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On classification of simple plane curve singularities

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a degree of Master of Science in Pure Mathematics**

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Declaration

Declaration by the Candidate

I the undersigned declare that this dissertation is my original research work and every effort has been made to indicate contributions of others. To the best of my knowledge has not been presented for the award of another degree in other university or other institution of learning.

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Dedication

This dissertation is dedicated to Joseph, Jane and my family.

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Abstract

In this project we aim to classify simple plane curve singularities by considering simple plane curve singularities as quotient singularities in \mathbb{C}^2 by a finite subgroup $G \leq SL(2, \mathbb{C})$. We also review McKay correspondence which gives a connection between classes of finite subgroups of $SL(2, \mathbb{C})$ and Dynkin diagrams.

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Chapter 1

Introduction

Throughout this project, we fix our ground field to \mathbb{C} . The main objectives of this project is to understand complex plane curves singularities. They appear throughout the classification of surfaces and in many other areas of algebraic geometry, singularity theory and group theory [MR85]. By means of representation theory and by so called A-D-E Dynkin diagrams we survey classification of these simple singularities by considering them as quotient singularities in \mathbb{C}^2 by a finite subgroup $G \leq SL(2, \mathbb{C})$. In particular, we review McKay correspondence which gives a connection between classification of finite subgroups of $SL(2, \mathbb{C})$ and A-D-E Dynkin diagrams.

In what will follow, we will see that there is a relation between quotient singularities with conjugacy classes of finite subgroups of $SL(2, \mathbb{C})$ and with A-D-E Dynkin diagrams as shown below.

$$\boxed{G \leq SL(2, \mathbb{C})} \longleftrightarrow \boxed{A - D - E \text{ Dynkin diagrams}} \longleftrightarrow \boxed{\text{resolution graphs of } \mathbb{C}^2/G}$$

Klein in [GT69] determined the structure of quotient space \mathbb{C}^2/G where G is any finite subgroup of $SL(2, \mathbb{C})$. For each such group G , the algebra of invariants polynomial has three generators related to a single equation. Thus \mathbb{C}^2/G can be realized as a space \mathbb{C}^3 defined by a single equation. These equations are isomorphic to equations of simple surface singularities also known as Du Val singularities [IR94]. For each case the origin is the singular point. The corresponding G being a cyclic group, binary dihedral group and binary groups of tetrahedral, octahedral and the icosahedral respectively.

Outline

This dissertation consists of four main chapters. In chapter 2 and chapter 3 we introduce and explain main objects and relevant theory which will be handy in later chapters. Below is the summary of each chapter.

Chapter 2

Here we do a quick review on complex plane curves and complex projective curves.

Chapter 3

This chapter is an introduction to quivers, quiver algebra and their representations.

Chapter 4

In chapter 3, we introduce quotient singularities and explain the structure of \mathbb{C}^2/G for G a finite subgroup of $SL(2, \mathbb{C})$.

Chapter 5

Finally, we construct McKay graph of $G \leq SL(2, \mathbb{C})$ and explain the McKay correspondence.

Chapter 2

Algebraic Plane Curves

We fix our ground field to be \mathbb{C}

2.1 Complex plane curves

Definition 2.1.1. A complex plane curve is a subset C of \mathbb{C}^2 of the form

$$C = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\},$$

where f is a non constant polynomial in $\mathbb{C}[x, y]$.

Definition 2.1.2. The degree d of the curve C is defined as the degree of $f \in \mathbb{C}[x, y]$ defining C i.e.

$$d := \max\{r + s : a_{r,s} \neq 0\},$$

where

$$f(x, y) = \sum a_{r,s} x^r y^s.$$

Definition 2.1.3. The singularity of curve C in \mathbb{C}^2 defined by $f \in \mathbb{C}[x, y]$ is the point $(a, b) \in C$ satisfying

$$\frac{\partial f(a,b)}{\partial x} = 0 = \frac{\partial f(a,b)}{\partial y}.$$

We denote the set of all singular points of a curve by $Sing(C)$.

Definition 2.1.4. The curve C is said to be nonsingular (or smooth) if $Sing(C) = \emptyset$

Definition 2.1.5. A curve defined by a linear equation

$$\alpha x + \beta y + \gamma = 0,$$

for $\alpha, \beta \in \mathbb{C}$ and $\alpha, \beta \neq 0$ is called a line.

Example 2.1.6. Let $f(x, y) = y^2 - x^3$ this is a cubic curve in \mathbb{C}^2 with a singular point $(0, 0)$.
 Since

$$\frac{\partial f}{\partial x}(0, 0) = 0 = \frac{\partial f}{\partial y}(0, 0).$$

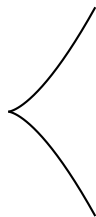


Figure 2.1: Real part of cubic curve defined by $y^2 = x^3$.

Example 2.1.7. $f(x, y) = y^2 - x^3 - x^2$ is a curve in \mathbb{C}^2 , which is a singularity at the origin.
 Since

$$\frac{\partial f}{\partial x}(0, 0) = 0 = \frac{\partial f}{\partial y}(0, 0).$$

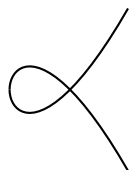


Figure 2.2: Real part of cubic curve defined by $y^2 = x^3 + x^2$.

Definition 2.1.8. A nonzero polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is homogeneous of degree d if

$$f(\lambda x_1, \dots, \lambda x_n) = \lambda^d f(x_1, \dots, x_n),$$

for all $\lambda \in \mathbb{C}$. Equivalently f has a form,

$$f(x_1, \dots, x_n) = \sum_{r_1 + \dots + r_n = d} a_{r_1, \dots, r_n} x_1^{r_1}, \dots, x_n^{r_n},$$

for some $a_{r_1, \dots, r_n} \in \mathbb{C}$.

Definition 2.1.9. The multiplicity of a curve C defined by f at a point $(a, b) \in \mathbb{C}^2$ is the smallest positive integer m such that

$$\frac{\partial^m f}{\partial x^i \partial y^j}(a, b) \neq 0,$$

for some $i \geq 0, j \geq 0$ such that $i + j = m$.

A point $(a, b) \in C$ is called a double point if its multiplicity is 2, and an ordinary point if C has distinct tangent lines at (a, b) .

Example 2.1.10. *The cubic curves in example 2.1.6 and 2.1.7 have double points at the origin. But the first is not ordinary but the second is an ordinary double point.*

Definition 2.1.11. *A curve C is called irreducible if the polynomial f is irreducible; that is if f has no other factors other than constant and scalar multiples of itself. Otherwise C is reducible.*

Example 2.1.12. *The curve $x^3 + y^3 + 1 = 0$ is irreducible in $\mathbb{C}[x, y]$.*

Definition 2.1.13. *A node is a singularity on the curve which is locally complex isomorphic to a neighborhood of the origin in the zero locus $xy = 0 \subset \mathbb{C}^2$.*

2.2 Projective plane curves

By a projective space \mathbb{P}^n we will simply be referring to a set of complex one dimensional subspace of complex vector space \mathbb{C}^{n+1} , Then any nonzero vector

$$(x_0, \dots, x_n) \in \mathbb{C}^{n+1}$$

represents an element $x \in \mathbb{P}^n$. We call (x_0, \dots, x_n) a homogeneous coordinate for x and write

$$x = [x_0, \dots, x_n].$$

Then

$$\mathbb{P}^n = \{[x_0, \dots, x_n] : (x_0, \dots, x_n) \in \mathbb{C}^{n+1} - \{0\}\}. \quad (2.1)$$

When $n = 1$ we have a projective line \mathbb{P}^1 .

Definition 2.2.1. *A projective plane \mathbb{P}^2 is the set of one dimensional complex subspaces of \mathbb{C}^3 denoted by $[x, y, z]$ the subspace spanned by $(x, y, z) \in \mathbb{C}^3 - \{0\}$, thus from equation 2.1 we have*

$$\mathbb{P}^2 := \{[x, y, z] : (x, y, z) \in \mathbb{C}^3 - \{0\}\}. \quad (2.2)$$

If $f \in \mathbb{C}[x, y, z]$ is homogeneous of degree d then

$$f(\lambda x, \lambda y, \lambda z) = \lambda^d f(x, y, z),$$

for all $\lambda \in \mathbb{C}^*$.

Definition 2.2.2. *Let f be a non constant homogeneous polynomial in $\mathbb{C}[x, y, z]$. Then the projective curve defined by f is*

$$C := \{[x, y, z] \in \mathbb{P}^2 : f(x, y, z) = 0\}. \quad (2.3)$$

Definition 2.2.3. The degree d of a projective curve $C \in \mathbb{P}^2$ defined by a homogeneous polynomial f is the degree d of the polynomial f . i.e.

$$d := \max\{r + s + t : a_{r,s,t} \neq 0\},$$

where

$$f(x, y, z) = \sum a_{r,s,t} x^r y^s z^t.$$

Definition 2.2.4. The singularity of curve C in \mathbb{P}^2 defined by the polynomial f is the point $(a, b, c) \in C$ satisfying

$$\frac{\partial f}{\partial x}(a, b, c) = 0 = \frac{\partial f}{\partial y}(a, b, c) = 0 = \frac{\partial f}{\partial z}(a, b, c). \quad (2.4)$$

The set of singular points of a curve C is denoted by $Sing(C)$. Again the curve C is said to be nonsingular projective curve (or smooth) if $Sing(C) = \emptyset$.

Definition 2.2.5. A projective curve defined by a linear equation

$$\alpha x + \beta y + \gamma z = 0,$$

for $\alpha, \beta, \gamma \neq 0$ is called a projective line.

The tangent line to a projective curve C in \mathbb{P}^2 defined by a homogeneous polynomial f at a singular point $[x, y, z]$ is the line

$$\frac{\partial f}{\partial x}(a, b, c)x + \frac{\partial f}{\partial y}(a, b, c)y + \frac{\partial f}{\partial z}(a, b, c)z = 0. \quad (2.5)$$

Example 2.2.6. Let $f(x, y, z) = zy^2 - x^3$, this is a cubic curve in \mathbb{P}^2 with a singular point $(0, 0, 1)$. Since

$$\frac{\partial f}{\partial x}(0, 0, 1) = \frac{\partial f}{\partial y}(0, 0, 1) = \frac{\partial f}{\partial z}(0, 0, 1) = 0$$

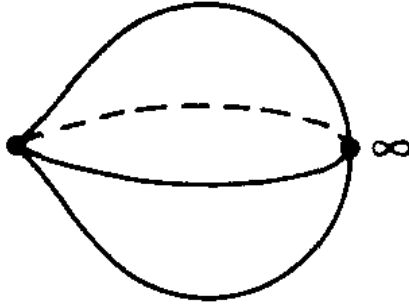


Figure 2.3: Complex algebraic curve with a point at infinity defined by $zy^2 = x^3$.

Example 2.2.7. $f(x, y, z) = zy^2 - x^3 - zx^2$ is a curve in \mathbb{P}^2 , which has a singularity at the point $(0, 0, 1)$. Since

$$\frac{\partial f}{\partial x}(0, 0, 1) = \frac{\partial f}{\partial y}(0, 0, 1) = \frac{\partial f}{\partial z}(0, 0, 1) = 0.$$

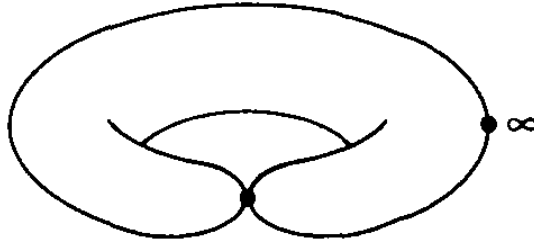


Figure 2.4: Complex algebraic curve with a point at infinity defined by $zy^2 = x^3 + zx^2$.

Chapter 3

Quivers and quiver representations

In this section, we review definitions and some relevant results on (*associative*) algebras, largely following [PO11].

3.1 Algebras and representations

Definition 3.1.1. *An associative algebra over \mathbb{C} is a \mathbb{C} -vector space A equipped with an associative bilinear map,*

$$\mu: A \times A \mapsto A, (a, b) \rightarrow ab = \mu(ab).$$

such that $(ab)c = a(bc)$.

We will always consider associative algebras with unit, i.e., with an element 1 such that $1a = a1 = a$ for all $a \in A$.

Example 3.1.2. *1. Let V be a \mathbb{C} -vector space, and let $A = \text{End}(V)$ be the space of \mathbb{C} -linear maps from a vector space V to itself. Then A is an associative algebra with multiplication the composition of maps.*

2. The ring of polynomials $\mathbb{C}[x]$ in one indeterminate x with coefficients in \mathbb{C} is an algebra whose unity is the constant polynomial 1 .

Definition 3.1.3. *Let A be an algebra such that $A = \bigoplus_{n=0}^{\infty} A_n$ for $(A_n)_{n>0}$ sub algebras of A and $A_m A_n \subseteq A_{m+n}$ for all $m, n \geq 0$ then A is said to be a graded algebra.*

Definition 3.1.4. *A representation of an algebra A is a pair (V, ρ) , where V is a \mathbb{C} -vector space and $\rho: A \rightarrow \text{End } V$ is a homomorphism of algebras.*

Equivalently a representation is a right A -module V equipped with an antihomomorphism $\rho: A \rightarrow \text{End } V$; i.e., ρ satisfies $\rho(ab) = \rho(b)\rho(a)$ and $\rho(1) = 1$. If A has an identity element 1 , then we require that $\rho(1)$ act as the identity map I_V on V i.e., a linear map preserving the multiplication $\rho(ab) = \rho(a)\rho(b)$ and unit. Equivalently we can view it as a left A -module.

We shall call V an A -module and write av for $\rho(a)v$. If V, W are both A -modules, then we make the vector space $V \oplus W$ into an A -module by the action $a \cdot (v \oplus w) = av \oplus aw$ for all $v \in V, w \in W$.

Example 3.1.5. 1. $V = A$, and $\rho: A \rightarrow \text{End } A$ is defined as follows: $\rho(a)$ is the homomorphism of left multiplication by a , so that $\rho(a)b = ab$ (the usual product). This representation is called the regular representation of A . Similarly, one can equip A with a structure of a right A -module by setting $\rho(a)b := ba$.

2. $A = \mathbb{C}$. Then a representation of A is simply a vector space over \mathbb{C} .

Definition 3.1.6. A subrepresentation of a representation V is a subspace $U \subset V$ which is invariant under all morphisms $\rho(a)$, from A to $\text{End } V$ for $a \in A$.

Definition 3.1.7. A nonzero representation (V, ρ) is irreducible if and only if the invariant subspaces of V are $\{0\}$ and V itself.

Definition 3.1.8. A nonzero representation (V, ρ) is indecomposable if it cannot be written as a direct sum of two nonzero subrepresentations.

Basic problems of representation theory includes, classification of irreducible representations of a given algebra A and classification of indecomposable representations of A .

3.2 Quiver and quiver representations

In this section, we assume our quiver is of a finite type.

Definition 3.2.1. A quiver is a directed graph Q where loops and multiple arrows between the vertices are allowed. Specifically a quiver Q is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is finite set of vertices's, Q_1 is finite set of arrows and $s, t: Q_1 \rightarrow Q_0$ are maps which assigns to each arrow $a \in Q_1$ its source and target respectively.

The set Q_0 usually will be identified with the set $i = \{1, \dots, n\}$ for $i \in Q_0$. The arrows will be denoted by initial letters of the alphabet. If $a \in Q_1$ is an arrow, then a has its source sa and target ta , both in Q_0 .

Definition 3.2.2. A quiver Q is said to be of finite type if Q_0 and Q_1 are finite set.

Example 3.2.3. Consider a quiver

$$Q = i \xrightarrow{a} j$$

with two vertices i and j in Q_0 and one arrow $a \in Q_1$. This quiver is of finite type.

Example 3.2.4. The r -arrow Kronecker quiver K_r is the quiver having two vertices's i and j and r arrows between the vertices's denoted by K_r . For instance

$$K_2: i \rightrightarrows j$$

is a Kronecker quiver with 2 arrows.

Example 3.2.5. The loop is the quiver denoted by L having a unique vertex i and a unique arrow a such that $(s(a) = t(a) = i)$.

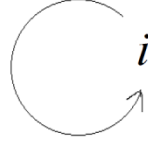


Figure 3.1: Loop quiver.

Definition 3.2.6. Let Q be quiver, a representation V of Q is a pair $\{V_i, g_a\}$, for all $i \in Q_0$ and $a \in Q_1$ where $\{V_i; i \in Q_0\}$ is a family of finite dimensional \mathbb{C} -vector spaces and \mathbb{C} -linear maps.

Definition 3.2.7. Let $V = \{V_i, g_a\}$ be a representation of a quiver Q with vertices's $\{1, 2, \dots, n\}$. Then the dimension vector of V over \mathbb{C} is

$$d_V := \dim V,$$

where

$$\dim V := \dim_{\mathbb{C}} V.$$

Example 3.2.8. A representation of the loop quiver in example 3.2.5 is a \mathbb{C} – vector space V together with an endomorphism $g_a: V \rightarrow V$

Example 3.2.9. Consider a quiver

$$Q = 1 \xrightarrow{a} 2$$

The representation of this quiver is a collection of two finite \mathbb{C} – dimensional vector spaces V_1 and V_2 , together with a \mathbb{C} -linear map:

$$g_a: V_1 \rightarrow V_2$$

Recall that if V and W are two representations of the same quiver Q , we define their direct sum $V \oplus W$ by:

$$(V \oplus W)_i = V_i \oplus W_i$$

for all $i \in Q_0$, and

$$(V \oplus W)_a := \begin{pmatrix} V_a & 0 \\ 0 & W_a \end{pmatrix} : V_i \oplus W_i \rightarrow V_j \oplus W_j,$$

for all $a \in Q_1$, and if V is a vector space with basis x_1, \dots, x_n and W is a vector space with basis y_1, \dots, y_m then $V \oplus W$ is the vector space with basis

$$\{x_1, \dots, x_n, y_1, \dots, y_m\}.$$

Recall a representation V is indecomposable if it cannot be expressed as a proper direct sum, i.e. from the decomposition $V = V_1 \oplus V_2$ it follows that either V_1 or V_2 is zero.

Example 3.2.10. A quiver in example 3.2.3 its indecomposable representations fall into three isomorphism classes. i.e

$$S_1: \mathbb{C} \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} 0$$

$$S_2: 0 \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} \mathbb{C}$$

$$P_1: \mathbb{C} \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} \mathbb{C}$$

of respective dimension vectors $(1, 0)$, $(0, 1)$, $(1, 1)$.

Definition 3.2.11. Suppose $V = \{V_i, g_a\}$ and $W = \{W_i, f_a\}$ are two representations of Q , then a morphism

$$\varphi: V \rightarrow W$$

is a collection of \mathbb{C} -linear maps such that the following diagram commutes.

$$\begin{array}{ccc} V_i & \xrightarrow{g_a} & V_j \\ \downarrow \varphi_i & & \downarrow \varphi_j \\ W_i & \xrightarrow{f_a} & W_j \end{array}$$

Remark 3.2.12. A morphism $\varphi_i: V_i \rightarrow W_i$ is an isomorphism if φ_i is invertible for every $i \in Q_0$.

Example 3.2.13. Consider a Kronecker quiver in example 3.2.4

$$K_2: i \rightrightarrows j.$$

A representation $M = (M_1, M_2)$ of Q is given by

$$M_1: \mathbb{C}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \mathbb{C}^2$$

$$M_2: \mathbb{C} \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \mathbb{C}^2.$$

We need to calculate the $\text{Hom}(M_1, M_2)$, suppose that $\varphi = (\varphi_1, \varphi_2)$ is a morphism from $M_1 \rightarrow M_2$, then φ_1 and φ_2 can be written as

$$\varphi_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \varphi_2 = \begin{pmatrix} x \\ y \end{pmatrix} \mid a, b, c, d, x, y \in \mathbb{C}$$

therefore

$$\varphi_2 g_a = f_a \varphi_1$$

using these commutative equation we find that

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

Therefore

$$f = \left[\begin{pmatrix} a & 0 \\ c & c \end{pmatrix}, \begin{pmatrix} a \\ c \end{pmatrix} \right],$$

$\text{Hom}(M_1, M_2) \simeq \mathbb{C}^2$ is a two dimensional vector space with basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

One naturally wants to classify all representations of a given quiver Q up to isomorphism. We denote the set of all representations of a given quiver Q by $\mathfrak{Rep} Q$.

Given two representations V and W as above, the set of all morphisms from V to W is a subspace of $\prod_{i \in Q_0} \text{Hom}(V_i, W_i)$; which we denote by

$$\text{Hom}_Q(V, W).$$

If $V = W$ then

$$\text{End}_Q V = \text{Hom}_Q(V, V)$$

is a subalgebra of the product algebra $\prod_{i \in Q_0} \text{End}(V_i)$

Definition 3.2.14. A category is a triple $\mathfrak{C} = (\text{Ob } \mathfrak{C}, \text{Hom } \mathfrak{C}, \circ)$, where $\text{Ob } \mathfrak{C}$ is called the class of objects of \mathfrak{C} , $\text{Hom } \mathfrak{C}$ is called the class of morphisms of \mathfrak{C} , and \circ is a partial binary operation on morphisms of \mathfrak{C} satisfying the following conditions:

- (a) to each pair of objects X, Y of \mathfrak{C} , we associate a set $\text{Hom}_{\mathfrak{C}}(X, Y)$, called the set of morphisms from X to Y , such that if $(X, Y) \neq (Z, U)$ then the intersection of the sets $\text{Hom}_{\mathfrak{C}}(X, Y)$ and $\text{Hom}_{\mathfrak{C}}(Z, U)$ is empty; and
- (b) for each triple of objects X, Y, Z of \mathfrak{C} , the operation $\circ: \text{Hom}_{\mathfrak{C}}(Y, Z) \times \text{Hom}_{\mathfrak{C}}(X, Y) \rightarrow \text{Hom}_{\mathfrak{C}}(X, Z)$, $(g, f) \mapsto g \circ f$ (called the composition of f and g).

Definition 3.2.15. Let \mathfrak{C} be a category. A category \mathfrak{D} is a subcategory of \mathfrak{C} if the following four conditions are satisfied:

- (a) the class $\text{Ob } \mathfrak{D}$ of objects of \mathfrak{D} is a subclass of the class $\text{Ob } \mathfrak{C}$ of objects of \mathfrak{C} ;
- (b) if X, Y are objects of \mathfrak{C} , then $\text{Hom}_{\mathfrak{D}}(X, Y) \subseteq \text{Hom}_{\mathfrak{C}}(X, Y)$;
- (c) the composition of morphisms in \mathfrak{C} is the same as in \mathfrak{D} ; and

(d) for each object X of \mathfrak{D} , the identity morphism 1_X in $\text{Hom}_{\mathfrak{D}}(X, X)$ coincides with the identity morphism 1_X in $\text{Hom}_{\mathfrak{C}}(X, X)$.

Definition 3.2.16. Let M and N be any two indecomposable representations of a quiver Q . The radical of M and N is a subspace of $\text{Hom}_Q(M, N)$ consisting of only non-isomorphic morphisms $f: M \rightarrow N$ and is denoted by;

$$\text{rad}_Q(M, N)$$

$\text{rad}_Q^2(M, N) = \sum \text{rad}(L, N) \text{rad}(M, L)$ where L ranges over subcategory of $\mathfrak{Rep} Q$. $\text{rad}_Q^2(M, N)$ is the sum of morphisms

$$f: M \longrightarrow L \longrightarrow N$$

and

$$\text{irr}(M, N) = \frac{\text{rad}(M, N)}{\text{rad}^2(M, N)}$$

If $f \in \text{irr}(M, N)$ the f is said to be an irreducible morphism between M and N .

Example 3.2.17. Consider example 3.2.3. For a \mathbb{C} -linear map $g_a: \mathbb{C} \rightarrow \mathbb{C}$ we can always choose bases in V_i and in V_j in which g_a is given by the block matrix A

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where r is the rank of A and I_r is the $r \times r$ identity matrix. Note that two representations V and W then $g_a: V_i \rightarrow V_j$ and $f_a: W_i \rightarrow W_j$ are isomorphic if and only if $\dim V_i = \dim W_i, \dim V_j = \dim W_j$, and g_a and f_a have the same rank.

3.3 Quiver Algebra

We fix a quiver $Q = (Q_0, Q_1, s, t)$. To any representation $V = (V_i, f_a)$ of Q , we associate the vector space

$$\bar{V} = \bigoplus_{i \in Q_0} V_i$$

equipped with two families of linear self maps: the projections

$$f: V \rightarrow V, (i \in Q_0)$$

(the compositions $V \rightarrow V_i \hookrightarrow V$ for the projections with the inclusions), and the linear maps

$$f_a: V \rightarrow V (a \in Q_1).$$

these maps satisfy the relations

$$f_i^2 = f_i, f_i f_j = 0 (i \neq j), f_{t(\alpha)} f_\alpha = f_\alpha f_{s(\alpha)} = f_\alpha$$

and all other products equals to zero. This leads to the following definitions.

Definition 3.3.1. A path in a quiver Q is a sequence

$$p := (i|a_1a_2\dots a_n|j)$$

of n arrows in Q such that $t(a_l) = s(a_{l+1})$ for all $l \leq n$. Its length is n . The path has source $s(p) = s(a_1)$ and target $t(p) = t(a_n)$. We also allow length zero paths, concentrated at a single vertex i of Q , which we denote e_i .

Definition 3.3.2. The path algebra $\mathbb{C}Q$ of a quiver Q is a \mathbb{C} -algebra with a basis labeled by paths in Q . It is associative and determined by the generators e_i , where $i \in Q_0$, and the following relations are satisfied

$$e_i e_j = 0 \ (i \neq j), \ e_i^2 = e_i, \ e_{t(a)} a = a e_{s(a)} = a.$$

e_i are orthogonal idempotents of $\mathbb{C}Q$, Also $\sum_{i \in Q_0} e_i = 1$

The product of basis elements is then extended to arbitrary elements of $\mathbb{C}Q$ by distributivity. In other words, there is a direct sum decomposition

$$\mathbb{C}Q = \mathbb{C}Q_0 \oplus \mathbb{C}Q_1 \oplus \mathbb{C}Q_2 \oplus \dots \oplus \mathbb{C}Q_l \oplus \dots$$

of the \mathbb{C} -vector space $\mathbb{C}Q$, where, for each $l \geq 0$, $\mathbb{C}Q_l$ is the subspace of $\mathbb{C}Q$ generated by the set Q_l of all paths of length l . We observe that $(\mathbb{C}Q_n) - (\mathbb{C}Q_m) \subseteq \mathbb{C}Q_{n+m}$ for all $n, m \geq 0$, because the product in $\mathbb{C}Q$ of a path of length n by a path of length m is either zero or a path of length $n + m$. This is expressed sometimes by saying that the decomposition defines a grading on $\mathbb{C}Q$ or that $\mathbb{C}Q$ is a graded \mathbb{C} -algebra.

Example 3.3.3. Consider a quiver consisting of a single point and a single loop. The defining basis of the path algebra $\mathbb{C}Q$ is $\{\epsilon_1, a, a^2, \dots, a^l, \dots\}$ and the multiplication of basis vectors is given by

$$\epsilon_1 a^l = a^l \epsilon_1 = a^l \text{ for all } l \geq 0$$

and

$$a^l a^k = a^{l+k} \text{ for all } l, k \geq 0,$$

where $a^0 = \epsilon_1$. Thus $\mathbb{C}Q$ is isomorphic to the polynomial algebra $\mathbb{C}[x]$ in one indeterminate x , the isomorphism being induced by the \mathbb{C} -linear map such that $\epsilon_1 \rightarrow 1$ and $a \rightarrow x$

Example 3.3.4. Consider a quiver

$$1 \xrightarrow{a} 2.$$

The path algebra $\mathbb{C}Q$ has as its defining basis the set $\{\epsilon_1, \epsilon_2, a\}$ with the multiplication table.

$$\begin{array}{c|ccc} \cdot & \epsilon_1 & \epsilon_2 & a \\ \hline \epsilon_1 & \epsilon_1 & 0 & a \\ \epsilon_2 & 0 & \epsilon_2 & 0 \\ a & 0 & a & 0 \end{array} \quad (3.1)$$

We can see, $\mathbb{C}Q$ is isomorphic to the 2×2 upper triangular matrix algebra,

$$T_2\mathbb{C} = \left(\begin{array}{cc} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{array} \right) := \left\{ \left(\begin{array}{cc} b & c \\ 0 & d \end{array} \right) \mid b, c, d \in \mathbb{C} \right\},$$

where the isomorphism is induced by the \mathbb{C} -linear map such that

$$\epsilon_1 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \epsilon_2 \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad a \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The representations of a quiver Q form a category where the morphism sets are denoted by $\text{Hom}_Q(V, W)$ for $V, W \in \mathfrak{Rep} Q$. For a finite-dimensional algebra A , the finite-dimensional (left) A -modules form a category $\mathfrak{Mod} A$, where the morphisms are called homomorphisms and form sets denoted by $\text{Hom}_A(M, N)$ for $M, N \in \mathfrak{Mod} A$.

Definition 3.3.5. Let Q be a quiver. A relation in Q with coefficients in \mathbb{C} is \mathbb{C} -linear combination of paths of length at least two having the same source and target. Thus, a relation ρ is an element of $\mathbb{C}Q$ such that $\rho = \sum_{i=1}^m \lambda_i w_i$, where the λ_i are scalars (not all zero) and the w_i are paths in Q of length at least 2 such that, if $i \neq j$, then the source (or the target, respectively) of w_i coincides with that of w_j .

3.3.1 Module over a path algebra

Assume that Q is a finite type quiver and V a representation of Q . Given any representation V of Q , we can define a left $\mathbb{C}Q$ -modules

$$\bar{V} = \bigoplus_{i \in Q_0} V_i$$

by defining the multiplication \bar{V}_w with a path $w = (j|a_l, \dots, a_1|i)$ on a family $(v_h)_{h \in Q_0}$ as the family having $V_{(a_1)} \dots V_{(a_l)}(v_i)$ in the j^{th} coordinate and zero elsewhere. Notice that \bar{V} is always finite dimensional, since Q is of finite type and by definition a representation as finite-dimensional vector spaces attached to each vertex. Conversely, given a finite-dimensional left $\mathbb{C}Q$ -module M , we define $M_i = e_i M = \{e_i m \mid m \in M\}$. We have then $M = \bigoplus_{i \in Q_0} M_i$ and can easily define a representation by setting $M_a: M_i \rightarrow M_j$, $e_i m \rightarrow (j|a|i)m$ for any arrow $a: i \rightarrow j$ in Q .

If $\phi: V \rightarrow W$ is a morphism of representations, then we define $\bar{\phi} = \bigoplus_{i \in Q_0} \phi_i: \bar{V} \rightarrow \bar{W}$, which is a homomorphisms of $\mathbb{C}Q$ -modules.

Conversely, any homomorphism $\psi: M \rightarrow N$ of finite-dimensional $\mathbb{C}Q$ -modules gives rise to a morphism of representations by $\psi_{(i)}: M_i \rightarrow N_i, e_i m \rightarrow \psi(e_i m) = e_i \psi(m)$. The direct sum of $\mathbb{C}Q$ -modules correspond to the direct sum of representations and therefore their indecomposable s correspond one-to-one (up to isomorphism).

Remark 3.3.6. *Every representation has a unique decomposition into indecomposable representations (up to isomorphism and permutation of components). The classification problem reduces to classifying the indecomposable representations.*

Definition 3.3.7. *Let Q be a quiver. Then the opposite quiver, denoted Q^{opp} , is the quiver with the same vertices as Q , and an arrow $a^*: j \rightarrow i$ for every arrow $a: i \rightarrow j$ of Q .*

Recall that if V is a vector space with basis x_1, \dots, x_n and W is a vector space with basis y_1, \dots, y_m then $V \oplus W$ is the vector space with basis $\{x_1, \dots, x_n, y_1, \dots, y_m\}$.

Theorem 3.3.8. *Let Q be a quiver. Then representations of the quiver Q are the same as left $\mathbb{C}Q$ -modules.*

Proof. Suppose we have a $\mathbb{C}Q$ -modules. Set

$$V_i = e_i M = e_i m | m \in M.$$

Then, for each arrow $a: i \rightarrow j$, define a linear map $g_a: V_i \rightarrow V_j$ by

$$g_a(m) = am.$$

We should check that $am \in V_j$. This is true because $a: i \rightarrow j$, so $a = e_j a e_i$ in $\mathbb{C}Q$. Now suppose we have representation V of Q . Define

$$M = \bigoplus_{i \in Q_0} V_i,$$

where the sum is over all the vertices i of Q . Define the module action as follows. For each vertex i , let

$$\pi_i: M \rightarrow V_i$$

be the obvious projection map and let

$$\iota_i: V_i \rightarrow M$$

be the obvious inclusion. Note that

$$\pi_i \iota_i: V_i \rightarrow V_i$$

is the identity map. Then define $e_i m = \iota_i \pi_i(m)$, and for each arrow a of $\mathbb{C}Q$, define $am = \iota_j V_a \pi_i(m)$. The action of a path is obtained by using the action of the arrows in the path. This action respects the multiplication because

$$\iota_i \pi_i: V_i \rightarrow V_i$$

is the identity map. □

3.4 Dynkin diagrams and extended dynkin diagrams

We fix $Q = (Q_0, Q_1, s, t)$.

Definition 3.4.1. *The Ringel form for Q is the bilinear on \mathbb{Z}^{Q_0} defined by*

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_j \beta_i,$$

and the quadratic form is

$$q(\alpha) = \langle \alpha, \alpha \rangle.$$

Then the corresponding symmetric bilinear form

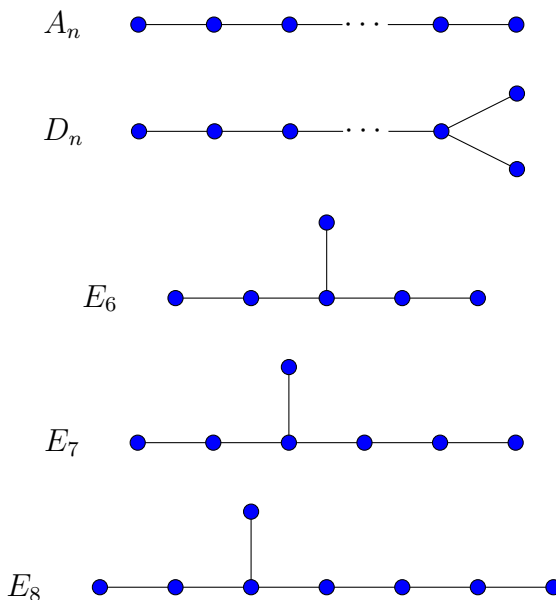
$$(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle.$$

Definition 3.4.2. *If $q(\alpha) > 0$ for all $0 \neq \alpha \in \mathbb{Z}^{Q_0}$ the q is said to be positive definite and if $q(\alpha) \geq 0$ then is said to be semi positive definite.*

Recall that a quiver is of finite type when it has finite number of vertices and arrows, equivalently a quiver with finitely many indecomposable representations is said to be of a finite type. Forgetting the orientations of the arrows yields the underlying undirected graph of a quiver.

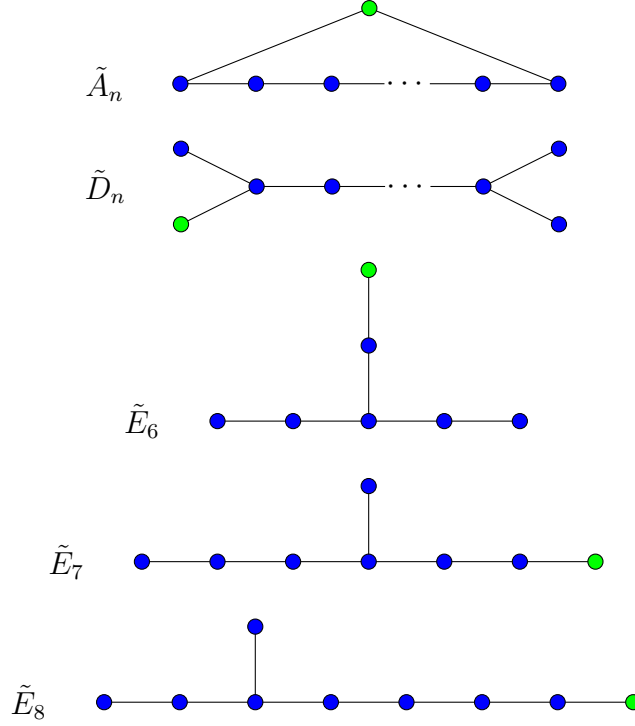
Theorem 3.4.3. *Let Q be a finite connected quiver without oriented cycles. The following are equivalent:*

- i). Q is representation-finite.
- ii). Q is a simply laced Dynkin diagram as shown below:



Theorem 3.4.4. *Let Q be a finite connected quiver without oriented cycles. The following are equivalent:*

- a). Q is representation-infinite and tame.
- b). Q is a simply laced extended Dynkin diagram as shown below:



3.4.1 Root system

Given a quiver $Q = (Q_0, Q_1, s, t)$, we introduce the associated root system ΔQ a subspace in \mathbb{Z}^{Q_0} , where \mathbb{Z}^{Q_0} is a vector space of dimension n determined by the finite number of vertices in Q as follows;

If k is a loop free vertex then there is a reflection

$$S_i: \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}^{Q_0}, S_k(\alpha) = \alpha - (\alpha, \varepsilon_i)\varepsilon_i,$$

where $\alpha = \sum k_i \varepsilon_i \in \mathbb{Z}^{Q_0}$. The group $W \subset \text{Aut}(\mathbb{Z}^{Q_0})$ generated by all reflections is called *Wely group* of the Quiver.

The real roots are the orbits of ε_k under W and the imaginary roots are the orbits of $+\alpha$ and $-\alpha$ under W . The real root $q(\alpha) = 1$ and the imaginary $q(\alpha) \leq 0$. Hence the root system ΔQ is defined by

$$\begin{aligned} \Delta Q &= \{\text{real root}\} \cup \{\text{imaginary root}\} \\ \Delta Q &= \left\{ \alpha \in \mathbb{Z}^{Q_0} \mid \alpha \neq 0, q(\alpha) \leq 1 \right\}. \end{aligned}$$

Theorem 3.4.5. *The indecomposable representations are in one-to-one correspondence with the positive roots of the corresponding root system. For a Dynkin quiver Q , the dimension vectors of indecomposable representations do not depend on the orientation of the arrows in Q .*

The following table shows the number of positive roots of (q) for each Dynkin diagram.

Δq	A_n	D_n	E_6	E_7	E_8
$P(q)$	$\frac{n(n+1)}{2}$	$(n-1)n$	36	63	120

Table 3.1: Number of positive roots of (q) for a given Dynkin diagram.

This gives the number of indecomposable representation for n , number of vertices's.

Theorem 3.4.6. *For an arbitrary quiver Q , the set of dimension vectors of indecomposable representations of Q does not depend on the orientation of arrows in Q . The dimension vectors of indecomposable representations correspond to positive roots of the corresponding root system.*

3.5 Variety of representations

In this part we let $Q = (Q_0, Q_1, s, t)$. We fix a dimension vector $d \in \mathbb{N}^{Q_0}$, where \mathbb{N}^{Q_0} is a vector space determined by the finite number of vertices in Q . Then each representation V with dimension vector d is isomorphic to a representation of the form

$$M = \left((\mathbb{C}^{d_i})_{i \in Q_0}, (M(a))_{a \in Q_1} \right), \quad (3.2)$$

where, for each arrow $a: i \rightarrow j$, $M(a)$ is a matrix defining a linear map

$$M(a): \mathbb{C}^{d_i} \rightarrow \mathbb{C}^{d_j},$$

that is $M(a) \in \mathbb{C}^{d_i \times d_j}$.

We can define

$$\mathfrak{Rep}(Q, d): = \oplus Hom(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}). \quad (3.3)$$

We can also view it as a vector space

$$\mathfrak{Rep}(Q, d) = \prod_{a \in Q_1} \mathbb{C}^{d_i \times d_j}.$$

Then $\mathfrak{Rep}(Q, d)$ is said to be a variety of representations of Q of dimensional vector $d \in \mathbb{N}^{Q_0}$. Two representations $M, N \in \mathfrak{Rep}(Q, d)$ are isomorphic if and only if there exist a family $(\phi_i: \mathbb{C}^{d_i} \rightarrow \mathbb{C}^{d_i})_{i \in Q_0}$ of invertible linear maps, such that for every arrow $a: i \rightarrow j$ in Q we have $N(a)\phi_i = \phi_j M(a)$. We let

$$GL(Q, d) = \prod_{i \in Q_0} GL(d_i, \mathbb{C}), \quad (3.4)$$

and define a group action of $GL(Q, d)$ on $\mathfrak{Rep}(Q, d)$ by

$$(g.M = (g_j M(a) g_i^{-1})(a: i \rightarrow j) \in Q_1.$$

Then M and N are isomorphic if and only if they lie in the same orbit under the action of $GL(Q, d)$. The orbit of M is denoted by $O(M)$.

Let $M, N \in \mathfrak{Rep}(Q, d)$, then the set of isomorphisms of representations $M \rightarrow N$ can be identified with

$$\{g \in GL(Q, d) | g.M = N\}.$$

It follows that there is a one-to-one correspondence between isomorphism classes of representations M with dimension vector d and the $GL(Q, d)$ orbit O_M . In particular the stabilizer $GL(d)_M$ of M in $GL(Q, d)$ is identified with the set of $\text{Aut}_{\mathbb{C}Q}(M)$ of automorphisms of M . The dimensions of $\mathfrak{Rep}(Q, d)$ and $GL(Q, d)$. Clearly

$$\dim \mathfrak{Rep}(Q, d) = \sum_{a \in Q_1} d_i d_j,$$

and viewing $GL(Q, d)$ as an open subset in $G = \prod_{i \in Q_0} \mathbb{C}^{d_i \times d_j}$. The

$$\dim GL(Q, d) = \sum_{i \in Q_0} d_i^2$$

Now define $\chi_Q: \mathbb{Z}^n \rightarrow \mathbb{Z}$ by

$$\chi_Q(d) = \sum_{\chi_i \in Q_0} d_i^2 - \sum_{\chi_a \in Q_1} d_i d_j.$$

This is a quadratic form, which satisfies

$$\chi_Q(d) = \dim GL(Q, d) - \dim \mathfrak{Rep}(Q, d), \tag{3.5}$$

for each $d \in \mathbb{N}_0^Q$.

There fore the

$$\dim O_M = \dim GL(Q, d) - \dim(\text{Aut}_{\mathbb{C}Q}).$$

Chapter 4

Classification of simple singularities

In this chapter, we classify simple singularities as quotient singularity by a finite subgroup of $SL(2, \mathbb{C})$.

4.1 Basic definitions

Definition 4.1.1. *An affine algebraic variety is a set $C \subset \mathbb{C}^n$ defined as a zero locus of some polynomial:*

$$C := \{(x_1, \dots, x_n) : f(x_1, \dots, x_n) = 0 \text{ for } f \in I_C\}$$

where $I_C \subset \mathbb{C}[x_1, \dots, x_n]$ is an ideal of functions vanishing along the set C .

Example 4.1.2. *If we fix $n = 2$ then*

$$C := \{(x, y) : f(x, y) = 0\}$$

is a variety in an plane \mathbb{C}^2 and is called an affine plane curve.

Now we translate the definition for singularities in chapter two to a modern language.

Definition 4.1.3. *If $C \subset \mathbb{C}^n$ is a variety, then we define the coordinate ring \mathcal{O}_C of functions on C , to be:*

$$\mathcal{O}_C = \mathbb{C}[x_1, \dots, x_n]/I_C.$$

For any point $p \in C$ we have a maximal ideal $m_p \subset \mathcal{O}_C$ of functions vanishing at p . The localization of \mathcal{O}_C with respect to m_p is the local ring $\mathcal{O}_{C,p}$ of regular functions at p . Local ring is isomorphic to the localization of a corresponding coordinate ring localized with respect to m_p .

Definition 4.1.4. *Let $\mathcal{O}_{C,p}$ be a local ring, then its Krull dimension is defined as the maximal length of a strictly decreasing sequence of prime ideals in $\mathcal{O}_{C,p}$.*

Definition 4.1.5. *A point $p \in C$ is smooth if the dimension $\dim_{\mathbb{C}}(m/m^2)$ is equal to Krull dimension of the local ring $\mathcal{O}_{C,p}$. Otherwise we say that C is singular at x .*

We can also check the smoothness of a point $x \in C \subset \mathbb{C}^n$ using Jacobian criterion. Namely, the point is smooth if and only if there exists a function $f_i \in I_C$ such that the Jacobi matrix of derivatives

$$\left(\frac{\partial f_i}{\partial x_j} \right)$$

evaluated at x is of rank $n - r$ where $r = \dim C$.

Example 4.1.6. Consider a variety where

$$f = y^2 - x^3$$

then Jacobi matrix of derivatives at a point $(0, 0)$ is

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) (0, 0) = (0, 0)$$

having a rank not equal to $n - r$ for $r \in \mathbb{N}$.

Definition 4.1.7. Let $U \subset \mathbb{C}^{n+1}$ be an open subset containing a point x_0 . Let f be a regular function defined on U and vanishing at x_0 .

Let us consider the algebraic set

$$C = \{x \in U : f(x) = 0\}$$

We will assume that f generates the ideal I_C in the neighborhood of x_0 , call it U . That is $I_C = f \cdot \mathcal{O}_U$ and C is called a hypersurface.

For curves we have simple singularities:

$$\begin{aligned} A_n : x^2 + y^{n+1} &= 0 & (n \geq 1) \\ D_n : x^2 y + y^{n-1} &= 0 & (n \geq 4) \\ E_6 : x^3 + y^4 &= 0 \\ E_7 : x^3 + x y^3 &= 0 \\ E_8 : x^3 + y^5 &= 0 \end{aligned} \tag{4.1}$$

They have their counterparts in dimension 2 [IR94]: $A - D - E$ singularities of surfaces (Du Val singularities):

$$\begin{aligned} A_n : x^2 + y^{n+1} + z^2 &= 0 & (n \geq 1) \\ D_n : x^2 y + y^{n-1} + z^2 &= 0 & (n \geq 4) \\ E_6 : x^3 + y^4 + z^2 &= 0 \\ E_7 : x^3 + x y^3 + z^2 &= 0 \\ E_8 : x^3 + y^5 + z^2 &= 0 \end{aligned} \tag{4.2}$$

4.2 Finite subgroup of $SL(2, \mathbb{C})$

In this section, we follow [JC14].

Definition 4.2.1. *The special linear group of degree 2 over \mathbb{C} is the set of 2×2 matrices with determinant 1 with complex entries, denoted by:*

$$SL(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb = 1 \right\}.$$

We aim to list the finite subgroups $G \leq SL(2, \mathbb{C})$, and describe their representation.

Theorem 4.2.2. *Let G be a finite subgroup of $SL(2, \mathbb{C})$. Then G is one of the following cases (up to conjugacy):*

- a) a cyclic group, of the form $\mathbb{Z}/n\mathbb{Z}$, with $n \in \mathbb{N}$;
- b) a binary dihedral group, of the form $B\mathbb{D}_{4n}$, with $n \in \mathbb{N}$;
- c) a binary group corresponding to one of the Platonic solids, that is $B\mathbb{T}_{24}$, $B\mathbb{O}_{48}$ or $B\mathbb{I}_{120}$

This theorem characterizes any finite subgroup of $SL(2, \mathbb{C})$, and hence it gives the classification of them. We give the proof later step by step.

Definition 4.2.3. *The special unitary group of degree 2 over \mathbb{C} is the set of 2×2 unitary matrices with determinant 1 with complex entries, Observe it is a subgroup of $SL(2, \mathbb{C})$, denoted by:*

$$SU(2, \mathbb{C}) = \{U \in SL(2, \mathbb{C}) : UU^* = U^*U = I_2\},$$

where U^* denotes the conjugate transpose of U .

Definition 4.2.4. *The special orthogonal group of dimension 3 over \mathbb{R} is the subgroup of the orthogonal group $O(3)$ whose elements have determinant 1, denoted by*

$$SO(3) = \{R \in O(3) : \det R = 1\},$$

where

$$O(3) = \{Q \in GL(3, \mathbb{R}) : Q^T = Q^{-1}\}.$$

It is called the rotation group of \mathbb{R}^3 because its elements are rotations around an axis passing through the origin.

Having these definitions we can now state a theorem without proof.

Theorem 4.2.5. *There is a surjective group homomorphism*

$$\eta : SU(2, \mathbb{C}) \rightarrow SO(3),$$

with

$$\ker(\eta) = \{\pm I_2\}.$$

Proof. Now we can apply the first isomorphism theorem on groups, so $SU(2, \mathbb{C})/\ker(\pi) \cong im(\pi)$, but since it is surjective, $im(\pi) = SO(3)$, then $SU(2)/\pm I_2 \cong SO(3)$. And using the properties of the group homomorphisms, we get that the finite subgroups of $SU(2, \mathbb{C})/\{\pm I_2\}$ are the preimage under the natural projection (the double cover $SU(2) \rightarrow SO(3)$) of the finite subgroups of $SO(3)$ as desired. So let $G \leq SU(2, \mathbb{C})$ be finite, it defines $\bar{G} \leq SO(3)$ finite, and let $\bar{G} \leq SO(3)$ be finite, it can be lifted to $G \leq SU(2, \mathbb{C})$ finite with kernel of order ≤ 2 . \square

We will use the following notation in this section: $G < SU(2, \mathbb{C})$ is a finite subgroup, and $H = \eta(G)$ is its homomorphic image in $SO(3)$. Then either $H \simeq G$, which happens if $G \cap \{\pm I\} = I$, or else $H \simeq G/\{\pm I\}$. Capital letters A, B, \dots will denote elements of $SU(2, \mathbb{C})$, thought of as 2×2 matrices; small letters g, h, \dots will denote elements of $SO(3)$, thought of as geometric rotations. In particular, $I \in SU(2, \mathbb{C})$ and $id \in SO(3)$ will denote the identity elements. Now we can proof theorem 4.2.2.

a) The cyclic case.

Suppose that H is just a cyclic group. Then it is generated by rotation g by angle $2\pi/r$ around some axis u , and $H \simeq C_r$ is the corresponding cyclic group of order r . To find the corresponding subgroup of $SU(2)$, let us fix u to be X-axis. Then using the geometric description of η , leading to the subgroup of $SU(2, \mathbb{C})$ generated by the diagonal matrix

$$A = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}^k : \xi^r = 1, r \in \mathbb{N}, k = 1, \dots, r-1 \right\} = C_r, r \in \mathbb{N},$$

the n^{th} primitive root of unity is denoted by $\xi = \exp 2\pi i/r$, and for a fixed r this group is called the cyclic group of order r , denoted by C_r , or $\mathbb{Z}/r\mathbb{Z}$. Which generates $G \simeq C_r$ inside $SU(2, \mathbb{C})$, then G is a subgroup of $SU(2)$ of type A_{r-1} , for $r \geq 2$.

b) The dihedral case. Geometrically, it is fairly clear that we can extend any cyclic group of rotation by a further rotation h whose axis is orthogonal to the original one by a rotation by angle π only. In this way, we will continue to have a finite subgroup of $SO(3)$. This group H has order $2r$. Let us now find the corresponding subgroup of $SU(2, \mathbb{C})$. Choose g as before, with axis $u = (1, 0, 0)$. Choose h to have axis $v = (0, 1, 0)$, and angle $2\phi = \pi$. So all of these rotations are given by

$$A = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}^k : \xi^r = 1, r \in \mathbb{N}, k = 1, \dots, r-1 \right\},$$

the n^{th} primitive root of unity is denoted by $\xi = \exp 2\pi i/r$, Therefore G is generated by

$$\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, k = 1, \dots, r \text{ and } \begin{pmatrix} 0 & \xi \\ -\xi^{-1} & 0 \end{pmatrix},$$

with ξ being a n th primitive root of unit, which is equivalent to be generated by by $A = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ where ξ is a $2n$ th primitive root of unity:

$$G = \langle A, B : A^r = B^2; B^4 = I_2; BAB^{-1} = A^{-1} \rangle,$$

for every $r \in \mathbb{N}$. The order of G is $|G| = 4r$. The group G of this type is called the binary dihedral group of type D_{r+2} , for $r \geq 2$, of order $4r$, denoted by $B\mathbb{D}_{4r}$

c) The exceptional cases.

It turns out that, up to conjugation, there are only three further finite subgroups in $SO(3)$. It can be seen in [JC14] that corresponding subgroup of $SU(2, \mathbb{C})$ can be generated by the following triple of matrices. This subgroups are results on study of rotations groups of the Platonic solids: the groups of rotations of the tetrahedron, octahedron and icosahedron.

(a) For the matrices

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, C = \frac{1}{2} \begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}.$$

We have

$$G_1 = \langle A, B, C : A^4 = (AC)^3 = (BC)^3 = I_2, A^2 = B^2 = (AB)^2 = C^3 \rangle.$$

This group is called the binary tetrahedral group.

(b) For the matrices

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, C = \frac{1}{2} \begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}, D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix},$$

then

$$G_2 = \langle B, C, D : D^4 = B^2 = (D^2B)^2 = C^3 = -I_2, (D^2C)^3 = (BC)^3 = I_2, D^8 = I_2 \rangle.$$

This group is called the binary octahedral group of order 48.

(c) Lastly for the matrices,

$$A = \begin{pmatrix} \xi^3 & 0 \\ 0 & \xi^2 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, C = \frac{1}{\sqrt{5}} \begin{pmatrix} -\xi + \xi^4 & \xi^2 - \xi^3 \\ \xi^2 - \xi^3 & \xi - \xi^4 \end{pmatrix},$$

with ξ a fifth primitive root of unity:

$$G_3 = \langle A, B, C, : C^2 = -I_2 = (AC)^3 = (BC)^3, A^5 = I_2 = B^4 \rangle.$$

The order of G is $|G| = 120$. This group is called the binary icosahedral group.

The subgroups G_1, G_2, G_3 of $SU(2, \mathbb{C})$, mapping to the groups H_1, H_2, H_3 of symmetries of the regular solid-pairs in $SO(3)$, is that they are the subgroups of $SU(2, \mathbb{C})$ of type E_6, E_7, E_8 .

4.2.1 Quotient singularities

Let G be a finite group of algebraic local automorphisms of \mathbb{C}^n at 0. Following (Cartan 1957) the action of G can be linearized, that is, in terms of a new coordinate system for $(\mathbb{C}^n, 0)$, G acts linearly. Thus we are not restricted to assume that $G \leq GL(n, \mathbb{C})$. Then G acts also on the ring of polynomials $\mathbb{C}[x_1, \dots, x_n]$ by

$$(g \circ f)(x_1, \dots, x_n) = f(g(x_1, \dots, x_n)),$$

for $g \in G \leq GL(n, \mathbb{C})$ and f a polynomial function.

Then G -invariant polynomials i.e. the polynomials $f \in \mathbb{C}[x, y]$ such that $g \cdot f = f$ for all $g \in G$ form a homogeneous sub algebra

$$\mathbb{C}[x_1, \dots, x_n]^G \subset \mathbb{C}[x_1, \dots, x_n],$$

that is finitely generated and closed. Thus we can choose a homogeneous generators ϕ_1, \dots, ϕ_N of $\mathbb{C}[x_1, \dots, x_n]^G$ of positive degree and define a polynomial map

$$(\phi_1, \dots, \phi_N): \mathbb{C}^n \rightarrow \mathbb{C}^N$$

whose image is the orbit space \mathbb{C}^n/G .

Then ϕ_1, \dots, ϕ_N are constant on the G -orbit and thus factor through the orbit space by the mapping

$$(\phi_1, \dots, \phi_N): \mathbb{C}^n/G \rightarrow \mathbb{C}^N.$$

We can say that the orbit space \mathbb{C}^n/G is an affine variety whose algebra of regular functions is $\mathbb{C}[x_1, \dots, x_n]^G$. Then \mathbb{C}^n/G at $G \cdot 0$ is called a quotient singularity.

Definition 4.2.6. *A non unit $g \in GL(n, \mathbb{C})$ is called complex reflection if it is of finite order and leaves hyperplane pointwise fixed.*

Remark 4.2.7. *\mathbb{C}^n/G is isomorphic to \mathbb{C}^n if and only if $\mathbb{C}[x_1, \dots, x_n]^G$ is a polynomial algebra. Chevalley (1955) gives a beautiful characterization of G , that is, $\mathbb{C}[x_1, \dots, x_n]^G$ is a polynomial algebra if and only if G is generated by complex reflections $g \in GL(n, \mathbb{C})$*

This reduces the study of quotient singularities to case where G contains no complex reflection. Thus we will consider G to be a finite subgroup of $SL(2, \mathbb{C})$.

- 1) Let m be a positive integer and consider a cyclic subgroup $G \leq SL(2, \mathbb{C})$ of order m generated by a diagonal matrix

$$G = \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} : \zeta^m = 1 \right\},$$

then the algebra of polynomial in 2 variables invariant under G posses three generators therefore $\mathbb{C}[x, y]^G$ is generated by

$$t_1 = xy, t_2 = x^m, t_3 = y^m,$$

using this as coordinate of

$$\phi: \mathbb{C}^2 \rightarrow \mathbb{C}^3$$

then the image of ϕ is in a hypersurface \mathbb{C}^3 defined by

$$t_1^m - t_2 t_3 = 0.$$

The germ $(\mathbb{C}^n/G, 0)$ is isomorphic to hypersurface singularity define by $t_1^m - t_2 t_3 = 0$, which is a singularity of type A_n for $n = m - 1$.

Example 4.2.8. We take a singularity A_2 as quotient singularity by consider the group $G = \mathbb{Z}/2$ acting on \mathbb{C}^2 by $(x, y) \mapsto (t_1, t_2, t_3)$. The quadratic monomials

$$\{x^2, xy, y^2\}$$

are G -invariant functions on \mathbb{C}^2 , so that the map $\mathbb{C}^2 \rightarrow \mathbb{C}^3$ given by

$$t_1 = xy, t_2 = x^2, t_3 = y^2$$

identifies the quotient space \mathbb{C}^2/G with

$$t_2 t_3 - t_1^2 = 0 \subset \mathbb{C}^3.$$

- 2) Considering G to be a dihedral group, we use result by Klein in 1884 [GT69]. Let $2 \leq p \leq q \leq r$ be integer such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1,$$

so that $(p, q, r) = ((2, 2, r), (2, 3, 3), (2, 3, 4), \text{ or } (2, 3, 5))$.

According to spherical geometry there exists a spherical triangle Δ on the unit sphere S^2 whose angles are $\pi/p, \pi/q, \pi/r$. Orthogonal reflection in the sides of Δ generates a subgroup Σ of the orthogonal group $O_3(\mathbb{C})$ which has Δ as a fundamental domain. Then $G.\Delta = S^2$ and the Δ is obtained by spherical projection of the shaded region.

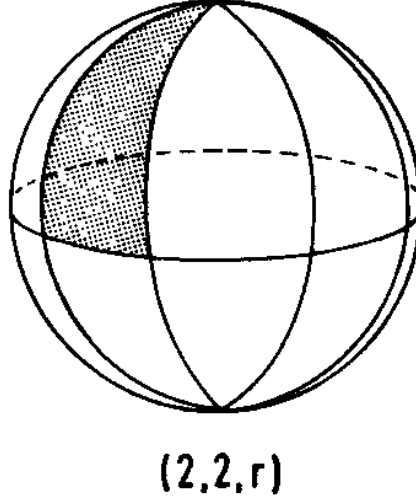


Figure 4.1: spherical polyhedral [EL83]

We consider a subgroup $\Sigma_+ := \Sigma \cap SO_3$. $SU(2, \mathbb{C})$ naturally acts on \mathbb{C}^2 and thereby on the complex projective line $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. The spherical projection defines an isomorphism between \mathbb{P}^1 and the unit sphere S^2 of \mathbb{C}^3 , which defines a homomorphism

$$\eta: SU(2, \mathbb{C}) \rightarrow SO_3$$

of lie groups. since both are compact connected and have $\dim 3$, then η is a double- fold covering. We take $G := \eta^{-1}(\Sigma_+)$. As a result we obtain Singularities of type D_n and E_n .

Lets us consider the case of $(p, q, r) = (2, 2, n)$ where $n \geq 4$ and a dihedral subgroup G of $GL(2, \mathbb{C})$ of order $2n$ generated by

$$\left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} : \zeta^{2n} = 1 \right\}, \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The first generator gives the invariants

$$t_1 = xy, \quad t_2 = x^n, \quad t_3 = y^n,$$

and the second has the invariants

$$t_1 = x^2 - y^2, \quad t_2 = xy(x^2 + y^2), \quad t_3 = x^2y^2.$$

We get G-invariants sub algebra of $\mathbb{C}[x, y]$ is generated by

$$t_1 = x^2y^2, \quad t_2 = x^{2n} + y^{2n}, \quad t_3 = xy^{2n+1} - x^{2n+1}y.$$

They span the whole of $\mathbb{C}[x, y]^G$ and satisfy the relation

$$t_3^2 - t_1 t_2^2 = 4(-t_1)^{n-1}.$$

Substituting $-t_1$ for t_1 , $2t_2$ for t_2 and $2t_3$ for t_3 we obtain the equation of type D_n which is

$$t_1^{n-1} + t_1 t_2^2 + t_3^2 = 0.$$

Example 4.2.9. We consider a singularity D_4 as a quotient singularity. Consider \mathbb{C}^2 with coordinates x, y and a binary dihedral group G of $|G| = 16$ generated by

$$\rho: x, y \mapsto ix, -iy,$$

and

$$\tau: x, y \mapsto y, -x.$$

Thus $\rho^2 = \tau^2 = -1$. Therefore

$$x, y \mapsto \begin{cases} t_3 = (x^4 - y^4)xy \\ t_2 = (x^4 + y^4) \\ t_1 = x^2 y^2. \end{cases}$$

defines a G -invariant map $\mathbb{C}^2 \rightarrow \mathbb{C}^3$, and that the image is the singular surface $X \subset \mathbb{C}^3$ defined by $4t_1^3 + t_1 t_2^2 + t_3^2$, which is D_4 up to a change of coordinates.

For the case $(p, q, r) = (2, 3, 3)$, $(2, 3, 4)$, or $(2, 3, 5)$.) The relations between generators of polynomial algebras yields equations of type E_6 , E_7 and E_8 . For each such group G , the algebra of invariants polynomial has three generators related to a single equation. Thus \mathbb{C}^2/G can be realized as a space \mathbb{C}^3 defined by a single equations. These equations are isomorphic to equations of Simple surface singularities also known as Du Val singularities [IR94].

G	(p, q, r)	equations	notation
\mathbb{D}_{4r}	$(2, 2, r \geq 2)$	$t_1^{n-1} + t_1 t_2^2 + t_3^2 = 0$	D_{r+2}
$B\mathbb{T}$	$(2, 3, 3)$	$t_1^4 + t_2^3 + t_3^2 = 0$	E_6
$B\mathbb{O}$	$(2, 3, 4)$	$t_1^3 t_2 + t_2^3 + t_3^2 = 0$	E_7
$B\mathbb{I}$	$(2, 3, 5)$	$t_1^5 + t_2^3 + t_3^2 = 0$	E_8

Table 4.1: Equations of Simple surface singularities.

Chapter 5

McKay correspondence

5.1 Representation theory

Fix a finite group G and a finite \mathbb{C} -vector space V . Denote $GL(V)$ the set of all invertible linear transformations of V to itself, called the general linear group of V .

Definition 5.1.1. *A representation of G over \mathbb{C} , is a group homomorphism*

$$\rho: G \rightarrow GL(V),$$

$$\rho(g)(v) := gv.$$

Equivalently, it is a structure on V of a left module over the group algebra $\mathbb{C}[G]$. Recall that $\mathbb{C}[G]$ is the linear space of functions on G with values in \mathbb{C} with the multiplication law defined by

$$(\phi * \psi)(x) = \sum \phi(g)\psi(\tilde{g}).$$

We call the dimension of V the degree of our ρ .

Let us recall some standard facts about representations of finite groups.

Theorem 5.1.2. *Assume that $|G|$ is coprime to the characteristic of \mathbb{C} . Every linear representation is isomorphic to a direct sum of irreducible subrepresentations.*

Proof. The space of linear maps $L(V, W)$ has a natural structure of a $\mathbb{C}[G]$ -module via

$$(g \circ f)(x) = g \cdot f(g^{-1} \cdot x).$$

We have

$$L(V, W)^G = \text{Hom}_{\mathbb{C}[G]}(V, W).$$

Let W be a $\mathbb{C}[G]$ -submodule of V and $p: V \rightarrow W$ be a projection operator. Let

$$\tilde{p} = \frac{1}{|G|} \sum_{g \in G} g \cdot p.$$

This is the standard averaging operation. It gives $g\tilde{p} = \tilde{p}$. Also, for any $w \in W$,

$$\tilde{p}(w) = \frac{1}{|G|} \sum_{g \in G} g \cdot p(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} g(g^{-1}w) = \frac{1}{|G|} |G|w = w.$$

Thus the kernel of \tilde{p} is a $\mathbb{C}[G]$ -submodule and $V = W \oplus K$ as $\mathbb{C}[G]$ -modules. Starting from an irreducible submodule, we find the complementary submodule, and proceed by induction on the dimension of V . \square

Lemma 5.1.3. *Let $f: V \rightarrow W$ be a nonzero homomorphism of irreducible representations. Then f is the composition of an isomorphism $\phi: V \rightarrow W$ and a scalar endomorphism $c \text{id}_V$.*

Proof. The image $f(V)$ is a submodule of W , and the kernel $\text{Ker}(f)$ is a submodule of V . Since V and W are irreducible, none of them is a proper submodule. Since f is nonzero, $\text{Ker}(f) = 0$ and $f(V) = W$. Thus f is an isomorphism. Obviously we may assume that $V = W$. Let c be an eigenvalue of f . The map $f - c \text{id}_V \in \text{Hom}_{\mathbb{C}[G]}(V, V)$ and has non-trivial kernel. Since V is irreducible, the kernel is equal to V . Thus $f - c \text{id}_V$ is the zero map. \square

corollary 5.1.4. *Let $\rho: G \rightarrow GL(V)$ be a linear irreducible representation. Then the image of the center of G is contained in the center of $GL(V)$.*

Proof. Let z be an element of the center of G . For any $g \in G$ we have

$$\rho(z) = \rho(g \cdot z \cdot g^{-1}) = \rho(g) \circ \rho(z) \circ \rho(g)^{-1}.$$

Thus $\rho(z): V \rightarrow V$ is an automorphism of the representation ρ . By Schurs Lemma, it must be a scalar automorphism, i.e. an element of the center of $GL(V)$. \square

Definition 5.1.5. *The character of a representation ρ is the function $\chi_\rho: G \rightarrow \mathbb{C}$ defined by;*

$$\chi_\rho(g) = \text{Tr}(\rho(g)).$$

Where Tr denotes the trace of the matrix $\rho(g)$ representing $g \in G$.

It is a central function on G , i.e. constant on conjugacy classes of G . We will identify a central function on G with a function on the set $C(G)$ of conjugacy classes of G . Thus a character is a special function in the linear space of central functions. A character of an irreducible representation is called an irreducible character.

Define a hermitian inner product on the space of central functions $\mathbb{C}C(G)$ by

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)},$$

where the over line denotes the complex conjugate. Obviously,

$$\langle \phi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} |\phi(g)|^2 > 0.$$

Thus the inner product is a unitary product on the space of central functions $\mathbb{C}C(G)$.

corollary 5.1.6. *The number of non-isomorphic irreducible representations is equal to the number of conjugacy classes of G .*

Proof. Let $Ir(G) = \{\rho_1, \dots, \rho_c\}$ and $\{\chi_1, \dots, \chi_c\}$ be the set of the corresponding irreducible characters. Let ρ be a linear representation. We know that

$$\rho \cong \bigoplus_{i=1}^c \rho_i^{\oplus m_i},$$

where m_i are non-negative integers. The corresponding element $[\rho] \in R(G)$ can be written in the form

$$[\rho] = \sum_{i=1}^c m_i [\rho_i].$$

The number m_i is called the multiplicity of ρ_i in ρ and is denoted by $mult_{\rho_i} \rho$. It is clear that

$$\dim \rho = \sum_{i=1}^c mult_{\rho_i} \rho \dim \rho_i.$$

Taking the characters, we get

$$\chi_\rho = \sum_{i=1}^c m_i \chi_i.$$

Since $(\chi_i)_{i=1, \dots, c}$ is an orthonormal basis, we obtain

$$m_i = \langle \chi_\rho, \chi_i \rangle.$$

□

corollary 5.1.7. *Let n_1, \dots, n_c be the dimensions of irreducible representations of G . Then*

$$|G| = n_1^2 + \dots + n_c^2.$$

5.2 The MacKay correspondence

Let G be a finite subgroup of $SU(2, \mathbb{C})$, ρ be the faithful representation of G obtained from the embedding $G \hookrightarrow SU(2, \mathbb{C})$. and let $\{\rho_i\}$ be the irreducible representations of G .

Definition 5.2.1. *McKay quiver of G is a directed multi-graph with vertices ρ_i and m_{ij} edges from ρ_i to ρ_j if ρ_j occurs m_{ij} times in the decomposition of $\rho_i \oplus \rho_{nat}$ into irreducible. In other words, we have*

$$\rho_i \oplus \rho_{nat} = \bigoplus_j \rho_j^{\oplus m_{ij}}.$$

Its vertices correspond to irreducible representations ρ_i of G . We put a label over the vertex to indicate the dimension of the representation. A vertex ρ_i is connected to the vertex ρ_j by an edge pointing to ρ_j if ρ_j is a direct summand of $\rho_{nat} \oplus \rho_i$. We put the label m_{ij} over this edge if

$$\langle \chi_{nat} \chi_i, \chi_j \rangle = m_{ij}.$$

Note that if $G \subset SL(2, \mathbb{C})$ then $m_{ij} = m_{ji}$. The McKay correspondence classifies the possible groups G via their MacKay quivers.

Definition 5.2.2. The MacKay graph of G is defined to be the graph consisting of vertices $v(\rho)$ for $\rho \in \text{Irr}G$, and simple edges connecting any pair of vertices ρ_i and ρ_j with $m_{ij} = 1$.

Example 5.2.3. Let $G = C_n$ be a cyclic group of order n . Every linear representation $\rho : C_n \rightarrow GL(V)$ decomposes into the direct sum of one dimensional representation. So, G has n irreducible representations of dimension 1. Let ρ_k be defined by sending g to $\exp 2\pi ik/n, k \in \mathbb{Z}/n\mathbb{Z}$. Take $\rho_0 = \rho_1$. Obviously $\rho_1 \otimes \rho_k = \rho_{k+1}$, Thus the McKay graph $\Gamma(C_n, \rho_1)$ is equal to the graph \tilde{A}_{n-1} with additional orientation by giving arrows all pointing in one direction. On the other hand, if we consider the representation $\rho_0 : C_n \rightarrow SL(2, \mathbb{C})$ given by the matrix

$$\begin{pmatrix} \xi_n & 0 \\ 0 & \xi_n^{-1} \end{pmatrix}.$$

Thus $\rho_0 \otimes \rho_k = \rho_{k-1} + \rho_{k+1}$. This gives us the Dynkin diagram of type \tilde{A}_{n-1} .

Example 5.2.4. Let $G = \mathbb{D}_3$ be a binary dihedral group. As we already know, $|\mathbb{D}_3| = 12$. The group \mathbb{D}_3 has two generators A, B which satisfy the following relations:

$$G = \langle A, B : A^3 = B^2, B^4 = I_2; BAB^{-1} = A^{-1} \rangle.$$

The group \mathbb{D}_3 has 4 1-dimensional representations

$$A = 1, B = 1; A = 1, B = -1; A = 1, B = i \text{ and } A = 1, B = -i.$$

The natural representation is also known, it is just

$$A = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

There is also another one irreducible 2-dimensional representation:

$$A = \begin{pmatrix} \cos \frac{2\pi}{3} & i \sin \frac{2\pi}{3} \\ i \sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We have found all 6 indecomposable representations of G . We can construct the character table.

ρ	$Tr(\rho)$	$\chi(A)$	$\chi(B)$	dim
ρ_0	χ_0	1	1	1
ρ_1	χ_1	1	-1	1
ρ_2	χ_2	1	0	2
ρ_3	χ_3	-1	0	2
ρ_4	χ_4	-1	i	1
ρ_5	χ_5	-1	$-i$	1

Table 5.1: Character table of \mathbb{D}_3 (of type D_5).

Then from Table we see that

$$\chi_{nat}^2 = \chi_0 + \chi_1 + \chi_3,$$

General representation theory says that an irreducible representation of G is uniquely determined up to equivalence by its character. Therefore $\rho_2 \otimes \rho_{nat} = \rho_0 + \rho_1 + \rho_3$. Hence $m_{ij} = 1$ for $j = 0, 1, 3$ and $m_{ij} = 0$ for $j = 2, 4, 5$. We also observe that,

$$\chi_0 \chi_{nat} = \chi_2, \chi_1 \chi_{nat} = \chi_2,$$

$$\chi_3 \chi_{nat} = \chi_2 + \chi_4 + \chi_5,$$

$$\chi_4 \chi_{nat} = \chi_3, \chi_5 \chi_{nat} = \chi_3.$$

We get the McKay graph of $|\mathbb{D}_3|$

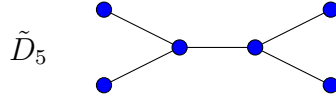


Figure 5.1: $\Gamma(\mathbb{D}_3)$

We have just obtained the dual graph of the D_5 -singularity. Note that the fundamental cycle of the D_5 -singularity. The coefficients of this decomposition are the same as the dimensions of the representations corresponding to the vertices's of the McKay quiver.

Example 5.2.5. Let us check the case of the binary tetrahedral group. let $T \leq SO(3)$, we know that $T \cong A_4$ of even permutations of a set $\{1, 2, 3, 4\}$. Let $\rho_i, i = 1, 2, 3$ be one-dimensional irreducible representations of G obtained as the compositions $G \rightarrow A_4$. The first one is the trivial representation. Let ρ_4 be the 3-dimensional representation obtained as the composition $G \rightarrow SO(3)$. It is easy to see that this representation is irreducible. Let $\rho_0 = \rho_5$ be the natural representation of G in $SU(2, \mathbb{C})$ and $\rho_6 = \rho_0 \otimes \rho_i, i = 2, 3$. These are 2-dimensional representations. Since ρ_0 is irreducible, it is easy to see that these representations are also irreducible. Now

$$|G| = 24 = 1 + 1 + 1 + 2^2 + 2^2 + 2^2 + 3^3,$$

so all irreducible representations are accounted for. This agrees with the number of conjugacy classes of G . The group A_4 has one conjugacy class of the identity, one class of elements of order 2, and two classes of elements of order 3. We will denote the irreducible character corresponding to the irreducible representation ρ_i by χ_i . We now turn to finding the conjugacy classes of G and in total they must 7 of them. Every conjugacy class of T is lifted through $\eta : G \rightarrow T$ to either two conjugacy classes with the same order or one conjugacy class with twice the order of the original conjugacy class. Since 1 is lifted to 1 and -1 and each conjugacy class of elements of order 2 is lifted to two conjugacy classes of elements of order 3 and 6. Thus $|C(G)| = 7$. We already know that

$$\rho_0 \otimes \rho_i = \rho_{4+i}, i = 1, 2, 3.$$

ρ	$Tr(\rho)$	c_1	c_2	c_3	c_4	c_5	c_6	c_7
ρ_0	χ_1	1	1	1	1	1	1	1
ρ_1	χ_2	1	-1	1	ξ_3	ξ_3^2	ξ_3	ξ_3^2
ρ_2	χ_3	1	0	2	ξ_3^2	ξ_3	ξ_3^2	ξ_3
ρ_3	χ_4	3	3	-1	0	0	0	0
ρ_4	χ_5	2	-2	0	-1	-1	1	1
ρ_5	χ_6	2	-2	0	$-\xi_3$	$-\xi_3^2$	ξ_3	ξ_3^2
ρ_6	χ_7	2	-2	0	$-\xi_3^2$	$-\xi_3$	ξ_3^2	ξ_3

Table 5.2: Character table of $B\mathbb{T}_{24}$.

We have

$$\chi_5\chi_4 = \chi_5 + \chi_6 + \chi_7.$$

This gives $m_{45} = m_{46} = m_{47} = 1$. We have

$$\chi_5\chi_5 = \chi_4 + \chi_1,$$

$$\chi_5\chi_6 = \chi_4 + \chi_2,$$

$$\chi_5\chi_7 = \chi_4 + \chi_3.$$

This gives $m_{54} = m_{51} = m_{62} = m_{64} = m_{73} = m_{74} = 1$. Getting all edges together we get the McKay graph for (G, ρ_5)

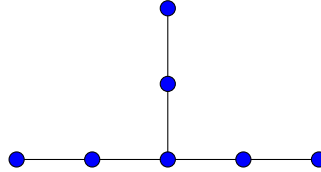


Figure 5.2: $\Gamma(B\mathbb{T}_{24})$.

Theorem 5.2.6. *Let G be a finite subgroup of $SU(2, \mathbb{C})$ and ρ_{nat} be its natural 2-dimensional representation defined by the inclusion $G \leq SU(2, \mathbb{C})$. Then the McKay graph of (G, ρ) is the following extended Dynkin diagram.*

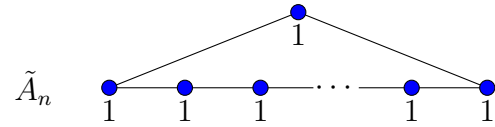


Figure 5.3: If G is cyclic of order n

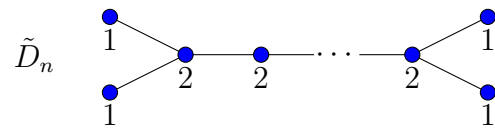


Figure 5.4: If G is binary dihedral of order $2n$, $n \geq 4$.

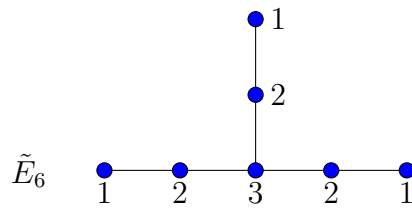


Figure 5.5: If G is binary tetrahedral group of order 24.

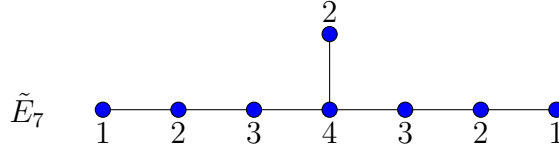


Figure 5.6: If G is binary octahedral group of order 48.

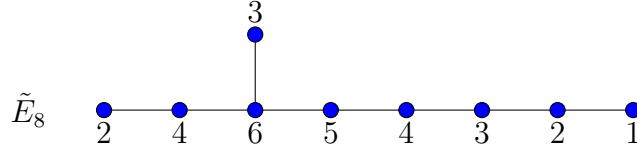


Figure 5.7: If G is binary icosahedral group of order 120.

This diagrams are the extended Dynkin diagrams of type A_n, D_n, E_6, E_7, E_8 . Then G is isomorphic to a finite subgroup of $SU(2, \mathbb{C})$ and ρ is its natural 2-dimensional representation.

5.3 Resolutions and dual graphs of quotient singularities

We want now to answer our next question: what are minimal resolutions and dual graphs of these singularities?

Definition 5.3.1. *The blowup surface \tilde{X} is the set of all pairs consisting of a point p in \mathbb{C}^n and a line from p through the origin of \mathbb{C}^n . Note that \tilde{X} is in the product space $\mathbb{C}^n \times \mathbb{P}^{n-1}$. Thus we have*

$$\tilde{X} = (p, \bar{p}) | p \in \mathbb{C}^n \subset \mathbb{C}^n \times \mathbb{P}^{n-1}.$$

Therefore the resolution of a singularity $X \in \mathbb{C}^n$ at 0 is the surface \tilde{X} together with the projection map,

$$\pi: \tilde{X} \rightarrow X$$

Thus, π is an isomorphism on the open set $\mathbb{C}^n - \{0\}$. The preimage of O is $\{0\} \times \mathbb{P}^{n-1}$, which is isomorphic to \mathbb{P}^{n-1} .

Definition 5.3.2. *The set $\pi^{-1}(O)$ is called the exceptional divisor of the blow up.*

Each line through the origin intersects the exceptional divisor at exactly one point, and hence the blown-up curve intersects the exceptional divisor exactly once for every time the original curve passes through the origin.

Example 5.3.3. *Consider the singularity at the origin of the $X = x^2 + y^2 - z^2$. The resolution \tilde{X} is a subset of $\mathbb{C}^3 \times \mathbb{P}^2$. If \mathbb{C}^3 has coordinates (x, y, z) and \mathbb{P}^2 has coordinates $[t : u : v]$, then \tilde{X} has the following defining equations:*

$$\{((x, y, z), (t : u : v)) \in \mathbb{C}^3 \times \mathbb{P}^2 | xu = ty, xv = ty, yv = zu\}.$$

$$\begin{aligned}xu - ty &= 0, & x^2 + y^2 &= z^2, \\ yv - uz &= 0, & t^2 + u^2 &= v^2, \\ xv - ty &= 0.\end{aligned}$$

On the coordinate chart $v = 1$ the equations reduce to the following:

$$y = uz, \quad x = ty, \quad x^2 + y^2 = z^2.$$

After eliminating the variables x and y , the set of points in \mathbb{C}^3 satisfying the three equations above is isomorphic to the cylinder $\{(z, t, u) \mid t^2 + u^2 = 1\}$,

Let $(X, 0)$ be a germ of a simple singularity, $\pi: \tilde{X} \rightarrow X$ its resolution, $E = \bigcup_{i=1}^n E_i = \pi^{-1}(0)$ and E_i for $1 \leq i \leq r$ the irreducible component of E . It is known that $E_i: = \mathbb{P}^1$ with self-intersection -2 .

Definition 5.3.4. Let E_1, \dots, E_n be the irreducible components of a divisor E . The symmetric matrix

$$A = (a_{ij})_{1 \leq i, j \leq n}$$

where $a_{ij} = E_i \cdot E_j$ is called the intersection matrix of the divisor E .

Remark 5.3.5. Let G finite subgroup of $SL(2, \mathbb{C})$. Let X be the surface defined by G and let E be the exceptional curve of the minimal resolution of x . We can assign an intersection matrix I_G to G .

We associate a vertex v_i to any irreducible component E_i of E , and join two vertices v_i and v_j if and only if $(E_i, E_j) = 1$. Thus we have a finite graph with simple edges. We call this graph the dual graph of E , and denote it by $\Gamma(\text{irr } E)$.

Definition 5.3.6. The minimal nonzero effective unique divisor of \tilde{X} denoted by E_{fund} such that $E_{fund}E_i \leq 0$ for all i is called the fundamental cycle of \tilde{X} .

Let $E_{fund}: = \sum_{i=1}^r E_i$ and $E_0: = -E_{fund}$. For the simple singularities, we have $E_0E_i = 0$ or 1 for any $E_i \in \text{irr } E$ (except for the case A_1 , when $E_0E_1 = 2$). Therefore we can draw a new graph $\tilde{\Gamma}$ after considering the vertex v_0 . In this case we denote $\text{irr } E \cup \{E_0\}$ by $\text{irr}^* E$. For a quotient singularity $(\mathbb{C}^2/G, 0)$, we have a Dynkin diagram as a dual graph $\Gamma(\mathbb{C}^2/G, 0)$ and $\tilde{\Gamma}(\mathbb{C}^2/G, 0)$.

Example 5.3.7. In the D_5 case, we have

$$E = E_1 + E_2 + E_3 + E_4 + E_5$$

with

$$E_i^2 = -2$$

and

$$-E_0 = E_{fund} = E_1 + 2E_2 + 2E_3 + E_4 + E_5.$$

Then

$$E_0E_2 = E_1E_2 = E_2E_3 = E_3E_4 = E_3E_5 = 1,$$

and all other $E_iE_j = 0$.

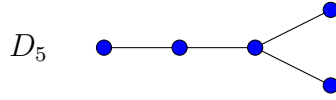


Figure 5.8: $\Gamma(\mathbb{C}^2/G, 0)$.

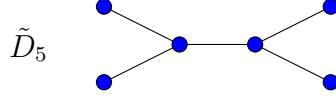


Figure 5.9: $\tilde{\Gamma}(\mathbb{C}^2/G, 0)$.

Observe that we obtained the dual graph of the D_5 -singularity. Note that the fundamental cycle of the D_5 -singularity is

$$E_{fund} = E_1 + 2E_2 + 2E_3 + E_4 + E_5.$$

Let us now consider E_6 -singularity $X = (x^2 + y^3 + z^4) \subset \mathbb{C}^3$. The calculation will be rather tedious as we will need multiple blowups, however we still carry through with it since it nicely illustrates how to handle more complicated singularities.

Let $\pi: \tilde{X} \rightarrow X$ be a minimal resolution, $E = \cup E_i = \pi^{-1}(0)$ the exceptional divisor. Then considering the blow-up of \mathbb{C}^3

$$B = \{((x, y, z), (u : v : w)) \in \mathbb{C}^3 \times \mathbb{P}^2 \mid xv = yu, xw = zu, yw = zv\},$$

and the first chart $v \neq 0$ (i.e. $v = 1$). We get equations

$$x = yu, \quad y = y, \quad z = yw.$$

To get the equation of the strict transform of \tilde{X} , we assume that $y \neq 0$ and

$$y^2 u^2 + y^3 + y^4 w^4 = 0, \quad \text{or} \quad u^2 + y + y^2 w^4 = 0.$$

In this chart \tilde{X}_i $i \in \mathbb{Z}$ is smooth: Jacobi criterium implies system has no solutions. In the chart $u = 1$ the strict transform \tilde{X}_i is again smooth. Consider finally the chart $w = 1$.

$$x = zu, \quad y = zv, \quad z = z.$$

The strict transform is

$$z^2 u^2 + z^3 v^3 + z^4 = 0, \quad \text{or} \quad u^2 + zv^3 + z^2 = 0.$$

Jacobi criterion implies that this surface has a unique singular point $u = 0, v = 0, z = 0$, or in the global coordinates $((0, 0, 0), (0 : 0 : 1))$. We see that this point indeed lies only in one of three charts of \tilde{X}_1 .

The exceptional fiber is by the definition the intersection $\tilde{X}_1 \cap \{((0, 0, 0), (u : v : w))\}$. To get its equation in the chart $w = 1$ we just have to set $z = 0$ in the equation of \tilde{X}_1 . $z = 0$ implies $u = 0$. Hence we get

$$E_0 = \{((0, 0, 0), (0 : v : 1))\} \cong \mathbb{C}^1.$$

Going to the other charts shows that $E_0 \cong \mathbb{P}^1$. Finally, the function f in this chart gets the form $f = zu$.

We have the following situation:

$$\begin{cases} x^2 + zy^3 + z^2 = 0 \\ E_0 \quad x = 0, \quad z = 0. \end{cases}$$

Consider again the blowing-up of this surface. We find that the interesting chart is

$$x = yu, \quad y = y, \quad z = yw,$$

with strict transform

$$y^2u^2 + yvy^3 + y^2v^2 = 0, y \neq 0, \quad \text{or} \quad u^2 + y^2v + v^2 = 0.$$

The exceptional fibre of this blowing-up has two irreducible components: $y = 0$ implies $u \pm iv = 0$ (we call this components E'_1 and E''_1). We have the following situation:

$$\begin{cases} x^2 + zy^2 + z^2 = 0 \\ E_0 \quad x = 0, \quad z = 0 \\ E_1 \quad y = 0 \quad x \pm iz = 0. \end{cases}$$

Consider again the blowing-up of this surface, we find that the interesting chart is

$$x = yu, \quad y = y, \quad z = yw,$$

with strict transform

$$y^2u^2 + yvy^2 + y^2v^2 = 0, y \neq 0, \quad \text{or} \quad u^2 + yv + v^2 = 0.$$

The exceptional fiber consists again of two irreducible components E'_2, E''_2 .

We now obtain the equation of preimage of E_0 by considering the chart

$$x = x \quad y = xu \quad z = xv,$$

we obtain the strict transform $1 + xu^2v + v^2$, for $x \neq 0$. So the equation of exceptional fiber E_2 in this chart are $x = 0$ and $v = \pm i$. Therefore the preimage of E_1 is given by $x \pm ixv = 0 \quad xu = 0$, thus we have 2 more coordinate charts,

$$\left\{ \begin{array}{l} x^2 + zy + z^2 = 0 \\ E_0 \quad x = 0, \quad z = 0 \\ E_2 \quad y = 0 \quad x \pm iz = 0. \end{array} \right. \quad \left\{ \begin{array}{l} 1 + xy^2z + z^2 = 0 \\ E_1 \quad y = 0, \quad z \pm i = 0 \\ E_2 \quad x = 0 \quad z \pm i = 0. \end{array} \right. \quad \text{Our}$$

next step is the blowing-up at the point $(0, 0, 0)$ in the different coordinate charts. To obtain equations of the preimages of E_0 and E_2 we have to consider two coordinate charts

$$x = yu, \quad y = y, \quad z = yw.$$

The strict transform is a cylinder $u^2 + v + v^2 = 0$. The preimage of E_0 is given by equations $u = 0, v = 0$, the exceptional fibre E_3 is given by $u^2 + v + v^2 = 0, y = 0$.

In the chart

$$x = x \quad y = ux \quad z = yw,$$

we obtain a strict transform $1 + uv + v^2 = 0$ The exceptional fiber E_3 is given by $1 + uv + v^2 = 0, x = 0$, the preimages of E'_2 and E''_2 are given by

$$u = 0, v = \pm i.$$

Hence our exceptional fiber E is given by the following configuration of projective lines:

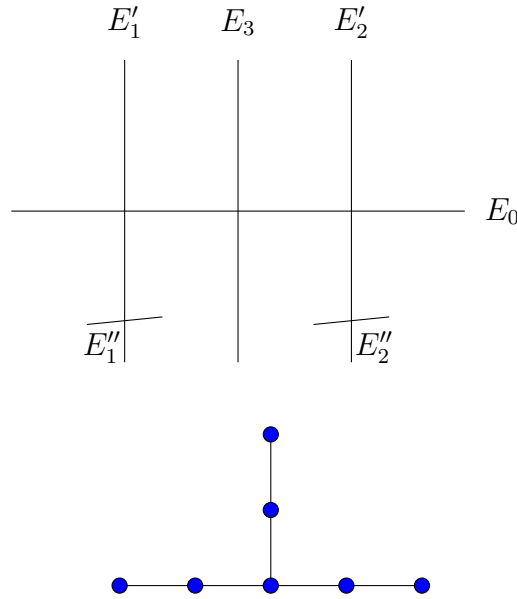


Figure 5.10: Dual graph for a minimum resolution of E_6 singularity.

We have to take into account three coordinate charts of a minimal resolution.

$$\left\{ \begin{array}{l} \tilde{X}_i : x^2 + z + z^2 = 0 \\ E_0 : x = 0, \quad z = 0 \\ E_3 : y = 0 \quad x^2 + yz + z^3 = 0. \end{array} \right. \quad \left\{ \begin{array}{l} \tilde{X}_i : 1 + yz + z^2 = 0 \\ E_3 : y = 0, \quad x^2 + yz + z^2 = 0 \\ E_2 : y = 0 \quad z \pm i = 0. \end{array} \right.$$

$$\begin{cases} \tilde{X}_i : 1 + xy^2z + z^2 = 0 \\ E_1 : y = 0, \quad z \pm i = 0 \\ E_2 : x = 0 \quad z \pm i = 0. \end{cases}$$

Remark 5.3.8. Let X be a surface singularity, $\pi: \tilde{X} \rightarrow X$ its resolution, $E = \bigcup_{i=1}^n E_i = \pi^{-1}(0)$ the exceptional divisor. Suppose that \tilde{X} is a good resolution, all $E_i \cong \mathbb{P}^1$ and $E_i^2 = -2$. Then X is a simple hypersurface singularity.

Let Γ be the dual graph of X . Then the quadratic form coincide with the Tits form of the dual graph:

$$Q(x_1, x_2, \dots, x_n) = -2\left(\sum_{i=1}^n x_i^2 - \sum_{1=i < j=n} a_{ij} x_i x_j\right),$$

where a_{ij} is the number of arrows connecting vertices's i and j . From the theorem of Gabriel we know that Q is negatively definite (and quiver is representation finite) if and only if $\Gamma = A - D - E$. Since our singularity is determined by its dual graph.

Bibliography

- [MR85] M. Reid. *Young persons guide to canonical singularities*, in *Algebraic geometry*, (Bowdoin, 1985), Proc. Sympos. Pure Math. 46 Part 1, AMS 1987, pp. 345—414.
- [ID07] I.V. Dolgachev. *McKay Correspondence*, <http://www.math.lsa.umich.edu/idolga/lecturenotes> 2007.
- [MR02] M. Reid. *La correspondance de McKay*, Seminaire Bourbaki, Vol.1999/2000, Asterisque 276 (2002) pp. 53—72.
- [IR94] Y. Ito and M. Reid. *The McKay correspondence for finite subgroups of $SL(3, \mathbb{C})$* , in *Higher dimensional complex varieties*, Trento, 1994, M. Andreatta and others Eds., de Gruyter, pp. 221—240, 1996.
- [IN99] Y. Ito and I. Nakamura. *Hilbert schemes and simple singularities*, New Trends in Algebraic Geometry, Proc. of Euro conference of Algebraic Geometry, Warwick 1996, pp. 151—233, CUP 1999.
- [EL83] E.J.N Looijenga. *Isolated singular points on complete intersections*, London mathematical society lecture notes series 77 (1983) pp.10—13.
- [GT69] G.N Tjurina. *Locally semi universal flat deformations of isolated singularities of complex space*, maths USSR Ivesta 3 (1969) 967—999.
- [PD44] P.Du Val. *On absolute and non-absolute singularity of algebraic surfaces*, Istanbul university 1944 159—215.
- [MB00] Michel Brion, *Representation of Quivers*.
- [PO11] Pavel Etingof, Oleg Goldberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, and Elena Yudovina. *Introduction to representation theory*, January 2011.
- [CW04] C.T.C Wall. *singular Points of a plane curve*, London mathematical society Cambridge University Press, 2004.
- [JC14] Javier Carrasco. *Finite subgroups of $SL(2)$ and $SL(3)$* , Undergraduate project, May 2014.