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Subdegrees and suborbital graphs of symmetric group $S_n$ ($n = 3, 4, 5$) acting on unordered pairs

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In this research paper, we compute the ranks and subdegrees of the symmetric group $S_n$ ($n = 3, 4, 5$) acting on unordered pairs from the set $X = \{1, 2, ..., n\}$. When $S_n$ ($n \geq 4$) acts on unordered pairs from $X$, the rank is 3. Therefore the main study will be on the subdegrees of the suborbital graphs. The suborbital graphs corresponding to the suborbitals of these actions are also constructed. The graph theoretic properties of these suborbital graphs are also discussed. When $S_n$ ($n \geq 4$) acts on unordered pairs the suborbital graphs corresponding to the non-trivial suborbits $\Delta_1$ and $\Delta_2$, are connected, regular and complementary.

Keywords: Subdegrees, suborbital graphs of symmetric group, unordered pairs

INTRODUCTION

In this paper we investigate some properties of the symmetric group $S_n$ ($n = 3, 4, 5$) acting on unordered pairs from $X = \{1, 2, ..., n\}$. We also find suborbits and suborbitals of $S_n$ ($n = 3, 4, 5$) and construct suborbital graphs corresponding to these suborbitals. We shall also discuss some of the graph theoretic properties of these suborbital graphs.

This paper is divided into three parts; with our main results in part two.
In part one, we give definitions and preliminary results needed throughout the paper.
In part two, we investigate some properties of the action of $S_n$ ($n = 3, 4, 5$) on unordered pairs. Next, we find the ranks, suborbits and construct suborbital graphs, corresponding to suborbitals of $S_n$ ($n = 3, 4, 5$). We also discuss the graph theoretic properties of these suborbital graphs.

Finally in part three, we give conclusions.

Definitions and preliminaries

We establish background information and results that will be used throughout this paper.

Notations

$\sum_i$ – Sum over $i$.
$\binom{a}{b}$ - a combination of $a$.
$S_n$ - Symmetric group of degree $n$ and order $n!$.
$|G|$ - The order of a group $G$.
$X^{(2)}$ - The set of unordered pairs from the set $X = \{1, 2, ..., n\}$.
$\{t,q\}$ - Unordered pair.
$X \times Y$ – Cartesian product of $X$ and $Y$. 
Permutation groups

Definition 1.2.1

Let X be a non-empty set. A permutation of X is a one-to-one mapping of X onto itself.

Definition 1.2.2

Let X be the set \( \{1, 2, ..., n\} \), then the symmetric group of degree n is the group of all permutations of X under the binary operation of composition of maps. It is denoted as \( S_n \) and has an order \( n! \).

Definition 1.2.3

A permutation of a finite set is even or odd according as it can be expressed as the product of an even or odd number of 2-cycles (transpositions).

Group actions

Definition 1.3.1

Let X be a non-empty set. The group G acts on the left on X if for each \( g \in G \) and each \( x \in X \) there corresponds a unique element \( gx \in X \) such that:

(i) \( (g_1g_2)x = g_1(g_2x) \) for all \( g_1, g_2 \in G \) and \( x \in X \).

(ii) For any \( x \in X \), \( 1x = x \), where 1 is the identity in G. The action of G from the right on X can be defined in a similar way. In fact it is merely a matter of taste whether one writes the group element on the left or on the right.

Definition 1.3.2

Let G act on a set X. Then X is partitioned into disjoint equivalence classes called orbits or transitivity classes of the action. For each \( x \in X \), the orbit containing x is called the orbit of x and is denoted by \( \text{Orb}_G(x) \).

Definition 1.3.3

If a finite group G acts on a set X with \( n \) elements, each \( g \in G \) corresponds to a permutation \( \sigma \) of X, which can be written uniquely as a product of disjoint cycles. If \( \sigma \) has \( \alpha_1 \) cycles of length 1, \( \alpha_2 \) cycles of length 2, \( \alpha_3 \) cycles of length 3, ..., \( \alpha_n \) cycles of length n; then we say that \( \sigma \) and hence \( g \) has a cycle type \( (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n) \).

Definition 1.3.4

If the action of a group G on a set X has only one orbit, then we say that G acts transitively on X. In other words, G acts transitively on X if for every pair of points \( x, y \in X \), there exists \( g \in G \) such that \( gx = y \).

Theorem 1.3.5 [Krishnamurthy, 1985, p. 68].

Two permutations in \( S_n \) are conjugate if and only if, they have the same cycle type, and if \( g \in S_n \) has a cycle type \( (\alpha_1, \alpha_2, ..., \alpha_n) \), then the number of permutations in \( S_n \) conjugate to \( g \) is \( n! \).

Theorem 1.3.6 [Cauchy—Frobenius Lemma—Rotman, 1973, p.45].

Let G be a group acting on a finite set X. Then the number of G-orbits in X is

\[
\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|.
\]

Graphs

Definition 1.4.1

A graph is a diagram consisting of a set V whose elements are called vertices, nodes or points and a set E of unordered pairs of vertices called edges or lines. We denote such a graph by \( G(V,E) \) or simply by \( G \) if there is no ambiguity of V and E.

Definition 1.4.2

A graph consisting of one vertex and no edges is called a trivial graph.

Definition 1.4.3

A graph whose edge set is empty is called a null graph.

Definition 1.4.4

If we allow the existence of loops (edges joining vertices, to themselves) and multiple edges (more
than one edge joining two distinct vertices), then we get a multigraph.

**Definition 1.4.5**

A graph with no loops or multiple edges is called a simple graph.

**Definition 1.4.6**

The degree (valency) of a vertex $v$ of $G(V,E)$ is the number of edges incident to $v$.

**Definition 1.4.7**

Any vertex of degree zero is called an isolated vertex.

**Definition 1.4.8**

A graph $G(V,E)$ is said to be connected if there is a path between any two of its vertices.

**Definition 1.4.9**

The girth of a graph $G(V,E)$ is the length of the shortest cycle if any in $G(V,E)$.

**Definition 1.4.10**

A graph in which every vertex has the same degree is called a regular graph.

**Suborbits and suborbital graphs**

**Definition 1.5.1**

Let $G$ be transitive on $X$ and let $G_x$ be the stabilizer of a point $x \in X$. The orbits $\Delta_0 = \{x\}, \Delta_1, \Delta_2, \ldots, \Delta_{r-1}$ of $G_x$ on $X$ are called the suborbits of $G$. The rank of $G$ is $r$ and the sizes $n_i = |\Delta_i|$ ($l = 0, 1, \ldots, r-1$) often called the lengths of the suborbits, are known as subdegrees of $G$.

It is worth while noting that both $r$ and the cardinalities of the suborbits $\Delta_i$ ($i = 0, 1, \ldots, r-1$) are independent of the choice of $x \in X$.

**Theorem 1.5.2** [Wielandt, 1964, Section 16.5]

$G_x$ has an orbit different from $\{x\}$ and paired with itself if and only if $G$ has even order. Observe that $G$ acts on $X \times X$ by $g(x, y) = (gx, gy), g \in G, x, y \in X$.

If $O \subseteq X \times X$ is a $G$-orbit, then for a fixed $x \in X$, $\Delta = \{y \in X | (x, y) \in O\}$ is a $G_x$ orbit.

Conversely, if $\Delta \subseteq X$ is a $G_x$-orbit, then $O = \{(gx, gy) \mid g \in G, y \in \Delta\}$ is a $G$-orbit on $X \times X$. We say $\Delta$ corresponds to $O$.

**Actions of the symmetric group $S_n$ on unordered pairs**

We investigate some properties of the action of $S_n$ on the set of all unordered pairs from $X = \{1, 2, \ldots, n\}$. We shall also construct and discuss the suborbital graphs associated with this action. Let $G = S_n$ act naturally on $X$. Then $G$ acts on $X^{(2)}$, the set of all unordered pairs from $X$ by the rule; $g(x, y) = (gx, gy)$, $\forall g \in G$ and $\{x, y\} \in X^{(2)}$.

**Some general results of permutation groups acting on $X^{(2)}$**

The following two Theorems, whose proofs are given, will be very useful in this part, for the calculations of the number of unordered pairs fixed by $g$, that is $|\text{Fix}(g)|$ and the number of permutations in $G$ fixing $\{a, b\}$ and having the same cycle type as $g \in G$ respectively.

**Theorem 2.1.1**

Let $G$ be a symmetric group $S_n$ acting on a set $X = \{1, 2, \ldots, n\}$ and $g \in G$ have cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then $|\text{Fix}(g)|$ in $X^{(2)}$ is given by $\left(\frac{\alpha_1}{2}\right)^{+} + \alpha_2$.

**Proof**

For $g \in G$ with cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ to fix an unordered pair $\{a, b\}$, then either both $a$ and $b$ come from cycles of length $1$ in $g$ or both come from a $2$-cycle in $g$.

From the first case, the number of unordered pairs fixed by $g$ is $\left(\frac{\alpha_1}{2}\right)^{+}$ and from the second case the number of unordered pairs fixed by $g$, is the number of $2$-cycles in $g$; that is $\alpha_2$. Therefore $|\text{Fix}(g)|$ in $g$ is $\left(\frac{\alpha_1}{2}\right)^{+} + \alpha_2$.  

Njagi et al., 335.
Theorem 2.1.2

Let $G$ be symmetric group $S_n$ acting on the set $X = \{1, 2, \ldots, n\}$ and let $g \in G$ have, say cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then the number of permutations in $G$ fixing $\{a, b\}$ and having the same cycle type as $g$ is given by:

$$\frac{(n-2)!}{1^{\alpha_1-2} \alpha_1! \prod_{i=2}^{\alpha_1} i^{\alpha_i-1} \alpha_i!} + \frac{(n-2)!}{2^{\alpha_2-1} \alpha_2! \prod_{i=3}^{\alpha_2} i^{\alpha_i-1} \alpha_i!}.$$

Proof

For a permutation in $G$ having cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ to fix $\{a, b\}$, either

(i) $a$ and $b$ are in single cycle and in this case the number of permutations in $S_n$ of cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ fixing $\{a, b\}$ is the same as the number of permutations in $S_{n-2}$ of cycle type $(\alpha_1 - 2, \alpha_2, \ldots, \alpha_n)$. By Theorem 1.3.10, this number is

$$\frac{(n-2)!}{1^{\alpha_1-2} \alpha_1! \prod_{i=2}^{\alpha_1} i^{\alpha_i-1} \alpha_i!}.$$

Or (ii) $a$ and $b$ are in a $2-\text{cycle}$ and in this case the number of permutations in $S_n$ of cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ fixing $\{a, b\}$ is the same as the number of permutations in $S_{n-2}$ of cycle type $(\alpha_1, \alpha_2 - 1, \ldots, \alpha_n)$. By Theorem 1.3.10, this number is

$$\frac{(n-2)!}{2^{\alpha_2-1} \alpha_2! \prod_{i=3}^{\alpha_2} i^{\alpha_i-1} \alpha_i!}.$$

Adding (1) and (2) we get the number of permutations in $G$ fixing $\{a, b\}$ and having the same cycle type as $g \in G$. That is

$$\frac{(n-2)!}{1^{\alpha_1-2} \alpha_1! \prod_{i=2}^{\alpha_1} i^{\alpha_i-1} \alpha_i!} + \frac{(n-2)!}{2^{\alpha_2-1} \alpha_2! \prod_{i=3}^{\alpha_2} i^{\alpha_i-1} \alpha_i!}.$$

Suborbits of $S_n$ $(n = 3, 4, 5)$ acting on $X^{(2)}$ and the corresponding suborbital graphs

Suborbits of $G = S_3$ acting on $X^{(2)}$ and the corresponding suborbital graphs

Lemma 2.2.1.1

$G$ acts transitively on $X^{(2)}$.

Proof

By Definition 1.3.4, it suffices to show that the action of $G$ has only one orbit. We do this by use of Cauchy – Frobenius lemma (Theorem 1.3.6).

Let $g \in G$ have cycle type $(\alpha_1, \alpha_2, \alpha_3)$, then the number of permutations in $G$ having the same cycle type as $g$ is given by Theorem 1.3.5. The number of elements in $X^{(2)}$ fixed by $g$ is given by Theorem 2.1.1. We have the following table 1;
Now applying Cauchy – Frobenius Lemma, we get the number of orbits of $G$ acting on $X^{(2)}$
\[
\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)| = \frac{1}{6} \left( (1 \times 3) + (3 \times 1) + (2 \times 0) \right)
\]
\[
= \frac{1}{6} \times 6 = 1
\]
Thus $G = S_3$ acts transitively on $X^{(2)}$.

**Lemma 2.2.1.2**

The number of orbits of $G_{(1,2)}$ acting on $X^{(2)}$ is 2.

**Proof**

To prove this, we apply the Cauchy – Frobenius Lemma (Theorem 1.3.6).

The second and the third columns of the following table 2 can be got by applying Theorems 2.1.2 and 2.1.1 respectively in table 2.

| Permutation in $G_{(1,2)}$ | No. of permutations | $|\text{Fix}(g)|$ in $X^{(2)}$ Cycle type $(\alpha_1, \alpha_2, \alpha_3)$ |
|-----------------------------|----------------------|-------------------------------------------|
| I                           | 1                    | 3 (3,0,0)                                 |
| (12) (c)                    | 1                    | 1 (1,1,0)                                 |

Therefore $|G_{(1,2)}| = 2$.

Now, applying Cauchy – Frobenius Lemma, the number of orbits of $G_{(1,2)}$ on $X^{(2)}$
\[
\frac{1}{|G_{(1,2)}|} \sum_{g \in G_{(1,2)}} |\text{Fix}(g)|
\]
\[
= \frac{1}{2} \left( (1 \times 3) + (1 \times 1) \right)
\]
\[
= \frac{1}{2} \times 4 = 2.
\]
The two orbits of $G_{(1,2)}$ acting on $X^{(2)}$ found in the immediate lemma above are;
The trivial orbit and $\text{Orb}_{G_{(1,2)}}{1,2} = \{1,2\}$

$\text{Orb}_{G_{(1,2)}}{1,3} = \{1,3\} \cup \{2,3\} = \Delta_1$, the set of all unordered pairs containing exactly one of 1 and 2.

The rank of $G$ on $X^{(2)}$ is 2 and the subdegrees are 1 and 2.

Next, we discuss the suborbits $\Delta_0$ and $\Delta_1$ and the corresponding suborbital graphs.

The suborbital graph corresponding to $\Delta_0$ is the null graph which is not as such interesting. We now consider the non-trivial suborbits $\Delta_1$. Since the $|G| = 6$ is even, by Theorem 1.5.2, $\Delta_1$ is self-paired and hence, the corresponding suborbital graph $\Gamma_1$ is undirected.
Since the suborbital $O_1$ corresponding to the suborbit $\Delta_1$ is $O_1 = \{(g[1, 2], g[1, 3]) | g \in G\}$, (see Section 1.5) the suborbital graph $\Gamma_1$ corresponding to suborbital $O_1$ has two 2-element subsets $V$ and $W$ from $X = \{1, 2, 3\}$ adjacent if and only if $|V \cap W| = 1$. $\Gamma_1$ is connected and regular of degree 2. The properties discussed above can be clearly seen by construction of the suborbital graph $\Gamma_1$ as follows in figure 1.

![Figure 1: The suborbital graph $\Gamma_1$ corresponding to the suborbit $\Delta_1$ of $G$ on $X^{(2)}$](image)

**Suborbits of $G = S_4$ acting on $X^{(2)}$ and the corresponding suborbital graphs**

**Lemma 2.2.2.1**

$G$ acts transitively on $X^{(2)}$.

**Proof**

By Definition 1.3.4, it is enough to show that the action of $G$ has only one orbit. We do this by use of Cauchy-Frobenius Lemma (Theorem 1.3.6). Let $g \in G$ have cycle type $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, then the number of permutations in $G$ having the same cycle type as $g$ is given by theorem 1.3.5 and the number of elements in $X^{(2)}$ fixed by $g$ is given by Theorem 2.1.1.

We now have the following table 3.
Table 3: Permutations in G and the number of fixed points

| Permutation | No. of permutations | $|\text{Fix}(g)|$ in $X^{(2)}$ | Cycle type $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ |
|-------------|---------------------|-------------------------------|-----------------------------------------|
| 1           | 1                   | 6                             | $(4,0,0,0)$                              |
| (ab)        | 6                   | 2                             | $(2,1,0,0)$                              |
| (abc)       | 8                   | 0                             | $(1,0,1,0)$                              |
| (abcd)      | 6                   | 0                             | $(0,0,0,1)$                              |
| (ab)(cd)    | 3                   | 2                             | $(0,2,0,0)$                              |
| Total       |                     | 24                            |                                          |

Now applying the Cauchy-Frobenius Lemma, we get

Number of orbits of G acting on $X^{(2)} = \frac{1}{|G|} \sum |\text{Fix}(g)|$

$= \frac{1}{24} [(1 \times 6) + (6 \times 2) + (8 \times 0) + (6 \times 0) + (3 \times 2)]$

$= \frac{1}{24} (6 + 12 + 6) = \frac{1}{24} (24) = 1.$

Thus G acts transitively on $X^{(2)}$.

Lemma 2.2.2.2

The number of orbits of $G_{(1,2)}$ acting on $X^{(2)}$ is 3.

Proof

To prove this, we apply the Cauchy- Frobenius Lemma (Theorem 1.3.6). The second and the third columns of the following table 4 can be got by applying Theorems 2.1.2 and 2.1.1 respectively.

Table 4: Permutations in $G_{(1,2)}$ and the number of fixed points in $X^{(2)}$

| Permutation g in $G_{(1,2)}$ | No. of permutations | $|\text{Fix}(g)|$ in $X^{(2)}$ | Cycle type $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ |
|------------------------------|---------------------|-------------------------------|-----------------------------------------|
| 1                            | 1                   | 6                             | $(4,0,0,0)$                              |
| (1) (2) (cd)                 | 1                   | 2                             | $(2,1,0,0)$                              |
| (12) (c) (d)                 | 1                   | 2                             | $(2,1,0,0)$                              |
| (12) (cd)                    | 1                   | 2                             | $(0,2,0,0)$                              |
| Total                        |                     | 4                             |                                          |

Applying Cauchy-Frobenius Lemma, we get the number of orbits of $G_{(1,2)}$ on $X^{(2)}$. That is

$= \frac{1}{|G_{(1,2)}|} \sum_{g \in G_{(1,2)}} |\text{Fix}(g)|$

$= \frac{1}{4} [(1 \times 6) + (1 \times 2) + (1 \times 2) + (1 \times 2)]$
The three orbits of $G_{(1,2)}$ acting on $X^{(2)}$ determined above are;

1. $\text{Orb}_{G_{(1,2)}}(1,2) = \{(1,2)\}$, the trivial orbit
2. $\text{Orb}_{G_{(1,2)}}(1,3) = \{(1,3), (1,4), (2,3), (2,4)\} = \Delta_1$, the set of all unordered pairs containing exactly one of 1 and 2.
3. $\text{Orb}_{G_{(1,2)}}(3,4) = \{(3,4)\} = \Delta_2$, the set of all unordered pairs containing neither 1 nor 2.

Therefore the rank of $G$ on $X^{(2)}$ is 3. And the subdegrees are 1,4,1.

We now discuss the suborbits $\Delta_0, \Delta_1$ and $\Delta_2$ and the corresponding suborbital graphs. The suborbital graph corresponding to $\Delta_0$ is the null graph which is not interesting as such. We therefore discuss the non-trivial suborbits $\Delta_1$ and $\Delta_2$.

Since $|G| = 4! = 4 \times 3 \times 2 \times 1 = 24$ is even, by Theorem 1.5.2, $\Delta_1$ and $\Delta_2$ are self-paired and hence their corresponding suborbital graphs $\Gamma_1$ and $\Gamma_2$ are undirected.

The suborbital $O_1$ corresponding to the suborbit $\Delta_1$ is $O_1 = \{ (g(1,2), g(1,3)) | g \in G \}$

(see Section 1.5). The suborbital graph $\Gamma_1$ corresponding to the suborbital $O_1$ has two 2-element subsets $V$ and $W$ from $X = \{1,2,3,4\}$ adjacent if and only if $|V \cap W| = 1$.

Similarly, the suborbital $O_2$ corresponding to the suborbit $\Delta_2$ is $O_2 = \{ (g(1,2), g(3,4)) | g \in G \}$.

The suborbital graph $\Gamma_2$ corresponding to $O_2$ has two 2-element subsets $V$ and $W$ adjacent if and only if $|V \cap W| = 0$.

Clearly $\Gamma_1$ and $\Gamma_2$ are complementary. It can also be easily seen that $\Gamma_1$ is regular of degree 4 and its girth is 3. $\Gamma_2$ is regular of degree 1.

The properties mentioned above can easily be seen by constructing the suborbital graphs $\Gamma_1$ and $\Gamma_2$ as follows in figure 2, 3.

![Figure 2: The suborbital graph $\Gamma_1$ corresponding to the suborbit $\Delta_1$ of $G$ on $X^{(2)}$](image)
Suborbits of $G = S_5$ acting on $X^{(2)}$ and the corresponding suborbital graphs

Lemma 2.2.3.1

$G$ acts transitively on $X^{(2)}$.

Proof

By Definition 1.3.4, it suffices to show that the action of $G$ on $X^{(2)}$ has only one orbit. We do this using Cauchy–Frobenius Lemma (Theorem 1.3.6).

Let $g \in G$ have cycle type $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, then the number of permutations in $G$ having the same cycle type as $g$ is given by Theorem 1.3.5. The number of elements in $X^{(2)}$ fixed by $g$ is given by Theorem 2.1.1.

We now have the following table 5:

Table 5: Permutations in $G$ and the number of fixed points

| Permutation | No. of permutations | $|\text{Fix}(g)| \mid X^{(2)}$ | Cycle type $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ |
|-------------|---------------------|-----------------|--------------------------------------------------|
| 1           | 1                   | 10              | (5,0,0,0,0)                                      |
| (ab)        | 10                  | 4               | (3,1,0,0,0)                                     |
| (abc)       | 20                  | 1               | (2,0,1,0,0)                                     |
| (abcd)      | 30                  | 0               | (1,0,0,1,0)                                     |
| (abcde)     | 24                  | 0               | (0,0,0,0,1)                                     |
| (ab)(cd)    | 15                  | 2               | (1,2,0,0,0)                                     |
| (ab)(cde)   | 20                  | 1               | (0,1,1,0,0)                                     |
| Total       | 120                 |                 |                                                  |

Applying the Cauchy–Frobenius Lemma, we get

Number of orbits of $G$ acting on $X^{(2)} = \frac{1}{|G|} \sum |\text{Fix}(g)|$

$= \frac{1}{120} \{(1 \times 10) + (10 \times 4) + (20 \times 1) + (30 \times 0) + (24 \times 0) + (15 \times 2) + (20 \times 1)\}$

$\Gamma_1$ and $\Gamma_2$ are complementary.

Figure 3: The suborbital graph $\Gamma_2$ corresponding to the suborbit $\Delta_2$ of $G_{(1,2)}$ on $X^{(2)}$
Thus $G$ acts transitively on $X^{(2)}$.

**Lemma 2.2.3.2**

The number of orbits of $G_{(1,2)}$ acting on $X^{(2)}$ is 3.

**Proof**

To prove this we apply the Cauchy-Frobenius Lemma (Theorem 1.3.6). From Theorems 2.1.2 and 2.1.1, the second and the third columns of the following table 6 can be obtained as follows:

Table 6: Permutations in $G_{(1,2)}$ and the number of fixed points in $X^{(2)}$

| Permutation $g$ in $G_{(1,2)}$ | No. of permutations | $|\text{Fix}(g)|$ in $X^{(2)}$ | Cycle type ($\alpha_1, \alpha_2, \alpha_3, \alpha_4$) |
|-------------------------------|-------------------|-------------------|-----------------|
| (1)(2)(cde)                   | 2                 | 1                 | (2,0,1,0,0)     |
| (1)(2)(cd)(e)                 | 3                 | 4                 | (3,1,0,0,0)     |
| (12)(c)(d)(e)                 | 1                 | 4                 | (3,1,0,0,0)     |
| (12)(cde)                     | 2                 | 1                 | (0,1,1,0,0)     |
| (12)(cd)(e)                   | 3                 | 2                 | (1,2,0,0,0,)    |
| Total                         | 12                |                   |                 |

Now applying the Cauchy-Frobenius Lemma we get the number of orbits of $G_{(1,2)}$ on $X^{(2)}$.

\[
\frac{1}{|G_{(1,2)}|} \sum |\text{Fix}(g)| = \frac{1}{12} \left( (1 \times 10) + (2 \times 1) + (3 \times 4) + (1 \times 4) + (2 \times 1) + (3 \times 2) \right)\\
= \frac{1}{12} (10 + 2 + 12 + 4 + 2 + 6)\\
= \frac{1}{12} (36) = 3.
\]

The three orbits of $G_{(1,2)}$ acting on $X^{(2)}$ determined above are:

- $\text{Orb}_{G_{(1,2)}} \{1,2\} = \{(1,2)\} = \Delta_0$, the trivial orbit.
- $\text{Orb}_{G_{(1,2)}} = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\} = \Delta_1$, the set of all unordered pairs containing exactly one of 1 and 2.
- $\text{Orb}_{G_{(1,2)}} = \{(3,4), (3,5), (4,5)\} = \Delta_2$, the set of all unordered pairs containing neither 1 nor 2.

Therefore the rank of $G$ on $X^{(2)}$ is 3 and the subdegrees are 1, 6 and 3.

We now discuss the suborbits $\Delta_0, \Delta_1, \Delta_2$ and the corresponding suborbital graphs. The suborbital graph corresponding to $\Delta_0$ is the null graph.

We now discuss the non-trivial suborbits $\Delta_1$ and $\Delta_2$.

Since the $|G| = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ is even, by Theorem 1.5.2, $\Delta_1$ and $\Delta_2$ are self paired and hence their corresponding suborbital graphs $\Gamma_1$ and $\Gamma_2$ are undirected.

The suborbital $O_1$ corresponding to the suborbit $\Delta_1$ is $O_1 = \{(g(1,2), g(1,3)) | g \in G\}$ (see Section 1.5). The suborbital graph $\Gamma_1$ corresponding to the suborbital has two 2-element subsets $V$ and $W$ from $X = \{1,2,3,4,5\}$ adjacent if and only if $|V \cap W| = 1$.

Similarly the suborbital $O_2$ corresponding to the suborbit $\Delta_2$, is $O_2 = \{(g(1,2), g(3,4)) | g \in G\}$.
The suborbital graph corresponding to $O_2$ has two 2-element subsets $V$ and $W$ adjacent if and only if $|V \cap W| = 0$.

Clearly $\Gamma_1$ and $\Gamma_2$ are complementary. It can easily be seen that $\Gamma_1$ is regular of degree 6. Its girth is 3. $\Gamma_2$ is regular of degree 3 and its girth is 5. $\Gamma_2$ is the famous Petersen graph.

The properties discussed above can easily be seen by constructing the suborbital graphs $\Gamma_1$ and $\Gamma_2$ as follows in figure 4, 5.

$\Gamma_1$ is regular of degree 6. Its girth is 3.

**Figure 4:** The suborbital graph $\Gamma_1$ corresponding to the suborbit $\Delta_1$ of $G$ on $X^{(2)}$
$\Gamma_2$ is the famous Petersen graph.
It is regular of degree 3.
Since $\Gamma_1$ and $\Gamma_2$ are connected, $G$ acts primitively on $X^{(2)}$.

**Figure 5**: The suborbital graph $\Gamma_2$ corresponding to the suborbit $\Delta_2$ of $G$ on $X^{(2)}$.
CONCLUSION

In this paper, we have discussed some properties of $S_n$, $(n = 3, 4, 5)$ acting on unordered pairs. We found out that $S_3 (n = 3, 4, 5)$ acts transitively and primitively on $X^{(2)}$.

$S_3$ is of rank 2 and has subdegrees 1 and 2.
$S_4$ and $S_5$, each is of rank 3 and have subdegrees 1, 1, 3 and 1, 3, 6, 3 respectively.

The suborbital graph corresponding to the non-trivial suborbit of $S_3$ is connected and regular of degree 2.

The suborbital graphs corresponding to the non-trivial suborbits of $S_4$ and $S_5$ are regular and complementary to each other.

In $S_5$, the second suborbital graph, $\Gamma_2$ is the famous Petersen graph that is regular and of degree 3. Its girth is 5.

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