MULTIVARIATE MARKOV CHAIN MODEL FOR CREDIT RISK MEASUREMENT

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DECLARATION

Candidate's Declaration

This thesis is my original work and has not been presented for a degree in any other university or

for any other award.

SIGNATURE: _____ DATE: _____

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Approval

I confirm that this project has been submitted for examination for the degree of Master of science (Actuarial Science) of the University of Nairobi with my approval as the candidates supervisor.

SIGNATURE: _____ DATE: _____

 DR. IVIVI MWANIKI

DEDICATION

This thesis is dedicated to the Almighty God and to my family for their help and support as I took up the study. Without your help, I would not have made it this far.

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ABSTRACT

The aim of the study is to use a multivariate Markov model to simulate the dynamics of correlated credit ratings of multiple firms. Credibility theory is used to get the weighting used to estimate the transition matrix and other unknown model parameters in the multivariate Markov chain model. The study presents a higher-order Markov chain models for forecasting the various credit risk rating dynamics. The results are transition matrices from various risk ratings.

The most important step in analyzing data is the selection of an appropriate mathematical model for the data as it helps in predictions and hypothesis testing. The research design used for the study was a descriptive research design that basically involves obtaining information concerning the current status of the phenomena to describe, "What exists" with respect to variables or conditions in a situation Gardner et al (2004). This design was appropriate for this study, as it gave the relevant information as it was.

The traditional reduced-form approach assumes that the losses from a credit risk at each particular rating class can be evaluated based on accounting information and principles. With the data entered and parameters estimated, a prediction table of transition probabilities from one grade to another was created which indicated that the Ching et al (2002) model follows a 3rd order polynomial to be able to come up with the transition matrices.

In conclusion the transition matrices were estimated by the use of a linear combination of the empirical transition matrix and prior estimate of the transition matrix. By incorporating both the historical rating data and another source of information about rating data which includes expert opinion or subjective views, one is able to come up with the next period's rating of a particular credit risk to that depends on its current rating and also the current ratings of other credit risks in the portfolio.

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<center>**CHAPTER ONE**</center>

1 INTRODUCTION

1.0 Background

Credit risk management is a major concern for the bank industry and other financial intermediaries. Credit ratings are used by investors to asses a firm's ability to meet its debt obligations. The interest at hand is to model the dependencies of credit risks that have implications when measuring and managing credit portfolios.

The dependencies of credit risky securities can be modelled through copula which was introduced by Li (2000), Monte Carlo simulation, Poisson mixture model of Credit Risk developed in 1997 by Credit Suisse Financial Products and infectious default model for Binomial Expansion Technique (BET) developed by Davis and Lo (2001).

Jarrow et al. (1997) was first to provided an arbitrage-free Markov model for the term structure of credit risk spreads which explicitly incorporates credit rating information into the valuation methodology so as to price and hedge options on risky debt. This contingent claims model explicitly incorporates credit rating information into the valuation methodology This credit risk modelby is useful in practice because the parameters of the process are easily estimated using observable data only.

Ching et al. (2002) considers modelling of the dynamics of transitions between credit ratings important for credit risk analysis. The discrete-time homogeneous Markov chain model has been used to model these transitions over time. The transition matrix indicates the likelihood of the future changes of the ratings. The transition matrix can be estimated from empirical data for credit ratings in practice.

Siu et al. (2005) constructed different parameters for the multivariate Markov chain to model dependent credit risks, as well as proposing the use of credibility theory in actuarial science to calibrate the model.

Kijima et al. (2002) propose the multivariate Markov chain model to simulate the evolution of the dependent ratings for several credit risks. Their approach is an extension of the credit risk model developed by Jarrow et al. (1997).

Kijima et al. (2002) model considers the changes in credit rating over time due to changes in the micro and macro-economic conditions. These are common across firms as they depend on common economic factors and could be linked through being subsidiaries of each other.

The approaches of measuring transition matrices is use of historical data as in Lucas (1995), which in the case of corporate bonds would be hard to measure for high rated ones as they are not likely to default. The time series data thus would not be sufficient to estimate the transition matrices. Thomas et al. (2002) suggested that historical rating data alone is not adequate to reflect future movements in ratings when the future may not evolve smoothly from the past experience.

The second approach would be through Credit Metrics (Gupton et al. 1997) provides a comprehensive account of practical implementation of default processes. For measuring and managing the risk of a credit portfolio, it is importance to develop quantitative models that can describe the dependencies between the credit ratings of the individual assets in the portfolio, as the losses incurred in individual assets depend on their credit ratings. A multivariate Markov chain model provides a suitable way to describe these dependencies.

In addition to credit risk measures such as credit spread and expecteddefault probability, recent developments in credit risk management have highlightedthe use of credit value at risk (VaR) as an important approach to measure credit risk.(VaR) is derived from the framework of value at risk (VaR), which measures market risk. (VaR) assigns probability to credit losses that could occur in a portfolio over a given period.

Ching et al. (2005) used a discrete-time homogenous Markov chain model which has been used by academic researchers and market practitioners to model credit ratings transitions over time. The transition matrix represents the likelihood of the future evolution of the ratings.

Ching et al. (2005) considered the market to have a mixture of views which are determined by the historical movements in the rating and other views such as expert opinion on future movements or the outlook of pessimistic risk managers and regulators. Ching et al. (2005) approach provides a consistent way to incorporate both historical rating data and another source of information, for instance, expert opinion or a subjective view. While the estimation method in Kijima et al. (2002) only incorporated historical rating data. Ching et al. (2005) provided an estimate of the transition matrix as a linear combination of the empirical transition matrix and a prior transition matrix.

The empirical transition matrix can be specified based on historical rating data while the prior transition matrix can be determined by expert opinion or some other subjective view. Hence, their model is more suitable for measuring and managing the risk of credit portfolios when the market view is a mixture of beliefs based on both historical data and expert opinion.

The model formulates an estimation problem as a set of linear programming (LP) problems, which is more efficient to compute compared with the minimization of the square error in Kijima et al. (2002). This estimation procedure can be implemented using Excel spreadsheets.

The study will incorporate credibility theory to obtain the credit scores of firms and by using a multivariate Markov chain model, a transition matrix will be constructed which will represent the financial efficiency of firms. The proposed model will assist financial institutions in manage credit risk and also ensure that the organization complies with the Basil Capital Accord. Credit risk management is an important issue in the financial industry. This has led to sophisticated methods for approaching credit risk management. This was confirmed by the Basel Committee on Banking Supervision when they formalized a universal approach to credit risk for financial institutions in 1988. In 2001, the Bank for International Settlements (BIS) issued a consultative document (Basel II) that updated the credit risk assessment methods first proposed under a 1993 agreement (Basel I). The Basel committee updated the guidelines to Basel III: A global regulatory framework for more resilient banks and banking systems (2010). One of the key lessons of the economic crisis has been the need to strengthen the riskcoverage of the capital

framework. Failure to capture major on- and off-balance sheet risks,as well as derivative related exposures, was a key destabilizing factor during the crisis.

In response to these shortcomings, the Committee in July 2009 completed a number of critical reforms to the Basel II framework. These reforms will raise capital requirements for the trading book and complex securitization exposures, a major source of losses for many internationally active banks. The enhanced treatment introduces a stressed value-at-risk (VaR) capital requirement based on a continuous 12-month period of significant financial stress. In addition, the Committee has introduced higher capital requirements for securitization in both the banking and the trading book.

The reforms also raise the standards of the Pillar 2 supervisory review process and strengthen Pillar 3 disclosures. The Pillar 1 and Pillar 3 enhancements must be implemented by the end of 2011; the Pillar 2 standards became effective when they were introduced in July 2009. The Committee is also conducting a fundamental review of the trading book. The work on the fundamental review of the trading book is targeted for completion by year-end 2011.

This Basel III document also introduces measures to strengthen the capital requirements for counterparty credit exposures arising from banks' derivatives, repo and securities financing activities. These reforms will raise the capital buffers backing these exposures, reduce the way in which economic quantity is related to economic fluctuations and provide additional incentives to move OTC derivative contracts to central counterparties, thus helping reduce systemic risk across the financial system. They also provide incentives to strengthen the risk management of counterparty credit exposures.

The Committee is introduced the following reforms where banks must determine their capital requirement for counterparty credit risk using stressed inputs. This will address concerns about capital charges becoming too low during periods of compressed market volatility and help address how economic quantity is related to economic fluctuations. The approach, which is similar to what has been introduced for market risk, will also promote more integrated management of market and counterparty credit risk.

Banks will be subject to a capital charge for potential mark-to-market losses (ie credit valuation adjustment – CVA – risk) associated with deterioration in the credit worthiness of counterparties. While the Basel II standard covers the risk of a counterparty default, it does not address such CVA risk, which during the financial crisis was a greater source of losses than those arising from outright defaults.

1.1 Statement of the problem

Credit risk is the chance it is the chance that the issuer will default. The credit risk of an issuer can vary due to interest rates changes, changes in technology or regulatory changes. Rating agencies analyze portfolios to measure the credit risks on particular securities and publish the ratings.

Credit risk management is a major concern for the bank industry and other financial intermediaries. The interest at hand is to model the dependencies of credit risks that have implications when measuring and managing credit portfolios. A multivariate Markov model is used to simulate the dynamics of credit ratings of a portfolio and come up with transition probabilities. The transition probabilities estimate the risks of downgrading and of default from the current credit rating.

1.2   Objectives

1. To model the dependencies of credit risks that is used in measuring and managing credit portfolios through the use of Markov Chain model.

2. To estimate the transition matrix and other unknown model parameters in the Markov chain model using actuarial credibility theory to combine historical data and expert opinion

1.3 Significance of the Study

The study is used to estimate the transition matrix that shows the probability of a portfolio at a specified credit rating moving to another rating. Credit risk ratings assist in the pricing and hedging of corporate bonds that depend on credit rating. The probability of default is also noted which is of importance to financial institutions to be able to set aside provisions in case of default.

1.4 Definition of Terms

Credit Risk: The probability that a loan will not be repaid according to the terms of the contract. In the case of a bond it is the chance that a bond issuer will not make the coupon payments or principal repayment to its bondholders.

Value at Risk (VAR): This is the maximum loss not exceeded with a given probability defined as the confidence level, over a given period of time.

The methodology adopted for measuring credit risk over a portfolio is called Credit Value at Risk (CvaR). It shows the relationship between a loss level and its probability of occurrence -loss probability distribution. . If the losses are larger than a given threshold, then default occurs. The cornerstone of this methodology is the knowledge of the probability distribution of the portfolio. This means that the probability that the portfolio suffers losses larger than the sum of expected and unexpected losses is equal to the confidence level, let's say 99.9%.

Bank for International Settlements (BIS): Committee that comes up with the Basel regulations.

CHAPTER TWO

2.1 LITERATURE REVIEW

The development of the model over the years can be shown as below;

The Jarrow–Turnbull Model

The model by Jarrow and Turnbull (1995), with the bankruptcy process following a discrete state space Markov chain was used for credit ratings. The parameters of this process are estimated using observable data. This model is useful for pricing and hedging corporate debt with imbedded options, for pricing and hedging OTC derivatives with counterparty risk, for pricing and hedging (foreign) government bonds subject to default risk (e.g., municipal bonds), for pricing and hedging credit derivatives, and for risk management.

The model by Jarrow and Turnbull (1997) can be used to define the current credit exposure and to generate the distribution of credit exposure over the life of a contract. Two commonly used statistics can be computed: the maximum exposure and expected exposure time profiles. That is, starting from a particular credit class one can compute the probability of being in a given credit class after a fixed time interval. For pricing purposes, it is necessary to use the "risk-neutral" probabilities. For risk management purposes, however, it is necessary to use both the "risk neutral" and the empirical probabilities. This model can also be extended to portfolios of contracts such as interest rate and foreign currency swaps.

The model by Jarrow and Turnbull (1997) was used for the pricing of an individual defaultable discount bond. The multivariate Markov model can be used in finance in credit risk management and the pricing of similar credit derivatives based on credit rating information. We assume that the portfolio consists of defaultable discount bonds only and that the prices are determined by their credit ratings. The evolution of credit rating of the nth bond follows the Markov chain $X_t^{(j)}$ defined by the model below and we assume that all the parameters of the model have been estimated.

$$X_t^{(j)} = (x_t^{(j)}(1), x_t^{(j)}(2), \ldots\ldots, x_t^{(j)}(m))$$

The Discrete Time Case

The distribution for the default time is modelled via a discrete time, time-homogeneous Markov chain on a finite state space S= {1,………,K}. The state space *S* represents the possible credit classes, with 1 being the highest and *K*-1 being the lowest. The last state, *K*, represents bankruptcy.

The discrete time, time-homogeneous finite state space Marko chain ($\acute{\eta}_i$: $0 \leq t \leq \tau$}is specified by a *K* *K transition matrix where the q_{ij} represents the actual probability of going from state *i* to state *j* in one time step. We assume that bankruptcy (state K) is an absorbing state, so that $q_{ki}=0$ for all values of I and $q_{kk}=1$.

Estimates of this transition matrix, with a time step of 1 year, were obtained from Standard &Poor Credit Review. The nonzero entries of the historical matrix tend to be concentrated around the diagonal, with movements of two credit ratings (or more) in a year being rare or nonexistent

The proportionality adjustments λ has the interpretation of being risk premiums.

The Continuous Time Case

The distribution for the default time is modelled via a continuous time, time-homogeneous Markov chain on a finite state space S= {1,………,K}. As in the discrete time setting, these states represent the various credit classes, with 1 being the highest and state *K* being bankruptcy.

The continuous time, time-homogeneous finite state space Marko chain ($\acute{\eta}_i$: $0 \leq t \leq \tau$} is specified by a *K* *K transition matrix where the λ_{ij} represents the transition rates from credit class *i* to credit class *j*. We assume that bankruptcy (state K) is an absorbing state, so that $\lambda_i=0$ for all values of *i* and that bankruptcy is an absorbing state.

Kijima et al. (2002)

Kijima et al. (2002) used a multivariate Markov chain model to simulate the evolution of correlated ratings of several firms. They applied their model to questions of pricing and risk measurement. They estimated the unknown parameters in their model by minimizing the squared

error based on historical rating data only. The model is based on the single index model in modern portfolio theory and is an extension of Jarrow and Turnbull (1997) model to the multivariate asset case.

Ching et al. (2005) model

Ching et al. (2005) employed actuarial credibility theory to combine two sources of information for estimating the transition matrix and other unknown model parameters in the multivariate Markov chain model. Their approach provided a consistent and convenient way to incorporate both historical rating data and another source of information, for instance, expert opinion or a subjective view. They provided an estimate of the transition matrix as a linear combination of the empirical transition matrix and a prior transition matrix. The empirical transition matrix can be specified based on historic rating data while the prior transition matrix can be determined by expert opinion or some other subjective view.

Ching et al. (2005) model is more suitable for measuring and managing the risk of credit portfolios when the market view is a mixture of beliefs based on both historic data and expert opinion. In practice, it is difficult to obtain plentiful historic rating data, in which, the role of the prior transition matrix becomes more important.

In actuarial the term credibility was originally attached to experience rating formulas that were convex combinations or weighted averages of individual and class estimates of the individual risk premium. Credibility theory was the branch of insurance mathematics that explored model based principles for construction of these formulas. The theory has developed and now goes beyond the original scope so that in today's usage credibility covers more broadly linear estimation and prediction in latent variable models Norberg, R. (2004).

The origin of credibility dates back to Whitney (1918) who addressed the issue of assessing the risk premium (m). The risk premium was defined as the expected claims expenses per unit of risk exposed, for an individual risk selected from a portfolio of similar risks. The combination of individual risk experience and class risk experience, would give the premium rate using the weighted average in the form

$$\overline{m} = z\,\widehat{m} + (1 - z)\,\mu \,, \tag{1}$$

Here ˆm is the observed mean claim amount per unit of risk exposed for the individual contract and μ is the corresponding overall mean in the insurance portfolio. Whitney (1918) viewed the risk premium as a random variable i.e. as a function m (θ) of a random element θ representing the unobservable characteristics of the individual risk. The random nature of θ expressed the notion of heterogeneity; the individual risk is a random selection from a portfolio of similar but not identical risks, and the distribution of θ describes the variation of individual risk characteristics across the portfolio. The weight z in (1) was soon to be named credibility (factor) since it measures the amount of confidence attached to the individual experience, and $\overline{m}$ was called the credibility premium.

In this case, credibility theory determines the weight that should be assigned to each source of information. Mowbray (1914), Buhlmann (1967) and Klugman et al. (1997) together provide excellent accounts of actuarial credibility theory. Buhlmann (1967) introduced a least-squares approach for the estimation of credibility premiums without imposing stringent parametric assumptions for the claim models.

Ching et al. (2005) formulated the transition matrix and other unknown model parameters in the multivariate Markov chain model as a set of Linear Programming (LP) problems as it is more computationally efficient compared to other approaches.

Norberg, R. (2004) considered the classification results of the different methods applied on a data set concerning the credit scoring for the private individuals which showed that, in the majority of cases, the classification result of linear programming models is more efficient than the result of statistical methods. The study showed that, the linear programming models are more flexible as it allows the analyst to incorporate some a priori information in the models.

Youssef et al. (2007) paper compared parametric methods and nonparametric methods to resolve the classification problems. They considered the parametric approach illustrated by the use of Fisher's linear discriminant function, Smith's quadratic discriminate function and the logistic discriminant model against the non-parametric approach in this case linear programming models

as alternatives to the parametric approach. The study showed linear programming being more flexible as it allowed the analyst to incorporate some a priori information in the models. Thus the non-parametric approach outperforms the parametric one. The linear programming models are justified by the violation of certain assumptions in the statistical discriminant methods. Such assumptions include the distribution's normality, the homogeneity of variance-covariance matrix, the sample's size and the absence of outliers in data.

CHAPTER THREE

3 RESEARCH METHODOLOGIES

3.1 Introduction

The most important step in analyzing data is the selection of an appropriate mathematical model for the data as it helps in predictions and hypothesis testing.

The research design used for the study was a descriptive research design that basically involves obtaining information concerning the current status of the phenomena to describe, "What exists" with respect to variables or conditions in a situation Gardner et al (2004). This design was appropriate for this study, as it gave the relevant information as it was.

3.2 The model

Ching et al. (2004) considers a Markov model as useful tool to model and analyze a categorical data sequence. Take a Markov process X_t with values in a finite state-space M= {1, 2, . . . ,m } over discrete time intervals t=1, 2, . . . , where m is finite. For each j=1,2,…,n and time t, we define a state probability vector of jth borrower,$X_t^{(j)}$denoted as follows

$$X_t^{(j)} = (x_t^{(j)}(1), x_t^{(j)}(2), \ldots\ldots, x_t^{(j)}(m))$$

The probability distribution of X_{t+1} depends on X_t only.

$$X_{t+1}^{(j)} = \sum_{k=1}^{n} \lambda_{jk} \, Q^{(jk)} X_t^{(k)} \tag{2}$$

where $\lambda_{jk} \geq 0$, $1 \leq j$, $k \leq n$ and $\sum_{i=1}^{n} \lambda_i = 1$, for j=1,2,…..,n. $Q^{(jk)}$ is the transition matrix from states in the kth sequence to the states in the jth sequence. Equation (2) denotes that the state probability distribution of the jth borrower, $X_{t+1}^{(j)}$, at time $t+1$, depends on the weighted average of $Q^{(jk)} X_t^{(k)}$ at time t. Equation (2) can be written as:

$$X_{t+1} \equiv \begin{bmatrix} X_{t+1}^{(1)} \\ X_{t+1}^{(2)} \\ \vdots \\ X_{t+1}^{(n)} \end{bmatrix} = \begin{bmatrix} \lambda_{11}Q^{(11)} & \lambda_{12}Q^{(12)} & \cdots & \cdots & \lambda_{1n}Q^{(1n)} \\ \lambda_{21}Q^{(21)} & \lambda_{22}Q^{(22)} & \cdots & \cdots & \lambda_{2n}Q^{(2n)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_{n1}Q^{(n1)} & \lambda_{n2}Q^{(n2)} & \cdots & \cdots & \lambda_{nn}Q^{(nn)} \end{bmatrix} * \begin{bmatrix} X_t^{(1)} \\ X_t^{(2)} \\ \vdots \\ X_t^{(n)} \end{bmatrix} \equiv QX_t$$

$$X_{t+1} = QX_t$$

However, in practice X_{t+1} may also depend on the other previous observations ($X_{t-1}, X_{t-2}, \ldots$) of the process as well. Therefore it is natural for one to consider higher-order Markov chain models. The conventional model for an nth order Markov chain has $(m-1)m^n$ parameters. The major problem in using such a model is that the number of parameters (transition probabilities) increases exponentially with respect to the order of the model. Large number of parameters discourages the use of a higher-order Markov chain directly.

Raftery,A.(1985) proposed a higher-order Markov chain model which involves only one additional parameter for each extra lag. Later Ching et al. (2004) proposed another higher-order Markov chain model based on Raftery's model and the number of model parameters is only O (nm^2). Their model is a generalization of Raftery's model and they propose an efficient method for estimating the model parameters.

Let X_{t+i} be the state vector at time $(t+i)$. If the system is instate $j \in M$ at time $(t+i)$, i.e. $X_{t+i} = j$ then

$$X_{t+i} = (0, \ldots, 0, \underset{jth\ entry}{\underbrace{1}} \ldots, 0)^t$$

The higher-order Markov chain model is as,

$$X_{t+n+1} = \sum_{i=1}^{n} \lambda_i\, Q_i\, X_{t+n+1-i} \tag{3}$$

In the model we assume that X_{t+n+1} depends on $X_{t+n+1-i}$ ($i = 1, 2, \ldots, n$) via the matrix Q_i and weight λ_i. The matrix Q_i is the ith step transition matrix of the process and is a non-negative stochastic matrix with column sums equal to one. The weight λ_i is assumed to be non-negative

such that $\sum_{i=1}^{n} \lambda_i = 1$. These conditions guarantee that the right hand side of the Markov chain model is a probability distribution.

The present efficient methods to estimate the parameters Q_i and $\lambda_i = (1, 2, \ldots, n)$ for an nth order Markov chain model. To estimate Q_i, we regard Q_i as the ith step transition matrix of the categorical data sequence $\{X_t\}$. Given the categorical data sequence $\{X_t\}$, one can count the transition frequency $f_{jk}^{(i)}$ in the sequence from state k to state j in the ith step. Hence one can construct the ith step transition frequency matrix for the sequence $\{X_t\}$ as:

$$F^{(i)} = \begin{bmatrix} f_{11}^{(i)} & \cdots & \cdots & f_{m1}^{(i)} \\ f_{12}^{(i)} & \cdots & \cdots & f_{m2}^{(i)} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1m}^{(i)} & \cdots & \cdots & f_{mm}^{(i)} \end{bmatrix}$$

From $F^{(i)}$ one can obtain the estimate for $Q_i = [q_{kj}^{(i)}]$ as follows:

$$\hat{Q}_i = \begin{bmatrix} \hat{q}_{11}^{(i)} & \cdots & \cdots & \hat{q}_{m1}^{(i)} \\ \hat{q}_{12}^{(i)} & \cdots & \cdots & \hat{q}_{m2}^{(i)} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{q}_{1m}^{(i)} & \cdots & \cdots & \hat{q}_{mm}^{(i)} \end{bmatrix} \quad \text{where } \hat{q}_{kj}^{(i)} = \begin{cases} \frac{f_{jk}^{(i)}}{\sum_{k=1}^{m} f_{jk}^{(i)}}, & if \ \sum_{k=1}^{m} f_{jk}^{(i)} \neq 0 \\ 0, & otherwise \end{cases}$$

It has been shown that for Q_i, the process $\mathbf{X}_t(2)$ converges to a unique stationary distribution $\overline{X}$ as t goes to infinity. The stationary distribution $\overline{X}$ can be estimated from the sequence $\{X_t\}$ by computing the proportion of the occurrence of each state in the sequence, denoted by $\hat{X}$. One would expect that

$$\sum_{i=1}^{n} \lambda_i \hat{Q}_i \hat{\mathbf{X}} \approx \mathbf{X}$$

Ching et al. (2004) assumed that the estimate of each transition matrix in the Q-matrix can be represented as a linear combination of a prior transition matrix and the empirical transition matrix, where the empirical transition matrix is based on the frequencies of transitions between

ratings. There exists a vector X of stationary probability distributions such that X = QX, we can estimate the Q-matrix based on the vector X of the stationary distributions for the ratings.

This suggests a natural method to estimate the parameters $\lambda = (\lambda_i, \ldots, \lambda_n)$. We consider the following optimization problem:

$$min_\lambda max_k \left\| \left[\sum_{i=1}^{n} \lambda_i \widehat{Q}_i \widehat{X} - \widehat{X} \right] \right\|$$

Subject to

$$\sum_{i=1}^{n} \lambda_i = 1 \qquad \text{and } \lambda_i \geq 0 \quad \forall i$$

Where $[.]_k$ denotes the *k*th entry of the vector. The constraints in the optimization problem guarantee the existence of the stationary distribution $\overline{X}$. The above optimization problem can be formulated as a linear programming problem. The linear programming problem is solved to obtain the parameters λ_i by using the function 'Solver' in the EXCEL worksheet.

Subject to:

$$\min_\lambda w$$

$$\begin{pmatrix} w \\ w \\ \vdots \\ \vdots \\ w \end{pmatrix} \geq \widehat{X} - \left[\widehat{Q_1 X} | \widehat{Q_2 X} | \ldots \ldots | \widehat{Q_n X} | \right] \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \vdots \\ \lambda_n \end{pmatrix}$$

$$\begin{pmatrix} w \\ w \\ \vdots \\ \vdots \\ w \end{pmatrix} \geq -\widehat{X} + \left[\widehat{Q_1 X} | \widehat{Q_2 X} | \ldots \ldots | \widehat{Q_n X} | \right] \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \vdots \\ \lambda_n \end{pmatrix}$$

$$w \geq 0, \qquad \sum_{i=1}^{n} \lambda_i = 1 \qquad \text{and } \lambda_i \geq 0 \quad \forall i$$

Ching et al. (2004) consider the problem of evaluating measures of risk, such as value at risk (VaR) and expected shortfall (ES), for a portfolio of credit risks with correlated ratings.

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Nagpal, K., & Bahar, R. (2001) states that observation of historical default rates supports the idea that default events (and, more generally, all indicators of credit quality and transition) are correlated. Default correlations are caused by similar economic conditions and, within a sector, by industry-specific reasons. However, incorporating default correlation in any portfolio credit risk analysis is difficult because of the lack of good data on default correlation, and the complexity of developing realistic models of default correlations that capture its dependence on credit quality, region, industry and time horizon.

Siu et al. (2005) came up with the method of estimating CVar, using the one-step a head forecast Profit/Loss distribution. The Profit/Loss distribution can be generated by the Markov chain model to obtain the Credit Var. Assume L_t^j represents the losses from the jth borrower at time t. The aggregate loss L_{t+1} of the bank loans at time $t+1$ is given by

$$L_{t+1} = L_{t+1}^{(1)} + L_{t+1}^{(2)} + \ldots\ldots + L_{t+1}^{(j)}$$

CHAPTER FOUR

4.1 DATA ANALYSIS

4.1.1 Data

The data in this study is a portfolio of default-free zero-coupon bonds of all maturities, a default-free money market account, and risky zero-coupon bonds of all maturities, their credit ratings for a period of 17 years of an asset as shown in Table 1.The data will illustrate the use of our multivariate Markov chain model for evaluating credit VaR and ES for a portfolio of credit risks with dependent credit ratings.

Table 1

Year	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Asset(X_t)	2	2	4	3	2	1	3	2	1	2	4	3	2	1	1	3	4

The credit ratings are classified into four groups according using a numerical definition from 1 to 4 that is Low-L, Low-H, High- L and High-H, respectively. These categories are then categorized in terms of their default risk and the likelihood of payment from each portfolio. Portfolios rated 1 (Low-L groups) were considered to have the lowest default risk. Portfolios rated 4 (High-H groups) were considered to have the highest default risk.

The data sequence $\{X_t\}$,is used to come up with the transition frequency $f_{jk}^{(i)}$.

$$F^{(12)} = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 3 & 1 & 0 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$F^{(21)} = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

We normalize the columns; in practice it is possible to get transition matrices with many columns of zero.

$$\widehat{Q}_1 = \begin{bmatrix} 1/4 & 1/5 & 1/2 & 0 \\ 3/4 & 1/5 & 0 & 2/3 \\ 0 & 3/5 & 0 & 1/3 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$$

$$\widehat{Q}_2 = \begin{bmatrix} 0 & 1/4 & 1/4 & 2/3 \\ 1/4 & 1/4 & 3/4 & 1/3 \\ 3/4 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{bmatrix}$$

Suppose that $Q^{(jk)}$ ($1 \leq j, k \leq n$) are irreducible and $\lambda jk > 0$ for $1 \leq j, k \leq n$. Then, there is a vector $X=(X^{(1)}, X^{(2)}, \ldots\ldots,X^{(n)})^{T}$ such that X=QX and

$$\sum_{i=1}^{n} X^{(j)i} = 1 \qquad\qquad 1 \leq j \leq n$$

The vector X contains the stationary probability distributions for the ratings of all credit risks in the portfolio. That is, for each j, $X^{(j)}$ represents the probability distribution for the ratings of the jth credit risk in the long-run Ching et al. (2002).

The vector X denoted as x-head give the values below;

Table 2

x head
0.235294118
0.352941176
0.235294118
0.176470588

Table 3

(Q1)*xhead	(Q2)*xhead
-0.247058824	-0.264705882
-0.364705882	-0.382352941
-0.270588235	-0.176470588
-0.117647059	-0.176470588
0.247058824	0.264705882
0.364705882	0.382352941
0.270588235	0.176470588
0.117647059	0.176470588

Ching et al. (2002) model relaxed the stringent assumption that the credit transition matrix is given in advance. Credibility theory is used to estimate the transition matrix and other unknown model parameters in the multivariate Markov chain model. Credibility theory has a long history in actuarial science. It has been widely applied in the actuarial discipline for calculating a policyholder's premium through experience rating of the policyholder's past claims. The main idea of credibility theory is to provide consistent and convenient way to combine two different sources of information for the premium calculation.

Using the solver in excel we are able to get the values of λ_i which is non-negative and $\sum_{i=1}^{n} \lambda_i = 1$.

Table 4

Target Cell (Max)

Cell	Name	Original Value	Final Value
$E$16		0	0

Adjustable Cells

Cell	Name	Original Value	Final Value
$B$16	(Q1)*xhead	0	0
$C$16	(Q2)*xhead	0	0

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$B$17	(Q1)*xhead	-0.247058824	$B$17<=$F$17	Not Binding	0.752505447
$C$17	(Q2)*xhead	-0.264705882	$C$17<=$F$17	Not Binding	0.247494553
$B$16	(Q1)*xhead	0	$B$16<=0	Binding	0
$C$16	(Q2)*xhead	0	$C$16<=0	Binding	0

Table 5

(Q1)*xhead*λ	(Q2)*xhead*λ
-0.251426374	-0.235294118
-0.369073433	-0.352941176
-0.24729463	-0.235294118
-0.132205562	-0.176470588
0.251426374	0.235294118
0.369073433	0.352941176
0.24729463	0.235294118
0.132205562	0.176470588

The traditional reduced-form approach assumes that the losses from a credit risk at each particular rating class can be evaluated based on accounting information and principles. With the data entered and parameters are estimated, a prediction table of transition probabilities from one grade to another can be created.

Table 6

	Higher order Markov Model			
	1	2	3	4
1	0.301002	0.061874	0.637124	0.000000
2	0.061874	0.563544	0.250835	0.123747
4	0.623747	0.000000	0.000000	0.376253
3	0.362876	0.185621	0.451503	0.000000
2	0.250000	0.626253	0.000000	0.123747
1	0.376253	0.061874	0.185621	0.376253
3	0.362876	0.185621	0.451503	0.000000
2	0.250000	0.626253	0.000000	0.123747

1	0.301002	0.061874	0.637124	0.000000
2	0.061874	0.563544	0.250835	0.123747
4	0.623747	0.000000	0.000000	0.376253
3	0.362876	0.185621	0.451503	0.000000
2	0.250000	0.626253	0.000000	0.123747
1	0.188126	0.626253	0.185621	0.000000
1	0.376253	0.061874	0.185621	0.376253
3	-0.077206	-0.437500	0.250835	0.000000
4	-0.323529	-0.225490	0.250835	0.000000

There are four possible states. Examining the first row, the probability of staying in the first credit class over a small period of time Δt is approximately 0.301002. The transition rate from the first credit class to the second credit class is 0.061874, the possibility of downgrading occurring with rate 0.637124 and the rate of default from the first credit class is 0.000000.
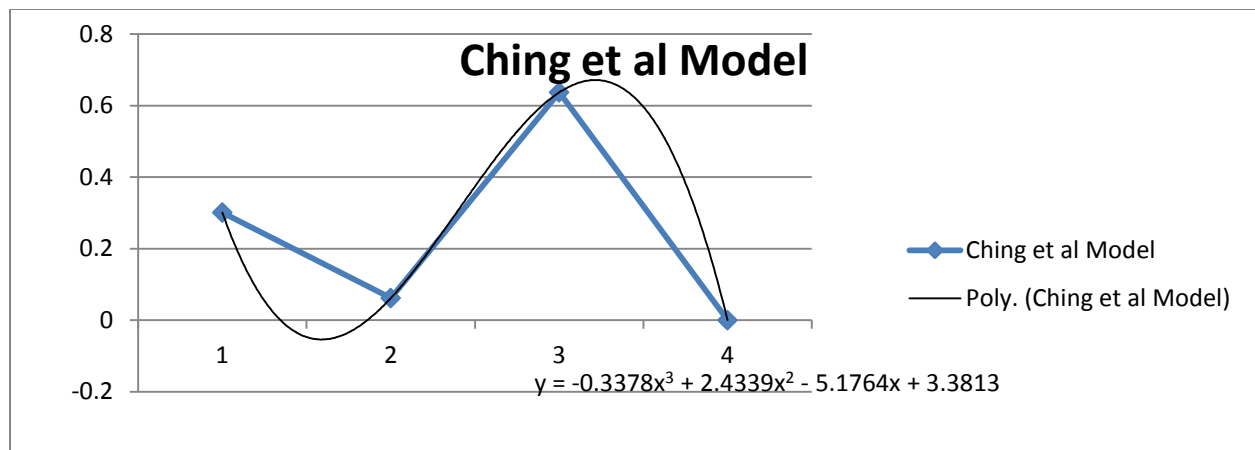
Figure 1

Figure 1 illustrates the graphing of the transition probability as a function of time (Ching et al model) and is one based on the rate of default directly from the highest credit rating i.e. 1.
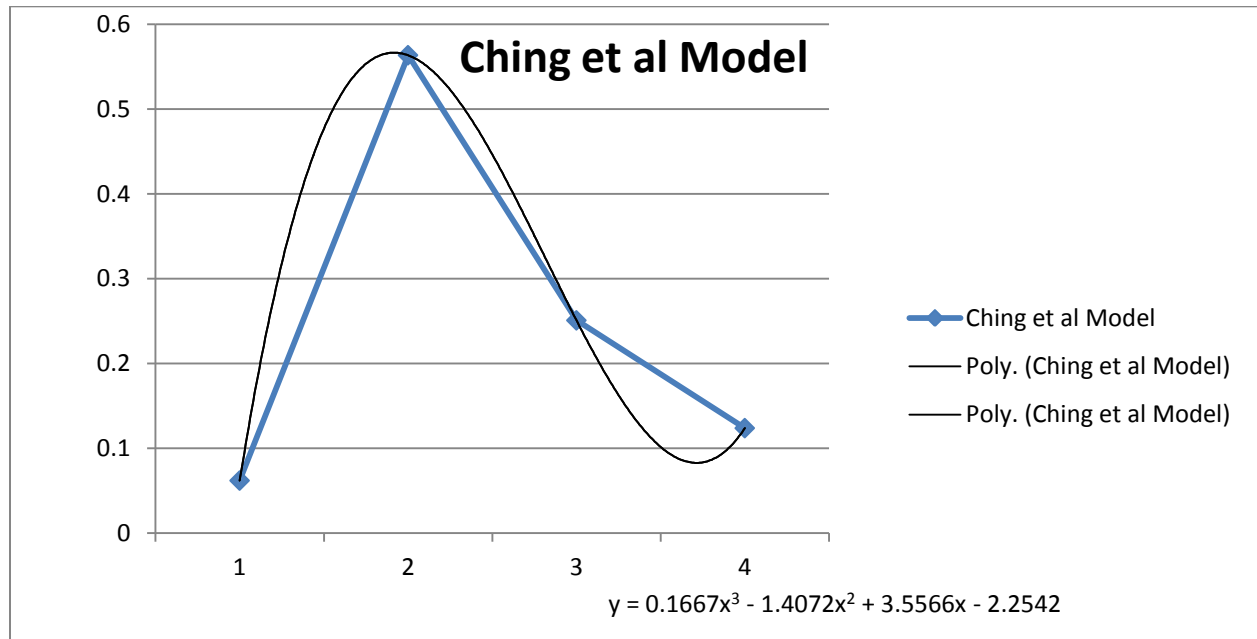
Figure 2

Figure 2 illustrates the graphing of the transition probability as a function of time (Ching et al model) and is one based on the rate of default directly from the second credit rating i.e. 2.

With this we can conclude that the Ching et al model follows a 3rd order polynomial to be able to come up with the transition matrices.

CHAPTER FIVE

5.1 CONCLUSION

The transition matrix is estimated by the use of a linear combination of the empirical transition matrix and prior estimate of the transition matrix. The estimation method incorporates both the historical rating data and another source of information about rating data which includes expert opinion or subjective views. The multivariate Markov chain model has incorporated the dependency of the ratings of credit risks in a portfolio. The estimates of the unknown parameters and the transition matrices were obtained by solving a set of LP problems. The estimation method is done through excel and is analytically tractable, easy to implement and computationally efficient.

The model shows the next period's rating of a particular credit risk that depend on its current rating and also the current ratings of other credit risks in the portfolio.

The application of the model is limited by the number of parameters involved, which depends on the dimension of the multivariate categorical time series and the number of possible credit ratings.

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