

A call option gives the owner the right, but not the obligation, to buy a particular security at a pre-specified price within a pre-specified time period. The value of such an option will be intimately related to the distribution of the price of the underlying instrument at the time of maturity. Specifically the more volatile the underlying price process is, the more valuable the option. The standard approach for pricing options rely on risk neutral valuation methods. In this risk-neutralized probability measure, the price of a call option, that does not allow for early exercise and pays no dividends, will be qual to the discounted expected value of the payoffs at the maturity date. Our analysis is meant to illustrate a possibility of modeling volatility dependency when calculating option prices.

To that end, we compare the performance of three lattice models for short time and long term maturity level at the money and out of the money European call options priced in Black and Scholes (1973) world,i.e.

$$\begin{aligned} C^{BS}(t,K) &= S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \\ d_1 &= \frac{\ln(S_t/Ke^{r(T-t)})}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} \\ d_2 &= \frac{\ln(S_t/Ke^{r(T-t)})}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t} \end{aligned}$$

Where we let t refer to the time that the option is written, i.e. October 31,2008 in the analysis reported figure6.2. The maturity time $\tau = T - t$ of the options in days. An option is said to be at the money if the exercise price, K, equals the current value of the underlying security; i.e. here $S_T = 92.97$. Lattices calibrated in data generating process $\mathbb{P}$ and transformed to $\mathbb{Q}$ minimal entropy martingale measure. In all the models, same parameters are used and results plotted against real market data. Here we recall the classic Cox et al. (1979) scheme (CRR), which has a constant logarithmic spacing between nodes on the same level. The following parameters are used

$$\begin{aligned} u &= \exp(\sigma\sqrt{\Delta t}), \quad d = 1/u, \quad p = 1/2 + (\mu/2\sigma)\sqrt{\Delta t}, \quad q = 1/2 - (\mu/2\sigma)\sqrt{\Delta t} \in \mathbb{Q}, \\ \hat{\sigma}^2 &= \frac{1}{N-1}\sum_{j=1}^{N}\left[\ln\left(\frac{S_{j\Delta t}}{S_{(j-1)\Delta t}}\right) - \mathbb{E}X_j\right]^2 \end{aligned} \tag{6.4.1}$$

Moreover, we use transition probabilities as in Derman and Kani (1994) for constructing trinomial lattices, where the middle transition probability is equal to $1 - p - q$ provided that

$$p = \left(\frac{e^{r\Delta t/2} - e^{\sigma}\sqrt{\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{\sigma}\sqrt{\Delta t/2}}\right)^2 \tag{6.4.2}$$

$$q = \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{r\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{\sigma\sqrt{\Delta t/2}}}\right)^2 \tag{6.4.3}$$

$$u = 1/d = e^{\sigma\sqrt{2\Delta t}} \tag{6.4.4}$$

Adjusted pentanomial was arrived at, by adjusting the dynamics of the proposed underlying model (6.2.1), here

$$\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}} = \exp\left(\mu + \varpi(i\Delta t) - \frac{1}{2}\sigma^2_{i\Delta t} + \sum_{k=1}^{d}\phi_k \ln\left(\frac{S_{(i-k)\Delta t}}{S_{(i-k-1)\Delta t}}\right) + \sigma_{i\Delta t}\left[Z_i + \sum_{j=1}^{N_i(\Delta t)} W_i^{(j)}(\Delta t)\right]\right)$$

where $Z_i \sim N(0,1)$ and $W_i^{(j)}(\Delta t) \sim N(\mu(\Delta t), \gamma^2(\Delta t))$, for $j = 1, 2, ...,$ and $N_i(\Delta t), i = 1, 2, ..., n$ are sequences of independent Poisson random variables with parameter $\lambda\Delta t$. The random variables $W_i^{(j)}$ are independent for $j = 0, 1, 2, ...,$ and $i = 1, 2, ..., n$.

Note that we could estimate the value of $\varpi(n\Delta t)$ by adjusting the present market value of the stock S_0^{mrkt}

$$S_0^{mrkt} + h = S_0^{mrkt}\exp(\varpi(n\Delta t)), 0 < h \in \mathbb{R}., n = 1, 2, ..., \tag{6.4.5}$$

6.5 Conclusion

In this chapter, the asset dynamics under the physical probability measure $\mathbb{P}$ in incomplete markets is not only established, but also applied to minimal entropy martingale measure, to change dynamics to risk neutral. The residuals of GARCH filtered daily log stock returns, conditioned on normal distributions, showed excess kurtosis and skewed positively and negatively. This occurrence, violates the normality assumption.

The valuation of contingent claims whose value depend on multiple sources of uncertainty is an important problem in financial economics. Since numerical methods for valuing such claims can be computationally expensive, the need for an efficient algorithm is clear. We made simplifying assumptions in that direction, even though there is more to be refined.

Although the pentanomial lattice provided in this chapter is as tractable as the standard binomial and trinomial lattices and may be extended to the multinomial case, the computational effort might increase exponentially with respect to dimension, similar to other lattice models. However pentanomial lattices can be considered useful for relatively short term contracts which can be used to solve problems related American type options.

The relatively GARCH(1,1)-normal model removes second order serial dependance in both markets in line with what is in the literature. The empirical evidence about spectral density of standardized GARCH(1,1)-normal filtered from other financial markets deserves further investigation.

Since option prices may react sensitively to changes in volatility, a proper specification of the conditional means at each step may play a crucial role in the proposed pentanomial model. It is well documented in literature, out-of-the money options with short times to maturity react strongly to volatility changes when measuring this sensitivity in relative terms (regarding the elasticity of option price with respect to volatility).

Under the proposed framework, the market is in general incomplete, which is challenging to handle for the implication is a multitude of equivalent martingale measures and thus, a multitude of no-arbitrage prices.

We note that under the proposed underlying dynamics, far in the money, at the money and out of the money options are valued slightly higher than the binomial and Black and Scholes (1973) model and in most cases can be adjusted to match market prices especially if prices are not highly volatile. We leave model refinement and extensions for future research.

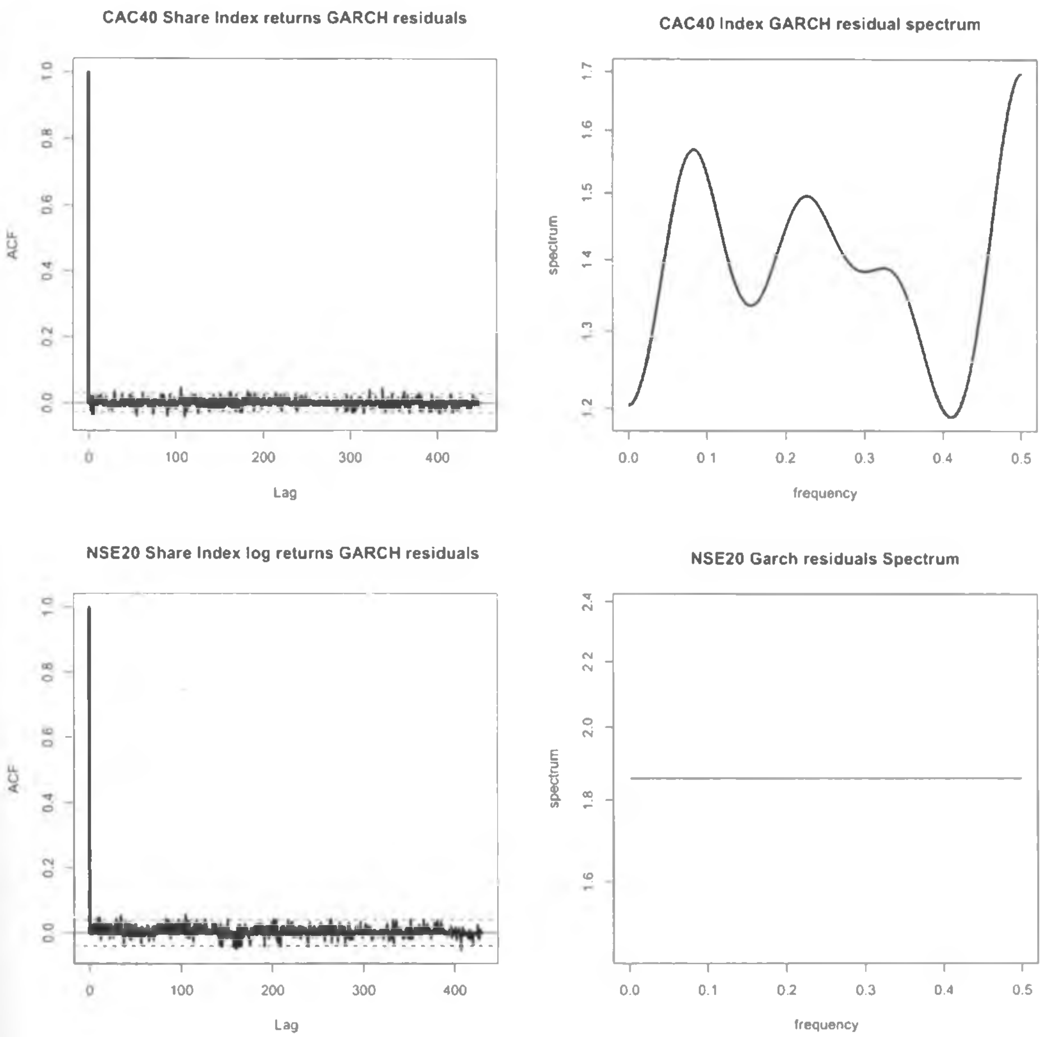


Figure 6.1: **Left:** The GARCH(1,1) filtered residuals of CAC40 daily share index(from March 1, 1990 to September 1,2008) and NSE20 share index (from March 2,1998 to July 11,2007)log returns. The residual distribution as in general skewed, leptokurtic, and more peaked at its mean than the distribution of a comparable normal variate see Mwaniki (2007). **Right:** Spectral density of such a distribution do strongly suggest that it is a form white noise for NSE20 index and kind of Poisson mixture of normal distributions for CAC40 index. The 95% confidence bands for no serial dependence are indicated in the figure.

Table 6.1: AR(3)GARCH(2,1) models for daily returns.

$$X_i = \phi_1 X_{i-1} + \phi_2 X_{i-2} + \phi_3 X_{i-3} + \varepsilon_i, \quad \varepsilon_i = \sigma_{i\Delta t} Z_i, \quad Z_i \sim N(0,1)$$

$$\sigma^2_{i+1\Delta t} = \alpha_0 + \alpha_1 \sigma^2_{i\Delta t} Z_i^2 + \sum_{j=1}^{2} \beta_j \sigma^2_{(i-j+1)\Delta t}$$

	NSE20	IBM Inc.	S&P500
Parameter	AR(3)GARCH (1,1)	GARCH(1,2)	AR(1)GARCH(1,1)
ϕ_1	0.18915(.024496)		-0.06623(0.02234)
ϕ_2	0.16451(.023785)		
ϕ_3	0.11388(.023413)		
$\omega * 10^4$	0.03549(.006902)	0.047289(0.014958)	0.009854(0.003105)
α	0.15023(.017978)	0.125707(0.025539)	0.072821(.01073)
β_1	0.78763(.024753)	0.619592(0.14557)	0.92185(.0074845)
β_2		0.245621(0.13237)	
$Q(10)$	9.3468(0.2287)	8.8337(0.2648)	14.00(0.0816)
$Q^2(10)$	7.1689(0.5739)	5.97965(0.542128)	17.56(0.1747)
lgl	-8363.5	-6057.773	-6692.37
n	2316	2316	2201

Notes: Standard errors are in parenthesis. lgl is the log likelihood.

Table 6.2: Probabilities $\mathbb{P}$ and risk-neutral probabilities $\mathbb{Q}$ of IBM Inc. Company daily log returns. We assume case *I* model to price options. $r = 5\% p.a.$

$$\min_{q \in \mathbb{R}^5, q>0} \left(\sum_{i=1}^{5} q_i \ln\left(\frac{q_i}{p_i}\right) \right), \quad s.t. \sum_{j=1}^{5} q_j = 1, \quad \sum_{j=1}^{5} q_j a_j = 1 + (r/250),$$

$$\widehat{q}_j = \frac{p_j e^{\gamma a_j}}{\sum_{i=1}^{5} p_i e^{\gamma a_i}}, \ j = 1, \ldots, 5$$

$\mathbb{P}$		$\mathbb{Q}$		**a**	
	γ =0.82552183419466				
p_1	0.06549397748689899	q_1	0.06161617410166113	a1	0.9258510638937044
p_2	0.10136302169003963	q_2	0.09822320397549678	a2	0.9616683145198932
p_3	0.65147353500424354	q_3	0.65098245271220745	a3	0.9988711826524486
p_4	0.13098795497379800	q_4	0.13513187777678223	a4	1.0375132719555373
p_5	0.05068151084501980	$q5$	0.05404629143385248	a5	1.0776502597917310

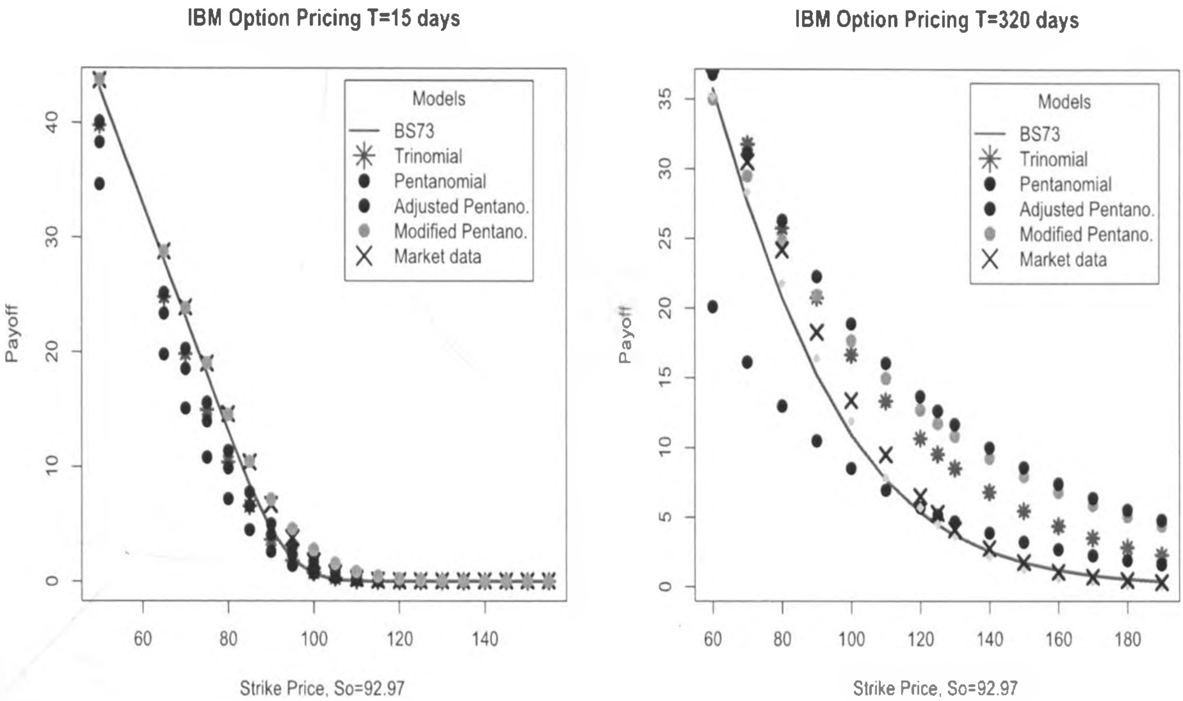


Figure 6.2: **Left:** IBM European call options price comparison between Black Scholes model, proposed pentanomial lattice, market data. Expiry date was on 21 November 2008 as of 31st October 2008. Daily adjusted share price from January 03,2000 to October 31,2008 were used. $S_0 = 92.97$, $r = 2.45\%$ p.a., $\tau = 15$. For the proposed pentanomial lattice model case 1, we varied $S_0 \in (92.97, 98.97)$ as strike price was fixed as it appears in the graph. **Right:** In the second figure, sketch comparison of two model and volatile market for European option due 15th of January 2010, were calculated using proposed model and Black and Scholes (1973) model.

Chapter 7

On APARCH Lévy Filter: A Closed-form Option Pricing Model

Popular models such as Black-Scholes-Merton(BSM73) lack most of empirically found stylized features of financial data, such as volatility clustering, leptokurtic in nature of log returns, joint covariance structure, aggregational Gaussianity, e.tc., hence it may not consistently price all European and exotic options that are quoted in one specific market. This could be as a result of assuming a stochastic process which do not describe the underlying asset price dynamics. Moreover, such simplifying assumptions in real financial markets, may translate to the implied volatility curves typically skewed, with smiley shapes or even more complex structures. A closed-form option pricing model, APARCH Lévy filter, which nests BSM73 model, minimizes the "consistent volatility smiles" and incorporates most of the stylized features observed in developed and emerging economies is presented. An extensive empirical analysis based on S&P500 index options and Nairobi stock index NSE20 index is used to compare performance of proposed model against BSM73 and GARCH option pricing model of Duan(1995).

7.1 Introduction

Exotic options are often hedged with European options. This hedging is called static replication. To improve the hedging performance, exotic and standard options need to be valued consistently. This is done by first assuming stochastic process which describes the underlying asset dynamics. The process is calibrated to the observed market prices of exchange-traded options. Then, the resulting process is used to price over the counter options. The need to minimize (or even eliminate) persistent smiles in options data, hence consistent valuation and hedging of exotic options with the prices of standard European contingent claims in developed and emerging markets is the aim of this chapter.

A common assumption underlying most option pricing models are that the logarithm of stock price are normally distributed. The well known and widely used in industry i.e. Black and Scholes (1973) and Merton (1973) option pricing model was derived under such an assumption. An extensive empirical literature in finance has documented not only the presence of anomalies in Black-Scholes model, but also the term structure of these anomalies (for instance, the behavior of the volatility smile or the unconditional returns at different maturities, riding on smile Derman and Kani (1994), pricing with a smile Dupire (1994), Duan (1996) cracking the smile, Das and Sundaram (1999), Bringo and Mercurio (2000), Meziou (2001) on adaptive mixture for a controlled smile). Our main focus will be to understand the source of the "grimaced or smile of option" and propose APARCH Lévy filter as an alternative for option pricing in any economy (be it emerging or developed).

The deviations from normality become more severe when more frequent data are used to calculate stock returns. Various studies have shown that the normal distribution does not accurately describe observed stock return data. Over the past several decades, some stylized facts have emerged about the statistical behavior of speculative market returns such as aggregational Gaussianity, volatility clustering, etc see Rydberg (2000), Cont (2001), Tsay (2002). On the same note, most of the literature for example Eberlein and Keller (1995), Carr and Madan (1998), Barndorff-Nielsen (1998), Carr et al. (2002) and references therein, make a simplifying assumptions, that daily log returns can be modeled by exponential Lévy processes, finding a number of explicit formulaes for pricing derivatives (see also Carr et al. (2003), Schoutens (2003), Carr and Wu (2004)) or modeling stock price process by a geometric Lévy process (see, Chan (1999)) in exact analogy with the ubiquitous geometric Brownian motion model.

There are two important directions in the literature regarding these type of stochastic volatility models. Continuous-time stochastic volatility process represented in general by a bivariate diffusion process, and the discrete time autoregressive conditionally heteroscedastic (ARCH) model of Engle (1982) or its generalization (GARCH) as first defined by Bollerslev (1986). In the last few years, much interest has been given to the discrete-time GARCH option pricing models. The most important papers which study the empirical fitting of these model include Pagan and Schwert (1990), Glosten et al. (1993), Bollerslev et al. (1994). Option pricing in GARCH models has been typically done using the local risk neutral valuation relationship (LRNVR) pioneered by Duan (1995). The crucial assumptions in his construction are the conditional, normality distribution of the asset returns under the underlying probability space and the invariance of the conditional volatility to the change of measure. The empirical performance of these normal option pricing models has been studied by many authors, for example Duan

(1996), Härdle and Hafner (2000), Heston and Nandi (2000), Christoffersen and Jacobs (2004).

This chapter presents two main contributions: a new closed form APARCH Lévy filter option model and an in-depth empirical study in developed and emerging market. We undertake an extensive empirical analysis using European options on the S&P500 index from December 2008 to December 2009. We compare the pricing performance of our approach, GARCH option pricing model of Duan (1995), and the classical Black and Scholes (1973) model. Interestingly, our APARCH Lévy filter outperforms all other pricing models, in all comparisons. In our pricing model, all parameters are estimated from historical data, i.e. for S&P500 index from January 03,1990 to November 21,2008 and NSE20 index from March 02, 1998 to July 11, 2007.

The chapter is organized as follows. In Section 2, a review of suggested models for pricing options is outlined, from the arithmetic Brownian motion of Bachelier, through geometric Brownian motion hence BSM73 model, Bates model and Barndorff-Nielson and Shephard model (BNS). In Section 3 the proposed model is presented after a brief introduction of the model building blocks. Empirical findings are summarized in Section 4. We draw conclusions and suggest directions for future research in section 5.

7.2 Modeling the underlying

In this section, an incomplete chronological review of modeling the dynamics of the underlying risky asset is outlined.

7.2.1 Bachelier's model

It is the pride of mathematical finance that Louis Bachelier was the first to analyze Brownian motion mathematically and he did so to develop a theory of option pricing. In his doctoral dissertation, Louis Bachelier (1900) introduced arithmetic Brownian motion i.e.

$$dS(t) = \mu dt + \sigma dB(t)$$

to model the evolution of a risky asset $S(t)$, say stock. This implied that

$$S_t = S_0(1 + \sigma B_t)$$

is normally distributed with mean S_0 and variance $S_0^2\sigma^2 T$. The price of the European call option at time $t = 0$

$$\begin{aligned} C(0, S_0) &= \mathbb{E}^{\mathbb{Q}}\left[(S_T^B - K)^+\right] && (7.2.1)\\ &= \int_{S_0-K}^{\infty} (S_0 + x - K)\frac{1}{S_0\sigma\sqrt{2\pi T}} e^{-\frac{1}{2}\frac{x^2}{S_0^2 T}} dx && (7.2.2)\\ &= (S_0 - K)\Phi\left(\frac{S_0 - K}{S_0\sigma\sqrt{T}}\right) + S_0\sigma\sqrt{T}\phi\left(\frac{S_0 - K}{S_0\sigma\sqrt{T}}\right) && (7.2.3)\\ &= (S_0 - K)\Phi\left(\frac{S_0 - K}{S_0\sigma\sqrt{T}}\right) + S_0\sigma\sqrt{T}\phi\left(\frac{S_0 - K}{S_0\sigma\sqrt{T}}\right) && (7.2.4) \end{aligned}$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \tag{7.2.5}$$

$$\phi'(x) = -x\phi(x). \tag{7.2.6}$$

denotes the standard density of a normal distribution. Sixty five years later, Samuelson (1965) proposed classical model, commonly known as geometric Brownian motion (GBM)

$$dS(t) = rS(t)dt + \sigma S(t)dB_t \tag{7.2.7}$$

which, applying Itô's lemma can be written as

$$S(t) = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t\right\} \tag{7.2.8}$$

where $S_0 > 0$ is the initial value of the asset, $B = \{B_t : t \geq 0\}$ is the standard Brownian motion, $\sigma > 0$ and $\mu \in \mathbb{R}$. Empirical evidence show squared returns as a measure of volatility, displaying positive autocorrelation over several days which contradict the constant volatility assumption. Non constant volatility can be observed as well in the options market where smiles in implied volatility decrease across strikes. Black and Scholes (1973) model is based on the GBM.

7.2.2 Merton Jump Diffusion Model

Three years after 1973, Merton (1976) proposed a model(Merton Jump Diffusion Model,hereafter MJD76) that allows discontinuous trajectories of asset prices. The model was to cope with the unusual random information at random times whose impact on the stock price may be treated as a random variable. It was an extension to GBM model by adding jumps to the stock price dynamics

$$\frac{dS(t)}{S(t)} = (\alpha - \lambda k)dt + \sigma dB(t) + (Y_t - 1)dN_t \tag{7.2.9}$$

where , α, is the instantaneous expected return on the asset. σ is the instantaneous volatility of the asset return, B_t is a standard Brownian motion process, N_t is a Poisson process with intensity λ. Standard assumption is that (B_t), (N_t) and (Y_t) are independent, $Y_t - 1$ is lognormally distributed with mean

$$\mathbb{E}[Y_t - 1] = e^{\mu + \frac{\delta^2}{2}} - 1 := k.$$

The relative jump size is assumed to be $Y_t - 1$. Note that there are two sources of randomness in the jump-diffusion process. The first source is the Poisson process dN_t which causes the asset price to jump randomly. The other source is the Wiener process. It follow that

$$\ln \mathbb{E}(Y_t - 1) = \mathbb{E}(\ln Y_t).$$

The expected relative price change $\mathbb{E}(dS_t/S_t)$ from the jump part dN_t in the time interval dt is $\lambda k dt$. This is the prediction part of the jump, since

$$\mathbb{E}[(Y_t - 1)dN_t] = \mathbb{E}[Y_t - 1]\mathbb{E}dN_t \tag{7.2.10}$$

$$= k\lambda dt. \tag{7.2.11}$$

This is why the instantaneous expected return on asset αdt is adjusted by $-\lambda k dt$ in the drift term of the jump diffusion process to make the jump part an unpredictable innovation.

$$\begin{aligned}\mathbb{E}\frac{dS_t}{S_t} &= \mathbb{E}[(\alpha-\lambda k)dt]+\mathbb{E}[\sigma dB_t]+\mathbb{E}[(Y_t-1)dN_t] \\ &= (\alpha-\lambda k)dt+0+k\lambda dt \\ &= \alpha dt\end{aligned} \tag{7.2.12}$$

Note that if the asset price does not jump in small time interval dt (i.e $dN_t=0$) then the jump diffusion process is simply a Brownian motion with a drift process. To find explicit solution to the proposed dynamic one would apply Itô formula for a jump diffusion process i.e.

$$\begin{aligned}df(X_t,t) &= \frac{\partial f}{\partial t}(X_t,t)+b_t\frac{\partial f}{\partial X}(X_t,t)dt+\frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial X^2}(X_t,t)dt \\ &+ \sigma_t\frac{\partial f}{\partial X}(X_t,t)dB_t+[f(X_{t-}+\Delta X)-f(X_{t-})]\end{aligned} \tag{7.2.13}$$

$$X_t = X_0+\int_0^t b_s ds+\int_0^t \sigma_s B_s+\sum_{i=1}^{N_t}\Delta X_i \tag{7.2.14}$$

therefore,

$$\begin{aligned}d\ln S_t &= \frac{\partial \ln S_t}{\partial t}dt+(\alpha-\lambda k)S_t\frac{\partial \ln S_t}{\partial S_t}+\sigma_t^2S_t^2\frac{\partial^2\ln S_t}{\partial S_t^2} \\ &+ \sigma S_t\frac{\partial \ln S_t}{\partial S_t}dB_t+[\ln Y_tS_t-\ln S_t]\end{aligned} \tag{7.2.15}$$

$$d\ln S_t = (\alpha-\lambda k)dt-\frac{\sigma^2}{2}dt+\sigma_t dB_t+\ln Y \tag{7.2.16}$$

$$\begin{aligned}S_t &= S_0\exp\left((\alpha-\frac{\sigma^2}{2}-\lambda k)t-\sigma_tB_t\right)\exp\left(\sum_{j=1}^{N}\ln Y_j\right) \\ &= S_0\exp\left((\alpha-\frac{\sigma^2}{2}-\lambda k)t+\sigma B_t+\sum_{j=1}^{N_t}\ln Y_j\right)\end{aligned} \tag{7.2.17}$$

This means that the asset price process $\{S_t, 0\le t\le T\}$ is modeled as an exponential Lévy process of the form

$$S_t = S_0e^{X_t},$$

where X_t is a a Lévy process which is categorized as a Brownian motion with drift(continuous part) plus a compound Poisson process(jump part). The probability density of log return $x_t=\ln(S_t/S_0)$ is obtained as a series of the following form

$$\mathbb{P}(x_t\in A) = \sum_{j=0}^{\infty}\mathbb{P}(x_t\in A|N_t=j) \tag{7.2.18}$$

$$\mathbb{P}(x_t) = \sum_{j=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^j}{j!}N\left(x_t;(\alpha-\frac{\sigma^2}{2}-\lambda k)t+j\mu,\sigma^2t+i\delta^2\right) \tag{7.2.19}$$

where

$$N\left(x_t;(\alpha-\frac{\sigma^2}{2}-\lambda k)t+j\mu,\sigma^2t+j\delta^2\right)=\frac{1}{\sqrt{2\pi(\sigma^2t+j\delta^2)}}\exp\left(-\frac{\left[x_t-\left\{\left(\alpha-\frac{\sigma^2}{2}-\lambda k\right)t+j\mu\right\}\right]^2}{2(\sigma^2t+j\delta^2)}\right)$$

The characteristic function of the Merton log -return density function is calculated as

$$\begin{aligned}\phi(\omega) &= \int_{-\infty}^{+\infty} \exp(i\omega x_t)\mathbb{P}(x_t)dx_t & (7.2.20)\\ &= \exp\left[\lambda t \exp\left\{\frac{1}{2}\omega(2i\mu - \sigma^2\omega)\right\} - \lambda t(1 + i\omega k) - \frac{1}{2}t\omega\left\{-2j\omega + \sigma^2(i+\omega)\right\}\right].\end{aligned}$$

After simplification

$$\phi(\omega) = \exp[t\psi(\omega)]$$

with the cumulant generating function

$$\psi(\omega) = \lambda\left\{\exp\left(j\omega\mu - \frac{\delta^2\omega^2}{2}\right) - 1\right\} + j\omega\left(\alpha - \frac{\sigma^2}{2} - \lambda k\right) - \frac{\sigma^2\omega^2}{2}. \quad (7.2.21)$$

This generates the following cumulants

$$\begin{aligned}\mathbb{K}_1 &= \alpha - \frac{\sigma^2}{2} - \lambda\left(e^{(\mu+\frac{\delta^2}{2})} - 1\right) + \mu\lambda & (7.2.22)\\ \mathbb{K}_2 &= \sigma^2 + \lambda\delta^2 + \lambda\mu^2 & (7.2.23)\\ \mathbb{K}_3 &= \lambda(3\sigma^2\mu + \mu^3) & (7.2.24)\\ \mathbb{K}_4 &= \lambda(3\sigma^4 + 6\mu^2\delta^2 + \mu^4). & (7.2.25)\end{aligned}$$

Annualized (per unit of time) mean variance skewness and excess kurtosis of the log return density are computed as follows

$$\begin{aligned}\frac{1}{T}\mathbb{E}\left[\log(S_T/S_0)\right] &= \mathbb{K}_1 & (7.2.26)\\ \frac{1}{T}var\left[\log(S_T/S_0)\right] &= \mathbb{K}_2 & (7.2.27)\\ \frac{1}{T}skewness\left[\log(S_T/S_0)\right] &= \frac{\mathbb{K}_3}{\mathbb{K}_2^{3/2}} & (7.2.28)\\ \frac{1}{T}\text{excess kurtosis} &= \frac{\mathbb{K}_4}{\mathbb{K}_2^2}. & (7.2.29)\end{aligned}$$

Using the first four cumulants of the distribution, parameters can be estimated by cumulant matching (see Press (1967) for detailed parameter estimation of compound poisson model). The jumps follow a homogeneous Poisson process N_t with intensity λ, which is independent of $B(t)$. We note that jump components add mass to the tails of the returns distribution. Increasing δ adds more mass to both tails.

The only difference between the BSM73 model and the Merton jump diffusion is the addition of the term $\sum_{j=1}^{N_t} Y_j$. A compound Poisson jump process $\sum_{j=1}^{N_t} Y_j$ contains two sources of randomness. The first is Poisson process dN_t with intensity λ(i.e. average number of jumps per unit time) which causes the price to jump randomly. It is assumed that these two sources of randomness are independent of each other. By introducing three extra parameters to the original BSM73 model, MJD76 model tries to capture the (negative) skewness and excess kurtosis of the log return density $\mathbb{P}\left[\ln(S_t/S_0)\right]$, which deviates from BSM73 lognormal return density.

7.2.3 Heston Model

In BSM73 model, a contingent claim is dependent on one or more tradable assets. The randomness in the option is solely due to the randomness of these assets. Since the assets are tradable, the option can be hedged by continuously trading the underlyings, hence every contingent claim can be replicated. In a stochastic volatility model a contingent claim is dependent on the randomness of the asset $\{S_t\}_{t\geq 0}$ and the randomness associated with volatility of the assets returns $(\{V_t\}_{t\geq 0})$ and the randomness associated with the volatility is not a traded asset.

Heston (1993) model assumes that the underlying asset $S(t)$, evolves according to

$$dS(t) = \mu S dt + \sqrt{v(t)}S(t)dB_1(t) \tag{7.2.30}$$

and volatility , $v(t)$, evolves according to

$$dv(t) = \kappa[\theta - v(t)]dt + \sigma\sqrt{v(t)}dB_2(t) \tag{7.2.31}$$

where the two Wiener process innovations $dB_1(t)$ and $dB_2(t)$ are allowed to be ρ correlated.

For a risk neutral valuation in this model, there is need to change measure from real world to empirical martingale measure. This can be achieved by Girsanov's theorem. In particular

$$\begin{aligned}
d\bar{B}_1(t) &= dB_1(t) + \vartheta_t dt && (7.2.32)\\
d\bar{B}_2(t) &= dB_2(t) + \wedge(S,V,t)dt && (7.2.33)\\
\frac{d\mathbb{Q}}{d\mathbb{P}} &= exp\left\{-\frac{1}{2}\int_0^t(\vartheta_s^2 + \wedge(S,V,s)^2)ds - \int_0^t \vartheta dB_1(s) - \int_0^t \wedge(S,V,s)dB_2(s)\right\} \\
\vartheta_t &= \frac{\mu - r}{\sqrt{V_t}}, && (7.2.34)
\end{aligned}$$

where the volatility $\wedge(V,S,t)$ is not constant for any asset F. The closed-form solution of a European call on a non dividend paying asset for the Heston model is

$$C(S_t, V_t, t, T) = S_t P_1 - Ke^{-r(T-t)}P_2$$

where

$$\begin{aligned}
P_j(x, V_t, T, K) &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-i\phi\ln(K)}}{i\phi}f_j(x,V_t,T,\phi)\right)d\phi && (7.2.35)\\
x &= \ln(S_t) && (7.2.36)\\
f_j(x,V_t,T,\phi) &= \exp\{C(T-t,\phi) + D(T-t,\phi)V_t + i\phi x\} && (7.2.37)\\
C(T-t,\phi) &= r\phi ir + \frac{a}{\sigma^2} + \left[(b_j - \rho\sigma\phi i + d)T - 2\ln\left(\frac{1-ge^{dr}}{1-g}\right)\right] && (7.2.38)\\
D(T-t,\phi) &= \frac{b_j - \rho\sigma\phi i + d}{\sigma^2}\left(\frac{1-e^{dr}}{1-ae^{dr}}\right) && (7.2.39)\\
g &= \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d} && (7.2.40)\\
d &= \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)} && (7.2.41)
\end{aligned}$$

for $j = 1,2$ where $u_1 = 0.5,\ \ u_2 = -u_1,\ a = k\theta$

$$b_2 = K\lambda,\ \ b_1 = b_2 - \rho\sigma.$$

The integral (7.2.35) cannot be evaluated exactly, but can be approximated by using some numerical integration technique or fast Fourier transform method of Carr and Madan (1998).

Note that Hull and White (1987) had earlier proposed a GBM for modeling volatility which might increase exponentially. Stein and Stein (1991) suggested the use of an Ornstein-Uhlenbeck type process (which might admit negative values for the variance)

$$dv(t) = \kappa(\theta - v(t))dt + \beta dB(t) \tag{7.2.42}$$

These deficiencies were eliminated in the Heston (1993) model.

7.2.4 Bates Model

The Merton (1976) and Heston (1993) models, were combined by Bates (1996), who proposed a model with stochastic volatility and jumps:

$$\begin{aligned} dS(t) &= rS(t)dt + \sqrt{v_t}S(t)dB_1(t) + S(t)dJ_t & (7.2.43)\\ dv_t &= k(\theta - v_t)dt + \sigma_t\sqrt{v_t}dB_2(t) & (7.2.44)\\ \mathrm{Cov}(dB_1(t), dB_2(t)) &= \rho dt & (7.2.45) \end{aligned}$$

Where $J(t)$ is a compound Poisson process with intensity γ and log-normal distribution of jump sizes independent of $B_1(t)$ and $B_2(t)$. Under the risk neutral probability an equation for the logarithm of the asset price is obtained.

$$dX(t) = (r - \gamma\bar{\kappa} - \frac{1}{2}v_t)dt + \sqrt{v_t}dB_1(t) + d\tilde{J}_t \tag{7.2.46}$$

where $\tilde{J}$ is a compound poisson process with normal distribution of jump magnitudes.

Since the jumps are independent of the diffusion part, the characteristic function for the log prices can be obtained as

$$\phi_{X_t}(z) = \phi^D_{X_t}(z)\phi^J_{X_t}(z)$$

where

$$\phi^D_{X_t}(z) = \frac{\exp\left\{\frac{\kappa\theta t(\kappa - i\rho\sigma z)}{\sigma^2} + izt(r - \lambda\bar{\kappa}) + izx_0\right\}}{\left(\cosh\frac{\gamma t}{2} + \frac{\kappa - i\rho\sigma z}{\gamma}\sinh\frac{\gamma t}{2}\right)^{\frac{2\kappa\theta}{\sigma^2}}} \exp\left\{-\frac{(z^2 + iz)v_0}{\gamma\coth\frac{\gamma t}{2} + \kappa - i\rho\sigma z}\right\} \tag{7.2.47}$$

is the characteristic function of the diffusion part and

$$\phi^J(z) = \exp\left(t\lambda\left[e^{-\delta^2\frac{z^2}{2} + i\left\{\ln(1+\bar{\kappa}) - \frac{\sigma^2}{2}\right\}z}\right]\right) \tag{7.2.48}$$

is the characteristic function of the jump part. Where x_0 and v_0 are the initial values for the long price process and the volatility process respectively. Option pricing can be done using fast Fourier transform (FFT) as suggested by Carr and Madan (1998) and Nicolato and Venardos (2003).

7.2.5 Barndorff-Nielsen and Shephard Model (BNS)

Under exponential Lévy model, the stock price dynamics are assumed to be driven by Lévy processes instead of Brownian process. Note all Lévy processes except for Brownian motion have jumps. Familiar special cases of Lévy processes are Brownian motion and the compound Poisson process (see for example Sato (1999), Bertoin (1996),Raible (2000a), Cont and Tankov (2004) for theoretical and numerics of Lévy processes). Most of the proposed exponential Lévy model miss the changing volatility in time. In view of this, Barndorff-Nielsen, Shephard and co-workers in their article Barndorff-Nielsen and Shepard (2001), introduced Non-Gaussian processes of Ornstein-Uhlenbeck type or OU processes which the background driving Lévy process (BDLP) is a subordinator.

Barndorff-Nielsen et al. (2002) show that the dynamics of the log price under such an equivalent martingale measure $\mathbb{Q}$ are given by,

$$dX(t) = (r - q - \eta\kappa(-\rho) - \frac{1}{2}\sigma_t^2)dt + \sigma_t dB(t) + \rho dZ_{\eta t}, X_0 = \log S_0 \qquad (7.2.49)$$

$$d\sigma_t^2 = -\eta\sigma_t^2 dt + dZ_{\eta t} \qquad (7.2.50)$$

where $B = \{B(t), t \geq 0\}$ is a Brownian motion under $\mathbb{Q}$ independent of background driving Lévy process (BDLP) $Z = \{Z_t, t \geq 0\}$. The Brownian process and the BDLP Z are independent and $(\mathcal{F}_t)$ is assumed to be the usual augmentation of the filtration generated by the pair (W, Z). Note that the

(i) instantaneous variance of the log returns is given by

$$\begin{aligned} var(dX_t) &= var\left\{(r - q - \eta\kappa(-\rho) - \frac{1}{2}\sigma_t^2)dt + \sigma_t dB(t) + \rho dZ_{\eta t}\right\} && (7.2.51)\\ &= \left[\sigma_t^2 + \rho^2\eta var(Z_1)\right] dt, && (7.2.52) \end{aligned}$$

(ii) integrated variance $\sigma_{t,T}^{2*}$ over the time period $[t, T]$ is

$$\begin{aligned} \sigma_{t,T}^{2*} &= \int_t^T \sigma_s^2 ds && (7.2.53)\\ &= \int_t^T \left(e^{-\lambda s}\sigma_0^2 + \int_0^t e^{-\lambda(t-s)} dZ_{\lambda s}\right) ds \\ &= \frac{1}{\lambda}\left[1 - e^{-\lambda(T-t)}\right]\sigma_t^2 + \lambda^{-1}\int_t^T \left(1 - e^{-\lambda(T-s)}\right) dZ_{\eta s} && (7.2.54) \end{aligned}$$

A European style contract with payoff $h(X_T)$ can be valued according to the fundamental theorem of asset pricing. Therefore it's arbitrage free price at time $t \leq T$ is given by

$$C_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}h(X_T)|\mathcal{F}_t\right]$$

Note that the expectation is taken with respect to $\mathbb{Q} \in \mathcal{M}$, where $\mathcal{M}$ is an equivalent martingale measure. Typically, financial markets where investors can only trade in a riskless asset is an underlying stock with stochastic volatility are incomplete and BN-S models are no exception. An alternative representation of derivative price can be obtained as an exception of the BSM73 formula with respect to particular functionals of the BDLP.

Define the effective spot log-price and volatility by

$$X_{\text{eff}} = X_t + \rho(Z_{\lambda T} - Z_{\lambda t}) - \lambda k \rho(T - t) \tag{7.2.55}$$

$$V_{\text{eff}} = \left(\frac{\sigma^{2*}_{t,T}}{T-t}\right)^{2} \tag{7.2.56}$$

Let $C_t^{BS}(x, v)$ denote the Black-Scholes price at time t of the claim $h(X_T)$ when the spot log price is x and the volatility is v. Due to independence between the Brownian motion and BDLP, Z in the Q dynamics of X and σ^2, the arbitrage free price at $t \leq T$ can be computed

$$C_t = \mathbb{E}^Q \left[C_t^{BS}(X_{\text{eff}}, V_{\text{eff}}) | X_t, \sigma_t^2\right]. \tag{7.2.57}$$

If the BSM73 price $C_t^{BS}(x, v)$ is known in closed form, as in the case of European options, expression (7.2.57) can be evaluated as a sample average across simulations of the pair $(X_{\text{eff}}, V_{\text{eff}})$. A self contained exposition of the required result, the reader is referred to Barndorff-Nielsen and Shepard (2001), Nicolato and Venardos (2003). Moreover, for more information about Lévy models with stochastic volatility see Barndorff-Nielsen et al. (2002), and Schoutens (2003).

7.3 Model construction

Consider a frictionless financial market where a riskless asset with constant return rate r and a risky asset (a stock) are traded upto a fixed horizon date T. We assume that the price process of the stock $S = (S_t)$ is defined on some filtered probability space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}).$$

We state selected basic tools required to construct the closed form valuation formula.

7.3.1 ARCH type models

Two of the most common empirical findings in the finance literature are that the distributions of asset returns display tails heavier than those of the normal distribution and the squared returns are highly correlated. The aforemention stylized empirical regularities led to some econometricians to develop models which can accommodate and account for these phenomena. The ARCH model have excess kurtosis accommodating the empirical findings. Furthermore, they show some persistence in the squared autocorrelations. Engle (1982) introduced the Autoregressive Conditional Heteroscedasticity (ARCH) model and Bollerslev (1986) generalized the ARCH to GARCH model (see Engle and Bollerslev (1986), Bollerslev et al. (1992), Bollerslev et al. (2002), Karanasos (1999))

$$\varepsilon_t \sim N(0, \sigma_t^2), \quad \sigma_t^2 = \sum_{i=1}^{q} \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2, t = 1, ...T. \tag{7.3.1}$$

Neither the ARCH nor the GARCH models take the asymmetry into account. Volatility is negatively correlated with changes in stock returns in the sense that "bad news" tend to give an increase in volatility and "good news" a decrease in volatility.

7.3.2 Compound Poisson process

It is assumed $J(t)$ is a process of stationary and independent increments whose basic mechanism is composed of a compound Poisson process augmented by Wiener process. Specifically

$$J(t) = Y_{N(0)} + \sum_{j=1}^{N(t)} Y_j, \quad j = 1, ..., N(t) \tag{7.3.2}$$

$(Y_1, ..., Y_k, ...)$ is a sequence of mutually independent random variables obeying the common law $N(\vartheta, \sigma_2^2)$. $N(t)$ is a poisson counting process with parameter ηt, which represents the number of random events occurring in time t; $\{N(t), t \geq 0\}$ is independent of the Y_k and $Y_{N(0)} \sim N(0, \sigma_1^2 t)$. For detailed exposition on Poisson intensity estimation, see Press (1967), Simar (1976), Vlaar,P. and Palm,F. (1993), Chan and Maheu (2002).

7.3.3 Lévy processes

A Lévy process $X = (X_t)_{t\geq 0}$ is a process with stationary and independent increments. Underlying is a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ to which the process is adapted. The distribution of a Lévy process is completely determined by any of its marginal distributions say generalized hyperbolic (GH) i.e. $X_t - X_s \sim GH, \quad 0 < s < t$.

7.3.4 Generalized Hyperbolic distribution

The method of modeling stock prices by subordinated processes generalizes the classical log-normal asset price model in continuous time. The physical time is substituted by an intrinsic time which provides a long tail effect observed in the market. A random variable W is said to have a generalized inverse gaussian distribution if its probability density function is given by

$$f_{GIG}(w; \lambda, \gamma, \delta) = \frac{(\gamma/\delta)^\lambda}{2K_\lambda(\gamma\delta)} w^{\lambda-1} \exp\left(-\frac{1}{2}(\frac{\delta^2}{w} + \gamma^2 w)\right), w > 0$$

where K_λ is a modified Bessel function of the third kind with the index λ.

$$K_\lambda(\omega) = \frac{1}{2}\int_0^\infty \exp\left[-\frac{\omega}{2}(v^{-1} + v)\right] v^{\lambda-1} dv \tag{7.3.3}$$

The parameters $\lambda \in \mathbb{R}, \gamma \geq 0, \delta \geq 0$ are such that $\gamma \neq \delta$ if either of the them takes the value zero. We note that if $\gamma > 0$ and $\delta > 0$ then

$$\mathbb{E}W^k = \left(\frac{\delta}{\gamma}\right)^k \frac{K_{\lambda+k}(\gamma\delta)}{K_\lambda(\gamma\delta)}, \quad k \in \mathbb{Z}$$

Suppose $B = \{B_t, t \geq 0\}$ is a Brownian process and $W = \{W(t), t \geq 0\}$ is a nonnegative stochastic process, then the new process $\{B_{W_t} := \sqrt{W_t}Z, \quad Z \sim N(0,1), \quad t \geq 0\}$ is said to be subordinated to B by intrinsic time process W.

(Normal Mean-Variance Mixture) A random variable X is said to have a normal mean-variance distribution if

$$X = \mu + \beta W + \sigma\sqrt{W}Z$$

where $Z \sim N(0,1)$, W is a positive random variable independent of Z; μ, β and $\sigma > 0$. From the definition, we can see that the conditional distribution of X given W is normal with mean $\mu + \beta W$ and variance $\sigma^2 W$. Note that if the mixture variable W is $GIG(\lambda, \gamma, \delta)$ distributed, then X is a Generalized Hyperbolic distribution with the $(\lambda, \alpha, \beta, \delta, \mu)$ parametrization, where $\alpha^2 = \gamma^2 + \beta^2$.

The probability density function of the one-dimensional Generalized Hyperbolic distribution is given by the following:

$$f_{GH}(x; \alpha, \beta, \delta, \mu, \lambda) = \frac{(\gamma/\delta)^\lambda}{\sqrt{2\pi} K_\lambda(\delta\gamma)} \frac{K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2 + (x-\mu)^2})}{(\sqrt{\delta^2 + (x-\mu)^2}/\alpha)^{\frac{1}{2}-\lambda}} e^{\beta(x-\mu)} \tag{7.3.4}$$

According to Barndorff-Nielsen (1977), the parameters domain is given by

$$\alpha > 0 \quad \alpha^2 > \beta^2 \quad \delta \geq 0 \quad \text{for} \quad \lambda > 0,$$

$$\alpha > 0 \quad \alpha^2 > \beta^2 \quad \delta > 0 \quad \text{for} \quad \lambda = 0,$$

$$\alpha > 0 \quad \alpha^2 \geq \beta^2 \quad \delta > 0 \quad \text{for} \quad \lambda < 0.$$

In all cases, μ is the location parameter and can take any real value, δ is a scale parameter; α and β determine the distribution shape and λ defines the subclasses of GH and is related to the tail flatness.
Characteristic function of the GH is given by

$$\phi_{GH}(u) = e^{iu\mu} \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2} \right)^{\lambda/2} \frac{K_\lambda\left(\delta\sqrt{\alpha^2 - (\beta + iu)^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2 - \beta^2}\right)}, \tag{7.3.5}$$

while mean and variance are given respectively by the following

$$E(X) = \mu + \frac{\beta\delta}{\sqrt{\alpha^2 - \beta^2}} \frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} \tag{7.3.6}$$

and

$$\text{Var}(X) = \delta^2 \left(\frac{K_{\lambda+1}(\zeta)}{\zeta K_\lambda(\zeta)} + \frac{\beta^2}{\alpha^2 - \beta^2} \left[\frac{K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)} - \left(\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} \right)^2 \right] \right) \tag{7.3.7}$$

where $\zeta = \delta\sqrt{\alpha^2 - \beta^2}$. Note that, if $X \sim GH(\lambda, \alpha, \beta, \delta, \nu)$, then

$$X \sim GH(-\frac{1}{2}, \alpha, \beta, \delta, \mu) \quad \text{has} \quad \text{Normal-Inverse Gaussian distribution}$$

$$X \sim GH(1, \alpha, \beta, \delta, \mu) \quad \text{Hyperbolic distribution} \tag{7.3.8}$$

$$X \sim GH(\lambda, \alpha, \beta, 0, \mu) \quad \text{Variance Gamma distribution} \tag{7.3.9}$$

7.3.5 Option pricing models

The Black-Scholes-Merton option pricing formula

The most celebrated of all models used in finance is probably the Black-Merton-Scholes model, suggested by Black and Scholes (1973), and Merton (1973)(hereafter BMS73). In BMS73 model under risk neutral probability measure $\mathbb{Q}$, stock simply follows a geometric Brownian motion,

$$dS(t) = rS(t)dt + \sigma S(t)dB(t),$$

which implies that log returns $\log(\frac{S(t)}{S(t-1)})$ are normally distributed with mean $r - \frac{\sigma^2}{2}$ and variance σ^2.

A European call option is an asset which gives the buyer the right, but not the obligation, to purchase the stock $S(t)$ at a prenegotiated price K at time T. The value of the European call option, at time of maturity, is given by

$$C(t, K) = \max(0, S(T) - K)$$

The price of this derivative is given by the BMS73 option pricing formula which is

$$\begin{aligned} C(t, K) &= S(t)\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \\ d_1 &= \frac{\ln(S(t)/Ke^{r(T-t)})}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} \\ d_2 &= \frac{\ln(S(t)/Ke^{r(T-t)})}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t} \end{aligned}$$

GARCH option pricing model

According to Duan (1995), pricing contingent payoffs requires temporally aggregating one period asset returns to arrive at a random terminal asset price at some future point in time T. He provided sufficient conditions to apply a locally risk-neutral valuation methodology which is applied in the following theorem;

Theorem 7.3.1. *Under the locally risk-neutral probability measure $\mathbb{Q}$, the process for asset price is*

$$\log S(t) - \log S(t-1) = r - \frac{1}{2}\sigma_t^2 + \sigma_t\xi_t \tag{7.3.10}$$

where $\xi_t|\phi_{t-1} \sim N(0, \sigma_t)$ and $\sigma_t^2 = \omega + \sum_{j=1}^{q} \alpha_j (\xi_{t-j} - \lambda\sigma_t)^2 + \sum_{i=1}^{p} \beta_i\sigma_{t-i}^2$

Proof: One can refer to the proof of Theorem 2.2 of Duan (1995).
Then from Theorem 2.2we have the following corollary

$$S(T) = S(t)\exp\left[(T-t)r - \frac{1}{2}\sum_{s=t+1}^{T}\sigma_s^2 + \sum_{s=t+1}^{T}\xi_s\right] \quad \text{then} \tag{7.3.11}$$

$$C^{D95}(t, K) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0)|\mathcal{F}_t] \tag{7.3.12}$$

Because there are no analytic expression for the expectation in (7.3.12), we use numerical techniques to simulate the option price $C^{D95}(t, K)$.

$$\bar{C}^{D95}(t, K) = e^{-r(T-t)}\frac{1}{m}\sum_{j=1}^{m}\max(S_T^{(j)} - K, 0) \tag{7.3.13}$$

7.3.6 The APARCH Lévy filter option pricing formula

Asymmetric Power Autoregressive Conditional Heteroscedasticity APARCH(p,q) model can be defined as;

$$\begin{aligned}
\log S(t) - \log S(t-1) &= u + \varepsilon_t, t = 1, ..., T \\
\varepsilon_t &= \sigma_t Z_t, \quad Z_t \sim i.i.d.N(0,1), \forall \ t, \\
\sigma_t^\delta &= \omega + \sum_{i=1}^{q} \alpha_i \kappa(\varepsilon_{t-i})^\delta + \sum_{j=1}^{p} \beta \sigma_{t-j}^\delta \\
\kappa(\varepsilon_{t-i}) &= |\varepsilon_{t-i}| - \gamma_i \varepsilon_{t-i} \text{where} \\
u, \omega > 0, \gamma_i, \alpha_i (1, .., q), & \quad \beta_j, (j = 1, ..., p) \text{are parameters} \\
\delta > 0 \text{ Box-Cox transformation} & \quad \gamma_i (i = 1, ..., q) \text{leverage effect}
\end{aligned}$$

Under the historical measure $\mathbb{P}$, the asymmetric GJR GARCH model is

$$\begin{aligned}
\log S(t) - \log S(t-1) &= u + \varepsilon_t \\
\sigma_t^2 &= \omega + \beta \sigma_{t-1}^2 + \alpha \varepsilon_{t-1}^2 + \gamma I_{t-1} \varepsilon_{t-1}^2
\end{aligned}$$

where $\varepsilon_t = \sigma_t z_t, \quad z_t \sim f(0,1)$ and $I_{t-1} = 1$ when $\varepsilon_{t-1} < 0$ and $I_{t-1} = 0$ otherwise. When $\gamma > 0$, the model accounts for the leverage effect, i.e., bad news ($\varepsilon_{t-1} < 0$) raises the future volatility more than good news $\varepsilon_{t-1} \geq 0$. In this chapter, we focus on a single lag version of the APARCH specification where $p = q = 1, \quad \kappa = 0$ and $\mathfrak{s} = 2$, for GARCH(1,1) and $p = q = 1, \quad \kappa \neq 0, \quad \mathfrak{s} = 2$ for GJR-GARCH(1,1) while for GJR-GARCH model, the process, is covariance stationary provided

$$\sum_{i=1}^{p} (\alpha_i + \frac{\gamma_i}{2}) + \sum_{j=1}^{q} \beta_j < 1. \tag{7.3.14}$$

For more information on GARCH families nesting on both symmetric and asymmetric GARCH models see Hentschel (1995) and Sebastien (2004).

Proposition 7.3.2. *(Dynamics under $\mathbb{P}$) The dynamics of the stochastic process $S(T)$ is given by*

$$S(T) = S(t) \exp\left(\left[(b + \eta \vartheta)\sigma + u - \frac{1}{2}\sigma^2 (1 + \varpi\varphi) \right| (T - t) + \sigma\sqrt{1 + \varpi\varphi}\sqrt{T - t} Z \right) \tag{7.3.15}$$

Proof:

$$
\begin{aligned}
\log \frac{S(j)}{S_{(j-1)}} &= u + \varepsilon_j + \text{noise}, \quad \varepsilon_j \sim i.i.d(0, \sigma_j^2) \\
&= u + \sigma_j \left[Z_j^{(1)} + X_j \right], \quad Z_j^{(1)} \sim i.i.dN(0,1), \quad \text{for all} \quad j = 1, ..., N. \\
&= u + \sigma_j \left[Z_j^{(1)} + \psi(L) Z_\alpha + \xi_j \right], \quad Z_\alpha \sim N(0, \sigma_2^2), \quad \psi(L) = \sum_{j=0}^{\infty} \psi_j L^j \\
&= u + \sigma_j \left[Z_j^{(1)} + \sqrt{(} Var^{\mathbb{P}} \psi(L) Z_\alpha) \left\{ Z_j^{(2)} + \xi_j \right\} \right] \quad \text{where} \quad X_j \sim GH, \quad Y_k \sim N(\vartheta, \sigma_2^2) \\
&= u + \eta \vartheta \sigma_j + \\
&\quad \sigma_j \left[Z_j^{(1)} + \sqrt{(} Var^{\mathbb{P}} \psi(L) Z_\alpha \left\{ Z_j^{(2)} + \left(Y_{N_0} Z_j^{(3)} + \sqrt{Var^{\mathbb{P}} \left(\sum_{k=N_{(t-1)+1}}^{N_{(t)}} Y_k \right) Z_j^{(4)}} \right) \right\} \right] \\
\therefore \quad \Delta \log S(j) &= (u + \eta \vartheta \sigma_j) \Delta t + \sigma_j \left[\Delta Z_j^{(1)} + \sqrt{V} ar^{\mathbb{P}} \psi(\mathbb{L}) Z_\alpha \left\{ \Delta Z_j^{(2)} + \left(\sigma_1^2 \Delta Z_j^{(3)} + \eta (\sigma_2^2 + \vartheta^2) \Delta Z_j^{(4)} \right) \right\} \right]
\end{aligned}
$$

Note that $\mathbb{L}$ is a back-shift operator say, $Z_{j-4} = \mathbb{L}^4 Z_j$.

An important application of the stochastic volatility model is the pricing of option. The large standard errors of the volatility estimates do not necessary carry over to option prices (see Mahieu and Schotman (1998)). Option prices depend on the average expected volatility over the length of the option contract and this averaging should reduce standard errors. In the limit, the average volatility over a long horizon converges to the unconditional variance, which is known without error when conditioning on the parameters of the process, hence the unconditional variance for GARCH(p,q) is given by

$$
\sigma^2 = \omega \left(1 - \sum_{j=1}^{p} \alpha_j - \sum_{k=1}^{q} \beta_k \right)^{-1}, \omega > 0, \quad 0 < \sum_{j=1}^{p} \alpha_j + \sum_{k=1}^{q} \beta_k < 1 \tag{7.3.16}
$$

The mean and variance of daily increment can be estimated by

$$
\mathbb{E}^{\mathbb{P}} \log \frac{S(j)}{S_{(j-1)}} = u + \eta \vartheta \sigma \qquad \text{Var}^{\mathbb{P}} \log \frac{S(j)}{S_{(j-1)}} = \sigma^2 (1 + \varpi \varphi) \tag{7.3.17}
$$

where

$$
\begin{aligned}
\varpi &= \text{Var}^{\mathbb{P}} X \\
&= \delta^2 \left(\frac{K_{\lambda+1}(\zeta)}{\zeta K_\lambda(\zeta)} + \frac{\beta^2}{\alpha^2 - \beta^2} \left[\frac{K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)} - \left(\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} \right)^2 \right] \right)
\end{aligned} \tag{7.3.18}
$$

and $\varphi \in \mathbb{R}$ can be easily estimated from historical data set, preferably at least 15 years of daily increments. For simplicity, let $\mu^* := \mathbb{E}^{\mathbb{P}} \log \frac{S(j)}{S_{(j-1)}}$ and $\sigma^* = \sqrt{(}\text{Var}^{\mathbb{P}} \log \frac{S(j)}{S_{j-1}})$ Let $0 \leq h < t < T$ such that $u = t - h \quad \forall \quad u \in \mathbb{R}$ then

$$
\log S(h+u) - \log S(u) = \mu^* u + \sigma^* (Z(h+u) - Z(h)), \quad Z(u) \sim N(0, u) \tag{7.3.19}
$$

w.l.o.g. let $Z(h) := B(h)$ such that $\mathbb{E}^{\mathbb{P}}[B^2(h)|\mathcal{F}_h] = h$

$$
\begin{aligned}
\lim_{u \to 0} (\log S(h+u) - \log S(u)) &= \mu^* \lim_{u \to 0} u + \sigma^* \lim_{u \to 0} (B(h+u) - B(h)) && (7.3.20) \\
\frac{1}{S(t)} dS(t) &= \mu^* dt + \sigma^* dB(t) && (7.3.21) \\
dS(t) &= \mu^* S(t) dt + \sigma^* S(t) dB(t) && (7.3.22)
\end{aligned}
$$

In our model, the coefficients μ^* and σ^* being constants, we know that the process has a Lognormal law. Here if the initial time point is t, then the logarithm of $S(t)$ is distributed under $\mathbb{P}$ according to

$$N[\log S(t) + (\mu^* - \frac{1}{2}(\sigma^*)^2)(T-t), (\sigma^*)^2(T-t)]$$

For clarity, use the notation $S(t) := S_t$

$$\therefore S_T = S_t \exp\left[\left((u + \eta\vartheta\sigma) - \frac{1}{2}\sigma^2(1+\varpi\varphi)\right)(T-t) + \sigma\sqrt{1+\varpi\varphi}\sqrt{T-t}Z\right], Z \sim N(0,1)$$

Proposition 7.3.3. *(Risk neutral valuation $\mathbb{Q}$) The implicit value of contingent claim $g(S(T))$ is given by*

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}g(S_T)|\mathfrak{F}_t\right] \tag{7.3.23}$$

where $\mathbb{Q}$ is the risk-neutral measure.

Proof:
Under $\mathbb{Q}$ the risky asset's price satisfies

$$dS_t = (r - \sigma\eta\vartheta - \frac{1}{2}(\sigma^*)^2)S_t dt + \sigma^* S_t dB_t$$

This implies that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[S_T|\mathfrak{F}_t] &= S_t \exp(r) && (7.3.24) \\ \therefore \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}g(S_T)|\mathfrak{F}_t\right] &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[g(S_T)\right] && (7.3.25) \\ &= e^{-r(T-t)}\int_{-\infty}^{\infty} g(e^u) f_{T-t}(u)du && (7.3.26) \\ & && (7.3.27) \end{aligned}$$

where $f_{T-t}(u)$ is the probability density function of the normal distribution with mean

$$\log S_t + \left(r - \sigma\eta\vartheta - \frac{1}{2}(\sigma^*)^2\right)(T-t)$$

and variance

$$(\sigma^*)^2(T-t)$$

When g has an explicit form say European option,

$$g(u) = (u-K)^+,$$

we can develop the explicit formulaes.

Example 7.3.1 (European Call option). *Let $g(u) = \max(0, u-K)$.*

Using the symmetric property of normal distribution, let

$$\begin{aligned} \phi(u) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-u^2/2}du && (7.3.28) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-u}^{+\infty} e^{-u^2/2}du && (7.3.29) \end{aligned}$$

therefore

$$\int_{-\infty}^{+\infty} g(e^u) f_{T-t} du = \int_{\{u>\ln K\}} e^u f_{T-t}(u) du - K \int_{\{u>\ln K\}} f_{T-t}(u) du \qquad (7.3.30)$$

Let $C^{1,2}([0,T]\times\mathbb{R}_+,\mathbb{R})$ be the set of functions f from $([0,T]\times\mathbb{R}_+)$ into $\mathbb{R}$, and of class C^1 with respect to t and C^2 with respect to x.

$$\begin{aligned} C(t,x) &= x\varphi\left(\frac{1}{\sigma\sqrt{1+\varpi\varphi}\sqrt{T-t}}\left\{\ln\left(\frac{x}{K}\right)+(T-t)\left(r-\eta\vartheta\sigma+\frac{\sigma^2(1+\varpi\varphi)}{2}\right)\right\}\right) \\ &- Ke^{-(r-\eta\vartheta\sigma)(T-t)}\phi\left(\frac{1}{\sigma\sqrt{1+\varpi\varphi}\sqrt{(T-t)}}\left\{\ln\left(\frac{x}{K}\right)+(T-t)\left(r-\sigma\eta\vartheta-\frac{\sigma^2}{2}(1+\varpi\varphi)\right)\right\}\right) \end{aligned}$$

Theorem 7.3.4. *The price of a European call is given by*

$$C(t,x) = x\phi(\varrho_1(t)) - Ke^{-r(T-t)}\phi(\varrho_2(t)), \qquad (7.3.31)$$

where

$$\begin{aligned} \varrho_1(t) &= \frac{1}{\sigma\sqrt{(1+\varpi\varphi)}}\left\{\ln\left(\frac{x}{K}\right)+(T-t)\left(r-\sigma\vartheta\eta+\frac{\sigma^2}{2}(1+\varpi\varphi)\right)\right\}, \\ \varrho_2(t) &= \varrho_1 - \sigma\sqrt{(1+\varpi\varphi)}\sqrt{(T-t)} \end{aligned} \qquad (7.3.32)$$

Remark 7.3.1. For purposes of fitting historical data set, one can vary index by some powers, for brevity i.e. Let $h(j,\varphi)=\sqrt{(1+\varpi^j\varphi)}$, $j=0,\pm1,\pm2,\pm3,\ldots$ then

$$\begin{aligned} \varrho_1(t) &= \frac{1}{\sigma h(j,\varphi)}\left\{\ln\left(\frac{x}{K}\right)+(T-t)\left(r-\sigma\vartheta\eta+\frac{1}{2}h(j,\varphi)\right)\right\}, \\ \varrho_2(t) &= \varrho_1 - \sigma\sqrt{h(j,\varphi)}\sqrt{(T-t)} \end{aligned} \qquad (7.3.33)$$

Example 7.3.2. *(Lookback put) Lookback options provide opportunities for the holders to realize attractive gains in the event of substantial movements of the under lying process during the life of the options. Let $\{S(t),\ 0\le t<\infty\}$ continuous stochastic process, say APARCH Levy motion under $\mathbb{P}$, i.e.*

$$dS(t) = S(t)(\mu+\sigma_t\eta\vartheta)dt + g(\sigma_t, h(j,\varphi))dB(t), S(t) = s \qquad (7.3.34)$$

then the price of a lookback put,

$$LB(x,t) = \max_{t\le T} S(t) - S(T)$$

option is

$$
\begin{aligned}
LB(t,s) &= s\left(\phi\left[-\varrho+\sigma h(j,\varphi)\sqrt{T-t}\right]-\sigma^2 h^2(j,\varphi)\phi\left[-\varrho+\sigma h(j,\varphi)\sqrt{T-t}\right]\right)\\
&\quad +s\left(\frac{\sigma^2 h^2(j,\varphi)}{2(r-\eta\vartheta\sigma)}\phi(\varrho)-\phi(\varrho)\right) \qquad (7.3.35)
\end{aligned}
$$

where

$$
\varrho=\frac{\left[(r-\theta\vartheta\sigma)(T-t)+\frac{1}{2}\sigma^2 h(j,\varphi)(T-t)\right]}{\sigma h(j,\varphi)\sqrt{(T-t)}} \qquad (7.3.36)
$$

Remark 7.3.2. For different markets one can vary level of variation accounted by ϖ. i.e. Let $h(j,\varphi):=\sqrt{(1+\varpi^j\varphi)}$, $j=0,\pm1,\pm2,\pm3,...$ then

$$
\begin{aligned}
\varrho_1(t,j) &= \frac{1}{\sigma h(j,\varphi)}\left\{\ln\left(\frac{x}{K}\right)+(T-t)\left(r-\sigma\vartheta\eta+\frac{1}{2}h(j,\varphi)\right)\right\},\\
\varrho_2(t,j) &= \varrho_1-\sigma\sqrt{h(j,\varphi)}\sqrt{(T-t)}
\end{aligned}
$$

7.4 Empirical example

Prices of Options on the S&P500 Index

We use European options on the S&P500 index (symbol: SPX) to test our model. The market for these standard (European) vanilla call options is one of the most active index option market in the world. Consequently, these options have been the focus of many empirical investigations. The data set consists of bid and ask price at the close of the market as at 21 November 2008. On this day the S&P500 closed at 800.3. Historical daily adjusted closing index as recorded from 03 January 1990 to 21 November 2008 was used. The option prices consist of options corresponding to options expiring in December 2008, January 2009, March 2009, June 2009, September 2009 and December 2009 and the following days to maturity $20, 36, 81, 145, 276, 302$ and 430 respectively. We consider closing prices of the out-of-the money (OTM)call options. It is well known that OTM options are actively traded than in-the-money options. Options data are downloaded from Market Watch webpage. The average of bid and ask prices are taken as options prices, while options with time to maturity less than 10 days or more than 430 days are discarded.

Options data is divided into two categories, according to either time to maturity, or moneyness, defined as the ratio of the strike price over the asset price,K/S. A call option is said to be out-of-the money if $1\le K/S<1.15$; and deep out-of-the-money if $K/S\ge1.15$. An option contract can be classified by the time to maturity: short (<60 *days*), median ($60-160$ *days*), or long maturity (> 160 *days*). It is worthy noting that by the time we were collecting these data sets, the markets world over were very nervous, following financial meltdown of one of the leading banks (LeeMan Brothers Inc.) in U.S.

Historical data was collected from Yahoo finance. Surprisingly, all necessary information about options market is retrieved from historical data, especially for more than 30 days to maturity. At such a time, the market dictates that there is a fixed interest rate $r \geq 0$. We can borrow and deposit money on this same continuously compounded interest rate of $r = 2.83\% p.a.$. Interestingly, the proposed model can be adjusted to follow the market price options, by varying the parameter ϖ accordingly see for example Figure 7.1 and Figure 7.3. Once the model price corresponds to the market price the parameters can be of use especially for pricing derivatives, such as OTC options, whose prices are not available in the market.

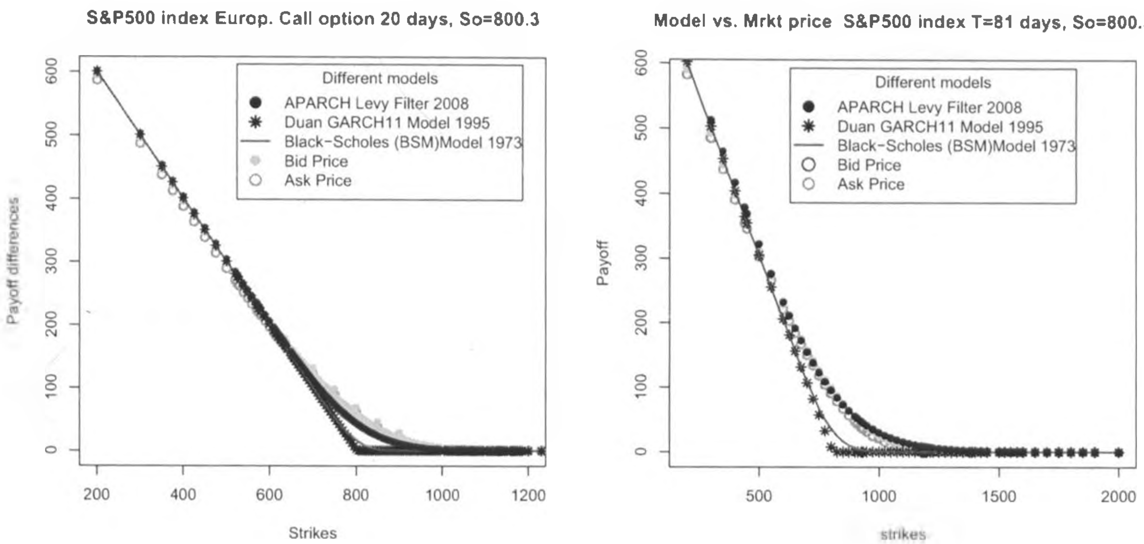


Figure 7.1: European call option

7.5 Conclusions

We use spectral analysis in frequency domain to study distributions of variance across the entire spectrum, which is most effective way of studying variance in any market(developed or emerging). Since options are derivative assets, the key to the success of any option pricing model is whether or not the process assumed is consistent with distributional and time series properties underlying asset. The proposed model takes care of most of the observed stylistic facts about financial time series data i.e. skewness and leptokurtic nature of log returns, aggregational Gaussianity, and presence of jumps.

Over the last decades, a major challenge for the equity derivative business, has been to build a model consistent with European option prices and able to handle exotic products. Presence of persistent smile across strikes has been a drawback in pricing consistently many exotic derivatives. To minimize or even overcome this problem an arbitrage free model has been constructed that seems to explain a significant portion of the persistent smile or smirks in benchmarks options pricing model(s). As it is stated in literature, the presence of volatility smile/smirk in the market should not be regarded as an abnormality; rather, it should be viewed as an indication of the failure of the standard model. The APARCH Lévy filter is very tractable compared to other jump-diffusion or stochastic volatility models. It addresses the drawbacks of local volatilities. Once calibrated, the model can be used to price European options, other vanilla options and standard exotic derivatives. It offers a convenient framework to observe and control volatility surface evolution.

The results have an important practical implication. Many exotic options, such as barrier and average price, are either traded over the counter or embedded in structured financial products. In pricing these exotic options, using the right model to describe the underlying process is critically important. Unlike the traded European style options, pricing exotic options can be conducted as a simple interpolation exercise. An inconsistent application of a pricing model can lead to unpredictable consequences.

Practitioners do not apply the BMS73 model mechanically in its original form. Volatilities are adjusted for moneyness and maturity combinations. For comparison purposes we used (GARCH(1,1) option pricing model, BMS73 model, and the constructed APARCH Levy filter) whereby they all drew information from historical data. In other words they fix (erroneously though) the shape of volatility smile/smirk but permit the curve to go up or down, depending on how well they fit the out-of-the sample option prices. The constructed model can be quite robust if more option data are used for more finetuning. The S&P500 example give some insights into how flexible and robust the model can be as illustrated in figures 7.7, 7.8, 7.6, etc.

We find that AR-APARCH Lévy filter deliver promising empirical performance and hope to provide additional empirical support in future work.

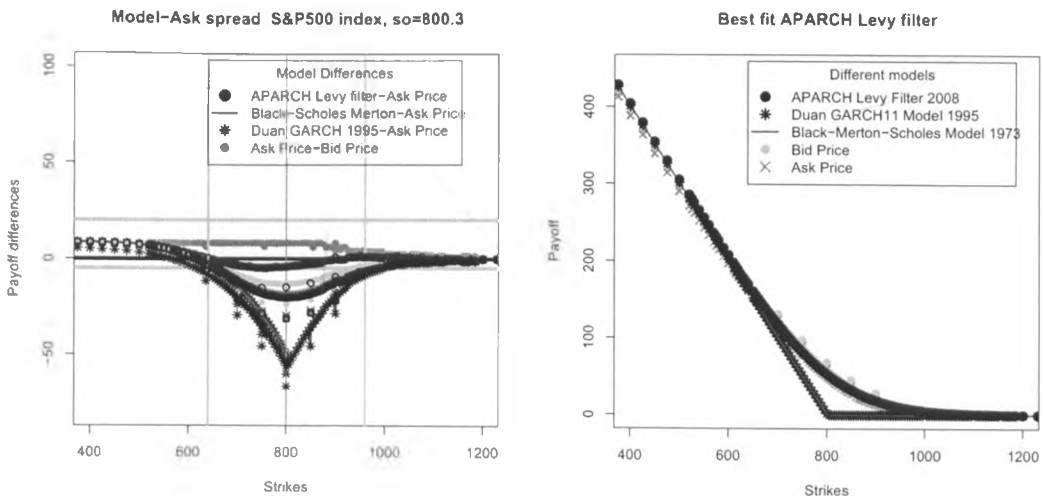


Figure 7.2: APARCH Lévy filter pricing model compare for 20 days

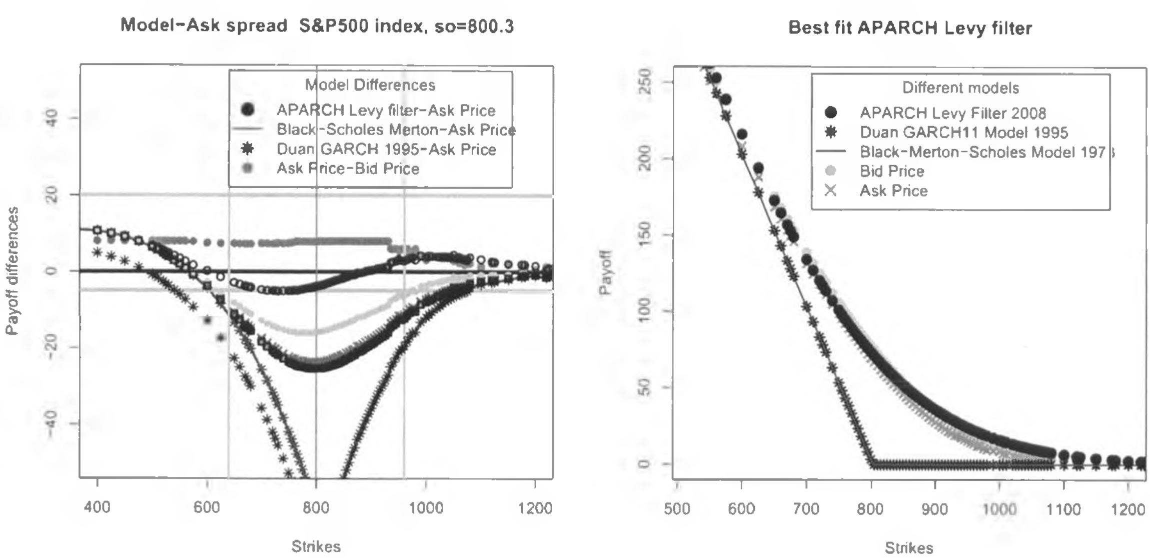


Figure 7.3: APARCH Lévy filter pricing model 36 days

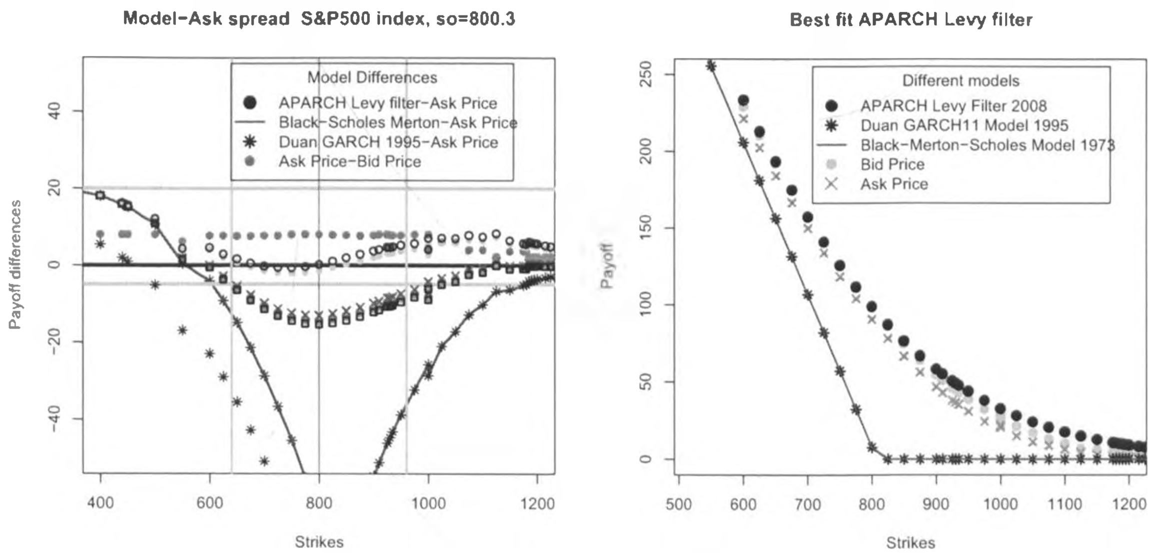


Figure 7.4: APARCH Levy filter pricing model compare for 81 days

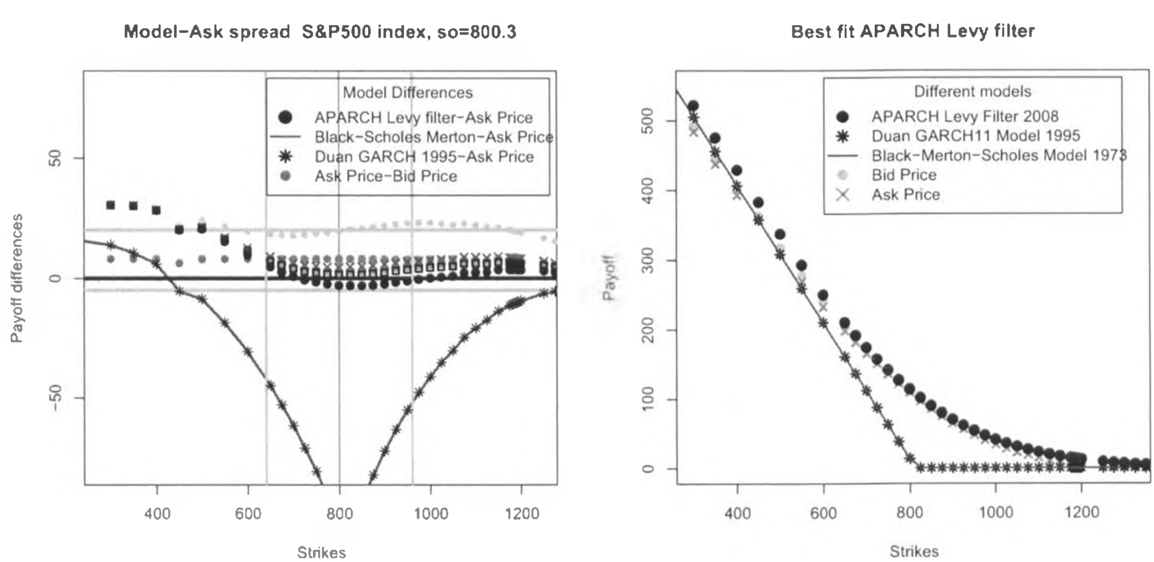


Figure 7.5: APARCH Levy filter pricing model 145 days

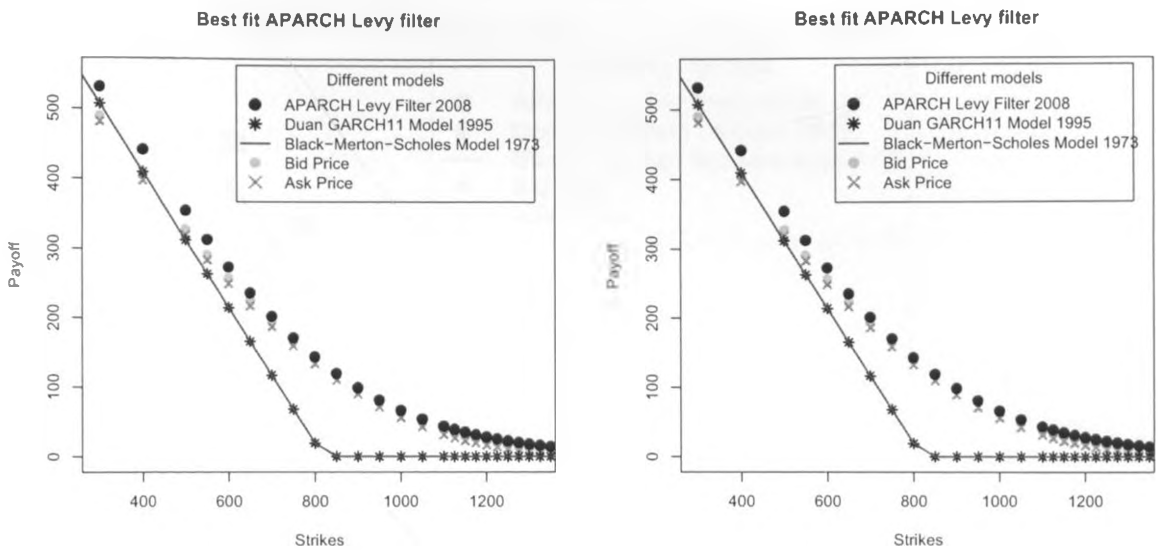


Figure 7.6: APARCH Lévy filter pricing model 211 days $h(j,\varphi) = \varpi^{-j}\varphi, j = 1$

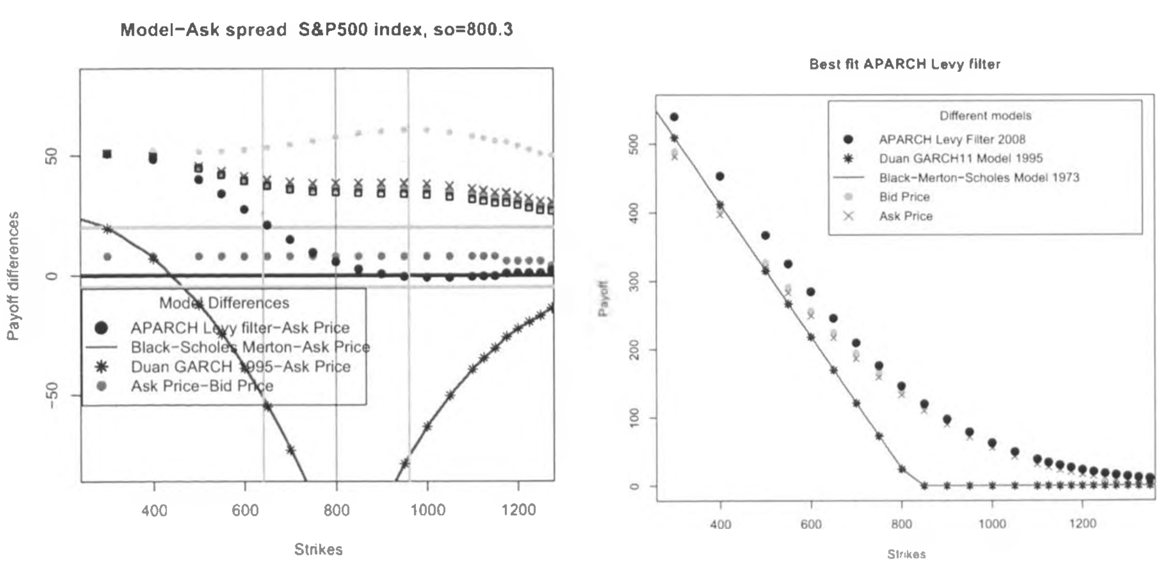
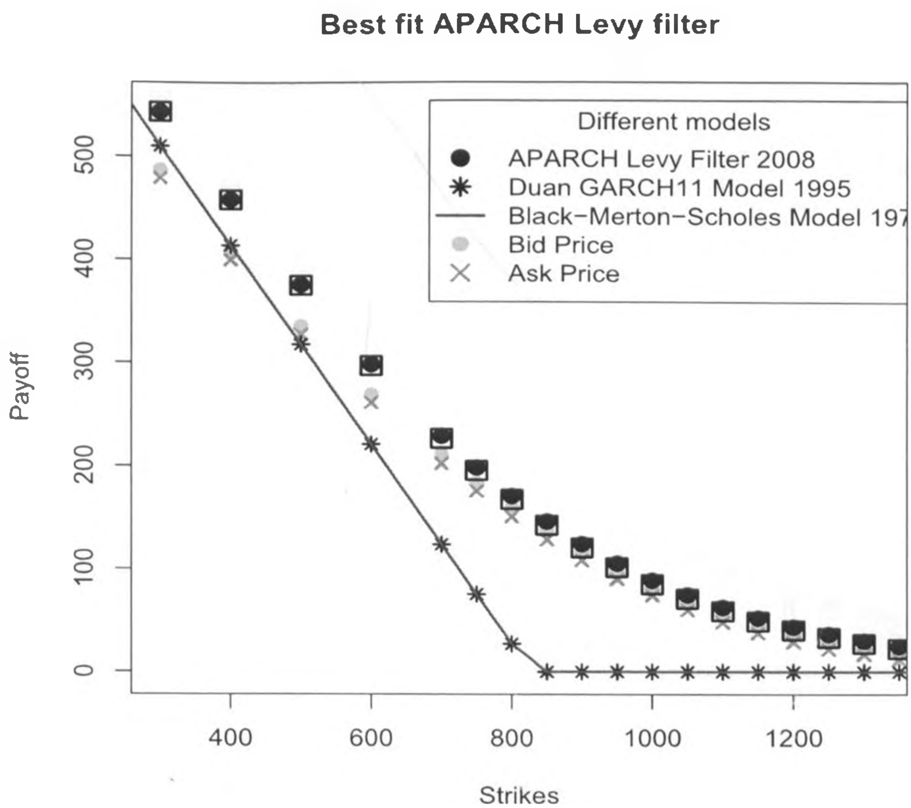


Figure 7.7: APARCH Levy filter pricing model 276 days $h(j,\varphi) = \varpi^{-j}\varphi, j = 4, 6$



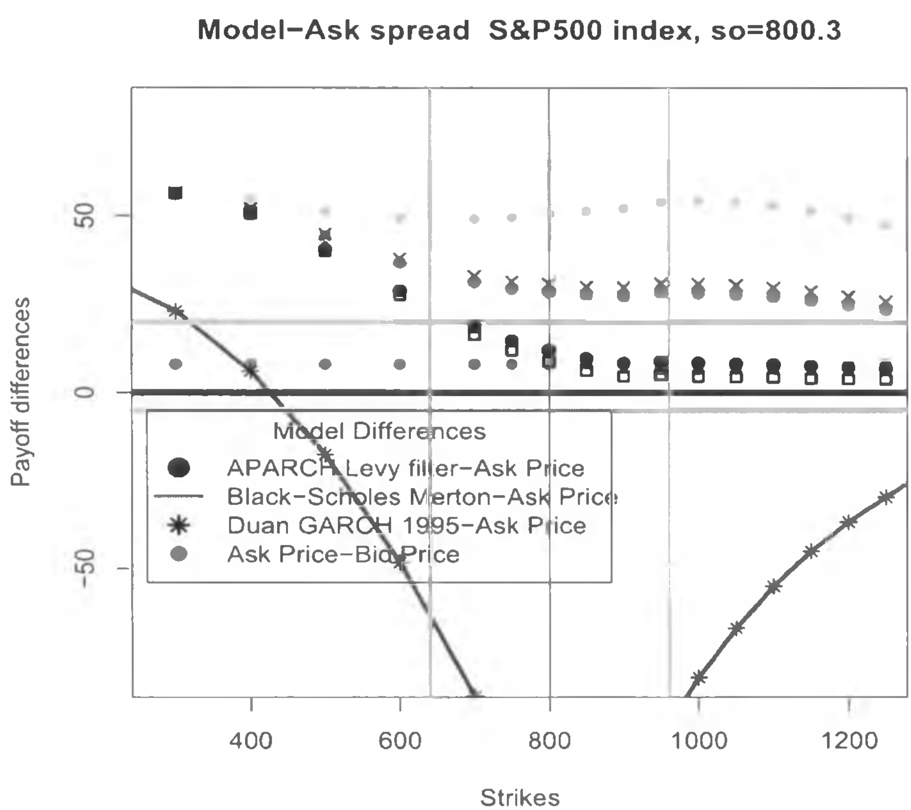


Figure 7.8: APARCH Lévy filter pricing model 302 $h(j,\varphi) = \varpi^{-2}\varphi$days

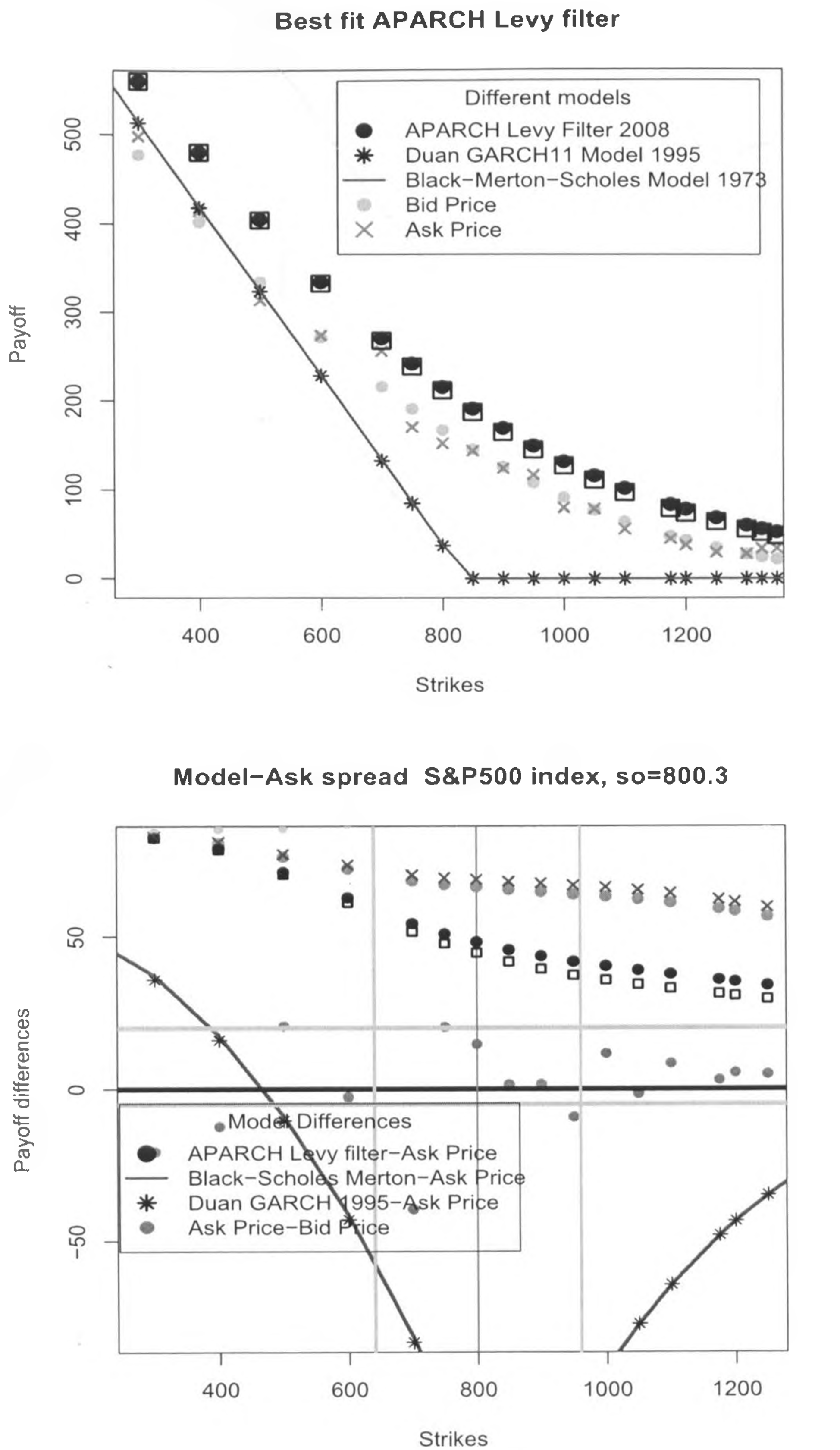


Figure 7.9: APARCH Levy filter pricing model 430 days $h(j,\varphi) = \varpi^{-2}\varphi$

Chapter 8

Conclusions and Recommendations

8.1 Conclusions

The Black-Scholes formula; one of the major breakthroughs of modern finance allows for an easy and fast computation of option prices but some of its assumptions, like constant volatility or lognormal distribution of asset prices do not find justification in the market. In view of this we set out in this study to investigate salient features of the underlying asset prices observed in developed and emerging markets in relation to BSM73 model.

Moreover, it is widely recognized in literature and from our study, that there is a non-negligible discrepancy between the BSM73 model and real market behavior, which appears as the "smile effect" or "implied volatility smile" in options market and the assumed dynamics of the underlying asset. With empirical evidence that implied volatility increases for in-the-money or out-of-the money options. Several models were investigated before arriving to the conclusion that the proposed APARCH Lévy filter, explains partly the presence of smile effect.

For complex models such exponential Lévy and Geometric Lévy which take into account some of the the empirical stylistic facts of log returns, often lead to more computations and thus time burden can become a severe problem when computation of many option prices is required. Luckily, Car and Madan developed a Fast Fourier transform method to compute a whole range of strikes. Such model(s) were proposed and investigated. Numerical results were compared contemporaneously with other bench mark models such as BSM73, and Duan GARCH (hereafter DG95) option pricing model. Exponential Lévy model which was assumed to incorporate leptokurtic nature of log returns, in comparison with BSM73 and DG95, seemed not to have any significant difference, especially, explaining at-the-money deviation from the observed (bid-ask-spread) market values.

Although the class of Lévy processes, say for example exponential Lévy model, is quite rich, it is sometimes insufficient for multiperiod modeling due to the stationarity of increments. The stock price returns for a fixed time horizon always have the same law. It is therefore impossible to incorporate any kind of new market information into the distribution. Secondly, for a Lévy process, the law of X_t for a given time horizon is completely determined by the law of X_1. Therefore; moments and cumulants depend on a well defined manner which does not coincide with the empirically observed time dependence of these quantities. Bearing this in mind, we thought of removing any second moments dependency while investigating the nature of residuals left after removing varying volatility. To this end, it was necessary to investigate the true dynamics of the underlying asset, say index.

Dynamics that govern developed and emerging stock markets daily index log returns are investigated in view of valuing financial instruments and computing Value at Risk. AR-APARCH models conditioned on Student t and Gaussian distribution, are used to filter first and second moment serial correlation of log returns. The white independent noise residuals are calibrated using Generalized Hyperbolic distribution. We identify appropriate models for estimating and forecasting daily volatility for four stock indices, **SP500**, **DAX**, **MASI** and **NSE20**. Univariate daily log returns turns out decompose into three components namely, ARMA filter (an object analogous to drift), GARCH filter (time dependent related to Brownian motion part) and compound poisson process closely related to jump density of Lévy increments.

One can infer that the accuracy of volatility estimation is most critical as it applies to pricing standard options with path dependent options. For contingent claims that depend on the extremum of the process, the prices are quite sensitive to the specification of the process. BSM73 prices differ significantly from APARCH-Lévy model in that they tend to be lower if the option is in and out of the money. These differences indicate that an appropriate choice of the model is of great importance.

Although the pentanomial lattice investigated in this study is as tractable as the standard binomial and trinomial lattices, the computational effort might increase exponentially with respect to dimension, similar to other lattice models. However pentanomial lattices can be considered useful for relatively short term contracts which can be used to solve American options problems GARCH(1,1) conditioned on normal distribution removes second order serial dependance in both markets in line with what is in the literature. The empirical evidence about spectral density of standardized GARCH(1,1)-normal filtered residuals from different financial markets deserves further investigation. However, since option prices may react sensitively to changes in volatility, a proper specification of the conditional means at each step may play a crucial role in the proposed pentanomial model. It is well documented in literature, out-of-the money options with short times to maturity react strongly to volatility changes when measuring this sensitivity in relative terms.

Presence of persistent smile across strikes has been a drawback in pricing consistently many exotic derivatives. To minimize or even overcome this problem an arbitrage free model has been constructed that seems to explain a significant portion of the persistent smile or smirks of benchmarks options pricing model(s). As it is stated in literature, the presence of volatility smile/smirk in the market should not be regarded as an abnormality; rather, it should be viewed as an indication of weakness of the standard BSM73 model, which can be extended to AR-APARCH-Lévy filter model as we have successfully shown in this study.

The APARCH Lévy Filter is very tractable compared to other jump-diffusion or stochastic volatility models. It addresses the drawbacks of local volatilities. Once calibrated, the model can be used to price European options, other plain vanilla options and standard exotic derivatives. It offers a convenient framework to observe and control volatility surface evolution.

8.2 Future Research

Over the last three decades, a major challenge for the equity derivative business has been to build a model consistent with European options prices and able to handle exotic products such as compound options. In this study we have tried to address the problem by constructing APARCH Lévy filter model. My approach is based on assuming an alternative explicit dynamics for the stock-price process which could match volatility smile and its implied smile. Moreover there is need for further research especially for high frequency data or real time data.

The valuation and hedging of ever increasing number of exotic and vanilla options is a topic of interest to many practitioners. One can price most function of plain vanilla options in a univariate case setting under APARCH Lévy Filter. Exotic options or path dependent options whose payoffs depends on the behavior of the price of the underlying process between time $t = 0$ to $t = T$ (maturity) can be priced accordingly. APARCH Lévy Filter can applied to foreign exchange market and it's related derivatives say Quanto products, etc, bond market, interest rate derivatives, credit markets, and weather derivatives to say the least.

It would be interesting to study discrete and continuous multivariate APARCH Lévy Filter, hidden Markov chain processes in relation to Bond market derivatives.

The process governing the arrival of jumps may be heterogeneous with respect to the type of news. Therefore, jump dynamics may differ across different types of news events. The infrequent occurrence of jumps makes this identification of different jump dynamics a challenging area of study.

Abandoning the normality assumption for multidimensional problems is a more involved issue. The massive use of derivatives in asset management, in particular from hedge funds, has made the non-normality of returns an investment tool rather than a mere statistical problem. It is interesting to extend the APARCH Lévy filter as a tool to solve credit risk related instruments in both univariate and multivariate setting.

Up to this point, we have seen that the three main frontier problems in derivative pricing are the departure from normality, emerging from the smile effect, market incompleteness, corresponding to hedging error, linked to the bivariate relationship in Over-The-Counter (OTC) transactions. It will be interesting to apply APARCH Levy Filter and copula functions to address these problems.

Option pricing under market incompleteness is an equally interesting topic. In recent years there is a considerable interest in the application of regime switching models driven by a hidden Markov chain process to various financial problems. It would be encouraging to use Esscher transform for option valuation under incomplete markets induced by APARCH Levy type process. There is a relatively less amount of work on the use of the Esscher transform for option valuation under incomplete markets generated by other asset price dynamics such as regime switching process.

We may explore the application of our model to other types of exotic or hybrid financial products such as barrier option, passport option and option-embedded insurance products.

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