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A Study of W_6 - K-contact Riemannian Manifold

By

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Declaration

This dissertation is my own work and has not been presented in part or wholly for a degree in any other University.

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This dissertation has been submitted for examination with our approval as University supervisors.

Dr. S.K. Moindi. Signature..

Prof. G.P. Pokhariyal. Signature.................................

Abstract

The properties of W6-curvature tensor are studied in k-contact manifold and the following theorems were proved;
A W_6-flat K-contact Riemannian manifold is a flat space or manifold.
A W_6-Semisymmetric K-contact Riemannian manifold is a W_6-flat manifold.
A W_6-symmetric and W_6- semi symmetric K-contact Riemannian manifold is a W_6-flat manifold.

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Dedication

My parents and In-laws: Thanks you for your unconditional support with my studies. Thank you for giving me a chance to prove and improve myself through all my walks of life.

My brothers and sisters: Hoping that with this project i have proven to you that there is no mountain higher as long as God is on our side. Hoping that you will walk again and be able to fulfill your dreams.

My family: Thank you for believing in me; for allowing me to further my studies. Please do not ever doubt my dedication and love for you, Mrs Kimetto, Joddie, Mercy and Marvel.

Contents

Chapter 1

Introduction

Riemannian geometry was first put forward in generality by Bernhard Riemann in the nineteenth century. It deals with a broad range of geometries whose metric properties vary from point to point, including the standard types of Non-Euclidean geometry.

Any smooth manifold admits a Riemannian metric, which often helps to solve problems of differential topology. It also serves as an entry level for the more complicated structure of pseudo-Riemannian manifolds, which (in four dimensions) are the main objects of the theory of general relativity. Other generalizations of Riemannian geometry include Finsler geometry. There exists a close analogy of differential geometry with the mathematical structure of defects in regular crystals. Dislocations and Disclinations produce torsions and curvature.

1.1 Tensor Analysis

1.1.1 Tensor algebras

We fix a ground field F which will be the real number field R or the complex number field C in our applications. All vector spaces we consider are finite dimensional over F unless otherwise stated. We define the tensor product $U \otimes V$ of two vector spaces U and V as follows. Let M(U,V) be the vector space which has the set $U \times V$ as a basis, i.e., the free vector space generated by the pairs (u, v) where $u \in U$ and $v \in V$. Let N be the vector subspace of M(U,V) spanned by elements of the form.

$(u + u^{'}, v) - (u, v) - (u^{'}, v),$

$(u, v + v^{'}) - (u, v) - (u, v^{'})$

$(ru, v) - r(u, v), (u, rv) - r(u, v),$

Where

$u, u^{'} \in V$, v,v$^{'} \in V$ and $r \in F$.

We set

$U \otimes V = M(U, V)/N$ For every pair (u, v) considered as an element of $M(U, V)$, its image by the natural projection $M(U, V) \longmapsto U \otimes V$ will be denoted by $u \otimes v$. Define the canonical bilinear mapping φ of $U \times V$ into $U \otimes V$ by $\varphi(u, v) = U \otimes V$ for $(u, v) \epsilon U \times V$

Let W be a vector space and $\varphi : U \times V \longmapsto W$ a bilinear mapping. We say that couple (W, φ) has the universal factorization property for $U \times V$ if for every vector space S and every bilinear mapping $f : U \times V \longmapsto S$ there exists a unique linear mapping $g : W \longmapsto S$ such that $f = g \circ \Psi$.

i. Contravariant and covariant vectors.

If N quantities $A^1, A^2, \ldots, A^N$ in a coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N other quantities $\bar{A}^1, \bar{A}^2, \ldots, \bar{A}^N$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations $\bar{A}^p = \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{\partial x^q} A^q p = 1, 2, \ldots, N$ which by the convections adopted can simply be written as

$$\bar{A}^p = \frac{\partial \bar{x}^p}{\partial x^q} A^q$$

They are called components of a contravariant vector or contravariant tensor of the first rank or first order.

If N quantities $A_1, A_2, \ldots, A_N$, in a coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N other quantities $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_N$ in another coordinate system $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ by the transformation equations

$\bar{A}_p = \sum_{q=1}^{N} \frac{\partial x^q}{\partial \bar{x}^p} A_q p = 1, 2, \ldots, N$ or

$$\bar{A}_p = \frac{\partial x^q}{\partial \bar{x}^p} A_q$$

They are called components of a covariant vector or covariant tensor of the first rank or first order. Note that a superscript is used to indicate contravariant components whereas a subscript used to indicate covariant components; an exception occurs in the notation for coordinates. Instead of speaking of a tensor whose components are A^p or A_p we shall often refer simply to the tensor A^p or A_p. No confusion should arise from this.

ii Contravariant, covariant and mixed tensors.

If N^2 quantities A^{qs} in a coordinate system $(x^1, x^2, \ldots, x^N)$ are related to N^2 other quantities $\bar{A}^{pr}$ in another coordinate system $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N$ by the transformation equations

$$\bar{A}^{pr} = \sum_{s=1}^{N} \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^q s, p, r = 1, 2, , N$$

by the adopted conventions, they are called contravariant components of a tensor of the second rank or rank two.
The N^2 quantities A_{qs} are called covariant components of a tensors of the second rank if

$$\bar{A}_{pr} = \frac{\partial x^q}{\partial \bar{x}^p} \frac{\partial x^s}{\partial \bar{x}^r} A^{qs}$$

The N^2 quantities A^q_s are called components of a mixed tensors of the second rank if

$$\bar{A}^r_s = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial x^s}{\partial \bar{x}^r} A^q_s$$

iii Tensors of rank greater than two.

Tensors of rank greater tha two are easily defined. For example, A_{kl}^{qst} are components of a mixed tensor of rank 5,contravariant of order 3 and covariant of order 2, if they transform according to the relations

$$\bar{A}_{ij}^{prm} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} \frac{\partial \bar{x}^m}{\partial x^t} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} A_{kl}^{qst}$$

iv. Scalars or invariants.

Suppose ϕ is a function of a coordinate x^k and let $\bar{\phi}$ denote the functional value under a transformation to a new set of coordinates $\bar{x}^k$. Then ϕ is called a scalar or invariant with respect to the coordinate transformation if $\phi = \bar{\phi}$. A scalar or invariant also called a tensor of rank zero.

v. Tensor fields.

If to each point of a region in N dimensional space there corresponds a definite tensor,we say that a tensor field has been defined.This is a vector field or a scalar field according as the tensor is of rank one or zero.It should be noted that a tensor or tensor field is not just the set of its components in one special coordinate system but all the possible sets under any transformation of coordinates.

Let $T_x = T_x(M)$ be the tangent space to a manifold M at a point x and T_x the tensor algebra over $T_x : T_x = \sum T_s^r(x)$, where $T_s^r(x)$ is the tensor space of type (r,s) over T_x. A tensor field of type (r,s) on a subset N of M is an assignment of a tensor $K_x \epsilon T_s^r(x)$ to each point x of N. In a coordinate neighborhood U with a local coordinate system $x^1, \ldots, x^n$, we take $X_i = \frac{\partial}{\partial x^i}$i=1,...,n, as a basis for each tangent space $T_x, x\epsilon U$, and $w^i = \partial x^i, i = 1, \ldots n$, as the dual basis of T_x^*. A tensor field K of type (r,s) defined on U is then expressed by,

$$K_x = \sum K_{j_1 \ldots j_s}^{i_1 \ldots i_r} X_{i_1} \otimes \ldots \otimes X_{i_r} \otimes w^{j_1} \otimes \ldots \otimes w^{j_s} \tag{1.1}$$

Where, $K_{j_1 \ldots j_s}^{i_1 \ldots i_r}$ are functions U, called the components of K with respect to the local coordinate system $x^1, \ldots, x^n$. We say that K is of class C^k

if its components $K^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ are functions of class C^k; of course, it has to be verified that this notion is independent of a local coordinate system. This is easily done by means of the formula.

$$\bar{K}^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \sum A^i_k, \ldots, A^{i_r}_{k_r} B^{m_1}_{j_1}, \ldots, B^{m_s}_{j_j} B^{k_1 \ldots k_r}_{m_1 \ldots m_s}, \tag{1.2}$$

Where the matrix (A^i_j) is to be replaced by Jacobian matrix between two local coordinate systems.

vi. Symmetric and skew-symmetric tensors.

A tensor is called symmetric with respect to two contravariant or two covariant indices if its components remain unaltered upon interchange of the indices. Thus if $A^{mpr}_{qs} = A^{pmr}_{qs}$ the tensor is symmetric in m and p. If a tensor is symmetric with respect to any two contravariantand any two covariant indices, its called symmetric. A tensor is called skew-symmetric with respect to two contravariant or two covariant indices if its components change sign upon interchange of the indices. Thus if $A^{mpr}_{qs} = -A^{pmr}_{qs}$ the tensor is skew-symmetric in m and p. If a is skew-symmetric with respect to any contravariant and any two covariant indices it is called skew-symmetric.

vii. The line element and metric tensor.

In rectangular coordinates (x, y, z) the differential are length ds is obtained from $ds^2 = dx^2 + dy^2 + dz^2$. By transforming to general curvilinear coordinates this becomes $ds^2 = \sum_{p=1}^{3} \sum_{q=1}^{3} \sum g_{pq} du_p du_q$. Such spaces are called three dimensional Euclidean spaces.

A generalization to N dimensional space with coordinates $(x^1, x^2, ..., x^N)$ is immediate. We define the line element ds in this space to be given by the quadratic form, called the metric form or metric,

$$ds^2 = \sum_{p=1}^{N} \sum_{q=1}^{N} \sum g_{pq} dx^p dx^q \tag{1.3}$$

or, using the summation convention.

$$ds^2 = g_{pq} dx^p dx^q \tag{1.4}$$

In the special case where there exists a transformation of coordinates from x^j to $\bar{x}^k$ such that the metric form is transformed into $(d\bar{x}^1)^2 + (d\bar{x}^2)^2 + \ldots + (d\bar{x}^N)^2$ or $d\bar{x}^k d\bar{x}^k$, then the space is called N dimensional Euclidean space. In the general case, however, the space is called Riemannian.

The quantities g_{pq} are the components of a covariant tensor of rank two called the metric tensor or fundamental tensor. We can and always will choose this tensor to be symmetric.

viii. Conjugate or reciprocal tensors.

Let $g = \mid g_{pq} \mid$ denote the determinant with elements g_{pq} and suppose $g \neq 0$.

Define g^{pg} by

$$g^{pq} = \frac{cofactor of g_{pq}}{g}$$

Then g_{pq} is a symmetric contravariant tensor of rank two called the conjugate or reciprocal tensor g_{pq}. It can be shown that

$$g^{pq} g_{rq} = \delta^p_r$$

ix. Associated tensors.

Given a tensor, we can derive other tensors by raising or lowering indices.
For example, given the tensor A_{pq} we obtain by raising the index p, the, tensor $A^{p}_{\cdot q}$, the dot indicating the original position of the moved index. By raising the index q also we obtain $A^{pq}_{\cdot\cdot}$. Where no confusion can

arise we shall often omit the dots; thus $A^{pq}_{\cdot\cdot}$ can be written A^{pq}. These derived tensors can be obtained by forming inner products of the given tensor with the metric tensor g_{pq} or its conjugate g^{pq}. Thus, for example

$A^{p}_{\cdot q} = g^{rp} A_{rq}$, A^{pq}=g$^{rp} g^{sq} A_{rs}$, $A^{p}_{\cdot rs} = g_{rq} A^{pq}_{\cdot\cdot s}$

$A^{qm\cdot tk}_{\cdot\cdot n} = g^{pk} g_{sn} g^{rm} A^{q\cdot st}_{\cdot r\cdot\cdot p}$

These become clear if we interpret multiplication by g^{rp} as meaning: let $r = p$ (or p=r) in whatever follows and raise this index. Similarly we interpret multiplication by g_{rq} as meaning: let r= q (or q = r) in whatever follows and lower this index.

All tensors obtained from a given tensor by forming inner products with the metric tensor and its conjugate are called associated tensors of the given tensor. For example A^m and A_m. are associated tensors, the first are contravariant and the second covariant components. The relation between them is given by

$A_p = g_{pq} A^q or A^p = g^{pq} A_q$

For rectangular coordinates $g_{pq} = 1$ if p= q , and 0 if $p \neq q$, so that $A_p = A^p$.

x. Fundamental operations with tensors.

(a) Addition. The sum of two or more tensors of the same rank and type (i.e same number of contravariant indices and same number of covariant indices)is also a tensor of the same rank and type thus if A^{mp}_q and B^{mp}_q are tensors, then $C^{mp}_q = A^{mp}_q + B^{mp}_q$ is also a tensor. Addition of tensors is commutative and associative.

(b) Subtraction. The difference of two tensors of the same rank and type is also a tensor of the same rank and type. Thus A^{mp}_q and B^{mp}_q are tensors, then $D^{mp}_q = A^{mp}_q - B^{mp}_q$ is also a tensor.

(c) Outer Multiplication. The product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors. This product which involves ordinary multiplication of the components of the tensor is called the outer product. For example $A_q^{pr} B_s^m = C_{qs}^{prm}$ is the outer product of A_q^{pr} and B_s^m. However, not every tensor can be written as a product of two tensors of lower rank.

(d) Contraction. If one contravariant and one covariant index of a tensor are set equal, the result indicates that a summation over the equal indices is to be taken according to summation convection. This resulting sum is a tensor of rank two less than that of the original tensor. The process is called contraction. For example, in the tensor of rank 5, A_{qs}^{mpr}, set $r = s$ to obtain $A_{qr}^{mpr} = B_q^{mp}$ a tensor of rank 3.

(e) Quotient law. Suppose it is not known whether a quantity X is a tensor or not. If an inner product of X with an arbitrary tensor is itself a tensor, then X is also a tensor. This is called the quotient law.

1.1.2 Charts

In order that the usual operation of calculus should be possible in a space more structure is required. Let M be a topological m-manifold a chart on M comprises an open set P of M called a coordinate patch and a map $\Psi : P \longmapsto R^m$ which is a homomorphism of P onto open subset of R^m. If x lies in P then the pair (P, Ψ) is called **a chart** around x.

A homomorphism is a map that preserves selected structure between two algebraic structures, with the structure to be preserved being given by the naming of the homomorphism.

A topological manifold guarantees the existence of a chart around each point. We need to establish the criteria of mutual consistency of coordinate systems by specifying conditions to be satisfied when two charts overlap. It is at this point that the concept of differentiability is introduced into the structure.

Suppose that $(P1, \Psi 1)$ and $(P2, \Psi 2)$ are two charts on M with overlapping coordinates patches in the overlap $(P1 \cap P2)$ two maps to R^m are specified. Since there maps are homomorphism they are invertible and therefore maps between open subsets of R^m maybe specified by

$$\chi = \Psi 2 \circ \Psi 1^{-1}) : \Psi 1(P1 \cap P2) \longmapsto \Psi 2(P1 \cap P2)$$

$$\chi^{-1} = \Psi \circ \Psi 2^{-1} : \Psi 2(P1 \cap P2) \longmapsto \Psi 1(P1 \cap P2)$$

The question of smoothness is now reduced to consideration of the maps χ and χ^{-1} we invoke the condition that χ and χ^{-1} are both C infinity (C_∞) or both smooth which is to say C^k for every k.

This means that the functions relating the coordinates in two overlapping patches may be differentiated any number of times. Pair of charts related in this manner are said to be smoothly related. It is also possible to say that two charts are smoothly related also if their domains do not intersect.

Since smooth functions of smooth functions are smooth functions the composition of smooth maps yields a smooth map. This suggests that it makes sense to allow all charts which are smoothly related. This is the mathematical realization of the physical idea that all coordinate systems are equally good.

1.1.3 Atlas

Consider a topological manifold $M1$ a smooth atlas for M is a collection of pairwise smoothly related charts whose coordinate patches cover M. Thus every point of M must lie in some patch of the atlas and thereby acquire coordinates and where two sets of coordinates are both in operation they must be smoothly related.

An atlas is called complete if it is not a proper sub collection of any other atlas. This means that there is no chart smoothly related to all the charts in the atlas which is not itself already in the atlas.

Any atlas may be completed by adding to it all the charts not already in it which are smoothly related to those in it.

A manifold is developed in two stages first by establishing the topological properties and then the differentiability.

Let M be a topological space. If every point of M has a neighborhood homomorphic to an open subset of R^m and if further more it is the Hausdorff space with a countable basis then M is called topological manifold of dimension M or topological m-manifold.

1.1.4 Differentiable manifold.

An m-dimensional topological manifold M together with a complete atlas is called an m-dimensional C^∞ or smooth differentiable manifold.

A complete atlas is also sometimes called a differential structures for M. Consider two charts $(P1, \Psi1)$ and $(P2, \Psi2)$ on m-dimensional smooth manifold m with overlapping coordinates patches we may use the map χ and χ^-1 to express the relation between the coordinates belonging to $P1$ and $P2$ respectively, then $x_2^a = \chi^a(x_1^b)$, $x_1^a = (\chi^-1)^a(x_2^b)$, the inevitability of χ implies that its Jacobian matrix and Jacobian matrix of (χ^-1) are inverses of each other. This is usually expressed in the following forms.

$$\frac{\partial x_2^a}{\partial x_1^c}\frac{\partial x_1^c}{\partial x_2^b} = \delta_b^a, \frac{\partial x_1^a}{\partial x_2^c}\frac{\partial x_2^c}{\partial x_1^b} = \delta_b^a$$

1.1.5 Submanifold

A submanifold of a manifold M is a subset S which itself has the structure of a manifold, and for which the inclusion map $S \longmapsto M$ satisfies certain properties. There are different types of submanifolds depending on exactly which properties are required.

If $\phi : M \longmapsto N$ is an immersion then in the special coordinates the image $(M) \subset N$ is represented locally by a coordinate m-plane, and the first M of the coordinate on N serve as coordinates for it. It is therefore appropriate to consider $\phi(M)$ as,locally, a submanifold of N.

A subset of N which is the image of an immersion $M \longmapsto N$ is called an immersed submanifold of N, while a subset which is the image of an imbedding is know is an imbedded submanifold or simply a submanifold of N.

Suppose that S is a subset of a smooth manifold N with the property that about each point in S there is chart of N such that the part of S covered by the chart coincides with the coordinate m-plane $y^{m+1} = y^{m+2} =, \ldots, = y^n = 0$ Then the restriction of these charts to S define on it the structure of a smooth manifold of dimension m, and the injection $S \longmapsto N$ which maps each point of S (considered as a differential manifold in its own right), to the same point regarded as a point of the differentiable manifold N, is an immersion. Thus S is and immersed submanifold of N. In particular, if $f^{m+1}, f^{m+2}, \ldots, f^n$ are smooth of a function on N, then the subset S of N on which they simultaneously vanish in an immersed submanifold of N, provided that differential $\partial f^{m+1}, \partial f^{m+2}, \ldots, \partial f^n$ are linearly independent every where on S.

In this case matrix of partial derivatives of the coordinate representation of the $f^i(i = m+1, m+2, \ldots, n)$ with respect to any coordinates z^∞ on N has $n - m$, and so without loss of generality it may be assumed that the $(n-m) \times (n-m)$ matrix $\frac{\partial f^i}{\partial z^j}$ non-singulars. It then follows that if $y^1 = z^1, y^2 = z^2, \ldots, y^n, y^{m+1} = f^{m+1} z^\infty, \ldots, y^n = f^n z^\infty$ then the y^∞ form a coordinate system with respect to which S is given by $y^{m+1} = y^{m+2} =, \ldots, = y^n = 0$

1.1.6 Connection.

The theory of connections are developed starting from ideas of parallelism first starting by vector method and then by exterior calculus.

Let M be a C^∞-manifold, a connection, infinitely small connection or covariant differentiation on M is an operator D_Y that assigns to each pair of C^∞ vector fields X and Y with domain A, a C^∞ field $D_X Y$ with domain A. If Z is a C^∞ vector field on A and f is a C^∞ real-valued function on A, then D satisfies the following four properties:

(1) $D_{X+Y} Z = D_X Z + D_Y Z$
(2) $D_{fX} Y = f D_X Y$
(3) $\mathrm{D}_X (Y + Z) = D_X Y + D_X Z$
(4) $D_X (fY) = (Xf) Y + f D_X Y$

- the first two properties follows from linearity assumption and the remaining two from properties of parallellism.

- these properties imply that the vector $(D_X Y)_m$ at a point m on M depends only on X_m and the value of Y on some curve that fits X_m.

- the existence of many manifolds with connections illustrated by the natural induced connections on the hyper surfaces of R^N.

- let $\sigma(t)$ be a curve in M with tangent fields then a C^∞ vectors field Y and σ is parallel along σ iff $D_T Y = 0$ on σ.

- the curve σ is a geodesic iff $D_T T = 0$ on σ thus a curve is a geodesic iff its tangent fields is a parallel field along the curve.

1.1.7 Torsion Tensor.

The torsion tensor of a connection D is a vector value tensor usually denoted by $Tor(X, Y)$ or $T(X, Y)$ that assigns to each pair of C^∞ vectors X and Y with domain (M). A C^∞ vector field $Tor(X, Y)$ with domain (M) by $Tor(X, Y) = D_X Y - D_Y X - [X, Y]$

We notice that $Tor(X, Y) = -Tor(Y, X)$

$Tor(X + Y, Z) = Tor(X, Z) + Tor(Y, Z)$

$Tor(fX, Y) = fTor(X, Y)$

Where f in space M and z in (M)
Thus the value of $Tor(X, Y)$ at a point m depends on X_m and Y_m, but not on the fields X and y. If for $TorD \equiv 0$, then we say D is symmetric or torsion free.

1.1.8 Curvature and Torsion tensors.

The torsion field (torsion) T and the Curvature tensor field (Curvature) R.

We set

$$T(X,Y) = u(2\Theta(X^*,Y^*)) for X, Y \in T_\alpha(M), \tag{1.5}$$

Where u is any point of $L(M)$ with $\pi(u) = x$ and X^* and Y^* are vectors of $L(M)$ at u with $\pi(X^*) = x$ and $\pi(Y^*) = y$. We know that $T(X,Y)$ is independent of the choice of u, X^*, and Y^*.

Thus , at every point x of M, T defines a skew symmetric bilinear mapping $T_x(M) \times T_x(M) \longmapsto T_x(M)$. In other words T is a tensor field of type (1,2) such that $T(X,Y) = -T(Y,X)$. We shall call $T(X,Y)$ the torsion translation in $T_x(M)$ determined by X and Y. Similarly , we set

$$R(X,Y)Z = u((2\Omega(X^*,Y^*))(u^{-1}Z) for X, Y, Z \in T_x(M) \tag{1.6}$$

Where u, X^* and Y^* are chosen as above. Then $R(X,Y)Z$ depends only on X,Y and Z, not on u, X^* and Y^*. In the above definition,

$((2\Omega(X^*,Y^*))(u^{-1}Z)$ denotes the image of $u^{-1}ZR^n$ by the linear endomorphism $2\Omega(X^*,Y^*(n,R) of R^n$.
Thus $R(X,Y)$is an endomorphism of $T_x(M)$ and is called the Curvature transformation of $T_x(M)$ determined by X and Y. It follows that R is a tensor field of type (1,3) such that $R(X,Y) = -R(Y,X)$

In terms of covariant differentiation, the torsion T and the curvature R can be expressed as follows.

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{1.7}$$

and

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - X - \nabla_{[X,Y]}Z \tag{1.8}$$

Where X,Y and Z are vector fields on M

1.2 Length of a vector, angle between vectors.

The quantity A^pB_p, which is the inner product of A^p and B_p , is a scalar analogous to the scalar product in rectangular coordinates. We define the length L of the vector A^p or A_p as given by'

$$L^2 = A^pA_p = g^{pq}A_pA_q = g_{pq}A^pA^q \tag{1.9}$$

We can define the angle θ between A^p and B_p as given by

$$cos\theta = \frac{(A^pB_p)}{\sqrt{((A^pA_p)(B^pB_p))}} \tag{1.10}$$

1.2.1 The physical components

The physical components of a vector A^p or A_p, denoted by A_u, A_v , and A_w. are the projections of the vector on the tangents to the coordinate curves and are given in the case of orthogonal coordinates by

$A_u = \sqrt{g_{11}}A^1 = \frac{A_1}{\sqrt{g_{11}}}$, $A_v = \sqrt{g_{22}}A^2 = \frac{A_2}{g_{22}}$, $A_w = \sqrt{g_{33}}A^3 = \frac{A_3}{\sqrt{g_{33}}}$,

Similarly the physical components of a tensor A^{pq} or A_{pq} are given by

$A_{uu} = g_{11}A^{11} = \frac{A^{11}}{g_{11}}$

$A_{uv} = \sqrt{g_{11}g_{22}}A^{12} = \frac{A_{12}}{\sqrt{g_{11}g_{22}}}$

$A_{uw} = \sqrt{g_{11}g_{33}}A^{13} = \frac{A_{13}}{\sqrt{g_{11}g_{33}}}$

1.2.2 Christoffel's symbols

Christoffel's symbols. The symbols

$$[pq, r] = \frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^q} + \frac{\partial g_{qr}}{\partial x^q} - \frac{\partial g_{pq}}{\partial x^r}\right) \tag{1.11}$$

$$\left\{ {s \atop pq} \right\} = g^{sr}[pq, r] \tag{1.12}$$

are called the Christoffel symbols of the first and second kind respectively. Other symbols used instead of $\left\{ {s \atop pq} \right\}$ are $\{pq, s\}$ and Γ^s_{pq}.

The latter symbol suggests however a tensor character, which is not true in general.

1.2.3 Transformation laws of christoffel's symbols.

If we denote by a bar a symbol in a coordinate system $\bar{x}^k$, then

$$\overline{[jk, m]} = [pq, r] \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} \frac{\partial x^r}{\partial \bar{x}^m} + g_{pq} \frac{\partial x^p}{\partial \bar{x}^m} \frac{\partial^2 x^q}{\partial \bar{x}^j \bar{x}^k}$$

$$\overline{\left\{ {n \atop jk} \right\}} = \left\{ {s \atop pq} \right\} \frac{\partial \bar{x}^n}{\partial \bar{x}^s} \frac{\partial x^p}{\partial \bar{x}^j} \frac{\partial x^q}{\partial \bar{x}^k} + \frac{\partial \bar{x}^n}{\partial x^q} \frac{\partial^2 x^q}{\partial \bar{x}^j \partial \bar{x}^k}$$

are the laws of transformation of the Christoffel symbols showing that they are not tensors unless the second terms on the right are zero.

1.2.4 The covariant derivative

The covariant derivative of a tensor A_p with repect to x^q is denoted by $A_{p,q}$ and is defined by

$$A_{p,q} \equiv \frac{\partial A_p}{\partial x^q} - \left\{ {s \atop pq} \right\} A_s$$

a covariant tensor of rank two. The covariant derivative of a tensor A^p with respect to x^q is denoted by $A^p_{\cdot q}$ and is defined by

$$A^p_{\cdot q} \equiv \frac{\partial A^p}{\partial x^q} + \left\{ {p \atop qs} \right\} A^s$$

a mixed tensor of rank two. For rectangular systems, the Christoffel symbols are zero and the covariant derivatives are the usual partial derivatives. Covariant derivatives of tensors are also tensors . The above results can be extended to covariant derivatives of higher rank tensors. Thus

$$A^{p_1 \cdots p_m}_{r_1 \ldots r_n, q} \equiv \frac{\partial A^{p_1 \cdots p_m}_{r_1 \ldots rn}}{\partial x^q} - \left\{ {s \atop r_1, q} \right\} A^{p_1 \cdots p_m}_{s \quad r_2 \ldots rn} - \left\{ {s \atop r_2, q} \right\} A^{p_1 \cdots p_m}_{s r_1 r_3 \ldots r_n}$$

$$- \ldots -, \left\{ {s \atop r_n, q} \right\} A^{p_1 \ldots p_m}_{r_1 \ldots r_{(n-1)} s} + \left\{ {p1 \atop q^s} \right\} A^{s p_2 \ldots p_m}_{r_1 \ldots r_n} + \cdots + \left\{ {p_m \atop q^s} \right\} A^{p_1 \ldots p_{(m-1)} s}_{r_1 \ldots r_n}$$

is the covariant derivative of $A^{p_1 \cdots p_m}_{r_1 \ldots r_n, q}$ with respect to x^p.

The rules of covariant differentiation for sums and products of tensors are the same as those for ordinary differentiation. In performing the differentiations, the tensors g_{pq} ,g^{pq} and δ^p_q may be treated as constants since their covariant derivatives are zero . Since covariant derivatives express rates of change of physical quantities independent of any frames of reference, they are of great importance in expressing physical laws.

1.2.5 Permutation symbols and tensors

Permutation symbols and tensors. Define e_{pqr} by the relations

$$e_{123} = e_{231} = e_{312} = +1$$
$$e_{213} = e_{123} = e_{321} = -1$$

$e_{pqr} = 0$ if two or more indices are equal and define e^{pqr} in the same manner. The symbols e_{pqr} and e^{pqr} are called permutation symbols in three dimensional space.

Further, let us define $\epsilon_{pqr} = \frac{1}{\sqrt{g}} e_{pqr}$,$\epsilon^{pqr} = \sqrt{g} e^{pqr}$

It can be shown that ϵ_{pqr} and ϵ^{pqr} are covariant and contravariant tensors respectively, called permutation tensors in three dimensional space. Generalizations to higher dimensions are possible.

1.2.6 Tensor form of gradient, divergence and curl.

1. Gradient. If Φ is a scalar or invariant the gradient of Φ is defined by

$$\nabla\Phi = grad\Phi = \Phi_{.p} = \frac{\partial\Phi}{\partial x^p}$$

where Φ, p is the covariant derivative of Φ with respect to x^p.

2. **Divergence.** The divergence of A^p is the contraction of its covariant derivative with respect to x^p, i.e. the contraction of $A^p_{.q}$ Then $divA^p = A^p_{.q} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^K}\left(\sqrt{g}A^k\right)$

3 Curl. The curl of A_p is

$$A_{p,q} - A_{q,p} = \frac{\partial A_p}{\partial x^q} - \frac{\partial A_q}{\partial x^p}$$

a tensor of rank two. The curl is also defined as $-\epsilon^{pqr}A_{p,q}$

4. Laplacian. The Laplacian of is the divergence of grad Φ or

$$\nabla^2\Phi = div\Phi_{,p} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}g^{jk}\frac{\partial\Phi}{\partial x^k}\right)$$

In case $g < 0$, $\sqrt{g}$ must be replaced by $\sqrt{(-g)}$. Both cases $g > 0$ and $g < 0$ can be included by $\sqrt{\mid g \mid}$ in place of $\sqrt{g}$.

1.2.7 The intrinsic or absolute derivative

The intrinsic or absolute derivative of A_p along a curve $x^p = x^q(t)$ denoted by $\frac{\delta A_p}{\delta t}$ is defined as the inner product of the covariant derivative of A_p and $\frac{\partial x^q}{\partial t}$ i.e. $A_{p,q}\frac{\partial x^q}{\partial t}$ and is given by

$$\frac{\delta A_p}{\delta t} \equiv \frac{\partial A_p}{\partial t} - \left\{ {r \atop pq} \right\} A_r \frac{\partial x^q}{dt}$$

Similarly, we define

$$\frac{\delta A^p}{\delta t} \equiv \frac{\partial A^p}{\partial t} - \left\{ {r \atop pq} \right\} A^r \frac{\partial x^q}{dt}$$

The vectors A_p or A^p are said to move parallelly along a curve along the curve if their intrinsic derivatives are zero, respectively. Intrinsic derivatives of higher rank tensors are similarly defined.

1.3 Relative and absolute tensors.

A tensor $A^{p_1 \dots p_m}_{r_1 \dots r_n, q}$ is called a relative tensor of weight w if its components transform according to the equation

$$\bar{A}^{q_1 \dots q_m}_{s_1 \dots s_n} = |\frac{\partial x}{\partial \bar{x}}|^w A^{p_1 \dots p_m}_{r_1 \dots r_n,} \frac{\partial \bar{x}^{q1}}{\partial x^{p1}} \dots \frac{\partial \bar{x}^{qm}}{\partial x^{pm}} \frac{\partial x^{r1}}{\partial \bar{x}^{s1}} \dots \frac{\partial x^{rn}}{\partial \bar{x}^{sn}}$$

where $J = |\frac{\partial x}{\partial \bar{x}}|$ is the Jacobian of the transformation. If $w = 0$ the tensor is called absolute and is the type of tensor with which we have been dealing above. If $w = 1$ the relative tensor is called a tensor density. The operations of addition, multiplication, etc., of relative tensors are similar to those of absolute tensors.

1.3.1 The summation convection.

In writing an expression such as $a_1 x^1 + a_2 x^2 + \dots + a_N x^N$ we can use the short notation $\sum_{j=1}^{N} a_j x^j$. An even shorter notation is simply to write it as $a_j x^j$, where we adopt the convection that whenever an index (subscript or superscript) is repeated in a given term we are to sum over that index from 1 to N unless otherwise specified. This is called the summation convection. Clearly, instead of using the index j we could have used another letter, say p, and the sum could be written, $a_p x^p$. Any index which is repeated in a given term, so that the summation convention applies, is called a dummy index or umbral index. An index occurring only once in a given term is called a free index and can stand for any of the numbers $1, 2, \dots, N$.

1.4 Curves and functions

Let M be a smooth manifold. A curve in M is a map $R \longmapsto M$ or a map $I \longmapsto M$, where I in an open interval of R. A curve is smooth if it is defined by a smooth map of manifold. Let $\sigma : I \longmapsto M$ be a curve in M (I may be the whole of R), the the curve σ is said to be smooth on a subinterval J of I if these in a chart (P, Ψ) of M such that σ,$(\sigma^a) = (x^a \circ \sigma) : J \longmapsto R^n$ is given by smooth functions. If $\sigma(J)$ lies in two overlapping coordinate patches and σ is smooth in one chart then it will also be smooth in the other, because of the assumed smoothness of coordinate changes. Since the whole of M is covered by charts, so is the whole of $\sigma(I)$, and σ is called smooth if its domain is covered by overlapping intervals in each of which it is smooth. The definition depends on the differentiable structure of M but not on the choice of particular charts.

1.5 Immersions and Imbeddings

Let ϕ:$M \longmapsto N$ be a smooth map and (P, Ψ) is a chart about some point $\chi \in M$ and if (Q, ξ) is a chart about $\phi(x) \in N$, then we may form the jacobian matrix $\frac{\partial \phi^\alpha}{\partial x^a}$ of ϕ or strictly of its coordinate presentation.

The Jacobian matrix changes when the coordinates have changed but it does so by pre and post multiplication by non-singular matrices. Namely the jacobian matrices of the coordinate transformation in R^n and R^m respectively. It follows that the rank of the jacobian matrix of ϕ at any point is independent of change of coordinate and is therefore a property of ϕ itself we call it the rank of ϕ at that point.

A smooth map whose rank does not vary from point to point is rather easier to deal with those one whose rank does vary

If a smooth map ϕ has a constant rank K on M then coordinate charts may always be found on M and N with respect to which the coordinate presentation of ϕ is given by,

$$\phi^1(x^a) = x^1, \phi^2(x^a) = x^2, \ldots, \phi^K(x^a) = x^K \tag{1.13}$$

$$\phi^{k+1}(x^a) = \phi^{k+2}(x^a) =, \ldots, \phi^n(x^a) = 0 \tag{1.14}$$

The two particular extreme cases are of interest. First when $K = n \leqslant m$, the coordinate presentation of ϕ corresponds to projection of R^m onto the first n factors. A smooth map whose rank is everywhere equal to the dimension of its core domain is called Submersion.

At the other extreme when $K = m \leqslant n$, the coordinate presentation of ϕ corresponds to the injection of R^m into R^n as a coordinate m-plane.

A smooth map whose rank is everywhere equal the dimension of its domain is called an Immersion.

It is clear that an immersion is locally injective. No two points of m lying in the coordinate neighborhood in which ϕ has the coordinates presentation can have the same image, however an immersion need not be injective globally. Moreover an immersion may have other undesirable global features.

1.6 Parallelism and connections on manifolds

Parallelism on a manifold shall be defined with respect to a path.

- A path in a manifold M is a curve freed from its parametrization. The essential notion of parallelism is that one should be able to identity the tangent spaces at any two points once we know of a path joining them.

- this identification should preserve the linear structure of the tangent spaces.

- vectors at point x and y which are identified in this manner are parallel with respect to the given path.

- one would expect that if z is a point on the path intermediate between x and y then vectors at x only will be parallel if they are both parallel to the same vector at z.

- a rule of parallel transport along a path is defined as a collection of nonsingular maps $Y_{Y,X} : T_X M \longmapsto T_Y M$ one for every pair of points x, y on the path, such that for any path z on the path.

$$Y_{Y,Z}.T_{Z,X} = T_Y X \tag{1.15}$$

It follows from above description that $Y_{X,X}$ is the identity on $T_X M$ and $T_{X,Y} = (Y_{Y,X})^{-1}$ If a rule of parallel transport is given for each path of M

then we say that a rule for parallel transport is given in M. We assume that if one path is a subset of the other then a rule of parallel transport on the subset is that one obtained by restriction.

- A vector field given along a path is known as parallel field along the path or said to be parallel transport. If it may be obtained by parallel transport from a vector given at some point of the path thus W is a parallel field if $W_Y = Y_{Y,X} W_X$ for each y and some x on the path.

- if a subspace H_X of $T_X M$ is given one may define a field of subspaces along a given path through x by parallel transporting the vectors in H_x and there by constructing subspaces

$$H_Y : \{Y_{Y,X} V | V \in H_X\} \tag{1.16}$$

The field of subspaces obtained in this manner known as parallel along the path.

- parallel transports may also be extended to co-vector in a straight forward way defined as follows.

- **a non singular linear map**

$Y^*_{Y,X} : T^*_X M \longmapsto T^*_Y M$ is defined by $\left\langle W, Y^*_{Y,X^\alpha} \right\rangle = \langle Y_{X,Y} W, \alpha \rangle$ for each $\alpha \in T^*_X m$ and for all $W \in T_Y M$

- this rule ensures that parallel transport preserves pairing.

- we may employ parallel transport to construct along a curve on absolute derivative of a vector field which is not necessarily parallel. The result is another vector field along the curve.

- let W be a vector field defined along a curve σ let $W(t)$ denote the vector of W at $\sigma(t)$ and let $W(t+\delta)$ be the vector at $\sigma(t)$ obtained by parallel transporting $W(t+\delta)$ along the path obtained by σ from $\sigma(t+\delta)$ to $\sigma(t)$ such that $W(t+\delta),, = Y_\sigma(t), \sigma(t+\delta) W(t+\delta)$.

The absolute derivative of W along σ at $\sigma(t)$is

$\frac{Dw}{Dt}(t) = \lim_{\delta \longmapsto 0} \frac{1}{\delta}(W(t+\delta,, -W(t))) = \frac{d}{ds}\left\{Y_{\sigma(t)}, \sigma(s)W(s)\right\}$

It is now shown that if $\frac{Dw}{Dt} = 0$ along a path, then W is a parallel path.

- Fix a point x on the path. Take a curve σ which defines the path such that $X = \sigma(0)$.

- observe that $\frac{D}{Dt}\left\{Y_{X,\sigma(t)}W(t)\right\}$

$= \frac{d}{ds}\left\{Y_{X,\sigma(t+s)}W(t+s)\right\} s = 0$

$= \frac{d}{ds}\left\{Y_{x,\sigma(t)}.Y_{\sigma(t)}, \sigma(t+s)W(t+s)\right\} = s = 0$

$= Y_{X,\sigma(t)} \frac{d}{ds}\left\{Y_{\sigma(t),\sigma(t+s)}W(t+s)\right\} s = 0$

$= Y_{X,\sigma(t)} \frac{Dw}{Dt}(t)$

Thus if $\frac{Dw}{Dt}(t) = 0$ in some interval about x then $Y_{X,\sigma(t)}W(t)$ is a constant vector in $T_X M$. Thus, we say W and so $W(t) = Y_{\sigma(t),x}W$ is parallel.

- the rule of parallel transport is said to determine a linear connection on m. This term is also used for rule of parallelism, for the associated absolute derivative or for the covariant derivative operator.

Hausdorff topological space. A space (X, τ) is called Hausdorff if for every pair of distinct points $x, y \in X$ there exist disjoint open sets U and V with $x \in U$ and $y \in V$

If (X, τ) is a topological space. An open set U which contains a point $x \in X$is called a neighbourhood of x.

1.6.1 Pseudogroup

A pseudogroup is an extension of the group concept, but one that grew out of the geometric approach of Sophus Lie, rather than out of abstract algebra (such as quasigroup, for example). A theory of pseudogroups was developed by lie Cartan in the early 1900s. It is not an axiomatic algebraic idea; rather it defines a set of closure conditions on sets of homeomorphisms defined on open sets U of a given Euclidean space E or more generally of a fixed topological space S. The groupoid condition on those is fulfilled, in that homeomorphisms.

$h : U \longmapsto V$

and

$g : V \longmapsto W$

compose to a homeomorphism from U to W. The further requirement on a pseudogroup is related to the possibility of patching (in the sense of descent, transition functions, or a gluing axiom).

That is a pseudogroup of transformation on a topological space S is a set Γ of transformation satisfying the following axioms:

(1) Each $f \epsilon \Gamma$ is a homeomorphism of an open set (called the domain of f) of S onto another open set (called the range of f) of S;

(2) If $f \epsilon \Gamma$, then the restriction of f to an arbitrary open subset of the domain of f is in Γ;

(3) Let $U = U_i U_i$ where each U_i is an open set of S. A homeomorphism f of U onto an open set of S belongs to Γ if the restriction of f to U_i is in Γ for every i;

(4) For every open set U of S, the identity transformation of U is in Γ;

(5) If $f \epsilon \Gamma$, then $f^{-1} \epsilon \Gamma$;

(6) If $f \epsilon \Gamma$ is a homeomorphism of U onto V and $f^{'} \epsilon \Gamma$ is a homeomorphism of $U^{'}$ onto $V^{'}$ and if $V \cap U^{'}$ is non-empty, then the homeomorphism $f^{'} \circ f$ of $f^{-1} V \cap U^{'}$ onto $f^{'} \left(V \cap U^{'} \right)$ is in Γ.

Examples of pseudogroups are,

Let R^n be the space of $n-$tuples of real numbers $(x^1, x^2, , x^n)$ with the usual topology. A mapping f of an open set of R^n into R^m is said to be of $C^r, r = 1, 2, \ldots, \infty$, if f is continuously r times differentiable. By class C^w we mean that f is real analytic. The pseudogroup $\Gamma^r(R^n)$ of transformation of class C^r of R^n is the set of homeomorphisms f of an open set of R^n onto an open set of R^n such that both f and f^-1 are of class C^r. Obviously $\Gamma^r(R^n)$ is a pseudogroup of transformation of R^n. If $r < s$, then $\Gamma^s(R^n)$ is a subpseudogroup of $\Gamma^r(R^n)$. If we consider only those $f \epsilon \Gamma^r(R^n)$ whose Jacobians are positive everywhere, we obtain a subpseudodroup of $\Gamma^r(R^n)$. This subpseudogroup, denoted by $\Gamma_0^r(R^n)$ is called the pseudogroup of orientation-preserving transformations of class C^r of R^n. Let C^n be the space of $n-$tuples of complex numbers with the usual topology. The pseudogroup of holomorphic (i.e., complex analytic) transforms of C^n can be similarly defined and will be denoted by $\Gamma(C^n)$.

1.6.2 Riemannian Manifold

A manifold on which one has (defined) singled out a specific symmetric and positive define (or non-singular) 2-covariant tensor field, know as the metric tensor, is referred to as Riemannian (or semi-Riemannian) manifold.

Metric tensor allows one to define lengths, angles, and distances. Let M be a Riemannian manifold with metric tensors $<,>$. Let X and Y be in mm. we define the length of X by $\mid X \mid= \sqrt{< X, X >}$. We define the angle θ between X and Y (both non-zero) by $< X, Y >=\mid X \mid\mid Y \mid$ makes this possible.

The length of a curve is defined by integrating the length of its tangent vector field. Let σ from a to b, denoted by $\mid \sigma \mid_a^b$, is defined by,

$$\mid \sigma \mid_a^b= \int_a^b \sqrt{< T(t), T(t) > dt}. \tag{1.17}$$

The integral exist, since the integrand is continuous. The length of a broken C^∞ pieces. The number $\mid \sigma \mid_a^b$ is independent of the parameterization of its image set in the following sernse:

Let g be a C^1 map of $[c, d]$ into $[a, b]$ with end points mapping to end points, we assume $g(c) = a$ and $g(d) = b$, then,

$$\int_a^b (< T_\sigma(t), T_\sigma(t) >)^{\frac{1}{2}} dt = \int_c^d (< T_\sigma(g(t)), T_\sigma(g(t)) >)^{\frac{1}{2}} g^{'}(b) dt$$
$$= \int_c^d (< T_{\sigma \circ g}(t), T_{\sigma \circ g}(t) >)^{\frac{1}{2}} dt \qquad (1.18)$$

Since $\mathrm{T}_{\sigma \circ g}(t) = g^{'}(t) T_\sigma(g(t))$ by the chain rule. Thus we can write $\mid \sigma \mid_q^1 = \mid \sigma \mid_a^b$, where $q = \sigma(a)$ and $p = \sigma(b)$.

Classically, the metric tensor is almost always expressed by the notation $ds^2 = g_{ij} dx^i dx^j$. This means one is giving the inverse product on a coordinate domain $\cup$ with coordinate functions $x_1, x_2, \ldots, x_n$ in terms of the coordinate bases; i.e $x_i = \frac{\partial}{\partial x_i}$, then $g_{ij} =< x_i, x_j >$ is a C^∞ function on $\cup$. If $Y = \sum y_i x_i$ and $Z = \sum z_k x_k$, then $< Y, Z >= \sum_{i,k=1}^n y_i z_k g_{ik}$. thus , giving the matrix of functions g_{ij} on $\cup$ determines the inner product of $\cup$. The ds only makes sense when one is discussing a curve σ which maps into $\cup$, so that for $s(t) = \mid \sigma \mid_a^t$, we have,

$$\left(\frac{ds}{dt}\right)^2 =< T, T >= \sum g_{ij} \frac{d(x_i \circ \sigma)}{dt} . \frac{d(x_j \circ \sigma)}{dt} \qquad (1.19)$$

If M is connected, a pseudo-metric is defined on M by $d(p, m) = inf[\mid \sigma \mid : \sigma$ a broken C^∞ curve from p to $m]$. Trivially, $d(p, m) \geq 0$, $d(p, p) = 0$, $d(p, m) = d(m, p)$ and the triangle inequality $d(p, m) \leq d(p, q) + d(q, m)$ are satisfied.

1.7 Riemannian Connexion

A connexion D on a Riemannian manifold M is called Riemannian connexion on M if it satisfies the following properties

1.

$$D_X Y - D_Y X = [X, Y]$$

2.

$$Z < X, Y >=< D_Z X, Y > + < X, D_Z Y >$$

for all fields X,Y and Z with a common domain

1.8 Fundamental theorem of Riemannian Geometry

There exists a uniquely determined Riemannian connexion on a Riemannian manifold.

- We show that a Riemannian connexion D exists and is uniquie on every coordinate domain $\cup$. The uniqueness implies D must agree on overlapping domains; hence D exists and is unique on all of M.

Let $x_1, x_2, \ldots, x_n$ be the coordinate fields on $\cup$. Let $g_{ij} =< x_i, x_j >$ on $\cup$ and let g_{ij}^{-1} be the j^{th} entry of the inverse matrix of $g = g_{ij}$ (which is non-singular).

We know that giving D on $\cup$ is equivalent to giving functions Γ_{gk}^i with $D_{xk}(X_j) = \sum_{i=1}^n \Gamma_{gk}^i x_i$ and demanding that properties

1. $D_{fX} Y = f D_X Y$
2. $D_X(fY) = (Xf)Y + f D_X$

1.8.1 Complex manifolds

Definition: Complex manifolds are differentiable manifolds with a holomorphic atlas. They are necessarily of even dimension, say $2n$, and allow for a collection of charts (U_j, Z_j) that are one to one maps of the corresponding U_j to C^n such that for every non-empty intersection $U_j \cap U_k$ the maps are $z_j z_k^{-1}$ are holomorphic.

The (unit) two-sphere S^2, which is the subset of R^3, defined by

$$x^2 + y^2 + z^2 = 1,$$

is a complex manifold. We can use stereographic projection from the North pole to the real plane R^2 with coordinates X, Y given by,

$$(X, Y) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

This can be done for any point except the North Pole itself (corresponding to $z = 1$). To include the North Pole, we introduce a second chart, in which we stereographically project from the South pole:

$$(U, V) = \left(\frac{x}{1+z}, \frac{y}{1+z} \right)$$

which holds for any point on S^2 except for the South pole (at $z = -1$). In both patches, we can now define complex coordinates,

$$Z = X + iY, \bar{Z} = X - iY, W = U - iV, \bar{W} = U + iV$$

,
and show that on the overlap of the two patches, the transition is holomorphic indeed, on the overlap we compute that $W = \frac{1}{z}$.
This expression relates the coordinate W to Z in a holomorphic way. Hence the two-sphere is a complex manifold which can be identified with $C \cup \infty$.

1.8.2 Almost complex manifolds.

Definition: An almost Complex structure on a manifold M is an operator $I : TM \longmapsto TM$ such that $I^2 = -Id$. It is called integrable if I is induced by a complex structure. Let M be a Hausdorf topological space. In order to analyze M locally, we use open charts, that is to say, pairs of the type (U, φ) where U is an open subset of M, and $\varphi : U \longmapsto \varphi(U) \subset R^k$ is a homeomorphism of U onto an open subset of R^k. A collection of charts $\{(U_\infty, \varphi\alpha)\alpha\epsilon A\}$ gives M the structure of a smooth manifold of dimension k if the open sets U_∞ cover M, and if for all pairs of indices α, β the transition function $\varphi\beta^\circ\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longmapsto \varphi\beta(U_\alpha \cap U_\beta)$ is a smooth map. We then say that $(U_\infty, \varphi\alpha)\alpha\epsilon A$ is an atlas of M.

A complex structure on a topological space M consists of a family $(U_\infty, \varphi\alpha)\alpha\epsilon A$, where U_∞ is an open subset of M and $U_\infty : U_\infty \longmapsto C^n$ is a homeomorphism onto an open subset of C^n, such that

(a) $M = U_{\alpha\epsilon A} U_\infty$
(b) For each pair of indices $\alpha, \beta\epsilon A$, the function
$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longmapsto \varphi_\beta(U_\alpha \cap U_\beta)$ is holomorphic.

Each pair $(U_\infty, \varphi_\alpha)$ is called a complex chart, and the whole collection $\{(U_\infty, \varphi_\alpha)_\infty \epsilon A\}$ is called a complex atlas. The integer n is the complex dimension of M.

A complex manifold of dimension n is, in a natural way, a real manifold of dimension $2n$. For given a point $p\epsilon M$, let us consider a complex chart (U, ∞) with $p\epsilon U$ and $\varphi(q) = (z^1(q), \ldots, z^n(q))$. The complex valued function z^j can be decomposed in terms of their real and imaginary parts, $z^j(q) = x^j(q) + iy^j(q)$, decomposition that in turn induces a map.

$q \longmapsto (x^1(q), y^1(q), \ldots, x^n(q), y^n(q))$

from U onto an open subset of R^{2n}. This function defines a real local chart of M. It is easy to see that transition functions of these charts of M are smooth functions. Thus, the collection of all such charts on M as a real differentiable manifold of dimension $2n$.

The set $\{\partial_{x^j}|p, \partial_{y^j}|p\}$ forms a basis of the tangent space T_pM. Using it, we define a linear isomorphism,

$$J = J_p : T_pM \longmapsto T_pM$$

by

$$J(\partial_{x^j}|p) = \partial_{y^j}|p, J(\partial_{y^j}|p) = -\partial_{x^j}|p, \tag{1.20}$$

This map is in effect independent of the choice of coordinates made. For if $\bar{\varphi}$ (q)=($^{-1}(q), \ldots, \bar{z}^{-n}(q))$ is another local chart in a neighborhood of p such that,

$\bar{z} = v^j(q) + iw^j(q)$, the linear map

$$\bar{J}(\partial_{v^j}|p) = \partial_{w^j}|p, \bar{J}(\partial_{w^j}|p) = -\partial_{v^j}|p$$

Coincides with J. Indeed, we have that

$$\partial_{x^j} = \sum_k \left(\frac{\partial v^k}{\partial x^j} \frac{\partial}{\partial v^k} + \frac{\partial w^k}{\partial x^j} \frac{\partial}{\partial w^k} \right)$$
$$\partial_{y^j} = \sum_k \left(\frac{\partial v^k}{\partial y^j} \frac{\partial}{\partial v^k} + \frac{\partial w^k}{\partial y^j} \frac{\partial}{\partial w^k} \right)$$

Since the transition function $\bar{\varphi} \circ \varphi^{-1}$ is holomorphic, the functions v^j, w^j satisfy the Cauchy-Riemann equations.

$$\frac{\partial v^k}{\partial x^j} - \frac{\partial w^k}{\partial y^j} = 0, \frac{\partial v^k}{\partial y^j} - \frac{\partial w^k}{\partial x^j} = 0$$

Thus

$$\partial_{x^j} = \sum_k \left(\frac{\partial v^k}{\partial x^j} \frac{\partial}{\partial v^k} + \frac{\partial w^k}{\partial x^j} \frac{\partial}{\partial w^k} \right) = \sum_k \left(\frac{\partial x^k}{\partial v^j} \frac{\partial}{\partial v^k} - \frac{\partial v^k}{\partial y^j} \frac{\partial}{\partial w^k} \right)$$

$$\partial_{y^j} = \sum_k \left(\frac{\partial v^k}{\partial y^j} \frac{\partial}{\partial v^k} + \frac{\partial w^k}{\partial y^j} \frac{\partial}{\partial w^k} \right) = \sum_k \left(\frac{\partial v^k}{\partial y^j} \frac{\partial}{\partial v^k} + \frac{\partial v^k}{\partial x^j} \frac{\partial}{\partial w^k} \right))$$

It then follows easily that,

$$\bar{J}(\partial_{v^j}|p) = \partial_{w^j}|p, \bar{J}(\partial_{w^j}|p) = -\partial_{v^j}|p,$$

Which shows that $\bar{J}$ agrees with J on the basis elements $\left\{ \partial_{x^j}|p, \partial_{y^j}|p \right\}$.
Thus, $\bar{J} = J$
In this way, we obtain a globally defined tensor
$p \longmapsto J_p : T_pM \longmapsto T_pM$, that squares to minus the identity, $\mathrm{J}^2 = -1$.

1.8.3 A contact metric manifold

Let (M, ϕ, ξ, η, g) be an (n=2m + 1)-dimensional almost contact metric manifold consisting of a (1, 1) tensor ϕ, a vector ξ, a 1-form η and a Riemannian metric g. Let $\chi(M)$ be the Lie algebra of vector in M. Consider $X, Y, Z, V, W \epsilon \chi(M)$ throughout the paper, unless otherwise specifically stated. Then,

$$\emptyset^2 = -I + \eta \otimes \xi, \eta(\xi) = 0, \eta \circ \phi = 0 \tag{1.21}$$

$$g(\emptyset X, \emptyset Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.22}$$

From (1.21)and (1.22) we have,

$$g(X, \phi Y) = -g(X, Y), g(X, \xi) = \eta(X) \tag{1.23}$$

Hence it is contact contact metric manifold since,

$$g(X, \phi Y) = d\eta(X, Y)$$

1.8.4 Almost contact metric manifold.

Let $(M, , \xi, \eta, g)$ be an (n=2m + 1)-dimensional almost contact metric manifold consisting of a (1, 1) tensor ϕ, a vector ξ, a 1-form η and a Riemannian metric g. Let $\chi(M)$ be the Lie algebra of vector in M. Consider $X, Y, Z, V, W \epsilon \chi(M)$ throughout the paper, unless otherwise specifically stated. Then,

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 0, \eta \circ \phi = 0 \tag{1.24}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.25}$$

From (1.24)and (1.25) we have,

$$g(X, \phi Y) = -g(\phi X, Y), g(X, \xi) = \eta(X) \tag{1.26}$$

Hence an almost contact metric manifold is a contact metric manifold if

$$g(X, \phi Y) = d\eta(X, Y)$$

1.8.5 K-contact metric manifold.

Let (M, ϕ, ξ, η, g) be an (n=2m + 1)-dimensional almost contact metric manifold consisting of a (1, 1) tensor ϕ, a vector ξ, a 1-form η and a Riemannian metric g. Let $\chi(M)$ be the Lie algebra of vector in M. Consider $X, Y, Z, V, W \epsilon(M)$ throughout the paper, unless otherwise specifically stated. Then, If M^n is a k-contact Riemannian manifold, then besides From (1.4) and (1.5) the following relations hold,

$$\nabla_x \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \nabla_x \xi = -\emptyset X \tag{1.27}$$

$$(\nabla_x \eta)(Y) = -g(\phi X, Y) \tag{1.28}$$

$$S(X, \xi) = (n-1)\eta(X) \tag{1.29}$$

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \tag{1.30}$$

For any vector fields X, Y where R and S denote respectively the curvature tensor of (1,3) and the Ricci tensor of type (0,2).

1.8.6 Almost Hermitian manifolds

Definition: Let (M, J) be an almost complex manifold. A Riemannian metric g is said to be J-Hermitian if $g(JX, JY) = g(X, Y)$ for all pair of vectors fields X, Y. we say that (J, g) is almost Hermitian structure on M which, when provided with one structure, will be called an almost Hermitian manifold.

On an almost Hermitian manifold (M, J, g), we may defined tensors that tie up properties of J and g. Indeed, let us start by introducing a $J-$Ricc tensor. Let us recall that the usual Ricci tensor $r(X, Y)$ of a Riemannian manifold (M, g) is the trace of the linear map $L \longmapsto R(L, X)Y$, where R is the Riemann curvature tensor $R(X, Y)Z = (\nabla_X \nabla_Y - \nabla_{[X,Y]})Z$, ∇ the levi-Civita connection of g. If g is Hermitian relative to J, we reproduce this concept with a J-twist, and define the J-Ricci tensor by,

$$r^J(X, Y) = traceL \longmapsto -J(R(L, X)JY),$$

The tensor defined above is essentially the only new tensor we can obtain by computing the trace of a J-twisting of R in two different positions. Indeed, varying the type of trace we take and using the symmetries of the curvature tensor R, up to a constant factor or a permutation of the arguments, we only obtain the expression r^J, $J^* r^J$ or $J^* r$, respectively.

Unlike r, r^J does not turn out to be a symmetric tensor in general.

The usual scalar curvature s is the total contraction of the curvature tensor, that is to say, the metric trace of the Ricci tensor r. Analogously, we define the J-scalar curvature s^J as the metric trace of r^J.

A straightforward calculation shows that in terms of the components of R and J, we have that,

$s = R^{li}_{li}$, $s^J = -J^i{}_t R^t_{ilm} J^l m$

For an almost Hermitian manifold (M, J, g), consider the tensor

$\omega(X, Y) = \omega^J_g(X, Y) = g(JX, Y)$.

The invariance of g and J makes this an alternate tensor, which is referred to at the fundamental form of (M, J, g). This form is J-invariant, but does not have any other special property unless we impose further conditions on the metric g. On the other hand, despite the fact that generally speaking r^J is neither symmetric nor J-invariant, the tensor

$p^J(X, Y) = -r^J(X, JY)$, is alternate. This 2-form will be called the J-Ricci form of the almost Hermitian structure (J, g).

1.8.7 The pseudo-metric topology on M equals the manifold topology

Consider a point m in M and $x_1, x_2, \ldots, x_n$ be a coordinate system about m will domain $\cup$. Now p in $\cup$ let $d(p) = d(m, p)$ and $d(p) = [\sum x_j(p)^1]^{\frac{1}{2}}$, where we assume $x_i(m) = 0$. Choose $a > 0$, so $A = [p : d^1(p) \leq a]$ is contained in $\cup$. On the compact set $B = [(p, xp) : p$ in A and $1 = \sum dx_i(p)^2]$, the form function, $\mid X_p \mid = \sum_{ij} g_{ij}(p) dx_i(Xp) dx_j(X_p)]^{\frac{1}{2}})$, is a continous function which takes on a maximum R and a minimum $r > 0$.

Let σ be any broken C^{∞} curve in A with $\sigma(0) = m$, $\sigma(b) = p$ and $(\sigma(t), T(t)$ always in B. Then $|\sigma| = \int_0^b |T_\sigma(t)|dt \geqslant rb \geqslant rd^{'(p)}$. From a broken curve σ from m to p that leaves A, one has $|\sigma| \geqslant ra \geqslant rd^{'(p)}$. Hence, we have'

(1) $d(p) \geqslant rd^{'p}$. But if σ curve with $x_i \circ \sigma(t) = \frac{tp_i}{d^{'}(p)}$, where $x_i(p) = p_i$, then $|\sigma| = \int_0{}^{d'p} |T_\sigma(t)dt \leq Rd^{'p}|$. Hence we have
(2) $d(p) \leq Rd^{'}(p)$. The inequalities (1) and (2) prove the theorem.

1.8.8 A Sasakian manifold

Let (M, ϕ, ξ, η, g) be an $(n = 2m+1)$-dimensional almost contact metric manifold consisting of a $(1,1)$ tensor ϕ, a vector ξ, a 1-form η and a Riemannian metric g. Let $\chi(M)$ be the Lie algebra of vector in M. Consider $X, Y, Z, V, W \epsilon \chi(M)$ throughout the paper, unless otherwise specifically stated. Then,

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \eta \circ \phi = 0, \phi\xi = 0 \tag{1.31}$$

$$g(,) = g(X, Y) - \eta(X)\eta(Y), g(X, \xi) = \eta(X) \tag{1.32}$$

$$({}_X\phi)Y = g(X, Y)\xi - \eta(Y)X, ({}_X\xi)Y = -\phi X \tag{1.33}$$

Thus M is a sasakian manifold. Further the following relation hold.

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \tag{1.34}$$
$$R(X, Y)\xi = (Y)X - \eta(X)Y, \tag{1.35}$$
$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{1.36}$$
$$R()\xi = \eta(X)\xi - X, \tag{1.37}$$
$$S(X, \xi) = (n-1)\eta(X), \tag{1.38}$$
$$\phi\xi = (n-1)\xi, \tag{1.39}$$

For all vectors fields X, Y, Z and where $\triangledown$ denotes the operator of covariant differentiation with respect to g, ϕ is a (1,1) tensor field, S is the Ricci tensor of type (0,2) and R is the Riemannian curvature tensor of the manifold.

Chapter 2

Literature Review

2.1 The tensor defined by different authors

Set of new curvature tensors was defined on the line of Weyl tensor by Pokhariyal amd Mishra (1970), and Pokhariyal (1979); to study Relativistic significance of curvature tensors. The Weyl's projective curvature tensor was defined on the basis of geodesic correspondence due to a particular type of distribution of vector fields contained in it.These new tensors were not necessary due to its invariance in two spaces V_n and $\bar{V_n}$, but showed that the "distribution"(order in which the vectors in question are arranged before being acted upon by the tensor in question), of vector field over the metric potentials and matter tensors plays an important role in shaping the various physical and geometrical properties of a tensor, viz the formulation of gravitational waves, reduction of electromagnetic field to a purely electric field, vanishing of the contracted tensor in an Einstein space and the cyclic property.

The Weyl's projectiive curvature tensor is given by;

$$W(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}[g(X,Z)Ric(Y,T) - g(X,T)Ric(Y,Z)]$$

The relativistic significance of Weyl's projective curvature tensor has also been explored by Singh et..al(1965)

On-τ-Curvature Tensor in K-contact and Sasakian manifolds by Mukut Mani Tripaths and Gupta (2011), studied properties of quasi$-\tau-$ flat, $\xi-\tau-$ flat and $\varphi-\tau-$ flat K-contact and Sasakian manifolds. They gave neccessary and suffient condition for the K-contact manifold to be $\varphi-\tau-$ flat under some algebraic condition. Among others, they proved that a compact $\xi-\tau-$ flat K-contact manifold with regular contact vector field, under an algebraic condition, is a principal S^1-bundle over an almost Kaehler space of constant holomorphic section curvature.They further defined a semi-Riemannian manifold M has flat if R(X, Y)Z = 0. It is also said to be $\xi-$flat if $R(X,Y)\xi = 0$, where ξ is a non-null unit vector field in M.

De and Biswas (2006) studied the $\xi-$conformally flat contact metric manifolds with $\xi \in N(k)$. They proved that a contact metric manifold with $\xi \in N(k)$ is $\xi-$conformally flat if and only if it is $\eta-$Einstein manifold. Dwivedi and Kim (2010) proved that a Sasakian manifold is $\xi-$conharmonically flat if and only if it is η Einstein.

A semi-Riemannian manifold M is said to be semisymmetric by Szbo (1982) if it satisfies $R(X,Y)\cdot R = 0$, where $R(X,Y)$ acts as a derivation on R. Semisymmetric manifold is a generalization of manifold of constant curvature and symmetric manifold ($\nabla R = 0$). A semi-Riemannian manifold is said to be recurrent by Walken (1950) if it satisfies $\nabla R = \alpha \otimes R$, where α is 1-form. Takagi (1972) gave an example of Riemannian manifolds satisfying $R(X,Y)\cdot R = 0$ but not $\nabla R = 0$.

A semi-Riemannian manifold M is said to be Ricci-semisymmetric by Deszcs (1989) if its Ricci tensor S satisfies $R(X,Y)\cdot S = 0$, where R(X, Y) acts as a derivation on S. Ricci-semisymmetric manifold is a generalization of manifold of constant curvature, Einstein manifold, Ricci symmetric manifold, symmetric manifold and semisymmetric manifold.

Ricci-semisymmetric manifolds were studied by Adati and Miyazawa (1979), On some Curvature Properties of K-contact Manifolds by Manuel de Leon in 2011 talked about Semisymmetry of a Riemannian manifold by Cartan (1926). A general study of semisymmetric Riemannian manifolds was made by Szabo (1982). Semisymmetric manifolds have been studied by other authors such as Sekigawa and Tanno (1970), Sekigawa and Takagi (1971) and

Sekigawa (1969). Also they studied about projectively semisymmetric K-contact manifolds and prove that a projectively semisymmetric K-contact manifold is Sasakian. The notion of pseudosymmetric manifolds has been introduced by Deszcz (1992).

The other tensors defined by (Pokhariyal and Mishra) (1970)are given as;

$$W_1(X,Y,Z,T) = R(X,Y,Z,T)+\frac{1}{n-1}[g(X,T)Ric(Y,Z)-g(Y,T)Ric(X,Z)]$$

$$W_2(X,Y,Z,T) = R(X,Y,Z,T)+\frac{1}{n-1}[g(X,Z)Ric(Y,T)-g(Y,Z)Ric(X,T)]$$

$$W_3(X,Y,Z,T) = R(X,Y,Z,T)+\frac{1}{n-1}[g(Y,Z)Ric(X,T)-g(Y,T)Ric(X,Z)]$$

$$W_4(X,Y,Z,T) = R(X,Y,Z,T)+\frac{1}{n-1}[g(X,Z)Ric(Y,T)-g(X,Y)Ric(Z,T)]$$

$$W_5(X,Y,Z,T) = R(X,Y,Z,T)+\frac{1}{n-1}[g(X,Z)Ric(Y,T)-g(Y,T)Ric(X,Z)]$$

$W_6(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)Y - s(Y,Z)X]$

The tensor W_6 has been studied in this project.

2.2 Curvature Tensor.

In a $(2n+1)$-dimensional Riemannian manifold M, the $\tau-$ curvature tensor is given by Tripathi and Gupta (2011).

$$\tau(X,Y)Z = a_0R(X,Y)Z + a_1S(Y,Z)X + a_2S(X,Z)Y$$
$$+a_3S(X,Y)Z + a_4g(Y,Z)QX + a_5g(X,Z)QY$$

$$+a_6 g(X,Y)QZ + a_7 r(g(Y,Z)X - g(X,Z)Y) \tag{2.1}$$

Where R,S,Q and r are the curvature tensor, the Ricci tensor,the Ricci operator and the scalar curvature, respectively

In particular, the τ-curvature tensor is reduced to be

1. the quasi-conformal curvature tensor C_* if

$a_1 = -a_2 = a_4 = -a_5, = a_3 = a_6 = 0 \ a_7 = -\frac{1}{2n+1}(\frac{a_0}{2n} + 2a_1)n$

2. the conformal curvature tensor C if

$a_0 = 1a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2n-1},\ a_3 = a_6 = 0, a_7 - \frac{1}{2n(2n-1)}$

3. the conharmonic curvature tensor if

$a_0 = 1, a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{(2n-1)}, = a_3 = a_6 = a_7 = 0$

4. the concircular curvature tensor υ if

$a_0 = 1, a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0, a_7 = -\frac{1}{2n(2n+1)},$

5. the pseudo-projective curvature tensor P_* if

$a_1 = -a_2, a_3 = a_4 = a_5 = a_6 = 0, a_7 = -\frac{1}{(2n+1)}(\frac{a_0}{2n} + a_1),$

6. the projective curvature tensor P if

$a_0 = 1, a_1 = -a_2 = -\frac{1}{2n}, a_3 = a_4 = a_5 = a_6 = a_7 = 0,$

7. the m-projective curvature tensor if

$a_0 = 1, a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{4n}, a_3 = a_6 = a_7 = 0,$

The w_0 -curvature tensor if in equation (2.1)

$a_0 = 1$ $a_1 = -a_5 = -\frac{1}{2n}$ $a_2 = a_3 = a_4 = a_6 = a_7 = 0$

$$w_0(X,Y)Z = R(X,Y)Z - \frac{1}{2n}S(Y,Z)X + \frac{1}{2n}g(X,Z)QY$$

$$= R(X,Y)Z + \frac{1}{2n}(g(X,Z)QY - S(Y,Z)X) \tag{2.2}$$

$$g(w_0(X,Y,Z),T) = g(R(X,Y,Z),T) + \frac{1}{2n}(g(X,Z)g(QY,T) - S(Y,Z)g(X,T))$$

$$w_0^{'}(X,Y,Z,T) = R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(X,Z)Ric(Y,T) - g(X,T)Ric(Y,Z)$$

$$= R^{'}(X,Y,Z,T) + \frac{1}{2}(g(X,Z)Ric(Y,T) - g(X,T)Ric(Y,Z) \tag{2.3}$$

The w_0^*-curvature tensor if in equation (2.1)

$a_0 = 1$ $a_1 = -a_5 = \frac{1}{2n}n$ $a_2 = a_3 = a_4 = a_6 = a_7 = 0$

$$w_0^*(X,Y)Z = R(X,Y)Z + \frac{1}{2n}S(Y,Z)X - \frac{1}{2n}g(X,Z)QY$$

$$= R(X,Y)Z + \frac{1}{2n}(S(Y,Z)X - g(X,Z)QY) \tag{2.4}$$

$$g(w_0^*(X,Y,Z),T) = g(R(X,Y,Z),T) + \frac{1}{2n}(S(Y,Z)g(X,T) - g(X,Z)g(QY,T))$$

$$w_0^{*'}(X,Y,Z,T) = R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(X,T)Ric(Y,Z) - g(X,Z)Ric(Y,T))$$

$$= R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(X,T)Ric(Y,Z) - g(X,Z)Ric(Y,T)) \tag{2.5}$$

The w_1 -curvature tensor if in equation (2.1)

$a_0 = 1$ $a_1 = a_2 = \frac{1}{2n}$ $a_3 = a_4 = a_5 = a_6 = a_7 = 0$

$$w_1(X,Y)Z = R(X,Y)Z + \frac{1}{2n}S(Y,Z)X - \frac{1}{2n}S(X,Z)Y$$
$$= R(X,Y)Z + \frac{1}{2n}(S(Y,Z)X - S(X,Z)Y) \tag{2.6}$$
$$g(w_1(X,Y,Z),T) = g(R(X,Y,Z),T) + \frac{1}{2n}(Ric(Y,Z)g(X,T) - Ric(X,Z)g(Y,T))$$
$$w_1^{'}(X,Y,Z,T) = R^{'}(X,Y,Z,T) + \frac{1}{2n}(Ric(Y,Z)g(X,T) - Ric(Y,T)g(X,Z))$$
$$= R^{'}(X,Y,Z,T) + \frac{1}{2n}(Ric(Y,Z)g(X,T) - Ric(Y,T)g(X,Z)) \tag{2.7}$$

The w_1^* -curvature tensor if in equation (2.1)

$a_0 = 1$ $a_1 = -a_2 = -\frac{1}{2n}$ $a_3 = a_4 = a_5 = a_6 = a_7 = 0$

$$w_1^*(X,Y)Z = R(X,Y)Z - \frac{1}{2n}S(Y,Z)X + \frac{1}{2n}S(X,Z)Y$$
$$= R(X,Y)Z - \frac{1}{2n}(S(Y,Z)X + S(X,Z)Y) \tag{2.8}$$
$$g(w_1^*(X,Y,Z),T) = g(R(X,Y,Z),T) + \frac{1}{2n}(S(X,Z)g(Y,T) - S(Y,Z)g(X,T))$$
$$w_1^{*'}(X,Y,Z,T) = R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(Y,T)Ric(X,Z) - g(X,T)Ric(Y,Z))$$
$$= R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(Y,T)Ric(X,Z) - g(X,T)Ric(Y,Z)) \tag{2.9}$$

The w_2 -curvature tensor if in equation (2.1)

$a_0 = 1$ $a_4 = -a_5 = -\frac{1}{2n}$ $a_1 = a_2 = a_3 = a_6 = a_7 = 0$

$$w_2(X,Y)Z = R(X,Y)Z - \frac{1}{2n}g(Y,Z)QX + \frac{1}{2n}g(X,Z)QY$$
$$= R(X,Y)Z + \frac{1}{2n}(g(X,Z)QY - g(Y,Z)QX) \tag{2.10}$$

$$g(w_2(X,Y,Z),T) = g(R(X,Y,Z),T)+\frac{1}{2n}(g(X,Z)g(QY,T)-g(Y,Z)g(QX,T))$$

$$w_2^{'}(X,Y,Z,T) = R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(X,Z)g(QY,T) - g(Y,Z)g(QX,T))$$

$$= R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(X,Z)Ric(Y,T) - g(Y,Z)Ric(X,T)) \qquad (2.11)$$

The w_3 -curvature tensor if in equation (2.1)

$a_0 = 1$ $a_2 = -a_4 = -\frac{1}{2n}$ $a_1 = a_3 = a_5 = a_6 = a_7 = 0$

$$w_3(X,Y)Z = R(X,Y)Z - \frac{1}{2n}nS(X,Z)Y + \frac{1}{2n}g(Y,Z)QX$$

$$= R(X,Y)Z + \frac{1}{2n}(g(Y,Z)QX - S(X,Z)Y) \qquad (2.12)$$

$$g(w_3(X,Y,Z),T) = g(R(X,Y,Z),T)+\frac{1}{2n}(g(Y,Z)g(QX,T)-S(X,Z)g(Y,T))$$

$$w_3^{'}(X,Y,Z,T) = R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(Y,Z)Ric(X,T) - g(Y,T)Ric(X,Z))$$

$$= R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(Y,Z)Ric(Y,T) - g(Y,T)Ric(X,Z)) \qquad (2.13)$$

The w_4 -curvature tensor if in equation (2.1)

$a_0 = 1$ $a_5 = -a_6 = \frac{1}{2n}$ $a_1 = a_2 = a_3 = a_4 = a_7 = 0$

$$w_4(X,Y)Z = R(X,Y)Z + \frac{1}{2n}g(X,Z)QY - \frac{1}{2n}g(X,Y)QZ$$

$$= R(X,Y)Z + \frac{1}{2n}(g(X,Z)QY - g(X,Y)QZ)$$

$$g(w_4(X,Y,Z),T) = g(R(X,Y,Z),T)+\frac{1}{2n}(g(X,Z)g(QY,T)-g(X,Y)g(QZ,T))$$

$$w_4^{'}(X,Y,Z,T) = R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(X,Z)Ric(Y,T) - g(X,Y)Ric(Z,T))$$

$$= R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(X,Z)Ric(Y,T) - g(X,Y)Ric(Z,T)) \qquad (2.14)$$

The w_5 -curvature tensor if in equation (2.1)

$a_0 = 1$ $a_2 = -a_5 = -\frac{1}{2n}$ $a_1 = a_3 = a_4 = a_6 = a_7 = 0$

$$w_5(X,Y)Z = R(X,Y)Z - \frac{1}{2n}S(X,Z)Y + \frac{1}{2n}g(X,Z)QY$$

$$= R(X,Y)Z + \frac{1}{2n}(g(X,Z)QY - S(X,Z)Y)$$

$$g(w_5(X,Y,Z),T) = g(R(X,Y,Z),T) + \frac{1}{2n}(g(X,Z)g(QY,T) - S(X,Z)g(Y,T))$$

$$w_5^{'}(X,Y,Z,T) = R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(X,Z)Ric(Y,T) - g(Y,T)Ric(X,Z))$$

$$= R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(X,Z)Ric(Y,T) - g(Y,T)Ric(X,Z)) \qquad (2.15)$$

The w_6 -curvature tensor if in equation (2.1)

$a_0 = 1$ $a_1 = -a_6 = -\frac{1}{2n}$ $a_2 = a_3 = a_4 = a_5 = a_7 = 0$

$$w_6(X,Y)Z = R(X,Y)Z - \frac{1}{2n}s(Y,Z)X + \frac{1}{2n}g(X,Y)QZ$$

$$= R(X,Y)Z + \frac{1}{2n}(g(X,Y)QZ - s(Y,Z)X)$$

$$g(w_6(X,Y,Z),T) = g(R(X,Y,Z),T) + \frac{1}{2n}(g(X,Y)g(QZ,T) - s(Y,Z)g(X,T))$$

$$w_6^{'}(X,Y,Z,T) = R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(X,Y)Ric(Z,T) - g(X,T)Ric(Y,Z))$$

$$= R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(X,Y)Ric(Z,T) - g(X,T)Ric(Y,Z)) \qquad (2.16)$$

The w_7 -curvature tensor if in equation (2.1)

$a_0 = 1$ $a_1 = -a_4 = -\frac{1}{2n}$ $a_2 = a_3 = a_4 = a_5 = a_7 = 0$

$$w_7(X,Y)Z = R(X,Y)Z - \frac{1}{2n}S(Y,Z)X + 1/2ng(Y,Z)QX$$

$$= R(X,Y)Z + \frac{1}{2n}(g(Y,Z)QX - S(Y,Z)X)$$

$$g(w_7(X,Y,Z),T) = g(R(X,Y,Z),T) + \frac{1}{2n}(g(Y,Z)g(QX,T) - S(Y,Z)g(X,T))$$

$$w_7^{'}(X,Y,Z,T) = R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(Y,Z)Ric(X,T) - g(X,T)Ric(Y,Z))$$

$$= R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(Y,Z)Ric(X,T) - g(X,T)Ric(Y,Z)) \qquad (2.17)$$

The w_8 -curvature tensor if in equation (2.1)

$a_0 = 1$ $a_1 = -a_3 = -\frac{1}{2n}$ $a_2 = a_4 = a_5 = a_6 = a_7 = 0$

$$w_8(X,Y)Z = R(X,Y)Z - \frac{1}{2n}S(Y,Z)X + \frac{1}{2n}S(X,Y)Z$$

$$= R(X,Y)Z + \frac{1}{2n}(S(X,Y)Z - S(Y,Z)X)$$

$$g(w_8(X,Y,Z),T) = g(R(X,Y,Z),T) + \frac{1}{2n}(S(X,Y)g(Z,T) - S(Y,Z)g(X,T))$$

$$w_8^{'}(X,Y,Z,T) = R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(Z,T)Ric(X,Y) - g(X,T)Ric(Y,Z))$$

$$= R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(Z,T)Ric(X,Y) - g(X,T)Ric(Y,Z)) \qquad (2.18)$$

The w_9 -curvature tensor if in equation (2.1)

$a_0 = 1$ $a_3 = -a_4 = \frac{1}{2n}$ $a_1 = a_2 = a_5 = a_6 = a_7 = 0$

$$w_9(X,Y)Z = R(X,Y)Z + \frac{1}{2n}S(X,Y)Z - \frac{1}{2n}g(Y,Z)QX$$

$$= R(X,Y)Z + \frac{1}{2n}(S(X,Y)Z - g(Y,Z)QX)$$

$$g(w_9(X,Y,Z),T) = g(R(X,Y,Z),T) + \frac{1}{2n}(S(X,Y)g(Z,T) - g(Y,Z)g(QX,T)) \qquad (2.19)$$

$$w_9^{'}(X,Y,Z,T) = R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(Z,T)Ric(X,Y) - g(Y,Z)Ric(X,T))$$
$$= R^{'}(X,Y,Z,T) + \frac{1}{2n}(g(Z,T)Ric(X,Y) - g(Y,Z)Ric(X,T)) \qquad (2.20)$$

The M-curvature tensor if in equation (2.1)

$a_0 = 1$ $a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{4n}$ $a_3 = a_6 = a_7 = 0$

$$W_M(X,Y)Z = R(X,Y)Z - \frac{1}{4n}S(Y,Z)X - \frac{1}{4n}g(Y,Z)QX + \frac{1}{4n}g(X,Z)QY$$
$$= R(X,Y)Z + \frac{1}{4n}(S(X,Z)Y + g(X,Z)QY - S(Y,Z)X - g(Y,Z)QX) \qquad (2.21)$$
$$g(W_M(X,Y,Z),T) = g(R(X,Y,Z),T) + \frac{1}{4n}(S(X,Z)g(Y,T)$$
$$+g(X,Z)g(QY,T) - S(Y,Z)g(X,T) - g(Y,Z)QX,T) \qquad (2.22)$$
$$w_9^{'}(X,Y,Z,T) = R^{'}(X,Y,Z,T) + \frac{1}{4n}(g(Y,T)Ric(X,Z)$$
$$+g(X,Z)Ric(Y,T) - g(X,T)Ric(Y,Z) - g(Y,Z)Ric(X,T)) \qquad (2.23)$$

Chapter 3

Methodology

By studying various properties of the defined tensors, W_6-curvature tensors results were derived to by making use of the Semisymmetric and symmetric propeties.

A W_6-flat K-contact Riemannian manifold is a flat space or manifold;
A W_6-Semisymmetric K-contact Riemannian manifold is a W_6-flat manifold.
A W_6-symmetric and W_6- semi symmetric K-contact Riemannian manifold is a W_6-flat manifold.

The K-contact Riemannian manifold M_n is said to be flat, if the Riemannian curvature tensor vanishes identically i.e

$$R(X,Y)Z = 0, \tag{3.1}$$

A K-contact Riemannian manifold M_n is said to be W_6- flat, if W_6-curvature tensor vanishes identically i.e.

$$W_6(X,Y)Z = 0, \tag{3.2}$$

A W_6-flat K-contact Riemannian manifold is a flat space or manifold.
Let W_6 be a (1,3)-type tensor. Then the K-contact Riemannian manifold said to be a W_6 -semisymmetric, if it satisfies

$$R(X,Y)W_6 = 0, \tag{3.3}$$

where R(X,Y) acts as the derivation on W_6.

Based on the properties of covariant derivation we were able to derive the results on symmetric, semi-symmetric and flatness of the spaces.

We obtainted the results based on embending theorem's, that K-contact Riemannain manifolds are simply, the subspace of contact manifolds. In this connection K-contact manifolds form the broader group of infinitely differentiable manifolds, so by using infinitesimal transformation; one gets the very odd dimensinal K-contact manifolds.

Chapter 4

A Study of W_6 - K-contact Riemannian Manifold

4.1 Results

Theorem 1. A W_6-Semisymmetric K-contact Riemannian manifold is a W_6-flat manifold. Let R be a (1,2)-type tensor. Then the K-contact Riemannian manifold is said to be R symmetric if

$$\nabla R = 0 \tag{4.1}$$

Theorem 2. A K-contact Riemannian manifold is said to be W_6-symmetric if it satisfies,

$$\nabla_{(X)} W_6(Y, Z, U) = 0. \tag{4.2}$$

Theorem 3. A W_6-symmetric and W_6- semi symmetric K-contact Riemannian manifold is a W_6-flat manifold.

4.2 Preliminaries

Let M_n be an $(n = 2m + 1)$ dimensional contact Riemannian manifold with the structure tensors (ϕ, ξ, η, g).

Then the following formulas holds:

$$\Phi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, \Phi\xi = 0, \tag{4.3}$$

$$g(X,\xi) = \eta(X) \tag{4.4}$$

$$g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{4.5}$$

$$F(X,Y) = -g(\Phi X, Y) = +g(X, \Phi Y) = (\nabla X\eta)(Y) = -(\nabla_Y \eta)(X), \tag{4.6}$$

$$d\eta(X,Y) = g(X, \Phi Y)$$

[see[1]] for any vector fields X and Y in M.

If we defined an operator $\bar{h}$ by $\bar{h} = \frac{1}{2} L_\xi \Phi$ where $\bar{h}$ is the lie derivative, then any contact Riemannian manifold satisfies the condition that $\bar{h}$ and $\Phi\bar{h}$ are symmetric operators, $\bar{h}$ anti-commutes with Φ (i.e. $\Phi\bar{h} + \bar{h}\Phi), \eta \circ \bar{h} = 0$ see [2] and [3] $\bar{h}\xi = 0$ and e $(\nabla_X \xi = -\Phi X - \Phi\bar{h}X)$, A contact Riemannian manifold is said to be K-contact if,

If $\nabla_X \xi = -\Phi X$ also in K-contact we have,

$$(\nabla_Y F)(Z,X) = R(Z,X,Y,\xi), \tag{4.7}$$

$$(\nabla_Z F)(\Phi X, \Phi Y) + (\nabla_Z F)(X,Y) - \eta(Y)\eta(\nabla_Z \Phi X) + \eta(X)\eta(\nabla_Z \Phi Y) = 0, \tag{4.8}$$

$$R(X,Y,Z,\xi) + R(\Phi X, \Phi Y, \Phi Z, \Phi\xi) = \eta(Y)\eta(\nabla_Z \Phi X - \eta(X)\eta(\nabla_Z \Phi Y)), \tag{4.9}$$

$$\eta(\nabla_Y \Phi X) = \eta(X)\eta(Y) - g(X,Y), \tag{4.10}$$

$$S(\xi,\xi) = Ric(\xi,\xi) = n-1,$$

where R is the Riemannian (0,4) curvature tensors $S = Ric(.,.)$ is the Ricci tensor and $F(X,Y) = g(\Phi X, Y)$.

4.3 W_6-tensor in K-contact Riemannian

Mishra and Pokhariya [4] gave the definition of W_6-tensor as

$$W_6(X,Y)Z = R(X,Y)Z + \frac{1}{(n-1)}[g(X,Z)Y - S(Y,Z)X]$$

where Q is the Ricci operator. or

$$W_6^{'}(X,Y,Z,U) = R^{'}(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)S(Z,U) - S(Y,Z)g(X,U)] \tag{4.11}$$

Definition 1: A K-contact Riemannian manifold M_n is said to be flat if the Riemannian curvature tensor vanishes identically i.e $R(X,Y)Z = 0$.

Definition 2: A K-contact Riemannian manifold M_n is said to be W_6-flat if W_6-curvature tensor vanishes identically i.e. $W_6(X,Y)Z = 0$.

Theorem 1. A W_6-flat K-contact Riemannian manifold is a flat space or manifold.

Proof; If $W_6 = 0$ i.e. flat in

$$W_6(X,Y,Z,U) = R^{'}(X,Y,Z,U) + \frac{1}{(n-1)}[g(X,Y)S(Z,U) - S(Y,Z)g(X,U)] \tag{4.12}$$

then we have,

$$R^{'}(X,Y,Z,U) = \frac{1}{(n-1)}[S(Y,Z)g(X,U) - S(Z,U)g(X,Y)] \tag{4.13}$$

using

$$S(Y,Z) = g(\phi Y,Z) = (n-1)g(Y,Z)$$

We have

$$R^{'}(X,Y,Z,U) = \frac{1}{(n-1)}(n-1)g(Y,Z)g(X,U) - (n-1)g(Z,U)g(X,Y)$$

$$= g(Y,Z)g(X,U) - g(Z,U)g(X,Y) \tag{4.14}$$

Since

$$R^{'}(X,Y,U)(Y,Z)g(X,U) - g(X,Y)g(Z,U) \Rightarrow R^{'}(X,Y,Z,U) = 0$$

and thus the theorem.

Definition 3. Let W_6 be a (1,3)-type tensor. Then the K-contact Riemannian manifold said to be a W_6

-semisymmetric if it satisfies $R(X,Y)W_6 = 0$ where $R(X,Y)$ acts as the derivation on W_6.

Theorem 2. A W_6-Semisymmetric K-contact Riemannian manifold is a W_6-flat manifold.

Proof : If we let R be a (1,2) tensors field and also W_6 i.e $W_6(U,V)Z$ then

$$R(X,Y)W_6(U,V)Z = 0$$

$$\Rightarrow R(X,Y)W_6 = g(Y,W_6)X - g(X,W_6)Y = W_6^{'}(Y,U,V)Z.X - W_6^{'}(X,U,V)Z.Y \tag{4.15}$$

$\Rightarrow$R(X,Y) $W_6 = 0$ iff $W^{'}(X,U,V)Z = 0$

$\Rightarrow$ that a W_6-semisymmetric manifold is a W_6-flat manifold. Hence the theorem.

4.4 W_6-symmetric K-contact Riemannian manifold M_n

Definition 4. Let R be a (1,2)-type tensor. Then the K-contact Riemannian manifold is said to be R symmetric if $\nabla R = 0$

Definition 5. A K-contact Riemannian manifold is said to be W_6-symmetric if it satisfies,

$$\nabla_X W_6(Y,Z,U) = 0.$$

Theorem 3. A W_6-symmetric and W_6- semi symmetric K-contact Riemannian manifold is a W_6-flat manifold.

Proof: If K-contact Riemannian is a W_6-symmetric then the following is satisfied.

$$R(X,Y,W_6(Z,U,V)) - W_6(R(X,Y,Z),U,V) - W_6(Z,R(X,Y,U),V)$$
$$-W_6(Z,U,R(X,Y,V)) = 0 \qquad (4.16)$$

Expanding the above equation $R^{'}(X,Y,W_6(Z,U,V),\xi) = g(R(X,Y,W_6(Z,U,V)))$

$$= \eta(X)W_{6'}(Z,U,V,X) - \eta(Y)W_6^{'}(Z,U,V,X) \qquad (4.17)$$

$$W_6'(R(X,Y,Z),U,V,\xi) = R^{'}(R(X,Y,Z),U,V,\xi)$$
$$+\frac{1}{(n-1)}[R^{'}(X,Y,Z,U)S(V,\xi) - S(U,V)(R(X,Y,Z))] \qquad (4.18)$$

using $S(U,V) = (n-1)g(U,V)$,

We have

$W_6'(R(X,Y,Z),U,V,\xi) = R^{'}(R(X,Y,Z),U,V,\xi)+\frac{(n-1)}{(n-1)}[R^{'}(X,Y,Z,U)\eta(V)-$
$g(U,V)(R(X,Y,Z))] = g(U,V)\eta(R(X,Y,Z))-\eta(U)R^{'}(X,Y,Z)+\eta(V)R^{'}(X,Y,Z)$

$$-g(U,V)\eta(R(X,Y,Z)) = \eta(V)R^{'}(X,Y,Z,U) - \eta(U)R^{'}(X,Y,Z,V) \qquad (4.19)$$

$$W_6'(Z,R(X,Y,U),V,\xi) = R^{'}(Z,R(X,Y,U)V,\xi)$$
$$+\frac{1}{(n-1)}[g(Z,R(X,Y,U)S(V,\xi)) - S(R(X,Y,U),V)g(Z,\xi)] \qquad (4.20)$$

using

$S(Y,Z) = (n-1)g(Y,Z)$ we have $= R^{'}(Z,R(X,Y,U),V,\xi)+\frac{(n-1)}{n-1})[\eta(V)R^{'}(X,Y,U,Z)-$

$\eta(Z)R^{'}(X,Y,U,V)] = \eta(Z)R^{'}(X,Y,U,V) - g(Z,V)\eta(R(X,Y,U)) + \eta(V)R^{'}(X,Y,U,Z) -$
$\eta(Z)R^{'}(X,Y,U,V) = \eta(V)R^{'}(X,Y,U,Z) - g(Z,V)\eta(R(X,Y,U))$
$\mathrm{W}_6^{'} = (Z,U,R(X,Y,V),\xi)$

$$= R^{'}(Z,U,R(X,Y,V),\xi) + \frac{1}{(n-1)}[g(Z,U)S(R(X,Y,V),\xi) - S(U,R(X,Y,V)g(Z,\xi))] \tag{4.21}$$

then with

$$\begin{aligned} S(Y,Z) = (n-1)g(Y,Z) &= \eta(Z)R^{'}(X,Y,V,U) - \eta(U)R^{'} \\ &= g(Z,U)\eta(R(X,Y,V)) - \eta(U)R^{'}(X,Y,V,Z) \end{aligned} \tag{4.22}$$

Put together (2.14-2.20) we get

$$\begin{aligned} &\eta(X)W_6'(X,Y,Z,U) - \eta(Y)W_6'(X,Z,U,V) + \eta(V)R^{'}(X,Y,Z,U) \\ &- \eta(U)R^{'}(X,Y,Z,V) + \eta(V)R^{'}(X,Y,U,Z) - g(Z,V)\eta(R(X,Y,U)) \\ &\qquad + g(Z,U)\eta(R(X,Y,V)) - \eta(U)R^{'}(X,Y,V,Z) = 0 \end{aligned} \tag{4.23}$$

or

$$\begin{aligned} &\eta(X)W_6'(Y,Z,U,V) - \eta(Y)W_6'(X,Z,U,V) + \eta(V)(R'(X,Y,Z,U) \\ &+ \mathrm{R}^{'}(X,Y,U,Z)) - \eta(U)R'(X,Y,Z,V) + R^{'}(X,Y,V,Z) + g(Z,U)\eta(R(X,Y,V)) \\ &\qquad - g(Z,V)\eta(R(X,Y,U)) = 0 \end{aligned} \tag{4.24}$$

Terms which are coefficients of $\eta(V)$ and $\eta(U)$ cancelled out since they are skew symmetric with respect to the last variable i.e. U,Z and Z,V. and also the K-contact manifold being W_6-semisymmetric means $R(X,Y,Z) = 0$ and thus follows the theorem.

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