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Full Length Research Paper

Subdegrees and suborbital graphs of symmetric group s_n (n = 3, 4, 5) acting on unordered pairs

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In this research paper, we compute the ranks and subdegrees of the symmetric group S_n(n = 3, 4, 5) acting on unordered pairs from the set X = $\{1, 2, \ldots, n\}$. When S_n (n ≥ 4) acts on unordered pairs from X, the rank is 3.Therefore the main study will be on the subdegrees of the suborbitals. The suborbital graphs corresponding to the suborbitals of these actions are also constructed. The graph theoretic properties of these suborbital graphs are also discussed. When S_n (n ≥ 4) acts on unordered pairs the suborbital graphs Γ_1 and Γ_2 corresponding to the non-trivial suborbits Δ_1 and Δ_2, are connected, regularand complementary.

Keywords: Subdegrees, suborbital graphs of symmetric group, unordered pairs

INTRODUCTION

In this paper we investigate some properties of the symmetric group S_n (n = 3, 4, 5) acting on unordered pairs from X = $\{1, 2, \ldots, n\}$.We also find suborbits and suborbitals of S_n (n = 3, 4, 5) and construct suborbital graphs corresponding to these suborbitals. We shall also discuss some of the graph theoretic properties of these suborbital graphs.

This paper is divided into three parts; with our main results in part two.

In part one, we give definitions and preliminary results needed throughout the paper.

In part two, we investigate some properties of the action of S_n (n = 3, 4, 5) on unordered pairs.

Next, we find the ranks, suborbits and construct suborbital graphs, corresponding to suborbitals of S_n (n = 3, 4, 5). We also discuss the graph theoretic properties of these suborbital graphs.

Finally in part three, we give conclusions.

Definitions and preliminaries

We establish background information and results that will be used throughout this paper.

Notations

$\sum_i$ – Sum over i.
$\binom{a}{b}$ - a combination b.
S_n - Symmetric group of degree n and order n!.
$|G|$ - The order of a group G.
$X^{(2)}$ - The set of unordered pairs from the set X = $\{1, 2, \ldots, n\}$.
$\{t, q\}$ - Unordered pair.
X×Y – Cartesian product of X and Y.

Permutation groups

Definition 1.2.1

Let X be a non-empty set. *A* permutation of X is a one-to-one mapping of X onto itself.

Definition 1.2.2

Let X be the set $\{1, 2, \ldots, n\}$, then the symmetric group of degree n is the group of all permutations of X under the binary operation of composition of maps. It is denoted as S_n and has an order n!.

Definition 1.2.3

A permutation of a finite set is even or odd according as it can be expressed as the product of an even or odd number of 2-cycles (transpositions).

Group actions

Definition 1.3.1

Let X be a non-empty set. The group G acts on the left on X if for each $g \in G$ and each $x \in X$ there corresponds a unique element $gx \in X$ such that:
(i) $(g_1 g_2)x = g_1(g_2 x), \forall g_1, g_2 \in G$ and $x \in X$.
(ii) For any $x \in X$, 1x = x, where 1 is the identity in G. The action of G from the right on X can be defined in a similar way. In fact it is merely a matter of taste whether one writes the group element on the left or on the right.

Definition 1.3.2

Let G act on a set X. Then X is partitioned into disjoint equivalence classes called orbits or transitivity classes of the action. For each $x \in X$, the orbit containing x is called the orbit of x and is denoted by $\mathrm{Orb}_G(x)$.

Definition 1.3.3

If a finite group G acts on a set X with n elements, each $g \in G$ corresponds to a permutation σ of X, which can be written uniquely as a product of disjoint cycles. If σ has $\propto_1$ cycles of length 1, $\propto_2$ cycles of length 2, $\propto_3$ cycles of length 3, …, $\propto_n$ cycles of length n; then we say that σ and hence g has a cycle type $(\propto_1, \propto_2, \propto_3, \ldots, \propto_n)$.

Definition 1.3.4

If the action of a group G on a set X has only one orbit, then we say that G acts transitively on X. In other words, G acts transitively on X if for every pair of points $x, y \in X$, there exists $g \in G$ such that gx = y.

Theorem 1.3.5 [Krishnamurthy, 1985, p. 68].
Two permutations in S_n are conjugate if and only if, they have the same cycle type, and if $g \in S_n$ has a cycle type $(\propto_1, \propto_2, \ldots \propto_n)$, then the number of permutations in S_n conjugate to g is

$$\frac{n!}{\prod_{i=1}^{n} \propto_i !\, i^{\propto_i}}.$$

Theorem 1.3.6 [Cauchy – Frobenius Lemma – Rotman,1973,p.45.
Let G be a group acting on a finite set X. then the number of G-orbits in X

$$\frac{1}{|G|}\sum_{g \in G} |\mathrm{Fix}\ (g)|.$$

Graphs

Definition 1.4.1

A graph is a diagram consisting of a set V whose elements are called vertices, nodes or points and a set E of unordered pairs of vertices called edges or lines. We denote such a graph by G(V,E) or simply by G if there is no ambiguity of V and E.

Definition 1.4.2

A graph consisting of one vertex and no edges is called a trivial graph.

Definition 1.4.3

A graph whose edge set is empty is called a null graph.

Definition 1.4.4

If we allow the existence of loops (edges joining vertices, to themselves) and multiple edges (more than one edge joining two distinct vertices), then we get a multigraph.

Definition 1.4.5

A graph with no loops or multiple edges is called a simple graph.

Definition 1.4.6

The degree (valency) of a vertex v of G(V,E) is the number of edges incident to v.

Definition 1.4.7

Any vertex of degree zero is called an isolated vertex.

Definition 1.4.8

A graph G(V,E) is said to be connected if there is a path between any two of its vertices.

Definition 1.4.9

The girth of a graph G(V,E) is the length of the shortest cycle if any in G(V,E).

Definition 1.4.10

A graph in which every vertex has the same degree is called a regular graph.

Suborbits and suborbital graphs

Definition 1.5.1

Let G be transitive on X and let G_x be the stabilizer of a point x∈ X. The orbits $\triangle_0 = \{x\}, \triangle_1, \triangle_2, \dots, \triangle_{r-1}$ of G_x on X are called the suborbits of G. The rank of G is r and the sizes $n_i = |\triangle_i|$ (I = 0,1,2,…,r-1) often called the lengths of the suborbits, are known as subdegrees of G.
It is worth while noting that both r and the cardinalities of the suborbits $\triangle_i$(i = 0, 1, …, r-1) are independent of the choice of x ∈ X.

Theorem 1.5.2 [Wielandt, 1964, Section 16.5]
G_x has an orbit different from $\{x\}$ and paired with itself if and only if G has even order. Observe that G acts on X×X by
g(x, y) = (gx, gy), g∈ G, x,y ∈ X.

If O ⊆ X×X is a G-orbit, then for a fixed x ∈ X, $\triangle = \{y \in X | (x, y) \in O\}$ is a G_x- orbit.
Conversely, if $\triangle \subseteq X$ is a G_x- orbit, then O = $\{gx, gy) | g \in G, y \in \triangle\}$ is a G-orbit on X×X. We say $\triangle$ corresponds to O.

Actions of the symmetric group S_n on unordered pairs

We investigate some properties of the action of S_n on the set of all unordered pairs from
X = $\{1, 2, \dots, n\}$. We shall also construct and dicuss the suborbital graphs associated with this action. Let G = S_n act naturally on X. Then G acts on $X^{(2)}$, the set of all unordered pairs from X by the rule;
$g\{x, y\} = \{gx, gy\}$, ∀g ∈ G and $\{x, y\} \in X^{(2)}$.

Some general results of permutation groups acting on $X^{(2)}$

The following two Theorems, whose proofs are given, will be very useful in this part, for the calculations of the number of unordered pairs fixed by g, that is $|\mathrm{Fix}\,(g)|$ and the number of permutations in G fixing $\{a, b\}$ and having the same cycle type as g ∈ G respectively.

Theorem 2.1.1

Let G be a symmetric group S_n acting on a set X = $\{1, 2, \dots, n\}$ and g∈G have cycle type $(\propto_1, \propto_2, \dots, \propto_n)$. Then $|\mathrm{Fix}\,(g)|$ in $X^{(2)}$ is given by $\binom{\propto_1}{2} + \propto_2$.

Proof

For g∈G with cycle type $(\propto_1, \propto_2, \dots, \propto_n)$ to fix an unordered pair $\{a, b\}$, then either both a and b come from cycles of length 1 in g or both come from a 2-cycle in g.
From the first case, the number of unordered pairs fixed by g is $\binom{\propto_1}{2}$ and from the second case the number of unordered pairs fixed by g, is the number of 2-cycles in g; that is $\propto_2$. □
Therefore$|\mathrm{Fix}\,(g)|$ in g is $\binom{\propto_1}{2} + \propto_2$.

Theorem 2.1.2

Let G be symmetric group S_n acting on the set X = $\{1, 2, \dots, n\}$ and let g∈G have, say cycle type $(\propto_1, \propto_2, \dots, \propto_n)$. Then the number of permutations in G fixing $\{a, b\}$ and having the same cycle type as g is given by;

$$\frac{(n-2)!}{1^{\propto_1-2}\,(\propto_1-2)!\prod_{i=2}^{n}\propto_i!\,i^{\propto_i}1^{\propto_1}\,\propto_1!} + \frac{(n-2)!}{2^{\alpha_2-1}(\propto_2-1)!\prod_{i=3}^{n}\propto_i!\,i^{\propto_i}}$$

Proof

For a permutation in G having cycle type $(\propto_1, \propto_2, \dots, \propto_n)$ to fix $\{a, b\}$, either

(i) a and b are in single cycle and in this case the number of permutations in S_n of cycle type $(\propto_1, \propto_2, \dots, \propto_n)$ fixing $\{a, b\}$ is the same as the number of permutations in S_{n-2} of cycle type $(\propto_1-2, \propto_2, \dots, \propto_n)$. By Theorem 1 . 3. 10, this number is

$$\frac{(n-2)!}{1^{\propto_1-2}\,(\propto_1-2)!\,\prod_{i=2}^{n}\propto_i!\;i^{\propto_i}} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 1$$

Or (ii) a and b are in a 2 – cycle and in this case the number of permutations in S_n of cycle type $(\propto_1, \propto_2, \dots, \propto_n)$ fixing $\{a, b\}$ is the same as the number of permutations in S_{n-2} of cycle type $(\propto_1, \propto_2-1, \dots, \propto_n)$. By theorem 1.3.10, this number is

$$\frac{(n-2)!}{1^{\propto_1}\,\propto_1!\,2^{\propto_2}-1\,(\propto_2-1)!\prod_{i=3}^{n}\propto_i!\;i^{\propto_i}} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2)$$

Adding (1) and (2) we get the number of permutations in G fixing $\{a, b\}$ and having the same cycle type as g∈G. That is

$$\frac{(n-2)!}{1^{\propto_1-2}\,(\propto_{1-2})!\prod_{i=2}^{n}\propto_i!\,i^{\propto_i}1^{\propto_1}\,\propto_1!} + \frac{(n-2)!}{2^{\alpha_2-1}(\propto_{2-1})!\prod_{i=3}^{n}\propto_i!\,i^{\propto_i}}.$$

Suborbits of S_n (n = 3,4,5) acting on $X^{(2)}$ and the corresponding suborbital graphs

Suborbits of G = S_3 acting on $x^{(2)}$ and the corresponding suborbital graphs

Lemma 2.2.1.1

G acts transitively on $X^{(2)}$.

Proof

By Definition 1.3.4, it suffices to show that the action of G has only one orbit. We do this by use of Cauchy – Frobenius lemma (Theorem 1.3.6).

Let g ∈ G have cycle type $(\propto_1, \propto_2, \propto_3)$, then the number of permutations in G having the same cycle type as g is given by Theorem 1.3.5. The number of elements in $X^{(2)}$ fixed by g is given by Theorem 2.1.1.

We have the following table 1;

Table 1: Permutations in G and the number of fixed points

Permutation g in G	No. of permutations	$\lvert\mathrm{Fix}\,(g)\rvert$ in $X^{(2)}$	cycle type $(\propto_1, \propto_2, \propto_3)$
1	1	3	(3,0,0)
(ab)	3	1	(1,1,0)
(abc)	2	0	(0,0,1)
Total	6		

Now applying Cauchy – Frobenius Lemma, we get the number of orbits of G acting on $X^{(2)}$

$= \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)| = \frac{1}{6}\{(1x3) + (3x1) + (2x0)\}$

$= \frac{1}{6}$ x 6 = 1

Thus G = S_3 acts transitively on $X^{(2).}$

Lemma 2.2.1.2

The number of orbits of $G_{\{1,2\}}$ acting on $X^{(2)}$ is 2.

Proof

To prove this, we apply the Cauchy – Frobenius Lemma (Theorem 1.3.6).
The second and the third columns of the following table 2 can be got by applying Theorems 2.1.2 and 2.1.1 respectively in table 2.

Table 2: Permutations in $G_{\{1,2\}}$ and the number of fixed points in $X^{(2)}$

Permutation in $G_{\{1,2\}}$	No. of permutations	$\lvert\mathrm{Fix}\,(g)\rvert$ in $X^{(2)}$	Cycle type $(\propto_1, \propto_2, \propto_3)$
I	1	3	(3,0,0)
(12) (c)	1	1	(1,1,0)

Therefore$\left|G_{\{1,2\}}\right|$ = 2.

Now, applying Cauchy – Frobenius Lemma, the number of orbits of $G_{\{1,2\}}$ on $X^{(2)}$

$$= \frac{1}{\left|G_{\{1,2\}}\right|}\sum_{g\in G_{\{1,2\}}}|\mathrm{Fix}(g)|$$

$= ½\big((1x3) + (1x1)\big)$

$=$ ½ x 4 = 2.

The two orbits of $G_{\{1,2\}}$ acting on $X^{(2)}$found in the immediate lemma above are;

$\mathrm{Orb}_{G_{\{1,2\}}}\{1,2\} = \big\{\{1,2\}\big\} = \Delta_0$, the trivial orbit and

$\mathrm{Orb}_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\},\{2,3\}\} = \Delta_1$, the set of all unordered pairs containing exactly one of 1 and 2.

Therefore the rank of G on$X^{(2)}$ is 2 and the subdegrees are 1 and 2.

Next, we discuss the suborbitsΔ_o and Δ_1 and the corresponding suborbital graphs.

The suborbital graph corresponding to Δ_o is the null graph which is not as such interesting. We now consider the non-trivial suborbit Δ_1. Since the $|G|$ = 6 is even, by Theorem 1.5.2, Δ_1 is self-paired and hence, the corresponding suborbital graph Γ_1 is undirected.

Since the suborbital O_1 corresponding to the suborbit Δ_1 is $O_1 = \{(g\{1,2\},)g\{1,3\} | g \epsilon G\}$, (see Section 1.5) the suborbital graph Γ_1 corresponding to suborbital O_1 has two 2-element subsets V and W from $X = \{1,2,3\}$ adjacent if and only if $|V \cap \mathcal{W}| = 1$. Γ_1 is connected and regular of degree 2.
The properties discussed above can be clearly seen by construction of the suborbital graph Γ_1 as follows in figure 1.

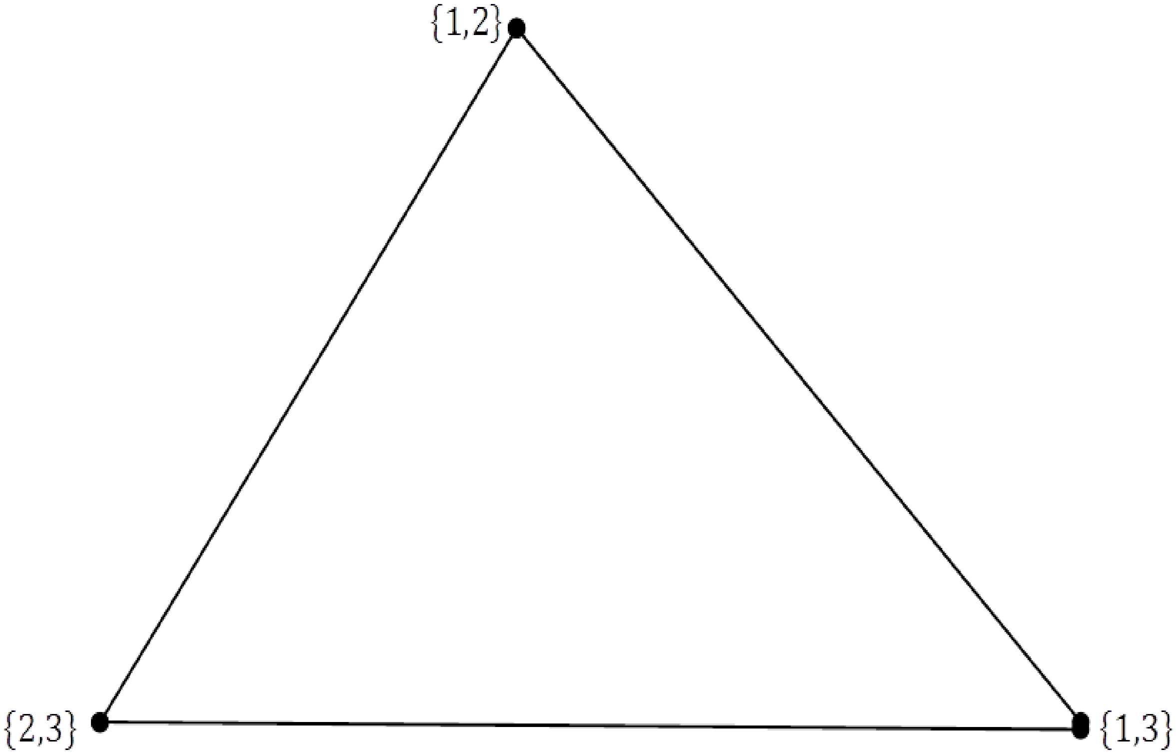


Figure 1: The suborbital graph Γ_1 corresponding to the suborbit Δ_1 of G on $X^{(2)}$

Suborbits of G = S_4 acting on $X^{(2)}$ and the corresponding suborbital graphs

Lemma 2.2.2.1

G acts transitively on $X^{(2)}$.

Proof

By Definition 1.3.4, it is enough to show that the action of G has only one orbit. We do this by use of Cauchy-Frobenius Lemma (Theorem 1.3.6).
Let $g \in G$ have cycle type $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, then the number of permutations in G having the same cycle type as g is given by theorem 1.3.5 and the number of elements in $X^{(2)}$ fixed by g is given by Theorem 2.1.1.
We now have the following table 3.

Table 3: Permutations in G and the number of fixed points

Permutation	No. of permutations	$\lvert\mathrm{Fix}\,(g)\rvert$ in $X^{(2)}$	Cycle type $(\propto_1, \propto_2, \propto_3, \propto_4)$
1	1	6	(4,0,0,0)
(ab)	6	2	(2,1,0,0)
(abc)	8	0	(1,0,1,0)
(abcd)	6	0	(0,0,0,1)
(ab)(cd)	3	2	(0,2,0,0)
Total	24		

Now applying the Cauchy-Frobenius Lemma, we get

Number of orbits of G acting on $X^{(2)} = \frac{1}{|G|}\sum|\mathrm{Fix}\,(g)|$

$= \frac{1}{24}[(1\text{x}6) + (6\text{x}2) + (8\text{x}0) + (6\text{x}0) + (3\text{x}2)]$

$= \frac{1}{24}(6 + 12 + 6) = \frac{1}{24}(24) = 1.$

Thus G acts transitively on $X^{(2)}$.

Lemma 2.2.2.2

The number of orbits of $G_{\{1,2\}}$ acting on $X^{(2)}$ is 3.

Proof

To prove this, we apply the Cauchy- Frobenius Lemma (Theorem 1.3.6).
The second and the third columns of the following table 4 can be got by applying Theorems 2.1.2 and 2.1.1 respectively.

Table 4: Permutations in $G_{\{1,2\}}$ and the number of fixed points in $X^{(2)}$.

Permutation g in $G_{\{1,2\}}$	No. of permutations	$\lvert\mathrm{Fix}\,(g)\rvert$ in $X^{(2)}$	Cycle type $(\propto_1, \propto_2, \propto_3, \propto_4)$
1	1	6	(4,0,0,0)
(1) (2) (cd)	1	2	(2,1,0,0)
(12) (c) (d)	1	2	(2,1,0,0)
(12) (cd)	1	2	(0,2,0,0)
Total	4		

Applying Cauchy-Frobenius Lemma, we get the number of orbits of $G_{\{1,2\}}$ on $X^{(2)}$. That is

$$= \frac{1}{|G_{\{1,2\}}|}\sum_{g\in G_{\{1,2\}}}|\mathrm{Fix}(g)|$$

$= \frac{1}{4}[(1x6) + (1x2) + (1x2) + (1x2)]$

$= \frac{1}{4}(12) = 3.$

The three orbits of $G_{\{1,2\}}$ acting on $X^{(2)}$ determined above are;

$\mathrm{Orb}_{G_{\{1,2\}}}\{1,2\} = \{\{1,2\}\} = \Delta_o$, the trivial orbit

$\mathrm{Orb}_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\} = \Delta_1$, the set of all unordered pairs containing exactly one of 1 and 2.

$\mathrm{Orb}_{G_{\{1,2\}}}\{3,4\} = \{\{3,4\}\} = \Delta_2$, the set of all unordered pairs containing neither 1 nor 2.

Therefore the rank of G on $X^{(2)}$ is 3. And the subdegrees are 1,4,1.

We now discuss the suborbits Δ_o, Δ_1 and Δ_2 and the corresponding suborbital graphs. The suborbital graph corresponding to Δ_o is the null graph which is not interesting as such.

We, therefore discuss the non-trivial suborbits Δ_1 and Δ_2.

Since $|G| = 4! = 4\times3\times2\times1 = 24$ is even, by Theorem 1.5.2, Δ_1 and Δ_2 are self-paired and hence their corresponding suborbital graphs Γ_1 and Γ_2 are undirected.

The suborbital O_1 corresponding to the suborbit Δ_1 is $O_1 = \{(g\{1,2\}, g\{1,3\}) | g \in G\}$;

(see Section 1.5). The suborbital graph Γ_1 corresponding to the suborbital O_1 has two 2- element subsets V and W from $X = \{1,2,3,4\}$ adjacent if and only if $|V \cap \mathcal{W}| = 1$.

Similarly, the suborbital O_2 corresponding to the suborbit Δ_2 is

$O_2 = \{(g\{1,2\}, g\{3,4\}) | g \in G\}$.

The suborbital graph Γ_2 corresponding to O_2 has two 2- element subsets V and W adjacent if and only if $|V \cap w| = 0$.

Clearly Γ_1 and Γ_2 are complementary. It can also be easily seen that Γ_1 is regular of degree 4 and its girth is 3. Γ_2 is regular of degree 1.

The properties mentioned above can easily be seen by constructing the suborbital graphs Γ_1 and Γ_2 as follows in figure 2, 3.

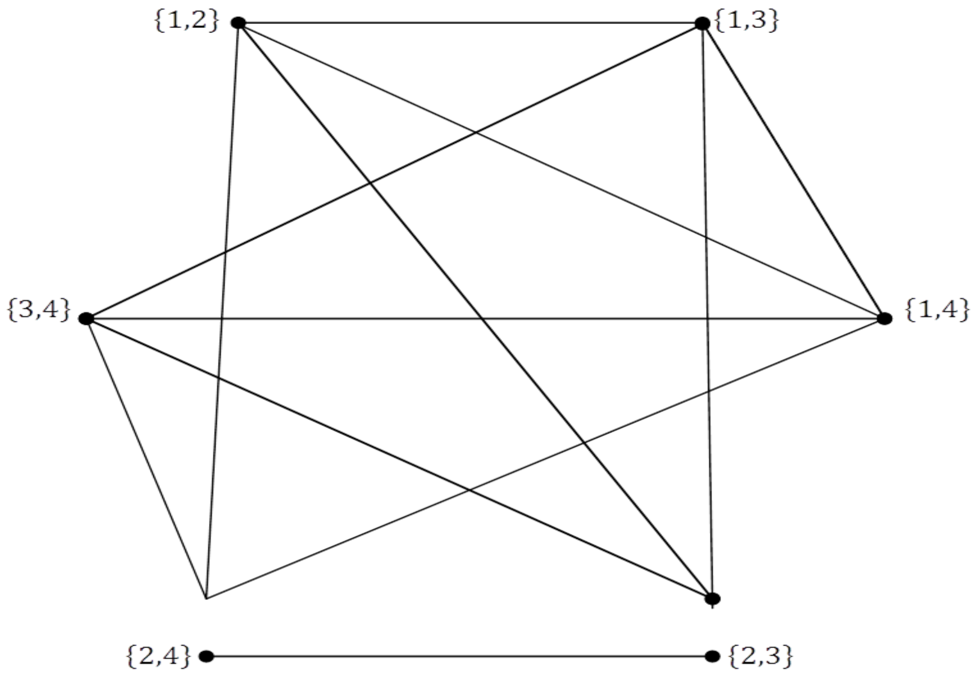
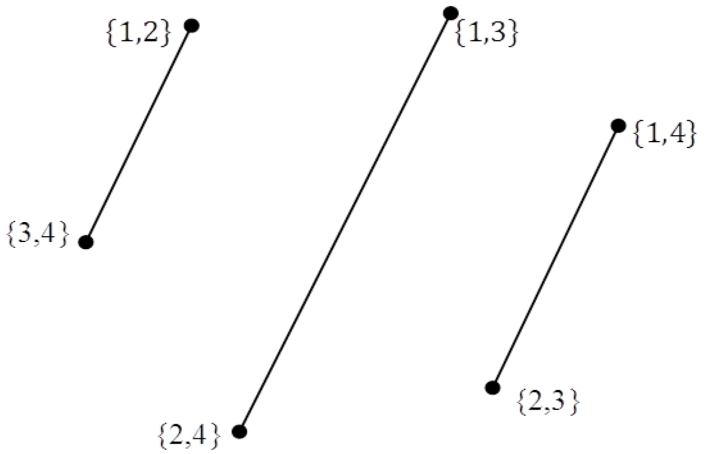


Figure 2: The suborbital graph Γ_1 corresponding to the suborbit Δ_1 of G on $X^{(2)}$



Γ_1 and Γ_2 are complementary.

Figure 3: The suborbital graph Γ_2 Corresponding to the suborbit Δ_2 of $G_{\{1,2\}}$ on $X^{(2)}$

Suborbits of G = S_5 acting on $X^{(2)}$ and the corresponding suborbital graphs

Lemma 2.2.3.1

G acts transitively on $X^{(2)}$.

Proof

By Definition 1.3.4, it suffices to show that the action of G on $X^{(2)}$ has only one orbit. We do this using Cauchy – Frobenius Lemma (Theorem 1.3.6).
Let g ∊ G have cycle type $(\propto_1, \propto_2, \propto_3, \propto_4, \propto_5)$, then the number of permutations in G having the same cycle type as g is given by Theorem 1. 3. 5. The number of elements in $X^{(2)}$ fixed by g is given by Theorem 2.1.1.
We now have the following table 5;

Table 5: Permutations in G and the number of fixed points

Permutation	**No. of permutations**	$\mathbf{|Fix(g)|\ in X^{(2)}}$	**Cycle type** $(\propto_1, \propto_2, \propto_3, \propto_4, \propto_5)$
I	1	10	$(5,0,0,0,0,)$
(ab)	10	4	$(3,1,0,0,0)$
(abc)	20	1	$(2,0,1,0,0)$
$(abcd)$	30	0	$(1,0,0,1,0)$
$(abcde)$	24	0	$(0,0,0,0,1)$
$(ab)(cd)$	15	2	$(1,2,0,0,0,)$
$(ab)(cde)$	20	1	$(0,1,1,0,0,)$
Total	120		

Applying the Cauchy –Frobenius Lemma, we get
Number of orbits of G acting on $X^{(2)} = \frac{1}{|G|}\sum|\mathrm{Fix}(g)|$
$= \frac{1}{120}\{(1x10) + (10x4) + (20x1) + (30x0) + (24x0) + (15x2) + (20x1)\}$

$= \frac{1}{120}(10 + 40 + 20 + 30 + 20)$
$= \frac{120}{120} = 1.$
Thus G acts transitively on$\mathrm{X}^{(2)}$.

Lemma 2.2.3.2

The number of orbits of $\mathrm{G}_{\{1,2\}}$ acting on $\mathrm{X}^{(2)}$ is 3.

Proof

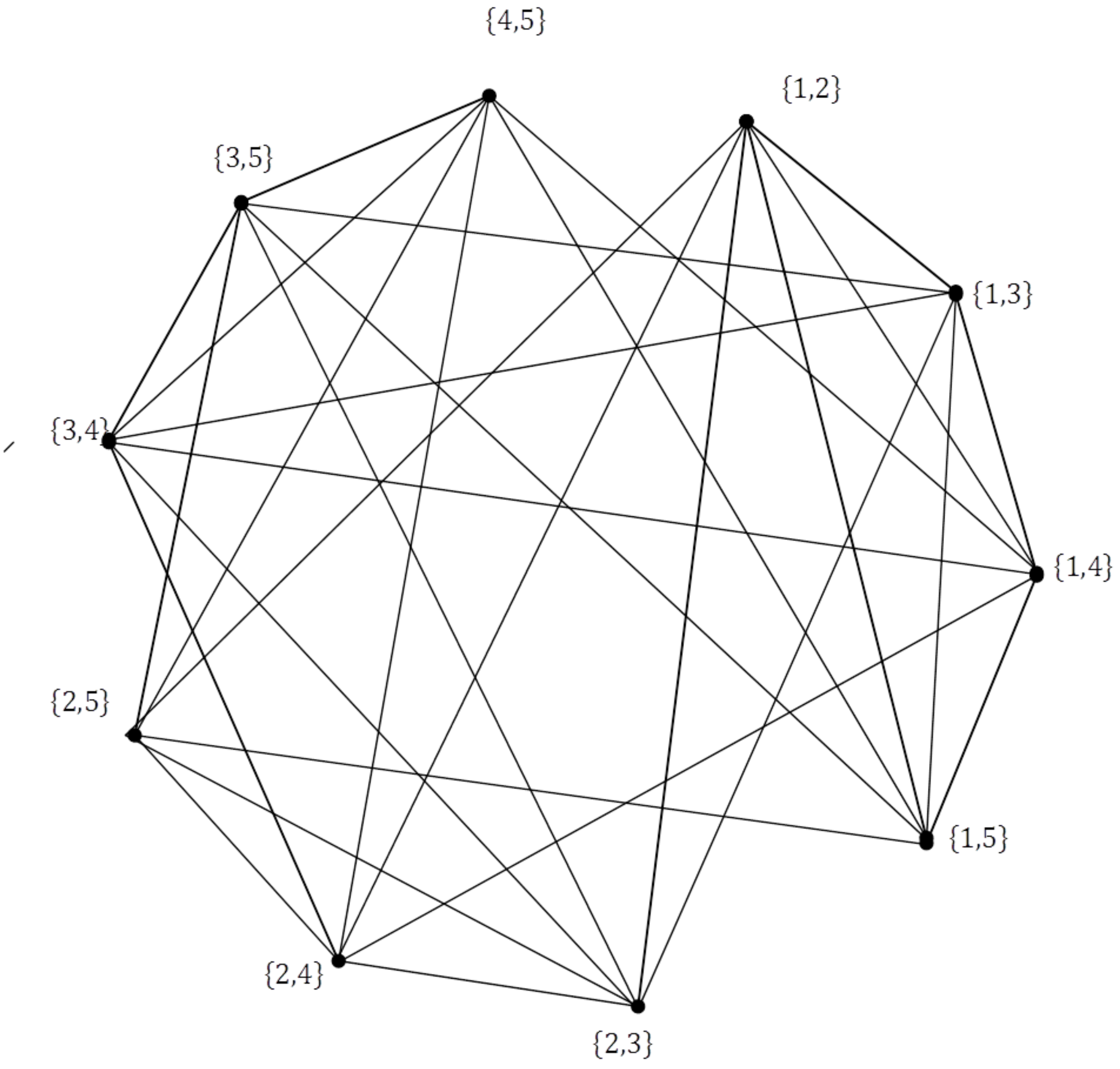
To prove this we apply the Cauchy- Frobenius Lemma (Theorem 1.3.6).
From Theorems 2.1.2 and 2.1.1, the second and the third columns of the following table 6 can be obtained as follows;

Table 6: Permutations in $\mathrm{G}_{\{1,2\}}$ and the number of fixed points in $\mathrm{X}^{(2)}$

Permutation g in $\mathrm{G}_{\{1,2\}}$	**No .of permutations**	**$\lvert\mathrm{Fix}(g)\rvert$ in $\mathrm{X}^{(2)}$**	**Cycle type$(\propto_1, \propto_2, \propto_3, \propto_4 \propto_5)$**
I	1	10	$(5,0,0,0,0,)$
$(1)(2)(\mathrm{cde})$	2	1	$(2,0,1,0,0)$
$(1)(2)(\mathrm{cd})(\mathrm{e})$	3	4	$(3,1,0,0,0)$
$(12)(\mathrm{c})(\mathrm{d})(\mathrm{e})$	1	4	$(3,1,0,0,0)$
$(12)(\mathrm{cde})$	2	1	$(0,1,1,0,0)$
$(12)(\mathrm{cd})(\mathrm{e})$	3	2	$(1,2,0,0,0,)$
Total	12		

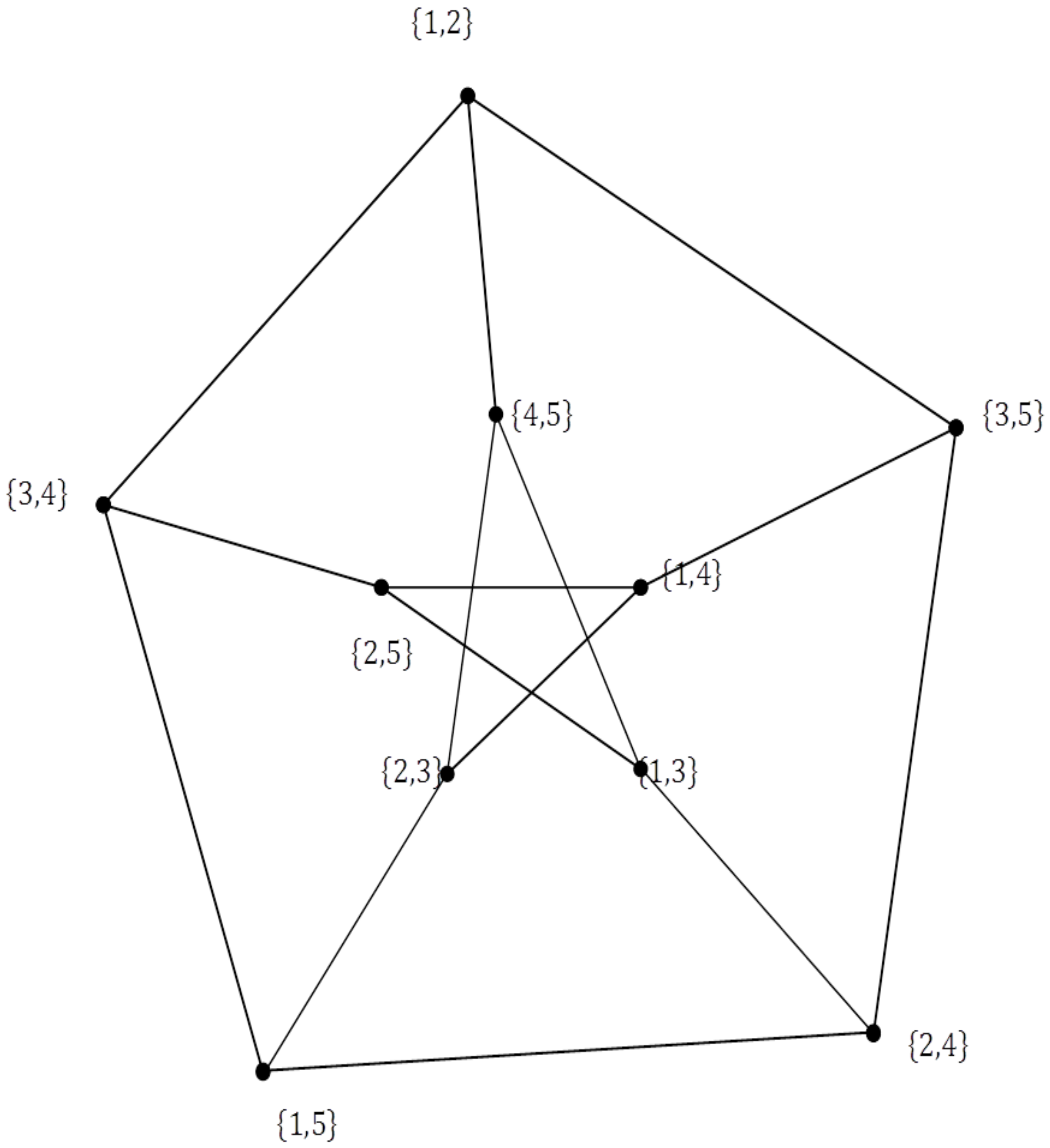
Now applying the Cauchy-Frobenius Lemma we get the number of orbits of $\mathrm{G}_{\{1,2\}}$ on $\mathrm{X}^{(2)}$.
$\frac{1}{\mathrm{G}_{\{1,2\}}} \sum |\mathrm{Fix}\ (\mathrm{g})| = \frac{1}{12}\{(1\mathrm{x}10) + (2\mathrm{x}1) + (3\mathrm{x}4) + (1\mathrm{x}4) + (2\mathrm{x}1) + (3\mathrm{x}2)\}$
$= \frac{1}{12}(10 + 2 + 12 + 4 + 2 + 6)$
$= \frac{1}{12}(36) = 3.$
The three orbits of $\mathrm{G}_{\{1,2\}}$ acting on $\mathrm{X}^{(2)}$ determined above are;
$\mathrm{Orb}_{\mathrm{G}_{\{1,2\}}}\{1,2\} = \{\{1,2\}\} = \Delta_o$, the trivial orbit.
$\mathrm{Orb}_{\mathrm{G}_{\{1,2\}}} = \{\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\}\} = \Delta_1$, the set of all unordered pairs containing exactly one of 1 and 2.
$\mathrm{Orb}_{\mathrm{G}_{\{1,2\}}} = \{\{3,4\},\{3,5\},\{4,5\}\} = \Delta_2$, the set of all unordered pairs containing neither 1 nor 2
Therefore the rank of G on $\mathrm{X}^{(2)}$ is 3 and the subdegrees are 1, 6 and 3.
We now discuss the suborbits $\Delta_0, \Delta_1 \mathrm{and} \Delta_2$ and the corresponding suborbital graphs.
The suborbital graph corresponding to Δ_0 is the null graph.
We now discuss the non-trivial suborbits Δ_1 and Δ_2 .
Since the $|\mathrm{G}| = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ is even, by Theorem 1.5.2, Δ_1 and Δ_2 are self paired and hence their corresponding suborbital graphs Γ_1 and Γ_2 are undirected.
The suborbital O_1 corresponding to the suborbit Δ_1, is $\mathrm{O}_1 = \{(\mathrm{g}\{1,2\}, \mathrm{g}\{1,3\}) | \mathrm{g} \in \mathrm{G}\}$
(see Section 1.5). The suborbital graph Γ_1 corresponding to the suborbital has two 2-element subsets V and W from X = $\{1,2,3,4,5\}$ adjacent if and only if $|V \cap W| = 1$.
Similarly the suborbital O_2 corresponding to the suborbit Δ_2, is$\mathrm{O}_2 = \{(\mathrm{g}\{1,2\}, \mathrm{g}\{3,4\}) | \mathrm{g} \in \mathrm{G}\}$

The suborbital graph corresponding to O_2 has two 2- element subsets V and W
adjacent if and only if $|V \cap W| = 0$.
Clearly Γ_1 and Γ_2 are complementary. It can easily be seen that Γ_1 is regular of degree 6. Its girth is 3. $\Gamma_2$is regular of degree 3 and its girth is 5.Γ_2 isthe famous Petersen graph.
The properties discussed above can easily be seen by constructing the suborbital graphs Γ_1 and Γ_2 as follows in figure 4, 5.



Γ_1 is regular of degree 6.Its girth is 3.

Figure 4: The suborbital graph Γ_1 corresponding to the suborbit Δ_1 of G on $X^{(2)}$



Γ_2 is the famous Petersen graph.
It is regular of degree 3.
Since Γ_1 and Γ_2 are connected, G acts primitively on $X^{(2)}$.

Figure 5: The suborbital graph Γ_2 corresponding to the suburbit Δ_2 of G on $X^{(2)}$

CONCLUSION

In this paper, we have discussed some properties of S_n, (n = 3, 4, 5) acting on unordered pairs.

We found out that S_n (n = 3, 4, 5) acts transitively and primitively on $X^{(2)}$.

S_3 is of rank 2 and has subdegrees 1 and 2.

S_4 and S_5, each is of rank 3 and have subdegrees 1,4,1 and 1,6,3 respectively.

The suborbital graph corresponding to the non-trivial suborbit of S_3 is connected and regular of degree 2.

The suborbital graphs corresponding to the non-trivial suborbits of S_4 and S_5 are regular and complementary to each other.

In S_5, the second suborbital graph, Γ_2 is the famous Petersen graph that is regular and of degree3. Its girth is 5.

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