EXPONENTIAL DISTRIBUTION: ITS CONSTRUCTIONS, CHARACTERIZATIONS AND RELATED DISTRIBUTIONS

By

Kennedy Karume Karakacha

I/56/74301/2009

Supervised by

Prof. J.A. M. Ottieno

School of Mathematics, University of Nairobi

A thesis submitted in partial fulfillment of the requirements for the degree in Master of Science in Mathematical Statistics of the University of Nairobi

DECLARATION

This is my original work and has not been presented for any of the study programme in any university

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KENNEDY KARUME KARAKACHA **Date**

This thesis has been submitted for examination with my approval as the university supervisor

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PROFESSOR J.A. M OTTIENO **Date**

SCHOOL OF MATHEMATICS

UNIVERSITY OF NAIROBI

ABSTRACT

Exponential distribution has been constructed in this thesis using transformations of uniform and Pareto distributions. It has also arisen from the Poison process and it is a special case of the gamma distribution.

The basic properties of the exponential distribution considered are the r-th moments in general.

Derived from the moments are mean, variance, skewness and kurtosis. The moment generating function, cumulant generating function and characteristic function have been stated.

There are many ways of characterizing the exponential distribution. In this work we have concentrated on characterization by lack of memory property and its extensions, and, three cases involving order statistics. These are: minimum and spacing between two order statistics, spacing between adjacent order statistics and the k-th order statistic. We have, however, stated other forms of characterizations including many also based on order statistics.

Distributions of sum, difference, quotient and product of exponential distributions have been derived. The beta-exponential and the exponentiated exponential distributions have also been derived. These are generalizations of the exponential distribution.

Exponential mixtures have been obtained for nine discrete mixing distributions-the Bernoulli, binomial, geometric types I and II, negative binomial types I and II, Poisson, discrete uniform and logarithmic distributions. Mixtures for thirteen continuous mixing distributions have also been determined. These are: beta, exponential, one-parameter gamma, two-parameter gamma, chi-square, inverse gamma, Erlang, inverse Gaussian, generalized inverse-Gaussian, half-normal, Rayleigh, uniform(rectangular) and chi distributions.

The survival-time function, hazard rate function, cumulative distribution function and the probability density function have been obtained for each mixture by using the moment generating function technique. Some of the mixture distribution functions were obtained explicitly. Others were obtained in terms of special functions such as modified Bessel, generalized hyper geometric and parabolic cylindrical functions.

Density curves with arbitrary parameter values have been sketched for each mixture. An exponential curve, also with arbitrary parameter value, has been superimposed on each mixture density for a rapid visual comparison.

ACKNOWLEDGEMENTS

I would like to thank Technical University of Mombasa (formerly The Mombasa Polytechnic) for financial support and study time extended to me. I would like to especially thank my supervisor, Professor J.A.M Ottieno whose valued advice and time enabled the writing of this thesis. I would like to thank all members of staff of the Statistics Department at The School of Mathematics, University of Nairobi for their ever-present assistance. Finally, I would like to thank my family for all their love and support.

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Chapter 1

Introduction

1.0 Background

The exponential distribution is one of the most extensively applied life-time and reliability analyses distributions. The distribution finds applications in diverse areas when events occur independently, at random but with mean rate λ, per unit of time, distance, volume etc, for instance, in modeling

(i) the distance of encountering a particular type of wild growing plant along a transect line in a forest as an ecologist may be interested in finding;

(ii) the length of time between emissions of a radio-active substance;

(iii) the failure time of manufactured items;

(iv) the inter-arrival times at ticket counters;

(v) the length of queues at particular sections of highway at various times;

(vi) in insurance, the amount of insurance losses.

The distribution therefore, plays a crucial role in probability and statistics and an organized study of its properties is necessary.

Characterization of a distribution is an important tool in its application. In this study characterization of the exponential distribution by the lack of memory property and three cases involving order statistics have been examined in detail. Characterization by use of other properties have also been briefly covered.

The study also looks at resulting processes when exponential distributions are manipulated among themselves, for instance as sums, differences, products or quotients.

Very often however, populations are not homogeneous, so that the appropriate distributions to handle them are mixtures. This study examines mixtures involving the exponential distribution with nine discrete distributions and thirteen continuous

distributions. The mixtures were derived by use of an innovative method based on moment generating functions. It is noted that this method of mixture derivation only applies to the exponential distribution due the special form of its function. This makes it possible to derive its mixtures with other distributions through the moment generating functions of the mixing distributions.

The rest of the study is organized as follows: constructions, moments and definitions concerning the exponential distribution are dealt with in chapter two. Chapter three deals with characterizations. Chapter four deals with distributions resulting from the actions of the exponential distribution on other exponential distributions. Chapter five deals with the derivation of the moment generating functions of the mixing distributions used in the study. Chapter six deals with the survival-time and related functions of the mixtures. Chapter seven is the conclusion.

The appendices contain extracts of pages from the sources quoted in the footnotes. These include theorems, lemmas, examples or table values that were needed by the proofs, other theorems or procedures used in the study.

1.1 Objective of the study

The objective of this study is to bring together the various properties of the exponential distribution that underlie its applications. This information is required of researchers in diverse fields such as engineering, biology, medicine, economics, epidemiology and demography.

1.2 Literature review

Characterization: Many researches have been conducted in the area of characterization of the exponential distribution. A number of the research findings have been new. Others have, however, been different forms of already available findings.

Furguson (1964), characterized the exponential distribution by the conditional independence between min(X, Y) and Y-X when X and Y were two exponential random variables with different means. This characterization was extended by Basu(1965) when both X and Y were exponential random variables with the same mean. Dallas, A.C.(1981) [8] accomplished a similar characterization by requiring that only one of the random variables be exponential. The other variable only needed to be continuous and have a positive real line support.

Srivastava, M.S.(1967) [23]has given characterization by considering the independence of functions of order statistics for a given population. Epstein, B. and Sobel, M. (1953) [11] have characterized the exponential distribution by the independence of spacings between adjacent order statistics. This property had however, already been identified by Sukhatme, P.V.(1937)[24].

Ahsanulla, M. (1977)[3] and Tavangar. M & Asadi.M.(1977) [24] characterized the exponential distribution by considering functions of order statistics having identical distributions. Khan, A. H., Faizan, M. and Haque, Z.(2009) [18] characterized the distribution by expected values of functions of order statistics. By assuming the existence

of a finite first moment for a continuous random variable, X, Wang, Y. H. and Srivastava, R. C.(1980)[27] have characterized the exponential distribution via linear regression of two functions of order statistics.

Ahsanulla, M. & Rahman, M. (1972) [2] characterized the exponential distribution by the k-th order statistic. Galambos, J. & Kotz, S. (1978) [12] have characterized the distribution by the lack of memory property. They have shown extensions of the lack of memory property and how other forms of characterization of the exponential distribution such as via constancy of hazard rate, constancy of residual life and the equivalency of the distribution of the first order statistic to that of the parent population are equivalents of the lack memory property.

LAU, K. S. & RAO, P. B. L. S. (1990) [19] provided a characterization by the relevation property.

Tavangar. M & Asadi.M.(2010) [25] characterized the distribution by residual life through the use of a Cauchy functional equation. Huang, W.J. & Li, S.H. A.C.(1993)[17] have characterized the distribution via the variance of the residual life.

Hamdan, M. A.(1972) [15] Characterized the exponential distribution via conditional expectation of a function of a random variable. Chang, T.(2001). [6] has provided another form of characterization via conditional expectation of a function of random variable but with additional conditionality. Galambos, J. & Kotz, S. (1978) [12] provide another characterization in terms of conditional expectation of moments of a random variable about some fixed value on the positive real axis. Chong, M.(1977). [7] has given a characterization by means of the distributions truncated from below at various points.

Dallas, A.C.(1981) [8] has characterized the exponential distribution via the independence between the spacing of two adjacent records and their minimum. Huang, W.J. & Li, S.H. A.C.(1993). [17] characterize the exponential distribution by use of the expectation of a function of a backward spacing between two record values conditioned on their minimum. They also show a characterization based on a forward spacing between two record values conditioned on their minimum. Ahsanullah, M. (1991) [4] characterizes the exponential distribution by the equivalence of the expectation of the spacing between two record values and the expectation of the record value corresponding to the difference between their record times. Gupta, R.C.(1984) [14] has characterized the exponential distribution by deriving a general theorem on the independence of the expectation of the spacing between two adjacent record values and their minimum and showing that theorems by

Dallas, A.C.(1981) [8], Huang, W.J. & Li, S.H. A.C.(1993) and Ahsanullah, M. (1991) [4] are its special cases.

Mixtures. Mixture distributions provide more flexibility for modelling populations than simple distributions. A number of researches have been conducted in this area.

Xekalaki, E. & Karlis, D. (2005)[28] have ably illustrated the underlying principles of mixture distributions in their paper. They have singled out mixtures of the Poisson distribution for special attention.

Saralees Nadarajah and Samuel Kotz(2006)[20] have listed the probability densities of a number of mixtures of the exponential distribution with the reciprocal rate (section 5.1). This form of the exponential distribution does not lend itself well to direct integration or other methods in obtaining its mixtures. The authors therefore, relied wholly on special functions to obtain all the mixture densities. Some of the mixture densities have very complex forms and it would be difficult to obtain related survival-time functions from them.

Miroslav Drodzenko and Mikhael Yadrenko [10]have in their paper considered mixtures of the exponential distribution from two angles: one in which they consider the exponential distribution parameter λ to be a linear function of the random variable X of the form $\lambda(x) = b + h(x)$, and two, a linear function of the form $\lambda(x) = bh(x)$. b is some positive constant.

Roy Kirk (1997)[22]has, in his thesis paper provided a probabilistic interpretation of the Laplace transform and its discrete equivalent, the z-tranform. He shows how the Laplace and Z-transforms of functions can be considered as mixtures of the exponential distribution with the said functions. The transforms acquire probabilistic meaning when the mixing functions are probability functions.

Enrique R. Villa and Luis A. Escobar[26] have shown, in their paper the efficiency of using moment generating functions, when available, to obtain mixtures for a varied number of distributions. This process, however, requires knowledge of moment generating functions both of the conditional distribution and the mixing distribution. One also needs to be able deduce a distribution from its moment generating function. They have also highlighted the advantages of mixtures in applications over simple distributions. In addition, they cite a number of situations that give rise to mixture distributions.

Ole Hesselar, Shaun Wang and Gordon Willmot(1998)[16] have also demonstrated the relationship between mixtures of the exponential distribution and the Laplace transform of any probability density that has support on $[0, \infty)$.

Chapter 2

Exponential distribution Constructions, moments and definitions

In this chapter a number of construction methods, moments and some other basic statistical definitions relating to the exponential distribution have been examined.

2.1 Construction

The exponential distribution can be constructed by various methods.

2.1.1 Construction from the uniform distribution

The exponential distribution can be obtained as a function of a uniform random variable.

Let $Y = -\frac{1}{\lambda} lnX$ where λ> 0 is a constant and X has a uniform distribution on the interval [0, 1]. We find the pdf of Y.

Using the method of variable transformation;

$$X = e^{-\lambda Y}$$

$$\therefore \frac{dx}{dy} = -\lambda e^{-\lambda y}$$

Thus the pdf of Y, g(y), is given by

$$g(y) = f(x)\left|\frac{dx}{dy}\right| = 1.\lambda e^{-\lambda y}$$

$$= \lambda e^{-\lambda y} \qquad (1)$$

which is an the exponential distribution of rate λ.

2.1.2 Construction from the Pareto distribution

The exponential distribution can be obtained as a function of a Pareto random variable.

Let $Y = -ln\frac{X}{\beta}$ where β is a constant and X has a Pareto distribution given by

$$f(x) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}, \qquad x > \beta$$

We find the pdf of Y using:

(i) variable transformation technique
(ii) cumulative distribution technique

Using the variable transformation technique;

$$X = \beta e^{Y}$$

$$\therefore \frac{dx}{dy} = \beta e^{y}$$

Thus the pdf of Y, g(y), is given by

$$g(y) = f(x)\left|\frac{dx}{dy}\right| = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}\ \beta e^{y}$$

$$= \frac{\alpha\beta^{\alpha}}{(\beta e^{y})^{\alpha+1}}\ \beta e^{y}$$

$$= \alpha e^{-\alpha y}$$

Putting $\alpha = \lambda$,

$$g(y) = \lambda e^{-\lambda y}$$

This is an exponential distribution with rate λ.

Using the cumulative distribution technique;

Let $G(y) = P(Y \leq y)$ where $G(y)$ is the cumulative distribution of Y.

$$= P\left(ln\frac{X}{\beta} \leq y\right)$$

$$= P\left(ln\frac{X}{\beta} \leq y\right)$$

$$= P(X \leq \beta e^{y})$$

$$= \int_{0}^{\beta e^{y}} f(x)dx$$

$$= \int_{0}^{\beta e^{y}} \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}\ dx$$

$$\therefore \quad 1 - G(y) = \int_{\beta e^{y}}^{\infty} \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}\ dx = -\beta^{\alpha}\left[\frac{1}{x^{\alpha}}\right]_{\beta e^{y}}^{\infty}$$

$$= -\beta^{\alpha}\left[0 - \frac{1}{(\beta e^{y})^{\alpha}}\right]$$

$$= e^{-\alpha y}, \alpha > 0,\ y > 0$$

$\therefore G(y) = 1 - e^{-\alpha y}$

$$g(y) = \frac{d[G(y)]}{dy}$$

$$= \frac{d[1 - e^{-\alpha y}]}{dy}$$

$= \alpha e^{-\alpha y}$, which is an exponential distribution with parameter α.

Let $\lambda = \alpha$,

$\therefore f(x) = \lambda e^{-\lambda y}, \ \lambda > 0$, $\quad y > 0$

2.1.3 Construction from the gamma function

The exponential distribution can be obtained from the gamma function.

For a constant $\alpha > 0,\ and\ variable\ t > 0$,

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

Dividing by $\Gamma(\alpha)$,

$$1 = \int_0^\infty \frac{t^{\alpha-1} e^{-t}}{\Gamma(\alpha)} dt$$

Thus,

$$f(t) = \frac{t^{\alpha-1} e^{-t}}{\Gamma(\alpha)}, \quad t > 0, \quad \alpha > 0$$

Let $X = \frac{T}{\beta}$

Thus,

$$f(x) = \frac{(x\beta)^{\alpha-1} e^{-x\beta}}{\Gamma(\alpha)}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-x\beta}, \quad x > 0, \quad \alpha > 0, \quad \beta > 0$$

Putting $\alpha = 1$,

$f(x) = \beta e^{-x\beta}$, $\quad x > 0, \ \beta > 0$

This is exponential distribution with parameter β

$Let\ \lambda = \beta,$

$\therefore f(x) = \lambda e^{-x\lambda}$, $\quad x > 0, \quad \lambda > 0$

2.1.4 Construction from a stochastic process

Consider a finite time interval $(0, t)$.

Dividing the interval into n sub-intervals each of length h, then,

$$t = nh$$

Let $P_n(h) = P(number\ of\ events\ in\ a\ time\ interval\ h)$

Let: $P_0(h) = 1 - \lambda h + 0(h)$

$$P_1(h) = \lambda h + 0(h)$$

$$P_n(h) = 0(h)\ for\ n > 1$$

where $0(h)$ means a term ψ(h) with property $\lim_{h\to 0} \frac{\psi(h)}{h} = 0$

Let $P_n\{T > t\} = P\{no\ event\ in\ (0, t)\}$

$$\therefore T = time\ for\ 1^{st} event$$

Let the probability of events in any sub-interval be independent of each other.

Then,

$$P_n\{T > t\} = [1 - \lambda h + 0(h)]^n$$

$$= \left[1 - \frac{\lambda t}{n} + 0(h)\right]^n$$

$$= \left(1 - \frac{\lambda t}{n}\right)^n + n.0(h)\left(1 - \frac{\lambda t}{n}\right)^{n-1} + \cdots$$

$$= e^{-\lambda t}$$

$$\therefore\ P\{T > t\} = lim_{h\to 0}\ P_n\{T > t\}$$

$$= e^{-\lambda t}$$

The pdf of T is,

$$-\frac{d}{dt}\{P\ \{T > t\}\} = \lambda e^{-\lambda t}$$

This is an exponential distribution with parameter λ.

Fig. 2.1 shows the probability curves for an exponential distribution with three rate parameters, 0.5, 1 and 2.

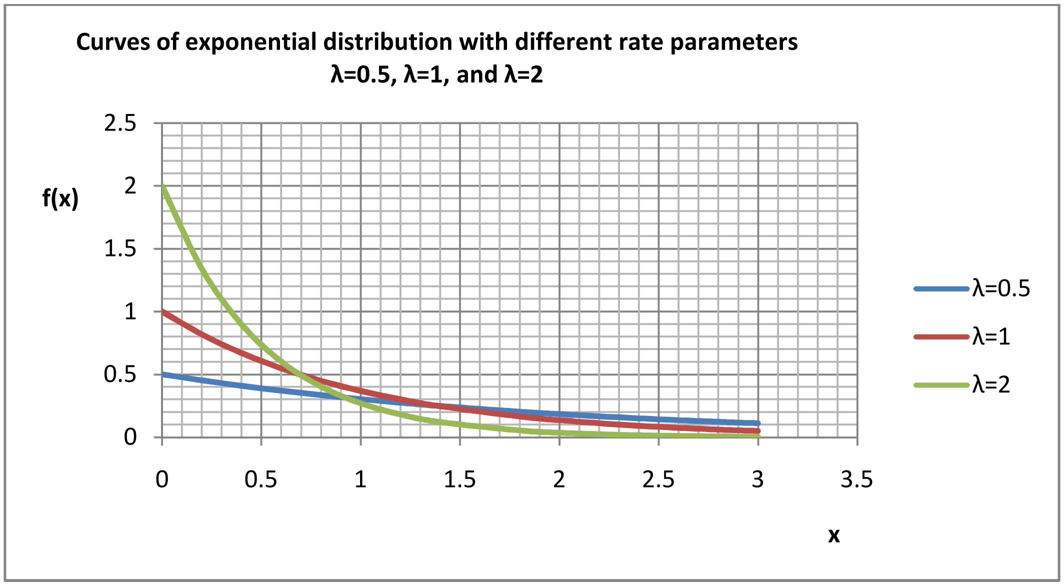

Fig. 2.1

2.2 Moments

The r-th moment of a random variable X is given by

$$E[X^r] = \int_0^\infty x^r f(x) dx$$

Thus when X is exponentially distributed,

$$E[X^r] = \int_0^\infty x^r \lambda \mathrm{e}^{-x\lambda}\ dx$$

$$Let\ u = \lambda x, \qquad dx = \frac{1}{\lambda} du$$

$$E[X^r] = \lambda \int_0^\infty \left(\frac{u}{\lambda}\right)^r \mathrm{e}^{-u}\ \frac{1}{\lambda} du$$

$$= \frac{1}{\lambda^r}\int_0^\infty u^r \mathrm{e}^{-\mathrm{u}}\ du$$

$$= \frac{1}{\lambda^r}\Gamma(r+1)$$

$$= \frac{r!}{\lambda^r}$$

for positive integer r.

$$Thus\ E[X^r] = \frac{r!}{\lambda^r}$$

Hence,

$$E[X] = \frac{1}{\lambda} = \mu$$

$$E[X^2] = \frac{2!}{\lambda^2} = \frac{2}{\lambda^2}$$

Thus $\mu_2 = E[X-\mu]^2 = \sigma^2$

$$= E[X^2] - 2\mu E[X] + \mu^2$$

$$= E[X^2] - \mu^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \qquad \therefore \sigma = \frac{1}{\lambda}$$

Also,

$$E[X^3] = \frac{3!}{\lambda^3} = \frac{6}{\lambda^3}$$

$$\therefore\ \mu_3 = E[X-\mu]^3$$

$$= E[X^3] - 3\mu E[X^2] + 3\mu^2 E[X] - \mu^3$$

$$= E[X^3] - 3\mu E[X^2] + 3\mu^2 E[X] - \mu^3$$

$$= E[X^3] - 3\mu E[X^2] + 3\mu^2 E[X] - \mu^3$$

$$= \frac{3!}{\lambda^3} - 3.\frac{1}{\lambda}.\frac{2}{\lambda^2} + 3.\frac{1}{\lambda^2}.\frac{1}{\lambda} - \frac{1}{\lambda^3}$$

$$= \frac{2}{\lambda^3}$$

$$E[X^4] = \frac{4!}{\lambda^4} = \frac{24}{\lambda^4}$$

$$\therefore \; \mu_4 = E[X - \mu]^4$$

$$= E[X^4] - 4\mu E[X^3] + 6\mu^2 E[X^2] - 4\mu^3 E[X] + \mu^4$$

$$= \frac{4!}{\lambda^4} - 4.\frac{1}{\lambda}.\frac{3!}{\lambda^3} + 6.\frac{1}{\lambda^2}.\frac{2}{\lambda^2} - 4.\frac{1}{\lambda^3}.\frac{1}{\lambda} + \frac{1}{\lambda^4}$$

$$= \frac{9}{\lambda^4}$$

2.3 Definitions

2.3.1 Mean

The mean

$$\mu = E[X] = \frac{1}{\lambda}$$

2.3.2 Variance

The variance

$$\sigma^2 = E[X^2] - \mu^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

2.3.3 Skewness

The skewness of a curve γ_1, is a measure of the symmetry of the curve in comparison with the normal curve and is given by

$$\gamma_1 = \frac{\mu_3}{\sigma^3}$$

Thus for an exponential curve,

$$\gamma_1 = \frac{\frac{2}{\lambda^3}}{\left(\frac{1}{\lambda}\right)^3} = 2$$

The normal curve has $\gamma_1 = 0$

2.3.4 Kurtosis

The kurtosis of a curve, γ_2, is a measure of the sharpness of its peak and the width and length of its tail in comparison with the normal curve and is given by

$$\gamma_2 = \frac{\mu_4}{\sigma^4}$$

Thus for an exponential curve,

$$\gamma_2 = \frac{\frac{9}{\lambda^4}}{\left(\frac{1}{\lambda}\right)^4} = 9$$

The normal curve has $\gamma_2 = 3$

2.3.5 Moment generating function

The moment generating function is

$$M(t) = E[e^{tx}] = \frac{\lambda}{\lambda - t} \qquad (section\ 5.5.2)$$

2.3.6 Cumulant generating function

The cumulant generating function is

$$K(t) = logE[e^{tx}] = log\lambda - \log{(\lambda - t)}$$

2.3.7 Characteristic function

The characteristic function is

$$\varphi(t) = E[e^{itx}] = \frac{\lambda}{\lambda - it}$$

2.3.8 Survival function

Let T denote the time from a well-defined starting point until some event called "failure", occurs. T is referred to as survival time and let $f(t)$ denote its probability density function.

For an exponentially distributed process with parameter λ

$$f(t) = \lambda e^{-\lambda t}$$

The probability that the survival time T exceeds some value t, $S(t)$, is given by

$$S(t) = P(T > t)$$

$$= 1 - F(t), \text{ where } F(t) \text{ is the cumulative distribution function.}$$

For an exponential distribution,

$$F(t) = \int_0^t \lambda e^{-\lambda x} dx \; = -\left[e^{-\lambda x}\right]_0^t$$

$$\therefore \; F(t) \; = 1 - e^{-\lambda t}$$

Thus the survival function is

$$S(t) = 1 - F(t) \quad = e^{-\lambda t}$$

2.3.9 Hazard rate function

The hazard rate function is

$$h(t) = \frac{f(t)}{S(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda, a\ constant.$$

2.3.10 Memoryless property

Consider two survival times $t > 0, \quad s > 0$.

$$P(T > t + s | T > t) = \frac{P(T > t + s, T > t)}{P(T > t)}$$

$$= \frac{P(T > t + s)}{P(T > t)}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}}$$

$$= e^{-\lambda s} \; = P(T > s)$$

This is the memory-less property of the exponential distribution which shows that the life-time of an exponentially distributed object is independent of its current age. The exponential distribution is the only continuous distribution with this unique property.

Chapter 3

Characterization

3.1 Introduction

There are many ways of characterizing the exponential distribution. We have characterizations based on:

Lack of memory property;
order statistics;
record values;
convolution and relevation;
residual lifetime;
moment properties, and
conditional expectations

In this chapter we shall examine the lack of memory property and sample three characterizations by order statistics. A brief discussion of the other characterizations is also given.

3.2 Memory-less and related properties

Definition (Ross, 2000).
A random variable X is said to be without memory or memory-less if

$$\Pr(X > s + t | X > t) = \Pr(X > s) \; for \; all \; s, t > 0 \tag{3.1}$$

If we think of *X* being the life-time of some instrument, then (3.1) states that the probability that the instrument lives for at least *s+t* hours given that it has survived *t* hours is the same as the initial probability that it lives for at least *s* hours. In other words, if the instrument is alive at time *t*, then the distribution of the remaining amount of time that it survives is the same as the original life-time distribution, that is, the instrument does not remember that it has already been in use for a time *t*.

The condition (3.1) is equivalent to

$$\frac{\Pr(X > s + t, \;\; X > t)}{\Pr(X > t)} = \Pr(X > s)$$

or

$$\Pr(\mathrm{X} > s + t, \; X > t) = \Pr(X > s)\Pr\,(X > t) \tag{3.2}$$

or

$$\Pr(\mathrm{X} > s + t) = \Pr(X > s)\Pr\,(X > t) \tag{3.3}$$

Since (3.2) or (3.3) is satisfied when *X* is exponentially distributed (for $e^{-\lambda(s+t)} = e^{-\lambda s}e^{-\lambda t}$), it follows that exponentially distributed random variables are memory-less. It turns out that not only is the exponential distribution "memoryless" but it is the unique continuous distribution possessing this property. The following theorem characterizes the exponential distribution by the lack of memory property.

Theorem 3.2.1

The exponential distribution is the only continuous distribution which is memory-less

Proof:

For a random variable X having a cdf $F(x) = \Pr(X \leq x)$, its survival function is $1 - F(x) = \Pr(X > x)$

Re-writing (3.3) in terms of survival functions,

$$1 - F(s + t) = [1 - F(s)][1 - F(t)] \qquad (3.4)$$

We show that the exponential distribution is the only continuous distribution to satisfy (3.4).

Re-arranging (3.4), we have

$$F(s + t) = 1 - [1 - F(s)][1 - F(t)]$$

$$= 1 - \{1 - F(t) - F(s) + F(s)F(t)\}$$

$$= F(t) + F(s) - F(s)F(t)$$

$$\therefore F(s + t) - F(t) = F(s) - F(s)F(t)$$

$$= F(s)[1 - F(t)] \qquad (3.5)$$

Dividing (3.5) by *s* we have

$$\frac{F(t + s) - F(t)}{s} = \frac{F(s)}{s}\,[1 - F(t)] \qquad (3.6)$$

Taking limits of (3.6) as $s \to 0^+$

$$\lim_{s \to 0^+} \frac{F(t + s) - F(t)}{s} = F'(t) = f^+(t)$$

$$and \quad \lim_{s \to 0^+} \frac{F(s)}{s} = b, a\ constant$$

$$\therefore f^+(t) = b[1 - F(t)] \qquad (3.7)$$

Equation (3.5) can also be written as

$$F(t) - F(t - s) = F(s)[1 - F(t - s)] \qquad (3.8)$$

Dividing (3.8) by s,

$$\frac{F(t) - F(t-s)}{s} = \frac{F(s)}{s}\ [1 - F(t-s)]$$

$$\therefore \lim_{s \to 0^-} \frac{F(t) - F(t-s)}{s} = F'(t) = f^-(t)$$

$$and \lim_{s \to 0^-} \frac{F(s)}{s} = b, a\ constant$$

$$\therefore f^-(t) = b[1 - F(t)] \tag{3.9}$$

Combining (3.7) and (3.9) (since both LHS and RHS derivatives exist),

$$f(t) = b[1 - F(t)] \tag{3.10}$$

But

$$f(t) = \frac{d[F(t)]}{dt}$$

Therefore, (3.10) can be written as

$$\frac{d[F(t)]}{dt} = b[1 - F(t)] \tag{3.11}$$

Solving (3.11) by variable separation,

$$\frac{d[F(t)]}{1 - F(t)} = bdt$$

$$\therefore \int \frac{d[F(t)]}{1 - F(t)} = \int bdt$$

$\Leftrightarrow -ln[1 - F(t)] = bt + c$, c is a constant of integration

$ln[1 - F(t)] = -bt + c_1, \quad c_1 = -c$

$\therefore 1 - F(t) = e^{-bt+c_1}$

$$\therefore F(t) = 1 - e^{-bt+c_1} = 1 - Ae^{-bt} \tag{3.12}$$

$F(t)$ is a cdf on $[0, \infty]$

$\therefore F(t) = 0\ at\ t = 0$

Substituting this initial condition in (3.12)

$0 = 1 - A \Rightarrow A = 1$

Hence $F(t) = 1 - e^{-bt}, b > 0,\ t \geq 0$, which is the distribution function of an exponential distribution.

3.2.2 Characterization by extension of the lack of memory property

Other characterizations of the exponential distribution are extensions of its lack of memory property.

The following theorem extends the lack of memory property of the exponential distribution [12]. The theorem requires the following definition:

Definition
A function is said to be **absolutely continuous** in an interval iff it is continuous and differentiable at every point in the interval.

Theorem 3.2.2

Consider a sample of size n of independent random variables $X_1,\ X_2, . . . \ , X_n$ taken from a population with an absolutely continuous distribution function$F(x)$.

Let G(x) = 1-F(x).

If $G(x_1+x_2+...+x_n) = G(x_1)\ G(x_2)...\ G(x_n)$, then

$$F(x) = 1 - e^{-bx}, \quad x \geq 0, \quad with\ some\ b > 0.$$

Proof.

The lack of memory property $1 - F(s + t) = [1 - F(s)][1 - F(t)]$ in **theorem 3.3.1** can be restated as:

G(x+z) = G(x).G(z). for all x, z ≥ 0.

Substituting for x and z with x_1 and x_2, we have
$G(x_1 + x_2) = G(x_1)\ G(x_2)$, and by induction,

$G(x_1+x_2+...+x_n) = G(x_1)\ G(x_2)...\ G(x_n)$.

Hence the theorem is proved.

3.2.3 An extension of the lack of memory property through order statistics

In this section an extension of the lack of memory property has been used in conjunction with order statistics to characterize the exponential distribution [12].

(Section 3.3 gives a more expanded overview of some properties of order statistics)

Let $\mathrm{X}_1, \mathrm{X}_2 . . . \mathrm{X}_\mathrm{n}$ be a sample of independent and identically distributed random variables from a population X. The associated sample order statistics are obtained by

re-arranging the variables X_i's in ascending order as $\mathrm{X}_{(1)}, \mathrm{X}_{(2)} . . . \mathrm{X}_{(\mathrm{n})}$, so that $\mathrm{X}_{(1)}$ is the smallest and $\mathrm{X}_{(\mathrm{n})}$ is the largest of the X_i's.

The cumulative distribution of the *i-th* order statistic $X_{(i)}$ is given by

$$F\left(x_{(i)}\right) = \sum_{r=i}^{n} [F(x)]^r [1 - F(x)]^{n-r} . \tag{1}$$

By putting *i*=1 in (1) we obtain the distribution of the minimum of the sample values $\mathrm{X}_{(1)} = \min\{ \mathrm{X}_1, \mathrm{X}_2 . . . \mathrm{X}_\mathrm{n}\}$ as

$$F\left(x_{(1)}\right) = \sum_{r=1}^{n} [F(x)]^r [1 - F(x)]^{n-r}$$

$$= 1 - [F(x)]^{r=0} [1 - F(x)]^{n-(r=0)}$$

$$= 1 - [1 - F(x)]^n \tag{2}$$

Theorem 3.2.3

Let $X_1, X_2 . . . X_n$ be a sample of independent and identically distributed random variables from a population X. Let F(x) = 1- e $^{-bx}$, be the distribution of X for some b > 0. If $\mathrm{X}_{(1)} = X_{1:n}$=min{ $X_1, X_2 . . . X_n$}, then

$$X_{1:n} \stackrel{d}{=} X/n$$

Proof:

From **Theorem 3.2.2**, we let $x_1 = x_2 = . . . x_n = x$

$$\therefore G(x_1 + x_2 + . . . + x_n) = G(nx) = [G(x)]^n$$

i.e. $1 - F(nx) = [1 - F(x)]^n$

$$\therefore F(nx) = 1 - [1 - F(x)]^n \tag{3}$$

Comparing equations (2) and (3) we can write

$$X_{1:n} \stackrel{d}{=} X/n$$

Hence the theorem is proved.

3.3 Characterization by order statistics

3.3.0 Introduction

In this section a brief overview of some properties of order statistics is given. These properties form the basis for the characterizations in **Theorem 3.31**, **Theorem 3.32**

and **Theorem 3.33.** A brief discussion of other characterizations by order statistics is also given.

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed values of a random variable X

of sample size n. Their associated order statistics are found by rearranging the variables in increasing order and are denoted by $X_{1:n}, X_{2:n}, \dots, X_{n:n}$.
Thus $X_{1:n}$ is the first order statistic and is the smallest of the $X_i{}'s$. $X_{n:n}$ is the *n*-th order statistic and is the largest of the $X_i's$.
There exist other notations for order statistics such as
$X_{1,n}, X_{2,n}, \dots, X_{n,n}$ or
$X_{(1)}, X_{(2)}, \dots, X_{(n)}$.
Let the sample of $X_i's$ be from a population with distribution function $F(x)$. Using the later notation, the cdf of the *i*-th order statistic is given by

$$F(x_{(i)}) = P(X_{(i)} \leq x) = P(at\ least\ i\ of\ the\ X_i's\ are\ less\ than\ x)$$

$$= \sum_{r=i}^{n} \binom{n}{r} [F(x)]^r [1 - F(x)]^{n-r} \qquad (1)$$

By substituting *i*=1 in (1) we obtain the cdf of the first order statistic $X_{(1)}$, as

$$F(x_{(1)}) = \sum_{r=1}^{n} \binom{n}{r} [F(x)]^r [1 - F(x)]^{n-r}$$

$$= 1 - \binom{n}{r=0} [F(x)]^{r=0} [1 - F(x)]^{n-(r=0)}$$

$$= 1 - [1 - F(x)]^n \qquad (2)$$

By substituting *i*=n in (1) we obtain the cdf of the n-th order statistic $X_{(n)}$, as

$$F(x_{(n)}) = \sum_{r=n}^{n} \binom{n}{n} [F(x)]^n [1 - F(x)]^{n-n} \qquad (3)$$

$$\therefore\ F(x_{(n)}) = \{F(x)\}^n \qquad (4)$$

Differentiating (1) w.r.t x, we can obtain the pdf of the *i*-th order statistic as follows:

$$f\left(\mathrm{x}_{(\mathrm{i})}\right) = \frac{d\left[F\left(\mathrm{x}_{(\mathrm{i})}\right)\right]}{dx} = \frac{d}{dx}\left\{\sum_{r=i}^{n} \binom{n}{r} [F(x)]^r [1 - F(x)]^{n-r}\right\}$$

$$= \sum_{r=i}^{n} \binom{n}{r} \frac{d}{dx}\{[F(x)]^r [1 - F(x)]^{n-r}\}$$

Using the product rule, let $u = [F(x)]^r$ and $v = [1 - F(x)]^{n-r}$

$$\frac{du}{dx} = r[F(x)]^{r-1} f(x), \qquad \frac{dv}{dx} = -(n-r)[1 - F(x)]^{n-r-1} f(x)$$

$$\therefore f\left(\mathrm{x}_{(\mathrm{i})}\right) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= \sum_{r=i}^{n} \binom{n}{r} \{[F(x)]^r [-(n-r)[1 - F(x)]^{n-r-1} f(x)] + [1 - F(x)]^{n-r} r [F(x)]^{r-1} f(x)\}$$

$$= \sum_{r=i}^{n} r\binom{n}{r} [1 - F(x)]^{n-r} [F(x)]^{r-1} f(x)$$

$$- \sum_{r=i}^{n} (n-r) \binom{n}{r} [F(x)]^r \, [1 - F(x)]^{n-r-1} f(x)$$

$$\therefore f\left(\mathrm{x}_{(\mathrm{i})}\right) = i\binom{n}{i} [1 - F(x)]^{n-i} [F(x)]^{i-1} f(x)$$

$$+ \sum_{r=i+1}^{n} r\binom{n}{r} [1 - F(x)]^{n-r} [F(x)]^{r-1} f(x)$$

$$- \sum_{r=i}^{n-1} (n-r) \binom{n}{r} [F(x)]^r \, [1 - F(x)]^{n-r-1} f(x)$$

Since

$$\sum_{r=n}^{n} (n-r) \binom{n}{r} [F(x)]^r \, [1 - F(x)]^{n-r-1} f(x) = 0$$

Also,

$$i\binom{n}{i} = \frac{n!}{i!\,(n-i)!} i = \frac{n!}{(i-1)!\,(n-i)!}$$

$$\therefore f\left(\mathrm{x}_{(\mathrm{i})}\right) = \frac{n!}{(i-1)!\,(n-i)!} [F(x)]^{i-1} [1 - F(x)]^{n-i} f(x)$$

$$+\sum_{r=i}^{n-1}(r+1)\binom{n}{r+1}[F(x)]^r[1-F(x)]^{n-r-1}f(x)$$

$$-\sum_{r=i}^{n-1}(n-r)\binom{n}{r}[F(x)]^r\,[1-F(x)]^{n-r-1}f(x)$$

$$But\ (r+1)\binom{n}{r+1}=\frac{n!}{(r+1)!\,(n-r-i)!}(r+1)=\frac{n!}{r!\,(n-r-i)!}=\frac{n!\,(n-r)}{r!\,(n-r)!}$$

$$=(n-r)\binom{n}{r}$$

$$\therefore\sum_{r=i}^{n-1}(r+1)\binom{n}{r+1}[F(x)]^r[1-F(x)]^{n-r-1}f(x)$$
$$=\sum_{r=i}^{n-1}(n-r)\binom{n}{r}[F(x)]^r\,[1-F(x)]^{n-r-1}f(x)$$

$$\therefore f(\mathrm{x}_{(\mathrm{i})})=\frac{n!}{(i-1)!\,(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x) \tag{5}$$

By substituting i=1 in (5), we obtain the pdf of $\mathrm{X}_{(1)}$ as

$$f(\mathrm{x}_{(1)})=\frac{n!}{0!\,(n-1)!}[F(x)]^0[1-F(x)]^{n-1}f(x)$$

$$\therefore f(\mathrm{x}_{(1)})=n[1-F(x)]^{n-1}f(x) \tag{6}$$

By substituting i=n in (5), we obtain the pdf of $\mathrm{X}_{(\mathrm{n})}$ as

$$f(\mathrm{x}_{(\mathrm{n})})=\frac{n!}{(n-1)!\,(n-n)!}[F(x)]^{n-1}[1-F(x)]^{n-n}f(x)$$

$$\therefore f(\mathrm{x}_{(\mathrm{n})})=n[F(x)]^{n-1}f(x) \tag{7}$$

The joint cumulative distribution function of $\mathrm{X}_{(\mathrm{r})}$ and $\mathrm{X}_{(\mathrm{s})}$ $(1\le \mathrm{r}\le \mathrm{s}\le \mathrm{n})$ is given by

$$F_{(r)(s)}(x,y)=\Pr(at\ least\ r\mathrm{X}_{\mathrm{i}}\le \mathrm{x},\text{at least }\ s\mathrm{X}_{\mathrm{i}}\le \mathrm{y}),\quad \mathrm{x}<y$$

$$=\sum_{j=s}^{n}\sum_{i=r}^{j}\frac{n!}{i!\,(j-i)!\,(n-j)!}[F(x)]^i[F(y)-F(x)]^{j-i}\,[1-F(y)]^{n-j} \tag{8}$$

The joint pdf of $\mathrm{X}_{(\mathrm{r})}$ and $\mathrm{X}_{(\mathrm{s})}$ $(1\le \mathrm{r}\le \mathrm{s}\le \mathrm{n})$ denoted by $f_{(r)(s)}(x,y)$ is given by

$$f_{(r)(s)}(x,y) = \frac{n!\,f(x)f(y)}{(r-1)!\,(s-r-1)!\,(n-s)!}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}$$

(9)

The joint pdf of $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is given by

$$f_{x_{(1)},x_{(2)}\ldots x_{(n)}}(x_1, x_2, \ldots x_n) = n!\,f(x_1)f(x_2)\ldots f(x_n), \quad x_1 < x_2 < \ldots < x_n$$

$$= 0, \quad otherwise \qquad (10)$$

We now apply the above properties to characterize the exponential distribution in the next three theorems.

3.3.1 Characterization by minimum and spacing between two order statistics

Let $X_{(m)}$ and $X_{(m+1)}$ be two adjacent order statistics from a sample of size n with

$X_{(m)} < X_{(m+1)}$, $1 \le m \le n$.

Thus $X_{(m)}$ is the minimum of the two order statistics and the spacing between them is $X_{(m+1)} - X_{(m)}$

The following theorem characterizes an exponential distribution of pdf

$$f(x) = \sigma e^{-\sigma(x-\theta)},\ x > \theta,\ \sigma > 0 \qquad (1)$$

by the independence between $X_{(m)}$ and $X_{(m+1)} - X_{(m)}$. [23]

Theorem 3.3.1

Let F be an absolutely continuous distribution function of the random variable X with F(θ)=0, $\theta > 0$, and with probability density function f(x). Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be order

statistics of a random sample of size n from this distribution. Then in order that the statistics $X_{(m+1)} - X_{(m)}$ and $X_{(m)}$ for fixed m, $1 \le m < n$, be independent, it is necessary and sufficient that the random variable X has the exponential distribution in (1).

Proof:

Let $U = X_{(m)}\ and\ \ V = X_{(m+1)}$.

Then the pdf of U is [by using **section 3.3.0** (5)]

$$\frac{n!}{(m-1)!\,(n-m)!}[F(u)]^{m-1}\,[1-F(u)]^{n-m}f(u)$$

And the joint pdf of U and V is [by using **section 3.3.0** (9)]

$$\frac{n!}{(m-1)!\,(n-m-1)!}[F(u)]^{m-1}\,[1-F(v)]^{n-m-1}f(u)f(v)$$

Hence the conditional pdf of $V|U$=u is

$$\frac{\frac{n!}{(m-1)!\,(n-m-1)!}[F(u)]^{m-1}\,[1-F(v)]^{n-m-1}f(u)f(v)}{\frac{n!}{(m-1)!\,(n-m)!}[F(u)]^{m-1}\,[1-F(u)]^{n-m}f(u)}$$

$$= \frac{(n-m)(1-F(v))^{n-m-1}}{\left(1-F(u)\right)^{n-m}}f(v) \qquad (2)$$

We show that the independence of V-U and U implies that X has the exponential distribution in (1).

Due to independence of V-U and U, E[V-U]=E[V-U|U=u] is free of u. Thus

$$E[V-U] = E[V-U|U=u]$$
$$= (n-m)\int_u^\infty (v-u)\frac{\left(1-F(v)\right)^{n-m-1}}{\left(1-F(u)\right)^{n-m}}f(v)dv \qquad (3)$$

and is free of u.

Differentiating (3) w.r.t. u, we obtain

$$0 = -\int_u^\infty \frac{\left(1-F(v)\right)^{n-m-1}}{\left(1-F(u)\right)^{n-m}}f(v)dv$$
$$+\frac{(n-m)f(u)}{1-F(u)}\int_u^\infty (v-u)\frac{\left(1-F(v)\right)^{n-m-1}}{\left(1-F(u)\right)^{n-m}}f(v)dv$$

$$= -\int_u^\infty \frac{\left(1-F(v)\right)^{n-m-1}}{\left(1-F(u)\right)^{n-m}}f(v)dv + \frac{(n-m)f(u)}{1-F(u)}\ \ E[V-U] \qquad (4)$$

Since $E[V-U]$ is independent of u and

$$\int_u^\infty \frac{\left(1-F(v)\right)^{n-m-1}}{\left(1-F(u)\right)^{n-m}}f(v)dv$$

is a constant, (4) can be written as

$$\frac{f(u)}{1-F(u)} = \sigma, \qquad (5)$$

where σ is some constant not equal to zero. (5) can be written as

$$\frac{dF(u)}{1-F(u)} = \sigma du$$

Solving,

$-\ln\,(1-F(u))= \sigma u + c$ where c is a constant of integration.

Therefore, $F(u) = 1 - e^{-\sigma u - c}$

Thus $f(u) = \sigma e^{-\sigma u - c}$

But f is a probability density function in the range $[\theta, \infty)$, it follows that $c = -\sigma\theta$

and $\theta > 0$.

Therefore, $f(u) = \sigma e^{-\sigma(u-\theta)},\ \theta > 0,\ u > \theta,\ \sigma > 0$.

3.3.2 Characterization by spacing between adjacent order statistics

Let $\mathrm{X_1, X_2, \dots, X_n}$ be independent and identically distributed values of a random variable X of sample size n.

Let $\mathrm{X_{(1)}, X_{(2)}, \dots, X_{(n)}}$ (1)

be the associated order statistics for the sample.

Also, let

$\mathrm{Y_1 = X_{(1)}}$

$\mathrm{Y_2 = X_{(2)} - X_{(1)}}$

$\mathrm{Y_3 = X_{(3)} - X_{(2)}}$

. . .

. . .

$\mathrm{Y_i = X_{(i)} - X_{(i-1)}\,,\ 2 \le i \le n}\,,$ (2)

be spacings between the respective adjacent order statistics.

The following theorem characterizes the exponential distribution of cdf

$$F(x) = 1 - e^{-x/\theta},\ x > 0,\ \theta > 0 \qquad (3)$$

by the mutual independence of the $\mathrm{Y_i}$'s [11].

Theorem 3.3.2

The random variables Y_i defined by (2) are mutually independent. Further, for each i, (n-i+1) $Y_{i:n}$ is distributed with common distribution (3)

Proof 1: (Mutual independence of $\mathrm{Y_i}$)

We shall prove the theorem by use of multivariate transformation and induction.

Let n = 2;

(1) becomes $\mathrm{X_{(1)}, X_{(2)}}$

The corresponding $\mathrm{Y_i}$ variables in (2) will be will be

$$\mathrm{Y_1} = \mathrm{X_{(1)}} \qquad \Longrightarrow \mathrm{X_{(1)}} \qquad = \mathrm{Y_1}$$

$$\mathrm{Y_2} = \mathrm{X_{(2)}} - \mathrm{X_{(1)}} \qquad \Longrightarrow \mathrm{X_{(2)}} = \mathrm{Y_2} + \mathrm{X_{(1)}} \qquad = \mathrm{Y_2} + \mathrm{Y_1}$$

Let $\overline{X} = \begin{bmatrix} \mathrm{X_{(1)}} \\ \mathrm{X_{(2)}} \end{bmatrix} = \begin{bmatrix} \mathrm{Y_1} \\ \mathrm{Y_2+Y_1} \end{bmatrix}$ and $\overline{Y} = \begin{bmatrix} \mathrm{Y_1} \\ \mathrm{Y_2} \end{bmatrix}$

$$\therefore \frac{d\overline{x}}{d\overline{y}} = \begin{bmatrix} \frac{\partial \mathrm{x_{(1)}}}{\partial \mathrm{y_1}} & \frac{\partial \mathrm{x_{(1)}}}{\partial \mathrm{y_2}} \\ \frac{\partial \mathrm{x_{(2)}}}{\partial \mathrm{y_1}} & \frac{\partial \mathrm{x_{(2)}}}{\partial \mathrm{y_2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Therefore, the Jacobian,

$$J = \left|\frac{d\overline{x}}{d\overline{y}}\right| = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

Assume the $\mathrm{X_i}$ have a distribution (3).
Therefore, the joint probability density function of the $\mathrm{X_{(i)}}$'s is[by using **section 3.3.0** (10)]

$$f_{X_{(1)},X_{(2)},}(x_1,\, x_2) = 2!\, f(x_1) f(x_2)$$

$$= 2! \frac{1}{\theta}\, e^{-\frac{1}{\theta}x_1} \frac{1}{\theta} e^{-\frac{1}{\theta}x_2}$$

$$= 2! \frac{1}{\theta^2}\, e^{-\frac{1}{\theta}(x_1+x_2)}$$

$$= 2! \frac{1}{\theta^2}\, e^{-\frac{1}{\theta}\sum_1^2 x_i}$$

Thus the joint distribution of $\mathrm{Y_i}$ is

$$f_{Y_1,Y_2,}(y_1,\, y_2) = f_{X_{(1)},X_{(2)},}(y_1,\, y_2) \left|\frac{d\overline{x}}{d\overline{y}}\right|$$

$$= 2!\frac{1}{\theta^2}\ e^{-\frac{1}{\theta}(y_1+y_2+y_1)}$$

$$= 2!\frac{1}{\theta^2}\ e^{-\frac{1}{\theta}(2y_1+y_2)} = \frac{2!}{\theta^2} e^{-\frac{1}{\theta}\sum_1^2(2-i+1)y_i}$$

$$= \left(\frac{2}{\theta}\ e^{-\frac{2y_1}{\theta}}\right)\left(\frac{1}{\theta}\ e^{-\frac{y_2}{\theta}}\right) \tag{4}$$

Let n=3;

(1) becomes $\mathrm{X}_{(1)}, \mathrm{X}_{(2)}, \mathrm{X}_{(3)}$

The corresponding $\mathrm{Y_i}$ variables in (2) will be will be

$\mathrm{Y}_1 = \mathrm{X}_{(1)} \quad \Rightarrow \mathrm{X}_{(1)} \quad = \mathrm{Y}_1$

$\mathrm{Y}_2 = \mathrm{X}_{(2)} - \mathrm{X}_{(1)} \quad \Rightarrow \mathrm{X}_{(2)} = \mathrm{Y}_2 + \mathrm{X}_{(1)} \quad = \mathrm{Y}_2+\mathrm{Y}_1$

$\mathrm{Y}_3 = \mathrm{X}_{(3)} - \mathrm{X}_{(2)} \quad \Rightarrow \mathrm{X}_{(3)} = \mathrm{Y}_3 + \mathrm{X}_{(2)} \quad = \mathrm{Y}_3 + \mathrm{Y}_2+\mathrm{Y}_1$

Let $\overline{X} = \begin{bmatrix} \mathrm{X}_{(1)} \\ \mathrm{X}_{(2)} \\ \mathrm{X}_{(3)} \end{bmatrix} = \begin{bmatrix} \mathrm{Y}_1 \\ \mathrm{Y}_2+\mathrm{Y}_1 \\ \mathrm{Y}_3 + \mathrm{Y}_2+\mathrm{Y}_1 \end{bmatrix}$ and $\overline{Y} = \begin{bmatrix} \mathrm{Y}_1 \\ \mathrm{Y}_2 \\ \mathrm{Y}_3 \end{bmatrix}$

$$\therefore \frac{d\overline{x}}{d\overline{y}} = \begin{bmatrix} \frac{\partial \mathrm{x}_{(1)}}{\partial \mathrm{y}_1} & \frac{\partial \mathrm{x}_{(1)}}{\partial \mathrm{y}_2} & \frac{\partial \mathrm{x}_{(1)}}{\partial \mathrm{y}_3} \\ \frac{\partial \mathrm{x}_{(2)}}{\partial \mathrm{y}_1} & \frac{\partial \mathrm{x}_{(2)}}{\partial \mathrm{y}_2} & \frac{\partial \mathrm{x}_{(2)}}{\partial \mathrm{y}_3} \\ \frac{\partial \mathrm{x}_{(3)}}{\partial \mathrm{y}_1} & \frac{\partial \mathrm{x}_{(3)}}{\partial \mathrm{y}_2} & \frac{\partial \mathrm{x}_{(3)}}{\partial \mathrm{y}_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Therefore, the Jacobian

$$J = \left|\frac{d\overline{x}}{d\overline{y}}\right| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1$$

Assuming the $\mathrm{X_i}$ have a distribution (3), the joint probability density function of the $\mathrm{X_{(i)}}'$s is

$$f_{X_{(1)},X_{(2)},X_{(3)},}(x_1,\ x_2, x_3) = 3!\, f(x_1)f(x_2)f(x_3)$$

$$= 3!\frac{1}{\theta}\ e^{-\frac{1}{\theta}x_1}\frac{1}{\theta}e^{-\frac{1}{\theta}x_2}\ \frac{1}{\theta}e^{-\frac{1}{\theta}x_3}$$

$$= 3!\frac{1}{\theta^2}\ e^{-\frac{1}{\theta}(x_1+x_2+x_3)}$$

$$= 3!\frac{1}{\theta^3}\ e^{-\frac{1}{\theta}\sum_1^3 x_i}$$

Thus the joint distribution of Y_i is

$$f_{Y_1,Y_2,Y_3,}(y_1,\ \ y_2,y_3) = f_{X_{(1)},X_{(2)},X_{(3)},}(y_1,\ \ y_2,y_3)\left|\frac{d\overline{x}}{d\overline{y}}\right|$$

$$= 3!\frac{1}{\theta^3}\ e^{-\frac{1}{\theta}(x_1+\ x_2+\ x_3)}$$

$$= 3!\frac{1}{\theta^3}\ e^{-\frac{1}{\theta}(y_1+\ y_2+y_1+y_3\ +\ y_2+y_1)}$$

$$= 3!\frac{1}{\theta^3}\ e^{-\frac{1}{\theta}(3y_1+\ 2y_2+y_3\ )} = \frac{3!}{\theta^3}e^{-\frac{1}{\theta}\sum_1^3(3-i+1)y_i}$$

$$= \left(\frac{3}{\theta}\ e^{-\frac{3y_1}{\theta}}\right)\left(\frac{2}{\theta}\ e^{-\frac{2y_2}{\theta}}\right)\left(\frac{1}{\theta}\ e^{-\frac{y_3}{\theta}}\right) \tag{5}$$

$$\therefore f_{Y_1,Y_2,\ .\ .\ .\ ,\ \ Y_n,}(y_1,\ \ y_2,.\ .\ .\ ,\qquad y_n) = f_{X_{(1)},X_{(2)},\ .\ .\ .\ ,\ \ X_{(n)},}(y_1,\ \ y_2,.\ .\ .\ ,y_n)\left|\frac{d\overline{x}}{d\overline{y}}\right|$$

$$= \left(\frac{n}{\theta}\ e^{-\frac{ny_1}{\theta}}\right)\left(\frac{n-1}{\theta}\ e^{-\frac{(n-1)y_2}{\theta}}\right)\ldots\left(\frac{3}{\theta}\ e^{-\frac{3y_{(n-2)}}{\theta}}\right)\left(\frac{2}{\theta}\ e^{-\frac{2y_{(n-1)}}{\theta}}\right)\left(\frac{1}{\theta}\ e^{-\frac{y_n}{\theta}}\right) \tag{6}$$

$$= \frac{n!}{\theta^n}\ e^{-\frac{\sum_1^n(n-i+1)y_i}{\theta}} \tag{7}$$

Since the joint pdf in (6) is a product of the individual pdfs of the Y_is, the Y_is are mutually independent.

Proof 2:(The distribution of the normalized spacings (n-i+1)Y_i is that of the population X)

$$Let\ \ Z_i = (n-i+1)Y_i \quad \therefore Y_i\ \ = \frac{Z_i}{n-i+1}$$

$$\therefore \frac{dy_i}{dz_i} = \frac{1}{n-i+1}$$

$$\therefore f_{Z_i}\ (z_i) = f_{Y_i}\ (z_i)\left|\frac{dy_i}{dz_i}\right|$$

From (4),(5) and (6),

$$\therefore f_{Y_i}\ (y_i) = \frac{n-i+1}{\theta}e^{-\frac{n-i+1}{\theta}y_i}$$

$$\therefore f_{Z_i}(z_i) = \frac{(n-i+1)}{\theta} e^{-\frac{(n-i+1)}{\theta}\frac{z_i}{(n-i+1)}} \quad \frac{1}{(n-i+1)}$$

$$= \frac{1}{\theta} e^{-\frac{z_i}{\theta}}$$

which is the pdf of the population X .

Remark: $Z_i = (n-i+1)Y_i$, where $Y_i = X_{(i)} - X_{(i-1)}$, $2 \leq i \leq n$, are i.i.d exponential random variables having the same distribution as the population X.

3.3.3 Characterization by k-th order statistics

Consider n independent random variables $X_1, X_2, \dots, X_n$ from an absolutely continuous distribution function $F(x)$.

Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the associated order statistics.

Also, let $Z_j = (n-j+1)\left(X_{(j)} - X_{(j-1)}\right)$ (1)

The $Z_j's$ are known as normalized spacings.

The following theorem characterizes an exponential distribution of pdf

$$f(x) = \begin{cases} \frac{1}{\sigma} e^{\frac{-x}{\sigma}}, & x > 0 \\ 0, & x \leq 0 \end{cases} \quad (2)$$

via the k-th order statistics [2].

Theorem 3.3.3

A necessary and sufficient condition that a non-negative r.v. X having absolutely continuous probability distribution $F(x)$ has the probability density function in (2) is that its kth-order statistics can be expressed as

$$X_{(k)} = \sum_{j=1}^{k} \frac{Z_j}{n-j+1} \quad (3)$$

for an integer k such that $1 \leq k \leq n$, where the $Z_j's$ are i.i.d. and have the probability distribution as X, $F(x)$.

Proof:

From **Theorem 3.3.2,** the normalized spacings $Z_j's$ are independent and identically distributed random variables having the same distribution as the population X, when X has the distribution in (2).

Expressing $X_{(j)}$ in terms of Z_j from (1), we have;

$$Z_1 = nX_{(1)} \qquad \Rightarrow X_{(1)} = \frac{Z_1}{n}$$

$$Z_2 = (n-1)\left(X_{(2)} - X_{(1)}\right)$$

$$\therefore X_{(2)} = \frac{Z_2}{n-1} + X_{(1)} \qquad \Rightarrow X_{(2)} = \frac{Z_2}{n-1} + \frac{Z_1}{n}$$

$$Z_3 = (n-2)\left(X_{(3)} - X_{(2)}\right)$$

$$\therefore X_{(3)} = \frac{Z_3}{n-2} + X_{(2)} \qquad \Rightarrow X_{(3)} = \frac{Z_3}{n-2} + \frac{Z_2}{n-1} + \frac{Z_1}{n}$$

$$Z_4 = (n-3)\left(X_{(4)} - X_{(3)}\right)$$

$$\therefore X_{(4)} = \frac{Z_4}{n-3} + X_{(3)} \qquad \Rightarrow X_{(4)} = \frac{Z_4}{n-3} + \frac{Z_3}{n-2} + \frac{Z_2}{n-1} + \frac{Z_1}{n}$$

$$\therefore X_{(k)} = \frac{Z_k}{n-k+1} + \frac{Z_{k-1}}{n-k+2} + \frac{Z_{k-2}}{n-k+3} + \ldots + \frac{Z_3}{n-2} + \frac{Z_2}{n-1} + \frac{Z_1}{n}$$

$$= \sum_{j=1}^{k} \frac{Z_j}{n-j+1},$$

by induction. The theorem is proved.

Remark: Theorem 3.3 implies that an order statistic from an exponential distribution can be represented by a weighted sum of i.i.d. exponential random variables.

3.3.4 Characterization by expected values of functions of order statistics

An exponential distribution with distribution function $F(x) = 1 - e^{-\lambda x}$ is characterized as a special case from the general class of distributions

$$F(x) = 1 - e^{-ah(x)}, a \neq 0, \qquad (1)$$

where $h(x)$ is a monotonic and differentiable function of x such that $h(x) \rightarrow 0 \; as \; x \rightarrow \alpha$ and $h(x)\{1 - F(x)\} \rightarrow 0 \; as \; x \rightarrow \beta$ [18]

Theorem 3.3.4

Let X be an absolutely continuous random variable with the distribution function F(x) and pdf f(x) in the interval (α, β) *where* α *and* β *may be finite or infinite, then for*

$$1 \le m < r < s \le n$$

$$E[h(X_{s:n}) - h(X_{r:n}) | X_{m:n} = x)] = \frac{1}{a}\sum_{j=r}^{s-1}\frac{1}{n-j} \tag{2}$$

iff X has the distribution in (1)

Proof.

Necessity.

Rearranging (1), we get

$$1 - F(x) = \frac{f(x)}{ah'(x)} \tag{3}$$

We need to prove that (1) implies (2).

Considering the case $1 \le r < s \le n$

Then

$$E[h(X_{s:n}) - h(X_{r:n}) | X_{r:n} = x)]$$
$$= \binom{n-r}{s-r-1}\frac{1}{[1-F(x)]^{n-r}}\int_x^\beta h'(y)[1 - F(y) - F(x)]^{s-r-1}\,[1 - F(y)]^{n-s+r}\,dy$$

Therefore, for

$$1 \le m < r < s \le n$$

$$E[h(X_{s:n}) - h(X_{r:n}) | X_{m:n} = x)]$$
$$= \sum_{j=r}^{s-1}\binom{n-m}{j-m}\frac{1}{[1-F(x)]^{n-m}}\int_x^\beta h'(y)[1 - F(y) - F(x)]^{j-m}\,[1 - F(y)]^{m-j}\,dy$$

$$= \frac{1}{a}\sum_{j=r}^{s-1}\frac{1}{n-j}\ ,$$

comparing with (3), proving the necessary part.

Sufficiency.

$$Let \qquad c = \frac{1}{a}\sum_{j=r}^{s-1}\frac{1}{n-j}$$

$Then\ E[h(X_{s:n}) - h(X_{r:n})|X_{m:n} = x)] = c$ implies

$$\frac{(n-m)!}{(s-m-1)!\,(n-s)!}\int_x^\beta h(y)[F(y)-F(x)]^{s-m-1}\,[1-F(y)]^{n-j}f(y)\,dy$$

$$-\frac{(n-m)!}{(r-m-1)!\,(n-r)!}\int_x^\beta h(y)[F(y)-F(x)]^{r-m-1}\,[1-F(y)]^{n-r}f(y)\,dy$$

$$= c[1-F(x)]^{n-m} \tag{3}$$

Differentiating (3) (r-m) times w.r.t. x, we get

$$\frac{(n-r)!}{(s-r-1)!\,(n-s)!}\int_x^\beta h(y)[F(y)-F(x)]^{s-r-1}\,[1-F(y)]^{n-s}f(y)\,dy$$

$$= \{h(x)+c\}[1-F(x)]^{n-r} \tag{4}$$

Integrating the LHS of (4) and simplifying, we obtain

$$\frac{(n-r)!}{(s-r-2)!\,(n-s+1)!}\int_x^\beta h(y)[F(y)-F(x)]^{s-r-2}\,[1-F(y)]^{n-s+1}f(y)\,dy$$

$$+\frac{(n-r)!}{(s-r-1)!\,(n-s+1)!}\int_x^\beta h'(y)[F(y)-F(x)]^{s-r-1}\,[1-F(y)]^{n-s+1}f(y)\,dy$$

$$= \{h(x)+c\}[1-F(x)]^{n-r} \tag{5}$$

From (4) and (5), it follows that

$$c_1 = \frac{1}{a}\sum_{j=r}^{s-2}\frac{1}{n-j}$$

$$\therefore \frac{[1-F(x)]^{n-r}}{a(n-s+1)!\,(n-s+1)!}$$

$$= \frac{(n-r)!}{(s-r-1)!\,(n-s+1)!} \int_x^{\beta} h'(y)[F(y)-F(x)]^{s-r-1}\,[1-F(y)]^{n-s+1}\,dy$$

Differentiating (s-r) times both sides w.r.t. x, we get

$$h'(x)[1-F(x)] = \frac{f(x)}{a}$$

Which is (3). Hence the theorem.

By substituting λ for a and x for $h(x)$, it can be seen that (3) characterizes the exponential distribution, $F(x) = 1 - e^{-\lambda x}$

3.3.5 Characterization by linear regression of functions of order statistics

In the following theorem, the exponential distribution is characterized by the linear regression between functions of order statistics.

Introduction. Let $X_1, \dots, X_n$ be $n \geq 2$ be independent observations on a random variable X having a two parameter distribution defined by [26]

$$F(x) = 1 - e^{-\frac{1}{\sigma}(x-\theta)},\ x > \theta > -\infty,\ \sigma > 0 \quad (1)$$

$$= 0, elsewhere$$

Also, let $Y_1, Y_2, \dots, Y_n$ be the corresponding order statistics from the sample of size n from this distribution.

Let

$U_k = Y_k - Y_{k-1},\ Y_0 = 0$, the corresponding spacings and $Z = \frac{1}{n-1}\sum_{i=2}^{n}(Y_{i:n} - X_{1:n})$

$$\begin{cases} Z_k = \frac{1}{n-k}\sum_{i=k+1}^{n}(Y_i - Y_k), & \text{for } k = 1, \dots, n-1 \\ W_k = \frac{1}{k-1}\sum_{i=1}^{k-1}(Y_k - Y_i), & \text{for } k = 2, \dots, k-1 \end{cases} \quad (2)$$

Theorem 3.3.5

If the distribution function F of X in (1) is continuous with finite first moment, then for some $1 \leq k \leq n-1$,

$E[Z_k|Y_k = y] = \beta$ *a.e. (dF)* (3) *where β is a constant, iff F is given by (1) with $\sigma = \beta > 0$ and some $\theta > -\infty$*

Proof: Fix $1 \leq k \leq n-1$

Let $s \in \Phi$ where Φ is a subset of size k of $\{1, . . . , n\}$

Let $\Phi_0 = \{1, \dots, k\}$

Let $A_{\Phi,s} = \{(X_1, \dots, X_n): X_j < y \ \ for\, j \in \Phi\ and\ \ j \neq s;\ X_s = y\ and X_j > y$

$for\ j \notin \Phi\ \}$ (4)

For each fixed *s*, there are $\binom{n-1}{k-1}$ disjoint $A_{\Phi,s}$'s and $\bigcup_{\Phi,s} A_{\Phi,s} = \{Y_k = y\}\ a.e.$

$P(.|Y_k = y)$. On $A_{\Phi,s}$, it can be written thus:

$\sum_{j=k+1}^{n}(Y_j - Y_k) = \sum_{j\notin\Phi}(X_j - y)$.

Therefore, since

$$P\left(A_{\Phi,s}\middle|Y_k = y\right) = \frac{1}{\binom{n-1}{k-1}n}$$

Is constant with respect to $\Phi\ and\ s$, we can write

$$E[Z_k|Y_k = y] = \frac{1}{\mathrm{n-k}}\mathrm{E}\left\{\sum_{\mathrm{i=k+1}}^{\mathrm{n}}(\mathrm{Y_i} - \mathrm{Y_k})\ |Y_k = y\right\}$$

$$= \frac{1}{\mathrm{n-k}}\sum\nolimits_{\Phi,s}\mathrm{E}\left\{\sum_{\mathrm{j}j\notin\Phi}^{\mathrm{n}}(\mathrm{X_j} - \mathrm{y})\ |A_{\Phi,s}\right\}\ P\left(A_{\Phi,s}\middle|Y_k = y\right)$$

$$= \frac{1}{\mathrm{n-k}}\sum_{\mathrm{j=k+1}}^{\mathrm{n}}\mathrm{E}\{(\mathrm{X_j} - \mathrm{y})\}\ |A_{\Phi_0,k})$$

$$= \mathrm{E}(\mathrm{X_n} - \mathrm{y}|A_{\Phi_0,k})$$

$$= \mathrm{E}(\mathrm{X_n} - \mathrm{y}|\mathrm{X_n} > y) \qquad (5)$$

Thus

$$E[Z_k|Y_k = y] = \int_{\mathrm{y}}^{\infty}(\mathrm{w} - \mathrm{y})\frac{\mathrm{F(dw)}}{1 - \mathrm{F(y)}} = \int_{\mathrm{y}}^{\infty}\frac{\mathrm{wF(dw)}}{1 - \mathrm{F(y)}} - \mathrm{y} \qquad (6)$$

From (6), it follows that $E[Z_k|Y_k = y] = \beta$ is equivalent to

$$\beta[1-F(y)] = \int_y^\infty wF(dw) \qquad \text{for almost all } y(dF) \tag{7}$$

From 7, we can write $\int_y^\infty wF(dw) = \int_y^\infty \int_0^w dt\, F(dw)$

$$\beta[1-F(y)] = \int_y^\infty \big(1-F(t)\big)dt \qquad \text{for almost all } y(dF) \tag{8}$$

Let $H(y) = \int_y^\infty \big(1-F(t)\big)dt$ (9)

Then $H(y)$ is a non-negative differentiable function with

$$H'(y) = -\big(1-F(y)\big) \tag{10}$$

Combining (8), (9) and (10),

$$\frac{H'(y)}{H(y)} = \frac{d[\ln H(y)]}{dy} = -\frac{\big(1-F(y)\big)}{\beta(1-F(y))} = -\frac{1}{\beta} \tag{11}$$

Solving (11),

$$\ln H(y) = \ln\{\beta(1-F(y)) = -\frac{1}{\beta}y + c,$$

Where c is a constant of integration.

$\therefore F(y) = 1 - e^{-\frac{1}{\beta}y+c}$

But $c = -\frac{1}{\beta}\theta$

Thus $F(y) = 1 - e^{-\frac{1}{\beta}(y-\theta)}$ which is (1)

3.4 Characterization by conditional independence

In the following theorem the exponential distribution has been characterized by the independence between min(X, Y) and Y-X with only Y as the exponential random variable and X a non-lattice random variable with a positive real line support [21].

Theorem 3.4 *Let X and Y be independent non-negative r.v.'s. Let Y be an r.v. with a continuous reliability function $R_y(y)$(= Pr(Y>y)). Let X be an r.v. which has a non-lattice distribution with a cumulative distribution function $F_x(x)$ such that $F_x(0)=0$, and $F_x(x) > 0$, for $x > 0$. Then Y - X and V =min(X, Y) (=X) are independent in their joint distribution given X <Y, if and only if Y has a negative exponential distribution.*

Proof.

If Y has a negative exponential distribution with $f_y(y) = \lambda e^{-\lambda \mathrm{y}}$ $, for\ y \geq 0$, where λ is a positive constant then

$$P(Y - X) > u|X < Y) = \frac{\int_0^\infty e^{-\lambda\,(\mathrm{x+u})}\ dF_x(x)}{\int_0^\infty e^{-\lambda\,\mathrm{x}}\ dF_x(x)}$$

$$= e^{-\lambda\,\mathrm{u}}\ \ for\ u \geq 0$$

$$P(X \leq v|X < Y) = \frac{\int_0^v e^{-\lambda\,\mathrm{x}}\ dF_x(x)}{P(X < Y)}\ , for\ v \geq 0$$

And

$$P(Y - X > u, X \leq v|X < Y) = \frac{\int_0^v e^{-\lambda\,(\mathrm{x+u})}\ dF_x(x)}{P(X < Y)}, for\ u \geq 0, v \geq 0$$

Thus the two random variables are independent.

If the two random variables are independent then

$P(Y - X > u, X \leq v|X < Y) = P(Y - X > u|X < Y)P(\,X \leq v|X < Y)\ for\ u \geq 0, v \geq 0.$

As $P(X < Y)$ is positive, then,

$$\int_0^v R_Y(x+u)dF_x(x) = P(Y - X > u|X < Y)\int_0^v R_Y(x)dF_x(x)\ , for\ u \geq 0, v \geq 0$$

Then there exist θ and $\theta\,'$ between 0 and v such that

$R_Y(u+\theta)(F_x(v) - F_x(0) = P(Y - X > u|X < Y)R_Y(\theta\,')\left(F_x(v) - F_x(0)\right),\ for\ u \geq 0, v \geq 0.$

As $F_x(v) - F_x(0) > 0\ for\ all\ v > 0$

Then $R_Y(u+\theta) = P(Y - X > u|X < Y)R_Y(\theta\,')\ \ for\ all\ u \geq 0, v \geq 0$

Letting $v \to 0$ gives

$$R_Y(u) = P(Y - X > u|X < Y), for\ u \geq 0$$

Hence, by applying Theorem 4 of Shimizu (1978), Y has a negative exponential distribution.[1] (APPENDIX 1)

3.5 Characterization by relevation-type equations

3.5.0 Introduction. The convolution of two distribution functions F and G is given by

$$(F * G)(x) = \int_{-\infty}^{\infty} F(x-u)G(u)du, \qquad -\infty < x < \infty$$

If the support of *F* and *G* are contained in *[0, ∞),* then

$$(F * G)(x) = \int_{0}^{x} F(x-u)G(u)du, \qquad x \geq 0$$

This is the distribution of the time to failure of the second of two components when the second component with life *G* is placed in service after the failure of the first component with life distribution *F*. When the second component with life distribution *G* is of the same

age as the first, the survival function of the time to system failure is called relevation of the survival function $\bar{F}(t) = 1 - F(t)\ with\ \ \bar{G}(t) = 1 - G(t)$. [19]

It is denoted by $\bar{F}\#\,\bar{G}(t)$

$$\bar{F}\#\,\bar{G}(t) = P(system\ survivves\ beyond\ time\ t)$$

$$= P(first\ component\ with\ survival\ function\ \bar{F}\ survives\ beyond\ time\ t)$$

$$+$$

$$P\begin{pmatrix} first\ component\ with\ survival\ function\ \bar{F}\ fails\ sometime\ before\ t\ and\ the\ second \\ component\ with\ survival\ function\ \bar{G}\ survives\ beyond\ time\ t\ given\ that\ it\ has \\ survived\ \ up\ to\ time\ to\ failure\ of\ the\ first\ component \end{pmatrix}(t)$$

$$= \bar{F}(t) + \int_0^t P\begin{bmatrix} second\ component\ survives\ beyond\ time\ t\ given\ that\ it\ has \\ survived\ beyond\ time\ u\ when\ the\ first\ component \\ failed \end{bmatrix} dF(u)$$

$$= \bar{F}(t) + \int_0^t \frac{P(life\ of\ second\ component > \ t)}{P(life\ of\ second\ component > \ u)} dF(u)$$

$$= \bar{F}(t) + \int_0^t \frac{\bar{G}(t)}{\bar{G}(u)} dF(u)$$

$$= \bar{F}(t) - \int_0^t \frac{\bar{G}(t)}{\bar{G}(u)} d\bar{F}(u)$$

$\bar{F}\#\,\bar{G}(t) = \overline{(F * G)(x)}$ iff

$$\int_0^t \frac{\bar{G}(t)}{\bar{G}(u)} d\bar{F}(u) = \int_0^t \bar{G}(t-u) d\bar{F}(u), \quad t \geq 0 \tag{1}$$

3.5.1 Characterization by the relevation transform

The following theorem characterizes the exponential distribution by relevation-type equations

Theorem 3.5.1

Suppose $\bar{F}$ and $\bar{G}$ are continuous survival functions and $\bar{G}'_+(0)$ exists. Further, suppose that for any $x > 0$, $\bar{F}$ has a point of increase in $(0, x)$. If $\bar{G}$ satisfies

$$\int_0^x \bar{G}(x-t) d\bar{F}(t) = \int_0^x \frac{\bar{G}(x)}{\bar{G}(t)} d\bar{F}(t), \tag{2}$$

$$for\ all\ x\ , where\ \ \bar{G}(x) \neq 0$$

then $\bar{G}$ is exponential, i.e. $\bar{G} = e^{-\alpha x},\ x \geq 0,\ for\ some\ \alpha \geq 0$

Proof:

Let $c = \sup\{x: \bar{G}(x) > 0\}$. Let h be the non-negative function such that *h(0)=0* and $e^{-hx} = \bar{G}(x),\ x \in I = [0, c)$. Then $h'_+(0)$ exists by hypothesis and

$$\int_0^x \{e^{-h(x-t)} - e^{-h(x)+h(t)}\} d\bar{F}(t) = 0, \qquad x \in I$$

by (2).

Since $g(t) = e^{-h(x-t)} - e^{-h(x)+h(t)},\ 0 \leq t \leq x$ and $l(t) = \bar{F}(t)$ satisfy the conditions in **Proposition 2.2** in [19], there exists $0 \leq \xi \leq x$, such that $g(\xi) = 0$ or

equivalently, $h(x) - h(x-\xi) - h(\xi) = 0$. This equation holds for all $x \in I$, by

Proposition 2.1 by LAU, K. S., PRAKASA RAO, B. L. S. (1990) [19] [004]. (APPENDIX 4)

Therefore, $h(x) = \alpha x \quad where\ \alpha = h'_+(0) \geq 0$. Hence

$\bar{G} = e^{-\alpha x}, x \in I$. Since by assumption, $\bar{G}$ is continuous, $I = [0, \infty)$.

3.6 Characterization via the residual lifetime

3.6.0 Introduction

In the following section the exponential distribution will be characterized via the residual lifetime through the application of the integrated Cauchy functional equation in (1) (Tavangar. M. & Asadi, M.(2010) [25])

We consider the functional equation

$F(x) = F(xy) + F(xQ(y)),\ x,\ xQ(y) \in [0, \vartheta),\ y \in [0, 1],$ (1)

where F and Q satisfy certain conditions.

Let X be a lifetime (non-negative) random variable with cumulative distribution function (cdf) F , and survival function $S = 1 - F$. The random variable X is said to have exponential distribution with mean λ if

$S(x) = e^{-x/\lambda},\ x \geq 0,\ \lambda > 0,$ (2)

3.6.1 Characterization by use Cauchy functional equation

The following theorem characterizes the exponential distribution via residual life using the Cauchy functional equation

Theorem 3.6.1

*Let F be any cdf with support **R+**, and S = 1 – F, the survival function .*

*Assume that Q : **R+** → **R+**. The functional equation*

$S(x) = S(x + y) + S(x + Q(y)),\ x, y \in [0, \infty),$ (3)

holds if and only if F is an exponential distribution with mean λ, for some λ > 0, and $Q(y) = -\lambda \log(1 - e^{-y/\lambda}),\ y > 0$.

Proof.

Let the cdf G be $G(z) = S(-\log z),\ z \in [0, 1)$, where S is the survival function defined in the theorem. Let $u = e^{-x}$, $v = e^{-y}$, and $Q*(v) = \exp\{-Q(-\log v)\}$.

Thus $Q*: [0, 1] \to R+$.

Eq. (3) implies that $G(u) = G(uv) + G(uQ*(v)),\ u,\ uQ*(v) \in [0, 1),\ v \in [0, 1]$.

That is, the pair of functions $(G, Q*)$ satisfies Eq. (1) with $\vartheta = 1$.

Therefore, using Theorem 2.1 by Tavangar. M. & Asadi, M.(2010) [25] ,(APPENDIX 5) we have $G(x) = x^{\alpha}$, *x ∈[0, 1)* and $Q^{*}(y) = (1 - y^{\alpha})^{1/\alpha}$, *y ∈[0, 1],* for some constant α > 0. This means that F is an exponential cdf with mean λ = 1/α, and Q is as stated in the theorem. The proof is complete.

3.6.2 Another application of the Cauchy functional equation

Theorem 3.6.2 *Let X be a non-negative random variable with the survival function S.*

Suppose that Q : ***R+*** *→* ***R+*** *is a strictly decreasing function. Let also*

$X_t = [X - t \mid X > t]$ *be the residual life random variable. Then*

$X_t \overset{d}{=} Q(X_t)$*, for almost all t ∈**R+** (with respect to Lebesgue measure) with S(t) > 0, if and only if S is the survival function of an exponential random variable with mean λ, for some constant λ > 0, and* $Q(y) = -\lambda \log(1 - e^{-\frac{y}{\lambda}})$*, y > 0.*

Proof. Since S(t) > 0, the conditional random variable X_t is defined. Also, since every monotone function is measurable, $Q(X_t\ )$ is a random variable. Thus,

$$\mathrm{P}\,[Q(X_t \geq \mathrm{x}] = 1 - \mathrm{P}\,[X_t > Q^{-1}(\mathrm{x})]$$
$$= 1 - \frac{\mathrm{P}\,[\mathrm{X} > t + Q^{-1}(\mathrm{x})]}{\mathrm{P}\,[\mathrm{X} > t]}$$

Let *U* be a random variable with uniform *U (0, 1)* distribution. Using Laplace trasforms, we have X $\overset{d}{=} F^{\leftarrow}(u)$, where $F^{\leftarrow}(.)$ is the quantile function. It follows from Lemma 3.1by Tavangar. M. & Asadi, M.(2010)[25] that

$$\mathrm{P}\,[Q(X_t \geq \mathrm{x}] = 1 - \frac{\mathrm{P}\,[\mathrm{U} > \mathrm{F}(\mathrm{t} + Q^{-1}(\mathrm{x}))]}{\mathrm{P}\,[\mathrm{U} > \mathrm{F}(\mathrm{t})]}$$

$$= 1 - \frac{\mathrm{S}(\mathrm{t} + Q^{-1}(\mathrm{x}))}{\mathrm{S}(\mathrm{t})}$$

and

$$\mathrm{P}\,[(X_t \geq \mathrm{x}] = \frac{\mathrm{P}\,[X \geq \mathrm{t} + \mathrm{x}]}{\mathrm{P}\,[X > t]}$$

$$= \frac{1}{\mathrm{S(t)}}\{\mathrm{S(t+x)} + \mathrm{F(t+x)} - \mathrm{F}\big((\mathrm{t+x})-\big)\}$$

We prove that

$$\mathrm{P}\,[X_t \geq \mathrm{x}] = \frac{\mathrm{S}\,(\mathrm{t}+\mathrm{x})}{\mathrm{S}\,(\mathrm{t})} \qquad (1)$$

Let D = {x ∈ **R+** | F has jump at x} denote the set of discontinuity points of F which is known to be countable. If D is an empty set, then the result is trivial. Therefore, letD = $\{d_1, d_2, \ldots\}$.

Let the sets

$E_i's$, i = 1, 2, . . ., be defined as E_i = {(t, x) ∈ **R+** × **R+** | t + x =d_i} = {(d_i- x, x) |x ∈ [0, d_i]}.

It can be observed that the E_i *'s* are measurable sets of planar Lebesgue measure zero which, in turn, implies that D is a set of planar Lebesgue measure zero. Therefore, Eq. (1), and consequently the following equation hold for almost all pairs (t, x) ∈ R+ × R+ with respect to planar Lebesgue measure:

$$\mathrm{S(t)} = \mathrm{S(t + x)} + \mathrm{S(\,t\ } + Q^{-1}\,\mathrm{(x))}$$

3.7 Characterization by conditional expectations

3.7.0 Introduction. Characterizations under conditional expectations also encompass characterization cases under left- or right-truncated distributions.

3.7.1 Characterization as a special case of characterizations by conditional expectations

In the following theorem the exponential distribution has been characterized as a special case of the general theorem of characterization by conditional expectations [15].Consider the exponential distribution function in (1).

$$F(x) = 1 - e^{-x/b}, \qquad x > 0, \qquad b > 0 \tag{1}$$

Theorem 3.7.1

An absolutely continuous random variable X has cumulative distribution function

$$P(X \le x) = 1 - e^{-h(x)/h(b)}, \ \ for\ x \in [\alpha, \beta)$$

$$= 0, \qquad for\ x \notin [\alpha, \beta) \tag{2}$$

Where b is a constant and h *is a strictly increasing differentiable function from* $[\alpha, \beta)$ *on to* $[0, \infty)$ *if and only if*

$$E\{h(X)|X > y\} = h(y) + h(b), \qquad for\ y \in [\alpha, \beta)[\alpha, \beta) \tag{3}$$

Proof:

(3) may be written as

$$\int_y^\beta \frac{h(x)dF(x)}{1 - F(y)} = h(y) + h(b)$$

Or

$$\{1-F(y)\}\{h(y)+h(b)\} = \int_y^\beta h(x)dF(x)$$

$$= E\{h(T)\} - \int_\alpha^y h(x)dF(x) \qquad (4)$$

Using integration by parts in (4),

$$\{1-F(y)\}\{h(y)+h(b)\} = E\{h(T)\} - [h(x)F(x)]_\alpha^y + \int_\alpha^y h'(x)F(x)\ dx$$

$$= E\{h(T)\} - [h(x)F(x)]_\alpha^y + \int_\alpha^y h'(x)F(x)\ dx$$

$$= E\{h(T)\} - h(y)F(y) + h(\alpha)F(\alpha) + \int_\alpha^y h'(x)F(x)\ dx$$

$$\therefore \{1-F(y)\}\{h(y)+h(b)\} = E\{h(T)\} - h(y)F(y) + \int_\alpha^y h'(x)F(x)\ dx \qquad (5)$$

since $h(\alpha) = F(\alpha) = 0$.

Differentiating (5) w.r.t.y,

$$\{1-F(y)\}h'(y) - \{h(y)+h(b)\}\frac{dF(y)}{dy} = -h'(y)F(y) - h(y)\frac{dF(y)}{dy} + h'(y)F(y) \qquad (6)$$

$$\therefore h(b)\frac{dF(y)}{dy} = \{1-F(y)\}h'(y) \qquad (7)$$

$$\Leftrightarrow \frac{dF(y)}{\{1-F(y)\}} = \frac{h'(y)dy}{h(b)} \qquad (8)$$

$$\Leftrightarrow -ln\{1-F(y)\} = \frac{h(y)}{h(b)} + c \qquad (9)$$

$$\Leftrightarrow 1-F(y) = Ae^{-h(y)/h(b)} \qquad (10)$$

$$\therefore F(y) = 1 - Ae^{-h(y)/h(b)} \qquad (11)$$

where A is a constant.

Substituting the boundary condition $F(\alpha) = 0$ in (11) yields $A = 1$.

Hence, $$F(y) = 1 - e^{-h(y)/h(b)} \qquad (12)$$

By substituting $h(y) = y$ and $h(b) = b$ it can be seen that (3) characterizes the exponential distribution in (1)

3.7.2 Characterization via conditional expectation of a function of a random variable on a fixed value

An exponential distribution with the following distribution function is characterized by the expectation of a random variable X being conditioned on some fixed value y$\in (0, \infty)$ [6]

$$f(x) = \lambda e^{-\lambda x}, \qquad x > 0, \qquad \lambda > 0 \tag{1}$$

Theorem 3.7.2

Let a<b be extended real numbers and g and h be real functions defined on (a, b). Assume g is continuous and h$\neq 0$, $\forall y \geq 0$, Then there exists an absolutely continuous random variable X with support C_X=(a, b) such that

$E[\, g(X)|X \leq y]$ is finite $\forall y \in C_X$, and

$$E[\, g(X)|X \leq y] = h(y)\frac{f(y)}{F(y)} \quad , \quad \forall\, y \in C_X \tag{2}$$

iff, for any $k \in (a, b)$ the following conditions hold:

(i) $\int_k^y \frac{g(u)}{h(u)} du$ *is finite*

(ii) $\int_a^b \exp\left\{\frac{g(u)}{h(u)} du\right\} / |h(y)| \; dy < \infty$

(iii) $lim_{y\to a} \int_k^y \frac{g(u)}{h(u)} du = -\infty$

Also, the p.d.f. of the random variable X which satisfies (2) with C_X=(a, b)

$$f(y) = \frac{1}{\alpha_k |h(y)|} \exp\{\int_k^y \frac{g(u)}{h(u)} du\}, \; \forall \; a < y < b$$

where $\alpha_k = \int_a^b \exp\left\{\frac{g(u)}{h(u)} du\right\} / |h(y)| \; dy.$

Proof of the theorem is provided by Chang, T.(2001) [6].[006](See APPENDIX 6)

We now show that the exponential distribution in (1) is characterized if

$$g(X)=X,\; h(y) = \frac{-\lambda y - 1 + e^{\lambda y}}{\lambda^2},\; C_X=(0,\infty),\; \lambda > 0 \tag{3}$$

If X is an exponential random variable,

$$E[\,X|X \le y] = \frac{\int_0^y xf(x)\,dx}{P(X \le y)}$$

$$= \frac{\int_0^y x\lambda e^{-\lambda x}\,dx}{F_X(y)}$$

(integrating by parts)

$$= \frac{-ye^{-\lambda y} - \frac{1}{\lambda}\left[e^{-\lambda x}\right]_0^y}{F_X(y)}$$

$$= \frac{-ye^{-\lambda y} - \frac{1}{\lambda}\left[e^{-\lambda y} - 1\right]}{F_X(y)}$$

$$= \frac{-\lambda y - 1 + e^{\lambda y}}{\lambda^2}\,\frac{e^{-\lambda y}}{F_X(y)}$$

$$= \frac{-\lambda y - 1 + e^{\lambda y}}{\lambda^2}\,\frac{f_X(y)}{F_X(y)}$$

Since conditions (i)-(iii) in (2) are satisfied by h and g, the theorem is proved.

3.7.3 Characterization via conditional expectation of moments on a fixed value

In the following theorem, the exponential distribution is characterized by conditional expectations of moments about some fixed value $z > 0$.

Theorem 3.7.3

Let $X \ge 0$ be a random variable with distribution function F(x). Assume that $E(X^k)$ is finite where $k \ge 2$ is a given integer. If

$E[(X-z)^k \mid X \ge z\,] = E(X^k)$ for all $z \ge 0$,

then $F(x) = 1-e^{-bx}$, $x \ge 0$, and $b > 0$.

Proof:

Let m = $E(x^k)$. Since

$$E[(X-z)^k\ |x \ge z] \ = \int_z^{+\infty} (y-z)^k\ dF(y)/[1-F(z)]$$

$$\int_z^{+\infty} (y-z)^{k-1} G(y) d(y) = \left[\frac{m}{k}\right] G(z)\ , \quad z \ge 0 \qquad (1)$$

Where G(u)=1-F(u). Denoting the LHS of (1) by H(z), then repeated differentiation of (1) gives

$$\frac{(-1)^m}{k!} H^k(z) = H(z) \qquad (2)$$

Manipulation as shown by Galambos, J. & Kotz, S. (1978) [12](theorem 2.3.2)[007] (APPENDIX 7) results in

$$H(z) = c_1 e^{-bz}$$

which leads to

$$F(z) = 1 - G(z) = 1 - (kc_1/m)e^{-bz}, \ \ b > 0, \ \ z \geq 0$$

But since m = E(x^k), (1) together with z = 0 and lemma 1.2.1 by Galambos, J. & Kotz, S. (1978) [12], imply that (APPENDIX 8)

G(0+)= 1 and F(0+) = 0 which leads to(kc_1/m) = 1.

This completes the proof.

3.7.4 Characterization by trunctated distributions

In the following theorem, the exponential distribution with the distribution function

$$F(x) = 1 - e^{-x/b}, \qquad x > 0, \qquad b > 0 \qquad (1)$$

is characterized by means of the distributions truncated from below at various points [7].

Theorem 3.7.4

A non-negative random variable X with finite expectation is exponentially distributed iff, for some constant $\alpha > 0$,

$$E[(X-s)^+]E[(X-t)^+] = \alpha E[(X-s-t)^+] \qquad (2)$$

$for \ \ all\ s, t$ belonging to a dense subset of $\Re^+$, where $(X-u)^+ = sup\{X-u, 0\}$, denotes the positive part of $X-u$, $\ and\ u \geq 0$.

Proof:

Using the Lebesgue dominated convergence theorem, the function

$u \to E[(X-s)^+], u \geq 0$ is a continuous function.

Thus , (2) holds for all $s, t \ \geq 0$.

(2) implies $E[T] = \alpha$

Let (2) hold. Then $f(s) = E[(X-s)^{+}]/\alpha$

satisfies the Cauchy equation

$$f(t)f(s) = f(s+t), \qquad s,t \geq 0 \tag{3}$$

Whose solution is

$f(s) = e^{-\lambda s}$ $for\ all\ s \geq 0$ and some constant λ, since the function $f(s)$ is continuous $for\ all\ s \geq 0$.

Thus

$$E[(X-s)^{+}] = \alpha e^{-\lambda s} \quad for\ \ s \geq 0 \tag{4}$$

Now, $\lambda > 0$ since $E[(X-s)^{+}] \to 0\ \ as\ \ s \to \infty$

Let $R(t) = P(X > t),\ \ t \geq 0$ (5)

Therefore, (5) is Lebesgue integrable and

$$E[(X-s)^{+}] = \int_{s}^{\infty} R(t)dt,\ \ s \geq 0. \tag{6}$$

Hence, $\int_{s}^{\infty} R(t)dt = \alpha e^{-\lambda s}$, $s \geq 0.$ (7)

And differentiating (7) w.r.t. s, gives

$$R(s) = \alpha\lambda e^{-\lambda s} \tag{8}$$

for almost all s with respect to Lebesgue measure.

(8) holds for $for\ \ s \geq 0$, since both sides of (8) are continuous.

Since $R(0) = P(X > 0) = 1$,

$$\alpha = \frac{1}{\lambda}$$

And $P(X > s) = e^{-s/\alpha}$ which proves the theorem.

3.8 Characterization based on record values

3.8.0 Introduction.

Let $X_{1,} X_{2}$, ... be a sequence of independent random variables with a common distribution function $F(x)$, and let Y_k = max $\{X_1, X_2, ..., X_k\}$. Then Xj, is a record value of the sequence if $Y_j > Y_{j-1}$. By definition X_1 is a record value. The sequence of indices at which records occur is defined by $L_0 = 1, L_n = \min\{j \mid j > L_{n-1}, X_j > X_{t\,n-1}\}$ for n = 1, 2,... Also, let $R_j = X_{t_1}$. The sequence $R_j, j = 0,1,...$ ... is called the sequence of (upper) records. Let $F(x)$ be a continuous function.

3.8.1 Characterization by minimum and spacing between two upper record values

In the following theorem, an exponential distribution of the form

$$F(x) = 1 - e^{-\lambda(x-a)}, \quad \lambda > 0, \qquad x \ge a$$

$$= 0, \qquad otherwise \tag{1}$$

is characterized by the independence of $Rj - Ri$ and R_i for some fixed i and j with $j > i \ge 0$ among the continuous distributions [8].

Theorem 3.8.1

Let R_o, R_1, . . . be a record sequence coming from a continuous distribution F(x). Then R_j and $R_j - R_i$ with $0 \le i < j$ arbitrary but fixed, are independent if and only if F(x) is the exponential distribution (1)

Proof.

When F(x) is the exponential distribution, using Tata's results, we arrive at the conclusion that $R_j - R_i$ and R_i are independent and this for $0 \le i < j$

Conversely, let R_i and R_j -R_i be independent.

Then the conditional probability element of R_j given $R_i = x_i$ is given by

$$dF_{R_j}(x_j | R_i = x_i) = \left\{-\log\frac{P(x_j)}{P(x_i)}\right\}^{j-i-1} \{(j-i-1)!\,P(x_i)\}^{-1} dF(x_i) \tag{2}$$

Thus, the conditional probability element of $D = R_j - R_i | R_i = x_i$

is a function of the values of D only.

Thus from (2) we get:

$$\left\{-\log\frac{P(x_i + d)}{P(x_i)}\right\}^{j-i-1} \{P(x_i)\}^{-1} dF(x_i + d) = w(d)\ \ for\ d \ge 0. \tag{3}$$

From (3) it can be observed that $x_i \ge a$ for some constant *a* since the RHS is fixed while the LHS tends to 0 as $x_i \to -\infty$.

Integrating (3) with respect to d from 0 to a value γ and setting

$$\frac{P(x_i + d)}{P(x_i)} = z$$

The equation becomes $\int_1^{\frac{P(x_i+\gamma)}{P(x_i)}} (log z)^{j-i-1} dz = W(\gamma),\ for\ all\ \gamma \geq 0, \quad x_i \geq 0$

Thus, $\frac{P(x_i+d)}{P(x_i)} = b(\gamma)\ , \quad for\ all\ \gamma \geq 0, x_i \geq 0$

which is the lack of memory property of the exponential distribution.

1This concludes the proof of the theorem.

3.8.2 Conditioning on a backward difference

In the following theorem, the exponential distribution is characterized by the expectation of a function of a backward spacing conditioned on a certain record value R_j[17]

Theorem 3.7.2

Assume $F(x)$ has density $f(x)$. Let G be a non-decreasing function such that for every $x > 0, G$ has a point of increase in $(0, x)$. Assume for some fixed interger $j \geq 1,$

$$E(G((R_j - R_{j-1}| \ R_j = x) = E(G((R_0)| \ R_j = x), \forall x > 0; \qquad (1)$$

Then X_1 has an exponential distribution.

Proof.

(1) implies

$$\int_0^x G(x-y) f_{Rj-1}| \ R_{j=x}(y)dy = \int_0^x G(y) fR_0| \ R_j = x(y)dy, \qquad (2)$$

Or, by equations (1.2) and (1.3) in Huang, W.J. & Li, S.H. A.C.(1993) [17],[9]

(See APPENDIX 9)

$$\int_0^x G(x-y) fR^{j-1}(y) r(y)dy = \int_0^x G(y) R(x) - R(y))^{j-1} r(y)dy \qquad (3)$$

Integration by parts implies $\int_0^x (R^j(x-y) - (R(x) - R(y))^j) dG(y) = 0, \forall x > 0.$. Hence, by Proportions 2.1 and 2.2 of Lau and Rao (1990),[4] (APPENDIX 4) $R(x) = \alpha x$, where $\alpha = r(0+)$. This in turn implies X_1 is exponentially distributed

3.8.3 Conditioning on a forward difference

In the following theorem, the exponential distribution is characterized by the expectation of the spacing between two adjacent record values after a certain record value R_j[17]

Theorem 3.8.3 *Assume that $F(x)$ has density function $f(x)$ and $F(x) > 0$ for $x > 0$ Let G be a non-decreasing function having non-lattice support on $x > 0$ with $G(0) = 0$ and $E(G(X_1)) < \infty$. If, for some fixed non-negative integers j and k ,*

$$E((R_{j+k+1} - R_{j+k})| \ R_j = x) = c \qquad (1)$$

for every $x > 0$, *where* $c > 0$ *is a constant, and if for some* $\xi > 0$ *((2.13) Huang, W.J. & Li, S.H. A.C.(1993) [17])*[10] (APPENDIX 10) *holds, then c=*$E(G(X_1))$ *and* X_1 *is exponentially distributed.*

Proof. Using equation (1.5) by Huang, W.J. & Li, S.H. A.C.(1993) [17] ,(APPENDIX 11) we have

$$E((R_{j+k+1} - R_{j+k})| \; R_j = x)$$

$$= \int_0^\infty P(R_{j+k+1} - R_{j+k} > y)| \; R_j = x) dG(y)$$

$$= \int_0^\infty \int_y^\infty \int_x^\infty f_{Ri+k+1,Rj+k| \; Rj=x}(z,w,w) dwdzdG(y)$$

$$= \int_0^\infty \int_x^\infty \; G(z) \frac{(R(w-R(x))^{k-1} r(w) f(w-z)}{\Gamma(k)\overline{F}(x)} dwdz$$

$$= \frac{1}{\Gamma(k)\overline{F}(x)} \int_x^\infty \; (R(w)-R(x))^{k-1} r(w) \int_0^\infty G(z) f(w+z) dzdw$$

$$= \frac{1}{\Gamma(k)\overline{F}(x)} \int_x^\infty \; (R(w)-R(x))^{k-1} r(w) \int_0^\infty \overline{F}(z) dG(z+w) dw. \tag{2}$$

Since $E(G(X_1)) < \infty$, we have

$$0 \le \lim_{z\to\infty} G(z-w)\overline{F}(z) \le \lim_{z\to\infty} G(z)\overline{F}(z) \le \lim_{z\to\infty} \int_z^\infty G(x) dF(x) = 0.$$

Using this and integrating by parts we have;

$$\int_0^\infty G(z) f(w+z) dz = -\int_w^\infty \; G(z-w) d\overline{F}(z) = \int_w^\infty \; \overline{F}(z) dG(z-w). \tag{3}$$

Thus, the last equation of (2) holds. (1) implies

$$\int_x^\infty R(w)-R(x))^{k-1} r(w) \int_w^\infty \; \overline{F}(z) dG(z-w) dw = c\Gamma(k)\overline{F}(x). \tag{4}$$

Differentiating both sides of (4) k times, with respect to x, we obtain

$$\int_x^\infty \overline{F}(z+x) dG(z) = c\,\bar{F}(x). \tag{5}$$

The solution of (5) is $F(x) = 1 - e^{-\lambda x}, x > 0,$ where λ is the positive number defined by $\int_0^\infty \; e^{-\lambda x} dG(x) = c$. By letting $x \to 0,$ in (5) we obtain $E(G(X_1)).$ This completes the proof.

3.8.4 Characterization by independence between functions of record values

Theorem 3.7.4 below characterizes the exponential distribution with the following results being special cases[14]:

i) The independence of $R_{j+1} - R_j$ and R_j characterizes the exponential distribution. Srivastava (1978). Ahsanullah (1979) and Pfeifer (1982).

ii) $E(R_{j+1} - R_j| \; R_j)$ is independent of R_j characterizes the exponential distribution, Srivastava (1978). Ahsanullah (1978) and Nagaraja(1977).

iii) $\text{Var}(R_{j+1} - R_j | \ R_j)$ is independent of R_j characterizes the exponential distribution (Ahsanullah (1981b)).

Theorem 3.8.4

$E((R_{j+1} - R_j)^r \mid R_j = y) = c$ *(independent of* y *) for fixed* j *and* $r, r \geq 1$ *if* F *is exponential.*

Proof. Since the survival function of $R_{j+1} - R_j$ given $Rj = y$ is $S(x+y)/S(y),$ where $S(x) = 1 - F(x)$, we have,

$$c = \int_0^x ru^{r-1} \frac{S(u-y)}{S(y)} du$$

or

$$cS(y) = \int_0^x r(x-y)^{r-1} S(x) dx \qquad (1)$$

Since $\int_0^\infty | \ S(y) | \ dy = E(X) < \infty$, the Mellin transforms of both sides of (1) exist

Thus we have,

$$cS*(s) = \left(\int_0^\infty r(x-y)^{r-1} S(x) dx \right)^*$$

$$= \frac{\Gamma(r+1)\Gamma(s)}{\Gamma(r+s)} S^*(s+r). \qquad (2)$$

Letting $S*(s)/\Gamma(s) = h(s),$ (2) can be written as

$$h(s+r) = Ah(s) \qquad (3)$$

where $A = c/\Gamma(r+1).$

Equation (3) can be written as

$$h(t) - Ah(t-r) = 0. \qquad (4)$$

This is a differential difference equation,(Bellman and Cooke (1963), p.54), with auxiliary equation $1 = Ae^{-r/s} = 0$ giving $s = (InA)/r = b$. [12] (APPENDIX 12)

Hence the solution of (4) is

$$h(x) = ke^{bx},$$

where k is an arbitrary constant. This gives $S*(s) = k\Gamma(s)e^{bs} = k\Gamma(s)/\lambda^3.$ where $\lambda = e^{-b} > 0$ This implies that $S(x) = ke^{-\lambda x},$ Since $S(0) = 1, S(x) = e^{-\lambda x}$ and hence exponential.

3.8.5 Characterization by conditional expectation of a spacing between two record values

Introduction.Let $\{X_n, n \geq 0\}$ be a sequence of independent and identically distributed absolutely continuous random variables with cumulative distribution function *F(x)* and corresponding density function (pdf), *f(x).* The ratio, r, given by $r(x) = f(x)/\bar{F}(x)$ where $\bar{F}(x) = 1 - F(x)$ on $D = \{x|F(x) < 1\}$ is called the hazard rate.

A distribution F of a random variable X is said to have an increasing (IHR)(or decreasing) hazard rate (DHR) if r is an increasing (decreasing) function on $D_+ = \{x|F(x) < 1,\ x > 0\}$.

$F \in c_1$ if r is monotone D_+

F is called ''New better than used'' (NBU) if $\bar{F}(x+y) \leq \bar{F}(x)\bar{F}(y),\ \ for\ x, y \geq 0$

F is called ''New worse than used''(NWU) if $\bar{F}(x+y) \geq \bar{F}(x)\bar{F}(y),\ \ for\ x, y \geq 0$

$F \in c_2$ if the distribution function F is either NBU or NWU.

Let X be a r.v. whose density is given by

$$f(x) = \theta^{-1} e^{-(\frac{x}{\theta})}, \qquad x \geq 0 \qquad (1)$$

$$= 0, \qquad otherwise$$

$X \in E(\theta)$ if the pdf of X is of the form in (1)

The following theorem characterizes the exponential distribution by the equivalence of the expectation of the spacing between two record values and the expectation of the record value corresponding to the difference between their record times[4].

Theorem 3.8.5

Let, $\{X_n, n \geq 1)$ *be a sequence of independent and identically distributed non-negative random variables with absolutely continuous (with respect to Lebesgue measure) distribution function* $F(x)$ *and the corresponding probability density function* $f(x)$.

If $F \in c_1$ *and for some* m, n *with* $1 \leq m < n, E(X_{L(n)} - X_{L(m)} = E(X_{L(n-m)},)$ *then* $X_n \varepsilon E(\sigma)$ *for some* $\sigma > 0$.

Proof. Let $f_4(x)$ and $f_5(x)$ be respectively the p.d.f's of $X_{L(n-m)}$ and $X_{L(n)} - X_{L(m)}$ then

with $1 - F_4(x) = \bar{F}_4(x) = 1 - \int_0^x f_4(u)du.$ and $\bar{F}_5(x) = 1 - F_5(x) = 1 - \int_0^x f_5(u)du.$ we have

$$E(X_{L(n-m)} = \int_0^\infty \bar{F}_4(x)dx, \qquad (1)$$

and

$$E(X_{L(m)} - X_{L(n)} = \int_0^\infty \overline{F}_5(x)dx, \qquad (2)$$

Writing $f_4(x)$ and $f_5(x)$ in terms of $R(x)$ and *f(x)*, we have

$$F_4(x) = 1 - g_4(x),$$

Where $$g_4(x) = \sum_{j=0}^{n-m-1} \frac{(R(x))^j}{J!} e^{-R(x)}, \qquad (3)$$

and

$$F_3(x) = 1\int_0^\infty g_3(x,u) \frac{1}{m!}(R(u))^m f(u)du, \qquad (4)$$

where

$$g_3(x,u) = \sum_{j=0}^{n-m-1} \frac{(R(u+x) - R(u))^j}{J!} e^{-R(u+x)-R(u)),},$$

Equating (2) and (3), we have on simplification,

$$\int_0^\infty \int_0^\infty \frac{1}{m!}(R(u))^m f(u) H_2(x,u)dudx = 0, \qquad (5)$$

Where

$$H_2(x,u) = g_3(x,u) - g_4(x)$$

But $H_2(x,0) = 0$ and

$$\frac{\partial}{\partial u} H_2(x,u) = \frac{(R(u+x) - (R(u))^{n-m-1}}{(n-m-1)!} \quad e^{-(R(u+x)-R(u))}(r(x) - (u+x)).$$

Since $F \epsilon c_1$ for (5) to be true, we must have $r(u+x) = r(u)$ for almost all $u \; and \; x$, $0 < u, x < \infty$. Hence $X_n \epsilon E(\sigma)$

CHAPTER 4

Related distributions

4.0 Introduction

In this chapter distributions resulting from interactions between and among exponential distributions are investigated. These include sums, differences, products and quotients of exponential distributions.

Exponential distributions have been generalized using the beta and exponentiated generators.

4.1 Sums of independent exponential random variables

Let $S_N = X_1 + X_2 + . . . + X_N$

We wish to obtain distributions for S_N for three cases.

Case (i): **X_i's are i.i.d exponential random variables with parameter λ for fixed *N=n***

Using the Laplace transform technique, we have the Laplace transform for S_n

$L_{S_n}(s) = E[e^{-sS_n}]$

$= E[e^{-s(X_1 + \ldots + X_n)}]$

$= E[e^{-sX_1}] \ldots E[e^{-sX_n}]$ since the the $X_i{}'s$ are independent

But the X_i's are also identical.

$\therefore L_{S_n}(s) = E[e^{-sX_i}]^n$

$\therefore L_{S_n}(s) = E[L_{X_i}(s)]^n$,

where

$$L_{X_i}(s) = \frac{\lambda}{\lambda + s}$$

is the Laplace transform of the density of X_i, X_i being an exponential random variable with parameter λ.

$$\therefore L_{S_n}(s) = \left[\frac{\lambda}{\lambda + s}\right]^n$$

which is the Laplace transform of a gamma distribution with parameters n and λ.

But n is a positive integer. Such a gamma distribution is called an Erlang distribution.

Using convolution approach and mathematical induction, let us start with

$$f_{X_1+X_2}(t) = \int_0^t f_{X_1}(s)\, f_{X_2}(t-s)ds$$

$$= \int_0^t \lambda e^{-\lambda s} \lambda\, e^{-\lambda(t-s)} ds$$

$$= \lambda^2 \int_0^t e^{-\lambda s - \lambda t + \lambda s}\, ds$$

$$= \lambda^2 e^{-\lambda t}.t$$

$$= \lambda e^{-\lambda t}.(\lambda t)$$

Next

$$f_{X_1+X_{2+X_3}}(t) = \int_0^t f_{X_1+X_2}(s)\, f_{X_3}(t-s)ds$$

$$= \int_0^t \lambda e^{-\lambda s} \lambda s . \lambda e^{-\lambda(t-s)} ds$$

$$= \lambda^2 . \lambda \int_0^t s e^{-\lambda s - \lambda t + \lambda s}\, ds$$

$$= \lambda^2 . \lambda \int_0^t s e^{-\lambda t}\, ds$$

$$= \lambda^2 . \lambda e^{-\lambda t} \left[\frac{s^2}{2}\right]_0^t$$

$$= \lambda e^{-\lambda t} \frac{(\lambda t)^2}{2!}$$

By induction, assume that

$$X_1+X_2+\ldots+X_{n-1}$$

has a pdf given by

$$f_{X_1+X_2+\cdots X_{n-1}}(t)=\lambda e^{-\lambda t}\frac{(\lambda t)^{n-2}}{(n-2)!}$$

Hence

$$f_{X_1+X_2+\cdots X_n}(t)=\int_0^t f_{X_1+X_2+\cdots X_{n-1}}(s)\,f_{X_n}(t-s)ds$$

$$=\int_0^t \lambda e^{-\lambda s}\frac{(\lambda s)^{n-2}}{(n-2)!}\lambda e^{-\lambda(t-s)}ds$$

$$=\frac{\lambda^n e^{-\lambda t}}{(n-2)!}\int_0^t s^{n-2}\,ds$$

$$=\frac{\lambda^n e^{-\lambda t}}{(n-2)!}\left[\frac{s^{n-1}}{n-1}\right]_0^t$$

$$=\lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$=\frac{\lambda^n t^{n-1}}{\Gamma(n)}e^{-\lambda t},\quad t\geq 0$$

$$=\frac{\lambda^n}{\Gamma(n)}e^{-\lambda t}t^{n-1},\quad t\geq 0$$

which is a gamma distribution with parameters *n* and λ

Case (ii): Independent non-identical exponential random variables with fixed *N=n*

Let X_i, i=1, 2, ...,n be independent exponential random variables with respective rates λ_i , $i=1,\ldots,n$, and suppose $\lambda_i \neq \lambda_j, for\ i\neq j$.

The random variable $\sum_{i=0}^{n} X_i$, is said to be a hypo-exponential random variable.

To compute its pdf, let us start with the case n=2. Now

$$f_{X_1+X_2}(t)=\int_0^t f_{X_1}(s)\,f_{X_2}(t-s)ds$$

$$= \int_0^t \lambda_1 e^{-\lambda_1 s} \lambda_2 \, e^{-\lambda_2 (t-s)} ds$$

$$= \lambda_1 \lambda_2 e^{-\lambda_2 t} \int_0^t e^{-(\lambda_1 - \lambda_2)s} \, ds$$

$$= \lambda_1 \lambda_2 e^{-\lambda_2 t} \left[\frac{e^{-(\lambda_1 - \lambda_2)s}}{-(\lambda_1 - \lambda_2)} \right]_0^t$$

$$= \frac{\lambda_1 \lambda_2 e^{-\lambda_2 t}}{-(\lambda_1 - \lambda_2)} \left(e^{-(\lambda_1 - \lambda_2)t} - 1 \right)$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \left(e^{-\lambda_2 t} - e^{-\lambda_1 t} \right)$$

$$= \frac{\lambda_1}{\lambda_1 - \lambda_2} \lambda_2 e^{-\lambda_2 t} + \frac{\lambda_2}{\lambda_2 - \lambda_1} \lambda_1 e^{-\lambda_1 t}$$

For *n=3*

$$f_{X_1+X_2+X_3}(t) = \int_0^t f_{X_1+X_2}(s) \, f_{X_3}(t-s) ds$$

$$= \int_0^t \left[\frac{\lambda_1}{\lambda_1 - \lambda_2} \lambda_2 e^{-\lambda_2 s} + \frac{\lambda_2}{\lambda_2 - \lambda_1} \lambda_1 e^{-\lambda_1 s} \right] \lambda_3 e^{-\lambda_3 (t-s)} \, ds$$

$$= \int_0^t \left[\frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_1 - \lambda_2} e^{-\lambda_2 s - \lambda_3 t + \lambda_3 s} + \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_2 - \lambda_1} e^{-\lambda_1 s - \lambda_3 t + \lambda_3 s} \right] ds$$

$$= \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_1 - \lambda_2} e^{-\lambda_3 t} \int_0^t e^{-(\lambda_2 - \lambda_3)s} \, ds + \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_2 - \lambda_1} e^{-\lambda_3 t} \int_0^t e^{-(\lambda_1 - \lambda_3)s} \, ds$$

$$= \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_1 - \lambda_2} e^{-\lambda_3 t} \left[\frac{e^{-(\lambda_2 - \lambda_3)s}}{-(\lambda_2 - \lambda_3)} \right]_0^t + \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_2 - \lambda_1} e^{-\lambda_3 t} \left[\frac{e^{-(\lambda_1 - \lambda_3)s}}{-(\lambda_1 - \lambda_3)} \right]_0^t$$

$$= \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_1 - \lambda_2} e^{-\lambda_3 t} \left[\frac{e^{-(\lambda_2 - \lambda_3)t} - 1}{-(\lambda_2 - \lambda_3)} \right] + \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_2 - \lambda_1} e^{-\lambda_3 t} \left[\frac{e^{-(\lambda_1 - \lambda_3)t} - 1}{-(\lambda_1 - \lambda_3)} \right]$$

$$= \lambda_2 e^{-\lambda_2 t} \frac{\lambda_1}{\lambda_1 - \lambda_2} . \frac{\lambda_3}{\lambda_3 - \lambda_2} + \lambda_3 e^{-\lambda_3 t} \frac{\lambda_1}{\lambda_1 - \lambda_2} . \frac{\lambda_2}{\lambda_2 - \lambda_3}$$

+

$$\lambda_1 e^{-\lambda_1 t}\frac{\lambda_2}{\lambda_2-\lambda_1}.\frac{\lambda_3}{\lambda_3-\lambda_1}+\lambda_3 e^{-\lambda_3 t}\frac{\lambda_1}{\lambda_2-\lambda_1}.\frac{\lambda_2}{\lambda_1-\lambda_3}$$

$$\therefore f_{X_1+X_{2+X_3}}(t) \quad = \lambda_1 e^{-\lambda_1 t}\frac{\lambda_2}{\lambda_2-\lambda_1}.\frac{\lambda_3}{\lambda_1-\lambda_3}+\lambda_2 e^{-\lambda_2 t}\frac{\lambda_1}{\lambda_1-\lambda_2}.\frac{\lambda_3}{\lambda_3-\lambda_2}$$
$$+\lambda_3 e^{-\lambda_3 t}\left[\frac{1}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}+\frac{1}{(\lambda_2-\lambda_1)(\lambda_1-\lambda_3)}\right]\lambda_1\lambda_2$$

But

$$\frac{1}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}+\frac{1}{(\lambda_2-\lambda_1)(\lambda_1-\lambda_3)}=\frac{(\lambda_1-\lambda_3)-(\lambda_2-\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)}$$

$$=\frac{\lambda_1-\lambda_2}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)}$$

$$=\frac{1}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}$$

$$\therefore f_{X_1+X_{2+X_3}}(t) = \lambda_1 e^{-\lambda_1 t}\frac{\lambda_2}{\lambda_2-\lambda_1}.\frac{\lambda_3}{\lambda_3-\lambda_1}+\lambda_2 e^{-\lambda_2 t}\frac{\lambda_1}{\lambda_1-\lambda_2}.\frac{\lambda_3}{\lambda_3-\lambda_2}$$
$$+\lambda_3 e^{-\lambda_3 t}.\frac{\lambda_1}{\lambda_1-\lambda_3}.\frac{\lambda_2}{\lambda_2-\lambda_3}$$

$$\therefore f_{X_1+X_{2+X_3}}(t) = \sum_1^3\left\{\lambda_i e^{-\lambda_i t}\prod_{j\neq i}\frac{\lambda_j}{\lambda_j-\lambda_i}\right\}$$

which suggests the general result to be

$$\therefore f_{X_1+X_{2+...+X_n}}(t) = \sum_1^n\left\{\lambda_i e^{-\lambda_i t}\prod_{j\neq i}\frac{\lambda_j}{\lambda_j-\lambda_i}\right\}$$

$$=\sum_1^n\left\{\left(\prod_{j\neq i}\frac{\lambda_j}{\lambda_j-\lambda_i}\right)\ \lambda_i e^{-\lambda_i t}\right\} \qquad (*)$$

$$=\sum_1^n C_{i,n}\,\lambda_i e^{-\lambda_i t}$$

where

$$C_{i,n} = \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}$$

We will now prove the formula in $(*)$ by induction on *n.*

Since the formula has already been established for *n=2* and *n=3*, assume it to be true for

n and consider n+1 arbitrary independent exponentials X_i with distinct rates λ_i, i= 1, 2, . . ., n+1. Now

$$\therefore f_{X_1+X_{2+...+}X_{n+1}}(t) = \int_0^t f_{X_1+X_{2+...+}X_n}(s)\, f_{X_{n+1}}(t-s)ds$$

$$= \int_0^t \sum_1^n \left(\prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i} \right) \lambda_i e^{-\lambda_i s} \,.\, \lambda_{n+1} e^{-\lambda_{n+1}(t-s)}\, ds$$

$$= \sum_1^n C_{i,n} \int_0^t \lambda_i \;\lambda_{n+1} e^{-\lambda_i s} e^{-\lambda_{n+1}(t-s)} ds$$

$$= \sum_1^n C_{i,n} \int_0^t \lambda_i \;\lambda_{n+1} e^{-\lambda_i s - \lambda_{n+1} t + \lambda_{n+1} s} ds$$

$$= \sum_1^n C_{i,n} \;\; \lambda_i\, \lambda_{n+1} e^{-\lambda_{n+1} t} \int_0^t e^{-(\lambda_i - \lambda_{n+1})s} ds$$

$$= \sum_1^n \left\{ C_{i,n} \;\; \lambda_i\, \lambda_{n+1} e^{-\lambda_{n+1} t} \left[\frac{e^{-(\lambda_i - \lambda_{n+1})s}}{-(\lambda_i - \lambda_{n+1})} \right]_0^t \right\}$$

$$= \sum_1^n \left\{ C_{i,n} \;\; \lambda_i\, \lambda_{n+1} e^{-\lambda_{n+1} t} \left[\frac{e^{-(\lambda_i - \lambda_{n+1})t} - 1}{-(\lambda_i - \lambda_{n+1})} \right] \right\}$$

$$= \sum_1^n \left\{ C_{i,n} \;\; \lambda_i\, \lambda_{n+1} \left[\frac{e^{-\lambda_i t} - e^{-\lambda_{n+1} t}}{-(\lambda_i - \lambda_{n+1})} \right] \right\}$$

$$= \sum_1^n C_{i,n} \left\{ \lambda_i e^{-\lambda_i t} \frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_i} + \lambda_{n+1} e^{-\lambda_{n+1} t} \frac{\lambda_i}{\lambda_i - \lambda_{n+1}} \right\}$$

$$= \sum_1^n C_{i,n} \frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_i} \;\; \lambda_i e^{-\lambda_i t} + \sum_1^n C_{i,n} \frac{\lambda_i}{\lambda_i - \lambda_{n+1}} \lambda_{n+1} e^{-\lambda_{n+1} t}$$

$$= \sum_1^n C_{i,n} \frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_i} \;\; \lambda_i e^{-\lambda_i t} + \; K_{n+1}\, \lambda_{n+1} e^{-\lambda_{n+1} t}$$

where

$$K_{n+1} = \sum_{1}^{n} C_{i,n} \frac{\lambda_i}{\lambda_i - \lambda_{n+1}}$$

But

$$C_{i,n} = \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i} = \frac{\lambda_1}{\lambda_1 - \lambda_i} \frac{\lambda_2}{\lambda_2 - \lambda_i} \cdots \frac{\lambda_n}{\lambda_n - \lambda_i}$$

$$\therefore \; C_{i,n} \cdot \frac{\lambda_{n+1}}{\lambda_{n+i} - \lambda_i} = \frac{\lambda_1}{\lambda_1 - \lambda_i} \frac{\lambda_2}{\lambda_2 - \lambda_i} \cdots \frac{\lambda_n}{\lambda_n - \lambda_i} \frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_i}$$

$$= \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i} \; for \; j = 1, 2, \dots, n+1$$

$$= \; C_{i,n+1}$$

$$\therefore f_{X_1+X_{2+...+X_{n+1}}}(t) = \sum_{1}^{n} C_{i,n+1} \; \lambda_i e^{-\lambda_i t} + \; K_{n+1} \lambda_{n+1} e^{-\lambda_{n+1} t} \qquad (**)$$

where K_{n+1} is a constant and does not depend on t.

Next,

$f_{X_1+X_{2+...+X_{n+1}}}(t)$ can also be written as

$$f_{X_1+X_{2+...+X_{n+1}}}(t) = \int_0^t f_{X_2+X_{2+...+X_{n+1}}}(s) \, f_{X_1}(t-s) ds$$

$$= \int_0^t \left[\sum_{2}^{n+1} C_{i,n+1}^{*} \, \lambda_i e^{-\lambda_i s} \right] \lambda_1 e^{-\lambda_1 (t-s)} \, ds$$

$$= \sum_{2}^{n+1} C_{i,n+1}^{*} \, \lambda_i \, \lambda_1 \int_0^t e^{-\lambda_i s - \lambda_1 t + \lambda_1 s} \; ds$$

$$= \sum_{2}^{n+1} C_{i,n+1}^{*} \, \lambda_i \lambda_1 e^{-\lambda_1 t} \int_0^t e^{-(\lambda_i - \lambda_1) s} \; ds$$

$$= \sum_{2}^{n+1} C_{i,n+1}^{*} \, \lambda_i \lambda_1 e^{-\lambda_1 t} \left[\frac{e^{-(\lambda_i - \lambda_1) s}}{-(\lambda_i - \lambda_1)} \right]_0^t$$

$$= \sum_{2}^{n+1} C_{i,n+1}^{*} \, \lambda_i \lambda_1 e^{-\lambda_1 t} \left[\frac{e^{-(\lambda_i - \lambda_1) t} - 1}{-(\lambda_i - \lambda_1)} \right]$$

$$= \sum_{2}^{n+1} C_{i,n+1}^{*} \lambda_i \lambda_1 e^{-\lambda_1 t} \left[\frac{e^{-\lambda_i t}}{-(\lambda_i - \lambda_1)} - \frac{e^{-\lambda_1 t}}{-(\lambda_i - \lambda_1)} \right]$$

$$= \sum_{2}^{n+1} \left[C_{i,n+1}^{*} \frac{\lambda_1}{\lambda_1 - \lambda_i} \lambda_i e^{-\lambda_i t} + C_{i,n+1}^{*} \frac{\lambda_i}{\lambda_i - \lambda_1} \lambda_1 e^{-\lambda_1 t} \right]$$

$$= \sum_{2}^{n+1} C_{i,n+1}^{*} \frac{\lambda_1}{\lambda_1 - \lambda_i} \lambda_i e^{-\lambda_i t} + \sum_{2}^{n+1} C_{i,n+1}^{*} \frac{\lambda_i}{\lambda_i - \lambda_1} \lambda_1 e^{-\lambda_1 t}$$

$$= \sum_{2}^{n+1} C_{i,n+1}^{*} \frac{\lambda_1}{\lambda_1 - \lambda_i} \lambda_i e^{-\lambda_i t} + K_1 \lambda_1 e^{-\lambda_1 t} \qquad (***)$$

where $K_1 = \sum_2^{n+1} C_{i,n+1}^{*} \frac{\lambda_i}{\lambda_i - \lambda_1}$

Equating the two formulae $(**)$ and $(***)$, we have

$$\sum_{1}^{n} C_{i,n+1} \; \lambda_i e^{-\lambda_i t} + \; K_{n+1} \, \lambda_{n+1} e^{-\lambda_{n+1} t} = \sum_{2}^{n+1} C_{i,n+1}^{*} \frac{\lambda_1}{\lambda_1 - \lambda_i} \lambda_i e^{-\lambda_i t} + K_1 \lambda_1 e^{-\lambda_1 t}$$

But

$$C_{i,n+1}^{*} \frac{\lambda_1}{\lambda_1 - \lambda_i} = \left[\prod_{j \neq i, j = 2,3,\dots,n+1} \frac{\lambda_j}{\lambda_j - \lambda_i} \right] \frac{\lambda_1}{\lambda_1 - \lambda_i}$$

$$= \left[\frac{\lambda_2}{\lambda_2 - \lambda_i} \frac{\lambda_3}{3 - \lambda_i} \dots \frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_i} \right] \frac{\lambda_1}{\lambda_1 - \lambda_i}$$

$$= \frac{\lambda_1}{\lambda_1 - \lambda_i} \frac{\lambda_2}{\lambda_2 - \lambda_i} \dots \frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_i} = C_{i,n+1}$$

Therefore,

$$\sum_{1}^{n} C_{i,n+1} \; \lambda_i e^{-\lambda_i t} + \; K_{n+1} \, \lambda_{n+1} e^{-\lambda_{n+1} t} = \sum_{2}^{n+1} C_{i,n+1} \lambda_i e^{-\lambda_i t} + K_1 \lambda_1 e^{-\lambda_1 t}$$

i.e,

$$C_{1,n+1} \; \lambda_1 e^{-\lambda_1 t} + \sum_{2}^{n} C_{i,n+1} \; \lambda_i e^{-\lambda_i t} + \; K_{n+1} \, \lambda_{n+1} e^{-\lambda_{n+1} t}$$
$$= \sum_{2}^{n} C_{i,n+1} \lambda_i e^{-\lambda_i t} + C_{n+1,n+1} \lambda_{n+1} e^{-\lambda_{n+1} t} + K_1 \lambda_1 e^{-\lambda_1 t}$$

$$C_{1,n+1} \; \lambda_1 e^{-\lambda_1 t} + K_{n+1} \lambda_{n+1} e^{-\lambda_{n+1} t} = C_{n+1,n+1} \lambda_{n+1} e^{-\lambda_{n+1} t} + K_1 \lambda_1 e^{-\lambda_1 t}$$

Comparing the coefficients of $\lambda_{n+1} e^{-\lambda_{n+1} t}$, we have

$$K_{n+1} = C_{n+1,n+1}$$

Comparing the coefficients of $\lambda_1 e^{-\lambda_1 t}$, we have

$$K_1 = C_{1,n+1}$$

Thus

$$f_{X_1+X_{2+...+X_{n+1}}}(t) = \sum_1^n C_{i,n+1}\ \lambda_i e^{-\lambda_i t} +\ K_{n+1}\,\lambda_{n+1} e^{-\lambda_{n+1} t}$$

$$= \sum_1^n C_{i,n+1}\ \lambda_i e^{-\lambda_i t} + C_{n+1,n+1}\ \lambda_{n+1} e^{-\lambda_{n+1} t}$$

$$= \sum_1^{n+1} C_{i,n+1}\ \lambda_i e^{-\lambda_i t}$$

$$= \sum_1^{n+1} \left(\prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i} \right)\ \lambda_i e^{-\lambda_i t}$$

Alternatively,

$$f_{X_1+X_{2+...+X_{n+1}}}(t) = \sum_2^{n+1} C_{i,n+1}^{*} \frac{\lambda_1}{\lambda_1 - \lambda_i} \lambda_i e^{-\lambda_i t} + K_1 \lambda_1 e^{-\lambda_1 t}$$

$$= \sum_2^{n+1} C_{i,n+1} \lambda_i e^{-\lambda_i t} + C_{1,n+1}\,\lambda_1 e^{-\lambda_1 t}$$

$$= C_{1,n+1}\,\lambda_1 e^{-\lambda_1 t} + \sum_2^{n+1} C_{i,n+1} \lambda_i e^{-\lambda_i t}$$

$$= \sum_1^{n+1} C_{i,n+1} \lambda_i e^{-\lambda_i t}$$

Remark: This proof has been given by Chiang (1980) and Ross (2000).

Case(iii) A random number of i.i.d exponential random variables

Let Let $S_N = X_1+X_2+...+X_N$

where the $X_i's$ are i.i.d random variables and N is also a random variable independent of $X_i's$

Suppose $X_i's$ are continuous random variables. Then, let

H(s)= the Laplace transform of S_N

$= E[e^{-sS_N}]$

F(s)=the probability generating function of N

$= E[S^N]$

And

L(s)=the Laplace transform of $X_i$for i=1, 2, . . ., N

$= E[e^{-sX_i}]$

$\therefore H(s) = E[e^{-sS_N}] = EE[e^{-sS_N}/N = n]$

$= EE\left[e^{-s(X_1+X_2+\ldots+X_N)}\right]$

$= E\{E(e^{-sX_1})\, E(e^{-sX_2})\ \ldots E\ (e^{-sX_N})\}$

$= E[L(s)]^N$

$= F[L(s)]$

$= F_N\left[L_{X_i}(s)\right]$

which is called a compound distribution.

If N is Poisson, then H(s) becomes a compound Poisson distribution. Suppose N is Poisson with parameterθ, then

$$H(s) = e^{-\theta[1-L(s)]}$$

If N is binomial, then $H(s)$ becomes a compound binomial distribution. Suppose N is binomial with parameters n and p, then

$$H(s) = [q + pL(s)]^n$$

where q= 1-p

If N is negative binomial, then $H(s)$ becomes a compound negative binomial distribution. Suppose N is binomial with parameters α and p, then

$$H(s) = \left[\frac{p}{1 - qL(s)}\right]^\alpha, \ \ \alpha > 0, 0 < p < 1, q + p = 1$$

If N is shifted geometric with parameter p, then

$$F(s) = \frac{ps}{1 - qs}, q = 1 - p$$

$$\therefore\ H(s) = F_N[L_X(s)]$$

$$= \frac{pL_X(s)}{1 - qL_X(s)}$$

Further, if X is exponential with parameterλ, then

$$L_X(s) = \frac{\lambda}{\lambda + s}$$

$$\therefore\ H(s) = \frac{p\lambda}{(\lambda + s)\left(1 - \frac{q\lambda}{\lambda + s}\right)}$$

$$= \frac{p\lambda}{(\lambda + s)\left(1 - \frac{q\lambda}{\lambda + s}\right)}$$

$$= \frac{p\lambda}{(\lambda + s) - q\lambda}$$

$$= \frac{p\lambda}{p\lambda + s}$$

which is the Laplace transform of an exponential distribution with parameter $p\lambda$.

$$\therefore Prob(S_N = y) = p\lambda e^{-p\lambda y}; y \geq 0$$

The distribution of S_N can also be looked at as a mixture as follows:

$$Prob(S_N = y) = \sum_n Prob(S_N = y, N = n)$$

$$= \sum_n Prob(S_N = y/\ N = n) Prob(N = n)$$

$$= \sum_n Prob\big(X_1 + X_{2+...+X_n} = y\big) Prob(N = n)$$

$$= \sum_n \{f_x\}^{*n} p_n$$

where $\{f_x\}^{*n}$ is the n-fold convolution of $X_i's$ and $p_n = Prob(N = n)$

Thus $Prob(S_N = y)$ is a mixed distribution with p_n as the mixing distribution.

If the $X_i's$ are i.i.d exponential random variables of parameterλ, then

$Prob(S_N = y/\ N = n)$ is a gamma distribution with parameters n and λ.

$$\therefore Prob(S_N = y) = \sum_n \left[\frac{\lambda^n}{\Gamma(n)} e^{-\lambda y} y^{n-1}\right] p_n$$

For shifted parametric,

$$p_n = pq^{n-1}, n = 1,2,3, \dots$$

$$\therefore Prob(S_N = y) = \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma(n)} e^{-\lambda y} y^{n-1} pq^{n-1}$$

$$\therefore Prob(S_N = y) = pe^{-\lambda y} \sum_{n=1}^{\infty} \frac{\lambda^n y^{n-1} q^{n-1}}{\Gamma(n)}$$

$$= p\lambda e^{-\lambda y} \sum_{n=1}^{\infty} \frac{(\lambda y q)^{n-1}}{\Gamma(n)}$$

$$= p\lambda e^{-\lambda y} \sum_{n=1}^{\infty} \frac{(\lambda y q)^{n-1}}{(n-1)!}$$

$$= p\lambda e^{-\lambda y} e^{\lambda y q}$$

$$= p\lambda e^{-\lambda y + \lambda y q}$$

$$= p\lambda e^{-\lambda y + \lambda(1-p) y}$$

$$= p\lambda e^{-\lambda y + \lambda y - p\lambda y}$$

$$= p\lambda e^{-p\lambda y},\ \ y > 0,$$

which is an exponential distribution with parameter $p\lambda$.

4.2 Distribution of the difference between two exponential random variables

Let X_1, and X_2 be two independently distributed exponential random variables with respective rates λ_i, i= 1, 2.

Let $Y = X_1 - X_2$

$\therefore -\infty < y < \infty$

Thus the CDF of Y, $F_Y(y)$, is piecewise, when $y \leq 0$ or $y \geq 0$.

The c.d.f. of Y is $F_Y(y) = P(Y \leq y)$.
case (i) $y \leq 0$;

$$F_Y(y) = P(Y \leq y)$$

$$= P(X_1 - X_2 \leq y)$$

$$= P(X_2 \geq X_1 - y)$$

$$= \int_0^\infty \int_{x_1-y}^\infty f_{X_1X_2}(x_1, x_2) d_{X_2} dx_1$$

$$= \int_0^\infty \int_{x_1-y}^\infty \lambda_1\lambda_2 e^{-\lambda_1 x_1} e^{-\lambda_2 x_2} \quad d_{X_2} dx_1$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2} e^{\lambda_2 y} \left[e^{-(\lambda_1+\lambda_2)\, x_2}\right]_0^\infty$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2} e^{\lambda_2 y}, \qquad y \leq 0.$$

Case (ii)$y \geq 0$;

$$F_Y(y) = P(Y \leq y)$$

$$= 1 - P(Y > y)$$

$$= 1 - P(X_1 - X_2 > y)$$

$$= 1 - P(X_2 \geq X_1 - y)$$

$$= 1 - \int_y^\infty \int_0^{x_1-y} f_{X_1,X_2}(x_1, x_2) d_{X_2} dx_1$$

$$= 1 - \int_y^\infty \int_0^{x_1-y} \lambda_1\lambda_2 e^{-\lambda_1 x_1} e^{-\lambda_2 x_2} \quad d_{X_2} dx_1$$

$$= 1 - \frac{\lambda_1}{\lambda_1+\lambda_2} e^{\lambda_2 y} \left[e^{-(\lambda_1+\lambda_2)\, x_2}\right]_0^{x_1-y}$$

$$= 1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} e^{-\lambda_1 y}, \qquad y > 0.$$

Differentiating $F_Y(y)$ in sections 4.3.1 ad 4.3.2 gives

$$f_Y(y) = \begin{cases} \dfrac{1}{\lambda_1 + \lambda_2} e^{\lambda_2 y}, & y \le 0 \\ \dfrac{1}{\lambda_1 + \lambda_2} e^{-\lambda_1 y}, & y > 0 \end{cases}$$

which is the pdf of a Laplace random variable with parameters $\lambda_1 \; and \; \lambda_2$

4.3 Distribution of the product of two exponential random variables

Let X_1 and X_2 be two independently distributed exponential random variables with respective rates λ_i, i= 1, 2.

Let $\qquad Y = X_1 X_2$

The CDF of Y is

$$G(y) = P(Y \le y) = P(X_1 X_2 \le y)$$

$$= P(X_1 \le y/x_2)$$

$$= P(X_1 \le \frac{y}{x_2}, 0 \le x_2 < \infty)$$

$$= \int_0^\infty F_1\left(\frac{y}{x_2}\right) f(x_2) dx_2$$

where $F_1(x_1) = Prob(X_1 \le x_1)$

$$\therefore G(y) = \int_0^\infty \left[1 - e^{-\lambda_1 \frac{y}{x_2}}\right] \lambda_2 e^{-\lambda_2 x_2} dx_2$$

$$g(y) = \frac{dG(y)}{dy}$$

$$= \int_0^\infty \frac{\lambda_1}{x_2} e^{-\lambda_1 \frac{y}{x_2}} \lambda_2 e^{-\lambda_2 x_2} dx_2$$

$$= \lambda_1 \lambda_2 \int_0^\infty \frac{1}{x_2} e^{-\lambda_1 \frac{y}{x_2} - \lambda_2 x_2} dx_2$$

$$= \lambda_1 \lambda_2 \int_0^\infty x_2^{-1} e^{-\lambda_2 x_2 - \lambda_1 y \frac{1}{x_2}} dx_2$$

$$= \lambda_1\lambda_2 \int_0^\infty {x_2}^{0-1} e^{-\lambda_2(x_2+\lambda_1\frac{y}{x_2}\frac{1}{x_2}).}\ dx_2$$

Let $x_2 = z\sqrt{\frac{\lambda_{1y}}{\lambda_2}}$, $\therefore dx_2 = \sqrt{\frac{\lambda_{1y}}{\lambda_2}}dz$

$$\therefore g(y) = \lambda_1\lambda_2 \int_0^\infty \left[z\sqrt{\frac{\lambda_{1y}}{\lambda_2}}\right]^{0-1} e^{-\lambda_2\sqrt{\frac{\lambda_{1y}}{\lambda_2}}(z+1/z))} \sqrt{\frac{\lambda_{1y}}{\lambda_2}}dz$$

$$= \lambda_1\lambda_2 \int_0^\infty z^{0-1}e^{-\sqrt{\lambda\lambda_2 y_1}(z+1/z))}dz$$

$$= \lambda_1\lambda_2 \int_0^\infty z^{0-1}e^{-2\frac{\sqrt{\lambda\lambda_2 y_1}}{2}(z+1/z))}dz$$

$$= 2\lambda_1\lambda_2\ K_0(2\sqrt{\lambda\lambda_2 y_1})$$

where $K_v(w)$ is the modified Bessel function of the third kind with index v=0 and

$w = 2\sqrt{\lambda\lambda_2 y_1}$

4.4 Distribution of the quotient of two exponential random variables

Let $Y = X_1/X_2$

where X_1 and X_2 are independently distributed exponential random variables with parameters $\lambda_1\ and\ \lambda_2\ respectively.$

The CDF of Y is

$$G(y) = P(Y \le y) = P(\frac{X_1}{X_2} \le y)$$

$$= P(\ X_1 \le x_2 y)$$

$$= P(X_1 \le y/x_2)$$

$$= P(X_1 \le x_2 y, 0 \le x_2 < \infty)$$

$$= \int_0^\infty F_1(x_2 y)\, f(x_2) dx_2$$

$$= \int_0^\infty \left[1 - e^{-\lambda_1 x_2 y}\right] \lambda_2 e^{-\lambda_2 x_2} dx_2$$

$$g(y) \quad = \int_0^{\infty} \lambda_1 x_2 e^{-\lambda_1 x_2 y}\, \lambda_2 e^{-\lambda_2 x_2} dx_2$$

$$= \lambda_1 \lambda_2 \int_0^{\infty} x_2 e^{-\lambda_1 x_2 y - \lambda_2 x_2}\, dx_2$$

$$= \lambda_1 \lambda_2 \int_0^{\infty} x_2 e^{-(\lambda_1 y + \lambda_2) x_2}\, dx_2$$

$Let\ z = (\lambda_1 y + \lambda_2) x_2,\ \therefore \frac{dz}{\lambda_1 y + \lambda_2} = dx$

$$\therefore \quad g(y) = \lambda_1 \lambda_2 \int_0^{\infty} \frac{z^{2-1}}{\lambda_1 y + \lambda_2} e^{-z} \frac{dz}{\lambda_1 y + \lambda_2} dx_2$$

$$= \frac{\lambda_1 \lambda_2}{(\lambda_1 y + \lambda_2)^2} \Gamma(2)$$

$$= \frac{\lambda_2}{\lambda_1 \left(y + \frac{\lambda_2}{\lambda_1}\right)^2}, \quad y > 0$$

,

4.5 Beta-Exponential distribution

The Beta-exponential distribution was introduced by Nadarajah and Kotz (2006) as a generalization of the exponential distribution.

The distribution based on what is called the beta generator approach, briefly discussed below.

The classical beta (type I) pdf is given by

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}, \qquad 0 < x < 1, \quad a, b > 0$$

where

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

The cdf is given by

$F(x) = \Pr\,(X \le x)$

$$= \int_0^x f(t)dt = \int_0^x \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt$$

For any random variable, Y, where $-\infty < Y < \infty$, its cdf, $G(y) = Prob(Y \leq y)$ has the property; $0 \leq G(y) \leq 1$.

Consider the case $0 < Y < \infty$.

So $0 \leq x \leq 1$ can be replaced by $G(y)$.

$$\therefore F[G(y)] = \int_0^{G(y)} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt$$

Let $W(y) = F[G(y)] = \int_0^{G(y)} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt$

$$\therefore w(y) = \frac{dW(y)}{dy} = \frac{d}{dy} \int_0^{G(y)} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt$$

$$= d \int_0^{G(y)} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}$$

$$= \left[\frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}\right]_0^{G(y)} \frac{d(G(y))}{dy}$$

$$= \frac{\left(G(y)\right)^{a-1}\left(1-G(y)\right)^{b-1}}{B(a,b)} g(y)$$

(Using Leibniz's theorem)

Thus

$$w(y) = \frac{\left(G(y)\right)^{a-1}\left(1-G(y)\right)^{b-1}}{B(a,b)} g(y)$$

This is called the beta-generated distribution.

From this distribution, distributions of order statistics can be obtained by letting

$$a = i \ and \ b = n - i + 1,$$

so that,

$$w(y) = \frac{\left(G(y)\right)^{i-1}\left(1-G(y)\right)^{n-i}}{B(a,b)} g(y)$$

where *i* and *n* are positive integers. This is the probability density of the *i*-th order statistic from a random sample of size *n*.

Let us consider Y to be an exponential random variable with parameter λ

$\therefore\ g(y) = \lambda e^{-\lambda y}\ and\ G(y) = 1 - e^{-\lambda y}, \quad y > 0,\ \ e^{-\lambda y}, \quad \lambda > 0$

The beta-exponential distribution is given by the pdf

$$w(y) = \frac{\left[1 - e^{-\lambda y}\right]^{a-1} \left[1 - (1 - e^{-\lambda y})\right]^{b-1} \lambda e^{-\lambda y}}{B(a,b)}$$

which simplifies to

$$w(y) = \frac{\lambda e^{-\lambda b y}\left(1 - e^{-\lambda y}\right)^{a-1}}{B(a,b)}$$

(Note: If *a*=*b*=1, $w(y) = \lambda e^{-\lambda y}$)

Let $Y_{(1)} < Y_{(2)} < \cdots . < Y_{(n)}$ be the order statistics from an exponential distribution.

The *i*-th order statistic has its pdf given by taking $a = i$ and $b = n - i + 1$ in the above formula. i.e.

$$g\left(y_{(i)}\right) = \frac{\lambda e^{-\lambda(n-i+1)y}\left[1 - e^{-\lambda y}\right]^{i-1}}{B(a,b)} \quad , \qquad y_{(i)} > 0$$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \frac{\Gamma(i)\Gamma(n-i+1)}{\Gamma(n+1)}$$

$$= \frac{(i-1)!\,(n-i)!}{n!}$$

$$\therefore \quad g\left(y_{(i)}\right) = \frac{n!}{(i-1)!\,(n-i)!} \ \ \lambda e^{-\lambda(n-i+1)y}\left[1 - e^{-\lambda y}\right]^{i-1}$$

Putting *i*=1, we obtain the distribution $y_{(1)} = \min\left(Y_{1,}Y_2\ , \dots Y_n\right)$, i.e,

$\therefore \quad g\left(y_{(1)}\right) = \ n\lambda e^{-\lambda n y}\ , \quad y_{(1)} > 0$

For max $(Y_{1,}Y_2\ , \dots Y_n)$, let *i*=*n*.

$$\therefore \quad g(y_{(n)}) = \frac{n!}{(n-1)!\,0!} \ \lambda e^{-\lambda y}\left[1 - e^{-\lambda y}\right]^{n-1}$$

$$= \ n\lambda e^{-\lambda y}\left(1 - e^{-\lambda y}\right)^{n-1} \quad , \quad y_{(n)} > 0.$$

4.6 Exponentiated exponential distribution

Let $F(x) = [G(x)]^{\alpha}$

where $G(x)$ is the cdf and $F(x)$ is the new cdf and $\alpha > 0$.

$$\therefore f(x) = \frac{dF(x)}{dx} = \alpha[G(x)]^{\alpha-1}g(x)$$

$g(x)\ and\ f(x)$ are old and new pdfs.

The new pdf $f(x)$ is called the exponentiated generated distribution.

For exponentiated exponential distribution, we have

$$f(x) = \alpha\left[1 - e^{-\lambda y}\right]^{\alpha-1}\lambda e^{-\lambda y}$$

$$= \alpha\lambda e^{-\lambda y}\left[1 - e^{-\lambda y}\right]^{\alpha-1}, \qquad x > 0, \qquad \alpha > 0, \qquad \lambda > 0$$

which is also a generalization of exponential distribution by Gupta and Kundu (1999).

When $\alpha = 1$, the exponential distribution is generated.

CHAPTER 5

Mixtures

5.0 Introduction

As mentioned in Chapter 1, in many situations involving the exponential distribution the populations may not be homogeneous. Such populations are therefore appropriately handled by the exponential mixture distributions. Mixture populations are modeled by considering the exponential distribution rate parameter as a random variable. The distribution of the rate parameter is called the mixing distribution.

In this chapter mixtures of the exponential distribution with nine discreet distributions and thirteen continuous distributions are constructed. In doing this, an innovative method employing the moment generating function is used first to determine the survival function of a mixture. This technique is employed due to the fact that the survival function of a mixture of the exponential distribution with any mixing distribution is the same as the moment generating function of the mixing distribution. Once the mixture's survival function has been determined, all of its other associated functions are then derived.

Sections 5.1and 5.2 highlight the method linking the survival function of an exponential distribution mixture with the moment generating function of a mixing distribution. Section 5.3 deals with the derivation of moment generating functions for the discreet and continuous mixing distributions covered in the study. Section 5.4 applies the moment generating functions in section5.3 in obtaining survival functions of the various mixtures together with the other functions associated with the mixtures. Frequency curves associated with the mixtures have also been drawn together with that of the exponential distribution for comparison. The parameters for the exponential distribution and its mixtures were arbitrarily selected to simulate a possible occurrence. Mixture distributions with more than one parameter have been depicted using more than one chart to provide a wider perspective of the effects of the various parameter combinations on the mixture curves.

Densities involving the following functions were computed with the assistance of the indicated packages in CRAN R:

confluent hypergeometric function –*hypergeo*

error function- *NORMT3*

modified Bessel function of the second kind-*Bessel* standard normal-*stats*

Plotting was done using Excel.

5.1 Methodology

A mixture of the exponential distribution will arise when the parameter λ in the exponential density

$$f(x) = \lambda e^{-\lambda x} \text{ , or} \quad (1)$$

$$f(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \quad (2)$$

is also a random variable. This means that the random variable X is conditioned on λ.

It should be noted that two-parameter formats exist for the distributions in (1) and (2). They are, respectively,

$$f(x) = \lambda e^{-\lambda(x-\theta)} \quad (1a)$$

$$and \qquad f(x) = \frac{1}{\lambda} e^{-\frac{x-\theta}{\lambda}} \quad (2a)$$

In both (1a) and (2a), the parameter θ represents the location while λ is the scale parameter.

We have restricted ourselves to mixtures of the exponential distribution in (1).

The general form of the pdf of the mixtures of (1) may be given as

$$f(x) = \int \lambda e^{-\lambda x} g(\lambda) d\lambda \quad (3)$$

where $g(\lambda)$ is called the mixing density.

When λ is a discreet random variable with pmf $p(\lambda)$ (3) becomes

$$f(x) = \sum \lambda\, e^{-\lambda x} p(\lambda) \quad (4)$$

The pdf of the distribution in (1) has a form that makes it convenient to find its mixtures simply by determining the moment generating function of the mixing density or mass function. This arises from the fact that for mixing distributions defined on [0,∞), the survival functions of their mixtures with the exponential distribution is the moment generating function of the mixing distribution with a negative parameter. This is the same as the Laplace transform of the mixing distribution. However, In this study we shall mostly refer to moment generating function because it is the most adopted term in statistical literature. This technique has been alluded to by Hasselager, O., Wang, S. & Gordon, W.(1998)[15].(APPENDIX 13)

5.2 Derivation of mixtures

The following terms are frequently used in this paper.

$$f(x) - the\ density\ function\ of\ a\ mixture$$

$$F(x) - the\ distribution\ \ function\ of\ a\ mixture$$

$$S(x) - the\ survival\ function\ of\ a\ mixture = 1 - F(x)$$

$$h(x) - the\ hazard\ rate\ function\ of\ a\ mixture = \frac{f(x)}{S(x)}$$

For a random variable X having an exponential distribution, its density function can be expressed as

$$f(x) = \lambda \mathrm{e}^{-\lambda \mathrm{x}},\ x > 0,\ \lambda > 0$$

When the parameter λ is also a random variable, then X can be considered to be

conditioned on λ so that the density of X is now a conditional density written as

$$f(x|\lambda) = \lambda \mathrm{e}^{-\lambda \mathrm{x}}$$

Thus the conditional cumulative distribution function is

$$F(x|\lambda) = \int_0^{\mathrm{x}} \lambda \mathrm{e}^{-\lambda \mathrm{y}} dy$$

$$= -\left[\mathrm{e}^{-\lambda \mathrm{y}}\right]_0^{\mathrm{x}}$$

$$= 1 - \mathrm{e}^{-\lambda \mathrm{x}}$$

Hence the conditional survival function is

$$S(x|\lambda) = 1 - \mathrm{F(x)} = \mathrm{e}^{-\lambda \mathrm{x}}$$

The marginal survival function of X which is also the survival function of the mixture of the exponential function and the mixing density $g(\lambda)$, is

$$S\ (x) = \int_0^{\infty} S(x|\lambda)\mathrm{g}(\lambda)\mathrm{d}\lambda$$

$$= \int_0^{\infty} \mathrm{e}^{-\lambda \mathrm{x}}\mathrm{g}(\lambda)\mathrm{d}\lambda = M_{\lambda}\ (-x) \tag{5}$$

Equation (1) represents the moment generating function (mgf) of $g(\lambda)$ with the mgf parameter $-x$.

When the distribution of λ is discreet, equation (1) is replaced by

$$\sum_{\lambda} \mathrm{e}^{-\lambda x}\, p(\lambda) \;\; = M_{\lambda}\,(-x) = S_{\lambda}\,(x) \qquad (6)$$

Where $p(\lambda)$ is the probability mass function of λ.

Thus by determining the moment generating function of $g(\lambda)$ or $p(\lambda)$ and using a negative mgf parameter, the survival function of the mixture of $g(\lambda)$ or $p(\lambda)$ with the exponential function is also determined.

5.3 Moment generating functions

The moment generating function (mgf) of a random variable Y is given by

$M_Y(s)$= E[e^{sY}] $= \sum e^{sY}\, p(y)$ for discreet Y (7)

$= \int e^{sY}\, f(y)dy$ for continuous Y (8)

where *s* is the mgf parameter, $p(y)$ is a probability mass function, (pmf), if y is discreet and $f(y)$ is the probability density function, (pdf), if y is continuous.

Equations (7) and (8) hold provided E[e^{sY}] exists for every real number *s* in the neighbourhood $-h < s < h$ for some positive number h

5.4 Moment generating functions of discreet distributions

Equation (7) will apply in the sections 5.3.1.1 - 5.3.1.9 with *x* replacing *s* and *λ* replacing *y*

5.4.1 Bernoulli distribution

When $p(\lambda;\theta) = \begin{cases} 1-\theta, & \lambda = 0 \\ \theta, & \lambda = 1 \end{cases}$

or $p(\lambda;\theta) = \theta^{\lambda}(1-\theta)^{1-\lambda}$ for $\lambda = 0, 1$ and $0 \le \theta \le 1$.

The mgf of a Bernoulli distribution is

$$M_{\lambda}\,(x) = E[e^{x\lambda}] = \textstyle\sum_0^1 e^{x\lambda}\, p(\lambda)$$

$$= \textstyle\sum_0^1 e^{x\lambda}\, \theta^{\lambda}(1-\theta)^{1-\lambda}$$

$$= 1 - \theta + \theta e^{x}$$

$$= 1 - \theta(1 - e^{x})$$

5.4.2 Binomial distribution

When $p(\lambda;\theta,n) = \binom{n}{\lambda}\theta^{\lambda}(1-\theta)^{\mathrm{n}-\lambda}$ $\quad\lambda = 0, 1,2 \ldots, n\,,\quad 0 \le \theta \le 1$

The mgf of a Binomial distribution is

$$M_{\lambda}\,(x) = E[e^{x\lambda}] = \textstyle\sum_0^n\, e^{x\lambda}\, p(\lambda)$$

$$= \textstyle\sum_0^n\,\binom{n}{\lambda}\,\theta^{\lambda}(1-\theta)^{\mathrm{n}-\lambda}\quad e^{x\lambda}$$

$$= \textstyle\sum_0^n\,\binom{n}{\lambda}\,(\theta e^{x})^{\lambda}(1-\theta)^{\mathrm{n}-\lambda}$$

$$= [(1-\theta) + \theta e^{x}]^{n}$$

5.4.3 Geometric distribution type I

when λ is the number of failures before a success

or $\quad p(\lambda;\theta) = \theta(1-\theta)^{\lambda-1}$ for $\lambda = 0, 1,2, \ldots$ and $0 \le \theta \le 1$.

The mgf of a geometric type I distribution is

$$M_{\lambda}\,(x) = E\left[e^{x\lambda}\right] = \sum_{1}^{\infty} e^{x\lambda}\, p(\lambda)$$

$$= \sum_{1}^{n} e^{x\lambda}\;\;\theta(1-\theta)^{\lambda-1}$$

$$= \frac{\theta}{1-\theta}\sum_{1}^{n} e^{x\lambda}\;\;(1-\theta)^{\lambda}$$

$$= \frac{\theta}{1-\theta}\sum_{1}^{n}\;[(1-\theta)\,e^{x}]^{\lambda}$$

$$= \frac{\theta}{1-\theta}\;\frac{(1-\theta)\,e^{x}}{1-(1-\theta)\,e^{x}}$$

$$= \frac{\theta e^{x}}{1-(1-\theta)\,e^{x}}$$

5.4.4 Geometric distribution type II

when λ is the number of trials for a first success

or $p(\lambda;\theta) = \theta(1-\theta)^{\lambda}$ for $\lambda = 0, 1,2, \ldots$ and $0 \leq \theta \leq 1.$

The mgf of a geometric type II distribution is

$$M_{\lambda}(x) = E\left[e^{x\lambda}\right] = \sum_{1}^{\infty} e^{x\lambda}\, p(\lambda)$$

$$= \sum_{1}^{n} e^{x\lambda} \;\; \theta(1-\theta)^{\lambda-1}$$

$$= \theta \sum_{1}^{n} \; [(1-\theta)\, e^{x}]^{\lambda}$$

$$= \frac{\theta(1-\theta)\, e^{x}}{1-(1-\theta)\, e^{x}}$$

5.4.5 Negative binomial distribution type I

When λ is the number of failures before the rth success

$$p(\lambda;\theta,r) = \binom{\lambda+n-1}{n-1}\theta^{\mathrm{r}}(1-\theta)^{\lambda}, \quad \lambda = 0,1,2 \ldots, n\,, \quad 0 \leq \theta \leq 1$$

The mgf of a negative binomial type I distribution is

$$M_{\lambda}(x) = E\left[e^{x\lambda}\right] = \sum_{0}^{\infty} e^{x\lambda}\, p(\lambda)$$

$$= \sum_{0}^{\infty} e^{x\lambda} \binom{\lambda+r-1}{r} \theta^{\mathrm{r}}(1-\theta)^{\lambda}$$

$$= \theta^{\mathrm{r}} \sum_{0}^{\infty} \binom{\lambda+r-1}{r} (1-\theta)\, e^{x}]^{\lambda}$$

$$= \left(\frac{\theta}{1-(1-\theta)\, e^{x}}\right)^{r}$$

5.4.6 Negative binomial distribution type II

When λ is the total number of trials to achieve r successes,

$$p(\lambda;\theta,n)=\binom{\lambda-1}{r-1}\theta^{\mathrm{r}}(1-\theta)^{\lambda-\mathrm{r}},\qquad \lambda=r\,,r+1,\dots,\quad 0\le\theta\le1$$

The mgf of a negative binomial type II distribution is

$$M_{\lambda}\,(x)=E\left[e^{x\lambda}\right]=\sum\nolimits_{0}^{\infty}e^{x\lambda}\,p(\lambda)$$

$$=\sum\nolimits_{\lambda=r}^{\infty}\,e^{x\lambda}\binom{\lambda-1}{r-1}\theta^{\mathrm{r}}(1-\theta)^{\lambda-\mathrm{r}}$$

$$=\sum_{\lambda=r}^{\infty}(-1)^{r}\binom{-r}{\lambda-r}(1-\theta)^{\lambda-r}\,\theta^{\mathrm{r}}e^{\lambda x}$$

$$=\sum_{\lambda=r}^{\infty}(-1)^{r}\binom{-r}{\lambda-r}(1-\theta)^{\lambda-r}(\,\theta\,e^{\mathrm{x}})^{\mathrm{r}}\;(e^{\mathrm{x}})^{\,\lambda-r}$$

$$=(\theta e^{\mathrm{x}})^{\,r}\sum_{\lambda=r}^{\infty}(-1)^{\lambda-r}\binom{-r}{\lambda-r}((1-\theta)e^{\mathrm{x}})^{\lambda-r}\;\;(\theta\,e^{\mathrm{x}})^{\mathrm{r}}$$

$$=(\theta\,e^{\mathrm{x}})^{\mathrm{r}}\;[1-(1-\theta)e^{\mathrm{x}}]^{-r}$$

$$=\left[\frac{\theta\,e^{\mathrm{x}}}{1-(1-\theta)e^{\mathrm{x}}}\right]^{r}$$

5.4.7 Poisson distribution

$$When\qquad p(\lambda;\theta)=\frac{e^{-\theta}\theta^{\lambda}}{\lambda!}$$

for $\lambda=0,1,2,\dots$ and $\theta>0$.

The mgf a Poisson distribution is,

$$M_{\lambda}\,(x)=E\left[e^{x\lambda}\right]=\sum\nolimits_{0}^{\infty}e^{x\lambda}\,p(\lambda)$$

$$= \sum_{\lambda=0}^{\infty} e^{x\lambda} \frac{e^{-\theta}\theta^{\lambda}}{\lambda!}$$

$$= e^{-\theta} \sum_{\lambda=0}^{\infty} \frac{(\theta\, e^{x})^{\lambda}}{\lambda!}$$

$$= e^{-\theta} e^{\theta\, e^{x}}$$

$$= e^{\theta(\, e^{x}-1)}$$

5.4.8 Discrete uniform distribution

$$When\ \ p(\lambda; n) = \frac{1}{n}\ , \qquad \lambda = 1,2,\dots,n\ , \qquad n \in N.$$

The mgf of a discrete uniform distribution is

$$M_{\lambda}\,(x) = E\left[e^{x\lambda}\right] = \sum_{1}^{n} e^{x\lambda}\, p(\lambda)$$

$$= \frac{1}{n} \sum_{1}^{n} e^{x\lambda}$$

$$= \frac{e^{x}}{n} \left(\frac{1 - e^{nx}}{1 - e^{x}} \right)$$

5.4.9 Logarithmic distribution

$$When\ p(\lambda; \theta\,) = -\frac{\theta^{\lambda}}{\lambda ln(1-\theta)}\ , \qquad \lambda = 0,1,2,\dots \quad 0 < \theta < 1$$

The mgf of a logarithmic distribution is

$$M_{\lambda}\,(x) = E\left[e^{x\lambda}\right] = \sum_{0}^{\infty} e^{x\lambda}\, p(\lambda)$$

$$= -\sum_{1}^{\infty} e^{x\lambda} \frac{\theta^{\lambda}}{\lambda ln(1-\theta)}$$

$$= -\frac{1}{ln(1-\theta)} \sum_{1}^{\infty} \frac{(\theta e^{x})^{\lambda}}{\lambda}$$

$$= -\frac{[-ln(1-\theta e^{x})]}{ln(1-\theta)} = \frac{ln(1-\theta e^{x})}{ln(1-\theta)}$$

5.5 Moment generating functions of continuous distributions

Equation (8) will apply in the sections 5.3.2.1 - 5.3.2.13 with *x* replacing s and λ replacing *y*

5.5.1 Beta distribution

$$When\ \ f(\lambda;\alpha,\beta)=\frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)},\qquad 0\le\ \lambda\ \le\ 1,\qquad \alpha>0,\qquad \beta\ >0$$

The mgf of a beta distribution is

$M_{\lambda}\ (x)=E\left[e^{x\lambda}\right]=\ \int_0^1\ e^{x\lambda}f(\lambda)d\lambda$

$$=\int_0^1\ e^{x\lambda}\frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)}d\lambda$$

$$=\frac{1}{B(\alpha,\beta)}\int_0^1\lambda^{\alpha-1}(1-\lambda)^{\beta-1}\ e^{x\lambda}d\lambda$$

$=\frac{1}{B(\alpha,\beta)}B(\alpha,\beta)\ {}_1F_1(\alpha;\alpha+\beta;x)$ [16] (APPENDIX 14)

$={}_1F_1(\alpha;\alpha+\beta;x)$, where for constants *a, b* and variable *z*,

${}_1F_1(a;b;z)$ is the confluent hypergeometric function defined by

${}_1F_1(a;b;z)=\sum_{k=0}^{\infty}\frac{(a)_k\,z^k}{(b)_k k!}$, and where for a parameter r , $(r)_k$ denotes

Pochhammer's symbol for increasing factorial given by

$(r)_k=r(r+1)(r+2)(r+3)\ldots(r+k)$

$$=\frac{\Gamma(r+k)}{\Gamma(r)}$$

5.5.2 Exponential distribution

When $g(\lambda;\theta\ )=\theta e^{-\theta\lambda},\ \ \theta>0,\quad \lambda\ >0,$

The mgf of an exponential distribution is

$$M_{\lambda}\ (x)=E\left[e^{x\lambda}\right]=\ \int_0^{\infty}\ e^{x\lambda}f(\lambda)d\lambda$$

$$=\int_0^{\infty}\ e^{x\lambda}\theta e^{-\theta\lambda}d\lambda$$

$$= \theta \int_0^\infty e^{-\lambda(\theta - x)} d\lambda$$

$$= -\frac{\theta}{(\theta - x)} \left[e^{-\lambda(\theta - x)}\right]_0^\infty$$

$$= \frac{\theta}{(\theta - x)}$$

5.5.3 One-parameter gamma distribution

$$When\ g(\lambda;\theta) = \frac{e^{-\lambda}\,\lambda^{\theta-1}}{\Gamma(\theta)}, \quad \lambda > 0, \quad \theta > 1$$

$$M_\lambda\,(x) = E\left[e^{x\lambda}\right] = \int_0^\infty e^{x\lambda} f(\lambda) d\lambda$$

$$= \int_0^\infty \mathrm{e}^{\lambda \mathrm{x}} \frac{e^{-\lambda}\,\lambda^{\theta-1}}{\Gamma(\theta)} \mathrm{d}\lambda$$

$$= \int_0^\infty \frac{\lambda^{\theta-1} e^{-\lambda(1-x)}}{\Gamma(\theta)} \mathrm{d}\lambda$$

Using substitution, let $u = \lambda(1 - x)$

$$\therefore d\lambda = \frac{du}{(1 - x)}$$

Thus,

$$M_\lambda\,(x) = \int_0^\infty \left(\frac{\mathrm{u}}{1 - \mathrm{x}}\right)^{\theta-1} \frac{e^{-u}}{\Gamma(\theta)} \frac{\mathrm{du}}{(1 - \mathrm{x})}$$

$$= \frac{1}{(1 - \mathrm{x})^\theta} \int_0^\infty \frac{\mathrm{u}^{\theta-1} e^{-u}}{\Gamma(\theta)} du$$

$$= \frac{1}{(1 - x)^\theta}$$

5.5.4 Two-parameter gamma distribution

$$When\ g(\lambda;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\ e^{-\lambda\beta} \lambda^{\alpha-1}, \quad \lambda > 0, \quad \alpha > 0, \quad \beta > 0.$$

The mgf of a two-parameter gamma distribution is

$$M_{\lambda}(x) = E\left[e^{x\lambda}\right] = \int_0^{\infty} e^{x\lambda} f(\lambda) d\lambda$$

$$= \int_0^{\infty} e^{x\lambda} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \ e^{-\lambda\beta} \lambda^{\alpha-1} d\lambda$$

$$= \beta^{\alpha} \int_0^{\infty} \frac{e^{-\lambda(\beta-x)}}{\Gamma(\alpha)} \lambda^{\alpha-1} d\lambda$$

Using substitution, let $u = \lambda(\beta - x)$,

$$\therefore d\lambda = \frac{du}{(\beta - x)}$$

Thus

$$M_{\lambda}(x) = \beta^{\alpha} \int_0^{\infty} \left(\frac{\mathrm{u}}{\beta - \mathrm{x}}\right)^{\alpha-1} \frac{e^{-u}}{\Gamma(\alpha)} \frac{\mathrm{du}}{(\beta - \mathrm{x})}$$

$$= \frac{\beta^{\alpha}}{(\beta - \mathrm{x})^{\alpha}} \int_0^{\infty} \frac{\mathrm{u}^{\alpha-1} e^{-u}}{\Gamma(\alpha)} du$$

$$= \left(\frac{\beta}{\beta - \mathrm{x}}\right)^{\alpha}$$

5.5.5 Chi-square distribution

$$When \quad g(\lambda; \mathrm{n}) = X^2{}_{(n)} = \frac{2^{\mathrm{n}/2}}{\Gamma(n/2)} \ e^{-2\lambda} \lambda^{(\mathrm{n}/2)-1}$$

The mgf of a gamma distribution is

$$M_{\lambda}(x) = E\left[e^{x\lambda}\right] = \int_0^{\infty} e^{x\lambda} f(\lambda) d\lambda$$

$$= \int_0^{\infty} e^{x\lambda} \frac{2^{\mathrm{n}/2}}{\Gamma\left(\frac{n}{2}\right)} \ e^{-2\lambda} \lambda^{\left(\frac{\mathrm{n}}{2}\right)-1} d\lambda$$

$$= \frac{2^{\mathrm{n}/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^{\infty} \lambda^{\left(\frac{\mathrm{n}}{2}\right)-1} e^{-\lambda(2-x)} \, d\lambda$$

Using substitution, let

$$\lambda(2-x) = u \qquad\qquad \therefore d\lambda = \frac{du}{2-x}$$

$$\therefore\ M_\lambda\,(x) = \frac{2^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{u^{(n/2)-1}}{(2-x)^{(n/2)-1}} e^{-u} \frac{du}{(2-x)}$$

$$= \frac{2^{n/2}}{(2-x)^{(n/2)}} \int_0^\infty \frac{u^{(n/2)-1}}{\Gamma(n/2)} e^{-u}\, du$$

$$= \left(\frac{2}{2-x}\right)^{n/2}, \qquad x < 2$$

5.5.6 Inverse gamma distribution

When $g(\lambda;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)\ \lambda^{\alpha+1}}\ e^{-\beta/\lambda}, \qquad \lambda > 0, \quad \alpha > 0, \qquad \beta > 0$

The mgf of an inverse gamma distribution is

$$M_\lambda\,(x) = E\left[e^{x\lambda}\right] = \int_0^\infty e^{x\lambda} f(\lambda) d\lambda$$

$$= \int_0^\infty e^{\lambda x} \frac{\beta^\alpha}{\Gamma(\alpha)\ \lambda^{\alpha+1}}\ e^{-\beta/\lambda}\, d\lambda$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{-\alpha-1} e^{\lambda x - \beta/\lambda}\ d\lambda$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} . 2 \left(\frac{\beta}{-x}\right)^{\frac{-\alpha}{2}} K_{-\alpha}(2\sqrt{(-\beta x)})$$.[17](APPENDIX 15)

$$= \frac{2\beta^\alpha(-x)^{\frac{\alpha}{2}}}{\Gamma(\alpha)} K_{-\alpha}(2\sqrt{(-\beta x)}),$$ where for a constant *v* and a parameter *z*,
$K_v(z)$ is the modified Bessel function of the second kind and order v defined by ;
$K_v(z) = \int_0^\infty\ e^{-z\cosh t} \cosh vt\ \ dt$.[18] (APPENDIX 16)

5.5.7 Erlang distribution

$when\ g(\lambda; n, \beta) = \frac{\beta^n}{(n-1)!} \lambda^{n-1}\ e^{-\lambda\beta}, \quad \lambda > 0, \quad \beta > 0,\ n \in \mathrm{N}$

The mgf of an Erlang distribution is

$$M_\lambda\,(x) = E\left[e^{x\lambda}\right] = \int_0^\infty e^{x\lambda} f(\lambda) d\lambda$$

$$= \int_0^\infty e^{x\lambda} \frac{\beta^n}{(n-1)!} \lambda^{n-1} \; e^{-\lambda\beta} \, d\lambda$$

$$= \frac{\beta^n}{(n-1)!} \int_0^\infty \lambda^{n-1} \; e^{-\lambda(\beta - x)} \, d\lambda$$

$$= \frac{\beta^n}{(n-1)!} \int_0^\infty \left(\frac{\mathrm{z}}{\beta - x}\right)^{\mathrm{n}-1} \; e^{-z} \frac{dz}{\beta - x},$$

$where\ z = \lambda(\beta - x),$

$$= \left(\frac{\beta}{\beta - x}\right)^n \int_0^\infty \frac{1}{\Gamma(n)} \mathrm{z}^{\mathrm{n}-1} \, e^{-z} dz,$$

$$= \left(\frac{\beta}{\beta - x}\right)^n$$

5.5.8 Inverse Gaussian distribution

$$When \;\; g(\lambda; \mu, \beta) = \left(\frac{\beta}{2\pi}\right)^{1/2} \lambda^{-3/2} e^{\frac{-\beta(\lambda - \mu)^2}{2\mu^2\lambda}}, \qquad \lambda > 0, \qquad \mu > 0, \qquad \beta > 0$$

The moment generating function of the Inverse Gaussian (Wald) Distribution is given by

$$M_\lambda(x) = E\left[e^{x\lambda}\right] = \int_0^\infty e^{x\lambda} f(\lambda) d\lambda$$

$$= \int_0^\infty e^{x\lambda} \left(\frac{\beta}{2\pi}\right)^{1/2} \lambda^{-3/2} e^{\frac{-\beta(\lambda - \mu)^2}{2\mu^2\lambda}} \, d\lambda$$

$$= \left(\frac{\beta}{2\pi}\right)^{1/2} \int_0^\infty \lambda^{-3/2} \, e^{x\lambda} e^{\frac{-\beta(\lambda - \mu)^2}{2\mu^2\lambda}} \, d\lambda$$

$$= \left(\frac{\beta}{2\pi}\right)^{1/2} \int_0^\infty \lambda^{-3/2} e^{\frac{-(\beta - 2x\mu^2)\lambda}{2\mu^2} - \frac{\beta}{2\lambda} + \frac{\beta}{\mu}} \, d\lambda$$

$$= \left(\frac{\beta}{2\pi}\right)^{1/2} e^{\frac{\beta}{\mu}} \int_0^\infty \lambda^{-3/2} e^{\frac{-(\beta - 2x\mu^2)\lambda}{2\mu^2} - \frac{\beta}{2\lambda}} \, d\lambda$$

$$= 2\left(\frac{\beta}{2\pi}\right)^{1/2} e^{\frac{\beta}{\mu}} \left(\frac{\beta\mu^2}{\beta - 2x\mu^2}\right)^{-1/4} K_{-1/2}\left[\frac{1}{\mu}\sqrt{\beta(\beta - 2x\mu^2)}\right].$$ [18] (APPENDIX 15)

$$= \left(\frac{2}{\pi\beta\mu}\right)^{1/2} (\beta - 2x\mu^2)^{1/4} e^{\frac{\beta}{\mu}} K_{-1/2}\left[\frac{1}{\mu}\sqrt{\beta(\beta - 2x\mu^2)}\right]$$

5.5.9 Generalized Inverse Gaussian distribution

$$When\ \ g(\lambda;\theta,\psi,\chi) = \left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}} \frac{\lambda^{\theta-1}}{2K_\theta\left(\sqrt{\chi\psi}\right)} e^{-\frac{1}{2}\left(\frac{\chi}{\lambda}+\psi\lambda\right)}\ , \qquad \lambda > 0,$$

$\chi > 0, \qquad \psi \geq 0\ \ when\ \theta < 0$

$\chi > 0, \qquad \psi > 0\ \ when\ \theta = 0$

$\chi \geq 0, \qquad \psi > 0\ \ when\ \theta > 0$

The mgf of the Generalized Inverse Gaussian distribution is given by

$$M_\lambda\ (x) = E\left[e^{x\lambda}\right] = \int_0^\infty e^{x\lambda} f(\lambda) d\lambda$$

$$= \int_0^\infty e^{x\lambda} \left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}} \frac{\lambda^{\theta-1}}{2K_\theta\left(\sqrt{\chi\psi}\right)} e^{-\frac{1}{2}\left(\frac{\chi}{\lambda}+\psi\,\lambda\right)} d\lambda$$

$$= \left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}} \frac{1}{2K_\theta\left(\sqrt{\chi\psi}\right)} \int_0^\infty \lambda^{\theta-1}\ e^{x\lambda} e^{-\frac{1}{2}\left(\frac{\chi}{\lambda}+\psi\,\lambda\right)} d\lambda$$

$$= \left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}} \frac{1}{2K_\theta\left(\sqrt{\chi\psi}\right)} \int_0^\infty \lambda^{\theta-1}\ e^{x\lambda} e^{-\frac{1}{2}\left(\frac{\chi}{\lambda}+\psi\,\lambda\right)} d\lambda$$

$$= \left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}} \frac{1}{2K_{\theta(\sqrt{\chi\psi})}} \int_0^\infty \lambda^{\theta-1} e^{-\lambda\left(\frac{1}{2}\psi - x\right) - \frac{\frac{1}{2}\chi}{\lambda}}\ d\lambda$$

$$= \left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}} \left(\frac{\chi}{\psi - 2x}\right)^{\frac{\theta}{2}}\ \frac{K_\theta\left(\sqrt{\chi(\psi - 2x)}\right)}{K_\theta\left(\sqrt{\chi\psi}\right)}$$

.[19] (APPENDIX 9)

5.5.10 Half-normal distribution

When $g(\lambda;\ \sigma) = \frac{1}{\sigma}\ \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\lambda^2}{2\sigma^2}},\ \lambda > 0,\ \ \sigma > 0$

The moment generating function of the half-normal distribution is given by

$$M_\lambda\ (x) = E\left[e^{x\lambda}\right] = \int_0^\infty e^{x\lambda} f(\lambda) d\lambda$$

$$= \int_0^\infty e^{x\lambda} \frac{1}{\sigma} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\lambda^2}{2\sigma^2}} d\lambda$$

$$= \frac{1}{\sigma} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty e^{-\frac{\lambda^2}{2\sigma^2}+x\lambda} d\lambda$$

$$= \frac{1}{\sigma} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty e^{-\frac{1}{2\sigma^2}(\lambda^2-2\sigma^2 x\lambda+\sigma^4x^2-\sigma^4x^2)} d\lambda$$

$$= \frac{1}{\sigma} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{\frac{\sigma^2x^2}{2}} \int_0^\infty e^{-\frac{1}{2}\left(\frac{\lambda-\sigma^2x}{\sigma}\right)^2} d\lambda$$

$Let\ z = \dfrac{\lambda - \sigma^2 x}{\sigma}$, $d\lambda = \sigma dz$

Limits

λ	Z
0	-σx
∞	∞

$$\therefore \quad M_\lambda(x) = 2e^{\frac{\sigma^2x^2}{2}} \frac{1}{\sqrt{(2\pi)}} \int_{-\sigma x}^\infty e^{-\frac{1}{2}z^2} dz ,$$

$$= 2e^{\frac{\sigma^2x^2}{2}}[1-\phi(-\sigma x)]$$

$$= 2e^{\frac{\sigma^2x^2}{2}}[1+\phi(\sigma x)]$$

5.5.11 Rayleigh distribution

When $g(\lambda;\ \sigma) = \frac{\lambda}{\sigma^2} e^{-\frac{\lambda^2}{2\sigma^2}}, \quad \lambda \geq 0,\ \ \sigma > 0$

The moment generating function of a Rayleigh Distribution is

$$M_\lambda(x) = E[e^{x\lambda}] = \int_0^\infty e^{x\lambda} f(\lambda) d\lambda$$

$$= \int_0^\infty e^{x\lambda} \frac{\lambda}{\sigma^2} e^{-\frac{\lambda^2}{2\sigma^2}} d\lambda$$

$$= \frac{1}{\sigma^2} \int_0^{\infty} \lambda \, e^{-\frac{\lambda^2}{2\sigma^2} + x\lambda} \, d\lambda$$

$$= \frac{e^{\frac{\sigma^2 x^2}{2}}}{\sigma^2} \int_{-\sigma x}^{\infty} (\sigma z + \sigma^2 x) e^{-\frac{1}{2}z^2} \sigma \, dz,$$

by introducing z as in **section 5. 5.10**

$$\therefore \quad M_\lambda(x) = \frac{e^{\frac{\sigma^2 x^2}{2}}}{\sigma^2} \int_{-\sigma x}^{\infty} (\sigma z + \sigma^2 x) e^{-\frac{1}{2}z^2} \sigma \, dz,$$

$$= e^{\frac{\sigma^2 x^2}{2}} \int_{-\sigma x}^{\infty} (z + \sigma x) e^{-\frac{1}{2}z^2} \, dz,$$

$$= e^{\frac{\sigma^2 x^2}{2}} \int_{-\sigma x}^{\infty} z e^{-\frac{1}{2}z^2} \, dz + \sigma x e^{\frac{\sigma^2 x^2}{2}} \int_{-\sigma x}^{\infty} e^{-\frac{1}{2}z^2} \, dz$$

$Let \; I_1 = e^{\frac{\sigma^2 x^2}{2}} \int_{-\sigma x}^{\infty} z e^{-\frac{1}{2}z^2} \, dz$ and $I_2 = \sigma x e^{\frac{\sigma^2 x^2}{2}} \int_{-\sigma x}^{\infty} e^{-\frac{1}{2}z^2} \, dz$

Considering I_1;

$Let \; y = \frac{1}{2} z^2, \quad dz = \frac{dy}{z}$

Limits

λ	**Z**
$-\sigma x$	$\frac{\sigma^2 x^2}{2}$
∞	∞

$$\therefore \; I_1 = e^{\frac{\sigma^2 x^2}{2}} \int_{\frac{\sigma^2 x^2}{2}}^{\infty} z e^{-y} \frac{dy}{z}$$

$$= -e^{\frac{\sigma^2 x^2}{2}} \left[e^{-y} \right]_{\frac{\sigma^2 x^2}{2}}^{\infty} = 1$$

Considering I_2;

$$I_2 = \sigma x e^{\frac{\sigma^2 x^2}{2}} \sqrt{(2\pi)} \frac{1}{\sqrt{(2\pi)}} \int_{-\sigma x}^{\infty} e^{-\frac{1}{2}z^2} \, dz$$

$$= \sigma x \sqrt{2\pi} \quad e^{\frac{\sigma^2 x^2}{2}} \left[1 - \phi(-\sigma x) \right]$$

$$= \sigma x\sqrt{2\pi} \quad e^{\frac{\sigma^2 x^2}{2}} [1 + \phi(\sigma x)]$$

$$\therefore \quad M_\lambda (x) = 1 + \sigma x\sqrt{2\pi} \quad e^{\frac{\sigma^2 x^2}{2}} [1 + \phi(\sigma x)]$$

5.5.12 Uniform (rectangular)distribution

When $g(\lambda; \mathrm{b}) = \frac{1}{\mathrm{b}}$, $\quad 0 < \lambda < b, \ b > 0$

The mgf of a uniform distribution is

$$M_\lambda (x) = E\left[e^{x\lambda}\right] = \int_0^\infty e^{x\lambda} f(\lambda) d\lambda$$

$$= \frac{1}{\mathrm{b}} \int_0^b e^{x\lambda} d\lambda$$

$$= \frac{e^{bx} - 1}{\mathrm{bx}}$$

5.5.13 Chi distribution

$$When\ gamma(\lambda; n) = \frac{\lambda^{\mathrm{n}-1}}{2^{\frac{\mathrm{n}}{2}-1}\Gamma(n/2)} \ e^{-\lambda^2/2} , \qquad \lambda \geq 0, \ n = 1,2, \ldots$$

The mgf of a chi distribution is

$$M_\lambda (x) = \int_0^\infty e^{\lambda x} \frac{\lambda^{\mathrm{n}-1}}{2^{\frac{\mathrm{n}}{2}-1}\Gamma(n/2)} \ e^{-\lambda^2/2} \ d\lambda$$

$$= \frac{1}{2^{\frac{\mathrm{n}}{2}-1}\Gamma(n/2)} \int_0^\infty \lambda^{\mathrm{n}-1} \ e^{-\frac{\lambda^2}{2}+\lambda x} \ d\lambda$$

$$= \frac{\Gamma(n)}{2^{\frac{\mathrm{n}}{2}-1}\Gamma\left(\frac{n}{2}\right)} \mathrm{e}^{x^2/4} \mathrm{D}_{-\mathrm{n}}(-x)$$ [20] (APPENDIX 17)

Where for a parameter *z* and a constant *p,* $D_p(z)$ is a parabolic cylindrical function given by

$$D_p(z) = 2^{-p/2} e^{-z^2/4} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)} \, {}_1F_1\left(\frac{-p}{2}, \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2\pi} z}{\Gamma\left(\frac{-p}{2}\right)} \, {}_1F_1\left(\frac{1-p}{2}, \frac{3}{2}; \frac{z^2}{2}\right) \right\}.$$ [21]

(APPENDIX 18)

Chapter 6

Exponential mixtures, survival-time and related functions

In this chapter all survival time and related functions of mixtures of exponential distribution with mixing distributions mentioned in **sections 5.4** and **5.5** are derived

6.1 Mixtures with discreet distributions

6.1.1 Exponential-Bernoulli mixture

For a Bernoulli distribution, $p(\lambda;\theta\,) = \theta^{\lambda}(1-\theta)^{1-\lambda}, \quad 0<\theta\leq 1, \quad \lambda = 0,1,$

The mgf of a Bernoulli distribution is

$M_{\lambda}\,(x) \;= 1-\theta(1-e^{x})$

The following are survival-time and related functions for the exponential-Bernoulli mixture.

Survival function:

$$S\,(x) = M_{\lambda}\,(-x) = 1-\theta(1-e^{-x})$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = \theta(1-e^{-x})$$

Probability density function:

$$f(x) = F'(x) = \theta e^{-x}$$

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{\theta e^{-x}}{1-\theta(1-e^{-x})} = \frac{\theta}{(1-\theta\,)e^{x}+\theta}$$

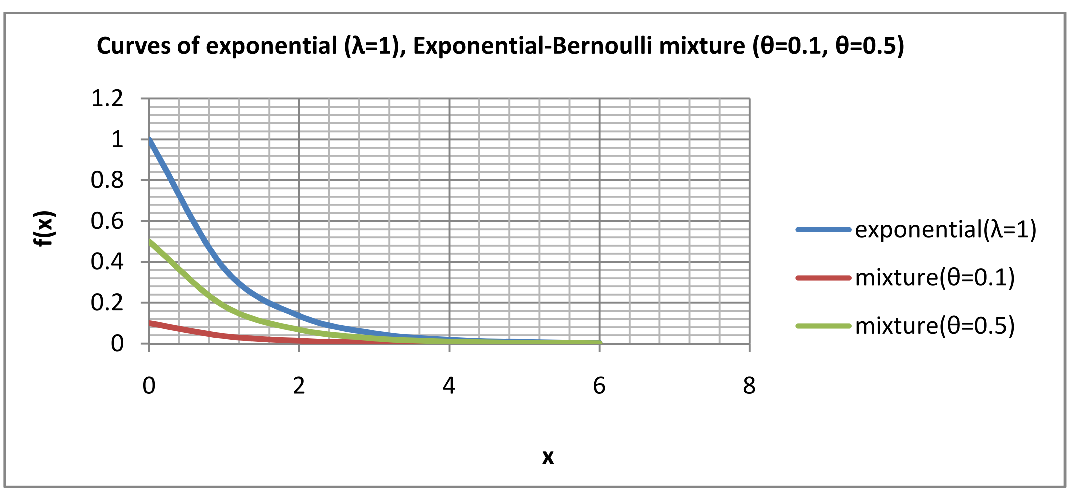

Fig. 6.1.1

6.1.2 Exponential-Binomial mixture

For a binomial distribution, $(\lambda; \theta, n) = \binom{n}{\lambda}\theta^{\lambda}(1-\theta)^{n-\lambda}$, $\lambda = 0, 1, 2 \ldots, n$, $0 \leq \theta \leq 1$

The mgf of a binomial distribution is

$M_{\lambda}\,(x) = [(1-\theta) + \theta e^{x}]^{n}$

The following are survival-time and related functions for the exponential-binomial mixture.

Survival function:

$$S\,(x) = M_{\lambda}\,(-x) = [(1-\theta) + \theta e^{-x}]^{n}$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - [(1-\theta) + \theta e^{-x}]^{n}$$

Probability density function:

$$f(x) = F'(x) = n\theta e^{-x}[(1-\theta) + \theta e^{-x}]^{n-1}$$

$$x \geq 0, \quad 0 \leq \theta \leq 1, \; n = 1, 2, \ldots$$

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{n\theta e^{-x}[(1-\theta) + \theta e^{-x}]^{n-1}}{[(1-\theta) + \theta e^{-x}]^{n}}$$

$$= \frac{n\theta}{\theta + (1-\theta)e^{x}}$$

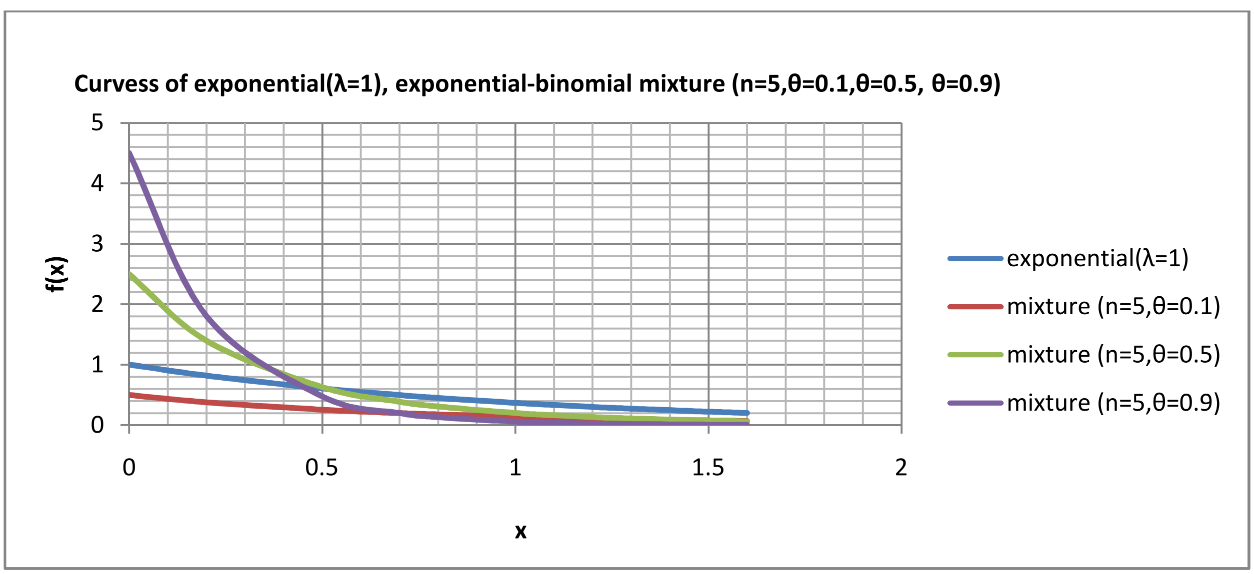

Fig.6.1.2(a)

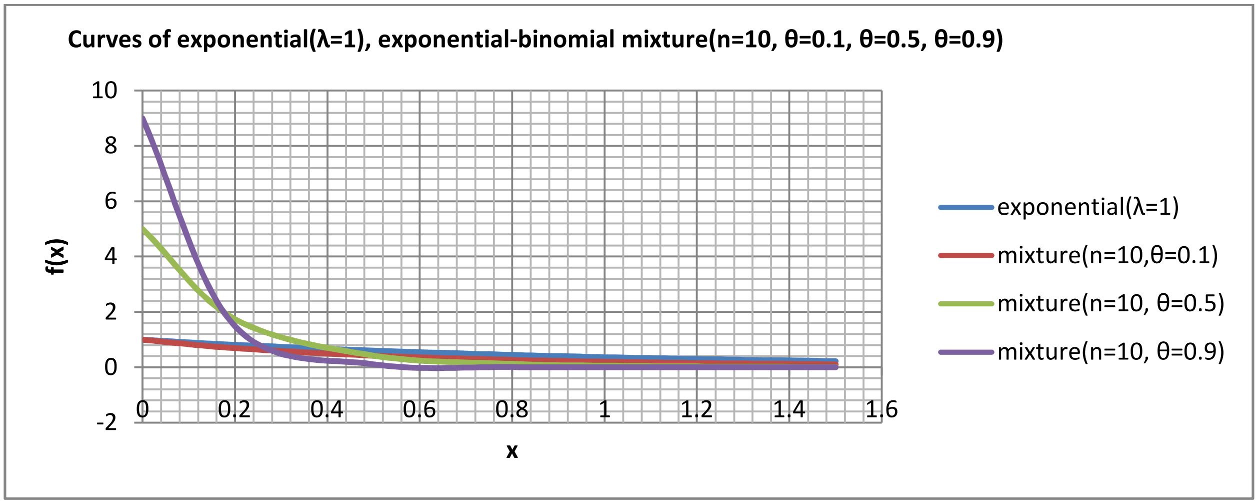

Fig. 6.1.2(b)

6.1.3 Exponential-Geometric type I mixture

when λ is the number of failures before a success,

$$p(\lambda;\theta) = \theta(1-\theta)^{\lambda-1} \quad \text{for } \lambda = 0, 1, 2, \ldots \text{ and } 0 \le \theta \le 1.$$

The mgf of a geometric type I distribution is

$$M_\lambda(-x) = \frac{\theta e^x}{1-(1-\theta)\, e^x}$$

The following are survival-time and related functions for the exponential-geometric type I mixture.

Survival function:

$$S(x) = M_\lambda(-x) = \frac{\theta e^{-x}}{1-(1-\theta)\, e^{-x}}$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \frac{\theta e^{-x}}{1-(1-\theta)\, e^{-x}} \qquad = \frac{e^x - 1}{e^x - 1 + \theta}$$

Probability density function:

$$f(x) = F'(x) = \frac{\theta e^x}{(e^x - 1 + \theta)^2}, \qquad x \ge 0, \quad 0 \le \theta \le 1$$

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{e^x}{e^x - 1 + \theta}$$

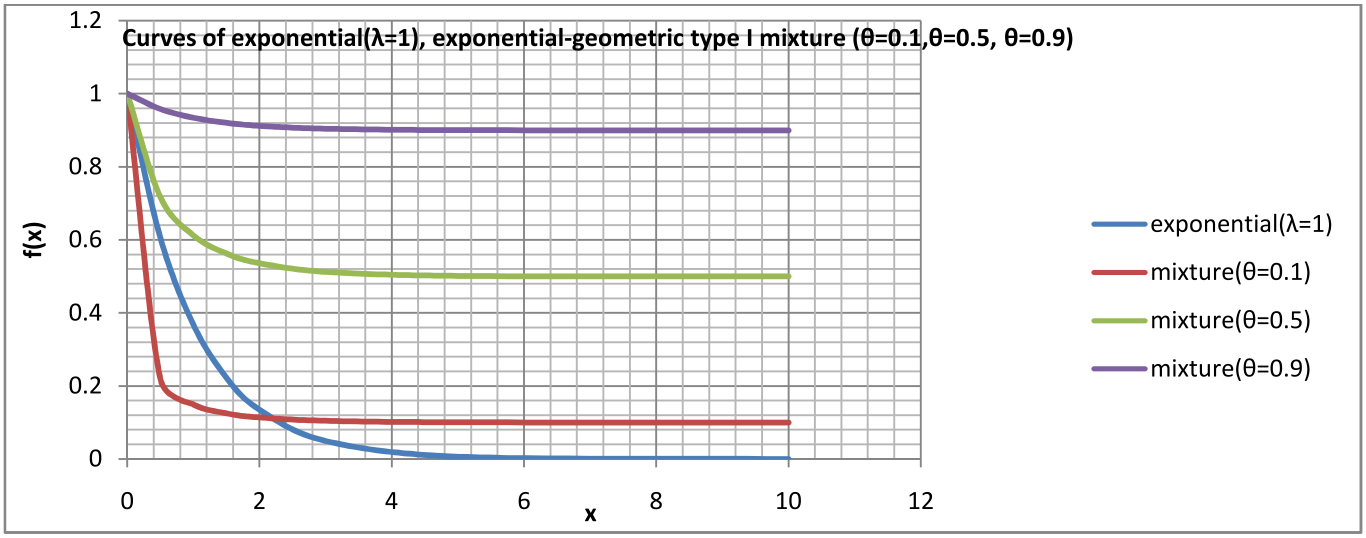

Fig. 6.1.3

Examining the curves of exponential ($\lambda = 1$), exponential-geometric type I (θ=0.1, θ=0.5, θ=0.9) in **Fig. 6.1.3** clearly shows that the mixtures approaches a uniform distribution as θ tends to 1

6.1.4 Exponential-Geometric type II mixture

when λ is the number of trials for a first success

$$p(\lambda;\theta) = \theta(1-\theta)^{\lambda} \quad \text{for } \lambda = 0, 1, 2, \ldots \text{ and } 0 \le \theta \le 1.$$

The mgf of a geometric type II distribution is

$$M_{\lambda}(-x) = \frac{\theta(1-\theta)\,e^{x}}{1-(1-\theta)\,e^{x}}$$

The following are survival-time functions for the exponential-geometric type II mixture.

Survival function:

$$S(x) = M_{\lambda}(-x) = \frac{\theta(1-\theta)\,e^{-x}}{1-(1-\theta)\,e^{-x}} = \frac{\theta(1-\theta)}{e^{x}-1+\theta}$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \frac{\theta(1-\theta)}{e^{x}-1+\theta} = \frac{e^{x}-1+\theta^{2}}{e^{x}-1+\theta}$$

Probability density function:

$$f(x) = F'(x) = \frac{(1-\theta)\,\theta e^{x}}{(e^{x}-1+\theta)^{2}}, \qquad x \ge 0, \quad 0 \le \theta \le 1$$

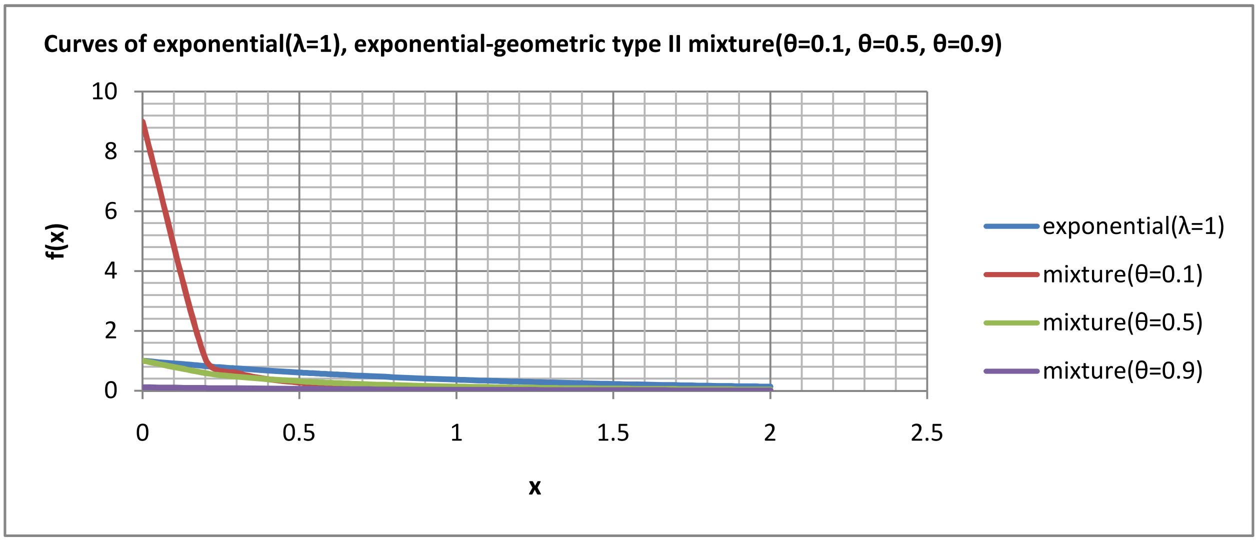

Fig.6.1.4

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{e^x}{e^x - 1 + \theta}$$

6.1.5 Exponential-Negative binomial type I mixture

When λ is the number of failures before the rth success

$$p(\lambda;\theta,r) = \binom{\lambda+n-1}{n-1}\theta^{\mathrm{r}}(1-\theta)^{\lambda}, \quad \lambda = 0,1,2\ldots,n\,, \quad 0 \le \theta \le 1$$

The mgf of a negative binomial type I distribution is

$$M_{\lambda}\,(x) = \left(\frac{\theta}{1-(1-\theta)\,e^{x}}\right)^{r}$$

The following are survival-time and related functions for the exponential-negative binomial type I mixture.

Survival function:

$$S(x) = M_{\lambda}\,(-x) = \left(\frac{\theta}{1-(1-\theta)\,e^{-x}}\right)^{r}$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \left(\frac{\theta}{1-(1-\theta)\,e^{-x}}\right)^{r}$$

Probability density function:

$$f(x) = F'(x) = \frac{r(1-\theta)(\theta e^x)^r}{(e^x - 1 + \theta)^{r+1}}, \quad x \geq 0, \quad 0 \leq \theta \leq 1$$

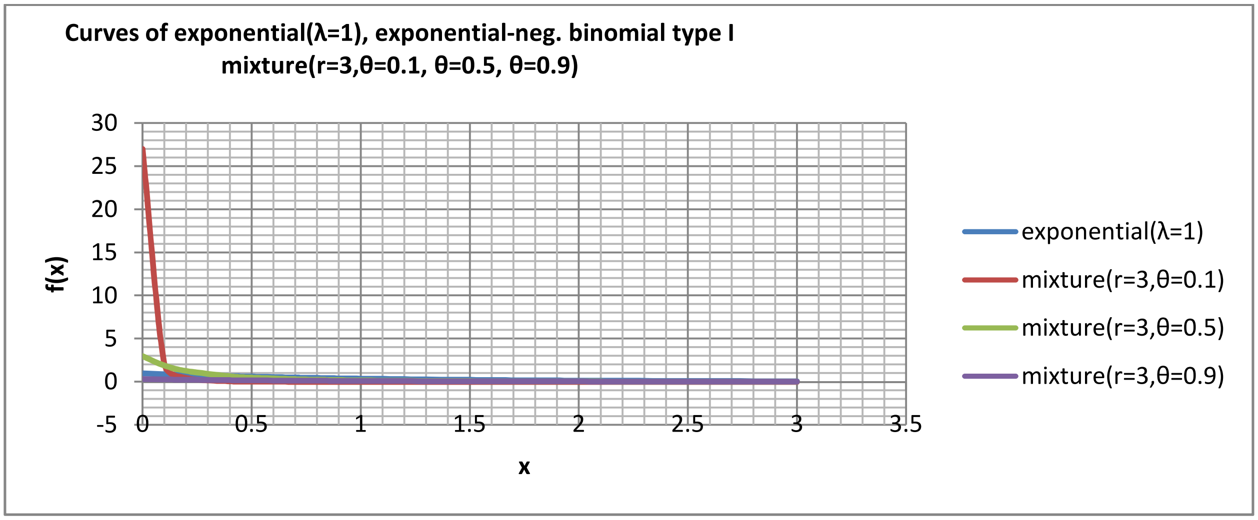

Fig. 6.1.5(a)

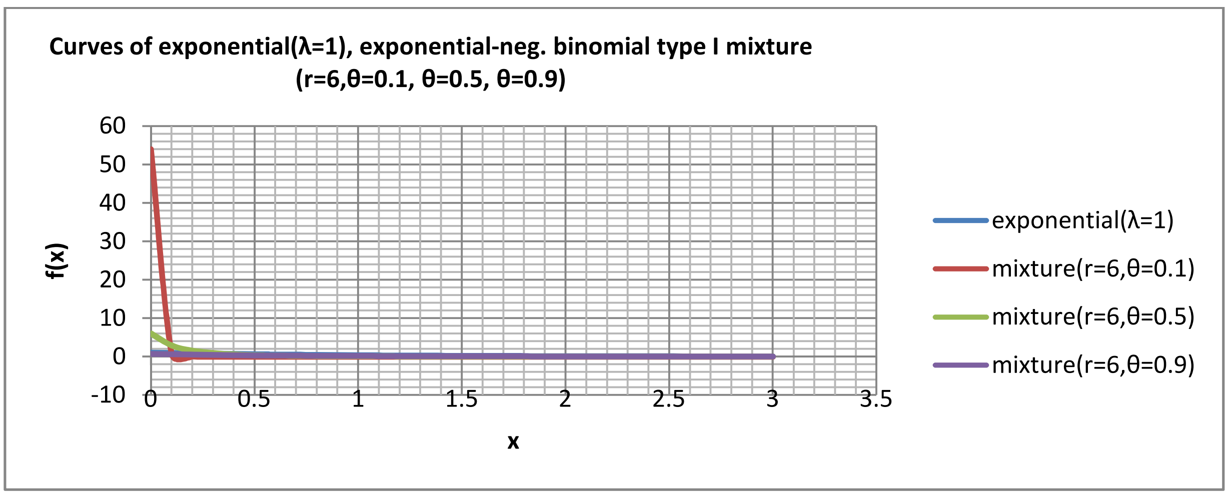

Fig. 6.1.5(b)

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{r(1-\theta)}{e^x - 1 + \theta}$$

6.1.6 Exponential-Negative binomial type II mixture

When λ is the total number of trials to achieve r successes

$$p(\lambda; \theta, n) = \binom{\lambda-1}{r-1}\theta^{\mathrm{r}}(1-\theta)^{\lambda-\mathrm{r}}, \quad \lambda = r, r+1, r+2 \ldots, \quad 0 \leq \theta \leq 1$$

The mgf of a negative binomial type II distribution is

$$M_\lambda\,(x) = \left[\frac{\theta\, e^{x}}{1-(1-\theta)e^{x}}\right]^{r}$$

The following are the survival-time and related functions for the exponential-negative binomial type II mixture.

Survival function:

$$S(x) = M_\lambda\,(-x) = \left(\frac{\theta e^{-x}}{1-(1-\theta)\,e^{-x}}\right)^{r}$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \left(\frac{\theta e^{-x}}{1-(1-\theta)\,e^{-x}}\right)^{r} = 1 - \left(\frac{\theta}{e^{x}-1+\theta}\right)^{r}$$

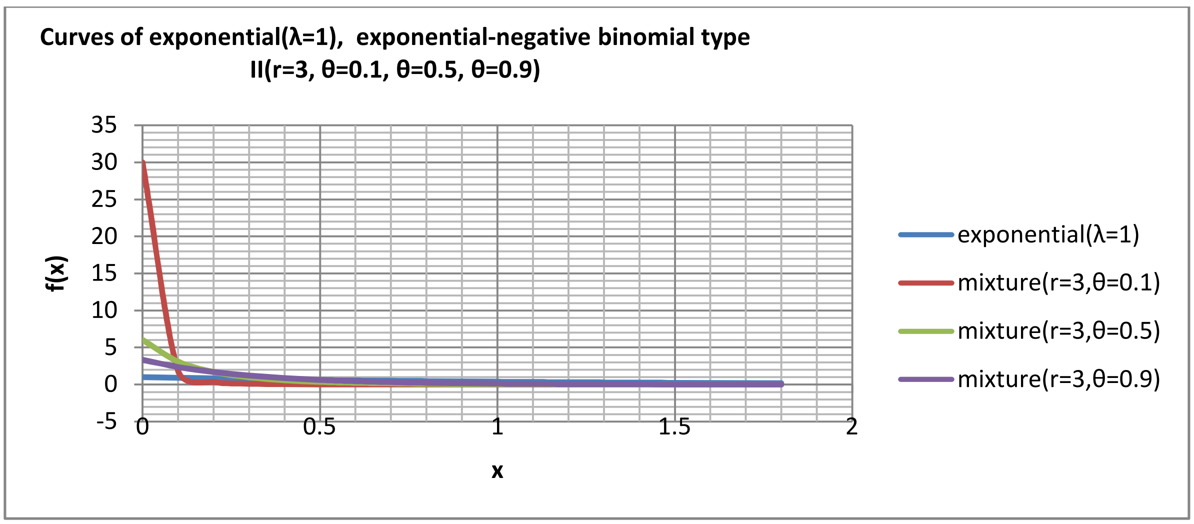

Fig.6.1.6(a)

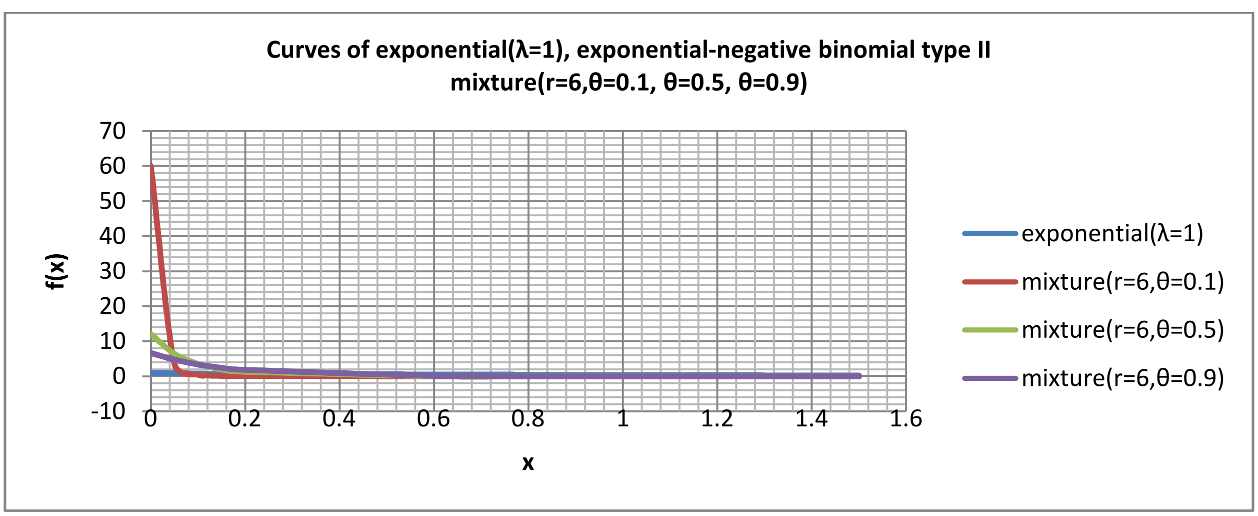

Fig. 6. 4.1.6(b)

Probability density function:

$$f(x) = F'(x) = \frac{r\ \theta^r e^x}{(e^x - 1 + \theta)^{r+1}}\ ,\ \ x \geq 0, \qquad r = 1,2,\ldots \quad 0 \leq \theta \leq 1$$

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{r\ e^x}{e^x - 1 + \theta}$$

6.1.7 Exponential-Poisson mixture

For a Poisson distribution,

$$p(\lambda;\theta) = \frac{e^{-\theta}\theta^{\lambda}}{\lambda!}, \quad \lambda = 0, 1,2, \ldots \ \text{ and } \ \theta > 0$$

The mgf of a Poisson distribution is

$$M_\lambda\,(x) = e^{\theta(e^x - 1)}$$

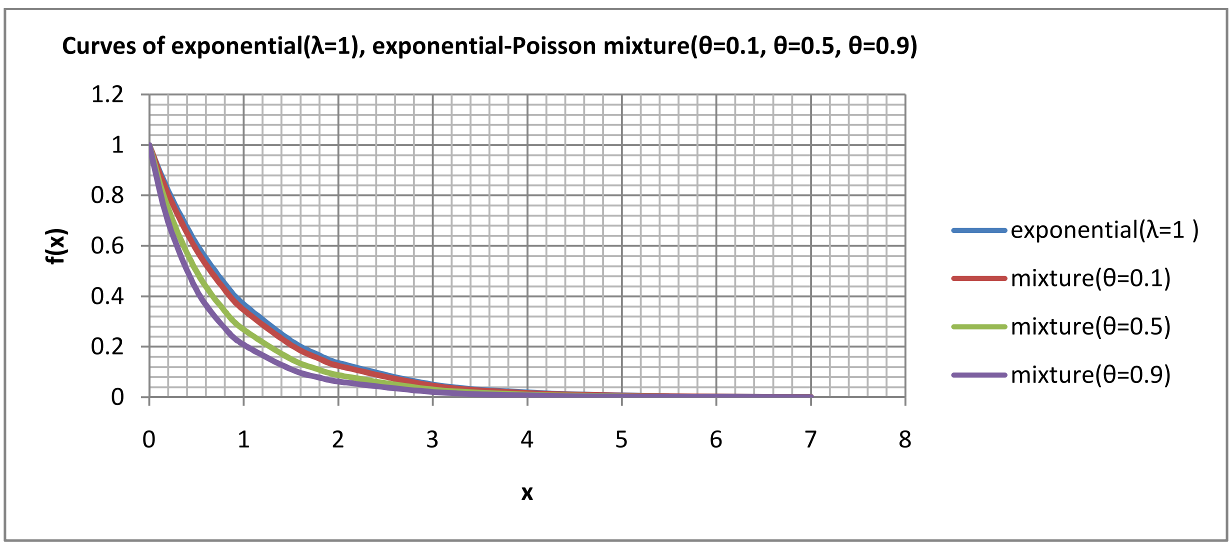

Fig. 6.1.7

The following are the survival-time and related functions for the exponential-Poisson mixture.

Survival function:

$$S(x) = M_\lambda\,(-x) = e^{\theta(e^{-x} - 1)}$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - e^{\theta(e^{-x} - 1)}$$

Probability density function:

$$f(x) = F'(x) = e^{\theta(e^{-x} - 1) - x} \qquad\qquad x > 0, \quad 0 \leq \theta \leq 1$$

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = e^{-x}$$

6.1.8 Exponential-discrete uniform mixture

when $p(\lambda; n) = 1/\mathrm{n}\,, \quad \lambda = 1,2,\ldots,n\,, \ \mathrm{n} \in N.$

The mgf of a discreet uniform distribution is

$$M_\lambda(x) = \frac{e^x(1-e^{nx})}{\mathrm{n}(1-e^x)}$$

The following are the survival-time and related functions for the exponential-discreet uniform mixture.

Survival function:

$$S(x) = M_\lambda(-x) = \frac{e^{-x}(1-e^{-nx})}{\mathrm{n}(1-e^{-x})} = \frac{e^{nx}-1}{\mathrm{n}e^{nx}(e^x-1)}$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \frac{e^{nx}-1}{\mathrm{n}e^{nx}(e^x-1)} = \frac{ne^{nx+1}-(n+1)e^{nx}+1}{\mathrm{n}e^{nx}(e^x-1)}$$

Probability density function:

$$f(x) = F'(x) = \frac{n(ne-n-1)e^{nx}(e^x-2)+(n+1)e^{(n+1)x}+(n+1)e^x-n}{\mathrm{n}e^{nx}(e^x-1)^2}$$

$$x > 0$$

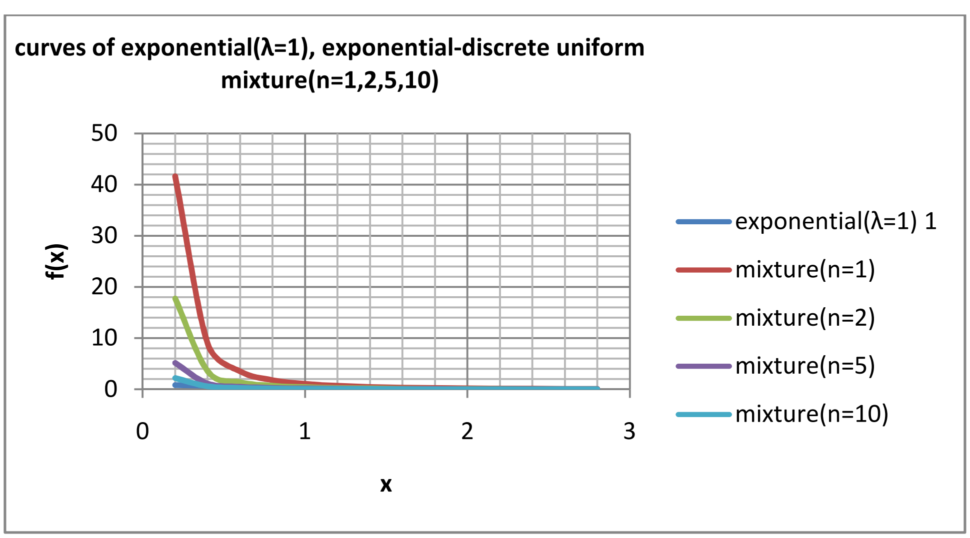

Fig. 6.1.8

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{n(ne - n - 1)e^{nx}(e^x - 2) + (n+1)e^{(n+1)x} + (n+1)e^x - n}{(e^x - 1)(e^{nx} - 1)}$$

6.1.9 Exponential-logarithmic mixture

$$When\ p(\lambda;\theta\ ) = -\frac{\theta^{\lambda}}{\lambda ln(1-\theta)}, \qquad \lambda = 1, 2, \ldots \quad 0 < \theta < 1$$

The mgf of a logarithmic distribution is

$$M_{\lambda}\ (x) \quad = \frac{log(1 - \theta e^x)}{\log(1 - \theta)}$$

The following are the survival-time and related functions for the exponential-logarithmic mixture.

Survival function:

$$S(x) = M_{\lambda}\ (-x) = \frac{log(1 - \theta e^{-x})}{\log(1 - \theta)}$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \frac{log(1 - \theta e^{-x})}{\log(1 - \theta)}$$

Probability density function:

$$f(x) = F'(x) = -\frac{\theta e^{-x}}{(1 - \theta e^{-x})\log(1 - \theta)} \qquad x > 0, \quad 0 < \theta < 1$$

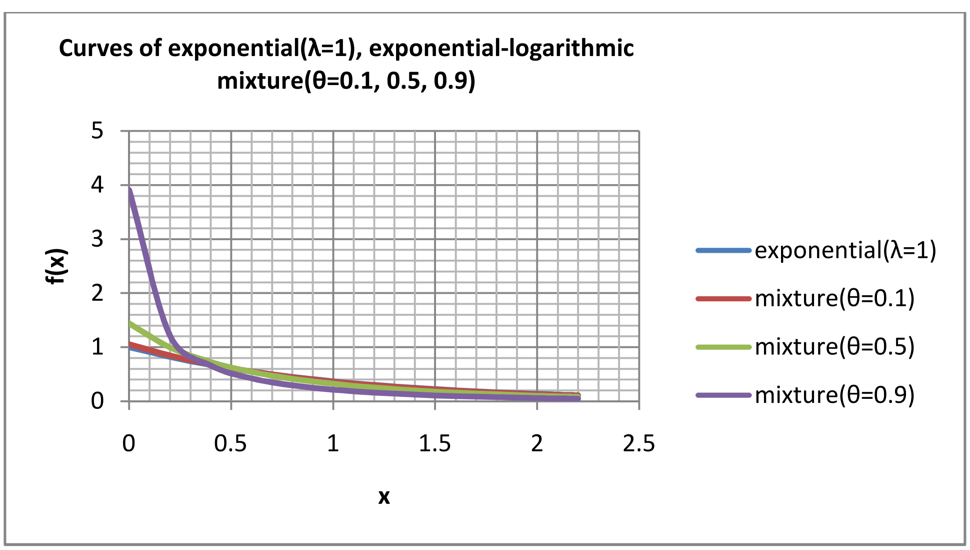

Fig.6.1.9

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = -\frac{\theta e^{-x}}{(1-\theta e^{-x})\log(1-\theta e^{-x})} = -\frac{\theta}{(e^{x}-\theta)\log(1-\theta e^{-x})}$$

6.2 Mixtures of the exponential distribution with continuous distributions

6.2.1 Exponential-beta mixture

$$When\ \ p(\lambda;\alpha,\beta) = \frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)}, 0 \le x \le 1, \qquad \alpha > 0, \qquad \beta > 0$$

The mgf of a beta distribution is

$$M_{\lambda}(x) = {}_1F_1(\alpha;\alpha+\beta;x)$$

The following are the survival-time and related functions for the exponential-beta mixture:

Survival function:

$$S(x) = M_{\lambda}(-x) = {}_1F_1(\alpha;\alpha+\beta;-x) = e^{-x}\,{}_1F_1(\beta;\alpha+\beta;x)$$.[22] (See APPENDIX 19)

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - e^{-x}\,{}_1F_1(\beta;\alpha+\beta;x)$$

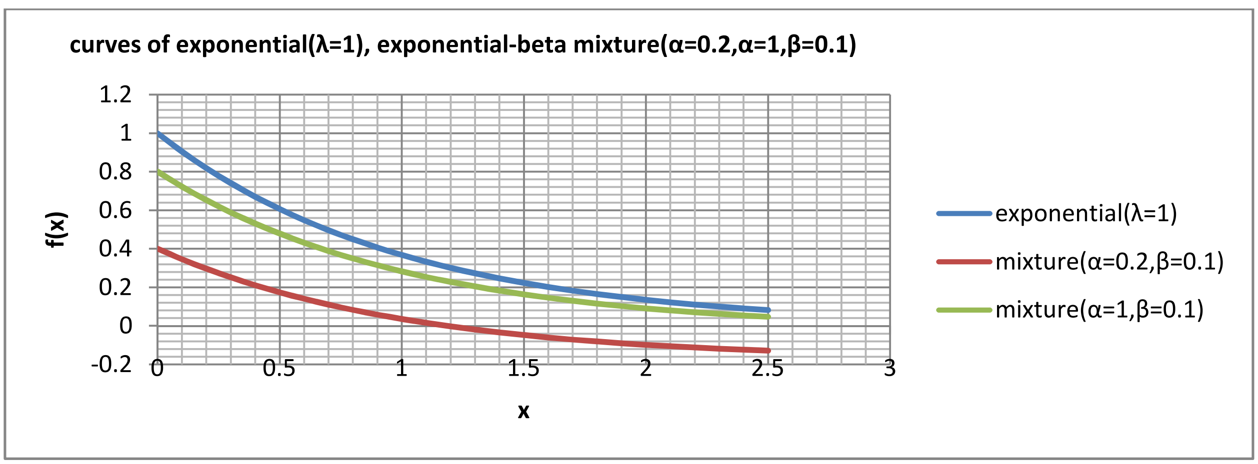

Fig.6.2.1

Probability density function:

$$f(x) = F'(x) = \frac{d}{dx}\left[\ 1 - e^{-x}\,{}_1F_1(\beta;\alpha+\beta;x)\right], \qquad 0 < x < 1$$

$$= e^{-x}\,{}_1F_1(\beta;\alpha+\beta;x) - e^{-x}\frac{\beta}{\alpha+\beta}\,{}_1F_1(\beta+1;\alpha+\beta+1;x)$$

$$= e^{-x}\left[\,{}_1F_1(\beta;\alpha+\beta;x) - \frac{\beta}{\alpha+\beta}\,{}_1F_1(\beta+1;\alpha+\beta+1;x)\right]$$

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{{}_1\mathrm{F}_1(\beta; \alpha+\beta; x) \;-\; \frac{\beta}{\alpha+\beta}\, {}_1\mathrm{F}_1(\beta+1; \alpha+\beta+1; x)}{{}_1\mathrm{F}_1(\beta; \alpha+\beta; x)}$$

6.2.2 Exponential-exponential mixture

When $g(\lambda; \theta\,) = \theta e^{-\theta\lambda}$, $\quad \theta > 0, \quad \lambda > 0,$

The mgf of an exponential distribution is

$$M_\lambda\,(x) = \frac{\theta}{\theta - x}$$

The following are the survival-time and related functions for the exponential-exponential mixture.

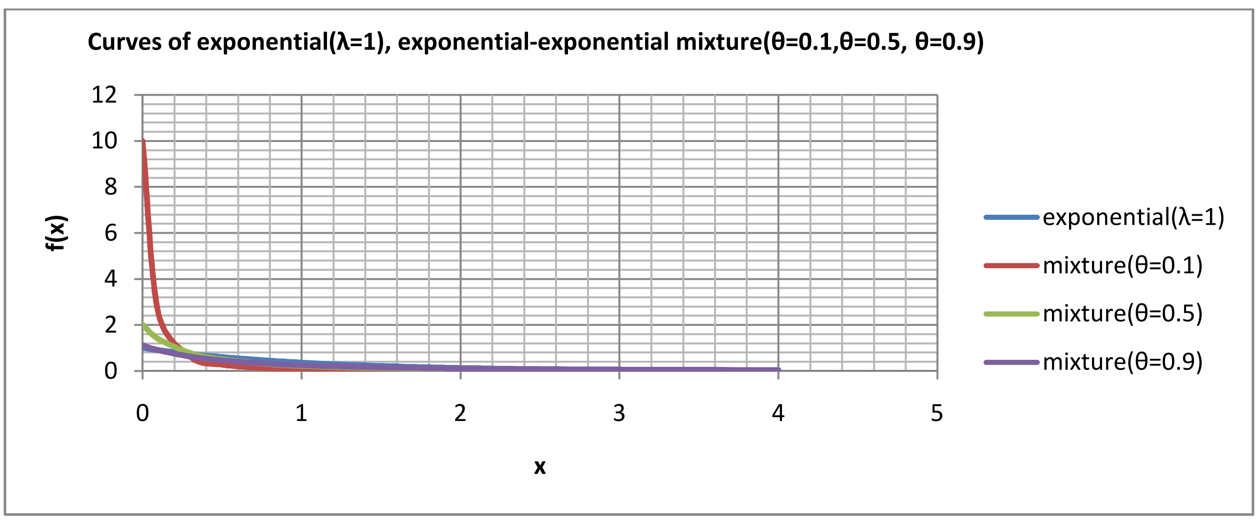

Fig. 6.2.2

Survival function:

$$S(x) = M_\lambda\,(-x) = \frac{\theta}{\theta + x}$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \frac{\theta}{\theta + x} \quad = \frac{x}{\theta + x}$$

Probability density function:

$$f(x) = F'(x) = \frac{d}{dx}\,[\,\frac{x}{\theta+x}\,], \quad x > 0$$

$$= \frac{\theta}{(\theta + x)^2}$$

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{\frac{\theta}{(\theta + x)^2}}{\frac{\theta}{\theta + x}} = \frac{1}{\theta + x}$$

6.2.3 Exponential-one parameter gamma mixture

When $g(\lambda;\theta) = \frac{e^{-\lambda}\lambda^{\theta-1}}{\Gamma(\theta)}, \qquad \lambda > 0, \quad \theta > 0$

The mgf of a one-parameter gamma distribution is

$$M_\lambda(x) = \frac{1}{(1-x)^\theta}$$

The following are the survival-time and related functions for the exponential-one-parameter gamma mixture.

Survival function:

$$S(x) = M_\lambda(-x) = \frac{1}{(1+x)^\theta}$$

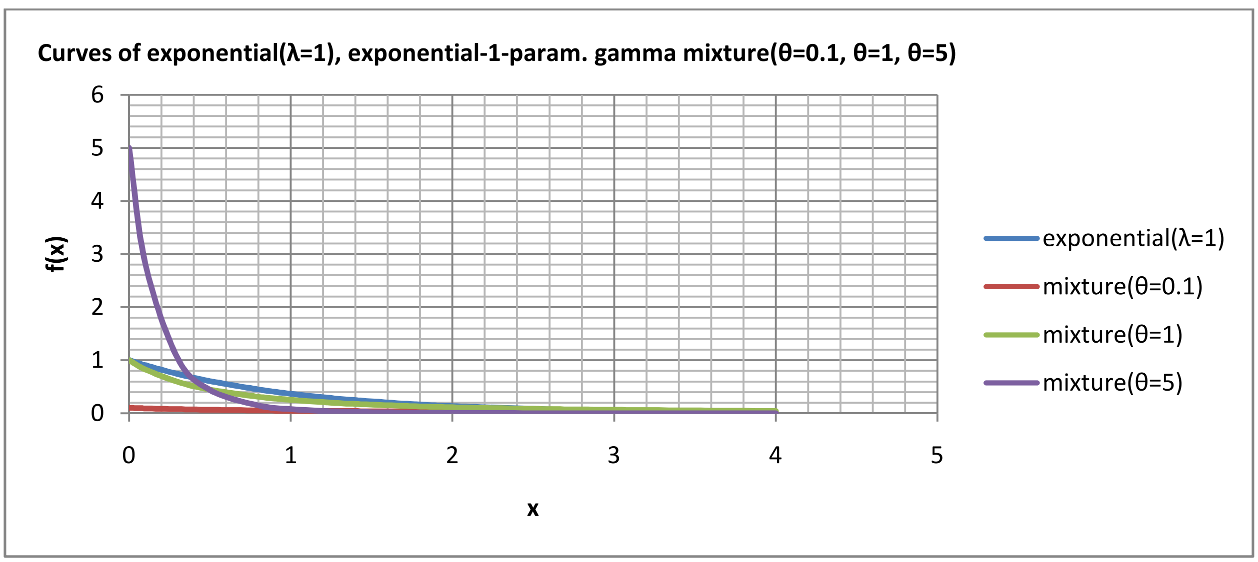

Fig. 6.2.3

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \frac{1}{(1+x)^\theta}$$

Probability density function:

$$f(x) = F'(x) = \frac{\theta}{(1+x)^{\theta+1}}, \qquad x > 0, \qquad \theta > 0$$

This is the pdf of a Pareto distribution of the second kind

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{\frac{\theta}{(1+x)^{\theta+1}}}{\frac{1}{(1+x)^{\theta}}} = \frac{\theta}{1+x}$$

6.2.4 Exponential-2-parameter gamma mixture

When $g(\lambda; \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \; e^{-\lambda\beta}\lambda^{\alpha-1}$, $\quad \lambda > 0, \quad \alpha > 0, \beta > 0$

The mgf of a 2-parameter gamma distribution is

$$M_{\lambda}(x) = \left(\frac{\beta}{\beta - x}\right)^{\alpha}$$

The following are the survival-time and related functions for the exponential-two parameter gamma mixture.

Survival function:

$$S(x) = M_{\lambda}(-x) = \left(\frac{\beta}{\beta + x}\right)^{\alpha}$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \left(\frac{\beta}{\beta + x}\right)^{\alpha}$$

Probability density function:

$$f(x) = F'(x) = \frac{\alpha\beta^{\alpha}}{(\beta + x)^{\alpha+1}} \quad , \qquad \mathrm{x} > 0, \quad \alpha > 0, \qquad \beta > 0$$

This is the pdf of a Pareto distribution of the second kind with parameters $\alpha \; and \; \beta$.

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{\alpha}{\beta + x}$$

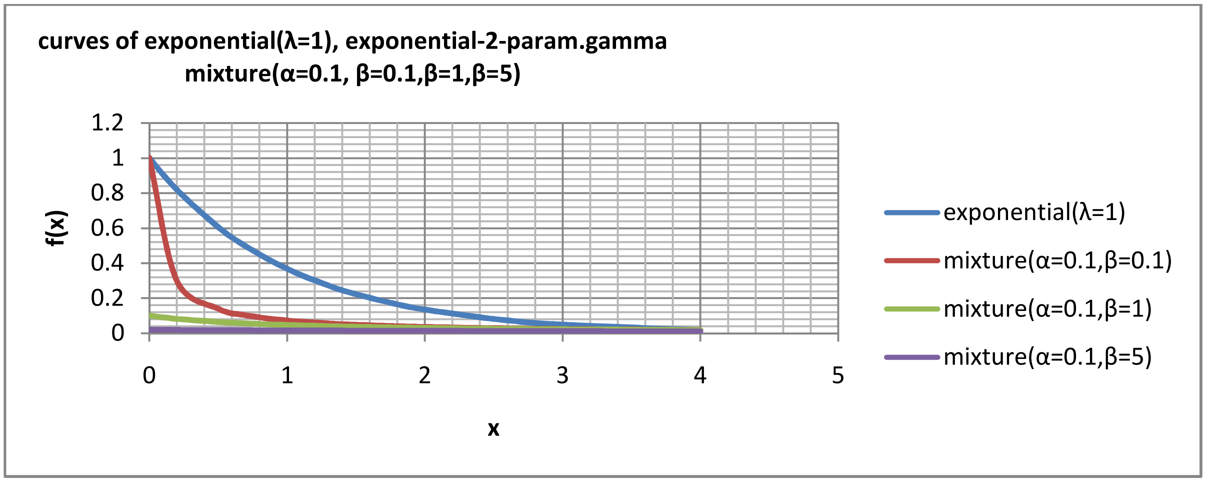

Fig. 6.2.4(a)

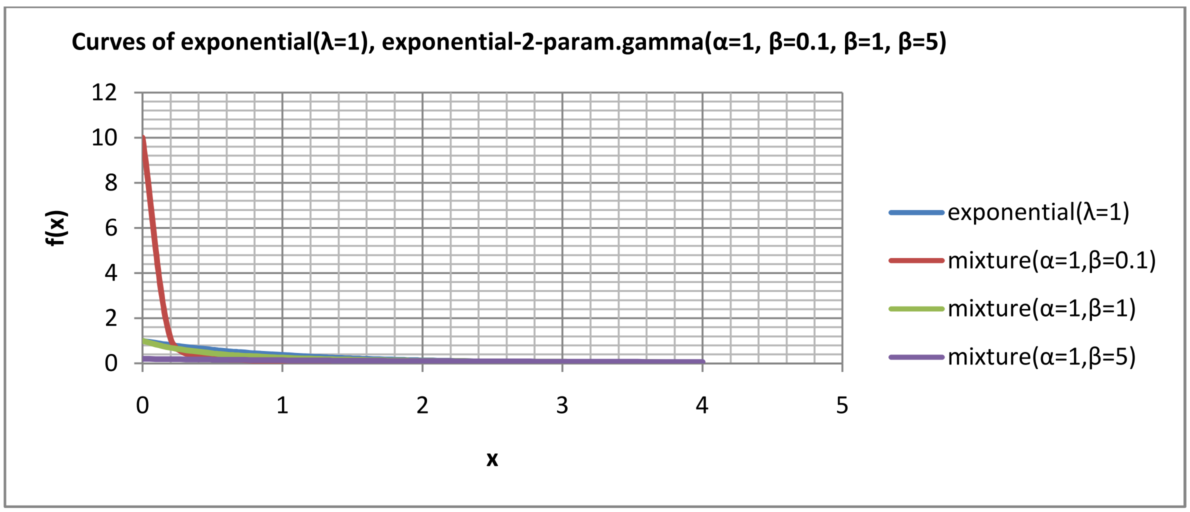

Fig. 6.2.4(b)

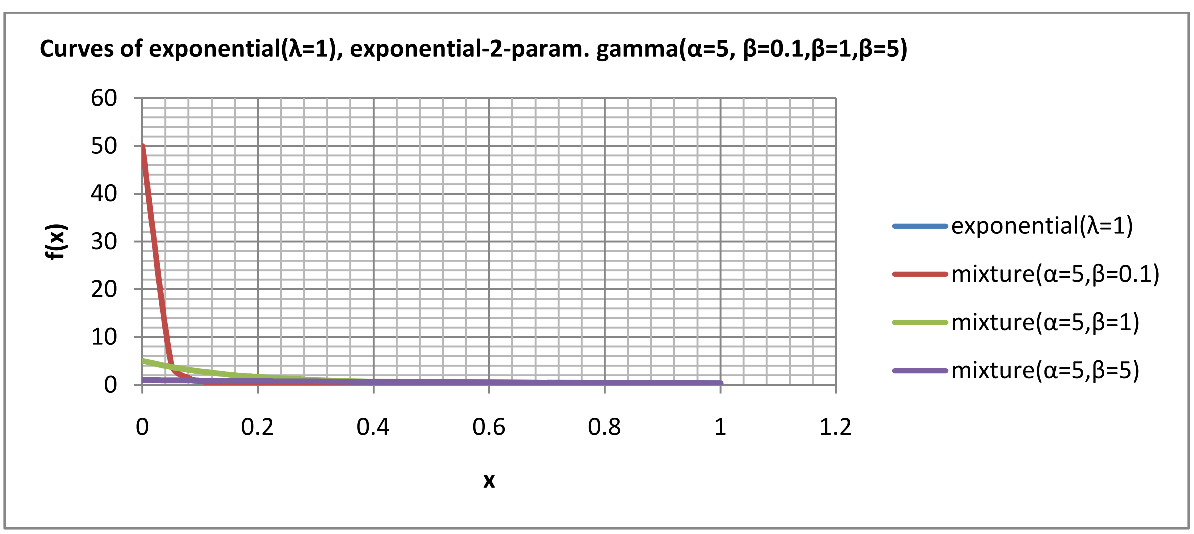

Fig. 6.2.4(c)

6.2.5 Exponential- chi-square mixture

Thus when $g(\lambda;\mathrm{n}) = X^2{}_{(n)} = gamma\left(\lambda;\frac{n}{2},\frac{1}{2}\right) = \frac{1}{2^{\frac{\mathrm{n}}{2}}\Gamma(n/2)}\ \ e^{-\lambda/2}\lambda^{(\mathrm{n}/2)-1}$,

The mgf of a chi-square distribution is

$$M_\lambda\,(-x) = \left(\frac{2}{2-x}\right)^{n/2}$$

The following are the survival-time and related functions for the exponential-chi-square mixture.

Survival function: $S(x) = M_\lambda\,(-x) = \left(\frac{2}{2+x}\right)^{n/2}$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \left(\frac{2}{2+x}\right)^{n/2}$$

Probability density function:

$$f(x) = F'(x) = \frac{n2^{\frac{n}{2}-1}}{(2+x)^{\frac{n}{2}+1}} \quad , \qquad \mathrm{x} > 0, \quad n = 1,2, \ldots$$

This is the pdf of a Pareto distribution of the second kind with parameter $\frac{n}{2}$.

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{n}{2(2+x)}$$

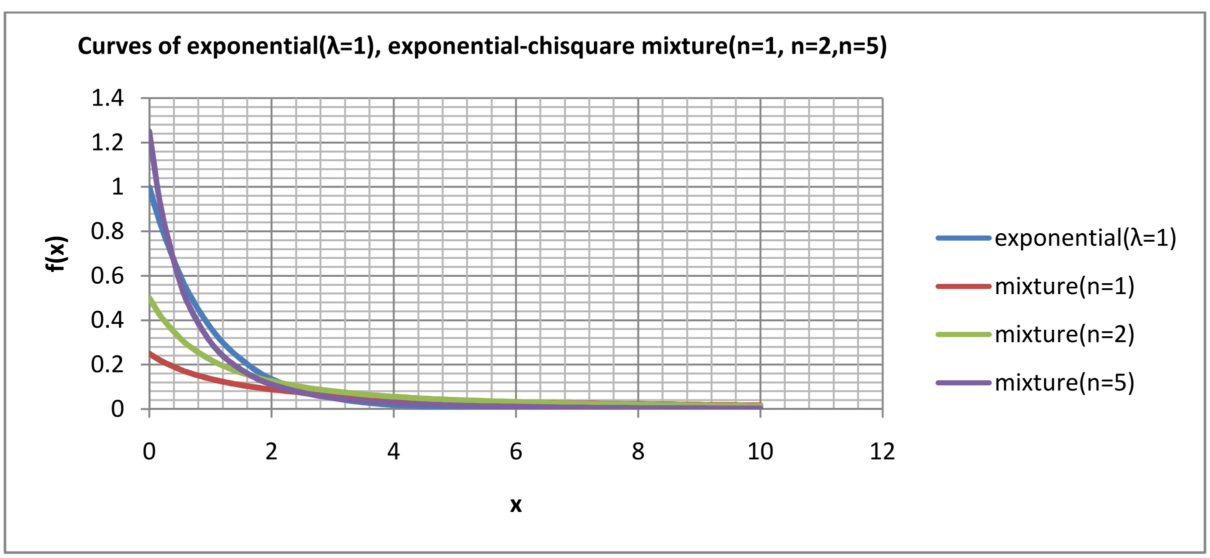

Fig. 6.2.5

6.2.6 Exponential-inverse gamma mixture

When $g(\lambda;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)\ \lambda^{\alpha+1}}\ e^{-\beta/\lambda}, \qquad \lambda > 0, \quad \alpha > 0, \quad \beta > 0$

The mgf of an inverse gamma distribution is

$$M_{\lambda}\,(x) = \frac{2\beta^{\alpha}(-x)^{\frac{\alpha}{2}}}{\Gamma(\alpha)} K_{-\alpha}(2\sqrt{(-\beta x)})$$

The following are the survival-time and related functions for the exponential-inverse gamma mixture.

Survival function:

$$S(x) = M_{\lambda}\,(-x) = \frac{2\beta^{\alpha}(x)^{\frac{\alpha}{2}}}{\Gamma(\alpha)} K_{-\alpha}(2\sqrt{(\beta x)})$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \frac{2\beta^{\alpha}(x)^{\frac{\alpha}{2}}}{\Gamma(\alpha)} K_{-\alpha}(2\sqrt{(\beta x)})$$

Probability density function:

$$f(x) = F'(x) = \frac{d}{dx}\left[-\frac{2\beta^{\alpha}(x)^{\frac{\alpha}{2}}}{\Gamma(\alpha)} K_{-\alpha}(2\sqrt{(\beta x)})\right] \quad , \qquad \mathrm{x} > 0, \quad \alpha = 1,2,\ldots$$

$$= -\frac{2\beta^{\alpha}}{\Gamma(\alpha)}\frac{d}{dx}\left[(x)^{\frac{\alpha}{2}}\ K_{-\alpha}(2\sqrt{(\beta x)})\right]$$

$\text{Using the product rule, let } u = x^{\frac{\alpha}{2}}\,, \qquad \frac{du}{dx} = \frac{\alpha}{2}x^{\frac{\alpha}{2}-1}$

$\text{And let } v = K_{-\alpha}\big(2\sqrt{(\beta x)}\big), \quad \frac{dv}{dx} = \frac{-\alpha}{2\sqrt{(\beta x)}}K_{-\alpha}\big(2\sqrt{(\beta x)}\big) - K_{1-\alpha}\big(2\sqrt{(\beta x)}\big).$[23]

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$$\therefore \qquad F'(x) = -\frac{2\beta^{\alpha}}{\Gamma(\alpha)}\frac{d(uv)}{dx}$$

$$= \frac{2\beta^{\alpha}}{\Gamma(\alpha)}\left[x^{\frac{\alpha}{2}}\left(\frac{\alpha}{2\sqrt{(\beta x}}K_{-\alpha}\big(2\sqrt{(\beta x)}\big) + K_{1-\alpha}\big(2\sqrt{(\beta x)}\big)\right) - K_{-\alpha}\big(2\sqrt{(\beta x)}\big)\frac{\alpha}{2}x^{\frac{\alpha}{2}-1}\right]$$

$$\therefore f(x) = \frac{2\beta^{\alpha}x^{\frac{\alpha}{2}}}{\Gamma(\alpha)}\left[\left(\frac{1}{\sqrt{\beta x}} - \frac{1}{x}\right)\frac{\alpha}{2}K_{-\alpha}\big(2\sqrt{(\beta x)}\big) + K_{1-\alpha}\big(2\sqrt{(\beta x)}\big)\right]$$

Hazard rate function:

$$h(x)=\frac{f(x)}{S(x)}=2\beta^{\alpha}x^{\frac{\alpha}{2}}\frac{\left[\left(\frac{1}{\sqrt{(\beta x}}-\frac{1}{x}\right)\frac{\alpha}{2}K_{-\alpha}\left(2\sqrt{(\beta x)}\right)+K_{1-\alpha}\left(2\sqrt{(\beta x)}\right)\right]}{\Gamma(\alpha)-2\beta^{\alpha}(x)^{\frac{\alpha}{2}}K_{-\alpha}(2\sqrt{(\beta x)})}$$

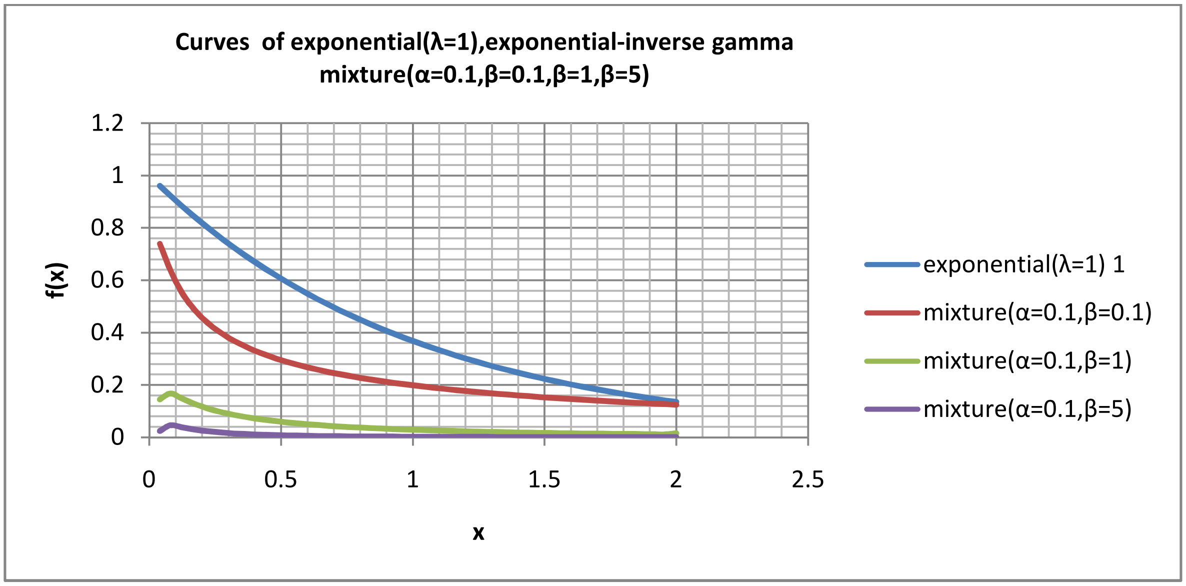

Fig. 6.2.6(a)

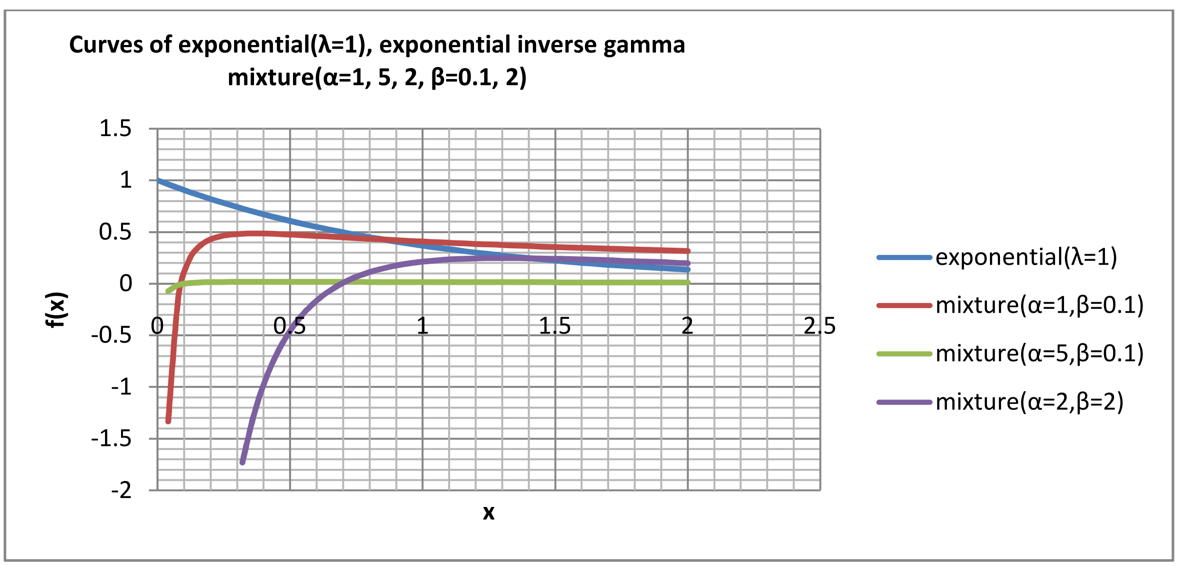

Fig.6.2.6(b)

6.2.7 Exponential-Inverse Gaussian mixture

$$When\;\; g(\lambda;\mu,\beta\,)=\left(\frac{\beta}{2\pi}\right)^{1/2}\lambda^{-3/2}e^{\frac{-\beta(\lambda-\mu)^2}{2\mu^2\lambda}}\;,\qquad \lambda>0,\qquad \mu>0,\qquad \beta>0$$

The mgf the Inverse Gaussian (Wald) distribution is

$$\mathrm{M}(x) = \left(\frac{2}{\pi\beta\mu}\right)^{1/2} (\beta - 2\mathrm{x}\mu^2)^{1/4} e^{\frac{\beta}{\mu}} K_{-1/2}\left[\frac{1}{\mu}\sqrt{\beta(\beta - 2\mathrm{x}\mu^2)}\right]$$

The following are the survival-time and related functions for the exponential-inverse Gaussian mixture.

Survival function:

$$S(x) = M(-x) = \left(\frac{2}{\pi\beta\mu}\right)^{1/2} (\beta + 2\mathrm{x}\mu^2)^{1/4} e^{\frac{\beta}{\mu}} K_{-1/2}\left[\frac{1}{\mu}\sqrt{\beta(\beta + 2\mathrm{x}\mu^2)}\right]$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \left(\frac{2}{\pi\beta\mu}\right)^{1/2} (\beta + 2\mathrm{x}\mu^2)^{1/4} e^{\frac{\beta}{\mu}} K_{-1/2}\left[\frac{1}{\mu}\sqrt{\beta(\beta + 2\mathrm{x}\mu^2)}\right]$$

Probability density function:

$$f(x) = F'(x) = -\left(\frac{2}{\pi\beta\mu}\right)^{1/2} e^{\frac{\beta}{\mu}} \frac{d}{dx}\left\{(\beta + 2\mathrm{x}\mu^2)^{1/4} K_{-1/2}\left[\frac{1}{\mu}\sqrt{\beta(\beta + 2\mathrm{x}\mu^2)}\right]\right\},\quad x > 0$$

Using the product rule, let $u = (\beta + 2\mathrm{x}\mu^2)^{1/4}$, $\frac{du}{dx} = \frac{\mu^2}{2}(\beta + 2\mathrm{x}\mu^2)^{-3/4}$

And let

$$v = K_{-1/2}\left[\frac{1}{\mu}\sqrt{\beta(\beta + 2\mathrm{x}\mu^2)}\right],$$

$$\frac{dv}{dx} = -\frac{\mu}{2\sqrt{\beta(\beta + 2\mathrm{x}\mu^2)}} K_{-1/2}\left[\frac{1}{\mu}\sqrt{\beta(\beta + 2\mathrm{x}\mu^2)}\right] - K_{1/2}\left[\frac{1}{\mu}\sqrt{\beta(\beta + 2\mathrm{x}\mu^2)}\right].$$[23]

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$$\therefore f(x) = F'(x) = -\left(\frac{2}{\pi\beta\mu}\right)^{1/2} e^{\frac{\beta}{\mu}} \frac{d(uv)}{dx}$$

$$= \left(\frac{2}{\pi\beta\mu}\right)^{1/2} e^{\frac{\beta}{\mu}} \left\{(\beta + 2\mathrm{x}\mu^2)^{1/4}\left(\frac{\mu}{2\sqrt{\beta(\beta + 2\mathrm{x}\mu^2)}} K_{-1/2}\left[\frac{1}{\mu}\sqrt{\beta(\beta + 2\mathrm{x}\mu^2)}\right] + K_{1/2}\left[\frac{1}{\mu}\sqrt{\beta(\beta + 2\mathrm{x}\mu^2)}\right]\right) - K_{-1/2}\left[\frac{1}{\mu}\sqrt{\beta(\beta + 2\mathrm{x}\mu^2)}\right]\frac{\mu^2}{2}(\beta + 2\mathrm{x}\mu^2)^{-3/4}\right\}$$

$$= \left(\frac{2}{\pi\beta\mu}\right)^{1/2} e^{\frac{\beta}{\mu}} \left\{\frac{\mu}{2\beta^{1/2}}(\beta+2x\mu^2)^{-3/4}\left[(\beta+2x\mu^2)^{1/2} - \mu\beta^{1/2}\right]K_{-1/2}\left(\frac{1}{\mu}\sqrt{\beta(\beta+2x\mu^2)}\right) + K_{1/2}\left(\frac{1}{\mu}\sqrt{\beta(\beta+2x\mu^2)}\right)\right\}$$

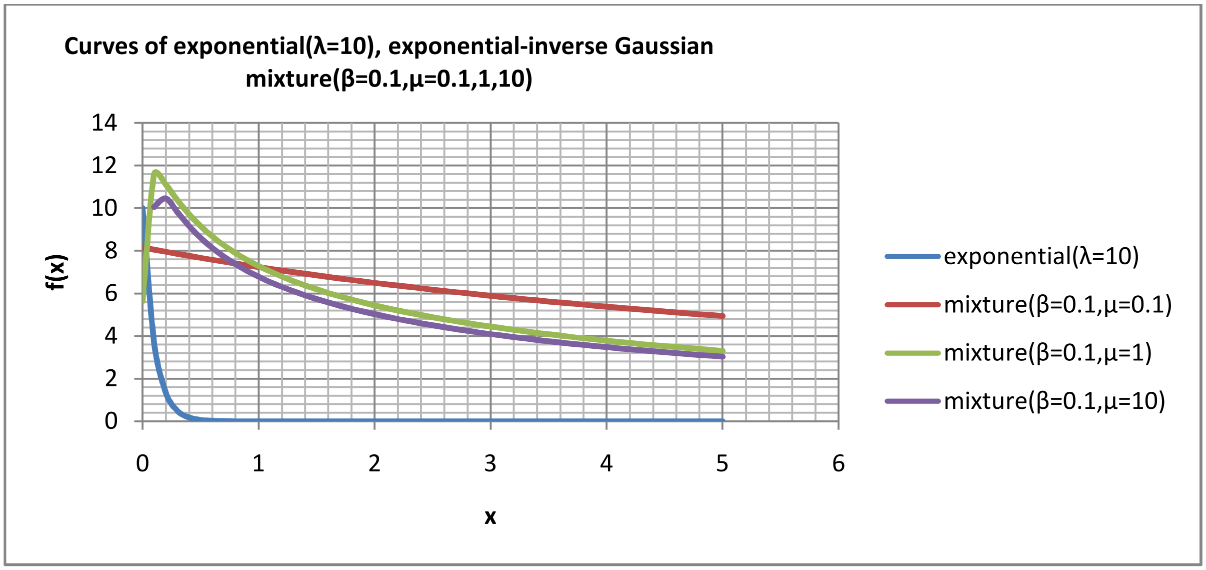

Fig. 6.2.7(a)

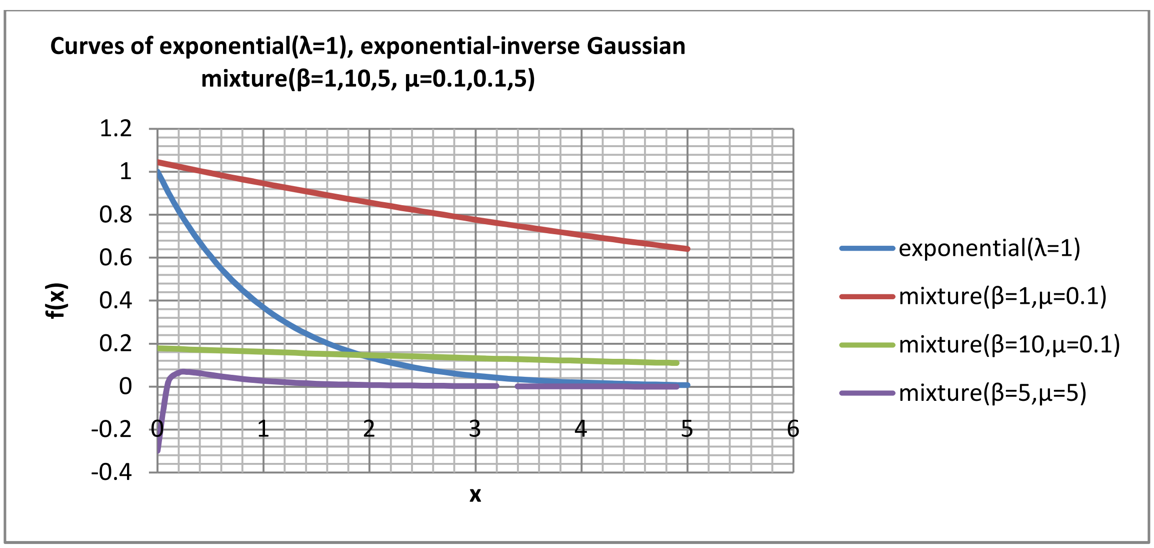

Fig.6.2.7(b)

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)}$$

$$= \frac{\left(\frac{2}{\pi\beta\mu}\right)^{1/2} e^{\frac{\beta}{\mu}} \left\{\frac{\mu}{2\beta^{1/2}}(\beta+2x\mu^2)^{-3/4}\left[-\mu\beta^{1/2}\right]K_{-1/2}\left(\frac{1}{\mu}\sqrt{\beta(\beta+2x\mu^2)}\right) + K_{1/2}\left(\frac{1}{\mu}\sqrt{\beta(\beta+2x\mu^2)}\right)\right\}}{1-\left(\frac{2}{\pi\beta\mu}\right)^{1/2}(\beta+2x\mu^2)^{1/4}e^{\frac{\beta}{\mu}}K_{-1/2}\left[\frac{1}{\mu}\sqrt{\beta(\beta+2x\mu^2)}\right]}$$

6.2.8 Exponential-generalized inverse Gaussian mixture

$$When\ \ g(\lambda;\theta,\psi,\chi) = \left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}} \frac{\lambda^{\theta-1}}{2K_\theta\left(\sqrt{\chi\psi}\right)} e^{-\frac{1}{2}\left(\frac{\chi}{\lambda}+\psi\lambda\right)}\ , \qquad \lambda > 0,$$

$\chi > 0, \qquad \psi \geq 0 \ \ when\ \theta < 0$

$\chi > 0, \qquad \psi > 0 \ \ when\ \theta = 0$

$\chi \geq 0, \qquad \psi > 0 \ \ when\ \theta > 0$

The mgf of a generalized Inverse Gaussian distribution is given by

$$M_\lambda(-x) = \left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}}\left(\frac{\chi}{\psi-2x}\right)^{\frac{\theta}{2}} \frac{K_\theta\left(\sqrt{\chi(\psi-2x)}\right)}{K_\theta\left(\sqrt{\chi\psi}\right)}$$

The following are the survival-time and related functions for the exponential-generalized inverse Gaussian mixture.

Survival function:

$$S(x) = M_\lambda(-x) = \left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}}\left(\frac{\chi}{\psi+2x}\right)^{\frac{\theta}{2}} \frac{K_\theta\left(\sqrt{\chi(\psi+2x)}\right)}{K_\theta\left(\sqrt{\chi\psi}\right)}$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}}\left(\frac{\chi}{\psi+2x}\right)^{\frac{\theta}{2}} \frac{K_\theta\left(\sqrt{\chi(\psi+2x)}\right)}{K_\theta\left(\sqrt{\chi\psi}\right)}$$

Probability density function:

$$f(x) = F'(x) = -\left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}} \frac{1}{K_\theta\left(\sqrt{\chi\psi}\right)}\ \frac{d}{dx}\left[\left(\frac{\chi}{\psi+2x}\right)^{\frac{\theta}{2}}\ K_\theta\left(\sqrt{\chi(\psi+2x)}\right)\right],$$

$$x > 0$$

Using the product rule, let

$$u = \left(\frac{\chi}{\psi + 2x}\right)^{\frac{\theta}{2}}, \qquad \frac{du}{dx} = \frac{-\theta\chi^{\frac{\theta}{2}}}{(\psi + 2x)^{\frac{\theta}{2}+1}}$$

And let

$v = K_\theta\left(\sqrt{\chi(\psi + 2x}\right),$

$$\frac{dv}{dx} = \frac{\theta}{\sqrt{\chi(\psi + 2x}} K_\theta\left(\sqrt{\chi(\psi + 2x}\right) - K_{\theta+1}\left(\sqrt{\chi(\psi + 2x}\right).^{24}$$

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Thus

$$f(x) = F'(x) = -\left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}} \frac{1}{K_\theta\left(\sqrt{\chi\psi}\right)} \frac{d(uv)}{dx}$$

$$= -\left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}} \frac{1}{K_\theta\left(\sqrt{\chi\psi}\right)} \left\{\left(\frac{\chi}{\psi + 2x}\right)^{\frac{\theta}{2}} \left(\frac{\theta}{\sqrt{\chi(\psi + 2x}} K_\theta\left(\sqrt{\chi(\psi + 2x}\right) - K_{\theta+1}\left(\sqrt{\chi(\psi + 2x}\right)\right) + K_\theta\left(\sqrt{\chi(\psi + 2x}\right) \frac{-\theta\chi^{\frac{\theta}{2}}}{(\psi + 2x)^{\frac{\theta}{2}+1}}\right\}$$

$$= \left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}} \frac{1}{K_\theta\left(\sqrt{\chi\psi}\right)} \left[\frac{(\theta\chi)^{\frac{\theta}{2}} K_\theta\left(\sqrt{\chi(\psi + 2x}\right)}{(\psi + 2x)^{\frac{\theta}{2}+1}} \left(\frac{1}{(\psi + 2x)^{\frac{1}{2}}} - \frac{1}{\chi^{\frac{1}{2}}}\right) + K_{\theta+1}\left(\sqrt{\chi(\psi + 2x}\right)\right]$$

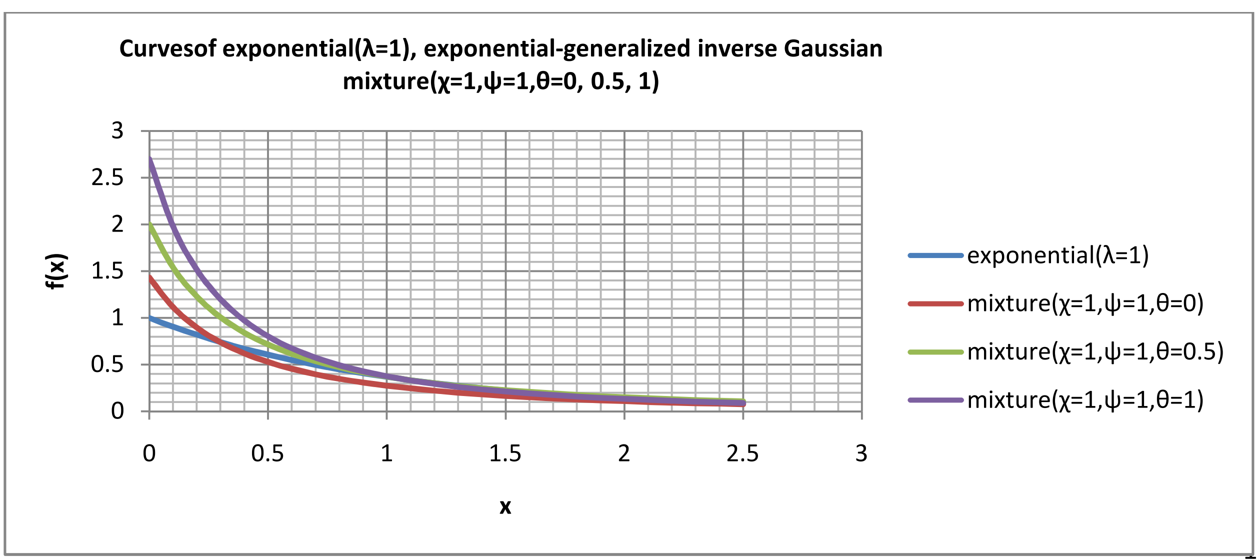

Fig.6.2.8(a)

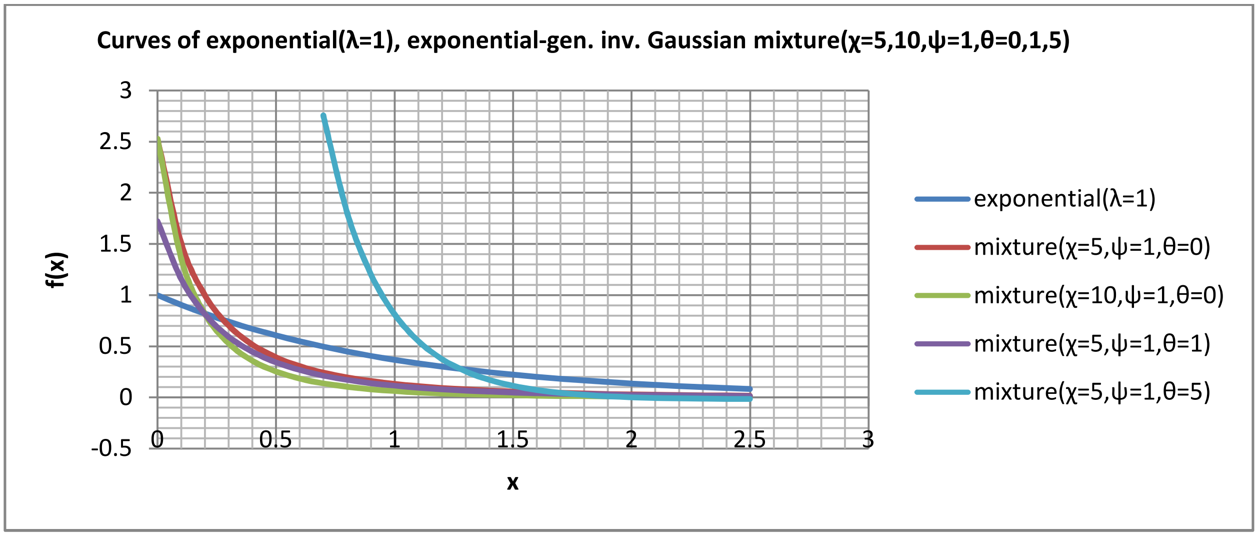

Fig.6.2.8(b)

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)}$$

$$= \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}} \frac{1}{K_\theta\left(\sqrt{\chi\psi}\right)} \left[\frac{(\theta\chi)^{\frac{\theta}{2}} K_\theta\left(\sqrt{\chi(\psi+2x)}\right)}{(\psi+2x)^{\frac{\theta}{2}+1}}\left(\frac{1}{(\psi+2x)^{\frac{1}{2}}} - \frac{1}{\chi^{\frac{1}{2}}}\right) + K_{\theta+1}\left(\sqrt{\chi(\psi+2x)}\right)\right]}{\left(\frac{\psi}{\chi}\right)^{\frac{\theta}{2}} \left(\frac{\chi}{\psi+2x}\right)^{\frac{\theta}{2}} \frac{K_\theta\left(\sqrt{\chi(\psi+2x)}\right)}{K_\theta\left(\sqrt{\chi\psi}\right)}}$$

$$= \frac{(\theta\chi)^{\frac{\theta}{2}}}{(\psi+2x)\chi^{\frac{\theta}{2}}}\left(\frac{1}{(\psi+2x)^{\frac{1}{2}}} - \frac{1}{\chi^{\frac{1}{2}}}\right) + \frac{(\psi+2x)^{\frac{\theta}{2}} K_{\theta+1}\left(\sqrt{\chi(\psi+2x)}\right)}{K_\theta\left(\sqrt{\chi(\psi+2x)}\right)}$$

6.2.9 Exponential-half normal mixture

When $g(\lambda;\, \sigma) = \frac{1}{\sigma}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\lambda^2}{2\sigma^2}}, \quad \lambda > 0,\ \sigma > 0$

The mgf a half-normal distribution is

$$M_\lambda(x) = 2e^{\frac{\sigma^2 x^2}{2}}[1 + \phi(\sigma x)]$$

The following are the survival-time and related functions for the exponential-half normal mixture.

Survival function:

$$S(x) = M_\lambda(-x) = 2e^{\sigma^2 x^2/2}[1 - \phi(\sigma x)]$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - 2e^{\sigma^2 x^2/2}[1 - \phi(\sigma x)]$$

Probability density function:

$$f(x) = F'(x) = -\ \frac{d}{dx}\left[2\ e^{\sigma^2 x^2/2}\ [1 - \phi(\sigma x)]\right], \quad x > 0,$$

Using the product rule, let

$$u = 2e^{\sigma^2 x^2/2} \quad , \qquad \frac{du}{dx} = 2\sigma^2 x e^{\sigma^2 x^2/2}$$

And let

$$v = [1 - \phi(\sigma x)]\,, \qquad and\ \ y = \ \sigma x$$

$$\frac{dy}{dx} = \sigma$$

$$\therefore\ v = 1 - \phi(y), \qquad \frac{dv}{dy} = -\frac{1}{\sqrt{2\pi}}e^{-y^2}$$

$$\frac{dv}{dx} = \frac{dv}{dy}.\frac{dy}{dx} = -\frac{1}{\sqrt{2\pi}}e^{-y^2}.\sigma$$

$$= -\ \frac{\sigma}{\sqrt{2\pi}}\ e^{\frac{-\sigma^2 x^2}{2}}$$

$$\therefore f(x) = F'(x) = \frac{d(uv)}{dx} = 2\ e^{\sigma^2 x^2/2}\ \frac{\sigma}{\sqrt{2\pi}}e^{\frac{-\sigma^2 x^2}{2}} - [1 - \phi(\sigma x)]2\sigma^2 x e^{\sigma^2 x^2/2}$$

$$= \ \sigma\sqrt{\left(\frac{2}{\pi}\right)} - 2\sigma^2 x[1 - \phi(\sigma x)]e^{\sigma^2 x^2/2}\,, x > 0, \sigma > 0$$

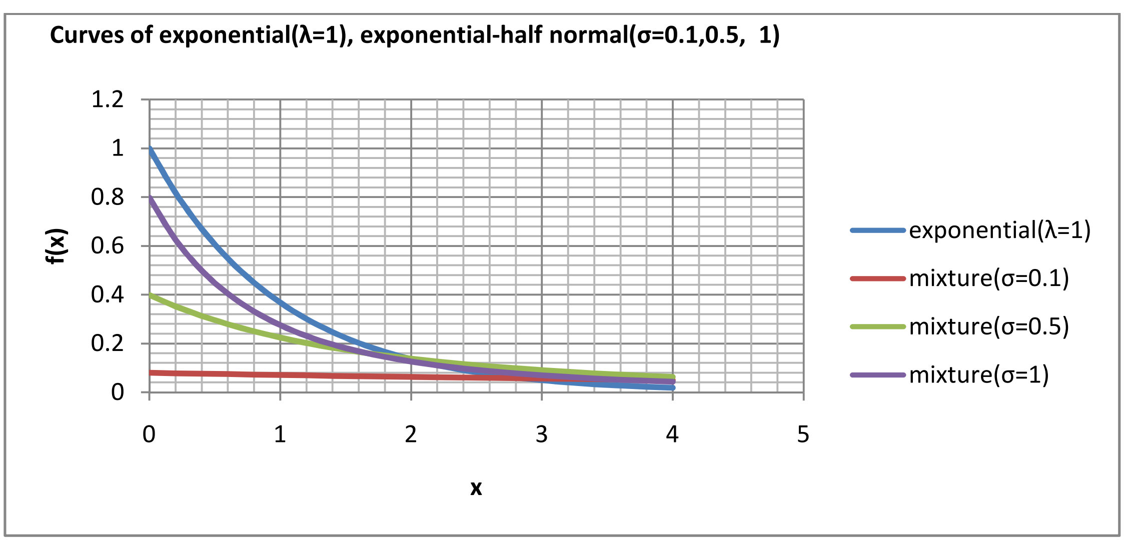

Fig.6.2.9

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{\sigma\sqrt{\left(\frac{2}{\pi}\right)} - 2\sigma^2 x[1-\phi(\sigma x)]e^{\sigma^2 x^2/2}}{2e^{\sigma^2 x^2/2}[1-\phi(\sigma x)]}$$

6.2.10 Exponential-Rayleigh mixture

When

$$g(\lambda;\ \sigma\ ) = \frac{\lambda}{\sigma^2} e^{-\frac{\lambda^2}{2\sigma^2}}, \qquad \lambda \geq 0,\ \ \sigma > 0$$

The mgf of the Rayleigh distribution is

$$M_\lambda\ (x) = 1 + \ \sigma x\sqrt{2\pi} \quad e^{\frac{\sigma^2 x^2}{2}}\,[1+\phi(\sigma x)]$$

The following are the survival-time functions for the exponential-Rayleigh mixture.

Survival function:

$$S(x) = M_\lambda\ (-x) = 1 - \ \sigma x\sqrt{2\pi} \quad e^{\frac{\sigma^2 x^2}{2}}\,[1-\phi(\sigma x)]$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = \ \sigma x\sqrt{2\pi} \quad e^{\frac{\sigma^2 x^2}{2}}\,[1-\phi(\sigma x)]$$

Probability density function:

$$f(x) = F'(x) = \sigma\sqrt{2\pi}\,\frac{d}{dx}\left\{x\ e^{\frac{1}{2}\sigma^2 x^2}[1-\phi(\sigma x)]\right\},\ \ x > 0$$

Using the product rule, let

$$u = x\ e^{\frac{1}{2}\sigma^2 x^2}\ , \qquad \frac{du}{dx} = (\sigma^2 x^2 + 1)\ e^{\frac{1}{2}\sigma^2 x^2}$$

And let

$$v = [1-\phi(\sigma x)], \quad \frac{dv}{dx} = -\ \frac{\sigma}{\sqrt{2\pi}}\ e^{-\frac{1}{2}\sigma^2 x^2}$$

Thus,

$$f(x) = F'(x) = \sigma\sqrt{2\pi}\ \frac{d(uv)}{dx} = \sigma\sqrt{2\pi}\left[u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$= \ \sigma\sqrt{2\pi}\left[-x\ e^{\frac{1}{2}\sigma^2 x^2}\ \frac{\sigma}{\sqrt{2\pi}}\ e^{-\frac{1}{2}\sigma^2 x^2} + (\sigma^2 x^2 + 1)[1-\phi(\sigma x)]\ e^{\frac{1}{2}\sigma^2 x^2}\right]$$

$$= \ \sigma\sqrt{2\pi}\ \ (\sigma^2 x^2 + 1)[1-\phi(\sigma x)]\ e^{\frac{1}{2}\sigma^2 x^2} - \sigma^2 x$$

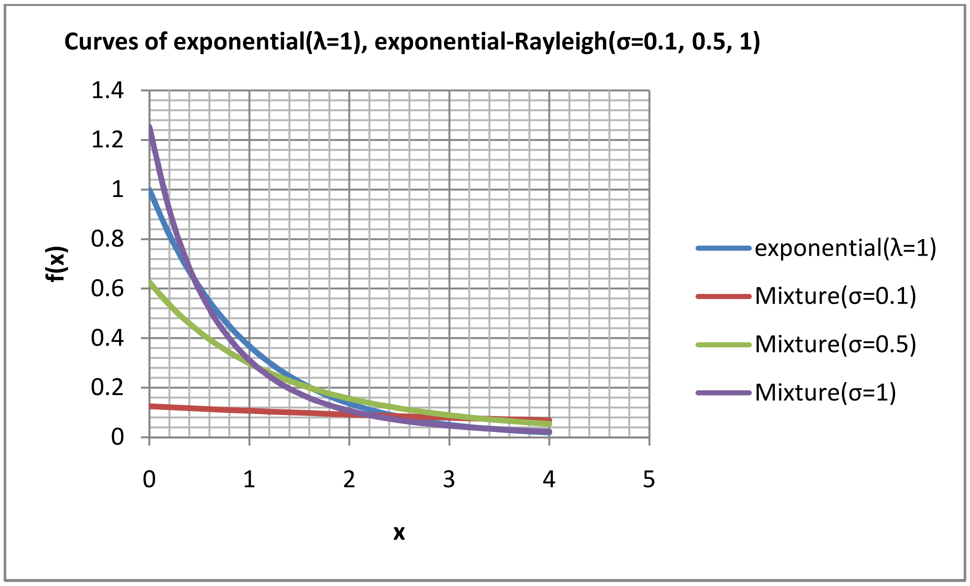

Fig. 6.2.10

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{\sigma\sqrt{2\pi}\ \ (\sigma^2x^2+1)[1-\phi(\sigma x)]\, e^{\frac{1}{2}\sigma^2x^2} - \sigma^2 x}{1-\ \sigma x\sqrt{2\pi}\quad e^{\frac{\sigma^2x^2}{2}}\, [1-\phi(\sigma x)]}$$

6.2.11 Exponential-uniform(rectangular) mixture

When $g(\lambda;\mathrm{b}) = \frac{1}{\mathrm{b}}$, $\qquad 0<\lambda<b,\ \ b>0$

The mgf of a uniform distribution is

$$M(x) = \frac{\mathrm{e}^{bx}-1}{bx}$$

The following are the survival-time and related functions for the exponential-uniform (rectangular) mixture.

Survival function:

$$S(x) = M(-x) = \frac{1-\mathrm{e}^{-bx}}{bx}$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \frac{1-\mathrm{e}^{-bx}}{bx}$$

$$= \frac{\mathrm{e}^{-bx}+bx-1}{bx}$$

Probability density function:

$$f(x) = F'(x) = \frac{1-(bx+1)e^{-bx}}{bx^2}, \quad x > 0, \qquad b > 0$$

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{\frac{1-(bx+1)e^{-bx}}{bx^2}}{\frac{1-e^{-bx}}{bx}} = \frac{1-(bx+1)e^{-bx}}{x(1-e^{-bx})}$$

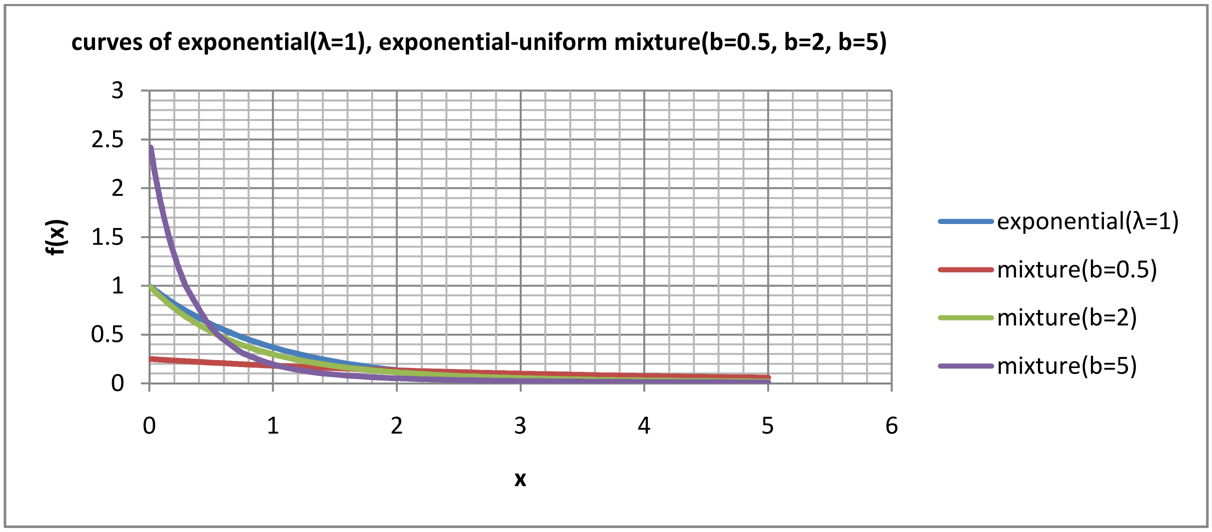

Fig.6.2.11

6.2.12 Exponential-Erlang mixture

$when\ g(\lambda; n, \beta) = \frac{\beta^n}{(n-1)!}\lambda^{n-1}\ e^{-\lambda\beta}, \quad \lambda > 0, \quad \beta > 0, \ n \in \mathrm{N}$

N=the set of natural numbers

The mgf of an Erlang distribution is

$$M_\lambda(x) = \left(\frac{\beta}{\beta - x}\right)^n$$

The following are survival-time and related functions for the exponential-Erlang mixture.

Survival function:

$$S(x) = M_\lambda(-x) = \left(\frac{\beta}{\beta + x}\right)^n$$

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \left(\frac{\beta}{\beta + x}\right)^n$$

Probability density function:

$$f(x) = F'(x) = \frac{\beta^n}{(\beta + x)^{n+1}}, \qquad x \geq 0, \quad n \in \mathrm{N}, \qquad \beta > 0$$

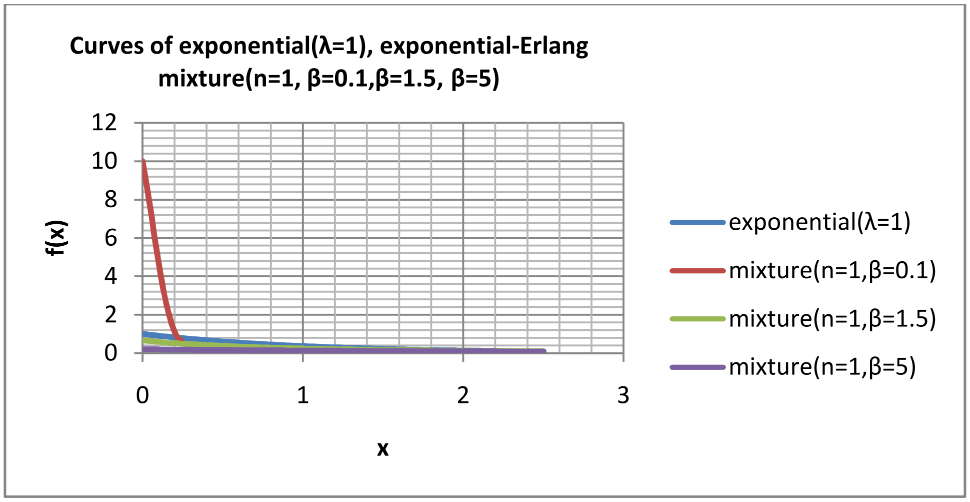

Fig. 6.2.12(a)

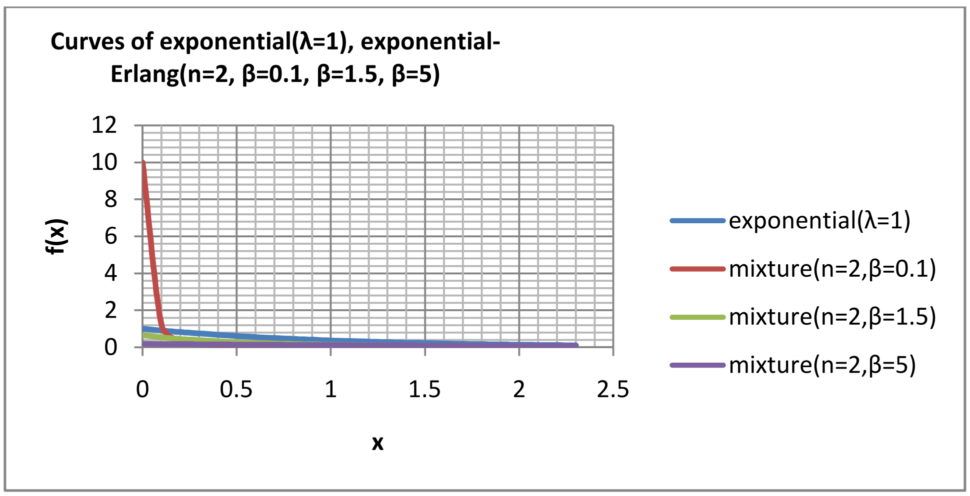

Fig.6.2.12(b)

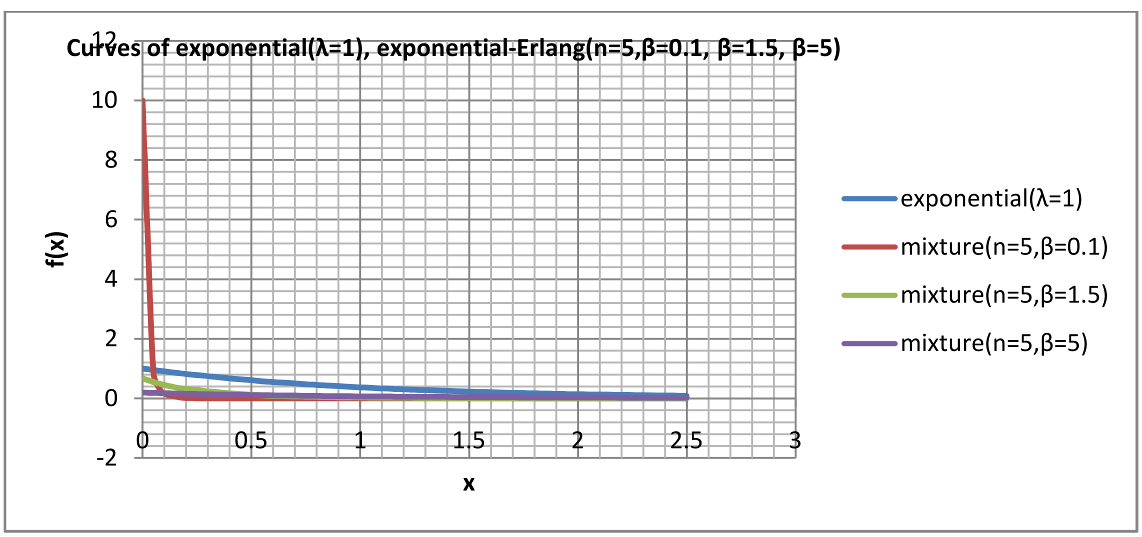

Fig. 6.2.12(c)

Hazard rate function:

$$h(x) = \frac{f(x)}{S(x)} = \frac{\frac{\beta^n}{(\beta+x)^{n+1}}}{\left(\frac{\beta}{\beta+x}\right)^n} \quad = \frac{1}{\beta+x}$$

6.2.13 Exponential-chi mixture

$$When\ gamma(\lambda; n) = \frac{\lambda^{n-1}}{2^{\frac{n}{2}-1}\Gamma(n/2)}\ e^{-\lambda^2/2}\ , \qquad \lambda \geq 0,\ \ n = 1,2,\ldots$$

The mgf of a chi distribution is

$$M_\lambda\,(x) = \frac{\Gamma(n)}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)} e^{x^2/4} D_{-n}(-x)$$

The following are the survival-time and related functions for the exponential-chi mixture.

Survival function:

$$S(x) = M_\lambda\,(-x) = \frac{\Gamma(n)}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)} e^{x^2/4} D_{-n}(x)$$

$$= \frac{\Gamma(n)}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)} e^{x^2/4} 2^{n/2} e^{-x^2/4} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1+n}{2}\right)}\, {}_1F_1\left(\frac{n}{2}, \frac{1}{2}; \frac{x^2}{2}\right) - \frac{\sqrt{2\pi}x}{\Gamma\left(\frac{n}{2}\right)}\, {}_1F_1\left(\frac{1+n}{2}, \frac{3}{2}; \frac{x^2}{2}\right) \right\}$$

$$= \frac{2\Gamma(n)\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)} \left\{ \frac{1}{\Gamma\left(\frac{1+n}{2}\right)}\, {}_1F_1\left(\frac{n}{2}, \frac{1}{2}; \frac{x^2}{2}\right) - \frac{\sqrt{2}x}{\Gamma\left(\frac{n}{2}\right)}\, {}_1F_1\left(\frac{1+n}{2}, \frac{3}{2}; \frac{x^2}{2}\right) \right\}$$

by substituting for $D_{-n}(x)$ from **section 5.5.13**

Cumulative distribution function:

$$F(x) = 1 - S(x) = 1 - \frac{\Gamma(n)}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)} e^{x^2/4} D_{-n}(x)$$

$$= 1 - \frac{2\Gamma(n)\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)} \left\{ \frac{1}{\Gamma\left(\frac{1+n}{2}\right)}\, {}_1F_1\left(\frac{n}{2}, \frac{1}{2}; \frac{x^2}{2}\right) - \frac{\sqrt{2}x}{\Gamma\left(\frac{n}{2}\right)}\, {}_1F_1\left(\frac{1+n}{2}, \frac{3}{2}; \frac{x^2}{2}\right) \right\}$$

Probability density function:

$$f(x) = F'(x) = -\frac{d}{dx}\left[\frac{\Gamma(n)}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)} e^{-x^2/4} D_{-n}(x)\right], \qquad x > 0, \quad n = 1,2,\ldots$$

$$= -\frac{d}{dx}\left[\frac{2\Gamma(n)\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)}\left\{\frac{1}{\Gamma\left(\frac{1+n}{2}\right)}\,{}_1F_1\left(\frac{n}{2},\frac{1}{2};\frac{x^2}{2}\right) - \frac{\sqrt{2}x}{\Gamma\left(\frac{n}{2}\right)}\,{}_1F_1\left(\frac{1+n}{2},\frac{3}{2};\frac{x^2}{2}\right)\right\}\right]$$

$$= -\frac{2\Gamma(n)\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)}\frac{d}{dx}\left[\left\{\frac{1}{\Gamma\left(\frac{1+n}{2}\right)}\,{}_1F_1\left(\frac{n}{2},\frac{1}{2};\frac{x^2}{2}\right) - \frac{\sqrt{2}x}{\Gamma\left(\frac{n}{2}\right)}\,{}_1F_1\left(\frac{1+n}{2},\frac{3}{2};\frac{x^2}{2}\right)\right\}\right]$$

$$Let\ v = \frac{1}{\Gamma\left(\frac{1+n}{2}\right)}\,{}_1F_1\left(\frac{n}{2},\frac{1}{2};\frac{x^2}{2}\right) - \frac{\sqrt{2}x}{\Gamma\left(\frac{n}{2}\right)}\,{}_1F_1\left(\frac{1+n}{2},\frac{3}{2};\frac{x^2}{2}\right)$$

By the making the substitution, $y = \frac{x^2}{2}, \quad \frac{dy}{dx} = x$

$$\therefore v = \frac{1}{\Gamma\left(\frac{1+n}{2}\right)}\,{}_1F_1\left(\frac{n}{2},\frac{1}{2};y\right) - \frac{\sqrt{2}\sqrt{2y}}{\Gamma\left(\frac{n}{2}\right)}\,{}_1F_1\left(\frac{1+n}{2},\frac{3}{2};y\right)$$

$$And\ \frac{dv}{dy} = \frac{n}{\Gamma\left(\frac{1+n}{2}\right)}\,{}_1F_1\left(\frac{1+n}{2},\frac{3}{2};y\right)$$

$$-\frac{2}{\Gamma\left(\frac{n}{2}\right)}\left[\sqrt{y}\ \frac{1+n}{3}\,{}_1F_1\left(\frac{3+n}{2},\frac{5}{2};y\right) + \frac{1}{2\sqrt{y}}\,{}_1F_1\left(\frac{1+n}{2},\frac{3}{2};y\right)\right].$$ [24]

(APPENDIX 21)

$$\therefore \frac{dv}{dx} = \frac{dv}{dy}.\frac{dy}{dx} = \frac{n}{\Gamma\left(\frac{1+n}{2}\right)}\,x\,{}_1F_1\left(\frac{1+n}{2},\frac{3}{2};y\right)$$

$$-\frac{2x}{\Gamma\left(\frac{n}{2}\right)}\left[\sqrt{y}\ \frac{1+n}{3}\,{}_1F_1\left(\frac{3+n}{2},\frac{5}{2};y\right) + \frac{1}{2\sqrt{y}}\,{}_1F_1\left(\frac{1+n}{2},\frac{3}{2};y\right)\right]$$

$$= \frac{nx}{\Gamma\left(\frac{1+n}{2}\right)}\,{}_1F_1\left(\frac{1+n}{2},\frac{3}{2};\frac{x^2}{2}\right)$$

$$-\frac{2}{\Gamma\left(\frac{n}{2}\right)}\left[\frac{x^2}{\sqrt{2}}\frac{1+n}{3}\,{}_1F_1\left(\frac{3+n}{2},\frac{5}{2};\frac{x^2}{2}\right)+\frac{1}{\sqrt{2}}\,{}_1F_1\left(\frac{1+n}{2},\frac{3}{2};\frac{x^2}{2}\right)\right]$$

$$Thus\,,f(x)=F'(x)=\frac{2\Gamma(n)\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)}\;\frac{dv}{dx}$$

$$=\frac{2\Gamma(n)\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)}\left\{\frac{2}{\Gamma\left(\frac{n}{2}\right)}\left[\frac{x^2}{\sqrt{2}}\frac{1+n}{3}\cdot{}_1F_1\left(\frac{3+n}{2},\frac{5}{2};\frac{x^2}{2}\right)+\frac{1}{\sqrt{2}}\cdot{}_1F_1\left(\frac{1+n}{2},\frac{3}{2};\frac{x^2}{2}\right)\right]-\frac{nx}{\Gamma\left(\frac{1+n}{2}\right)}\cdot{}_1F_1\left(\frac{1+n}{2},\frac{3}{2};\frac{x^2}{2}\right)\right\}$$

Hazard rate function:

$$h(x)=\frac{f(x)}{S(x)}=$$

$$\frac{\frac{n}{\Gamma\left(\frac{1+n}{2}\right)}x\cdot{}_1F_1\left(\frac{1+n}{2},\frac{3}{2};\frac{x^2}{2}\right)-\left(\frac{2}{\Gamma\left(\frac{n}{2}\right)}\left[\frac{x^2}{\sqrt{2}}\frac{1+n}{3}\,{}_1F_1\left(\frac{3+n}{2},\frac{5}{2};\frac{x^2}{2}\right)+\frac{1}{\sqrt{2}}\,{}_1F_1\left(\frac{1+n}{2},\frac{3}{2};\frac{x^2}{2}\right)\right]\right)}{\frac{1}{\Gamma\left(\frac{1+n}{2}\right)}\cdot{}_1F_1\left(\frac{n}{2},\frac{1}{2};\frac{x^2}{2}\right)-\frac{\sqrt{2}x}{\Gamma\left(\frac{n}{2}\right)}\cdot{}_1F_1\left(\frac{1+n}{2},\frac{3}{2};\frac{x^2}{2}\right)}$$

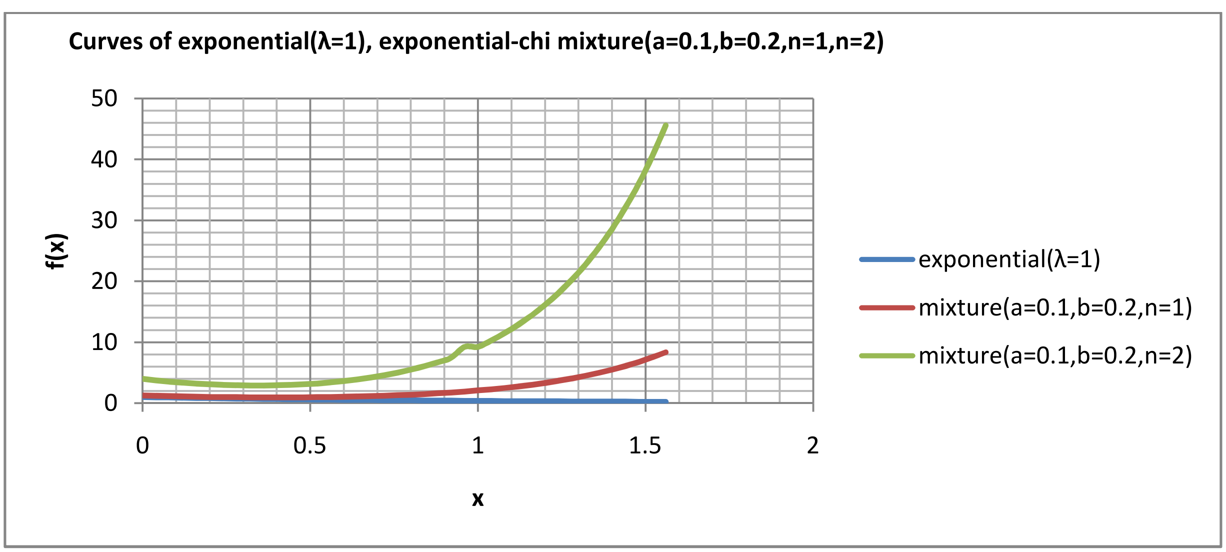

Fig. 6.2.13(a)

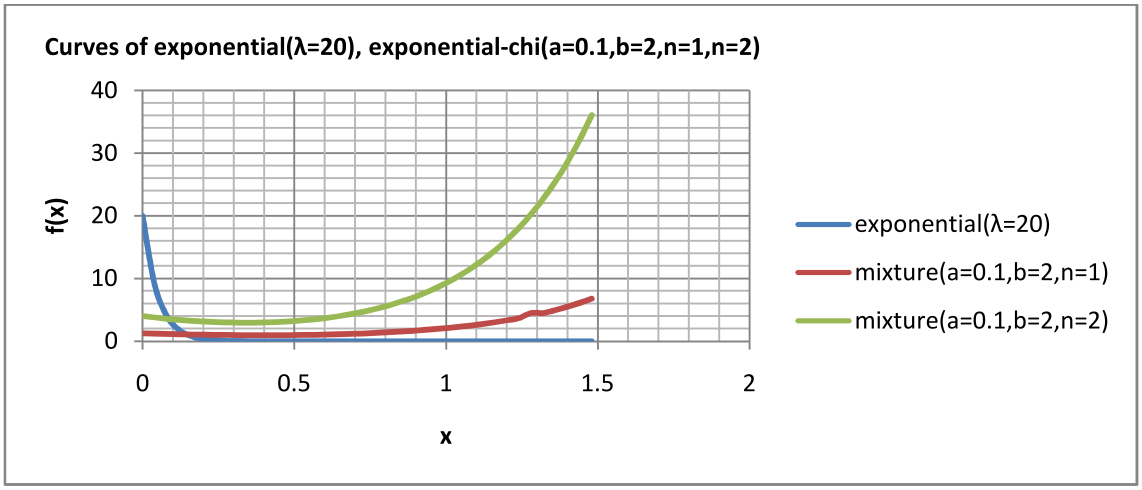

Fig. 6.2.13(b)

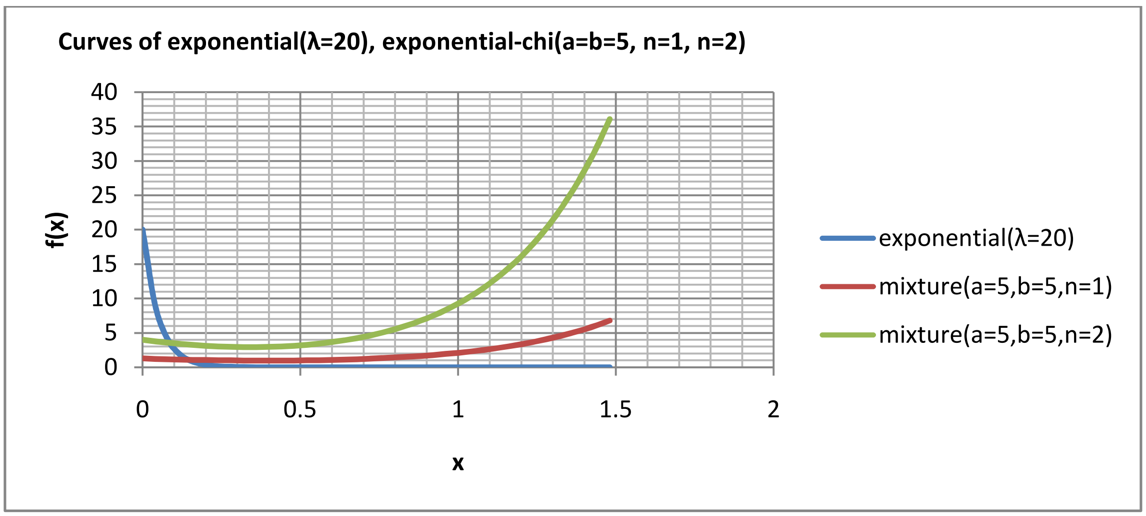

Fig. 6.2.13(c)

Chapter 7

Conclusion

Characterization. Characterization of a probability distribution is a powerful tool in enabling the usage of the distribution. A number of characterization procedures for the exponential distribution have been highlighted in this study. In particular, the principles underlying the characterizations by lack of memory and its extensions as well a sample of three characterizations by order statistics have been clearly analyzed in the study.

Characterization via other properties have also been briefly mentioned together with their accompanying proofs, lemmas and references. This provides a researcher using the exponential distribution a basis for the method to adopt in testing the exponentiality of his data.

Mixtures. Many of the mixtures of exponential distribution have curves whose right tails resemble that of the exponential distribution but with skewness coefficients that are clearly parameter dependent. This is true for all parameter values provided that the values lie in their workable ranges. All exponential mixtures involving discrete distributions fall in this category. Many of the mixtures involving continuous distributions also fall in the category.

However, it is also clear in some instances of mixtures of continuous distributions that certain combinations of parameters result in behavior that is not of a probability distribution. That means that in the event of application of these mixtures, careful attention must paid to determining the parameter combinations that are useful. This presents an area that requires further research.

Mixtures of the exponential-inverse gamma (Figs. 6.2.6 (a),(b)) and exponential-inverse Gaussian (Fig.6.2.7(a),(b)) are two examples that have this characteristic.

The exponential-chi mixture curves (Figs.6.2.13(a),(b),(c)), do not have the characteristic of a probability distribution and more investigation is required to determine whether there exist a band of parameter values and parameter combinations for which the mixture is a probability distribution.

Since the nature of a tail is an important factor in the applications of a continuous distribution, the parameter dependence of these tails in mixtures of the exponential distribution is an area that also requires further investigation for their applications to be suitably employed.

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APPENDIX 1

SUMMARY. The solution to a functional equation is given, which is in turn used to characterize the (semi-) stable distributions and the exponential distribution.

1. Introduction

The classes of non-degenerate characteristic functions $\phi(t)$ which satisfy the equation

$$\phi(t) = \prod_1^\infty \phi^{\gamma_{2j}}(a_{2j}\,t) \prod_1^\infty \phi^{\gamma_{2j+1}}(-a_{2j+1}t) \quad \ldots \quad (1)$$

where γ's and a's are non-negative numbers, were determined by Lévy (1937, $\phi(t) = \phi^\gamma(at)$), Laha and Lukacs (1965, all γ's and the sum of squares of a's are equal to one), Shimizu (1968, γ's are equal to one and the right hand side of (1) is a finite product), Ramachandran and Rao (1970, general case) and Davies and Shimizu (1976, general case). In the last two papers the authors assumed some additional conditions including

$$1 < \sum_{j=1} \gamma_j a_j^\beta \quad \text{for some } \beta > 0 \quad \ldots \quad (2)$$

and

$$a \equiv \sup_n a_n < 1. \quad \ldots \quad (3)$$

The equation (1) is expressed in terms of the function $f(t) = -\log\phi(t)$ as

$$f(t) = \Sigma\gamma_{2j}f\,(a_{2j}t) + \Sigma\,\gamma_{2j+1}\,f(-a_{2j}\,t),$$

$$|t| \leqslant t_0 < t_1 \equiv \inf\{t; t > 0, \phi(t) = 0\}. \quad \ldots \quad (4)$$

This equation can be generalized in the form

$$f(t) = \int_{(0,\,1]} f(at)\,dU_1(a) + \int_{(0,\,1]} f(-at)\,dU_2(a), \;\; |t| \leqslant t_0 \quad \ldots \quad (5)$$

where U's are monotone non-decreasing left continuous functions, possibly $U_k(1) = \infty$. The equation is a special case of the one considered in the book by Kagan, Linnik and Rao (1973). In the later section (Section 3) we shall give a complete solution to this equation assuming only that

$$1 = \int_{(0,\,1]} a^\alpha d(U_1(a) + U_2(a)) < \int_{(0,\,1]} a^\beta d(U_1(a) + U_2(a)) < \infty,$$

$$\text{for some} \quad \alpha > \beta > 0. \quad \ldots \quad (6)$$

Theorem 4 : *Suppose the characteristic function $\phi(t)$ of a non-degenerate distribution is such that $f(t) \equiv -\log \phi(t)$ satisfies the equation* (5), *and suppose U's satisfy the condition* (6). *Then $\phi(t)$ can be put in the form*

$$\phi(t) = exp\,\{i\mu t - |t|^{\alpha}\, \Gamma(log|t|) + i|t|^{\alpha}\, sign\,(t)\, \Delta\,(log|t|)\} \qquad \ldots \quad (46)$$

where α is as in (6) *and necessarily $0 < \beta < \alpha \leqslant 2$, $\mu = 0$ if $\alpha < 1$, and Γ is a positive constant while $\Delta \equiv 0$ if $\alpha = 2$, and where Γ and Δ are as follows if $\alpha < 2$.*

Case 1 : *Γ and Δ are constants, and $\Delta \equiv 0$ if $\Omega_2 \neq \phi$. $\phi(t)e^{-i\mu t}$ is stable.*

Case 2 : *Γ and Δ are periodic functions with period ρ. $\Delta \equiv 0$ if $\Omega_2 \neq \phi$.*

Case 3 : *Γ is a periodic function with period ρ and Δ satisfies $\Delta(x+\rho) = -\Delta(x)$ for all x.*

A4–3

Especially the characteristic function $\phi(t)e^{-i\mu t}$ satisfies (43) *with $\gamma = e^{2\alpha\rho}$ and $a = e^{-2\rho}$, ρ being arbitrary in Case* 1.

Proof of Theorem 4 : With the distribution function G as defined by (44) and (45) the function

$$H(x) \equiv -e^{\alpha x} \log |\phi(e^{-x})| = e^{\alpha x} \cdot \mathrm{Re}\, f(e^{-x})$$

satisfies the equation

$$H(x) = \int_0^{\infty} H(x+y)\, dG(y), \quad x \geqslant x_0 > x_1 \equiv -\log t_1. \qquad \ldots \quad (47)$$

Unfortunately, however, we do not know if it satisfies the condition (9) of Theorem 1. So we rather define $H(x)$ by

$$H(x) \equiv -e^{(\alpha+1)x} \int_0^{e^{-x}} \log|\phi(t)|\, dt$$

$$= e^{(\alpha+1)x} \int_0^{e^{-x}} \mathrm{Re}\; f(t)\; dt (\geqslant 0).$$

Then $H(x)$ satisfies (9) with $C(y) = e^{(\alpha+1)y}$, and $G(x)$ satisfies (8) with $0 < \delta < (\alpha-\beta)/2$. Also it is not difficult to show that the equation (47) is satisfied. We conclude from Theorem 1 that $H(x)$ is bounded. We may therefore apply Theorem 2 to obtain $H(x) = \Gamma_0(x)$, where $\Gamma_0(x+u) = \Gamma_0(x)$ for $u \in \Omega = \Omega_1 \bigcup \Omega_2$. This means that

$$|\phi(t)| = e^{-|t|^\alpha \Gamma(\log|t|)}, \quad |t| \leqslant t_0 \qquad \dots \quad (48)$$

where $\Gamma(x) = (\alpha+1)\,\Gamma_0(x) + \Gamma_0{}'(x)$ is again a periodic function with period $u \in \Omega$. Especially we have for all $0 < t \leqslant t_0 < t_1 = \inf\{t;\, t > 0,\ |\phi(t)| = 0\}$,

$$|\phi(t)|^2 = |\phi(at)|^{2\gamma},$$

where $a = e^{-u}$, $u \in \Omega$ and $\gamma = a^{-\alpha}$. As $0 < a < 1$, we conclude that $t_1 = \infty$ and hence that $|\phi(t)|^2$ is the semi-stable characteristic function with characteristic exponent α. It then follows that ϕ can be put in the form

$$\phi(t) = e^{-|t|^\alpha \Gamma(\log|t|) - iA(t)}, \quad -\infty < t < \infty$$

where $A(t)$ is a real valued continuous odd function.

We can apply the similar argument as in the proof of Theorem 2 in Davies-Shimizu (1976) to conclude that there exists a real number μ such that $I(t) \equiv (\mu t + A(t)) \cdot |t|^{-\alpha}$ satisfies $\sup_t |I(t) - I(\varepsilon t)| < \infty$ for all $\varepsilon > 0$, and

that $H(x) \equiv I(e^{-x})$ satisfies the equation (11). As $\sup |I(t) - I(\varepsilon t)| < \infty$ is equivalent to $\sup_x |H(x+y) - H(x)| < \infty$ for all $y > 0$, we can apply Theorem 3 to obtain $H(x) = \Delta(x)$, where $\Delta(x)$ is as stated in Theorem 4. Q.E.D.

Shimizu, R. (1978) Solution to a functional equation and its application to some characterization problems, *Sankhya A,* **40**, 319-332.

APPENDIX 2

Theorem 1.

$$f(x) = e^{cx} \qquad and \qquad f(x) = 0 \tag{5}$$

are the most general solutions of

$$f(x+y) = f(x)f(y) \tag{1}$$

(for all real or for all positive x, y), that are continuous at one point (or

2.1. Cauchy's Equations and Jensen's Equation **39**

that can be majorized by a measurable function on a set of positive measure), while (1) supposed for nonnegative x, y has, in addition to (5), the solution

$$f(0) = 1, \qquad f(x) = 0 \qquad (x > 0)$$

in these classes of functions.

This reasoning, combined with that in Sect. 2.1.1, can also be used for the definition of the exponential function (similarly, that of Theorem 2 below for the definition of the logarithmic function).

2. Aczel, J.(1966). Functional equations and their applications Academic Press New York

APPENDIX 3

Lemma 4.1 (Tavangar. M & Asadi.M.(2010) [24] **)**

Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the order statistics from any cdf F . Then,

(i) *The survival function of $[X_{r:n} \mid X_{1:n} > t]$, $1 \le r \le n$, is given by*

- $G_{r:n}(x|t) = \sum_{i=0}^{r-1}\binom{n}{i}\{1-\theta_t(x)\}^i\{\theta_t(x)\}^{n-i}, \; x > t,$

where $\theta_t(x) = S(x)/S(t)$, and $S(x) = 1 - F(x)$, and

(ii) *The cdf of $[X_{r:n} \mid X_{1:n} \le t]$ is given by*

$$H_{r:n}(x|t) = \sum_{i=r}^{n}\binom{n}{i}\{\varphi_t(x)\}^i\{1-\varphi_t(x)\}^{n-i}, \qquad x \le t,$$

where $\varphi_t(x) = F(x)/F(t)$.

Proof of the lemma is given in Tavangar. M & Asadi.M.(2010) [24] .

APPENDIX 4

Proposition 2.1

Let $I = [0, c) or\ [0, \infty)$. Suppose $h: I \to \Re$ is continuous and $h(0) = 0$, $h'_+(0) = \alpha$ and for any $x \in I$, there exists $0 < \xi < x$ such that

(2.1) $$h(x) = h(\xi) + h(x - \xi)$$

Then

$$h(x) = \alpha x \quad for\ all\ x \in I$$

Proof:

Let $x \in I,\ x \neq 0$, and let $h(x)/x = \beta$. Then there exists $0 < \xi' < x$ such that let $h(\xi')/\xi' = \beta$.

Setting ξ as in (2.1), we have

$$\frac{h(x) - h(\xi)}{x - \xi} = \frac{h(x - \xi)}{x - \xi}$$

Let $h(x - \xi) \geq \beta(x - \xi)$.Therefore (2.1) implies that

$$\frac{h(x) - h(\xi)}{x - \xi} \geq \beta = \frac{h(x)}{x}$$

A direct calculation shows that $\xi h(x) \geq x h(\xi)$

or $h(\xi) \leq\ \beta\xi$

Applying the intermediate value theorem to $h(t) - \beta t$ on the interval determined by ξ and $x - \xi$ there exists a ξ' such tha $h(\xi') - \beta\xi' = 0$. Therefore, the claim follows.

To show that $\beta = \alpha$:

Let $y_0 = inf \left\{ y \in I: y \neq 0, \frac{h(y)}{y} = \beta \right\}$

Then $y_0 = 0$, for if $y_0 > 0, then\ h(y_0)/y_0 = \beta$. And applying the claim made above to y_0, we can find$0 < \xi' < y_0$, such that $\frac{h(\xi')}{\xi'} = \beta$, which contradicts the fact that y_0 is the infimum. Thus there exists a sequence $x_n \to 0$ such that

$$\beta = \lim_{n\to\infty}(h(x)/x_n) = h'_+(0) = \alpha$$

Therefore $h(x) = \alpha x$

Proposition 2.2

Let g: [a, b]→$\Re$ be continuous, *l*:[a, b]→$\Re$ be increasing, and suppose the set of points of increase D of *l* is *not* contained in {a, b} . Then there exists

$a < \xi < b$ such that

(2.2) $$\int_a^b g(t)dl(t) = g(\xi)(l(b) - l(a))$$

Remark: By mean value theorem of integration, there exists $a \leq \xi \leq b$ such that the above holds.

To show that ξ can be chosen to be different from *a* or *b*.

Proof: Let $\alpha = \min_{x\in[a,b]} g(x) = g(x_1),\ \beta = \max_{x\in[a,b]} g(x) = g(x_2)$, and

$$G(x) = \int_a^b g(t)dl(t)$$

Assume $x_1 \ < x_2$. Then

$\alpha(l(b) - l(a)) \leq G(b) \leq \beta(l(b) - l(a)$

If $G(b) = \alpha(l(b) - l(a))$, then D is contained in $\{x: g(x) = \alpha)$. By assumption, D contains points other than *a* and *b.* We can choose $\xi \neq a, b$ such that $g(\xi) = \alpha$, and hence

$$g(\xi)\big(l(b) - l(a)\big) = \int_a^b g(t)dl(t)$$

Similarly $G(b) = \beta\big(l(b) - l(a)\big)$

Thus if $\alpha(l(b) - l(a) < G(b) < \beta\big(l(b) - l(a)\big)$

Then, by continuity of g, we can find x_1', x_2' in the neighbourhood of x_1, x_2, respectively such that $a < \ x_1', < x_2' < b$

And if $g(x_1')(l(b) - l(a) \leq G(b) \leq g(x_2')\big(l(b) - l(a)\big)$

The intermediate value theorem applied to g in $[x_1', x_2']$ implies that there exists $\xi \neq a, b\ \ (x_1' < \xi < x_2')$such that

$$G(\xi) = \int_a^b g(t)dl(t)$$

LAU, K. S., PRAKASA RAO, B. L. S. (1990). Characterization of the exponential distribution by the relevation transform. J. Appl. Prob. 27, 726-729.

APPENDIX 5

Theorem 2.1. *Let F be any cdf with support [0, ϑ), ϑ > 0. Suppose that*

Q : [0, ϑ) → R+. The functional equation

F (x) = F (xy) + F (xQ(y)), x, xQ(y) ∈ [0, ϑ), y ∈ [0, 1], (2.1)

holds if and only if F is a (rescaled) power function distribution with parameter vector (α, ϑ), for some constant α > 0, and Q(y) = (1 −yα)1/α , 0 ≤ y ≤ 1

Tavangar. M & Asadi.M.(2010) Some new characterization results on exponential and related distributions, *Bulletin of the Iranian Mathematical Society, Vol. 36 No.1*, pp 257-272.

APPENDIX 6

2. Characterization of univariate random variable by a relationship between the conditional expectation and the probability density function

In this section, according to Theorems 1 and 2 of Huang and Su (2000), we give the following characterization theorem based on a relationship between the conditional expectation and probability density function.

Theorem 2.1. Let $a < b$ be extended real numbers, and g and h be the real functions defined on (a, b). Assume g is continuous and $h(y) \neq 0$, $\forall\ y \in (a, b)$. Then there exists an absolutely continuous random variable X with $C_X = (a, b)$, such that

$E(g(X) \mid X \leq y)$ is finite, $\forall\ y \in C_X$, and

$$E(g(X) \mid X \leq y) = h(y)\frac{f(y)}{F(y)}, \qquad \forall\ y \in C_X, \tag{1}$$

if and only if for any fixed $\kappa \in (a, b)$, the following conditions hold.

(i) $\int_{\kappa}^{y} g(u)/h(u)\ du$ is finite,

(ii) $\int_{a}^{b} \exp\{\int_{\kappa}^{y} g(u)/h(u)\ du\}/ \mid h(y) \mid\ dy\ <\ \infty$,

(iii) $\lim_{y \to a} \int_{\kappa}^{y} g(u)/h(u)\ du = -\infty$.

Moreover, the $p.d.f.$ of the random variable X which satisfies (1) with $C_X = (a, b)$ is

$$f(y) = \frac{1}{\alpha_\kappa \mid h(y) \mid} \exp\{\int_{\kappa}^{y} \frac{g(u)}{h(u)}\ du\}, \qquad \forall\ a < y < b, \tag{2}$$

where $\alpha_\kappa = \int_{a}^{b}\ \exp\{\int_{\kappa}^{y} g(u)/h(u)\ du\}/ \mid h(y) \mid\ dy$.

Proof. First, we prove the necessity. From (1), we have

$$\int_{a}^{y} g(u)f(u)\ du = h(y)f(y), \qquad \forall\ a < y < b. \tag{3}$$

This in turn implies that

$$\int_{\kappa}^{y} \frac{g(u)}{h(u)} du = \int_{\kappa}^{y} \frac{g(u)f(u)}{\int_a^u\ g(\omega)f(\omega)\ d\omega} du \tag{4}$$

$$= \ ln \mid \int_{a}^{y} g(\omega)f(\omega)\ d\omega \mid -ln \mid \int_{a}^{\kappa} g(\omega)f(\omega)\ d\omega \mid$$

$$= \ ln \mid h(y)f(y) \mid -ln \mid h(\kappa)f(\kappa) \mid, \qquad \forall\ y, \kappa \in (a, b),$$

thus $\int_\kappa^y\ g(u)/h(u)\ du$ is finite, $\forall\ y, \kappa \in (a, b)$. For any $\kappa \in (a, b)$, from (4),

$$f(y) = \frac{\mid h(\kappa) \mid f(\kappa)}{\mid h(y) \mid} \exp\{\int_{\kappa}^{y} \frac{g(u)}{h(u)}\ du\}, \qquad \forall\ a < y < b, \tag{5}$$

since $\int_a^b\ f(y)\ dy = 1$, we have $\int_a^b\ \exp\{\int_\kappa^y\ g(u)/h(u)\ du\}/ \mid h(y) \mid\ dy < \infty$, and the $p.d.f.$ of the absolutely continuous random variable X satisfies (1), with $C_X = (a, b)$

is given in (2). Also it is easy to see that (3) implies $\lim_{y \to a} h(y)f(y) = 0$. Hence $\lim_{y \to a} \exp\{\int_{\kappa}^{y} g(u)/h(u)\ du\} = 0$, i.e. $\lim_{y \to a} \int_{\kappa}^{y} g(u)/h(u) = -\infty$.

Next, assume conditions (i)-(iii) hold. For any $\kappa \in (a, b)$, let f be defined as in (2). Conditions (i) and (ii) imply that f is a $p.d.f.$ of some random variable X with $C_X = (a, b)$. Also it can be shown that

$$\int_{\kappa}^{y} \frac{g(u)}{h(u)}\, du = ln\ \alpha_\kappa + ln(|\ f(y)h(y)\ |), \qquad \forall\ a < y < b. \tag{6}$$

The left side of (6) is differentiable with respect to y, taking the derivatives of both sides of (6) with respect to y, after some manipulations, we obtain

$$g(y)f(y) = d(f(y)h(y)), \qquad y \in (a, b). \tag{7}$$

As condition (iii) is equivalent to $\lim_{y \to a} h(y)f(y) = 0$, (7) implies

$$\int_{a}^{y} g(u)f(u)\, du = h(y)f(y), \qquad \forall\ a < y < b, \tag{8}$$

From (8), we obtain that $E(g(X) \mid X \leq y) = h(y)$, $\forall\ y \in C_X$. The sufficiency is proved.

Chang, T.(2001). Characterization of distributions by conditional expectation, *Department of Applied Mathematics*, National Sun Yat-Sen University, Kaohsiung, Taiwan, 804, R.O.C

APPENDIX 7

Theorem 2.3.2. Let $X \geq 0$ be a random variable with distribution function $F(x)$. Assume that $E(X^k)$ is finite where $k \geq 2$ is a given integer. If

(33) $E[(X-z)^k | X \geq z] = E(X^k)$ for all $z \geq 0$,

then $F(x) = 1 - e^{-bx}$, $x \geq 0$, and $b > 0$.

Proof: Let $m = E(X^k)$. Since

$$E[(X-z)^k | X \geq z] = \int_z^{+\infty} (y-z)^k dF(y)/[1-F(z)] ,$$

an argument similar to the one applied in Lemma 1.2.1 and (32) yields

(34) $\int_z^{+\infty} (y-z)^{k-1} G(y)\, dy = \frac{m}{k} G(z)$, $z \geq 0$,

where $G(u) = 1-F(u)$. If we denote the left hand side of (34) by $H(z)$, then repeated differentiation in (34) gives

$$(35) \quad \frac{(-1)^k m}{k!} H^{(k)}(z) = H(z) \ .$$

It is well known from the elementary theory of linear differential equations that the general solution of (35) is

$$(36) \quad H(z) = \sum_{j=1}^{k} c_j e^{b_j z} \ ,$$

where the c_j are arbitrary constants and b_j, $1 \le j \le k$, are the k (complex) solutions of the equation

$$\frac{(-1)^k m}{k!} t^k = 1 \ .$$

Let its negative solution be denoted by $b_1 = -b$, $b > 0$ (evidently, b is the positive real root of $(k!)/m$). Because, by definition and by (34), $H(z) = (m/k)G(z)$, $H(z) \ge 0$, decreases and $H(z) \to 0$ as $z \to +\infty$. We shall show that these properties can be satisfied only if

$$(37) \quad \sum_{j=2}^{k} c_j \ e^{b_j z} = 0$$

identically. First let $k = 3$. Considering the conjugate complex roots, we have $\mathrm{Re}b_2 = \mathrm{Re}b_3 = r > 0$ and $r < b$. Then $|\exp(b_j z)| \to +\infty$ as $z \to +\infty$ for $j = 2,3$. Hence, $H(z) \ge 0$ and $H(z) \to 0$ can be satisfied only if $c_2 = c_3 = 0$. If $k = 4$, then $\mathrm{Re}b_2 = \mathrm{Re}b_3 = 0$ and $b_4 = b$. In a similar manner as above, we easily verify that $c_4 = 0$. On the other hand, since

$$c_2 e^{b_2 z} + c_3 e^{b_3 z}$$

has to be real, $H(z)$ cannot be decreasing for large z unless $c_2 = c_3$. But then the above expression is identically zero. Now the general case can be handled similarly. If one combines two roots b_j and b_{j+1}, which are complex conjugates of each other, step by step one indeed arrives at the validity of (37). Hence

$$H(z) = c_1 e^{-bz} \ , \ b > 0 \ ,$$

from which

$$F(z) = 1-G(z) = 1 - \frac{kc_1}{m} e^{-bz} \ , \ b > 0, \ z \ge 0 \ .$$

But since $m = E(X^k)$, (34) with $z = 0$ and Lemma 1.2.1 imply that $G(0+) = 1$ or $F(0+) = 0$. Consequently, $(kc_1)/m=1$, which completes the proof. □

Galambos ,J. & Kotz, S. (1978). *Characterizations of Probability Distributions*. Springer-Verlag Berlin Heidelberg New York

APPENDIX 8

The following formula based on integration by parts will be utilized on several occasions in the sequel.

Lemma 1.2.1. Let $X \geq 0$ be a random variable with a distribution function $F(x)$ Assume that $E(X^a)$ is finite for some $a > 0$. Then, for any $0 \leq T < +\infty$,

$$(1) \quad \int_T^{+\infty} x^a dF(x) = T^a[1-F(T)] + a \int_T^{+\infty} x^{a-1}[1-F(x)]dx \ .$$

In particular, if $E(X)$ is finite, then

$$(2) \quad \int_T^{+\infty} (x-T)dF(x) = \int_T^{+\infty} [1-F(x)]dx \ .$$

Proof: Let $T < N < +\infty$ be arbitrary. Then integrating parts yields

$$\int_T^N x^a dF(x) = N^a F(N) - T^a F(T) - a \int_T^N x^{a-1} F(x)dx$$

$$= T^a[1-F(T)] - N^a[1-F(N)] + a \int_T^N x^{a-1}[1-F(x)]dx \ .$$

Since, for any $T \leq N < +\infty$,

$$\int_T^N x^a dF(x) + \int_N^{+\infty} x^a dF(x) = \int_T^{+\infty} x^a dF(x)$$

and the integral on the right hand side is finite, the second term on the left hand side converges to zero as $N \to +\infty$. Hence, in view of the inequality

$$\int_N^{+\infty} x^a dF(x) \geq N^a \int_N^{+\infty} dF(x) = N^a[1-F(N)] \ ,$$

$N^a[1-F(N)] \to 0$ as $N \to +\infty$. Formula (1) thus follows. On the other hand, (2) is a special case of (1) with $a = 1$, hence the proof is completed. □

Galambos ,J. & Kotz, S. (1978). *Characterizations of Probability Distributions*. Springer-Verlag Berlin Heidelberg New York

APPENDIX 9

$$f_{R_n}(y) = \frac{1}{\Gamma(n+1)} R^n(y) f(y), \quad y > 0. \tag{1.2}$$

For $n > m$, the joint density of R_m and R_n is

$$f_{R_m,R_n}(x,y) = \frac{1}{\Gamma(n-m)\Gamma(m+1)} R^m(x)(R(y)-R(x))^{n-m-1} r(x) f(y), \quad 0 < x < y, \tag{1.3}$$

and the conditional density of R_n given $R_m = x$ is

$$f_{R_n|R_m=x}(y) = \frac{(R(y)-R(x))^{n-m-1} f(y)}{\Gamma(n-m)\bar{F}(x)}, \quad 0 < x < y. \tag{1.4}$$

Huang, W.J. & Li, S.H. A.C.(1993) .Characterization results based on record values. *Statistica Sinica* 3, pp583-599

APPENDIX 10

Theorem 2. *Let G be a non-decreasing function having non-lattice support on $x \geq 0$ with $G(0) = 0$ and $E(G(X_1)) < \infty$.*
(i) *If*

$$E(G(\gamma_t^*)) = c, \quad \forall t > 0, \tag{2.12}$$

and if

$$c < \int_0^\infty e^{-\xi x} dG(x) < \infty \tag{2.13}$$

for some $\xi > 0$, then $c = E(G(X_1))$ and $\{N(t), t \geq 0\}$ is a homogeneous Poisson process.

Proof. (i) First, $c = E(G(X_1))$ is obvious. Next, since

$$E(G(\gamma_t^*)) = \int_0^\infty P(\gamma_t^* > x) dG(x) = \int_0^\infty e^{-R(t+x)+R(t)} dG(x), \tag{2.15}$$

(2.12) implies

$$ce^{-R(t)} = \int_0^\infty e^{-R(t+x)} dG(x), \tag{2.16}$$

or (since $R_0 = X_1$)

$$cP(X_1 > t) = \int_0^\infty P(X_1 > t + x) dG(x). \tag{2.17}$$

This, together with (2.13) implies, by Shimizu (1978) or (1979), X_1 is exponentially distributed. This completes the proof of part (i).

Huang, W.J. & Li, S.H. A.C.(1993) .Characterization results based on record values. *Statistica Sinica* 3, pp583-599

APPENDIX 11

$$f_{R_{j+l},R_{j+k}|R_j=x}(u,v) = \frac{(R(u)-R(v))^{l-k-1}(R(v)-R(x))^{k-1}f(u)r(v)}{\Gamma(l-k)\Gamma(k)\bar{F}(x)}, \quad 0<x<v<u. \quad (1.5)$$

Huang, W.J. & Li, S.H. A.C.(1993) .Characterization results based on record values. *Statistica Sinica* 3, pp583-599

APPENDIX 12

in the latter theory the simple solutions are exponentials, it is not surprising that we can also find exponential solutions of differential-difference equations. We have

$$L(e^{st}) = (a_0 s + b_0 + b_1 e^{-\omega s})e^{st}. \quad (3.5.2)$$

Hence, $u = e^{st}$ is a solution of $L(u) = 0$, for all t, if and only if the number s is a zero of the transcendental function

$$h(s) = a_0 s + b_0 + b_1 e^{-\omega s}. \quad (3.5.3)$$

Definition. *The function $h(s)$ associated with the equation $L(u) = 0$ is called the characteristic function of L, the equation $h(s) = 0$ is called the characteristic equation of L, and the roots of $h(s) = 0$ are called the characteristic roots of L.*

Corresponding to each characteristic root there is a solution (which may be complex) of $L(u) = 0$, and to distinct roots correspond linearly independent solutions. As we shall see later, there are, in general, infinitely many roots. Moreover, a multiple root gives rise to several independent solutions, as we shall now show. We first observe that

$$h'(s) = a_0 - b_1 \omega e^{-\omega s},$$
$$h^{(k)}(s) = (-1)^k b_1 \omega^k e^{-\omega s}, \qquad k = 2, 3, \cdots. \tag{3.5.4}$$

For any $n \geq 1$, we have

$$L(t^n e^{st}) = a_0(t^n s e^{st} + n t^{n-1} e^{st}) + b_0 t^n e^{st} + b_1 (t - \omega)^n e^{s(t-\omega)}. \tag{3.5.5}$$

If $(t - \omega)^n$ is expanded by the binomial theorem, we see that the coefficient of $t^{n-k}e^{st}$ $(0 \leq k \leq n)$ in (3.5.5) is

$$\binom{n}{k} h^{(k)}(s).$$

Bellman, R and Cooke, K.L (1963) *Differential-difference equations*, Elsevier, p.54

APPENDIX 13

Since most insurance policies contain policy provisions such as deductibles or limits, the observed data are also subject to such constraints. Insurers (reinsurers) often observe only losses above the deductible (retention) and below the limit. In spite of the incomplete observations with censored or truncated data, actuaries are usually interested in estimating the ground-up losses, which is required in order to evaluate the effect of inflation or changing deductibles.

If the ground-up loss X is modeled by a distribution with survivor function $S(x)$, the distribution of the excess loss $(X - x_0 \mid X \geqslant x_0)$ is given by the survivor function $S_{X - x_0 \mid X > x_0}(x) = S(x + x_0)/S(x_0)$. In terms of the hazard rate function

$$h(x) = -\frac{d}{dx} \ln S(x),$$

we have that

$$S(x) = \exp\left\{ -\int_0^x h(y)\, dy \right\}$$

and

$$S_{X - x_0 \mid X > x_0}(x) = \exp\left\{ -\int_0^x h(y + x_0)\, dy \right\}.$$

If the hazard rate function has the form

$$h(x) = f_\vartheta(x + \lambda), \tag{1}$$

for some simple function $f_\vartheta(\cdot)$ which depends on one or more parameters ϑ, it follows that once the model parameters (λ, ϑ) have been estimated, it is possible to work with arbitrary deductibles x_0 simply by redefining the parameter λ to $\lambda + x_0$. A class of distributions with hazard function (1) is thus closed under the formation of excess losses.

A layer $(a, b]$ of a risk X is defined as an excess-of-loss cover

$$I_{(a,b]} = \begin{cases} 0, & 0 \leqslant X < a, \\ (X-a), & a \leqslant X < b, \\ b-a, & b \leqslant X, \end{cases}$$

where a is the deductible (or retention), and $(b-a)$ is the limit.

Given that a distribution model has been selected with best-estimate parameter values, an estimate of the expected loss (or net premium) of the layer $(a, b]$ can be calculated as

$$\mathsf{E} I_{(a,b]} = \int_a^b S(u)\, du.$$

It is often needed to impose risk margins to guard against adverse deviations or mis-estimations in the expected cost. Wang (1995) suggested that one adjust directly the survivor function $S(x)$ by a proportional hazard (PH) transform such that the risk-adjusted premium is calculated as

$$\mathsf{H}[I_{(a,b]}] = \int_a^b [S(u)]^{1/\rho}\, du, \quad \rho > 1,$$

which is the expected loss in the layer $(a, b]$ under the risk-adjusted distribution with survivor function $[S(x)]^{1/\rho}$.

The risk adjusted survivor function $[S(x)]^{1/\rho}$ has a hazard rate function $h(x)/\rho$. Whence, within a model with hazard rate function of the form

$$h(x) = \alpha f_\vartheta(x + \lambda), \tag{2}$$

one may change the deductible by redefining λ, and change the risk load by redefining the parameter α. A class of distributions (2) is thus closed under the formation of excess losses as well as risk-adjustment using the PH-transform.

3. EXPONENTIAL MIXTURES

Consider an exponential mixture with a conditional survivor function

$$S(x \mid \vartheta) = e^{-\vartheta x},$$

where the parameter ϑ has a distribution function $G(\vartheta)$. If G is absolutely continuous, denote the density by $g(\vartheta)$. The resulting exponential mixture has a survivor function

$$S(x) = \int_0^\infty S(x \mid \vartheta)\, dG(\vartheta) = \int_0^\infty e^{-\vartheta x}\, dG(\vartheta) = L_G(x), \tag{3}$$

where L_G is the Laplace transform of the mixing distribution $G(\vartheta)$.

Recall that a function f on $[0, \infty]$ is completely monotone (c.m.) if it possesses derivatives of all orders $f^{(n)}$ and $(-1)^n f^{(n)}(x) \geq 0$, and that a function f is c.m. with $f(0) = 1$ if and only if it is the Laplace transform for a distribution on $[0, \infty)$ (see e.g. Feller, 1971, p. 439). We summarize some properties for the class of mixed exponential distributions.

THEOREM 1. *1. A distribution is a mixed exponential distribution if, and only if, its survivor function is c.m.*

2. If X and Y are independent with mixed exponential distributions, then $Z = \min(X, Y)$ has a mixed exponential distribution. The mixing distribution for Z is the convolution of the mixing distributions for X and Y.

3. A distribution with c.m. hazard function is a mixed exponential distribution.

4. Mixed exponential distributions are DFR (decreasing failure rate), i.e. $S(x+t)/S(x)$ is a non-decreasing function of x.

5. Mixed exponential distributions are log-convex, i.e. the logarithm of the density function is convex.

6. Mixed exponential distributions are infinitely divisible.

Hasselager, O., Wang, S. & Gordon, W. (1988): Exponential and scale mixtures and equilibrium distributions, *Scandinavian Actuarial Journal*, 1998:2, 3-4

APPENDIX 14

3.382

1.[6] $\int_0^u (u-x)^\nu e^{-\mu x}\,dx = (-\mu)^{-\nu-1} e^{-u\mu}\,\gamma(\nu+1,-u\mu)$ $\quad [\operatorname{Re}\nu > -1, \quad u > 0]$ ET I 137(6)

2. $\int_u^\infty (x-u)^\nu e^{-\mu x}\,dx = \mu^{-\nu-1} e^{-u\mu}\,\Gamma(\nu+1)$ $\quad [u > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}\mu > 0]$ ET I 137(5), ET II 202(11)

3. $\int_0^\infty (1+x)^{-\nu} e^{-\mu x}\,dx = \mu^{\frac{\nu}{2}-1} e^{\frac{\mu}{2}}\,W_{-\frac{\nu}{2},\frac{(1-\nu)}{2}}(\mu)$ $\quad [\operatorname{Re}\mu > 0]$ WH

4. $\int_0^\infty (x+\beta)^\nu e^{-\mu x}\,dx = \mu^{-\nu-1} e^{\beta\mu}\,\Gamma(\nu+1,\beta\mu)$ $\quad [|\arg\beta| < \pi, \quad \operatorname{Re}\mu > 0]$ ET I 137(4), ET II 233(10)

5. $\int_0^u (a+x)^{\mu-1} e^{-x}\,dx = e^a[\gamma(\mu,a+u) - \gamma(\mu,a)]$ $\quad [\operatorname{Re}\mu > 0]$ EH II 139

6. $\int_{-\infty}^\infty (\beta+ix)^{-\nu} e^{-ipx}\,dx = 0$ $\quad$ [for $p > 0$]

$= \frac{2\pi(-p)^{\nu-1} e^{\beta p}}{\Gamma(\nu)}$ $\quad$ [for $p < 0$]

$[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\beta > 0]$ ET I 118(4)

7. $\int_{-\infty}^\infty (\beta-ix)^{-\nu} e^{-ipx}\,dx = \frac{2\pi p^{\nu-1} e^{-\beta p}}{\Gamma(\nu)}$ $\quad$ [for $p > 0$]

$= 0$ $\quad$ [for $p < 0$]

$[\operatorname{Re}\nu > 0, \quad \operatorname{Re}\beta > 0]$ ET I 118(3)

3.383

1.[11] $\int_0^u x^{\nu-1}(u-x)^{\mu-1} e^{\beta x}\,dx = \mathrm{B}(\mu,\nu) u^{\mu+\nu-1}\;{}_1F_1(\nu;\mu+\nu;\beta u)$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET II 187(14)

2.[11] $\int_0^u x^{\mu-1}(u-x)^{\mu-1} e^{\beta x}\,dx = \sqrt{\pi}\left(\frac{u}{\beta}\right)^{\mu-\frac{1}{2}} \exp\left(\frac{\beta u}{2}\right)\Gamma(\mu)\,I_{\mu-\frac{1}{2}}\left(\frac{\beta u}{2}\right)$

$[\operatorname{Re}\mu > 0]$ ET II 187(13)

3. $\int_u^\infty x^{\mu-1}(x-u)^{\mu-1} e^{-\beta x}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{u}{\beta}\right)^{\mu-\frac{1}{2}} \Gamma(\mu)\exp\left(-\frac{\beta u}{2}\right) K_{\mu-\frac{1}{2}}\left(\frac{\beta u}{2}\right)$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\beta u > 0]$ ET II 202(12)

Gradshteyn, I.S. & Ryzhik, I.M. (2007). *Tables of Integrals, Series and Products*, Elsevier Inc., Burlington, MA 01803, U.S.A[3.383 1] pp.347

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3. $$\int_0^u x^{-\mu-1}(u-x)^{\mu-1}e^{-\frac{\beta}{x}}\,dx = \beta^{-\mu}u^{\mu-1}\Gamma(\mu)\exp\left(-\frac{\beta}{u}\right)$$
$[\operatorname{Re}\mu > 0, \quad u > 0]$ ET II 187(16)

4. $$\int_0^u x^{-2\mu}(u-x)^{\mu-1}e^{-\frac{\beta}{x}}\,dx = \frac{1}{\sqrt{\pi u}}\beta^{\frac{1}{2}-\mu}e^{-\frac{\beta}{2u}}\Gamma(\mu)\,K_{\mu-\frac{1}{2}}\left(\frac{\beta}{2u}\right)$$
$[u > 0, \quad \operatorname{Re}\beta > 0, \quad \operatorname{Re}\mu > 0]$
ET II 187(17)

5. $$\int_u^\infty x^{\nu-1}(x-u)^{\mu-1}e^{\frac{\beta}{x}}\,dx = \mathrm{B}(1-\mu-\nu,\mu)u^{\mu+\nu-1}\,{}_1F_1\left(1-\mu-\nu;1-\nu;\frac{\beta}{u}\right)$$
$[0 < \operatorname{Re}\mu < \operatorname{Re}(1-\nu), \quad u > 0]$
ET II 203(15)

6. $$\int_u^\infty x^{-2\mu}(x-u)^{\mu-1}e^{\frac{\beta}{x}}\,dx = \sqrt{\frac{\pi}{u}}\beta^{\frac{1}{2}-\mu}\Gamma(\mu)\exp\left(\frac{\beta}{2u}\right)I_{\mu-\frac{1}{2}}\left(\frac{\beta}{2u}\right)$$
$[\operatorname{Re}\mu > 0, \quad u > 0]$ ET II 202(14)

7. $$\int_0^\infty x^{\nu-1}(x+\gamma)^{\mu-1}e^{-\frac{\beta}{x}}\,dx = \beta^{\frac{\nu-1}{2}}\gamma^{\frac{\nu-1}{2}+\mu}\,\Gamma(1-\mu-\nu)e^{\frac{\beta}{2\gamma}}\,W_{\frac{\nu-1}{2}+\mu,-\frac{\nu}{2}}\left(\frac{\beta}{\gamma}\right)$$
$[|\arg\gamma| < \pi, \quad \operatorname{Re}(1-\mu) > \operatorname{Re}\nu > 0]$
ET II 234(13)a

8. $$\int_0^u x^{-2\mu}\left(u^2-x^2\right)^{\mu-1}e^{-\frac{\beta}{x}}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\beta}\right)^{\mu-\frac{1}{2}}u^{\mu-\frac{3}{2}}\,\Gamma(\mu)\,K_{\mu-\frac{1}{2}}\left(\frac{\beta}{u}\right)$$
$[\operatorname{Re}\beta > 0, \quad u > 0, \quad \operatorname{Re}\mu > 0]$
ET II 188(23)a

9. $$\int_0^\infty x^{\nu-1}e^{-\frac{\beta}{x}-\gamma x}\,dx = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}}K_\nu\left(2\sqrt{\beta\gamma}\right)$$
$[\operatorname{Re}\beta > 0, \quad \operatorname{Re}\gamma > 0]$
ET II 82(23)a, LET I 146(29)

10. $$\int_0^\infty x^{\nu-1}\exp\left[\frac{i\mu}{2}\left(x-\frac{\beta^2}{x}\right)\right]dx - 2\beta^\nu e^{\frac{i\nu\pi}{2}}\,K_{-\nu}(\beta\mu)$$
$[\operatorname{Im}\mu > 0, \quad \operatorname{Im}\left(\beta^2\mu\right) < 0;$ note that $K_{-\nu} \equiv K_\nu]$ EH II 82(24)

Gradshteyn, I.S. & Ryzhik, I.M. (2007). *Tables of Integrals, Series and Products*, Elsevier Inc., Burlington, MA 01803, U.S.A. [3.471 9] pp.368

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5. $$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos\nu\theta\,d\theta - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}\,dt$$ $\left[|\arg z| \le \frac{\pi}{2}, \quad \operatorname{Re}\nu > 0\right]$ WA 201(4)

See also **3.383** 2, **3.387** 1, **3.471** 6, **3.714** 5.

For an integral representation of $I_0(z)$ and $I_1(z)$, see **3.366** 1, **3.534** **3.856** 6.

The function $K_\nu(z)$

8.432

1. $$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh\nu t\,dt$$ $\left[|\arg z| < \frac{\pi}{2} \text{ or } \operatorname{Re} z = 0 \text{ and } \nu = 0\right]$ MO 39

2. $$K_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty e^{-z\cosh t}\sinh^{2\nu} t\,dt$$ $\left[\operatorname{Re}\nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0; \text{ or } \operatorname{Re} z = 0 \text{ and } -\frac{1}{2} < \operatorname{Re}\nu < \frac{1}{2}\right]$ WA 190(5), WH

3. $$K_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_1^\infty e^{-zt}\left(t^2-1\right)^{\nu-\frac{1}{2}}\,dt$$ $\left[\operatorname{Re}\left(\nu+\frac{1}{2}\right) > 0, \quad |\arg z| < \frac{\pi}{2}; \text{ or } \operatorname{Re} z = 0 \text{ and } \nu = 0\right]$ WA 190(4)

4. $$K_\nu(x) = \frac{1}{\cos\frac{\nu\pi}{2}}\int_0^\infty \cos(x\sinh t)\cosh\nu t\,dt$$ $\left[x > 0, \quad -1 < \operatorname{Re}\nu < 1\right]$ WA 202(13)

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[8.432 1] pp.917

APPENDIX 17

[Re $a > 0$, Re $\nu > 0$]

3.462

1. $$\int_0^\infty x^{\nu-1} e^{-\beta x^2 - \gamma x}\, dx = (2\beta)^{-\nu/2}\Gamma(\nu)\exp\left(\frac{\gamma^2}{8\beta}\right) D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right)$$
[Re $\beta > 0$, Re $\nu > 0$]
EH II 119(3)a, ET I 313(13)

2.[8] $$\int_{-\infty}^\infty x^n e^{-px^2+2qx}\, dx = \frac{1}{2^{n-1}p}\sqrt{\frac{\pi}{p}}\frac{d^{n-1}}{dq^{n-1}}\left(qe^{q^2/p}\right) \quad [p > 0] \quad \text{BI (100)(8)}$$
$$= n! e^{q^2/p}\sqrt{\frac{\pi}{p}}\left(\frac{q}{p}\right)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{(n-2k)!(k)!}\left(\frac{p}{4q^2}\right)^k \quad [p > 0] \quad \text{LI (100)(8)}$$

3.[11] $$\int_{-\infty}^\infty (ix)^\nu e^{-\beta^2 x^2 - iqx}\, dx = 2^{-\frac{\nu}{2}}\sqrt{\pi}\beta^{-\nu-1}\exp\left(-\frac{q^2}{8\beta^2}\right) D_\nu\left(\frac{q}{\beta\sqrt{2}}\right)$$
$\left[\text{Re}\,\beta^2 > 0, \quad \text{Re}\,\nu > -1, \quad \arg ix = \frac{\pi}{2}\,\text{sign}\,x\right]$ ET I 121(23)

4. $$\int_{-\infty}^\infty x^n \exp\left[-(x-\beta)^2\right]\, dx = (2i)^{-n}\sqrt{\pi}\, H_n(i\beta)$$
EH II 195(31)

5.[11] $$\int_0^\infty xe^{-\mu x^2 - 2\nu x}\, dx = \frac{1}{2\mu} - \frac{\nu}{2\mu}\sqrt{\frac{\pi}{\mu}} e^{\frac{\nu^2}{\mu}}\left[1 - \text{erf}\left(\frac{\nu}{\sqrt{\mu}}\right)\right]$$
$\left[|\arg \nu| < \frac{\pi}{2}, \quad \text{Re}\,\mu > 0\right]$ ET I 146(31)a

6. $$\int_{-\infty}^\infty xe^{-px^2+2qx}\, dx = \frac{q}{p}\sqrt{\frac{\pi}{p}}\exp\left(\frac{q^2}{p}\right)$$
[Re $p > 0$] BI (100)(7)

7.[11] $$\int_0^\infty x^2 e^{-\mu x^2 - 2\nu x}\, dx = -\frac{\nu}{2\mu^2} + \sqrt{\frac{\pi}{\mu^5}}\frac{2\nu^2 + \mu}{4} e^{\frac{\nu^2}{\mu}}\left[1 - \text{erf}\left(\frac{\nu}{\sqrt{\mu}}\right)\right]$$
$\left[|\arg \nu| < \frac{\pi}{2}, \quad \text{Re}\,\mu > 0\right]$ ET I 146(32)

8. $$\int_{-\infty}^\infty x^2 e^{-\mu x^2 + 2\nu x}\, dx = \frac{1}{2\mu}\sqrt{\frac{\pi}{\mu}}\left(1 + 2\frac{\nu^2}{\mu}\right) e^{\frac{\nu^2}{\mu}}$$
[$|\arg \nu| < \pi$, Re $\mu > 0$] BI (100)(8)a

9.* $$\int_0^\infty e^{-\beta x^n \pm a}\, dx = \frac{e^{\pm a}}{n\beta^{1/n}}\Gamma\left(\frac{1}{n}\right)$$
[Re $\beta > 0$, Re $n > 0$]

Gradshteyn, I.S. & Ryzhik, I.M. (2007). *Tables of Integrals, Series and Products*, Elsevier Inc., Burlington, MA 01803, U.S.A.

[3.462 1] pp.365

APPENDIX 18

3. $$W_{l+\mu+\frac{1}{2},\mu}(z) = (-1)^l z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z}(2\mu+1)(2\mu+2)\cdots(2\mu+l)\,\Phi(-l, 2\mu+1; z)$$
$$= (-1)^l z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z}\, L_l^{2\mu}(z)$$
MO 116

9.238

1. $$J_\nu(x) = \frac{2^{-\nu}}{\Gamma(\nu+1)} x^\nu e^{-ix}\,\Phi\left(\tfrac{1}{2}+\nu, 1+2\nu; 2ix\right)$$ EH I 265(9)

2. $$I_\nu(x) = \frac{2^{-\nu}}{\Gamma(\nu+1)} x^\nu e^{-x}\,\Phi\left(\tfrac{1}{2}+\nu, 1+2\nu; 2x\right)$$ EH I 265(10)

3. $$K_\nu(x) = \sqrt{\pi} e^{-x}(2x)^\nu\,\Psi\left(\tfrac{1}{2}+\nu, 1+2\nu; 2x\right)$$ EH I 265(13)

9.24–9.25 Parabolic cylinder functions $D_p(z)$

9.240 $$D_p(z) = 2^{\frac{1}{4}+\frac{p}{2}}\, W_{\frac{1}{4}+\frac{p}{2},-\frac{1}{4}}\left(\frac{z^2}{2}\right) z^{-1/2}$$
$$= 2^{\frac{p}{2}} e^{-\frac{z^2}{4}} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)}\,\Phi\left(-\frac{p}{2}, \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2\pi}z}{\Gamma\left(-\frac{p}{2}\right)}\,\Phi\left(\frac{1-p}{2}, \frac{3}{2}; \frac{z^2}{2}\right)\right\}$$
MO 120a

are called *parabolic cylinder functions.*

Integral representations

9.241

1. $$D_p(z) = \frac{1}{\sqrt{\pi}} 2^{p+\frac{1}{2}} e^{-\frac{\pi}{2}pi} e^{\frac{z^2}{4}} \int_{-\infty}^{\infty} x^p e^{-2x^2+2ixz}\,dx$$ $[\operatorname{Re} p > -1; \quad \text{for } x < 0, \quad \arg x^p = p\pi i]$
MO 122

2. $$D_p(z) = \frac{e^{-\frac{z^2}{4}}}{\Gamma(-p)} \int_0^\infty e^{-xz-\frac{x^2}{2}} x^{-p-1}\,dx$$ $[\operatorname{Re} p < 0]$ (cf. **3.462** 1) MO 122

9.242

1.[10] $$D_p(z) = -\frac{\Gamma(p+1)}{2\pi i} e^{-\frac{1}{4}z^2} \int_\infty^{(0+)} e^{-zt-\frac{1}{2}t^2}(-t)^{-p-1}\,dt$$ $[|\arg(-t)| \le \pi]$ WH

2. $$D_p(z) = 2^{\frac{1}{2}(p-1)} \frac{\Gamma\left(\frac{p}{2}+1\right)}{i\pi} \int_{-\infty}^{(-1+)} e^{\frac{1}{4}z^2 t}(1+t)^{-\frac{1}{2}p-1}(1-t)^{\frac{1}{2}(p-1)}\,dt$$
$\left[|\arg z| < \frac{\pi}{4}; \quad |\arg(1+t)| \le \pi\right]$ WH

3. $$D_p(z) = \frac{1}{2\pi i} e^{-\frac{1}{4}z^2} \int_{-\infty i}^{\infty i} \frac{\Gamma\left(\frac{1}{2}t - \frac{1}{2}p\right)\Gamma(-t)}{\Gamma(-p)} \left(\sqrt{2}\right)^{t-p-2} z^t\,dt$$
$\left[|\arg z| < \frac{3}{4}\pi; \quad p \text{ is not a positive integer}\right]$ WH

Gradshteyn, I.S. & Ryzhik, I.M. (2007). *Tables of Integrals, Series and Products*, Elsevier Inc., Burlington, MA 01803, U.S.A. [9.240] pp.1028

Example 4

Show that $_1F_1(\alpha;\beta;x) = e^x\,{}_1F_1(\beta-\alpha;\beta;-x)$.

From theorem 9.7 we have

$$_1F_1(\alpha;\beta;x) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 (1-t)^{\beta-\alpha-1}t^{\alpha-1}\,e^{xt}\,dt.$$

If we make the change of variable $t = 1-\tau$, we have

$$\begin{aligned} {}_1F_1(\alpha;\beta;x) &= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 \tau^{\beta-\alpha-1}(1-\tau)^{\alpha-1}\,e^{x(1-\tau)}\,d\tau \\ &= e^x\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{\beta-\alpha-1}\,e^{-x\tau}\,d\tau \\ &= e^x\,{}_1F_1(\beta-\alpha;\alpha;-x) \end{aligned}$$

(using theorem 9.7 again).

Bell, W.W.(1968). *Special Functions for Scientists & Engineers*. D. Van Nostrand Company Ltd, London [Example 4] pp.215

which, on using the fact that $I_n(x) = i^{-n}J_n(ix)$, becomes

$$\frac{1}{i}\frac{d}{dx}\{i^n x^n i^n I_n(x)\} = i^n x^n i^{n-1} I_{n-1}(x)$$

and this equation, on cancelling a factor of i^{2n-1} throughout, gives

$$\frac{d}{dx}\{x^n I_n(x)\} = x^n I_{n-1}(x).$$

(ii) From theorem 4.8 (ii) we have

$$\frac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x),$$

which, on replacing x by ix, becomes

$$\frac{d}{d(ix)}\{i^{-n}x^{-n}J_n(ix)\} = -i^{-n}x^{-n}J_{n+1}(ix).$$

Thus

$$\frac{1}{i}\frac{d}{dx}\{i^{-n}x^{-n}i^n I_n(x)\} = -i^{-n}x^{-n}i^{n+1}I_{n+1}(x)$$

and hence

$$\frac{1}{i}\frac{d}{dx}\{x^{-n}I_n(x)\} = -ix^{-n}I_{n+1}(x)$$

so that

$$\frac{d}{dx}\{x^{-n}I_n(x)\} = x^{-n}I_{n+1}(x).$$

Theorem 4.16

(i) $\dfrac{d}{dx}\{x^n K_n(x)\} = -x^n K_{n-1}(x).$

(ii) $\dfrac{d}{dx}\{x^{-n} K_n(x)\} = -x^{-n} K_{n+1}(x).$

(iii) $K_n'(x) = -K_{n-1}(x) - \dfrac{n}{x}K_n(x).$

(iv) $K_n'(x) = \dfrac{n}{x}K_n(x) - K_{n+1}(x).$

(v) $K_n'(x) = -\frac{1}{2}\{K_{n-1}(x) + K_{n+1}(x)\}.$

(vi) $K_{n-1}(x) - K_{n+1}(x) = -\dfrac{2n}{x}K_n(x).$

Bell, W.W.(1968). *Special Functions for Scientists & Engineers*. D. Van Nostrand Company Ltd, London[Theorem 4.16 (iv)] pp.114

APPENDIX 21

13.4.6

$$(a-1+z)M(a,b,z)+(b-a)M(a-1,b,z)+(1-b)M(a,b-1,z)=0$$

13.4.7

$$b(1-b+z)M(a,b,z)+b(b-1)M(a-1,b-1,z)-azM(a+1,b+1,z)=0$$

13.4.8 $$M'(a,b,z)=\frac{a}{b}M(a+1,b+1,z)$$

13.4.9 $$\frac{d^n}{dz^n}\{M(a,b,z)\}=\frac{(a)_n}{(b)_n}M(a+n,b+n,z)$$

13.4.10 $$aM(a+1,b,z)=aM(a,b,z)+zM'(a,b,z)$$

13.4.11

$$(b-a)M(a-1,b,z)=(b-a-z)M(a,b,z)+zM'(a,b,z)$$

13.4.12

$$(b-a)M(a,b+1,z)=bM(a,b,z)-bM'(a,b,z)$$

13.4.13

$$(b-1)M(a,b-1,z)=(b-1)M(a,b,z)+zM'(a,b,z)$$

13.4.14

$$(b-1)M(a-1,b-1,z)=(b-1-z)M(a,b,z)+zM'(a,b,z)$$

13.4.15

$$U(a-1,b,z)+(b-2a-z)U(a,b,z)+a(1+a-b)U(a+1,b,z)=0$$

13.4.19

$$(a+z)U(a,b,z)-zU(a,b+1,z)+a(b-a-1)U(a+1,b,z)=0$$

13.4.20

$$(a+z-1)U(a,b,z)-U(a-1,b,z)+(1+a-b)U(a,b-1,z)=0$$

13.4.21 $$U'(a,b,z)=-aU(a+1,b+1,z)$$

13.4.22

$$\frac{d^n}{dz^n}\{U(a,b,z)\}=(-1)^n(a)_nU(a+n,b+n,z)$$

13.4.23

$$a(1+a-b)U(a+1,b,z)=aU(a,b,z)+zU'(a,b,z)$$

13.4.24

$$(1+a-b)U(a,b-1,z)=(1-b)U(a,b,z)-zU'(a,b,z)$$

13.4.25 $$U(a,b+1,z)=U(a,b,z)-U'(a,b,z)$$

13.4.26

$$U(a-1,b,z)=(a-b+z)U(a,b,z)-zU'(a,b,z)$$

13.4.27

$$U(a-1,b-1,z)=(1-b+z)U(a,b,z)-zU'(a,b,z)$$

13.4.28 $$2\mu M_{\kappa-\frac{1}{2},\mu-\frac{1}{2}}(z)-z^{\frac{1}{2}}M_{\kappa,\mu}(z)=2\mu M_{\kappa+\frac{1}{2},\mu-\frac{1}{2}}(z)$$

13.4.29

$$(1+2\mu+2\kappa)M_{\kappa+1,\mu}(z)-(1+2\mu-2\kappa)M_{\kappa-1,\mu}(z)=2(2\kappa-z)M_{\kappa,\mu}(z)$$

Abramowitz, M. & Stegun I.A.(1972). *Handbook of mathematical functions* . National Bureau of Standards Applied Mathematics Series 55, Washington, D.C. 20402 [13.4.8] pp 507

APPENDIX 22

3.324

1. $$\int_0^\infty \exp\left(-\frac{\beta}{4x} - \gamma x\right) dx = \sqrt{\frac{\beta}{\gamma}} K_1\left(\sqrt{\beta\gamma}\right) \qquad [\operatorname{Re}\beta \geq 0, \quad \operatorname{Re}\gamma > 0]$$ ET I 146(25)

2.[11] $$\int_{-\infty}^\infty \exp\left[-\left(x - \frac{b}{x}\right)^{2n}\right] dx = \frac{1}{n}\Gamma\left(\frac{1}{2n}\right) \qquad [b \geq 0]$$

3.325 $$\int_0^\infty \exp\left(-ax^2 - \frac{b}{x^2}\right) dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \exp\left(-2\sqrt{ab}\right) \qquad [a > 0, \quad b > 0]$$ FI II 644

3.326

1.[8] $$\int_0^\infty \exp\left(-x^\mu\right) dx = \frac{1}{\mu}\Gamma\left(\frac{1}{\mu}\right) \qquad [\operatorname{Re}\mu > 0]$$ BI (26)(4)

2.[10] $$\int_0^\infty x^m \exp\left(-\beta x^n\right) dx = \frac{\Gamma(\gamma)}{n\beta^\gamma} \quad \gamma = \frac{m+1}{n} \qquad [\operatorname{Re}\beta > 0, \quad \operatorname{Re}m > 0, \quad \operatorname{Re}n > 0]$$

3.* $$\int_0^\infty (x-a)\exp\left(-\beta(x-b)^n\right) dx = \frac{\Gamma\left(\frac{2}{n}, \beta(-b)^n\right)}{n\beta^{2/n}} - (a-b)\frac{\Gamma\left(\frac{1}{n}, \beta(-b)^n\right)}{n\beta^{1/n}}$$
$$[\operatorname{Re}n > 0, \quad \operatorname{Re}\beta > 0, \quad |\arg b| < \pi]$$

4.* $$\int_0^u (x-a)\exp\left(-\beta(x-b)^n\right) dx = \frac{\Gamma\left(\frac{2}{n}, \beta(-b)^n\right) - \Gamma\left(\frac{2}{n}, \beta(u-b)^n\right)}{n\beta^{2/n}} - (a-b)\frac{\Gamma\left(\frac{1}{n}, \beta(-b)^n\right) - \Gamma\left(\frac{1}{n}, \beta(u-b)^n\right)}{n\beta^{1/n}}$$
$$[\operatorname{Re}n > 0, \quad \operatorname{Re}\beta > 0, \quad |\arg b| < \pi, \quad |\arg(u-b)| < \pi]$$

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