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Determination Of Minimum Initial Capital Using The Cramer-Lundberg Model And Discrete Time Bisection Ruin Model.

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DECLARATION

This dissertation, as presented in this document, is my original work and has not been replicated, extracted or copied from any other published or unpublished sources.

Erastus Kimani Ndekele .... Date ....

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This project was supervised and approved as original work by:

Prof. Joseph A.M. Otieno .... Date ....

DEDICATION

To my dad Erastus G.M. Ndekele, and, my immediate family.

ACKNOWLEDGEMENT

I would, first and foremost, like to thank God Almighty for giving me the strength, will and grace to complete this dissertation.

I would also like to acknowledge Prof. Joseph A.M Otieno, my supervisor, for his support and great assistance in the writing of this dissertation. Also, to the school of mathematics for providing the necessary environment and mentoring.

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TABLE OF CONTENTS

DECLARATION **I**

DEDICATION **II**

ACKNOWLEDGEMENT **III**

ABBREVIATION **VII**

ABSTRACT **VIII**

1 GENERAL INTRODUCTION **9**

1.1 BACKGROUND OF THE PROBLEM AND LITERATURE REVIEW 9

1.2 PROBLEM STATEMENT 13

1.3 OBJECTIVES OF THE STUDY 14

1.4 SIGNIFICANCE OF THE STUDY 15

2 RUIN THEORY **17**

2.1 INTRODUCTION 17

2.2 INTEGRO-DIFFERENTIAL EQUATION 18

2.3 APPROXIMATION METHODS 21

2.3.1 DISCRETE TIME MODELS 23

2.3.2 CONTINUOUS TIME MODELS 28

3 APPROXIMATIONS **41**

3.1 ASSUMPTIONS AND LIMITATION 41

3.1.1 ASSUMPTIONS 41

3.1.2 LIMITATIONS 41

3.2 DATA USED 42

3.3 DESCRIPTION OF APPROXIMATION METHODS 43

3.3.1 DISCRETE TIME BISECTION METHOD 43

3.3.2 COMPOUND POISSON PROCESS 43

3.3.3 BROWNIAN MOTION APPROXIMATION 52

4 NUMERICAL ANALYSIS **56**

4.1 DISCRETE TIME MINIMUM INITIAL CAPITAL 56

4.2 COMPOUND POISSON PROCESS 60

4.2.1 USING LUNDBERG'S INEQUALITY 60

4.2.2 USING CRAMER'S APPROXIMATE FORMULA 62

4.3 BROWNIAN MOTION APPROXIMATION TO COMPOUND POISSON PROCESS 64

5 SUMMARY AND CONCLUSION **69**

REFERENCES **73**

APPENDIX **75**

TABLES AND FIGURES **79**

SYMBOLS AND ABBREVIATION

EU European Union

NAIC National Association of Insurance Commissioners

SCR Solvency Capital Requirement

MGF/mgf Moment Generating Function

i.i.d Independent and Identically Distributed

MIC/MIC$\{\alpha, N, c, S_N\}$ Minimum Initial Capital

u = U_0 MIC

c Constant Rate of Premium Income

λ Rate of Claim Number

μ Mean of Claim Severities

$\alpha = \psi(u)$ Probability of Ruin (Constant/Fixed)

N(t) Poisson Distributed Number of Claim

Theta/θ Premium Safety Loading

$S_N = \sum_{i=1}^{N(t)} X_i$ Aggregate Claims Random Variable

ABSTRACT

The dissertation uses the classical Cramer-Lundberg model to find the minimum initial capital (MIC) required by a hypothetical insurance company in launching a new product line, or, for investors wishing to open a new insurance company with their expectations likely to follow those of the classical ruin model. The Poisson and exponential rate were taken as one and the probability of ruin taken as fixed (0.1, 0.2 or 0.3). The safety coefficient for each of the probabilities of ruin was taken as either 0.1 or 0.25 of the premium. A Brownian motion approximation to the compound Poisson aggregate claims model was also used.

It was observed that there was a linear relationship between the the minimum initial surplus and the number of claims for the continuous time ultimate ruin model.Cramer's approximation was considered as providing the most correct MIC whereas the Lundberg model provided a ceiling. The Brownian motion approximation was slightly higher than the values provided by the Lundberg model and this can be explained by the variance (higher moments) effect taken into account by the Brownian motion approximation.

A discrete time model was also considered as that provided by Sattayatham et al (2013) where the bisection method was used to find the MIC. It was noticed that the MIC in this case was considerably smaller than those provided by the continuous case and this can be explained by the fact that the discrete case only requires non-ruin at integer durations. Therefore ruin can occur in between the intervals as long as there is no ruin at the integer duration. The curve of the discrete MIC against number of claims was curvilinear. There was no intersection of the curves for any of the methods.

CHAPTER 1. GENERAL INTRODUCTION

This dissertation aims to look into ruin theory and its application into the insurance industry. Although this may be extended to any other industry, the theory has been narrowed to look into a hypothetical insurance company to determine how much investment (capital or surplus) is required to ensure a given survival (or non-ruin) probability.

Therefore, the dissertation will be looking into a surplus process and trying to determine how much initial investment would keep the chances of ruin at or below a certain level.

1.1 BACKGROUND OF THE PROBLEM AND LITERATURE REVIEW

Ruin theory is based on one of the applications of probability theory. Its origin therefore is tied with the origin of probability. Probability theory developed and grew mainly in seventeenth and eighteenth centuries due to the appeal of games of chance during this period. The major contributors during this period were Jacob Bernoulli (1654-1705) and Abraham De Moivre (1667-1754). Bernoulli was a Swiss mathematician who was the first to use the term integral. His most original work was *'Ars Conjectandi'* published in 1713 in Basel.

De Moivre pioneered the modern approach to the theory of probability when he published *'The Doctrine of Chance: A method of calculating the probabilities of events in play'* in

1718. The definition of statistical independence appears first in this book. There were new expanded versions of this book in 1718,1738 and 1756. The birthday problem, in a slightly different form, appeared in the 1738 edition while the gambler's ruin problem featured in the 1756 edition. This was among the first initial use of ruin theory although it had not yet developed into a body of theory by this time.

The application of ruin theory in insurance is attributed to Filip Lundberg and Harald Cramer. Lundberg's 1903 thesis was largely written in Swedish and his notation was largely individual which meant that there were limited readers to his work and an even lower number of those who understood it. Cramer understood, mathematically developed and presented Lundberg's work into a coherent theory. Lundberg first described the total claim amount of an insurance company by what is currently known as a compound Poisson process, which was a concept that was not defined by then. Later in 1926, he studied the ruin probability for a risk process which describes the net surplus of a company. This net surplus was obtained by subtracting discontinuous compound Poisson outflow payments from continuous inflow of premia and an initial capital amount. Lundberg is considered the founder of mathematical risk theory.

After proving and presenting the work by Lundberg, Cramer became more interested in the practical work in the insurance industry. This could have been motivated by the interest rate crisis in the late 1930's caused by interest rates falling below the level used by insurance companies in premium calculations. This presented an avenue for Cramer to advance his research and to provide a solution to the problem without increasing premiums since such an increase was prohibited by law to policies already in force. Cramer thus advocated the use of the zero point method for premium calculation and for the use of mixed mortality for policies with benefits for both death and survival scenarios that would ensure a safe premium is calculated.

Since the original Cramer-Lundberg ruin model, there has been a lot of research done on ruin theory with applications in many areas (especially in finance and insurance) such as enterprise risk, barrier option pricing and various other areas in insurance. A prominent figure in the area of ruin has been Hans Gerber, who has published material related to ruin theory from 1968. He and other researchers have aimed to improve on the classical ruin model by making it more applicable to the real world and to various disciplines. This has over the years involved incorporating investment income, dividends, developing recursion formulae and algorithms, and, using other stochastic processes other than the compound Poisson process to model aggregate claims processes.

In actuarial science, there has been a growth of regulation in the insurance industry to ensure protection of consumers of insurance products from malpractices that end up in the insolvency (ruin) of insurance companies. It should be noted that most of the development of ruin theory has been to minimize the probability of ruin given an initial level of capital. The reverse question has of late become an important question for regulating agencies and regional trading blocs that use uniform policies.

The question of how much capital is required to ensure that there is a small likelihood of a company becoming insolvent and subsequently closing down has recently become a question of interest. In the European Union (EU), the insurance industry used a system analogous to that used by banks: the solvency system. The EU established the first non-life and life directives in 1973 and 1979 respectively. Later revisions to these directives led to the establishment of a single market for insurance among member nations in the EU. This meant that an insurance undertaking in one state would essentially be sold in any other member state without restrictions. This system came to be known as solvency I, and subsequently solvency II - which is currently being adopted and tested in various countries in the EU. Solvency II aims to take into consideration the varying nature of insurance risks based on the environments the companies operate in, affiliates, and, other factors such as individual underwriting risks of specific companies. Solvency II thus aims at better evaluation and management of risks and the determination of SCR for the insurance companies operating within the EU is one way of doing this. In the USA, NAIC uses a risk based capital system in its regulatory framework to determine such a capital amount. There has been no unified regulatory approach in Africa although in Kenya the IRA is planning to adopt the EU's solvency II framework. Whichever system that is being used, the main aim is to determine how much capital is required to ensure that an insurance company's chances of becoming insolvent are below a given measure i.e. risk management. The approaches would thus be concerned with a firms' assets, liabilities and shareholders contribution. Much more so, however, these approaches would be more interested in various risk measures associated with assets, liabilities, incomes and expenses. The insurance regulator would then be interested in each firm's accurate representation of their risk position and their ability to meet (or surpass) the regulators target. This will enhance good competition among insurance companies while at the same time provide a feeling of stability.

These types of regulatory frameworks allow for adoption of numerous risks and it is not inconceivable that propositions to unify such systems be developed to foster global business. In essence there is a possibility that the entire world could be a single market for insurance business.

1.2 PROBLEM STATEMENT

Early researches (such as works by Cramer, Lundberg, Sparre Andersen, Hans u. Gerber, Michael R. Powers and Shiu among others) have given more weight on the probability of ruin given initial capital (MIC/u). Hence ruin probability is a function of initial surplus and thus to improve the situation (reduce the ruin probability), one of the factors that may be considered is the initial capital. Although the business underwritten may also be changed to try and ensure that claims severity are not particularly damaging when they occur, the insurance firm may not have much flexibility in this area due to competition and demand for products.

A major challenge that occurs in the computation of ruin probabilities - and in essence computation of quantites in the ruin formula - is that the determination of accurate probabilities depends on the evaluation of an integro-differential equation that does not always have explicit solutions. The best bet in many cases is thus the use of approximate formulas or simulation.

It is also difficult in many cases to incorporate all the risk factors that may affect a specific class of business. A model that incorporates many risk factors has to make assumptions regarding the correlation of the risk factors and how to add up those factors. On the other hand, a very simplified model would not accurately represent reality. A risk measure derived from ruin probabilities known as value at risk (VaR) may be used to as an alternative to the MIC in enterprise risk management. VaR was developed mainly in the banking industry to check the derivative portfolio exposure by looking at what the loss in such a portfolio is over the the time it takes to get out of a taken position (Long or short position in derivative markets). Hence VaR is usually used over a short period of time. VaR may also be improved to take into account correlation of risks (mainly variances) or for the case of independent risks. The approach of VaR is however not tackled in this dissertation.

The problem of computing the MIC has been considered from the continuous and discrete time horizons. For the discrete time case, a bisection method developed by Sattayatham et al is adopted. The bisection method is carried out twenty times to find the required MIC. In the continuous time case, the pioneering work of Lundberg and Cramer is considered. Cramer's asymptotic formula gives the most accurate MIC while the upper limit provided by Lundberg's inequality provides an easy and prudent evaluation for regulators. A Brownian motion approximation of the classical Cramer-Lundberg model is also done to take into account extra variation.

The problem being considered is therefore that of determining an optimum starting capital (MIC) for an insurance company to have a ruin probability of at most a predetermined specified level for the classical Cramer-Lundberg model.

1.3 OBJECTIVES OF THE STUDY

The major objective of the study is to determine a surplus or capital requirement for a hypothetical insurance company that ensures the ruin probability is below a given measure.

The specific objectives are:

- Identifying approximate closed form expressions for the classical Cramer-Lundberg model of the probability of ultimate ruin and solving for u (MIC).

- Applying Brownian motion approximation to S_N as an approximation to the compound Poisson model.

- Applying a bisection approximation method for the case where time is at integer durations (discrete time model).

1.4 SIGNIFICANCE OF THE STUDY

The techniques used in this dissertation are applicable to various companies the result of which would be that different companies would have varying capital amounts based on risks that are specific to them. This concept is highly useful as it may be applied to insurance companies that provide different products and that are also in different geographical and cultural environments. This would enable the provision of similar regulation to companies in such varied environments. In Kenya, the methods used may be considered as an initial guideline to provide insurance regulation for the proposed East African Community integration regions. This will enable the companies in different countries to compute their proposed capital requirements which will in turn inform other regulatory policies such as margin requirements and the subdivision of the capital amounts into various categories.

Similarly, ruin methods may be applied to any other organization as a means of risk management. The main idea is to identify the major sources of income and expenses and model them to compute the probability of ruin. If the result is not desired, then one or more of the inputs (incomes or expenses) may be adjusted to obtain the desired ruin levels. It should be noted however that increasing charges to customers may interfere with income due to industrial competition. Therefore, in the instances that only part of expenses may be transfered to customers, unique measures may need to be adapted to reduce expenses or other sources of income identified (for example the capital markets).

The computation of initial capital also helps investors know when new companies are over/under capitalized. Over-capitalization leads to opportunity costs as funds that are tied up as equity could be used to purchase income generating assets. Under-capitalization on the other hand leads to high underwriting risk which may lead to insolvency.

CHAPTER 2. RUIN THEORY

2.1 INTRODUCTION

This section will aim to look at various methods of computing the initial surplus as a form of review of key papers. There will be a mention of how the Panjer recursion method may be used to compute the MIC/surplus in the form of an upper limit due to the importance of this model in the numerical computation of probabilities. However, Panjer's method is not included in the methodology and analysis.

The chapter also lays the foundation for the bisection method developed by Sattayatham et al in 2013 for the approximation of the initial surplus and this is the main model adopted for the discrete time ruin model.

In the continuous time ruin model, there is also the mention of the Erlang distribution and how it applies to the aggregate claims for certain assumptions of claim severity and claim number distribution. Again however, the Erlang distribution is not explored further in the paper but is only mentioned due to its importance in continuous time ruin models. The methods introduced here and further utilized are those of Cramer and Lundberg in the original ruin problem, the bisection method by Sattayatham et al and the Brownian motion approximation to the compound Poisson process.

2.2 INTEGRO-DIFFERENTIAL EQUATION

The probability of ruin is obtained by the observation of surplus over time. In the case of this dissertation the equation of surplus being observed is:

$$U_t = u + ct - \sum_{i=1}^{N(t)} X_i \tag{2.1}$$

Since $\sum_{i=1}^{N(t)} X_i$ is a random variable, whose randomness is brought about by the number of claims at a given time $N(t)$ and the claims severities $X_i's$, then the use of probabilities becomes inevitable. The claim numbers are modeled as being Poisson distributed (N(t) ~P(λ)) and claim severities as being exponentially distributed (X_i ~$exp(\mu)$). Therefore the new quantity of observation is:

$$Pr\{U_t\} = Pr\left\{u + ct - \sum_{i=1}^{N(t)} X_i\right\}$$

Ruin is said to occur if equation 2.1 ever becomes negative. Thus a more accurate definition of the probability being observed is:

$$\begin{aligned} Pr\{U_t < 0 \mid U_0 = u\} &= Pr\left\{u + ct - \sum_{i=1}^{N(t)} X_i < 0\right\} \\ &= Pr\left\{u + ct < \sum_{i=1}^{N(t)} X_i\right\} \end{aligned}$$

Which simply means that the probability of ruin occurs when there is a positive probability of the income and initial capital at a given time being less than the size of a claim. It should be noted that the probability is a cumulative distribution function (CDF).

To find this probability of ruin, or be able to determine its components (such as u for this dissertation), an integro differential equation [1] is developed. This equation is developed as in Klugman et al (2004).

Definition 2.1. Let G(u,y) = Pr{ruin occurs when the initial reserve is u and deficit immediately after ruin is at most y}

$\therefore$ the surplus immediately after ruin is between 0 and -y.

Then the probability that a claim amount x satisfies $u + ct < x \leq u + ct + y$ is $F(u + ct + y) - F(u + ct)$.

By the law of total probability we have:

$$G(u, y) = \int_0^\infty \left[\left\{ \int_0^{u+ct} G(u + ct - x,\, y) dF(x) \right\} + F(u + ct + y) - F(u + ct) \right] \lambda e^{-\lambda t} dt \tag{2.2}$$

The probability G(u, y) represents the sum of different possibilities after the occurrence of the first claim amount x. There is a likelihood that the first claim will not cause ruin to occur which means that the surplus now reduces to u + ct - x and that ruin can occur later on with the new probability G(u + ct - x). Alternatively, the first claim can cause ruin to occur and this ruin will not be greater than y. The time to ruin, if ruin occurs, is exponentially distributed since the claim numbers are assumed to be Poisson distributed.

Differentiating (2.2) with respect to u requires first a change of variable:

$$z = u + ct \Rightarrow dz = cdt \;\therefore\; dt = \frac{1}{c} dz$$
$$t = \frac{z - u}{c}$$

[1] An integro differential equation is an equation that involves both integrals and derivatives of a function

The integral now becomes:

$$
\begin{aligned}
G(u,y) &= \int_0^\infty \left[\int_0^z G(z - x\ y)dF(x) + F(z+y) - F(z)\right]\lambda e^{-\lambda\frac{z-u}{c}}\ \frac{1}{c}dz \\
&= \frac{\lambda}{c}\int_u^\infty \left[\int_0^z G(z - x\ y)dF(x) + F(z+y) - F(z)\right]e^{-\lambda\frac{z-u}{c}}\ dz \\
&= \frac{\lambda}{c}\, e^{\frac{\lambda u}{c}}\int_u^\infty e^{\frac{-\lambda z}{c}}\left[\int_0^z G(z - x\ y)dF(x) + F(z+y) - F(z)\right] dz
\end{aligned}
$$

From the fundamental theorem of calculus and Leibnitz theorem we have $\frac{d}{du}\int_u^\infty k(z)dz = -k(u)$. Applying this and using the product rule of differentiation gives:

$$
\begin{aligned}
\frac{\partial\, G(u,\ y)}{\partial\, u} &= \frac{\lambda}{c}\left\{\frac{\lambda}{c}\, e^{\frac{\lambda u}{c}}\int_u^\infty e^{\frac{-\lambda z}{c}}\left[\int_0^z G(z - x\ y)dF(x) + F(z+y) - F(z)\right]\right\} \\
&\quad + \frac{\lambda}{c}\, e^{\frac{\lambda u}{c}}\left\{-\, e^{\frac{\lambda z}{c}}\left[\int_0^z G(z - x,\ y)dF(x) + F(z+y) - F(z)\right]\right\}
\end{aligned}
$$

If we let z = u then this leads to the following theorem

Theorem 2.1.

$$
\frac{\partial\, G(u,\ y)}{\partial\, u} = \frac{\lambda}{c}\, G(u,\ y) - \frac{\lambda}{c}\left\{\int_0^u G(u - x,\ y)dF(x) + F(u+y) - F(u)\right\}
$$

NB. It should be noted that $lim_{y\to\infty}\, G(u,\ y) = \psi(u)$

Also, using the Lundberg inequality which will be developed later in the section, we see that:

$$0 \le G(u,\ y) \le \psi(u)$$

$$0 \le G(u,\ y) \le e^{-\kappa u}$$

Thus

$$0 \le G(\infty,\ y) = lim_{u\to\infty}\, G(u,\ y) \le lim_{u\to\infty} e^{-\kappa u}$$

$$0 \le G(\infty,\ y) \le 0$$

Therefore, with an infinite amount of initial capital the probability of ruin is 0. That is, if wealth is not a problem, then an insurance company adopting this model will almost surely never experience ruin.

Further

$$\begin{aligned}
\int_0^\infty G(u,\, y)du \;&\le\; \int_0^\infty e^{-\kappa u} \\
&\le\; \frac{1}{\kappa}\big[-e^{-\kappa u}\big]_0^\infty \\
&\le\; \frac{1}{\kappa}[-0\,+\,1] \\
\int_0^\infty G(u,\, y)du \;&\le\; \frac{1}{\kappa}
\end{aligned}$$

where κ is the adjustment/Lundberg coefficient as will be shown later on. A discrete time analogue may be obtained as:

$$\sum_{u=0}^{\infty} G(u,\, y) \;\le\; \frac{1}{\kappa} \tag{2.3}$$

Thus from theorem 2.1 and the note above, to obtain the infinite time ruin probability (or the MIC as desired in this dissertation), one must find a solution to the integro differential equation

$$\frac{\partial\, \psi(u)}{\partial\, u} \;=\; \frac{\lambda}{c}\,\psi(u) - \;\frac{\lambda}{c}\left\{\int_0^u \psi(u\,-\,x)dF(x)\,+\,F(u\,+\,y)\,-\,F(u)\right\} \tag{2.4}$$

2.3 APPROXIMATION METHODS

There have been various approximations to the integro differential equation (2.4). Some of those include:

I The Lundberg inequality/ceiling

II Cramer’s assymptotic formula

III Sparre Andersen model

IV Beekman’s approximate formula

V De Vylder approximation

VI Panjer recursions for ruin probability

VII Discrete time bisection method by sattayatham et al.

There are many other approximations and improvements to the basic ruin probability presented in this dissertation. However as mentioned earlier, the writing looks only at the discrete time bisection method, the Lundberg inequality, Cramer’s approximation and the Brownian motion approximation to the Lundberg Inequality. As such the dissertation chooses one discrete time method for computing the MIC, one continuous time method and an upper bound in continuous time.

Since the ruin probability is a function of u (MIC), then the analytical expressions have u in them which may be approximated. Methods of finding the ruin probability explicitly are rare and in many cases simulation is used.

In the discrete time case, the theory is developed that the ruin probability curve is a monotonically decreasing function and then it is illustrated how the MIC can be obtained given some set parameters.

2.3.1 DISCRETE TIME MODELS

Panjer Recursion

Panjer's model may be used as a consequence of (2.3). It should also be noted that Gerber came up with a discretization method which may be applied to continuous probability distributions to enable the use of Panjer's model. From (2.3) the idea is to obtain the probability $G(u, y)$ for various values of u using the Panjer recursion model. As this model was not used, then a brief mention is given.

Since

$$S_N = X_1 + X_2 + X_3 + ... + X_N$$

The probability generating function (pgf) may be defined as:

$$\begin{aligned} G_{S_N} &= E[s^j] \\ &= \sum_{j=0}^{\infty} s^j \; Pr\{S_N = j\} \\ \therefore \frac{d}{ds} G_{S_N}(s) &= \sum_{j=0}^{\infty} j s^{j-1} Pr\{S_N = j\} \end{aligned} \tag{2.5}$$

Similarly

$$
\begin{aligned}
G_{S_N}(s) &= G_N[G_x(s)] \\
&= e^{\lambda[G_x(s)\,-\,1]} \\
\therefore \frac{d}{ds}G_{S_N}(s) &= \lambda\, e^{\lambda[G_x(s)\,-\,1]}\, \frac{d}{ds}G_x(s) \\
&= \lambda\, G_{S_N}(s)\, \frac{d}{ds}G_x(s) \\
&= \lambda\Big[\sum_{j=0}^{\infty} s^j Pr\{S_N \,=\, j\}\Big]\Big[\sum_{x=0}^{\infty} s\; s^{x\,-\,1}\; Pr\{X \,=\, x\}\Big] \\
&= \sum_{j=0}^{\infty} \lambda s^j Pr\{S_N \,=\, j\}\; \sum_{x=1}^{\infty} x\; s^{x\,-\,1} Pr\{X \,=\, x\} \\
&= \sum_{j=0}^{\infty}\; \sum_{x=1}^{\infty}\; \lambda\; s^j\; x\; s^{x\,-\,1}\; Pr\{S_N \,=\, j\}\; Pr\{X \,=\, x\} \\
&= \sum_{j=0}^{\infty}\; \sum_{x=1}^{\infty}\; \lambda\; x\; s^{x\,+\,j\,-\,1}\; Pr\{S_N \,=\, j\}\; Pr\{X \,=\, x\}
\end{aligned}
$$

Let

$$j \,+\, x \,=\, k \,\Rightarrow\, j \,=\, k \,-\, x$$

Also

$$j \,=\, 0 \,\Rightarrow\, k \,-\, x = \, 0 \,\therefore\, k \,=\, x$$

Thus

$$\begin{aligned}
\frac{d}{ds}G_{S_N}(s) &= \sum_{k=x}^{\infty}\sum_{x=1}^{\infty}\lambda\, x\, s^{k\,-\,1}\, Pr\{S_N \,=\, k \,-\, x\}\, Pr\{X \,=\, x\} \\
&= \sum_{x=1}^{\infty}\sum_{k=x}^{\infty}\lambda\, x\, s^{k\,-\,1}\, Pr\{S_N \,=\, k \,-\, x\}\, Pr\{X \,=\, x\} \\
&= \sum_{x=1}^{\infty}\sum_{k=1}^{\infty}\lambda\, x\, s^{k\,-\,1}\, Pr\{S_N \,=\, k \,-\, x\}\, Pr\{X \,=\, x\} \\
&= \sum_{x=1}^{\infty}\sum_{k=1}^{\infty}\lambda\, x\, s^{k\,-\,1}\, p(k \,-\, x)\, p(x)
\end{aligned}$$

Replacing the left hand side with (2.6) and noting that the summation of j will give non zero values from 1, gives

$$\begin{aligned}
\sum_{j=1}^{\infty} j\, s^{j\,-\,1}\, Pr\{S_N \,=\, j\} &= \sum_{x=1}^{\infty}\sum_{k=1}^{\infty}\lambda\, x\, s^{k\,-\,1}\, p(k \,-\, x)\, p(x) \\
\Rightarrow \sum_{k=1}^{\infty} j\, s^{k\,-\,1}\, Pr\{S_N \,=\, k\} &= \sum_{x=1}^{\infty}\sum_{k=1}^{\infty}\lambda\, x\, s^{k\,-\,1}\, p(k \,-\, x)\, p(x) \qquad (2.6)
\end{aligned}$$

Comparing the LHS and RHS of (2.7) gives

$$\begin{aligned}
k\, p(k) &= \sum_{x=1}^{\infty}\lambda\, x\, p(x)\, p(k \,-\, x) \\
\therefore p(k) &= \frac{\lambda}{k}\sum_{x=1}^{\infty} x\, p(x)\, p(k \,-\, x) \qquad (2.7)
\end{aligned}$$

(2.8) is the Panjer recursion model where aggregate claims follow a compound Poisson distribution. It may also be written in terms of probability distribution functions as

$$g_k \,=\, \frac{\lambda}{k}\sum_{x=1}^{\infty} x\, f_x\, g_{k\,-\,x}$$

Bisection Method

The interest is to prove the existence of a minimum initial capital given some specific parameters i.e. MIC$\big(\alpha, N = n, c\{X_n,\ n\ \geq\ 1\}\big)$ such that the probability of ruin $\psi_n(u)$ is less than or equal to some pre-determined measure α. The first important theorem and its corollary is given below

Theorem 2.2. *Let* $N\ \in\ \{1,\ 2,\ 3,\ ...\},\ c\ >\ 0$ *and* $u\ \geq\ 0$*, be given. Then the ruin probability at the times 1, 2, 3,...,N satisfies the following equation*

$$\psi_N(u)\ =\ \psi_1(u)\ +\ \int_{-\infty}^{u\ +\ c} \psi_{N\ -\ 1}(u\ +\ c\ -\ x)dF_{X_1}(x)$$

The theorem can be seen as a variant of (2.4)

Corollary 2.1. *Let* $\alpha\ \in\ (0,1),\ N\ \in\ \{1,2,3,4,...\}$ *and* $c > 0$ *be given. If* $\{X_n, n \geq 1\}$ *is an i.i.d claim process, then there exists* $\tilde{u} \geq 0$ *such that for all* $u \geq \tilde{u}$*, u is an acceptable initial capital corresponding to* $\big(\alpha, N, c, \{X_n, n \geq 1\}\big)$

Proof. If we let $\psi_N(u)$ denote the ruin probability given surplus u, and $\phi_N(u) = 1 - \psi_N(u)$ be the corresponding survival probability, then we consider case by case. **Case 1:** If $\psi_N(0)\ \leq\ \alpha$, and since $\psi_N(u)$ is a decreasing function, then $\psi_N(u)\ \leq\ \psi_N(0)\ \leq\ \alpha$. This implies that 0 is the required minimum initial capital and thus no capital is needed. **Case 2** If $\psi_N(0) > \alpha$ and since $lim_{u\to\infty}\psi_N(u) \to 0$, then there exists $\tilde{u} > 0$ such that $\psi_N(\tilde{u}) < \alpha$. We conclude that for all $u\ \geq\ \tilde{u},\ \psi_N(u)\ \leq\ \psi_N(\tilde{u})\ <\ \alpha$. In this case $\tilde{u}$ is the required minimum initial capital. □

As a result of the above corollary, then $\{u \geq 0 : \psi_N(u) \leq \alpha\}$ is a non empty set. Therefore an initial capital can always be chosen to ensure that the value of ruin probability does not exceed the chosen measure α. The minimum initial capital is defined as:

$$MIC\big(\alpha, N = n, c, \{X_n, n \geq 1\}\big) = min_{u \geq 0}\{u : \psi_N(u) \leq \alpha\}$$

The proof of the existence of the minimum initial capital was then done using the following lemma

Lemma 2.3. *let $a,\ b$ and α be real numbers such that $a \leq b$. If a function f is decreasing and right continuous on $[a, b]$ and $\alpha \in [f(b), f(a)]$, then there exists $d \in [a, b]$ such that;*

$$d = min\{x \in [a, b] : f(x) \leq \alpha\}$$

Proof. Let

$$S = \{x \in [a, b] : f(x) \leq \alpha\}$$

Since $f(b) \leq \alpha \leq f(a)$ then $b \in S$ i.e. S is a non empty set. S is a subset of $[a, b]$, then there exists d (by the previous lemma) such that $d = \inf S$. **Case 1:** If $d = b$, then since $b \in S$, then $b = min\ S$. **Case 2:** If $a \leq d < b$, then we can $d_n \in S$ such that

$$d \leq d_n < d + 1/n$$

for all $n \in \mathbb{N}$ For each $\frac{1}{n} > \frac{b-d}{2}$ i.e $n > \frac{2}{b-d}$, we have

$$d < d + \frac{1}{n} < d + \frac{b-d}{2} = \frac{b+d}{2} < b$$

Thus, $d + \frac{1}{n} \in (d, b) \subset [a, b]$ for all $n > \frac{2}{b-d}$. Since f is right continuous at d we have

$$f(d) = lim_{n \to \infty} f\big(d + \frac{1}{n} \leq \alpha\big)$$

Thus $d \in S$ and $d = min\ S$. □

The existence of the minimum initial capital may thus be obtained from the theorem below.

Theorem 2.4. *Let* $\alpha \in (0,1),\ N \in \{1,2,3,4,...\}$ *and* $c > 0$. *Then there exists* $u^* \geq 0$ *such that*

$$u^* = MIC\big(\alpha, N = n, c, \{X_n, n \geq 1\}\big)$$

Proof. **Case 1:** $\psi_N(0) \leq \alpha$. This implies that

$$MIC\big(\alpha, N = n, c, X_n, n \geq 1\big) = 0$$

Case 2: $\psi_N(0) > \alpha$. By corollary 2.1 there exists $\tilde{u} > 0$ such that $\psi_N(\tilde{u}) < \alpha$. Since $\psi_N(u)$ is decreasing and right continuous, by lemma 2.1 there exists $u^* \in [0, u]$ such that:

$$u^* = min_{u\in[0,\tilde{u}]}\{u : \psi_n(u) \leq \alpha\}$$

That is:

$$u^* = MIC\big(\alpha, N = n, c, \{X_n, n \geq 1\}\big)$$

□

2.3.2 CONTINUOUS TIME MODELS

The Erlang Distribution

For the continuous time ruin probability, the classical Cramer-Lundberg process is used. For this model, the claim severities are considered to be exponential while the claim numbers Poisson distributed. Therefore, the sum of the i.i.d claim severities $S_N = \sum_{i=1}^{N(t)} X_i$ can be considered to form the Erlang distribution which is one way of evaluation. The Erlang distribution comes about as shown:

We will first use a gamma distribution with parameters α and β and then proceed to obtain the Erlang distribution

$$\begin{aligned}
f_X(x) &= \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \\
M_X(t) &= E[e^{tx}] \\
&= \int_0^\infty e^{xt} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-(\beta-t)x} x^{\alpha-1} dx \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\beta-t)^\alpha} \int_0^\infty \frac{(\beta-t)}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\beta-t)x} dx \\
&= \frac{\beta^\alpha}{(\beta-t)^\alpha} \\
&= \left(\frac{\beta-t}{\beta}\right)^{-\alpha} \\
M_x(t) &= \left(1 - \frac{t}{\beta}\right)^{-\alpha}
\end{aligned}$$

It can be seen that setting α to be n and β to be μ results in 3.1. We can use this to show that for integer values of α (or n) the values of $\Gamma(\alpha, x)$ can be calculated. We will show this by induction.

$$\Gamma(\alpha;x) = \frac{1}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}e^{-t}dt$$

For α=1

$$\Gamma(1;x) = \frac{1}{\Gamma(1)}\int_0^x e^{-t}dt$$

$$= (-e^{-t})_0^x$$

$$= 1 - e^{(-x)}$$

For α=2

$$\Gamma(2;x) = \frac{1}{\Gamma(2)}\int_0^x te^{-t}dt$$

using integration by parts

$u = t \ \Rightarrow du = dt,$ and $dv = e^{-t}dt \ \Rightarrow v = -e^{-t}$

Thus

$$\frac{1}{\Gamma(2)}\int_0^x te^{-t}dt = \frac{1}{\Gamma(2)}\left[(-e^{-t}t)_0^x + \int_0^x e^{-t}dt\right]$$

$$= \frac{1}{\Gamma(2)}[-xe^{-x} + 1 - e^{-x}]$$

$$= 1 - \left[e^{-x} + e^{-x}\frac{x}{\Gamma(2)}\right]$$

For α=3

$$\Gamma(3;x) = \frac{1}{\Gamma(3)}\int_0^x t^2e^{-t}dt$$

Using integration by parts

$u = t^2 \ \Rightarrow du = 2tdt$ and $dv = e^{-t}dt \ \Rightarrow v = -e^{-t}$

$$= \frac{1}{\Gamma(3)}\left[(-e^{-t}t^2)_0^x + 2\int_0^x te^{-t}dt\right]$$

$$= \frac{1}{\Gamma(3)} + \frac{2}{\Gamma(3)}[1 - e^{-x} - xe^{-x}]$$

$$= 1 - \left[e^{-x} + \frac{xe^{-x}}{1!} + \frac{x^2e^{-x}}{2!}\right]$$

Thus by induction we can see that:

$$\Gamma(n;x) = 1 - \left[\frac{e^{-x}}{0!} + \frac{xe^{-x}}{1!} + \frac{x^2e^{-x}}{2!} + \frac{x^3e^{-x}}{3!} + ... + \frac{x^ne^{-x}}{n!}\right]$$
$$\Gamma(n;x) = 1 - \sum_{j=0}^{n-1} e^{-x}\frac{x^j}{j!} \tag{2.8}$$

This is the cumulative distribution function of the Erlang distribution with rate equal to 1.

We now can get the distribution function of S_N.

$$\begin{aligned} F_{S_N}(x) &= Pr(S_N \leq x) \\ &= \sum_{n=0}^{\infty} p_n Pr(S_N \leq x | N = n) \\ F_{S_N}(x) &= p_n F_x^{*n}(x) \\ &= p_0 + \sum_{n=1}^{\infty} p_n \Gamma(n;x) \end{aligned} \tag{2.9}$$

Substituting 2.9 into 2.10

$$\begin{aligned} F_{S_N}(x) &= p_0 + \sum_{n=1}^{\infty} p_n \left[1 - \sum_{j=0}^{n-1} \frac{x^j e^{-x}}{j!}\right] \\ &= p_0 + \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} p_n \sum_{j=0}^{n-1} \frac{x^j e^{-x}}{j!} \\ &= 1 - \sum_{n=1}^{\infty} p_n \sum_{j=0}^{n-1} \frac{x^j e^{-x}}{j!} \\ F_{S_N}(x) &= 1 - \sum_{j=0}^{\infty} \frac{x^j e^{-x}}{j!} \sum_{n=j+1}^{\infty} p_n \end{aligned}$$

Thus we have a form of mixed Erlang distribution where p_n is $Pr(N = n)$.

For the compound Poisson distribution, N has a Poisson distribution say with rate λ. This leads to:

$$F_{S_N} = 1 - \sum_{j=0}^{\infty} \frac{x^j e^{-x}}{j!} \sum_{n=j+1}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!} \tag{2.10}$$

Brownian Motion / Wiener Approximation

The use of Brownian motion with drift term is used as an approximation to the compound Poisson process as done in Klugman et al (2004).

They first showed the link between the compound Poisson process and the Brownian motion process with drift term before using the Brownian motion process as an approximation to the compound Poisson process to find the ruin probabilities. The link between the two processes is shown here.

A continuous time stochastic process $\{W_t; t \geq 0\}$, is a Brownian motion risk process with drift process μt if:

1. $W_0 = 0$
2. $W_t, t \geq 0$ has stationary and independent increments
3. For every $t > 0, W_t$ is normally distributed with mean μt and variance $\sigma^2 t$.

The process $Z_t = U_t - u = ct - S_t \,; t > 0$ was considered. The idea was to show that in the limiting case, the process $\{Z_t, t > 0\}$ is a Brownian motion with local drift μt.

From the definition of Brownian it is seen that $Z_0 = 0$ and that $\{Z_t, t > 0\}$ has independent and stationary increments since the processes $\{U_t.t \geq 0\}$ and $\{S_t, t \geq 0\}$ also have stationary and independent increments. The remaining thing left to show is that $\{Z_t, t \geq 0\}$ is normally distributed with mean μt and variance $\sigma^2 t$. They did this using the MGF of Z_t.

$$
\begin{aligned}
M_{Z_t}(r) &= M_{ct-S_t}(r) \\
&= E\big[\exp(ct - S_t)r\big] \\
&= e^{ctr} E\Big(e^{-rS_t}\Big) \\
&= e^{ctr} e^{\lambda t(M_X(-r)-1)} \\
\therefore M_{Z_t}(r) &= \exp\{t(cr + \lambda[M_X(-r) - 1])\}
\end{aligned}
$$

Letting

$$\mu = c - \lambda E(x) \; and \; \sigma^2 = \lambda E(X^2)$$

$$\text{and} X = \alpha Y$$

Then

$$
\begin{aligned}
\frac{\ln M_{Z_t}(r)}{t} &= cr + \lambda[M_X(-r) - 1] \\
&= r[\mu + \lambda E(X)] + \lambda[E(e^{-rx}) - 1] \\
&= r\mu + r\lambda E(X) + \left[1 - rE(X) + \frac{r^2E(X^2)}{2!} - \frac{r^3E(X^3)}{3!} + ... - 1\right] \\
&= r\mu + \lambda\left[\frac{r^2E(X^2)}{2} - \frac{r^3E(X^3)}{3!} + ...\right] \\
&= r\mu + \left[\frac{\lambda r^2E(X^2)}{2} - \lambda\left(\frac{r^3E(X^3)}{3!} - \frac{r^4E(X^4)}{4!} + ...\right)\right]
\end{aligned}
$$

Using

$$X = \alpha Y$$

$$
\begin{aligned}
\frac{lnM_{Z_t}(r)}{t} &= r\mu + \frac{\lambda r^2\alpha^2 E[Y^2]}{2} - \lambda\left[\frac{r^3\alpha^3E(Y^3)}{3!} - \frac{r^4\alpha^4E(Y^4)}{4!} + ...\right] \\
&= r\mu + \frac{\sigma^2 r^2\alpha^2 E(Y^2)}{\alpha^2 E(Y^2)2} - \frac{\sigma^2}{\alpha^2 E(Y^2)}\left[\frac{r^3\alpha^3E(Y^3)}{3!} - \frac{r^4\alpha^4E(Y^4)}{4!} + ...\right] \\
\therefore \frac{lnM_{Z_t}(r)}{t} &= r\mu + \frac{\sigma^2 r^2}{2} - \frac{\sigma^2}{E(Y^2)}\left[\frac{r^3\alpha E(Y^3)}{3!} - \frac{r^4\alpha^2E(Y^4)}{4!} + ...\right]
\end{aligned}
$$

As $\alpha \to 0$

$$
lim_{\alpha\to 0}M_{Z_t}(r) = \exp\left\{t\left(r\mu + \frac{\sigma^2 r^2}{2}\right)\right\}
$$

Which is the MGF of a normal distribution with mean μt and variance $\sigma^2 t$.

This is the approximation to the compound Poisson process developed by Filip Lundberg and later Harald Cramer which leads to the approximation of the c.

The Brownian motion process with local drift term was used as an approximation to the compound Poisson process. The corollary to the following theorem was used to determine u:

Theorem 2.5. *For the process U_t where:*

- $U_t = u + W_t$
- $\{W_t; t \geq 0\}$ *is a Brownian motion with local drift*
- $U_0 = u$ *is the initial capital*

and the probability of ruin before time τ expressed as:

$$\psi(u,\tau) = Pr\{min_{0<t<\tau}U_t < 0\}$$

$$= Pr\{min_{0<t<\tau}U_0 + W_t < 0\}$$

$$= Pr\{min_{0<t<\tau}W_t < -u\}$$

the ruin probability is given by

$$\psi(u,\tau) = \Phi\left(\frac{-u+\mu\tau}{\sqrt{\sigma^2\tau}}\right) + \exp\left(\frac{-2\mu u}{\sigma^2}\right)\Phi\left(\frac{-u-\mu\tau}{\sqrt{\sigma^2\tau}}\right)$$

Where $\Phi(.)$ is the cumulative distribution function of the standard normal distribution.

When time (τ) tends to infinity, the following corollary is obtained

Corollary 2.2. *For the above process, the probability of ultimate ruin is given by*

$$\psi(u) = \exp\left(\frac{-2\mu u}{\sigma^2}\right)$$

From the corollary above, the initial capital may be obtained as

$$u = -ln\psi(u)\frac{\sigma^2}{2\mu}$$

Beekman's Approximation

Also, in 1969 John Beekman published a paper deriving a formula for approximating the ruin function of collective risk theory. In his paper in the transaction of actuaries, he explains the use of the ruin function in the setting of retention limits and for the determination of initial capital for a new line of business. The ruin model described by Beekman is:

$$U(t) = u + (p_1 + \lambda)t - \sum_{i=1}^{N(t)} X_i$$

Where:

- p_1 is the average claim amount.
- λ is the aggregate security loading.
- $N(t)$ is the number of claims that follows the Poisson process
- X_i represents the claim amounts each with probability P(z).

The ruin function is given by:

$$\psi(u) = lim_{T\to\infty} P\Big[minimum_{0\le t\le T} U(t) < 0\Big]$$

which represents the probability that the risk reserve eventually becomes zero. He explains that $\psi(u)$ is not a probability distribution function but is related to one as follows:

$$\psi^*(u) = \begin{cases} 1 \; - \; \psi(u) \quad , \quad u \; \geq \; 0 \\ 0 \qquad\qquad\quad , \quad u \; < \; 0 \end{cases}$$

Then since $\psi^*(u)$ is a probability distribution function, it involves the random variable Z defined by:

$$Z = maximum_{0\le t<\infty} \left[\sum_{i=1}^{N(t)} X_i \; - \; t(p_1 \; + \; \lambda) \right]$$

Roughly, Z is the maximum excess of claims over income examined at each point of very long time periods. The mean and variance of Z according to Beekman are given by:

$$E(Z) = \frac{E(X^2)}{2\lambda}$$
$$Var(Z) = \frac{E(X^3)}{3\lambda} + [E(Z)]^2$$

The main theorem the used by Beekman is based on the approximation of the incomplete gamma distribution based on Bower's paper (2). The theorem is stated here:

Theorem 2.6. *The distribution function $\psi^*(u)$ has a jump of $1 - \dfrac{p_1}{p_1 + \lambda}$ at $u = 0$, and for $u > 0$ has the approximate form:*

$$\psi^*(u) = \Gamma(\beta u, \alpha) = \int_0^{\beta u} \frac{w^{\alpha-1}e^{-w}}{\Gamma(\alpha)} dw$$

Where:

- $\beta = \frac{E(Z)}{Var(Z)}$ *and*
- $\alpha = \frac{[E(Z)]^2}{Var(Z)}$

With this theorem, then initial capital is obtained by setting the probability of ruin to desired levels and then solving for u.

Mirca's Approximation

In a conference paper in 2010, Mirca PHD et al, the authors considered the ruin probability of a risk process using approximating methods by: De Vylder, Beekman-Bowers, Cramer-Lundberg, Grandell, Willmot and the diffusion approximation. Since some of the methods are considered in this dissertation, a review is done for a few of those methods not considered.

The authors define the aggregate amount for the individual risk process as $D_n = \sum_{k=1}^{n} X_k$, where X_k represents the loss on on insured claim k. Ruin in this case is defined as $\psi(u) = P(D_n > u)$, u being a risk reserve (which can be considered also as the initial capital). This ruin quantity may then be obtained using the following theorem:

Theorem 2.7. *Let $\{X_k\}_k$ be independent random variables with $E(X_k) = \mu_k$, $Var(X_k) = \sigma_k^2 > 0$, $\rho_k^3 = E\Big(|X_k - \mu_k|^3\Big)$ finite and let $\rho^3(D_n) = \sum_{k=1}^{n} \rho_k^3$. Then, there exists a constant $0 < C_0 < \infty$ such that*

$$sup_{x \in R}\left| P\left(\frac{D_n - n\mu}{\sigma\sqrt{n}} < x\right) - \Phi(x)\right| < \frac{C_1}{\sqrt{n}}\left(\frac{\rho}{\sigma}\right)^3$$

Where $\Phi(.)$ is the cumulative distribution function of the normal distribution.

For the collective risk model, the authors considered a model in which the claim severities are i.i.d non negative random variables with cumulative distribution function F that is considered to be non- discrete. The number of claims N was taken to have a negative binomial distribution ($NB(\alpha, \beta)$) and $D = \sum_{i=1}^{N} X_i$. The probability of ruin is thus given by $\psi(u) = P(D > u)$.

The distribution of N is given by:

$$p_n = \binom{n + \alpha - 1}{n} \rho^n (1 - \rho)^\alpha, n \in N$$

Where $0 < \rho 1$ and $\alpha \in N$.

Assuming that there exists a constant $K > 0$ satisfying:

$$\int_0^\infty e^{Kx} dF(x) = \rho^{-1}$$

and

$$V = \rho \int_0^\infty x e^{Kx} dF(x) < \infty$$

The ruin probability may thus be obtained using the following proposition

Theorem 2.8 (Proposition). *If* $e^{Ku}.P(D > u)$ *is monotone, then* $\phi(u) \sim \Big(K.\Gamma(\alpha)\Big)^{-1} . \left(\frac{1 - \rho}{v}\right)^\alpha .u^{\alpha - 1} .e^{-Ku}$*, as* $r \to \infty$

approximations may be found for $\phi(u)$ and for the negative binomial distribution used in Mircea et al, this was found using Embrechts et al's result $\phi(u) = 1 - \sum_{x=0}^{u} f_D(x)$

The concept of ruin may be applied for many other situations. For instance inDickson et al (3), ruin theory is used in the case where an insurer has a fixed level of initial capital u. The problem that is considered in this case is whether the insurer can reduce ultimate ruin probability by allocating part of this initial capital to the purchase of a reinsurance contract. Th reinsurance contract would restore the insurers surplus to a positive level, say k, every time the surplus fell between o and k. Another application of ruin theory is as seen in Pranevicius H and Sutiene K (2008). They used the compound Poisson continuous time surplus process to consider how long a surplus process would remain below zero. This implies that an insurance company may not become ruined immediately the surplus drops below zero. In such situations a firm may be able to operate with a negative surplus for a period of time before this becomes impossible and the company has to eventually wind up operations. Therefore duration of ruin and negative surplus modeling is also important to insurance companies and ruin theory provides a framework to carry out such analyses.

CHAPTER 3. APPROXIMATIONS

3.1 ASSUMPTIONS AND LIMITATION

3.1.1 ASSUMPTIONS

i The rate of premium income is constant and continuous

ii The claim severities are exponentially distributed and are i.i.d

iii The number of claims follow a Poisson distribution and hence the aggregate claim is compound Poisson

iv The only factors affecting the probability of ruin are the initial surplus (MIC) the rate of premium income and the aggregate loss function

3.1.2 LIMITATIONS

i The dissertation only takes account of the case of exponential claim severities. This is because it was the case in the classical ruin model. However, there are more challenges for distributions other than the exponential distribution such as Pareto or Lognormal distributions.

ii The methods used take into account only the compound Poisson distribution and its

Brownian motion approximation. The dissertation has not taken into account other compound distributions such as the compound geometric distribution where the Tijms approximation may be used as an approximation.

iii The dissertation has also taken into account only the probability of ultimate ruin in the computation of the MIC. The probability of ruin in finite time has not been taken into account.

iv There has been no evaluation for single period models such as FFT (fast fourier transform) since the strength of ruin model is in tracing the probability of ruin over time.

v There has been no evaluation of acceptable ruin levels or, the modeling of negative surplus or, evaluation of the time to ruin. The assumption is that any amount of negative surplus causes ruin. In reality however, there exist facilities that assist firms in case of temporary shortfalls in operating capital.

3.2 DATA USED

The dissertation looks into the case of a hypothetical insurance company. Therefore the parameters used were arbitrarily chosen to demonstrate how the various methods chosen can be used to come up with a value for MIC to guide regulation or investors. The parameters chosen were: the ruin probability alpha, the number of claims N, the mean rate of claim severities and claim arrival and the premium loading factor.

3.3 DESCRIPTION OF APPROXIMATION METHODS

3.3.1 DISCRETE TIME BISECTION METHOD

Here the minimum initial capital is approximated using the bisection technique.

Theorem 3.1. *Let* $\alpha \in (0,1),\ N \in \{1,2,3,4,...\},\ u_0 \geq 0$*, and* v_0 *be such that* $v_0 < u_0$*. Let* $\{u_n\}_{n=1}^{\infty}$ *and* $\{v_n\}_{n=1}^{\infty}$ *be a real sequence defined by:*

$$\begin{cases} v_k = v_{k-1} \quad \text{and} \quad u_k = \dfrac{u_{k-1}+v_{k-1}}{2} \quad \text{if } \psi_N\left(\dfrac{u_{k-1}+v_{k-1}}{2}\right) \leq \alpha \\ v_k = \dfrac{v_{k-1}+u_{k-1}}{2} \quad \text{and} \quad u_k = u_{k-1} \quad \text{if } \psi_N\left(\dfrac{u_{k-1}+v_{k-1}}{2}\right) > \alpha \end{cases} \quad for \quad all\ k = 1,2,....$$

If $\psi_N(u_0) \leq \alpha < \psi_N(v_0)$*, then* $lim_{k\to\infty} u_k = MIC\big(\alpha, N, c, \{X_n, n \geq 1\}\big)$ *and* $0 \leq u_k - MIC\big(\alpha, N, c, \{X_n, n \geq 1\}\big) \leq \dfrac{u_0 - v_0}{2^k}$ $\quad for\ all\ k = 1,2,3,...$

The proof of this theorem can be found in Sattayatham et al, 2013.

3.3.2 COMPOUND POISSON PROCESS

. The case of a compound Poisson process is considered the classical ruin problem. It was first considered by Filip Lundberg in 1903 and later given rigorous mathematical proof by Swedish mathematician Harald Cramer. Thus, the risk process that utilizes the compound Poisson process is often known as the Cramer-Lundberg model or the classical ruin problem. The Poisson process used in this case is essentially a special case of a renewal process. The stationary and independent increments property of this process is what brings about the notion of the process starting over again once an event, ruin in this case, occurs

SURPLUS USING LUNDBERG'S INEQUALITY

A good indicator of the surplus requirements may be obtained using Lundberg's inequality. If we let $\psi(u)$ denote the probability of ultimate ruin with initial surplus u, then

$$\psi(u) \leq \exp\{-\kappa u\}$$

If we let α be the maximum ruin probability, we have

$$\psi(u) \ = \exp\{\ln \alpha\} \ \leq \exp\{-\kappa u\}$$

Thus

$$\therefore \ln \ \alpha \ = \ -\kappa u$$

$$u = \frac{-\ln \alpha}{\kappa}$$

Where κ is the adjustment coefficient

We can prove the Lundberg inequality using mathematical induction.

Proof of Lundberg Inequality. $\psi_n(u)$ is the probability that ruin occurs before claim number n. Also, $\psi(u) = lim_{n\to\infty}\psi_n(u)$. It will therefore be sufficient to show $\psi_n(u) \leq e^{-\kappa u}$ for n=0,1,2,...

If n=0 and $u \geq 0$, then $\psi_0(u) = 0$ which is less than or equal to $e^{-\kappa u}$

For n=1

$$\begin{aligned}
\psi_1(u) &= Pr[U(T_1) \leq 0 | U(0) = u] \\
&= \int_0^\infty Pr[U(T_1) < 0 | T_1 = t, U(0) = u] \lambda e^{-\lambda t} dt \\
&= \int_0^\infty Pr[u + ct - X_1 < 0] \lambda e^{-\lambda t} dt
\end{aligned}$$

Considering the first part of the integrand

$$\begin{aligned}
Pr[u + ct - X_1 \leq 0] &= Pr[u + ct > X_1] \\
&= Pr[X_1 > u + ct]
\end{aligned}$$

Taking $F(x)$ as the cdf of X_1

$$\begin{aligned}
Pr[X_1 \ \geq u \ + \ ct] &= \int_{u+ct}^\infty dF(x) \\
\therefore Pr[u + ct - X_1 \leq 0] &\leq \int_{u+ct}^\infty e^{-\kappa(u+ct-x) dF(x)} \\
&\leq e^{-\kappa(u+ct)} \int_{u+ct}^\infty e^{\kappa x} dF(x) \\
&\leq e^{-\kappa(u+ct)} \int_0^\infty e^{\kappa x} dF(x) \\
&\leq e^{-\kappa(u+ct)} M_X(\kappa)
\end{aligned}$$

Applying this inequality in the integrand

$$\psi_1(u) \le \int_0^\infty e^{-\kappa(u+ct)} M_X(\kappa)\lambda e^{-\lambda t} dt$$
$$\le e^{-\kappa u}\lambda M_X(\kappa)\int_0^\infty e^{-t(\kappa c+\lambda)} dt$$
$$\le e^{-\kappa u} M_X(\kappa)\frac{[-e^{-t(\kappa c+\lambda)}]_0^\infty}{\kappa c+\lambda}$$

Since $c = (1+\theta)\mu\lambda$

$$\psi_1(u) \le \frac{e^{-\kappa u} M_X(\kappa)\lambda}{(\kappa(1+\theta)\mu+1)\lambda}$$

From the definition of the adjustment coefficient

$$1 + (1+\theta)\mu\kappa = M_X(\kappa)$$

Thus

$$\psi_1(u) \le e^{-\kappa u}\frac{M_X(\kappa)}{M_X(\kappa)}$$

$$\psi_1(u) \le e^{-\kappa u} \text{for all } u \ge 0$$

Considering now when $n = 2$

We condition on both T_1 and X_1

$$\psi_2(u) = Pr[U(T_1) < 0 \text{ } or \text{ } U(T_2 < 0)|U(0) = u]$$
$$= \int_0^\infty \int_0^\infty Pr[U(T_1 < 0) or U(T_2) < 0|U(0) = u, T_1 = t, X_1 = x] dF(x)\lambda e^{-\lambda t} dt$$

Again looking at the probability in the integrand

$$Pr[U(T_1) < 0 \; or \; U(T_2) < 0 | U(0) = u, T_1 = t, X_1 = x] = \begin{cases} 1 & : u + ct - x < 0 \\ \psi_1(u + ct - x) & : u + ct - x \geq 0 \end{cases}$$

It can be observed that both the cases above are less than or equal to $e^{-\kappa(u+ct-x)}$

Thus

$$\psi_2(u) \leq \int_0^\infty \int_0^\infty e^{-\kappa(u+ct-x) dF(x)} \lambda e^{-\lambda t} dt$$

Thus

$$\begin{aligned}
\psi_2(u) &\leq \lambda e^{-\kappa u} \int_0^\infty \int_0^\infty e^{-\kappa(ct-x)} e^{-\lambda t} dF(x) \\
&\leq \lambda e^{-\kappa u} \int_0^\infty e^{-t(\kappa c+\lambda)} dt \int_0^\infty e^{\kappa x} dF(x) \\
&\leq \lambda e^{-\kappa u} \int_0^\infty e^{-t(\kappa c+\lambda)} M_X(\kappa) dt \\
&\leq \lambda e^{-\kappa u} M_X(\kappa) \frac{[-e^{-t(\kappa c+\lambda)}]_0^\infty}{\kappa c + \lambda} \\
&\leq \frac{\lambda e^{-\kappa u} M_X(\kappa)}{\kappa(1+\theta)\lambda\mu + \lambda}
\end{aligned}$$

The above follows from the expected value principle of premium calculation where there is an explicit loading for expenses in the premium calculation. In this case the premiums are the cash inflows that are expected to come in at a constant rate c.

Thus in our case.

$$c = (1 + \theta)\lambda\mu$$

Also from the definition of the adjustment coefficient as the solution κ (other than zero)to the equation

$$1 + (1+\theta)\kappa\mu = M_X(\kappa)$$

We have

$$\begin{aligned}
\psi_2(u) &\leq \frac{\lambda M_X(\kappa)e^{-\kappa u}}{\lambda(1+(1+\theta)\kappa\mu)} \\
&\leq \frac{\lambda M_X(\kappa)e^{-\kappa u}}{\lambda M_X(\kappa)}
\end{aligned}$$

$$\therefore \psi_2(u) \leq e^{-\kappa u}$$

For the general case we have:

$$Pr[U(T_{n\,+\,1}) < 0 \text{ for some } k\ \leq n\ +\ 1|U(0) = u, T_1 = t, X_1 = x]$$
$$= \begin{cases} 1 & : u + ct - x < 0 \\ \psi_n(u + ct - x) & : u + ct - x \geq 0 \end{cases}$$

$$\begin{aligned}
\psi_{n\,+\,1}\ &\leq\ \int_0^\infty \int_0^\infty e^{-\kappa(u\,+\,ct\,-\,x)} dF(x)\lambda e^{-\lambda t} dt \\
&\leq\ \int_0^\infty e^{-\kappa(u\,+\,ct)} M_x(\kappa)\lambda e^{-\lambda t} dt
\end{aligned}$$

and just as in the case where for n = 2

$$\psi_{n\,+\,1}\ \leq\ e^{-\kappa u}$$

We can therefore say by induction that

$$\psi_n(u) \leq e^{-\kappa u}$$

$$\Rightarrow lim_{n\rightarrow\infty}\psi_n(u) = \psi(u) \leq e^{-\kappa u}$$

$\square$

This inequality will be used as demonstrated before to obtain the initial surplus u, by making this quantity the subject of the formula. The fact that this inequality is an upper bound means that the value for u obtained will be prudent.

The task therefore is to obtain the adjustment coefficient for the distribution of S_N.

The adjustment coefficient used in this dissertation is obtained using the following approximation with the final inequality assumed to be an equality:

$$\begin{aligned}
1 + (1+\theta)\mu\kappa &= E(e^{\kappa x}) \\
&= E(1 + \kappa x + \frac{1}{2}\kappa^2 x^2 + ...) \\
&> E(1 + \kappa x + \frac{1}{2}\kappa^2 x^2) \\
&> 1 + \kappa E(x) + \frac{1}{2}\kappa^2 E(x^2) \\
\theta\kappa\mu &> \frac{1}{2}\kappa^2 E(x^2) \\
\frac{2\theta\mu}{E(x^2)} &> \kappa \\
\Rightarrow \kappa &< \frac{2\theta\mu}{E(x^2)}
\end{aligned}$$

(*Klugman et al 2004*)

SURPLUS USING CRAMER'S APPROXIMATION

Cramer's approximate formula for computing ruin probability is given below:

$$lim_{u\to\infty}e^{ku}\psi(u) = \frac{\theta\mu}{M'(k) - \mu(1+\theta)}$$

Which may also be written as

$$\psi(u) \sim Ce^{-ku}$$

Where :

$$C = \frac{\mu\theta}{M'(k) - \mu(1+\theta)}$$

and

$$\psi(u) \sim Ce^{-ku} \Rightarrow lim_{u\to\infty}\frac{e^{ku}\psi(u)}{C} = 1$$

The task again remains to compute the adjustment coefficient, κ, and to make u the subject of the formula for an independently determined ruin probability. If we choose a ruin probability of α then $\psi(u) = \alpha$ and make the asymptotic relationship an equality, we have:

$$\alpha = Ce^{ku}$$

$$u = -k\frac{\ln\alpha}{C}$$

Where C is as before and u is the initial surplus we want to determine. It can be seen that if $C \leq 1$ then Cramer's approximation is the same as the Lundberg inequality.

If the claim severities (x$'_i s$) have exponential densities (as was considered in this dissertation) then we can find the distribution function of the aggregate claims.

For one claim:

$$
\begin{aligned}
M_x(t) &= E[e^{tx}] = \int_0^\infty e^{tx}\mu e^{-\mu x}dx \\
&= \mu\int_0^\infty e^{-(\mu-t)x}dx \\
&= \frac{\mu}{\mu-t}\Big[-e^{-(\mu-t)}\Big]_0^\infty \\
&= \frac{\mu}{\mu-t}\Big[1-0\Big] \\
&= \left[\frac{\mu-t}{\mu}\right]^{-1} \\
M_x(t) &= \left[1-\frac{t}{\mu}\right]^{-1}
\end{aligned}
$$

Thus for all claim severities we have

$$
\begin{aligned}
M_{x_1+x_2+\ldots+x_N} &= E\Big[E\big[e^{t(x_1+x_2+\ldots+x_N)}\big]/N=n\Big] \\
&= \Big(E[e^{tx}]\Big)^n \\
&= \Big(M_x(t)\Big)^n \\
M_{S_N}(t) &= \left[1-\frac{t}{\mu}\right]^{-n} \qquad (3.1)
\end{aligned}
$$

The moment generating function for the aggregate claims is equal to the moment generating function of the gamma distribution with $\alpha = n$ and $\beta = \mu$ This was demonstrated earlier. It leads to an Erlang distribution that can be used as an alternative way of evaluation. This alternative is not explored further here however.

The moment generating function for the aggregate claims for n i.i.d exponential claims was the preferred method for evaluation of S_N.

3.3.3 BROWNIAN MOTION APPROXIMATION

The Brownian motion process with local drift term was used as an approximation to the compound Poisson process. Theorem 2.5 and corollary 2.2 are restated here again and a proof given.

Theorem: For the process U_t where:

- $U_t = u + W_t$
- $\{W_t; t \geq 0\}$ is a Brownian motion with local drift
- $U_0 = u$ is the initial capital

and the probability of ruin before time τ expressed as:

$$\begin{aligned}\psi(u,\tau) &= Prmin_{0<t<\tau}U_t < 0\\ &= Prmin_{0<t<\tau}U_0 + W_t < 0\\ &= Prmin_{0<t<\tau}W_t < -u\end{aligned}$$

the ruin probability is given by

$$\psi(u,\tau) = \Phi\left(\frac{-u + \mu\tau}{\sqrt{\sigma^2\tau}}\right) + \exp\left(\frac{-2\mu u}{\sigma^2}\right)\Phi\left(\frac{-u - \mu\tau}{\sqrt{\sigma^2\tau}}\right)$$

Where $\Phi(.)$ is the cumulative distribution function for the standard normal distribution.

Theorem When time (τ) tends to infinity the following corollary is obtained

Corollary: For the above process the probability of ultimate ruin is given by

$$\psi(u) \;=\; \exp\left(\frac{-2\mu u}{\sigma^2}\right)$$

Corollary From the corollary above, the initial capital may be obtained as

$$u \;=\; -\ln\;\psi(u)\,\frac{\sigma^2}{2\mu}$$

Brownian Motion Approximation of Compound Poisson Process. A sample path U_t with final level U_τ and crosses the barrier $U_t = 0$ in the time interval $(0, \tau)$ is one of two types:

1. Type A: where $U_\tau < 0$

2. Type B: where $U_\tau > 0$

A path of type B may be considered a reflection of another path of type A.

Let A_x and $B_x$denote the sets of all possible paths of type A and B respectively. Similarly, let $Pr\{A_x\}$ and $Pr\{B_x\}$ denote the total probability associated with the paths A_x and B_x respectively. Let also $U_\tau = x$ for A_x and $U_\tau = -x$ for B_x Then:

$$\begin{aligned}
\psi(u,\tau) &= Pr\{min_{0<t<\tau} U_t < 0 | U_0 = u\} \\
&= \int_{-\infty}^{0} Pr\{A_x\} + Pr\{B_x\} dx \\
&= \int_{-\infty}^{0} \frac{Pr\{U_\tau(x) = x\}}{Pr\{U_\tau(x) = x\}} Pr\{A_x\} + Pr\{B_x\} dx \\
&= \int_{-\infty}^{0} Pr\{U_\tau(x) = x\} \frac{Pr\{A_x\} + Pr\{B_x\}}{Pr\{A_x\}} dx \\
&= \int_{-\infty}^{0} Pr\{U_\tau(x) = x\} \left[1 + \frac{Pr\{B_x\}}{Pr\{A_x\}}\right] dx
\end{aligned}$$

The process $U_\tau - u$ is a Brownian motion then U_τ with mean $u + \mu\tau$ and variance $\sigma^2\tau$.

$$\therefore Pr\{U_\tau(x) = x\} = \frac{1}{\sqrt{2\pi\sigma^2\tau}} exp\left\{\frac{-(x-u-\mu\tau)^2}{2\sigma^2\tau}\right\}$$

To get $\frac{Pr\{B_x\}}{Pr\{A_x\}}$ we condition on all possible run times T.

$$\begin{aligned}
\frac{Pr\{B_x\}}{Pr\{A_x\}} &= \frac{\int_{-\infty}^{0} Pr\{B_x|T=t\}Pr\{T=t\}dt}{\int_{-\infty}^{0} Pr\{A_x|T=t\}Pr\{T=t\}dt} \\
&= \frac{\int_{-\infty}^{0} Pr\{U_\tau = -x\}Pr\{T=t\}dt}{\int_{-\infty}^{0} Pr\{U_\tau = x\}Pr\{T=t\}dt}
\end{aligned}$$

$$\begin{aligned}
Pr\{U_\tau = x|T=t\} &= Pr\{U_\tau - U_t = x\} \\
&= \frac{1}{\sqrt{2\pi\sigma^2(\tau-t)}} exp\left\{\frac{-(x-\mu(\tau-t))^2}{2\sigma^2(\tau-t)}\right\} \\
&= \frac{1}{\sqrt{2\pi\sigma^2(\tau-t)}} exp\left\{\frac{-(x^2-2x\mu(\tau-t)+\mu^2(\tau-t)^2)}{2\sigma^2(\tau-t)}\right\} \\
&= \frac{1}{\sqrt{2\pi\sigma^2(\tau-t)}} exp\left(\frac{x\mu}{\sigma^2}\right) exp\left\{\frac{-(x^2+\mu^2(\tau-t)^2)}{2\sigma^2(\tau-t)}\right\} \\
&= exp\left(\frac{x\mu}{\sigma^2}\right) \frac{1}{\sqrt{2\pi\sigma^2(\tau-t)}} exp\left\{\frac{-(x^2+\mu^2(\tau-t)^2)}{2\sigma^2(\tau-t)}\right\}
\end{aligned}$$

Similarly when we replace x with $-x$ we have:

$$\begin{aligned}
Pr\{U_\tau = -x|T=t\} &= Pr\{U_\tau - U_t = -x\} \\
&= exp\left(\frac{-x\mu}{\sigma^2}\right) \frac{1}{\sqrt{2\pi\sigma^2(\tau-t)}} exp\left\{\frac{-(x^2+\mu^2(\tau-t)^2)}{2\sigma^2(\tau-t)}\right\} \\
\therefore \frac{Pr\{B_x\}}{Pr\{A_x\}} &= exp\left\{-\frac{2\mu x}{\sigma^2}\right\}
\end{aligned}$$

Then:

$$\psi(u,\tau) = \int_{-\infty}^{0} Pr\{U_\tau(x) = x\}\left[1 + \frac{Pr\{B_x\}}{Pr\{A_x\}}\right]dx$$
$$= \int_{-\infty}^{0} Pr\{U_\tau = x\}dx + \int_{-\infty}^{0} Pr\{U_\tau = x\}\frac{Pr\{B_x\}}{Pr\{A_x\}}dx$$
$$\psi(u,\tau) = \Phi\left(\frac{-u+\mu\tau}{sqrt\sigma^2\tau}\right) + exp\left\{\frac{-2\mu u}{\sigma^2}\right\}\Phi\left(\frac{-u+\mu\tau}{sqrt\sigma^2\tau}\right)$$

And taking the limit as $\tau \ \rightarrow \ \infty$ we get

$$\psi(u) = \exp\left\{\frac{-2\mu u}{\sigma^2}\right\}$$

We note that $\Phi(.)$ is a defective cumulative normal distribution. □

Klugman, Panjer and Willmot(2004)

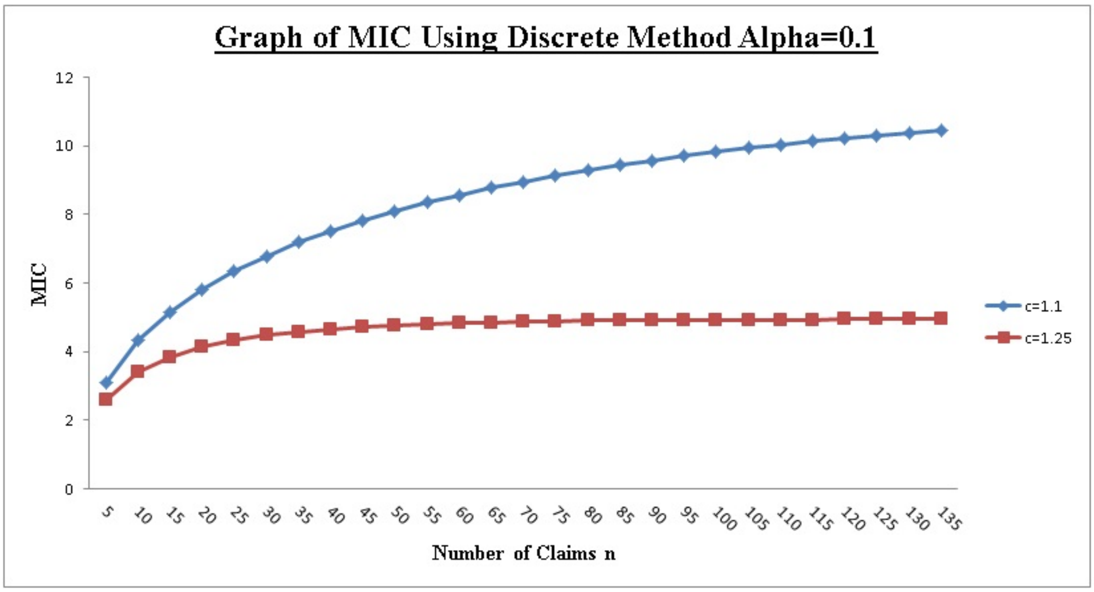
CHAPTER 4. NUMERICAL ANALYSIS

4.1 DISCRETE TIME MINIMUM INITIAL CAPITAL

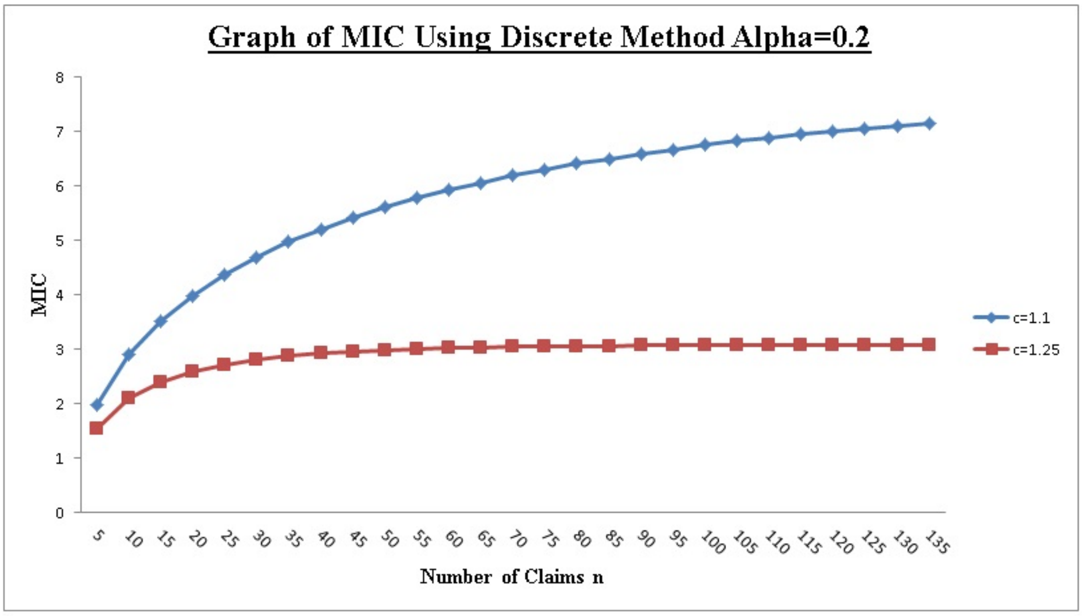
The graphs 4.1 and 4.2 are obtained from the results tabulated in table 4.1 below. Other tables are tabulated in the tables and figures sections. The MIC obtained as a result of the discrete method appears to be curvilinear. It is noticed that the higher the rate of premium income, then the lower the initial capital will be ceteris paribus. This is consistent with intuition as, if much of the risk is pushed to the consumers of the insurance products through a higher safety loading, then there should be more initial capital as opposed to the case where a product's inelasticity implies that the safety loading cannot be high. A high safety loading for inelastic products implies that the products will not be competitive and hence have low uptake. This is seen to be true for each of the predetermined ruin levels.

Table 4.1: MIC computed using discrete method

	alpha=0.1		alpha=0.2		alpha=0.3	
Number of Claims (n)	**c = 1.1**	**c = 1.25**	**c = 1.1**	**c = 1.25**	**c = 1.1**	**c = 1.25**
5	3.108841	2.608996	1.981775	1.533595	1.283336	0.877361
10	4.319788	3.397328	2.892986	2.093645	1.998658	1.298215
15	5.157427	3.843620	3.512266	2.397223	2.478046	1.519335
20	5.807574	4.132699	3.986288	2.587386	2.840990	1.654745
25	6.340315	4.332350	4.370061	2.715166	3.132105	1.744145
30	6.791095	4.475649	4.691300	2.804793	3.373776	1.805971
35	7.180778	4.581221	4.966256	2.869543	3.579084	1.850124
40	7.522858	4.660497	5.205403	2.917356	3.756431	1.882415
45	7.826648	4.720910	5.415947	2.953268	3.911574	1.906473
50	8.098894	4.767494	5.603086	2.980612	4.048656	1.924668
55	8.344659	4.803767	5.770712	3.001670	4.170761	1.938598
60	8.567860	4.832242	5.921820	3.018044	4.280255	1.949375
65	8.771593	4.854756	6.058772	3.030879	4.379002	1.957785
70	8.958358	4.872665	6.183465	3.041012	4.468489	1.964402
75	9.130206	4.886989	6.297449	3.049063	4.549926	1.969640
80	9.288840	4.898501	6.402008	3.055494	4.624313	1.973812
85	9.435694	4.907793	6.498217	3.060659	4.692482	1.977153
90	9.571986	4.915324	6.586986	3.064823	4.755138	1.979840
95	9.698762	4.921446	6.669092	3.068194	4.812877	1.982012
100	9.816929	4.926441	6.745204	3.070933	4.866212	1.983773
105	9.927272	4.930529	6.815904	3.073166	4.915586	1.985207
110	10.03048	4.933882	6.881695	3.074992	4.961385	1.986377
115	10.12717	4.936641	6.943025	3.076491	5.003945	1.987335
120	10.21788	4.938915	7.000285	3.077722	5.043562	1.988122
125	10.30308	4.940794	7.053822	3.078737	5.080498	1.988770
130	10.38321	4.942350	7.103944	3.079575	5.114982	1.989304
135	10.45865	4.943640	7.150931	3.080269	5.147223	1.989745



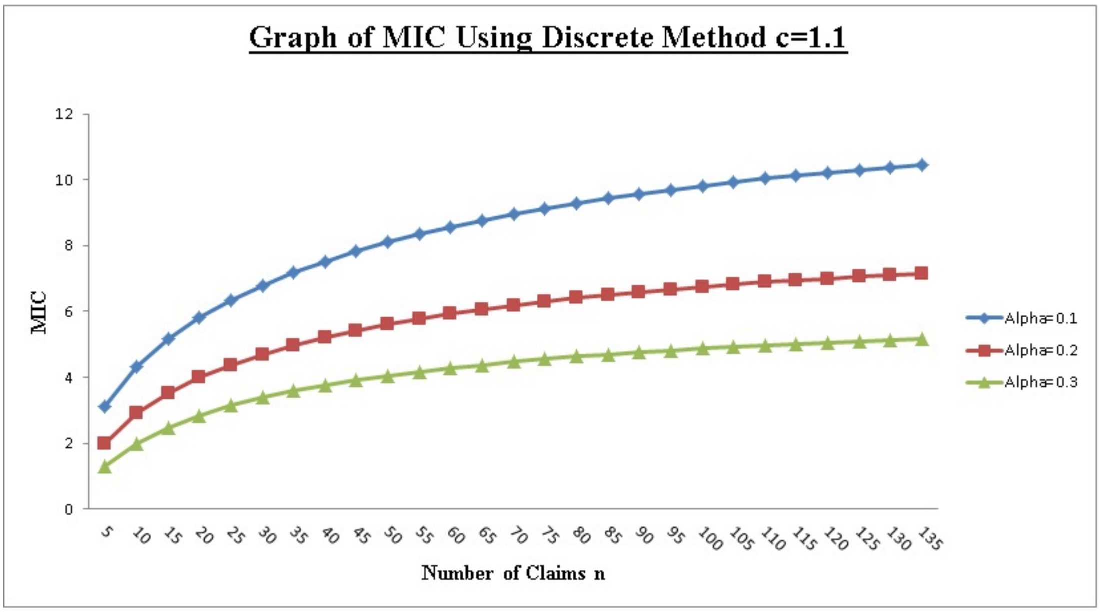
Graph 4.1



Graph 4.2

It is also seen that the higher the expected number of claims, the higher the initial capital should be to ensure that the given ruin probability is maintained. This may be explained by the intuition that if the company expects the number of claims to be large, then it must allow for a larger provision than if it expects claims during the period to be less in number. Since it may be difficult for a company to predict the expected number of claims for all the classes of business, it may be more practical to apply this system to individual portfolio of policies and maybe aggregate them later for the entire firm.

It should also be noticed that in comparison to continuous time cases (whose results will be seen later), the initial capital for the discrete cases is lower. This is because the probability of ruin is only required (or observed) at discrete time intervals.



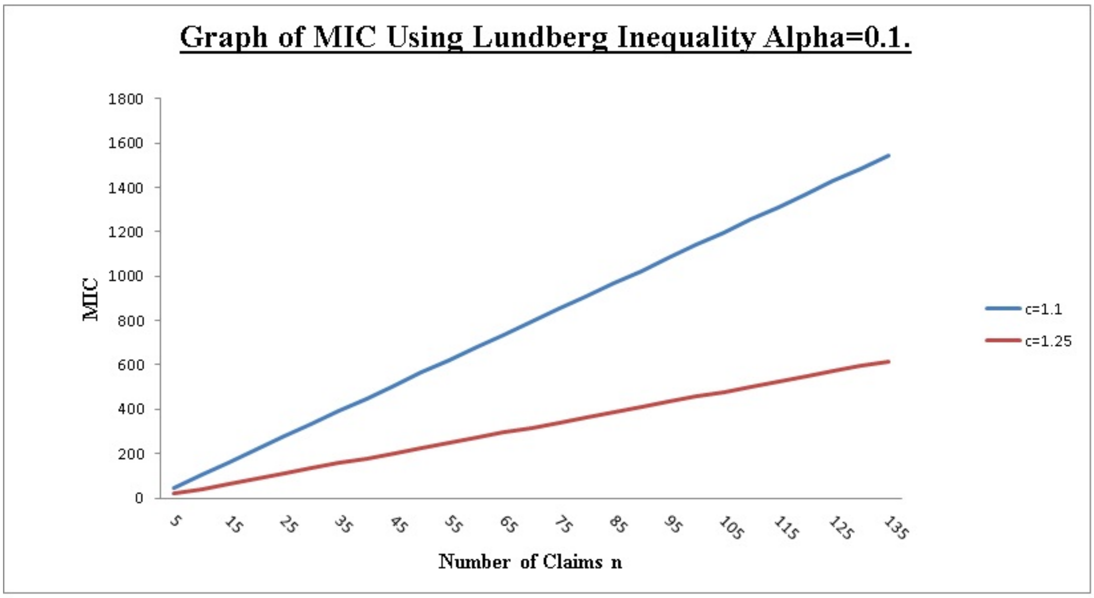
Graph 4.3

The graph 4.3 above demonstrates that for higher levels of ruin probability accommodated (i.e for a firm that does not want to tie up a lot of its initial capital as a provision for claims), then the required MIC is smaller for all number of claims than is the case for a more cautious firm which requires the probability of ruin to be small.

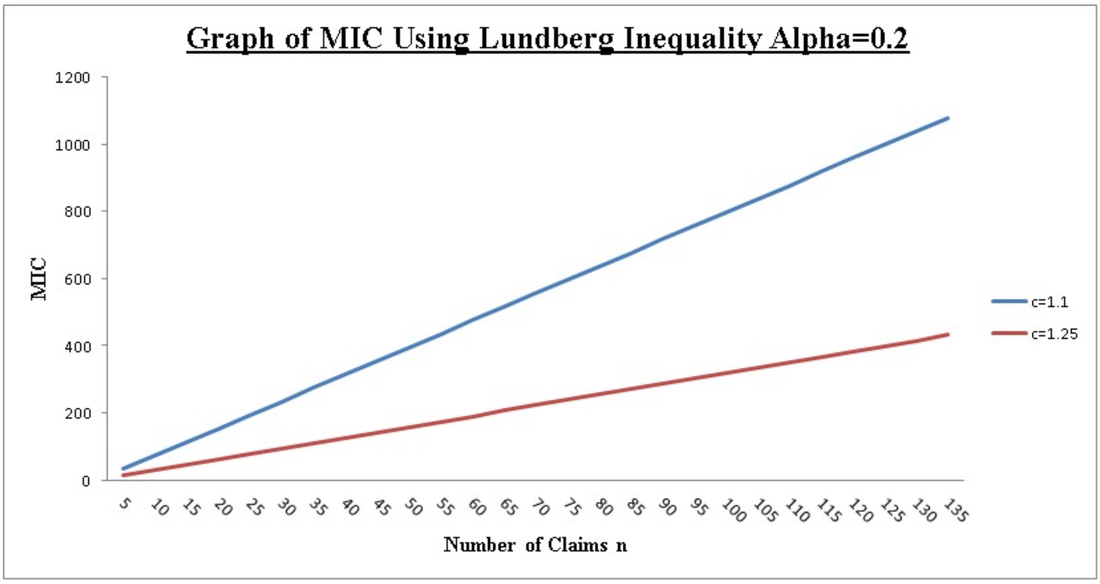
4.2 COMPOUND POISSON PROCESS

4.2.1 USING LUNDBERG'S INEQUALITY

The Lundberg inequality provides an upper limit to the probability of ruin and thus a more prudent MIC than exact methods. The graphs 4.4 and 4.5 indicate that the relationship between the number of claims and the MIC is a linear one. Therefore the higher the number of claims the higher will be the MIC.

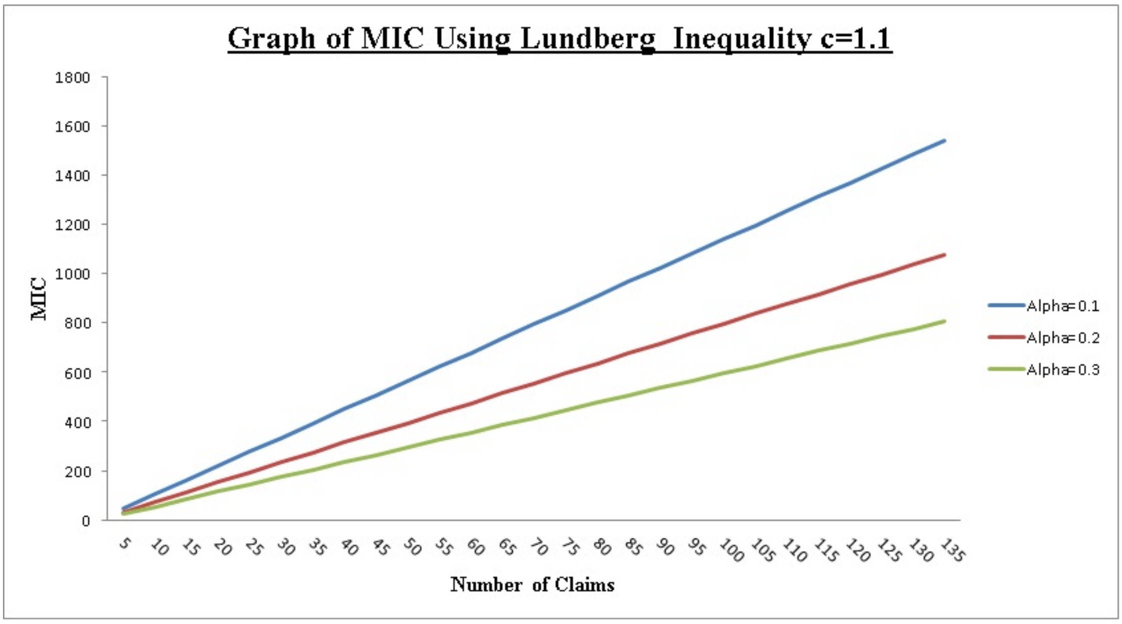


Graph 4.4



Graph 4.5

The graphs indicate that the higher the rate of income, which is directly influenced by the safety loading (θ), the lower the MIC. Similarly, the lower the rate of premium income, the higher the MIC to ensure the ruin probability is at the desired levels.

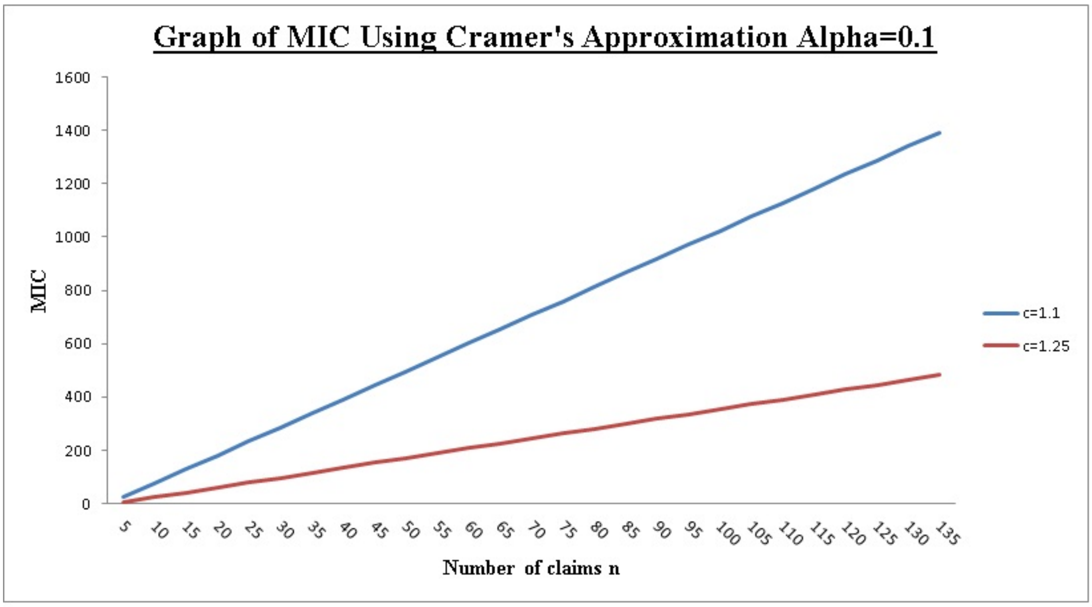


Graph 4.6

Just as was the case in the discrete case, the graph 4.6 (and consequently data on Table 5.2) also indicate that the lower the probability of ruin then the higher the MIC. Conversely, the higher the probability of ruin the lower the MIC.

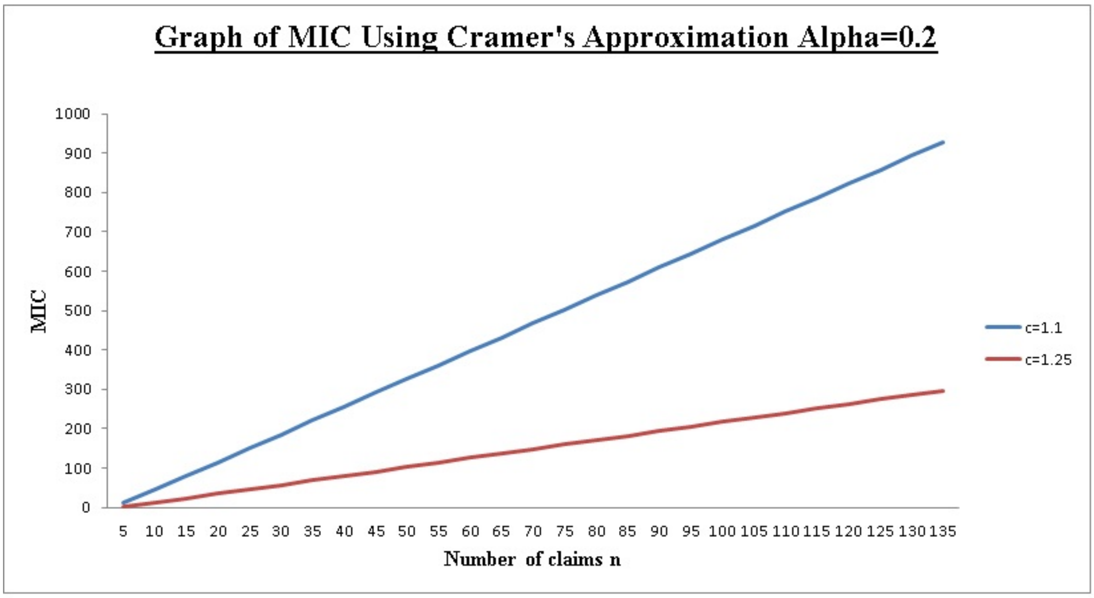
4.2.2 USING CRAMER'S APPROXIMATE FORMULA

As was the case when Lundberg's inequality was used to determine the MIC, Cramer's asymptotic formula demonstrates a linear relationship between the number of claims and the MIC. It also shows that for a higher safety loading coefficient and hence higher rate of premium income, the required MIC is small.



Graph 4.7

Cramer’s formula is more accurate (especially for exponential claim severities) as it is not an upper bound for the ruin probability. As a result, the associated MIC is smaller than that given by Lundberg inequality. The MIC obtained in this method may thus be of more interest to investors as it gives a more accurate value. This can be seen in the graphs 4.7 and 4.8 below.

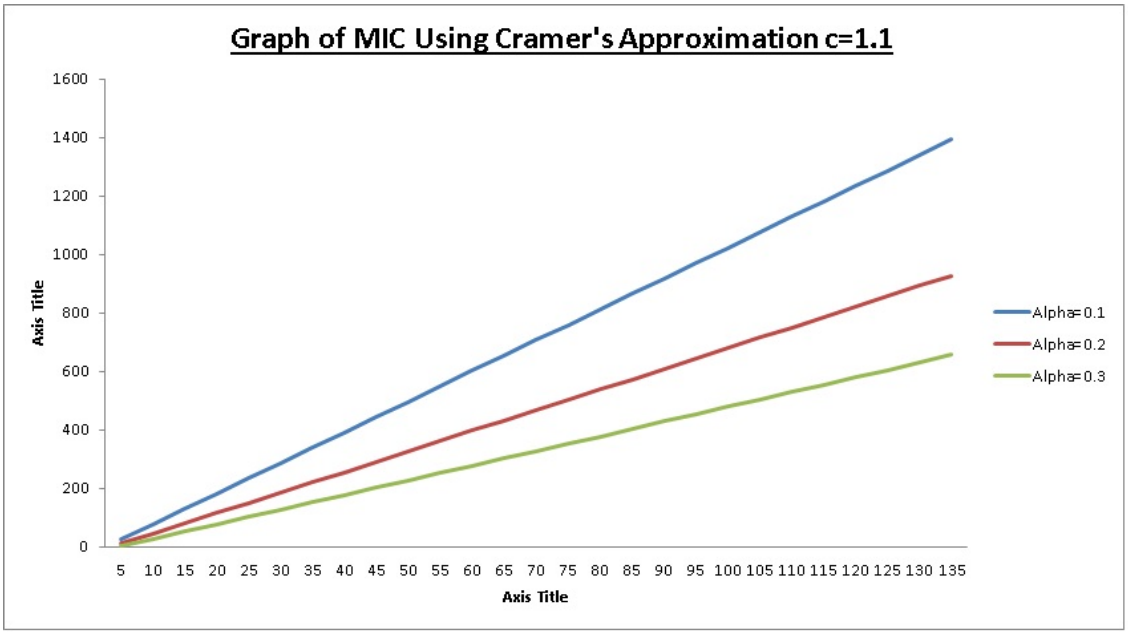


Graph 4.8

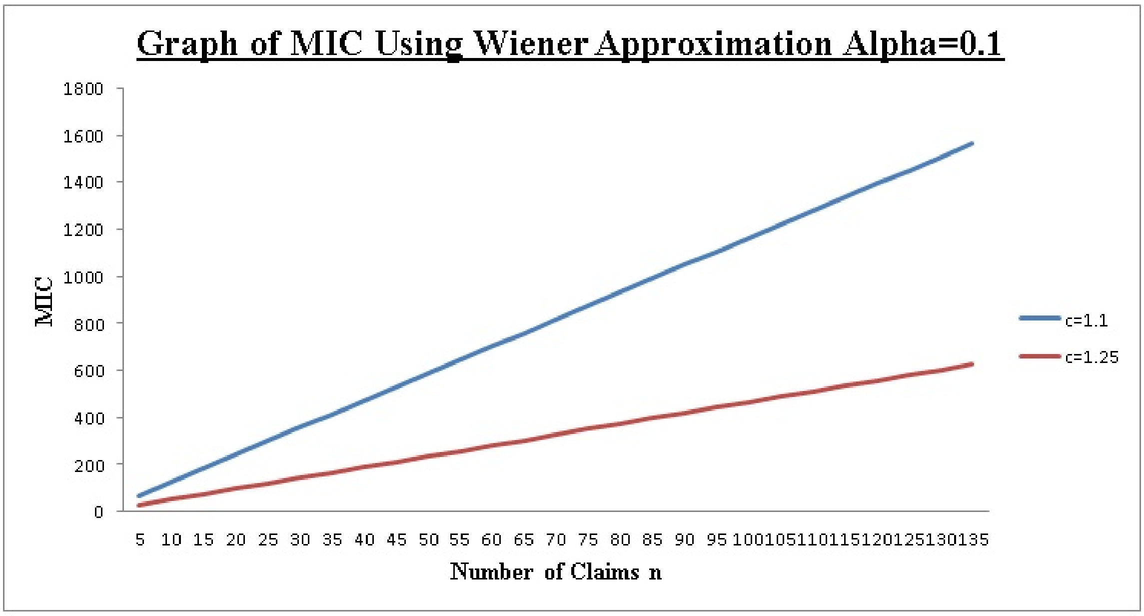
It is again seen, in the graph 4.9 below, that the MIC is greater for a lower probability of ruin than it is for a higher probability of ruin. These relationships may also be seen in table 5.4.

4.3 BROWNIAN MOTION APPROXIMATION TO COMPOUND POISSON PROCESS

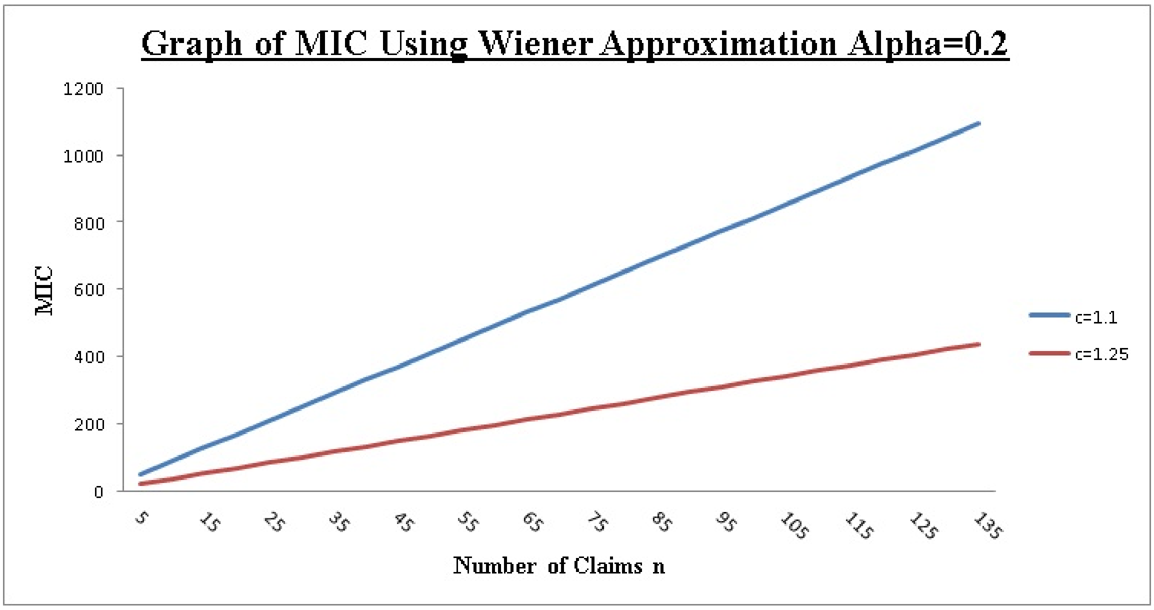
The graphs 4.10 and 4.11 of the Wiener/Brownian motion MIC also show a linear relationship between the number of claims and the initial MIC. Just like the case in the previous methods, the higher the rate of premium income the lower the MIC. Similarly, the lower the rate of premium income the lower the MIC. This can also be seen in table 5.3.



Graph 4.9

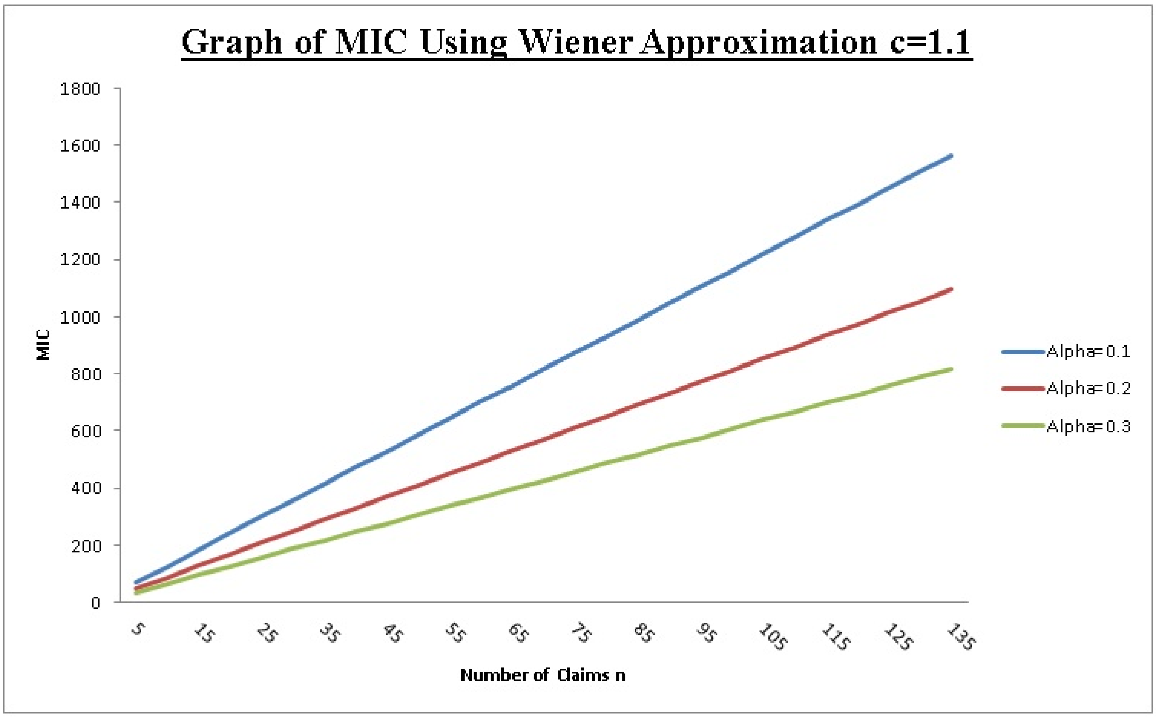


Graph 4.10



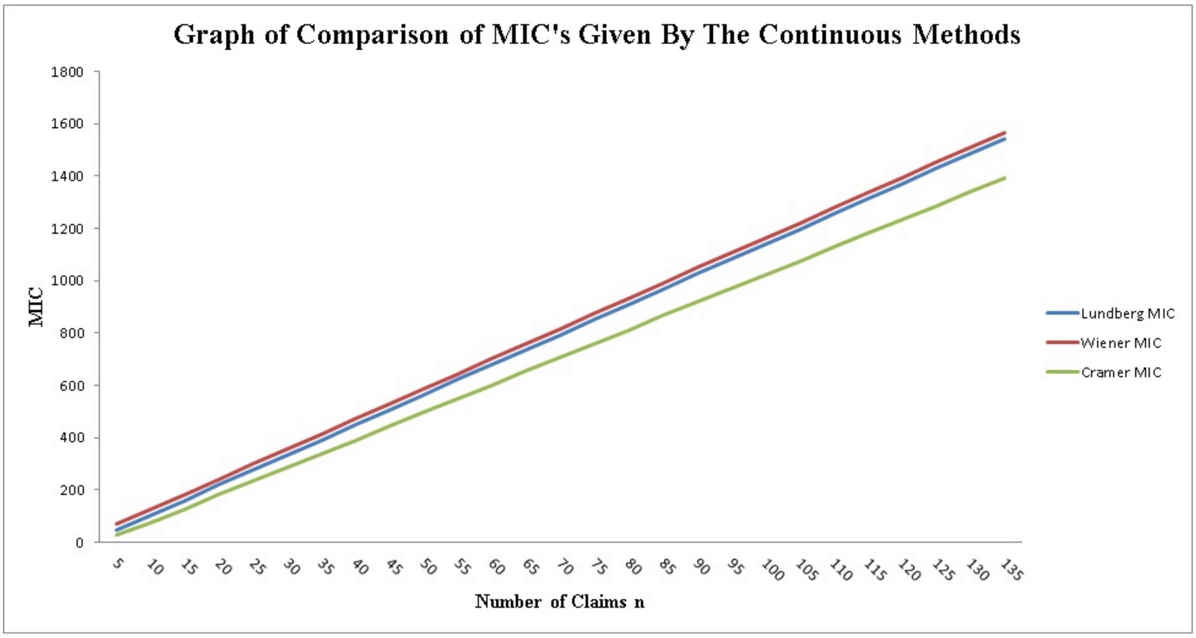
Graph 4.11

Again the graph 4.12 below shows that the higher the level of ruin probability the lower the MIC required. Similarly, the lower the level of ruin probability, the higher the required MIC.



Graph 4.12

Comparing all the continuous methods (Graph 4.13), it is clear that the Lundberg method is higher than the Cramer method of computing the MIC. This is because the Lundberg inequality provides an upper limit to ruin probability whereas Cramer's approximation is designed to be accurate for large values of the MIC. However, Cramer's asymptotic ruin formula is known to be accurate for even small MIC's especially in the case of exponential claim severities. Thus Cramer's formula provides the most accurate MIC. However, if prudence is require then the Lundberg MIC may be used especially for regulatory purposes or as a rough approximation.



Graph 4.13

The MIC given by the Brownian motion approximation of the compound Poisson process is even higher than that given by Lundberg's inequality. This may be attributed to the fact that the Brownian motion process takes into account the variation of the claim numbers. The slight difference between the Lundberg generated MIC and the Brownian motion generated MIC may thus be attributed to variance in claim numbers and hence the difference arises the evaluation of the aggregate claims. It should not be forgotten that the two processes (compound Poisson and Brownian motion) may be used as approximations to each other.

CHAPTER 5. SUMMARY AND CONCLUSION

It is seen from the dissertation that there is a big difference between the discrete time ruin MIC and the continuous time ruin MIC's. This can be attributed to the fact that in the discrete time the condition for ruin to occur is weaker than in the continuous time analogues. That is to say that in the discrete time case, ruin is checked only at integer time intervals and the requirement is that there is no ruin at the time ruin is checked for. This implies that ruin may have occurred in the interval between the checking times and then the surplus restored by the time of checking. The continuous time MIC requires that the surplus be monitored continuously and ensures that ruin does not occur at any point in time. Therefore the MIC given by the continuous time models provides more prudence in that they require higher initial reserves. This MIC may be seen as prudent and one which ensures that at the time of usage, the investors are highly guaranteed return on their investments. The continuous time models may also be used by regulatory bodies to protect customers who consume insurance products. However, in the case of an established firm that is launching a new product line, the higher MIC may be seen as providing an opportunity cost in that more funds are tied up to ensure desired levels of ruin are attained instead of being channeled into other income generating ventures.

It is also evident that in the continuous case, there is a direct linear relationship between the number of claims and the MIC. This means that a straight line may be used to forecast future MIC's barring any significant changes in the important parameters such as the safety loading, the mean size of the claim severities, the mean of claim numbers or even the probability distribution of the claim severities. In the case of the discrete time, it is observed that for a higher number of claim numbers the relationship is almost linear and thus can be approximated with the use of a regression line. However, for lower claim numbers, the graph of claims Vs MIC appears concave and probably fitting a distribution using moments or any other technique may be used to forecast the future MIC as long as there are no significant changes in the parameters of interest or the distribution of claim numbers and/or claim severities.

It is also noticed from the graphs and tables that the higher the rate of premium income c, which is essentially controlled by the premium loading factor theta, the lower the MIC required. The rationale is, it is expected that the constant trickle of income through premiums will be higher with a higher safety loading and thus the required MIC at the beginning of the period will be lower than when the safety loading is lower. The higher the expected flow of future income through premiums, the lower the MIC.

It is also observed that the higher the required level of ruin, the lower the MIC. It is easy to see this because a high level of ruin probability implies that there is a high risk appetite within the firm and therefore a low provision for the eventuality of insolvency is taken into account. However if the required ruin probability is low then there is a low risk attitude adopted and hence a higher provision for ruin required.

It can also be seen that the MIC derived in this dissertation is somehow related to if not equal to a value at risk (VaR) measure. The major difference between the MIC obtained and the VaR is that VaR is usually for a given unit period and in practice this is usually one year. In determination of the reserves required to meet the liabilities associated with the launch of a new product or in the opening of an insurance company, a time horizon of one year may not be ideal and thus using Var may not be as prudent as using MIC. Also, VaR is derived from ruin theory and the MIC uses a model that can incorporate many other factors that VaR may not be able to capture thus giving enhance flexibility. for instance, dividend payments and interest incomes may easily be incorporated in the VaR, negative surplus and time to ruin may also be evaluated to possibly come up with different MIC's.

The study shows that if the claims data can be properly fit into an exponential distribution for claim severities and into a Poisson distribution for the claim numbers, then the classical ruin model may be used to obtain a probability of ruin. If a company is starting a new line of business or investors want to start an insurance company, they may benchmark their expected losses and premium income with the market (if possible) or with another company to determine the important parameters that are used in the classical Cramer-Lundberg model and use that to determine their MIC.

It is not lost that the classical model is a starting point in the study of ruin theory and there have indeed been major improvements and advancements in this area. Some of the assumptions of this classical model may be relaxed or changed (such as using stochastic rate of premium income instead of a constant rate or the inclusion of dividend payments and/or interest income) to come up with more realistic models. Many of these models may be difficult to apply and hence there are equally as many approximations. The adjustment coefficient is also a big factor as it may not exist for sub exponential distributions such as the log normal distribution and the Pareto distribution among others. The aspect of zero inflation of probability distributions and the distribution of claim amounts is also important. Perhaps the most important aspect is stochastic claims process: whether they should be treated as aggregate or as individual claims and whether they should be i.i.d or not.

The dissertation therefore recognizes that although the classical model may be easier to understand and apply, it may not be the most realistic model to aid in regulation or in investment. It is however a good start especially for developing countries to understand and build models that are tailor made to suit the environment that the insurance companies operate in and have the ability to make changes (and improvements) when required.

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APPENDIX

R Code for Discrete MIC

```
rm(list=ls())

##The gmp package checks large factorials



library("gmp")

pdruin<-function(u,c,n,rate)

{

        if(n<=170){

        A<-(u+c)*(rate^(n-1))

        B<-(u+(n*c))^(n-2)

        C<-(A*B)/factorial(n-1)

        D<-rate*(u + (n*c))

        E<-exp(-D)

        G<-C*E

        return(G)


        }else if(n>170){

        A<-(u+c)*(rate^(n-1))

        B<-(u+(n*c))^(n-2)

        C<-(A*B)/factorialZ(n-1)
```

```
        D<-rate*(u + (n*c))

        E<-exp(-D)

        G<-C*E

        return(G)

                }

}



Bisec<-function(v,z,alpha){

        newu<-u

        u.up<-20

        u.down<-0

        it<-0

        ruin<-rep(n)

        for(i in 1:n){

        if(i<=0){0}

        if(i>0) {ruin[i]<-pdruin(newu,c,i,rate)}

        adruin<-sum(ruin)

        while(it<25 & adruin!=alpha){

        it<-it+1

        if(adruin<alpha){

                        u.up<-newu

                        newu<-(u.down+newu)/2

                        #ruin[i]<-pdruin(u=newu,c,i,rate)
```

```
                        #return(sum(ruin))


          }else if(adruin>alpha){
                 u.down<-newu
                 newu<-(u.up+newu)/2
                 #ruin[i]<-pdruin(u=newu,c,i,rate)
                 #return(sum(ruin))
                  }
         ruin<-rep(n)
         for(i in 1:n){
         if(i<=0){0}
         if(i>0) {ruin[i]<-pdruin(newu,c,i,rate)}
         adruin<-sum(ruin)
                 }
         }
         output<-data.frame(cat(c(newu,ruin[n],adruin,it=it),sep="\t"))
         return(output)
         }
}
#### An example using some of the parameters
c<-1.1
n<-10
alpha<-0.1
rate<-1
```

```
v<-0

z<-20

u<-z

Bisec(v,z,alpha)
```

TABLES AND FIGURES

Table 5.1: MIC computed using Lundberg inequality

	alpha=0.1		alpha=0.2		alpha=0.3	
Number of Claims (n)	**c = 1.1**	**c = 1.25**	**c = 1.1**	**c = 1.25**	**c = 1.1**	**c = 1.25**
5	46.05170	18.42068	32.18876	12.87550	24.07946	9.631782
10	103.6163	41.44653	72.42471	28.96988	54.17878	21.67151
15	161.1810	64.47238	112.6607	45.06426	84.27810	33.71124
20	218.7456	87.49823	152.8966	61.15864	114.3774	45.75097
25	276.3102	110.5241	193.1325	77.25302	144.4767	57.79069
30	333.8748	133.5499	233.3685	93.34740	174.5761	69.83042
35	391.4395	156.5758	273.6044	109.4418	204.6754	81.87015
40	449.0041	179.6016	313.8404	125.5362	234.7747	93.90988
45	506.5687	202.6275	354.0763	141.6305	264.8740	105.9496
50	564.1333	225.6533	394.3123	157.7249	294.9733	117.9893
55	621.6980	248.6792	434.5482	173.8193	325.0727	130.0291
60	679.2626	271.7050	474.7842	189.9137	355.1720	142.0688
65	736.8272	294.7309	515.0201	206.0081	385.2713	154.1085
70	794.3919	317.7567	555.2561	222.1024	415.3706	166.1482
75	851.9565	340.7826	595.4920	238.1968	445.4699	178.1880
80	909.5211	363.8084	635.7280	254.2912	475.5693	190.2277
85	967.0857	386.8343	675.9639	270.3856	505.6686	202.2674
90	1024.650	409.8601	716.1999	286.4799	535.7679	214.3072
95	1082.215	432.8860	756.4358	302.5743	565.8672	226.3469
100	1139.780	455.9118	796.6718	318.6687	595.9665	238.3866
105	1197.344	478.9377	836.9077	334.7631	626.0659	250.4263
110	1254.909	501.9636	877.1437	350.8575	656.1652	262.4661
115	1312.474	524.9894	917.3796	366.9518	686.2645	274.5058
120	1370.038	548.0153	957.6156	383.0462	716.3638	286.5455
125	1427.603	571.0411	997.8515	399.1406	746.4631	298.5853
130	1485.167	594.0670	1038.087	415.2350	776.5625	310.6250
135	1542.732	617.0928	1078.323	431.3294	806.6618	322.6647

Table 5.2: MIC computed using Brownian motion approximation

	alpha=0.1		alpha=0.2		alpha=0.3	
Number of Claims (n)	**c = 1.1**	**c = 1.25**	**c = 1.1**	**c = 1.25**	**c = 1.1**	**c = 1.25**
5	69.077553	27.631021	48.283137	19.313255	36.119184	14.447674
10	126.642180	50.656872	88.519085	35.407634	66.218504	26.487402
15	184.206807	73.682723	128.755033	51.502013	96.317824	38.527130
20	241.771435	96.708574	168.990981	67.596392	126.417144	50.566858
25	299.336062	119.734425	209.226929	83.690771	156.516465	62.606586
30	356.900689	142.760276	249.462876	99.785151	186.615785	74.646314
35	414.465317	165.786127	289.698824	115.879530	216.715105	86.686042
40	472.029944	188.811978	329.934772	131.973909	246.814425	98.725770
45	529.594571	211.837829	370.170720	148.068288	276.913745	110.765498
50	587.159199	234.863679	410.406668	164.162667	307.013065	122.805226
55	644.723826	257.889530	450.642615	180.257046	337.112385	134.844954
60	702.288453	280.915381	490.878563	196.351425	367.211705	146.884682
65	759.853081	303.941232	531.114511	212.445804	397.311025	158.924410
70	817.417708	326.967083	571.350459	228.540184	427.410346	170.964138
75	874.982335	349.992934	611.586407	244.634563	457.509666	183.003866
80	932.546963	373.018785	651.822355	260.728942	487.608986	195.043594
85	990.111590	396.044636	692.058302	276.823321	517.708306	207.083322
90	1047.676217	419.070487	732.294250	292.917700	547.807626	219.123050
95	1105.240845	442.096338	772.530198	309.012079	577.906946	231.162778
100	1162.805472	465.122189	812.766146	325.106458	608.006266	243.202506
105	1220.370099	488.148040	853.002094	341.200837	638.105586	255.242235
110	1277.934727	511.173891	893.238041	357.295217	668.204906	267.281963
115	1335.499354	534.199742	933.473989	373.389596	698.304227	279.321691
120	1393.063981	557.225593	973.709937	389.483975	728.403547	291.361419
125	1450.628609	580.251443	1013.945885	405.578354	758.502867	303.401147
130	1508.193236	603.277294	1054.181833	421.672733	788.602187	315.440875
135	1565.757863	626.303145	1094.417780	437.767112	818.701507	327.480603

Table 5.3: MIC computed using Cramer's asymptotic formula

	alpha=0.1		alpha=0.2		alpha=0.3	
Number of Claims (n)	**c = 1.1**	**c = 1.25**	**c = 1.1**	**c = 1.25**	**c = 1.1**	**c = 1.25**
5	26.91271	7.506759299	13.04977	1.961582	4.940468	-1.28214
10	77.05682	24.94606926	45.86519	12.46942	27.61926	5.171048
15	128.7739	42.94030073	80.25359	23.53218	51.87103	12.17916
20	180.9566	61.10315531	115.1076	34.76356	76.58844	19.35589
25	233.3406	79.33936577	150.1629	46.06830	101.5071	26.60598
30	285.8299	97.61407489	185.3236	57.41154	126.5311	33.89456
35	338.3813	115.9114958	220.5463	68.77749	151.6172	41.20586
40	390.9723	134.2234369	255.8086	80.15796	176.7429	48.53168
45	443.5902	152.5452240	291.0978	91.54827	201.8955	55.86734
50	496.2271	170.8739950	326.4060	102.9456	227.067	63.20999
55	548.8779	189.2078992	361.7282	114.3480	252.2526	70.55777
60	601.5394	207.5456869	397.0610	125.7543	277.4488	77.90944
65	654.2091	225.8864835	432.4020	137.1636	302.6531	85.26411
70	706.8852	244.2296589	467.7494	148.5753	327.8640	92.62116
75	759.5666	262.5747472	503.1021	159.9890	353.0800	99.98013
80	812.2522	280.9213971	538.4590	171.4041	378.3003	107.3407
85	864.9413	299.2693380	573.8195	182.8206	403.5241	114.7025
90	917.6334	317.6183586	609.1829	194.2382	428.7509	122.0654
95	970.3279	335.9682913	644.5487	205.6566	453.9801	129.4292
100	1023.025	354.3190014	679.9167	217.0759	479.2115	136.7938
105	1075.723	372.6703797	715.2865	228.4958	504.4446	144.159
110	1128.423	391.0223362	750.6579	239.9163	529.6794	151.5249
115	1181.125	409.3747967	786.0306	251.3372	554.9155	158.8912
120	1233.827	427.7276989	821.4046	262.7587	580.1528	166.258
125	1286.531	446.0809906	856.7796	274.1805	605.3912	173.6251
130	1339.235	464.4346274	892.1555	285.6027	630.6305	180.9927
135	1391.941	482.7885714	927.5323	297.0251	655.8707	188.3605

Table 5.4: Adjustment coefficient κ

Number of Claims (n)	**c=1.1**	**c=1.25**
5	0.050000	0.125000
10	0.022222	0.055556
15	0.014286	0.035714
20	0.010526	0.026316
25	0.008333	0.020833
30	0.006897	0.017241
35	0.005882	0.014706
40	0.005128	0.012821
45	0.004545	0.011364
50	0.004082	0.010204
55	0.003704	0.009259
60	0.003390	0.008475
65	0.003125	0.007813
70	0.002899	0.007246
75	0.002703	0.006757
80	0.002532	0.006329
85	0.002381	0.005952
90	0.002247	0.005618
95	0.002128	0.005319
100	0.002020	0.005051
105	0.001923	0.004808
110	0.001835	0.004587
115	0.001754	0.004386
120	0.001681	0.004202
125	0.001613	0.004032
130	0.001550	0.003876
135	0.001493	0.003731

Table 5.5: Cramer's coefficient C

			Cramer's C	
Number of Claims (n)	**M'(k) for c = 1.1**	**M'(k) for c = 1.25**	**c=1.1**	**c=1.25**
5	1.360374142	2.228187235	0.384063	0.2555748
10	1.280436985	1.875251066	0.554210	0.3998394
15	1.258876680	1.789403469	0.629419	0.4634750
20	1.248851459	1.750730604	0.671811	0.4992705
25	1.243058023	1.728732773	0.699017	0.5222120
30	1.239283919	1.714538630	0.717958	0.5381684
35	1.236630073	1.704621494	0.731903	0.5499080
40	1.234662162	1.697301434	0.742599	0.5589072
45	1.233144684	1.691676430	0.751063	0.5660252
50	1.231938873	1.687218826	0.757927	0.5717961
55	1.230957653	1.683599379	0.763606	0.5765691
60	1.230143623	1.680602012	0.768382	0.5805825
65	1.229457406	1.678079043	0.772455	0.5840043
70	1.228871091	1.675926107	0.775969	0.5869563
75	1.228364342	1.674067361	0.779033	0.5895290
80	1.227921993	1.672446362	0.781726	0.5917911
85	1.227532500	1.671020238	0.784114	0.5937957
90	1.227186923	1.669755840	0.786244	0.5955843
95	1.226878232	1.668627135	0.788157	0.5971901
100	1.226600821	1.667613392	0.789884	0.5986398
105	1.226350164	1.666697898	0.791451	0.5999550
110	1.226122568	1.665867030	0.792880	0.6011537
115	1.225914992	1.665109573	0.794187	0.6022506
120	1.225724905	1.664416209	0.795387	0.6032583
125	1.225550186	1.663779134	0.796494	0.6041871
130	1.225389044	1.663191760	0.797518	0.6050459
135	1.225239956	1.662648490	0.798467	0.6058425