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SCHOOL OF MATHEMATICS

ORDER STATISTICS OF UNIFORM, LOGISTIC AND EXPONENTIAL DISTRIBUTIONS

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Declaration

This is my original work and has not been presented for a degree in any other University.

Signature:... Date:.........................

Okoyo Collins Omondi

This project has been submitted for examination with my approval as the University Supervisor

Signature:... Date:.........................

Prof. J. A. M. Ottieno

Signature:... Date:.........................

Anne Wang'ombe

Dedication

I dedicate this project to my beautiful wife Caroline and daughter Sasha Malia.
A special feeling of gratitude to my loving parents, Charles Okoyo and Lucy Okoyo, my siblings Everline, Evans, Basyl, Sheilah and Oliver for their overwhelming social support and encouragement.

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Abstract

The term, order statistics, was introduced by Wilks in 1942. However, the subject is much older, as astronomers had long been interested in estimation of location beyond the sample mean. By early 19^{th} century, measures considered included the median, symmetrically trimmed means, the midrange and other related functions of order statistics.

In 1818, Laplace obtained (essentially) the distribution of the r^{th} order statistic in random samples and also derived a condition on the parent density under which the median is asymptotically more efficient than the mean.

Traditionally, distributions of order statistics have been constructed using the transformation method. Here we used both the transformation method and the new technique of beta generated distributions approach to construct distributions of order statistics.

We begin by studying the general properties and functions of order statistics from any continuous distribution. Specifically, we study the marginal and joint distributions, single and product moments of order statistics as well as distribution of the sample range and median.

We then apply these distributional properties of order statistics to the case of uniform, exponential and logistic distributions.

Even though, we have used the new technique of beta generated distribution approach in construction of order statistics distributions, we have not discussed this method in detail and we recommend further study on it. Finally, we hope that the knowledge summarized in this study will help in the understanding of order statistics.

Contents

Chapter 1

General Introduction

1.1 Background

For the last two decades, research in the area of order statistics has been steadily and rapidly growing. Gathering of results and presenting them in varied manner to suit diverse interests have been made possible due to the extensive role of order statistics in several areas of statistical inference. This project is an instance of such an attempt.

During this period, statistical inference theory has been developed for samples from populations having normal, binomial, poisson, multinomial and other specified forms of distribution functions depending on one or more unknown population parameters. These developments fall into two main categories: (i) statistical estimation, and (ii) the testing of statistical hypotheses.
The theory of statistical estimation deals with the problem of estimating values of the unknown parameters of distribution functions of specified form from random samples.
The testing of statistical hypotheses deals with the problem of testing, on the basis of a random sample, whether a population parameter has a specified value, or whether one or more specified functional relationships exist among two or more population parameters.

There are many problems of statistical inference in which one is unable to assume the functional form of the population distribution. Many of these problems are such that the strongest assumption which can be reasonably made is continuity of the cumulative distribution function of the population.

An increasing amount of attention is being devoted to statistical tests which hold for all populations having continuous cumulative distribution functions. Problems of this type in which the distribution function is arbitrary within abroad class are referred to as nonparametric problems of statistical inference.
In nonparametric problems it is being found that **order statistics**, that is, the ordered set of values in a random sample from least to greatest, are playing a fundamental role.

There are both theoretical and practical reasons for this increased attention to nonparametric problems and order statistics. Theoretically, it is desirable to develop methods of statistical inference which are valid with respect to broad classes of population distribution functions. This is indeed the case with statistical inference theory based on order statistics. Order statistics also permit very simple solutions of some of the more important parametric problems of statistical estimation and testing of hypotheses.

Historically, formal investigation in the sampling theory of order statistics dates back to 1902 when Karl Pearson solved the mathematical problem of finding the mean value of the difference between the r^{th} and $(r+1)^{th}$ order statistics in a sample of n observations from a population having a continuous probability density function.
Tippett (1925) extended the work of Pearson and found the mean value of the sample range (that is, the difference between the least and the greatest order statistics in a sample) and tabulated for certain sample sizes ranging from 3 to 1000, the cumulative distribution function (cdf) of the largest order statistic in a sample from a standard normal population.
Asymptotic results were first obtained by Fisher and Tippett (1928), who also derived under certain regularity conditions the limiting distributions of the largest and smallest order statistics as the sample size increases indefinitely by a method of functional equations. Mises (1936) made a precise determination of these reqularity conditions.
Further studies of these limiting distributions has been made by Gumbel (1935) and various applications to such problems like flood flows and maximum time intervals between successive emissions of gamma rays from a given source made by Gumbel (1941).
General expressions for the exact distribution functions of the median, quartiles, and range of a sample size of n was given by Allen (1932).

These early developments and subsequent researches carried out for a period of almost a quarter of a century have been summarized by Wilks (1948) in a survey paper. Moreover, exact distributions and properties of order statistics have been extensively studied in many articles and monographs e.g. Balakrishnan and Cohen (1991), David (1981) and Sarhan and Greenberg (1962).

Apart from the basic distribution theory and limit laws, attention has also been focused by various authors on problems involving order statistics in the theory of estimation and testing of hypotheses, and in multiple decision and comparison procedures. Most of these results are outlined in Gumbel (1958), Sarhan and Greenberg (1962) and Rupert Jr (2012).

Characterization of a distribution is an important tool in its application. In this study, characterization of the exponential distribution by order statistics and specifically by distributional properties, independence and moment assumption of order statistics have been examined in detail.

The aim of this project is to bring together various distributional properties of order statistics and inference based on them from any continuous distribution and from special cases of uniform, logistic and exponential distributions, and to describe how order statistics can be used to characterize exponential distribution.

The remaining parts of this study are organized as follows: In chapter 2, we give the general properties and functions of order statistics from any continuous population, and construct order statistics distributions based on both the transformational method and beta generated distributions approach. In chapters 3, 4 and 5 we apply these properties to the case of standard uniform, logistic and exponential distributions respectively. Characterization of exponential distribution based on order statistics is tackled in chapter 6. We give the conclusion and recommendation in chapter 7.

1.2 Notations, Terminologies and Definitions

Given random variables, $X_1, X_2, \cdots, X_n$, and arranging $X's$ in non-decreasing order, then $X_{1:n}$ denote the smallest observation, $X_{2:n}$ denote the second smallest and $X_{n:n}$ denote the largest observation. Hence, $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the ordered observations called *order statistics*. The focus of this study is on these order statistics.

Although, this notation of order statistics is widely used, some authors use $X_{(r)}$ to denote the r^{th} order statistic from a sample of size n. We have, however, used both notations interchangeably. Throughout this text, we assume that $X's$ are independent and identically distributed $(i.i.d)$ with cummulative distribution function $F(x)$ and density function $f(x)$.

More notations used herein are detailed below.

1. Given $a > 0,\ b > 0$ and $0 \leq p \leq 1$

$$I_p(a,b) = \frac{\int\limits_0^p t^{a-1}(1-t)^{b-1}\ dt}{\int\limits_0^1 t^{a-1}(1-t)^{b-1}\ dt}$$

   is the incomplete beta function

   Which results in;

$$\sum_{i=r}^{n} \binom{n}{i} p^i (1-p)^{n-i} = I_p(r, n-r+1)$$

2.

$$B(a,b) = \int\limits_0^1 t^{a-1}(1-t)^{b-1}\ dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\ a,b > 0$$

   is the complete beta function

   and

$$\Gamma(a) = \int\limits_0^{-\infty} t^{a-1}e^{-t},\ dt$$

   is the gamma function

3. cdf: cumulative distribution function

4. pdf: probability density function

5. $f(x)$: probability density function

6. $F(x)$: cumulative distribution function

7. $i.i.d$: independent and identically distributed

8. $R_n = X_{n:n} - X_{1:n}$: is the sample range

9. $U_{r:n}$: r^{th} order statistic from uniform $(0, 1)$ distribution

10. V_n : sample midrange $(X_{1:n} + X_{n:n})/2$

11. X : Population random variable

12. $X_{r:n}$ or $X_{(r)}$: r^{th} order statistic from a sample of size n

13. A cumulant K_n of a random variable X is defined using the cumulant-generating function (cgf) $K(t)$, which is the natural log of the moment generating function (mgf):

    $$K(t) = logE[e^{tX}]$$

    and mgf is defined as,

    $$M_X(t) = E[e^{tX}]$$

    where t are real values with the expected value being finite.

1.3 Problem Statement

In this project, we seek to construct order statistics distributions based on both transformation method and beta generated distribution approach.

1.4 Study Objective

The general objective was to study the general distributional properties and functions of order statistics from any continuous distribution and apply them to the uniform, logistic and exponential distributions.

1.4.1 Specific Objectives

1. Derive the expected values, moments, sample ranges and sample median of order statistics based on the standard uniform distribution.

2. Derive the expected values, moments, sample ranges and sample median of order statistics based on the standard logistic distribution.

3. Derive the expected values, moments, sample ranges and sample median of order statistics based on the standard exponential distribution.

4. Characterize exponential distribution based on distributional properties, independence and moment assumption of order statistics.

1.5 Literature Review

This section reviews various distributions of order statistics in general case from a continuous distribution and from specific distributions of uniform, logistic and exponential. We also review various characterization results of exponential distribution based on order statistics.

1.5.1 General distributions and functions of order statistics

Developments in the field of order statistics from the early $1960's$ are summarized in a book by Sarhan and Greenberg (1962).

Applications of order statistics in tests of hypotheses and estimation methods based on censored samples from lifetime distributions of interest have been widely brought forward by Harter (1969), Harter and Balakrishnan (1996) and Harter and Balakrishnan (1997).

David (1981) gave an exciting encyclopedic representation of order statistics. An introductory level of order statistics was prepared by Ahsanullah et al. (2013), while Galambos (1978) focused on the asymptomatic theory of extreme order statistics.

In this study, however, we simply give elementary description of order statistics presenting the marginal distributions, joint distributions and moments of order statistics. We also present brief details on sample ranges and median.

1.5.2 Order statistics from uniform distribution

The continuous uniform distribution or rectangular distribution is a family of symmetric probability distributions widely used in probability theory and statistics, such that for each member of the family, all intervals of the same length on the distribution's support are equally probable. This support is defined by the two parameters, a and b, which are the minimum and maximum values respectively. The distribution is usually abbreviated as $U(a, b)$.
Putting $a = 0$ and $b = 1$, the resulting distribution U(0,1) is called a *standard uniform distribution*, with an interesting property that, if u_1 has a standard uniform distribution, then so does $1 - u_1$.

Results for central order statistics from the uniform distribution were established by Weiss (1969), Ikeda and Matsunawa (1972) and Reiss (1976) while the extreme order statistics for uniform distribution was investigated by Pickands III et al. (1967), Weiss (1971), Ikeda and Matsunawa (1976), Reiss (1981), and De Haan and Resnick (1982), among others.

Here, we give a detailed description of the distributional properties from the uniform distribution. Specifically, we construct marginal and joint distributions, and moments of uniform order statistics. Detailed presentation of the distributions of ranges, midranges and median is also given.

1.5.3 Order statistics from logistic distribution

Initially, the *logistic growth function* was suggested as a tool for use in demographic studies and thereafter, the term "logistic distribution function" was developed by Reed and Berkson (1929). The logistic function has since then been used to estimate the growth of human population Pearl and Reed (1920), and to study income distributions Fisk (1961).

Order statistics can be applied to logistic distribution and a detailed discussion of order statistics from the logistic distribution and some of their properties is presented in Gupta and Balakrishnan (1990). They presented the exact and explicit expressions for the single and product moments in terms of gamma function.

Explicit expression of the cumulants of logistic order statistics were derived and their means and standard deviations tabulated by Birnbaum et al. (1963). Gupta et al. (1965) expressed cumulants in terms of polygamma functions. They also studied the sample range and provided a table of its percentage points for $n = 2$ and 3.
Malik (1980), generalized this result and derived cumulative distribution function of the r^{th} quasi-range in relation to $Y_{n-r:n} - Y_{r+1:n}$ for $r = 0, 1, \cdots, [\frac{n-1}{2}]$. The distribution of the sample median was studied in detail Gupta et al. (1965) and distribution of the sample mid-range in relation to $(Y_{1:n} + Y_{n:n})/2$ and the relationship in distribution between the mid-range and sample median of the logistic random variables was studied by George and Rousseau (1987).

1.5.4 Order statistics from exponential distribution

The exponential distribution is a model widely used in reliability theory and survival analysis with order statistics from exponential distribution widely applied in lifetesting and related areas.

Properties of order statistics and the use of resulting results in estimating parameters of exponential distribution has been studied by Balakrishnan and Cohen (1991), David (1981) and Sarhan and Greenberg (1962).

Expected values for $n \leq 100$ were given by Lieblein and Salzer (1957) for the extreme value distribution with common distribution function (cdf) $F(x) = \exp[-e^{-x}],\ -\infty < x < \infty$. Lieblein and Zelen (1956) also tabulated the covariances for $n \leq 6$. All means and variances for $n \leq 20$ (and separately for $n \leq 100$) were given by White (1969). Strictly, White dealt with $-X$, which he called a "reduced log-weibull" variate. Similarly working with $-X$, Balakrishnan and Chan (1992) provided 5D tables of all $\mu_{r:n}$ and $\sigma_{r,s:n}$ for $n = 1(1)15(5)30$. Further, Maritz and Munro (1967) gave 3D tables of $\mu_{r:n}$ for the generalized extreme-value distribution with cdf $F(x) = \exp[-(1-\gamma x)^{\frac{1}{\gamma}}],\ \gamma > 0,\ -\infty < x < \frac{1}{\gamma}$ and $5 \leq n \leq 10,\ \sigma = -0.10(0.05)0.40$

We, therefore, investigate the distributional and moment properties of order statistics from this exponential distribution and restricting ourselves to the case when $\lambda = 1$.

1.5.5 Characterizations of exponential distribution based on order statistics

The characterization theorems are increasingly becoming popular and since exponential distribution has wide applications, most characterization work had been focused towards this distribution.

Most of the results obtained from characterization of exponential distribution based on properties of order statistics were on independence of suitable functions of order statistics Ferguson (1967), Tanis (1964) and Govindarajulu (1966). Results based on the expected values of extreme order statistics were reported by Chan (1967).

Basu (1965) proved that if $F(x)$ is absolutely continuous with $F(0) = 0$, then the random variables $X_{1:n}$ and $(X_{2:n} - X_{1:n})$ are independent. Ferguson (1967) used the property of independence of $X_{1:n}$ and $(X_{1:n} - X_{2:n})$ to characterize the exponential distribution.

Therefore, in this study, we simply review these characterization results related to the exponential distribution based on order statistics.

1.6 Significance of the Study

Order statistics and related theory have many interesting and important applications in statistics, in modelling of empirical phenomena like climate characteristics, and in probability theory itself.

Below we list situations in which order statistics might have a significant role as outlined by Sarhan and Greenberg (1962).

1. **Robust location estimates.** Suppose that n independent measurements are available, and we wish to estimate their assumed common mean. It has long been recognized that the sample mean suffers from an extreme sensitivity to outliers and

model variations. Estimates based on the median or the average of central order statistics are less sensitive to model assumptions. A particular application of this observation is the accepted practice of using trimmed means (ignoring highest and lowest scores) especially in evaluating Olympic figure skating performances.

2. **Detection of outliers.** If one is confronted with a set of measurements and is concerned with determining whether some have been incorrectly made or reported, attention naturally focuses on certain order statistics of the sample. Usually the largest one (or two) and/or the smallest one (or two) are deemed most likely to be outliers. We may ask questions like: If the observations really were i.i.d, what is the probability that the largest order statistic would be as large as the suspiciously large value we have observed?

3. **Censored sampling.** Consider *life-testing* experiments, in which a fixed number n of items are placed on test and the experiment is terminated as soon as a prescribed number r have failed. The observed lifetimes are thus $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{r:n}$ whereas the lifetimes $X_{r+1:n} \leq X_{r+2:n} \leq \cdots \leq X_{n:n}$ remain unobserved.

4. **Natural disaster.** Disastrous floods and destructive earthquakes recur throughout history. Dam construction has long been focused on so called 100-year flood. Presumably the dams are built big enough and strong enough to handle any water flow to be encountered except for a level expected to occur only once every 100 years. Whether one agrees or not with the 100-year disaster philosophy, it is obvious that designers of dams and skycrapers, and even doghouses, should be concerned with the distribution of large order statistics from a possibly dependent, but possibly not identically distributed sequence.

5. **Strength of materials.** The adage that a chain is no longer than its weakest link underlines much of the theory of strength of materials, whether they are threads, sheets, or blocks. By considering failure potential in infinitely small sections of the material, quickly lead to strength distributions associated with limits of distributions of sample minima. Of course, if we stick to the finite chain with n links, its strength would be the minimum of the strengths of its n component links, again an order statistic.

6. **Reliability.** The example of a cord composed of n threads can be extended to lead us to reliability applications of order statistics. It may be that failure of one thread

will cause the cord to break (the weakest link), but more likely the cord will function as long as r (a number less than n) of the threads remains unbroken, as such it is an example of a r out of n system commonly discussed in reliability settings.
With regard to tire failure in automobile, is often an example of a 4 out 5 system (remember the spare).
Borrowing on terminology from electrical systems, the n out of n system is known as a *series system*, any component failure is disastrous. The 1 out of n system is known as a *parallel system*, it will function as long as any of the component survives.
The life of the r out of n system is clearly $X_{n-r+1:n}$, the $(n-r+1)^{th}$ largest observation of the component lifetimes, or equivalently, the time until less than r components are functioning. The study of system lifetime will necessarily involve distributions of order statistics.

7. **Quality control.** Here we use example of production of snickers candy bars passing through a conveyor belt. Each candy bar should weigh 2.1 ounces. No matter how well the pouring machine functions, minor fluctuation will occur, and potentially major aberrations might be encountered. We must be alert for correctable malfunctions causing unreasonable variation in the candy bar weight. In quality control, a sample of candy bars is weighted every hour, and close attention is paid to the order statistics of the weights so obtained. If the median (or perhaps the mean) is far from the target value, we must shaut down the line. Attention is also focused on the sample range, if it is too large, the process is out of control, and the widely fluctuating candy bar weights will probably cause problems further down the line. Hence, quality control clearly involve order statistics.

8. **Selecting the best.** Field trials of corn varieties involved carefully balanced experiments to determine which of several varieties is most productive. Obviously we are concerned with the maximum of a set of probability not identically distributed variables in such a setting. In this situation, the outlier (the best variety) is, however, good and merits retention (rather than being discarded as would be usual case with outlier setting).
There are other examples in which order statistics plays important role, for instance, in biology it helps in *selective breeding by culling.* Geneticists and breeders measure the effectiveness of a selection program by comparing the average of the selected group with the population average. Usually, the selected group consists of top or

bottom order statistics.

9. **Inequality of measurement.** The income distribution in most countries is clearly unequal. How does one make such statements precise? The usual approach involves order statistics of the corresponding income distributions. The particular device used is called a *Lorenz curve.* It summarizes the percent of total income accruing to the poorest p percent of the population for various values of p. Mathematically this is just the scaled integral of the empirical quantile function, a function with $X_{r:n}$ at the point r/n; $r = 1, 2, \cdots, n$ (where n is the number of individual incomes in the population). A high degree of convexity in the Lorenz curve signals a high degree of inequality in the income distribution.

10. **Olympic records.** Bob Beamons 1968 long jump remains on the olympic record book. Few other records last that long. If the best performances in each olympic games were modeled as independent identically distributed random variables, then records would become more and more scarce as time went by. Such is not the case. The simplest explanation involves improving and increasing populations, thus the 1968 high jumping champion was the best of, say, N_1 active international-caliber jumpers. In 1968 there were more high-caliber jumpers of probably higher caliber. So we are looking, most likely, at a sequence of not identically distributed random variables. But in any case we are focusing on maximum.

11. **Characterizations and goodness of fit.** The exponential distribution is famous for its so-called lack of memory. The usual model involves a light bulb or other electronic device. The argument goes that a light bulb that has been in service 20 hours is no more and no less likely to fail in the next minute than one that has been in service for, say, 5 hours, or even, than a brand new bulb. Such a curious distributional situation is reflected by the order statistics from exponential samples. For example, if $X_1, X_2, \cdots, X_n$ are $i.i.d$ exponential, then their spacings $X_i X_{i-1}$ are again exponential and, remarkably, are independent. It is only in the case of exponential random variables that such spacings properties are encountered. A vast literature of exponential characterizations and related goodness-of-fit tests has consequently developed.

    We remark in passing that most tests of goodness of fit for any parent distribution implicitly involve order statistics, since they often focus on deviations between the empirical quantile function and the hypothesized quantile function.

As a result of the above mentioned applications, it is of interest to study the theory of the distributional properties and functions of order statistics.

Moreover, this study seeks to contribute to the knowledge and comprehension of order statistics and how to characterize exponential distributions using order statistics.

Chapter 2

Distributions of Order Statistics and their Functions

2.1 Introduction

In this chapter we construct order statistics distributions using transformation method and using the beta generated distribution approach. Specifically, we obtain, in general case, the distributions of the single and joint order statistics and those of their functions. We also obtain the expected values and moments of order statistics.

2.2 Notations and Definitions

Let $X_1, X_2, ..., X_n$ be a random sample of size n from a continuous population having pdf f(x) and cdf F(x). The sample observations can be arranged in ascending order of magnitude such that $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$, where the numbers $i = 1, 2, \cdots, n$ in parenthesis indicate the rank of the observations in the sample.
Such an ordered set of new random variables constitutes the **order statistics**.
Where

$$\begin{aligned} X_{1:n} &= \text{the } 1^{st} \text{ order statistic} \\ &= \text{the smallest observation} \\ &= \min(X_1, X_2, ..., X_n) \end{aligned}$$

$$\begin{aligned} X_{n:n} &= \text{the } n^{th} \text{ order statistic} \\ &= \text{the largest observation} \\ &= \max(X_1, X_2, ..., X_n) \end{aligned}$$

and

$$\begin{aligned} X_{r:n} &= \text{the } r^{th} \text{ order statistic} \\ &= \text{the } r^{th} \text{ smallest value} \end{aligned}$$

Remark 1. *$X_1, X_2, ..., X_n$ is a random sample. They are therefore independent random variables; but $X_i^{'s}$ are dependent because of the inequality relation among them.*

Remark 2. *A more explicit notation of the order statistics is $X_{(1)}, X_{(2)}, ..., X_{(n)}$ or $X_{1,n}, X_{2,n}, ..., X_{n,n}$*

Remark 3. *Since X_r is a random variable, it's function is also a random variable.*

2.2.1 Functions of Order Statistics

Linear functions of order statistics are of the form

$$\sum_{i=1}^{n} \omega_i X_i \tag{2.1}$$

Certain functions of the order statistics $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ are important statistics themselves. A few of these are;

Range

A range is the distance between the smallest $X_{1:n}$ and the largest $X_{n:n}$ observations. It is a measure of the dispersion in the sample and hence reflect the dispersion in the population. The statistic

$$R_n = X_{n:n} - X_{1:n} \tag{2.2}$$

is known as the **sample range** of the random sample.
While the statistic

$$M = \frac{1}{2}\{X_{n:n} + X_{1:n}\} \tag{2.3}$$

is known as the **mid-range** of the random sample.

Median

The median is a measure of location that might be considered an alternative to the sample mean.
If $n = 2m + 1$ (i.e. n is odd), then the $(m + 1)^{th}$ observation, which is the middle value, is called **sample median** of the random sample, given as

$$X_m \text{ where } m = \frac{(n+1)}{2} \tag{2.4}$$

For $n = 2m$ (i.e. n is even), there is not a single middle observation but rather two middle observations. Thus the **sample median** becomes

$$Median = \frac{1}{2}\{X_m + X_{m+1}\} \tag{2.5}$$

Quantiles

Further, we can generalize the sample median to other **sample quantiles**.
If np is not an integer, we define the **sample quantile** of order p to be the order statistic

$$X_{k:n} \text{ where } k = ceil(np) \tag{2.6}$$

We note that $ceil(np)$ is the smallest integer greater than or equal to np.
If np is an integer k, then we define the sample quantile of order p to be the average of the order statistics.

$$\frac{[X_{k:n} + X_{k+1:n}]}{2} \tag{2.7}$$

The sample quantile of order p is a natural statistic that is analogous to the distribution quantile of order p.

Remark 4. *The sample quantile of order $\frac{1}{4}$ is known as the* ***first sample quartile*** *and is frequently denoted as Q_1.*
The sample quantile of order $\frac{3}{4}$ is known as the ***third sample quartile*** *and is frequently denoted as Q_3.*
Note that the sample median is the quantile of order $\frac{1}{2}$.
The ***interquartile range (IQR)*** *is defined to be*

$$IQR = Q_3 - Q_1$$

The IQR is a statistic that measures the spread of the distribution about the median, but of course this number gives less information than the interval $[Q_1, Q_3]$.

2.3 Distributions of Order Statistics Based on Transformation

In this section we derive using transformation the explicit form of the distribution functions and the density of a single and joint order statistics, also discussed is the distributions of the range and median.

Assumption $X_{1:n}, X_{2:n} \cdots , X_{n:n}$ are i.i.d with marginal cdf $F(.)$.

2.3.1 Distribution of Single Order Statistics

We derive the pdf $f(x)$ and cdf $F(x)$ of the largest observation $X_{n:n}$, smallest observation $X_{1:n}$ and the r^{th} observation $X_{r:n}$.

For the **largest observation**, the cdf of $X_{n:n}$ is given by,

$$\begin{aligned} F_n(x) &= Pr[X_{n:n} \leq x] \\ &= Pr[X_{1:n} \leq x, X_{2:n} \leq x, \cdots , X_{n:n} \leq x] \\ &= Pr(X_{1:n} \leq x)Pr(X_{2:n} \leq x) \cdots Pr(X_{n:n} \leq x) \\ &= [F(x)]^n \end{aligned}$$

Letting $f_n(x)$ denote the probability density function of $X_{n:n}$, and taking derivatives of both sides, we have;

$$f_n(x) = \frac{d}{dx}F_n(x) \ = n[F(x)]^{n-1}f(x), \ -\infty < x < \infty \tag{2.8}$$

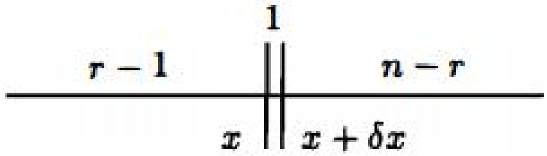
For the **smallest observation**, the cdf of $X_{1:n}$ is given by,

$$\begin{aligned} F_1(x) &= Pr[X_{1:n} \leq x] \\ &= 1 - Pr[X_{1:n} > x] \\ &= 1 - Pr[X_{1:n} > x, X_{2:n} > x, \cdots , X_{n:n} > x] \\ &= 1 - [1 - Pr(X_{1:n} \leq x)][1 - Pr(X_{2:n} \leq x)] \cdots [1 - Pr(X_{n:n} \leq x)] \\ &= 1 - [1 - F(x)]^n \end{aligned}$$

Thus, if $f_1(x)$ denotes the probability density function of $X_{1:n}$, differentiation of both sides of the last expression yields,

$$f_1(x) = \frac{d}{dx}F_1(x) = n[1 - F(x)]^{n-1}f(x), \ -\infty < x < \infty \tag{2.9}$$

For the r^{th} **order statistic**, we consider the following figure



Out of n observations, there are (r-1) less than x, one observation between x and x+dx, and (n-r) observations greater than x.
We are taking **dx** so small that the probability that more than one random variable falling between **x** and **x+dx** inclusive, i.e $[x, x + dx]$ is negligible and that

$$Pr[X_i > x] = Pr[X_i > x + dx], \text{ for } i = 1, 2, \cdots \tag{2.10}$$

Using multinomial probabilities, we get the pdf as follows;

$$\begin{aligned} f_{X_{r:n}}(x) &= Pr[X_{r:n} = x] \\ &= \frac{n!}{(r-1)!1!(n-r)!}[Pr\{X_i \leq x\}]^{r-1} Pr\{X_i = x\}[Pr\{X_i > x\}]^{n-r} \\ &= \frac{n!}{(r-1)!1!(n-r)!}[F(x)]^{r-1} f(x)[1 - F(x)]^{n-r}, \; -\infty < x < \infty \end{aligned} \tag{2.11}$$

In general, the cdf of $X_{r:n}$ may be obtained by integrating the pdf of $X_{r:n}$ in equation (2.11) as follows,

$$\begin{aligned} F_{r:n}(x) &= Pr(X_{r:n} \leq x) \\ &= \sum_{r=1}^{n} \binom{n}{r} [F(x)]^r [1 - F(x)]^{n-r} \end{aligned} \tag{2.12}$$

by using the identity that

$$\sum_{r=1}^{n} \binom{n}{r} p^r [1-p]^{n-r} = \int_0^p \frac{n!}{(r-1)!(n-r)!} t^{r-1}(1-t)^{n-r} \, dt, \; 0 < p < 1 \tag{2.13}$$

we write the cdf of $X_{r:n}$ from equation (2.12) as,

$$\begin{aligned} F_{r:n}(x) &= \int_0^{F(x)} \frac{n!}{(r-1)!(n-r)!} t^{r-1}(1-t)^{n-r} \, dt \\ &= I_{F(x)}(r, n-r+1), \; -\infty < x < \infty \end{aligned} \tag{2.14}$$

which is an incomplete beta function.

Pinsker et al. (1986) noted that cdf of $X_{r:n}$ can be written in terms of negative binomial probabilities instead of the binomial form given in equation (2.12) as,

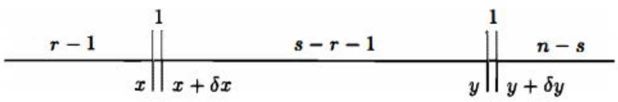
$$\begin{aligned} F_{r:n}(x) &= Pr(X_{r:n} \leq x) \\ &= \binom{r-1}{r-1}[F(x)]^r[1-F(x)]^0 + \binom{r}{r-1}[F(x)]^r[1-F(x)]^1 + \cdots \\ &+ \binom{n-1}{r-1}[F(x)]^r[1-F(x)]^{n-r} \\ &= \sum_{i=0}^{n-r}\binom{n-i-1}{r-1}[F(x)]^r[1-F(x)]^{n-r-i}, \; -\infty < x < \infty \end{aligned} \tag{2.15}$$

2.3.2 Joint Distribution of Two or More Order Statistics

We derive the joint pdf $f(x,y)$ and cdf $F(x,y)$ of two order statistics, $X_{r:n}$ and $X_{s:n}$, and use the same method to derive the joint density function of all n order statistics.

Joint distribution of two order statistics

In general, we consider two order statistics; namely $X_{r:n}$ and $X_{s:n}$ where $1 \leq r < s \leq n$. Further we consider the figure below



- $(r-1)$ of X'^s are less than x
- one X'^s lies between x and $x+\delta x$
- $(s-r-1)$ of X'^s lies between $x+\delta x$ and y
- one X'^s lies between y and $y+\delta y$
- $(n-s)$ of X'^s is larger than $y+\delta y$

Using multinomial probabilities, we get the joint pdf as follows,

$$
\begin{aligned}
&f_{X_{r:n}X_{s:n}}(x,y) \\
&= C_{r,s}[Pr\{X_i < x\}]^{r-1}Pr\{X_i = x\}[Pr\{x < X < y\}]^{s-r-1}Pr\{X_i = y\}[Pr\{X_i > y\}]^{n-s} \\
&= C_{r,s}[F(x)]^{r-1}f(x)[F(y) - F(x)]^{s-r-1}f(y)[1 - F(y)]^{n-s},\ -\infty < x_r < x_s < \infty
\end{aligned}
\tag{2.16}
$$

where

$$
C_{r,s} = \frac{n!}{(r-1)!1!(s-r-1)!1!(n-s)!} \tag{2.17}
$$

In particular, the **joint density of the maximum and the minimum**, $(X_{1:n}, X_{n:n})$ is

$$
f_{X_{1:n}X_{n:n}}(x,y) = n(n-1)[F(y) - F(x)]^{n-2}f(x)f(y),\ x < y \tag{2.18}
$$

Also, the **joint density of two consecutive order statistics**, $(X_{i:n}, X_{i+1:n})$ is

$$
f_{X_{(i)}X_{(i+1)}}(x,y) = \frac{n!}{(i-1)!(n-i-1)!}[F(x)]^{i-1}f(x)[F(y)-F(x)]^{n-i-1}f(y),\ x < y \tag{2.19}
$$

The joint cdf of $X_{r:n}$ and $X_{s:n}$ can in principle be obtained through integration of the joint pdf in equation (2.16)

$$
\begin{aligned}
F_{r,s:n}(x,y) &= Pr(X_{r:n} \le x, X_{s:n} \le y) \\
&= \sum_{i=s}^{n}\sum_{j=r}^{i} \frac{n!}{j!(i-j)!(n-i)!}[F(x)]^j[F(y) - F(x)]^{i-j}[1 - F(y)]^{n-i}
\end{aligned}
\tag{2.20}
$$

Thus, the joint cdf of $X_{r:n}$ and $X_{s:n}$, $(1 \le r < s \le n)$ is the tail probability [over the rectangular region $(s, r), (s, r+1), \cdots, (n, n)$] of a bivariate binomial distribution

$$
\begin{aligned}
&\sum_{i=s}^{n}\sum_{j=r}^{i} \frac{n!}{j!(i-j)!(n-i)!}p_1^j(p_2 - p_1)^{i-j}(1 - p_2)^{n-i} \\
&= \int_0^{p_1}\int_{t_1}^{p_2} C_{r,s}t_1^{r-1}(t_2 - t_1)^{s-r-1}(1 - t_2)^{n-s}\ dt_2dt_1,\ 0 < p_1 < p_2 < 1
\end{aligned}
\tag{2.21}
$$

Hence,

$$F_{r,s:n}(x,y) = \int_{0}^{F(x)} \int_{t_1}^{F(y)} C_{r,s} t_1^{r-1}(t_2 - t_1)^{s-r-1}(1-t_2)^{n-s}\, dt_2 dt_1,\ 0 < x < y < 1 \quad (2.22)$$

where $C_{r,s}$ is as in equation (2.17). The above equation (2.22) takes the form of an incomplete bivariate beta function.

Specifically, we now let $n = 2$ and find the joint density function for $X_{1:n}$ and $X_{2:n}$. The event $(X_{1:n} \leq x_1, X_{2:n} \leq x_2)$ means that either $(X_1 \leq x_1, X_2 \leq x_2)$ or $(X_2 \leq x_1, X_1 \leq x_2)$. [Notice that $X_{1:n}$ could be either X_1 or X_2, whichever is smaller]
Therefore, for $x_1 \leq x_2$,

$$Pr(X_{1:n} \leq x_1, X_{2:n} \leq x_2) = Pr[(X_1 \leq x_1, X_2 \leq x_2)\ \cup\ (X_2 \leq x_1, X_1 \leq x_2)]$$

Using the additive law of probability and recalling that $x_1 \leq x_2$, we see that

$$\begin{aligned} Pr(X_{1:n} \leq x_1, X_{2:n} \leq x_2) &= Pr(X_1 \leq x_1, X_2 \leq x_2)\ +\ Pr(X_2 \leq x_1, X_1 \leq x_2) \\ &-\ Pr(X_1 \leq x_1, X_2 \leq x_1) \end{aligned}$$

Because X_1 and X_2 are independent and $Pr(X_i \leq w) = F(w)$, for $i = 1, 2$, it follows that, for $x_1 \leq x_2$;

$$\begin{aligned} Pr(X_{1:n} \leq x_1, X_{2:n} \leq x_2) &= F(x_1)F(x_2)\ +\ F(x_2)F(x_1)\ -\ F(x_1)F(x_1) \\ &= 2F(x_1)F(x_2)\ -\ [F(x_1)]^2 \end{aligned} \quad (2.23)$$

Suppose now, $x_1 > x_2$ (recall that $X_{1:n} \leq X_{1:n}$), then we have;

$$\begin{aligned} Pr(X_{1:n} \leq x_1, X_{2:n}) \leq x_2) &= Pr(X_{1:n} \leq x_2, X_{2:n} \leq x_2) \\ &= Pr(X_1 \leq x_2, X_2 \leq x_2) \\ &= [F(x_2)]^2 \end{aligned} \quad (2.24)$$

Therefore, the joint cumulative density function of $X_{1:n}$ and $X_{2:n}$ is given as;

$$F_{X_{1:n},X_{2:n}}(x_1, x_2) = \begin{cases} 2F(x_1)F(x_2)\ -\ [F(x_1)]^2, & \text{for } x_1 \leq x_2 \\ [F(x_2)]^2, & \text{for } x_1 > x_2 \end{cases}$$

Letting, $g_{1,2}(x_1, x_2)$, denote the joint density function of $X_{1:n}$ and $X_{2:n}$, then on differentiating first with respect to x_2 and then with respect to x_1, we obtain

$$g_{1,2}(x_1, x_2) = \begin{cases} 2f(x_1)f(x_2), & \text{for } x_1 \leq x_2 \\ 0, & \text{elsewhere} \end{cases} \tag{2.25}$$

If we now consider the case $n = 3$, and find the joint density function for $X_{1:n}, X_{2:n}$ and $X_{3:n}$.

Considering a probability such as $Pr(a < X_1 = X_2 < b,\ a < X_3 < b)$, given by

$$\int_a^b \int_a^b \int_{x_2}^{x_2} f(x_1)f(x_2)f(x_3)dx_1dx_2dx_3 = 0$$

However, $\int_{x_2}^{x_2} f(x_1)dx_1$, is defined in calculus to be zero.

We may, without altering the distribution of X_1, X_2, X_3, define the joint p.d.f $f(x_1)f(x_2)f(x_3)$ to be zero at all points (x_1, x_2, x_3) that have at least two of their coordinates equal.

Then the set A, where $f(x_1)f(x_2)f(x_3) > 0$, is the union of the six mutually disjoint sets:

$$\begin{aligned}
A_1 &= \{(x_1, x_2, x_3); a < x_1 < x_2 < x_3 < b\}, \\
A_2 &= \{(x_1, x_2, x_3); a < x_2 < x_1 < x_3 < b\}, \\
A_3 &= \{(x_1, x_2, x_3); a < x_1 < x_3 < x_2 < b\}, \\
A_4 &= \{(x_1, x_2, x_3); a < x_2 < x_3 < x_1 < b\}, \\
A_5 &= \{(x_1, x_2, x_3); a < x_3 < x_1 < x_2 < b\}, \\
A_6 &= \{(x_1, x_2, x_3); a < x_3 < x_2 < x_1 < b\}.
\end{aligned}$$

There are six of these sets because we can arrange x_1, x_2, x_3 in precisely $3! = 6$ ways.

Consider the functions $y_1 =$ minimum of x_1, x_2, x_3; $y_2 =$ middle in magnitude of x_1, x_2, x_3 and $y_3 =$ maximum of x_1, x_2, x_3.

These functions define one-to-one transformations that map each of $A_1, A_2, \cdots, A_6$ onto the same set $B = \{(y_1, y_2, y_3); a < y_1 < y_2 < y_3 < b\}$.

The inverse functions are,

for points in A_1, $x_1 = y_1, \quad x_2 = y_2, x_3 = y_3;$

for points in A_2, $x_1 = y_2, \quad x_2 = y_1, x_3 = y_3;$

and so on for each of the remaining four sets.
Then we have that

$$\mathrm{J}_1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \qquad and \qquad \mathrm{J}_2 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$

It is easily verified that the absolute value of each of the $3! = 6$ Jacobians is $+1$.
Thus the joint pdf of the three order statistics $Y_1 =$ minimum of X_1, X_2, X_3; $\quad Y_2 =$ middle in magnitude of X_1, X_2, X_3 and $Y_3 =$ maximum of X_1, X_2, X_3 is

$$\begin{aligned} f(y_1, y_2, y_3) &= |J_1| f(y_1) f(y_2) f(y_3) + |J_2| f(y_2) f(y_1) f(y_3) + \cdots + |J_6| f(y_3) f(y_2) f(y_1) \\ &= \begin{cases} 3! f(y_1) f(y_2) f(y_3), & a < y_1 < y_2 < y_3 < b \\ 0, & \text{elsewhere} \end{cases} \end{aligned} \tag{2.26}$$

For $n = 4$, and considering $X_1 \leq X_2 \leq X_3 \leq X_4$
The joint pdf is similarly given as

$$f(x_1, x_2, x_3, x_4) = 4! f(x_1) f(x_2) f(x_3) f(x_4) \tag{2.27}$$

We can note that, this joint pdf can be used to obtain the marginal density function for any of the order statistics using integration.
For instance, we find the distribution of x_2, x_3 as follows;

$$f(x_2, x_3) = 4! \int_{x_{(3)}}^{\infty} \int_{-\infty}^{x_{(2)}} f(x_1, x_2, x_3, x_4) dx_1 dx_4$$

Joint distribution of n Order Statistics

The same method used to obtain joint density for $n = 3$ and $n = 4$ can be generalized to find the joint density of all $X_{1:n}, X_{2:n}, \cdots X_{n:n}$, which is

$$f_{1,2,\cdots,n}(x_1 x_2 \cdots x_n) = \begin{cases} n! f(x_1) f(x_2) \cdots f(x_n), & -\infty < x_1 \leq x_2 \leq \cdots \leq x_n < \infty \\ 0, & \text{elsewhere} \end{cases} \tag{2.28}$$

Remark 5. *The density of the r^{th} order statistic and the joint density of two order statistics are summarized in the below theorem.*

Theorem 2.3.1. *Let $X_1, X_2, \cdots X_n$ be independent identically distributed continuous random variables with common distribution function $F(x)$ and density function $f(x)$. If $X_{r:n}$ denotes the r^{th} order statistic, then the density function of $X_{r:n}$ is given by*

$$f_{r:n}(x) = \frac{n!}{(r-1)!1!(n-r)!}[F(x)]^{r-1} f(x)[1 - F(x)]^{n-r}, -\infty < x < \infty$$

If r and s are two integers such that $1 \leq r < s \leq n$, then the joint density of $X_{r:n}$ and $X_{s:n}$ is given by

$$f_{r,s:n}(x, y) = C_{r,s}[F(x)]^{r-1} f(x)[F(y) - F(x)]^{s-r-1} f(y)[1 - F(y)]^{n-s}, -\infty < x < y < \infty$$

where $C_{r,s}$ is as in equation (2.17)

2.4 Distributions of Order Statistics Based on Beta Generated Distribution

2.4.1 Introduction

Here we construct distributions of order statistics using the new family of generalized beta generated distribution approach. The new family of generalized beta generated distributions is based on beta generators classified as *beta generated distributions*

The beta generated distribution was first introduced by Eugene and Famoye (2002) through its cdf, and Jones (2004) called it the generalized order statistics.

2.4.2 Beta generated distribution

The cdf of a beta distribution is defined by,

$$F(t) = \int_0^x \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}\, dt,\ a > 0,\ b > 0,\ 0 < x < 1 \tag{2.29}$$

Replacing t by a cdf say $G(x)$, of any distribution, since $0 < G(x) < 1$, for $-\infty < x < \infty$ we have

$$F[G(x)] = \frac{1}{B(a,b)} \int_0^{G(x)} t^{a-1}(1-t)^{b-1}\, dt,\ a > 0,\ b > 0 \tag{2.30}$$

and where

$$\begin{aligned} F[G(x)] &= \text{A cdf of a cdf} \\ &= \text{A function of } x \\ &= F(x) \end{aligned} \tag{2.31}$$

Now, taking derivatives both sides (i.e.)

$$\frac{d}{dx}F(x) = \frac{d}{dx}F[G(x)] \tag{2.32}$$

gives

$$\begin{aligned} f(x) &= \{F^{'}[G(x)]\}G^{'}(x) \\ &= \{f[G(x)]\}g(x) \end{aligned} \tag{2.33}$$

Implying that

$$f(x) = \frac{g(x)[G(x)]^{a-1}[1-G(x)]^{b-1}}{B(a,b)},\ 0 < G(x) < 1,\ -\infty < x < \infty \tag{2.34}$$

where $a > 0, b > 0$ and $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function.

Equation (2.34) is the beta generator or beta generated distribution. It is also referred to as the generalized beta F-distribution (Kong and Sepanski (2007)).

The equation can be used to generate a new family of beta distributions usually referred to as beta generated distributions.

It can also be called the generalized r^{th} order statistic (Jones (2004)) because the order statistic distribution is a special case when $a = r, b = n - r + 1$, which gives the following density function

$$\begin{aligned} f[G(x)] &= \frac{g(x)[G(x)]^{r-1}[1-G(x)]^{(n-r+1)-1}}{B(r, n-r+1)} \\ &= \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)} g(x)[G(x)]^{r-1}[1-G(x)]^{n-r} \\ &= \frac{n!}{(r-1)!(n-r)!} g(x)[G(x)]^{r-1}[1-G(x)]^{n-r}, \ -\infty < x < \infty \end{aligned} \tag{2.35}$$

Therefore, equation (2.35) is the r^{th} order statistic generated from the beta distribution.

We can further obtain the pdf of the minimum ($r = 1$) and the maximum ($r = n$) values from equation (2.35) as below.

For $r = 1$ (minimum),

$$\begin{aligned} f_1(x) &= \frac{n!}{(1-1)!(n-1)!} g(x)[G(x)]^{1-1}[1-G(x)]^{n-1} \\ &= \frac{n!}{(n-1)!} g(x)[1-G(x)]^{n-1} \\ &= n[1-G(x)]^{n-1} g(x), \ -\infty < x < \infty \end{aligned} \tag{2.36}$$

For $r = n$ (maximum),

$$\begin{aligned} f_n(x) &= \frac{n!}{(n-1)!(n-n)!} g(x)[G(x)]^{n-1}[1-G(x)]^{n-n} \\ &= \frac{n!}{(n-1)!} g(x)[G(x)]^{n-1} \\ &= n[G(x)]^{n-1} g(x), \ -\infty < x < \infty \end{aligned} \tag{2.37}$$

2.4.3 Various beta generated distributions

Various beta generated distributions have been constructed by different authors, here we extend such work and show how beta generated distribution approach is used to find distributions of order statistics, especially, the r^{th}, minimum and maximum order statistics for various specific distributions. However, for the succeeding chapters, we will concentrate on the uniform, logistic and exponential distributions.

Standard uniform order statistics

The beta-standard uniform distribution can be obtained as follows.
Let

$$G(x) = u,\ 0 < u < 1 \tag{2.38}$$

be the cdf of the standard uniform distribution and the pdf given by

$$g(x) = 1 \tag{2.39}$$

Therefore, using equation (2.34), the density function of the beta-standard uniform distribution is given by

$$\begin{aligned} f(u) &= \frac{1 \cdot [u]^{a-1}[1-u]^{b-1}}{B(a,b)} \\ &= \frac{1}{B(a,b)} u^{a-1}(1-u)^{b-1} \end{aligned} \tag{2.40}$$

Hence, equation (2.40) is the beta-standard uniform distribution.

Letting $a = r$ and $b = n - r + 1$, we get the r^{th} order statistic for the standard uniform distribution as,

$$\begin{aligned} f_{r:n}(u) &= \frac{1}{B(r, n-r+1)} u^{r-1}(1-u)^{(n-r+1)-1} \\ &= \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)} u^{r-1}(1-u)^{n-r} \\ &= \frac{n!}{(r-1)!(n-r)!} u^{r-1}(1-u)^{n-r},\ 0 \leq u \leq 1 \end{aligned} \tag{2.41}$$

Subsequently, we get the minimum and the maximum order statistics for the standard uniform distribution as follows.

For $r = 1$ (minimum);

$$f_{1:n}(u) = n(1-u)^{n-1},\ 0 \leq u \leq 1 \tag{2.42}$$

For $r = n$ (maximum);

$$f_{n:n}(u) = nu^{n-1},\ 0 \leq u \leq 1 \tag{2.43}$$

Standard logistic order statistics

The beta-standard logistic distribution can be obtained as follows.
Let

$$G(x) = \frac{1}{(1+e^{-x})},\ -\infty < x < \infty \tag{2.44}$$

be the cdf of the standard logistic distribution and the pdf given by

$$g(x) = \frac{e^{-x}}{(1+e^{-x})^2},\ -\infty < x < \infty \tag{2.45}$$

Therefore, using equation (2.34), the density function of the beta-standard logistic distribution is given by

$$f(x) = \frac{1}{B(a,b)} \cdot \frac{e^{-x}}{(1+e^{-x})^2} \cdot \left[\frac{1}{(1+e^{-x})}\right]^{a-1} \cdot \left[1 - \frac{1}{(1+e^{-x})}\right]^{b-1} \tag{2.46}$$

Hence, equation (2.46) is the beta-standard logistic distribution.

Similarly, letting $a = r$ and $b = n - r + 1$, we get the r^{th} order statistic for the standard logistic distribution as,

$$\begin{aligned} f_{r:n}(x) &= \frac{1}{B(r, n-r+1)} \cdot \frac{e^{-x}}{(1+e^{-x})^2} \cdot \left[\frac{1}{(1+e^{-x})}\right]^{r-1} \cdot \left[1 - \frac{1}{(1+e^{-x})}\right]^{(n-r+1)-1} \\ &= \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)} \cdot \frac{e^{-x}}{(1+e^{-x})^2} \cdot \left[\frac{1}{(1+e^{-x})}\right]^{r-1} \cdot \left[1 - \frac{1}{(1+e^{-x})}\right]^{n-r} \\ &= \frac{n!}{(r-1)!(n-r)!} \frac{e^{-x}}{(1+e^{-x})^2} \left[\frac{1}{(1+e^{-x})}\right]^{r-1} \left[1 - \frac{1}{(1+e^{-x})}\right]^{n-r}, \ -\infty \leq x \leq \infty \end{aligned} \tag{2.47}$$

Subsequently, we get the minimum and the maximum order statistics for the standard logistic distribution as follows.

For $r = 1$ (minimum);

$$\begin{aligned} f_{1:n}(x) &= \frac{n!}{(1-1)!(n-1)!} \frac{e^{-x}}{(1+e^{-x})^2} \left[\frac{1}{(1+e^{-x})}\right]^{1-1} \left[1 - \frac{1}{(1+e^{-x})}\right]^{n-1} \\ &= \frac{ne^{-x}}{(1+e^{-x})^2} \left[1 - \frac{1}{(1+e^{-x})}\right]^{n-1}, \ -\infty \leq x \leq \infty \end{aligned} \tag{2.48}$$

For $r = n$ (maximum);

$$\begin{aligned} f_{n:n}(x) &= \frac{n!}{(n-1)!(n-n)!} \frac{e^{-x}}{(1+e^{-x})^2} \left[\frac{1}{(1+e^{-x})}\right]^{n-1} \left[1 - \frac{1}{(1+e^{-x})}\right]^{n-n} \\ &= \frac{ne^{-x}}{(1+e^{-x})^2} \left[\frac{1}{(1+e^{-x})}\right]^{n-1}, \ -\infty \leq x \leq \infty \end{aligned} \tag{2.49}$$

Standard exponential order statistics

The beta-standard exponential distribution can be obtained as follows.
Let

$$G(x) = 1 - e^{-x}, \ x \geq 0 \tag{2.50}$$

be the cdf of the standard exponential distribution and the pdf given by

$$g(x) = e^{-x}, \ x \geq 0 \tag{2.51}$$

Therefore, using equation (2.34), the density function of the beta-standard exponential distribution is given by

$$\begin{aligned} f(x) &= \frac{1}{B(a,b)} \cdot e^{-x} \cdot [1-e^{-x}]^{a-1} \cdot [1-(1-e^{-x})]^{b-1} \\ &= \frac{1}{B(a,b)}[1-e^{-x}]^{a-1}e^{-x}e^{-(b-1)x} \\ &= \frac{1}{B(a,b)}[1-e^{-x}]^{a-1}e^{-bx}, \ x \geq 0 \end{aligned} \tag{2.52}$$

Hence, equation (2.52) is the beta-standard exponential distribution.

Similarly, letting $a = r$ and $b = n - r + 1$, we get the r^{th} order statistic for the standard exponential distribution as,

$$\begin{aligned} f_{r:n}(x) &= \frac{1}{B(r, n-r+1)}[1-e^{-x}]^{r-1}e^{-x(n-r+1)} \\ &= \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)}[1-e^{-x}]^{r-1}e^{-x(n-r+1)} \\ &= \frac{n!}{(r-1)!(n-r)!}(1-e^{-x})^{r-1}e^{-(n-r+1)x}, \ 0 \leq x < \infty \end{aligned} \tag{2.53}$$

Subsequently, we get the minimum and the maximum order statistics for the standard exponential distribution as follows.

For $r = 1$ (minimum);

$$\begin{aligned} f_{1:n}(x) &= \frac{n!}{(1-1)!(n-1)!}(1-e^{-x})^{1-1}e^{-(n-1+1)x} \\ &= ne^{-nx}, \ 0 \leq x < \infty \end{aligned} \tag{2.54}$$

For $r = n$ (maximum);

$$\begin{aligned} f_{n:n}(x) &= \frac{n!}{(n-1)!(n-n)!}(1-e^{-x})^{n-1}e^{-(n-n+1)x} \\ &= ne^{-x}(1-e^{-x})^{n-1}, \ -\infty \leq x \leq \infty \end{aligned} \tag{2.55}$$

Pareto order statistics

The beta-pareto distribution can be obtained as follows.
Let

$$G(x) = 1 - (\frac{x_m}{x})^\alpha,\ x \geq x_m \tag{2.56}$$

be the cdf of pareto type 1 distribution and the pdf given by

$$g(x) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}},\ x \geq x_m \tag{2.57}$$

Therefore, using equation (2.34), we obtain the density function of the beta-pareto distribution as

$$\begin{aligned} f(x) &= \frac{1}{B(a,b)} \cdot \frac{\alpha x_m^\alpha}{x^{\alpha+1}} \cdot \left[1 - (\frac{x_m}{x})^\alpha\right]^{a-1} \cdot \left[1 - \{1 - (\frac{x_m}{x})^\alpha\}\right]^{b-1} \\ &= \frac{1}{B(a,b)} \frac{\alpha x_m^\alpha}{x^{\alpha+1}} \left[1 - (\frac{x_m}{x})^\alpha\right]^{a-1} \left[(\frac{x_m}{x})^\alpha\right]^{b-1},\ x \geq x_m \end{aligned} \tag{2.58}$$

Hence, equation (2.58) is the beta-pareto distribution.

Similarly, letting $a = r$ and $b = n - r + 1$, we get the r^{th} order statistic for the pareto distribution as,

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \frac{\alpha x_m^\alpha}{x^{\alpha+1}} \left[1 - (\frac{x_m}{x})^\alpha\right]^{r-1} \left[(\frac{x_m}{x})^\alpha\right]^{n-r},\ x \geq x_m \tag{2.59}$$

Subsequently, we get the minimum and the maximum order statistics for the pareto distribution as follows.

For $r = 1$ (minimum);

$$\begin{aligned} f_{1:n}(x) &= \frac{n!}{(1-1)!(n-1)!} \frac{\alpha x_m^\alpha}{x^{\alpha+1}} \left[1 - (\frac{x_m}{x})^\alpha\right]^{1-1} \left[(\frac{x_m}{x})^\alpha\right]^{n-1} \\ &= n \frac{\alpha x_m^\alpha}{x^{\alpha+1}} \left[(\frac{x_m}{x})^\alpha\right]^{n-1},\ x \geq x_m \end{aligned} \tag{2.60}$$

For $r = n$ (maximum);

$$\begin{aligned} f_{n:n}(x) &= \frac{n!}{(n-1)!(n-n)!}\frac{\alpha x_m^\alpha}{x^{\alpha+1}}\Big[1-(\frac{x_m}{x})^\alpha\Big]^{n-1}\Big[(\frac{x_m}{x})^\alpha\Big]^{n-n} \\ &= n\frac{\alpha x_m^\alpha}{x^{\alpha+1}}\Big[1-(\frac{x_m}{x})^\alpha\Big]^{n-1},\ x \geq x_m \end{aligned} \tag{2.61}$$

Normal order statistics

The pdf of beta-normal distribution was given by Eugene and Famoye (2002) as,

$$f(x; a, b, \mu, \sigma) = \frac{\sigma^{-1}\Big[\phi(\frac{x-\mu}{\sigma})\Big]^{a-1}\Big[1-\phi(\frac{x-\mu}{\sigma})\Big]^{b-1}}{B(a,b)} \cdot \phi(\frac{x-\mu}{\sigma}) \tag{2.62}$$

where $a, b, \sigma, \mu, x > 0$.

From equation (2.62), when $\mu = 0$ and $\sigma = 1$, we get the standard beta-normal distribution as,

$$f(x; a, b) = \frac{[\phi(x)]^{a-1}[1-\phi(x)]^{b-1}}{B(a,b)} \cdot \phi(x) \tag{2.63}$$

where $a, b, x > 0$.

We therefore, extend this work and get the r^{th} order statistic of the standard normal distribution by replacing $a = r$ and $b = n - r + 1$.

$$\begin{aligned} f_{r:n}(x) &= \frac{[\phi(x)]^{r-1}[1-\phi(x)]^{(n-r+1)-1}}{B(r, n-r+1)} \cdot \phi(x) \\ &= \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)}[\phi(x)]^{r-1}[1-\phi(x)]^{n-r}\phi(x) \\ &= \frac{n!}{(r-1)!(n-r)!}[\phi(x)]^{r}[1-\phi(x)]^{n-r},\ x > 0 \end{aligned} \tag{2.64}$$

Subsequently, we get the minimum and the maximum order statistics for the standard normal distribution as follows.

For $r = 1$ (minimum);

$$f_{1:n}(x) = \frac{n!}{(1-1)!(n-1)!}[\phi(x)]^1[1-\phi(x)]^{n-1} \\ = n[1-\phi(x)]^{n-1}\phi(x),\ x > 0 \tag{2.65}$$

For $r = n$ (maximum);

$$f_{n:n}(x) = \frac{n!}{(n-1)!(n-n)!}[\phi(x)]^n[1-\phi(x)]^{n-n} \\ = n[\phi(x)]^n,\ x > 0 \tag{2.66}$$

Weibull order statistics

The pdf of the beta-weibull distribution was given by Famoye et al. (2005) as below. This distribution was studied in detail by Lee et al. (2007) giving some of its properties and applications to censored data.

$$\begin{aligned} f(x;a,b,c,\beta) &= \frac{(\frac{|c|}{\beta^c})x^{c-1}e^{-(\frac{x}{\beta})^c}\left[1-e^{-(\frac{x}{\beta})^c}\right]^{a-1}\left[1-(1-e^{-(\frac{x}{\beta})^c})\right]^{b-1}}{B(a,b)} \\ &= \frac{(\frac{|c|}{\beta^c})x^{c-1}e^{-(\frac{x}{\beta})^c}\left[1-e^{-(\frac{x}{\beta})^c}\right]^{a-1}\left[e^{-(\frac{x}{\beta})^c}\right]^{b-1}}{B(a,b)} \\ &= \frac{(\frac{|c|}{\beta^c})x^{c-1}e^{-b(\frac{x}{\beta})^c}\left[1-e^{-(\frac{x}{\beta})^c}\right]^{a-1}}{B(a,b)}, \quad a,b,c,\beta,x > 0 \end{aligned} \tag{2.67}$$

Equation (2.67) is a four parameter beta-weibull distribution introduced by Famoye et al. (2005) and studied by Lee et al. (2007).

We therefore, extend this work and get the r^{th} order statistic of the weibull distribution by replacing $a = r$ and $b = n - r + 1$.

$$\begin{aligned} f_{r:n}(x) &= \frac{(\frac{|c|}{\beta^c})x^{c-1}e^{-(n-r+1)(\frac{x}{\beta})^c}\left[1-e^{-(\frac{x}{\beta})^c}\right]^{r-1}}{B(r,n-r+1)} \\ &= \frac{n!}{(r-1)!(n-r)!}(\frac{|c|}{\beta^c})x^{c-1}e^{-(n-r+1)(\frac{x}{\beta})^c}\left[1-e^{-(\frac{x}{\beta})^c}\right]^{r-1}, \quad c,\beta,x > 0 \end{aligned} \tag{2.68}$$

Subsequently, we get the minimum and the maximum order statistics for the weibull distribution as follows.

For $r = 1$ (minimum);

$$\begin{aligned} f_{1:n}(x) &= \frac{n!}{(1-1)!(n-1)!}(\frac{|c|}{\beta^c})x^{c-1}e^{-(n-1+1)(\frac{x}{\beta})^c}\left[1-e^{-(\frac{x}{\beta})^c}\right]^{1-1} \\ &= \frac{n!}{(n-1)!}(\frac{|c|}{\beta^c})x^{c-1}e^{-n(\frac{x}{\beta})^c} \\ &= n(\frac{|c|}{\beta^c})x^{c-1}e^{-n(\frac{x}{\beta})^c}, \quad c,\beta,x>0 \end{aligned} \tag{2.69}$$

For $r = n$ (maximum);

$$\begin{aligned} f_{n:n}(x) &= \frac{n!}{(n-1)!(n-n)!}(\frac{|c|}{\beta^c})x^{c-1}e^{-(n-n+1)(\frac{x}{\beta})^c}\left[1-e^{-(\frac{x}{\beta})^c}\right]^{n-1} \\ &= \frac{n!}{(n-1)!}(\frac{|c|}{\beta^c})x^{c-1}e^{-(\frac{x}{\beta})^c}\left[1-e^{-(\frac{x}{\beta})^c}\right]^{n-1} \\ &= n(\frac{|c|}{\beta^c})x^{c-1}e^{-(\frac{x}{\beta})^c}\left[1-e^{-(\frac{x}{\beta})^c}\right]^{n-1}, \quad c,\beta,x>0 \end{aligned} \tag{2.70}$$

Hyperbolic secant order statistics

Fischer and Vaughan (2004) introduced the beta-hyperbolic secant distribution. They gave its pdf as,

$$f(x;a,b,\pi) = \frac{\left[\frac{2}{\pi}arctan(e^x)\right]^{a-1}\left[1-\frac{2}{\pi}arctan(e^x)\right]^{b-1}}{B(a,b)\ \pi cosh(x)}, \quad a,b,x>0 \tag{2.71}$$

We therefore, extend this work and get the r^{th} order statistic of the hyperbolic secant distribution by replacing $a = r$ and $b = n - r + 1$.

$$
\begin{aligned}
f_{r:n}(x) &= \frac{\Big[\frac{2}{\pi}arctan(e^x)\Big]^{r-1}\Big[1-\frac{2}{\pi}arctan(e^x)\Big]^{(n-r+1)-1}}{B(r,n-r+1)\ \pi cosh(x)} \\
&= \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)}\frac{\Big[\frac{2}{\pi}arctan(e^x)\Big]^{r-1}\Big[1-\frac{2}{\pi}arctan(e^x)\Big]^{n-r}}{\pi cosh(x)} \\
&= \frac{n!}{(r-1)!(n-r)!}\frac{\Big[\frac{2}{\pi}arctan(e^x)\Big]^{r-1}\Big[1-\frac{2}{\pi}arctan(e^x)\Big]^{n-r}}{\pi cosh(x)}, \quad x>0
\end{aligned}
\tag{2.72}
$$

Subsequently, we get the minimum and the maximum order statistics for the hyperbolic secant distribution as follows.

For $r = 1$ (minimum);

$$
\begin{aligned}
f_{1:n}(x) &= \frac{n!}{(1-1)!(n-1)!}\frac{\Big[\frac{2}{\pi}arctan(e^x)\Big]^{1-1}\Big[1-\frac{2}{\pi}arctan(e^x)\Big]^{n-1}}{\pi cosh(x)} \\
&= \frac{n\Big[1-\frac{2}{\pi}arctan(e^x)\Big]^{n-1}}{\pi cosh(x)}, \quad x>0
\end{aligned}
\tag{2.73}
$$

For $r = n$ (maximum);

$$
\begin{aligned}
f_{n:n}(x) &= \frac{n!}{(n-1)!(n-n)!}\frac{\Big[\frac{2}{\pi}arctan(e^x)\Big]^{n-1}\Big[1-\frac{2}{\pi}arctan(e^x)\Big]^{n-n}}{\pi cosh(x)} \\
&= \frac{n\Big[\frac{2}{\pi}arctan(e^x)\Big]^{n-1}}{\pi cosh(x)}, \quad x>0
\end{aligned}
\tag{2.74}
$$

Gamma order statistics

The beta-gamma distribution was introduced by Kong and Sepanski (2007). They gave its pdf as,

$$
f(x;a,b,\rho,\lambda) = \frac{x^{\rho-1}e^{\frac{x}{\lambda}}\frac{\Gamma(x)}{\lambda}(\rho)^{a-1}\Big[1-\frac{\frac{\Gamma(x)}{\lambda}(\rho)}{\Gamma(\rho)}\Big]^{b-1}}{B(a,b)\ \Gamma(\rho)^a\lambda^\rho}, \quad a,b,\rho,\lambda,x>0 \tag{2.75}
$$

We therefore, extend this work and get the r^{th} order statistic of the gamma distribution by replacing $a = r$ and $b = n - r + 1$.

$$
\begin{aligned}
f_{r:n}(x) &= \frac{x^{\rho-1} e^{\frac{x}{\lambda}} \frac{\Gamma(x)}{\lambda}(\rho)^{r-1} \left[1 - \frac{\frac{\Gamma(x)}{\lambda}(\rho)}{\Gamma(\rho)}\right]^{(n-r+1)-1}}{B(r, n-r+1)\; \Gamma(\rho)^r \lambda^\rho} \\
&= \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)} \frac{x^{\rho-1} e^{\frac{x}{\lambda}} \frac{\Gamma(x)}{\lambda}(\rho)^{r-1} \left[1 - \frac{\frac{\Gamma(x)}{\lambda}(\rho)}{\Gamma(\rho)}\right]^{n-r}}{\Gamma(\rho)^r \lambda^\rho} \\
&= \frac{n!}{(r-1)!(n-r)!} \frac{x^{\rho-1} e^{\frac{x}{\lambda}} \frac{\Gamma(x)}{\lambda}(\rho)^{r-1} \left[1 - \frac{\frac{\Gamma(x)}{\lambda}(\rho)}{\Gamma(\rho)}\right]^{n-r}}{\Gamma(\rho)^r \lambda^\rho}, \quad \rho, \lambda, x > 0
\end{aligned} \tag{2.76}
$$

Subsequently, we get the minimum and the maximum order statistics for the gamma distribution as follows.

For $r = 1$ (minimum);

$$
\begin{aligned}
f_{1:n}(x) &= \frac{n!}{(n-1)!} \frac{x^{\rho-1} e^{\frac{x}{\lambda}} \frac{\Gamma(x)}{\lambda}(\rho)^{1-1} \left[1 - \frac{\frac{\Gamma(x)}{\lambda}(\rho)}{\Gamma(\rho)}\right]^{n-1}}{\Gamma(\rho)^1 \lambda^\rho} \\
&= n\lambda^{-\rho} x^{\rho-1} e^{\frac{x}{\lambda}} \frac{\Gamma(x)}{\lambda} \left[1 - \frac{\frac{\Gamma(x)}{\lambda}(\rho)}{\Gamma(\rho)}\right]^{n-1}, \quad \rho, \lambda, x > 0
\end{aligned} \tag{2.77}
$$

For $r = n$ (maximum);

$$
\begin{aligned}
f_{n:n}(x) &= \frac{n!}{(n-1)!} \frac{x^{\rho-1} e^{\frac{x}{\lambda}} \frac{\Gamma(x)}{\lambda}(\rho)^{n-1} \left[1 - \frac{\frac{\Gamma(x)}{\lambda}(\rho)}{\Gamma(\rho)}\right]^{n-n}}{\Gamma(\rho)^n \lambda^\rho} \\
&= \frac{n}{\Gamma(\rho)^n \lambda^\rho} x^{\rho-1} e^{\frac{x}{\lambda}} \frac{\Gamma(x)}{\lambda}(\rho)^{n-1}, \quad \rho, \lambda, x > 0
\end{aligned} \tag{2.78}
$$

Gumbel order statistics

The beta-gumbel distribution was introduced by Nadarajah and Kotz (2004) and gave its pdf as,

$$\begin{aligned} f(x; a, b, \mu, \sigma) &= \frac{1}{B(a,b)}[exp(-\mu)]^{a-1}[1 - exp(-\mu)]^{b-1} \cdot \frac{\mu}{\sigma}e^{-\mu} \\ &= \frac{\mu}{\sigma B(a,b)}e^{-\mu a}[1 - e^{-\mu}]^{b-1}, \quad a, b, \mu, \sigma, x > 0 \end{aligned} \tag{2.79}$$

We therefore, extend this work and get the r^{th} order statistic of the gumbel distribution by replacing $a = r$ and $b = n - r + 1$.

$$\begin{aligned} f_{r:n}(x) &= \frac{\mu}{\sigma B(r, n-r+1)}e^{-\mu r}[1 - e^{-\mu}]^{(n-r+1)-1} \\ &= \frac{\mu\Gamma(n+1)}{\sigma\Gamma(r)\Gamma(n-r+1)}e^{-\mu r}[1 - e^{-\mu}]^{n-r} \\ &= \frac{n!}{(r-1)!(n-r)!} \cdot \frac{\mu}{\sigma} \cdot e^{-\mu r}[1 - e^{-\mu}]^{n-r}, \quad \mu, \sigma, x > 0 \end{aligned} \tag{2.80}$$

Subsequently, we get the minimum and the maximum order statistics for the gumbel distribution as follows.

For $r = 1$ (minimum);

$$\begin{aligned} f_{1:n}(x) &= \frac{n!}{(1-1)!(n-1)!} \cdot \frac{\mu}{\sigma} \cdot e^{-\mu(1)}[1 - e^{-\mu}]^{n-1} \\ &= \frac{n\mu}{\sigma}e^{-\mu}[1 - e^{-\mu}]^{n-1}, \quad \mu, \sigma, x > 0 \end{aligned} \tag{2.81}$$

For $r = n$ (maximum);

$$\begin{aligned} f_{n:n}(x) &= \frac{n!}{(n-1)!(n-n)!} \cdot \frac{\mu}{\sigma} \cdot e^{-\mu(n)}[1 - e^{-\mu}]^{n-n} \\ &= \frac{n\mu}{\sigma}e^{-n\mu}, \quad \mu, \sigma, x > 0 \end{aligned} \tag{2.82}$$

Fréchet order statistics

The beta-fréchet distribution was introduced by Nadarajah and Gupta (2004) and gave its pdf as,

$$f(x; a, b, \lambda, \sigma) = \frac{\lambda\sigma^{\lambda} exp[-a(\frac{x}{\sigma})^{-\lambda}]\{1 - exp[-(\frac{x}{\sigma})^{-\lambda}]\}^{b-1}}{x^{1+\lambda}B(a, b)}, \quad a, b, \lambda, \sigma, x > 0 \tag{2.83}$$

We therefore, extend this work and obtain the r^{th} order statistic of the fréchet distribution by replacing $a = r$ and $b = n - r + 1$.

$$\begin{aligned} f_{r:n}(x) &= \frac{\lambda\sigma^{\lambda} exp[-(r)(\frac{x}{\sigma})^{-\lambda}]\{1 - exp[-(\frac{x}{\sigma})^{-\lambda}]\}^{(n-r+1)-1}}{x^{1+\lambda}B(r, n - r + 1)} \\ &= \frac{n!}{(r-1)!(n-r)!} x^{-(1+\lambda)} \lambda\sigma^{\lambda} exp[-r(\frac{x}{\sigma})^{-\lambda}]\{1 - exp[-(\frac{x}{\sigma})^{-\lambda}]\}^{n-r}, \quad \mu, \sigma, x > 0 \end{aligned} \tag{2.84}$$

Subsequently, we get the minimum and the maximum order statistics for the fréchet distribution as follows.

For $r = 1$ (minimum);

$$\begin{aligned} f_{1:n}(x) &= \frac{n!}{(1-1)!(n-1)!} x^{-(1+\lambda)} \lambda\sigma^{\lambda} exp[-(1)(\frac{x}{\sigma})^{-\lambda}]\{1 - exp[-(\frac{x}{\sigma})^{-\lambda}]\}^{n-1} \\ &= nx^{-(1+\lambda)} \lambda\sigma^{\lambda} exp[-(\frac{x}{\sigma})^{-\lambda}]\{1 - exp[-(\frac{x}{\sigma})^{-\lambda}]\}^{n-1}, \quad \mu, \sigma, x > 0 \end{aligned} \tag{2.85}$$

For $r = n$ (maximum);

$$\begin{aligned} f_{n:n}(x) &= \frac{n!}{(n-1)!(n-n)!} x^{-(1+\lambda)} \lambda\sigma^{\lambda} exp[-(n)(\frac{x}{\sigma})^{-\lambda}]\{1 - exp[-(\frac{x}{\sigma})^{-\lambda}]\}^{n-n} \\ &= nx^{-(1+\lambda)} \lambda\sigma^{\lambda} exp[-n(\frac{x}{\sigma})^{-\lambda}], \quad \mu, \sigma, x > 0 \end{aligned} \tag{2.86}$$

Maxwell order statistics

Amusan (2010) introduced the beta-maxwell distribution and gave its pdf as,

$$
\begin{aligned}
f(x; a, b, \alpha, \gamma) &= \frac{\left[\frac{2\gamma(\frac{3}{2}, \frac{x^2}{2\alpha})}{\sqrt{\pi}}\right]^{a-1}\left[1 - \frac{2\gamma(\frac{3}{2}, \frac{x^2}{2\alpha})}{\sqrt{\pi}}\right]^{b-1}\sqrt{\frac{2}{\pi}}\frac{x^2 e^{-(\frac{x^2}{2\alpha^2})}}{\alpha^3}}{B(a,b)} \\
&= \frac{1}{B(a,b)}\Big[\frac{2\gamma}{\sqrt{\pi}}\Big(\frac{3}{2}, \frac{x^2}{2\alpha}\Big)\Big]^{a-1}\Big[1 - \frac{2\gamma}{\sqrt{\pi}}\Big(\frac{3}{2}, \frac{x^2}{2\alpha}\Big)\Big]^{b-1}\sqrt{\frac{2}{\pi}}\frac{x^2 e^{-(\frac{x^2}{2\alpha^2})}}{\alpha^3} \\
&\text{where,} \quad a, b, \alpha, \gamma, x > 0
\end{aligned}
\tag{2.87}
$$

We therefore, extend this work and obtain the r^{th} order statistic of the maxwell distribution by replacing $a = r$ and $b = n - r + 1$.

$$
\begin{aligned}
f_{r:n}(x) &= \frac{1}{B(r, n-r+1)}\Big[\frac{2\gamma}{\sqrt{\pi}}\Big(\frac{3}{2}, \frac{x^2}{2\alpha}\Big)\Big]^{r-1}\Big[1 - \frac{2\gamma}{\sqrt{\pi}}\Big(\frac{3}{2}, \frac{x^2}{2\alpha}\Big)\Big]^{(n-r+1)-1}\sqrt{\frac{2}{\pi}}\frac{x^2 e^{-(\frac{x^2}{2\alpha^2})}}{\alpha^3} \\
&= \frac{n!}{(r-1)!(n-r)!}\Big[\frac{2\gamma}{\sqrt{\pi}}\Big(\frac{3}{2}, \frac{x^2}{2\alpha}\Big)\Big]^{r-1}\Big[1 - \frac{2\gamma}{\sqrt{\pi}}\Big(\frac{3}{2}, \frac{x^2}{2\alpha}\Big)\Big]^{n-r}\sqrt{\frac{2}{\pi}}\frac{x^2 e^{-(\frac{x^2}{2\alpha^2})}}{\alpha^3} \\
&\text{where,} \quad \alpha, \gamma, x > 0
\end{aligned}
\tag{2.88}
$$

Subsequently, we get the minimum and the maximum order statistics for the maxwell distribution as follows.

For $r = 1$ (minimum);

$$
\begin{aligned}
f_{1:n}(x) &= \frac{n!}{(1-1)!(n-1)!}\Big[\frac{2\gamma}{\sqrt{\pi}}\Big(\frac{3}{2}, \frac{x^2}{2\alpha}\Big)\Big]^{1-1}\Big[1 - \frac{2\gamma}{\sqrt{\pi}}\Big(\frac{3}{2}, \frac{x^2}{2\alpha}\Big)\Big]^{n-1}\sqrt{\frac{2}{\pi}}\frac{x^2 e^{-(\frac{x^2}{2\alpha^2})}}{\alpha^3} \\
&= n\Big[1 - \frac{2\gamma}{\sqrt{\pi}}\Big(\frac{3}{2}, \frac{x^2}{2\alpha}\Big)\Big]^{n-1}\sqrt{\frac{2}{\pi}}\frac{x^2 e^{-(\frac{x^2}{2\alpha^2})}}{\alpha^3} \\
&\text{where,} \quad \alpha, \gamma, x > 0
\end{aligned}
\tag{2.89}
$$

For $r = n$ (maximum);

$$f_{n:n}(x) = \frac{n!}{(n-1)!(n-n)!}\Big[\frac{2\gamma}{\sqrt{\pi}}\Big(\frac{3}{2},\frac{x^2}{2\alpha}\Big)\Big]^{n-1}\Big[1-\frac{2\gamma}{\sqrt{\pi}}\Big(\frac{3}{2},\frac{x^2}{2\alpha}\Big)\Big]^{n-n}\sqrt{\frac{2}{\pi}}\frac{x^2e^{-(\frac{x^2}{2\alpha^2})}}{\alpha^3}$$

$$= n\Big[\frac{2\gamma}{\sqrt{\pi}}\Big(\frac{3}{2},\frac{x^2}{2\alpha}\Big)\Big]^{n-1}\sqrt{\frac{2}{\pi}}\frac{x^2e^{-(\frac{x^2}{2\alpha^2})}}{\alpha^3} \tag{2.90}$$

where, $\alpha, \gamma, x > 0$

Rayleigh order statistics

Akinsete and Lowe (2009) introduced the beta-rayleigh distribution and gave its pdf as,

$$f(x;a,b,\alpha) = \frac{\Big[1-e^{-\frac{x^2}{2\alpha^2}}\Big]^{a-1}\Big[1-\Big(1-e^{-\frac{x^2}{2\alpha^2}}\Big)\Big]^{b-1}\frac{x}{\alpha^2}e^{-(\frac{x}{\alpha\sqrt{2}})^2}}{B(a,b)}$$

$$= \frac{x}{\alpha^2 B(a,b)}\Big[1-e^{-\frac{x^2}{2\alpha^2}}\Big]^{a-1}e^{-b(\frac{x}{\alpha\sqrt{2}})^2} \tag{2.91}$$

where, $a, b, \alpha, x > 0$

We therefore, extend this work and obtain the r^{th} order statistic of the rayleigh distribution by replacing $a = r$ and $b = n - r + 1$.

$$f_{r:n}(x) = \frac{x}{\alpha^2 B(r, n-r+1)}\Big[1-e^{-\frac{x^2}{2\alpha^2}}\Big]^{r-1}e^{-(n-r+1)(\frac{x}{\alpha\sqrt{2}})^2}$$

$$= \frac{n!}{(r-1)!(n-r)!}\frac{x}{\alpha^2}\Big[1-e^{-\frac{x^2}{2\alpha^2}}\Big]^{r-1}e^{-(n-r+1)(\frac{x}{\alpha\sqrt{2}})^2} \tag{2.92}$$

where, $\alpha, x > 0$

Subsequently, we get the minimum and the maximum order statistics for the rayleigh distribution as follows.

For $r = 1$ (minimum);

$$f_{1:n}(x) = \frac{n!}{(1-1)!(n-1)!}\frac{x}{\alpha^2}\Big[1-e^{-\frac{x^2}{2\alpha^2}}\Big]^{1-1}e^{-(n-1+1)(\frac{x}{\alpha\sqrt{2}})^2}$$

$$= n\frac{x}{\alpha^2}e^{-n(\frac{x}{\alpha\sqrt{2}})^2}, \quad \alpha, x > 0 \tag{2.93}$$

For $r = n$ (maximum);

$$
\begin{aligned}
f_{n:n}(x) &= \frac{n!}{(n-1)!(n-n)!}\frac{x}{\alpha^2}\left[1-e^{-\frac{x^2}{2\alpha^2}}\right]^{n-1}e^{-(n-n+1)(\frac{x}{\alpha\sqrt{2}})^2} \\
&= n\frac{x}{\alpha^2}\left[1-e^{-\frac{x^2}{2\alpha^2}}\right]^{n-1}e^{-(\frac{x}{\alpha\sqrt{2}})^2}, \quad \alpha, x > 0
\end{aligned}
\tag{2.94}
$$

Generalized-Logistic of type IV order statistics

Morais de Lemos (2009) introduced the beta-generalized logistic of type IV distribution and gave its pdf as,

$$
\begin{aligned}
f(x;a,b,p,q) = \frac{B(a,b)^{1-a-b}}{B(a,b)}\frac{e^{qx}}{(1-e^{-x})^{p+q}}\left[B_{\frac{1}{1+e^{-x}}}(p,q)\right]^{a-1}\left[B_{\frac{e^{-x}}{1+e^{-x}}}(q,p)\right]^{b-1} \\
\text{where,} \quad a,b,p,q,x > 0
\end{aligned}
\tag{2.95}
$$

We therefore, extend this work and obtain the r^{th} order statistic of the generalized logistic of type IV distribution by replacing $a = r$ and $b = n - r + 1$.

$$
\begin{aligned}
f_{r:n}(x) &= \frac{B(r,n-r+1)^{1-r-(n-r+1)}}{B(r,n-r+1)}\frac{e^{qx}}{(1-e^{-x})^{p+q}}\left[B_{\frac{1}{1+e^{-x}}}(p,q)\right]^{r-1} \\
&\left[B_{\frac{e^{-x}}{1+e^{-x}}}(q,p)\right]^{(n-r+1)-1} \\
&= \left[\frac{(r-1)!(n-r)!}{n!}\right]^{-n}\left[\frac{n!}{(r-1)!(n-r)!}\right]\frac{e^{qx}}{(1-e^{-x})^{p+q}}\left[B_{\frac{1}{1+e^{-x}}}(p,q)\right]^{r-1} \\
&\left[B_{\frac{e^{-x}}{1+e^{-x}}}(q,p)\right]^{n-r} \\
&= \left[\frac{n!}{(r-1)!(n-r)!}\right]^{n+1}\frac{e^{qx}}{(1-e^{-x})^{p+q}}\left[B_{\frac{1}{1+e^{-x}}}(p,q)\right]^{r-1}\left[B_{\frac{e^{-x}}{1+e^{-x}}}(q,p)\right]^{n-r} \\
&\text{where,} \quad p,q,x > 0
\end{aligned}
\tag{2.96}
$$

Subsequently, we get the minimum and the maximum order statistics for the generalized logistic of type IV distribution as follows.

For $r = 1$ (minimum);

$$\begin{aligned} f_{1:n}(x) &= \left[\frac{n!}{(1-1)!(n-1)!}\right]^{n+1} \frac{e^{qx}}{(1-e^{-x})^{p+q}} \left[B_{\frac{1}{1+e^{-x}}}(p,q)\right]^{1-1} \left[B_{\frac{e^{-x}}{1+e^{-x}}}(q,p)\right]^{n-1} \\ &= n^{n+1} \frac{e^{qx}}{(1-e^{-x})^{p+q}} \left[B_{\frac{e^{-x}}{1+e^{-x}}}(q,p)\right]^{n-1}, \quad p,q,x > 0 \end{aligned} \tag{2.97}$$

For $r = n$ (maximum);

$$\begin{aligned} f_{n:n}(x) &= \left[\frac{n!}{(n-1)!(n-n)!}\right]^{n+1} \frac{e^{qx}}{(1-e^{-x})^{p+q}} \left[B_{\frac{1}{1+e^{-x}}}(p,q)\right]^{n-1} \left[B_{\frac{e^{-x}}{1+e^{-x}}}(q,p)\right]^{n-n} \\ &= n^{n+1} \frac{e^{qx}}{(1-e^{-x})^{p+q}} \left[B_{\frac{1}{1+e^{-x}}}(p,q)\right]^{n-1}, \quad p,q,x > 0 \end{aligned} \tag{2.98}$$

Generalized-Logistic of type I order statistics

Morais de Lemos (2009) introduced the beta-generalized logistic of type I distribution which is a special case of the beta-generalized logistic of type IV distribution with $q = 1$. She gave its pdf as,

$$f(x; a, b, p) = \frac{pe^{-x}\left[(1+e^{-ax})^p - 1\right]^{b-1}}{B(a,b)(1+e^{-x})^{a+pb}}, \quad a,b,p,x > 0 \tag{2.99}$$

We therefore, extend this work and obtain the r^{th} order statistic of the generalized logistic of type I distribution by replacing $a = r$ and $b = n - r + 1$.

$$\begin{aligned} f_{r:n}(x) &= \frac{pe^{-x}\left[(1+e^{-rx})^p - 1\right]^{(n-r+1)-1}}{B(r, n-r+1)(1+e^{-x})^{r+p(n-r+1)}} \\ &= \frac{n!}{(r-1)!(n-r)!} pe^{-x}(1+e^{-x})^{-(r+p(n-r+1))}\left[(1+e^{-rx})^p - 1\right]^{n-r} \\ &\text{where,} \quad p, x > 0 \end{aligned} \tag{2.100}$$

Subsequently, we get the minimum and the maximum order statistics for the generalized logistic of type I distribution as follows.

For $r = 1$ (minimum);

$$\begin{aligned} f_{1:n}(x) &= \frac{n!}{(1-1)!(n-1)!} pe^{-x}(1+e^{-x})^{-(1+p(n-1+1))}\Big[(1+e^{-(1)x})^p - 1\Big]^{n-1} \\ &= npe^{-x}(1+e^{-x})^{-(1+np)}\Big[(1+e^{-x})^p - 1\Big]^{n-1}, \quad p, x > 0 \end{aligned} \tag{2.101}$$

For $r = n$ (maximum);

$$\begin{aligned} f_{n:n}(x) &= \frac{n!}{(n-1)!(n-n)!} pe^{-x}(1+e^{-x})^{-(n+p(n-n+1))}\Big[(1+e^{-(n)x})^p - 1\Big]^{n-n} \\ &= npe^{-x}(1+e^{-x})^{-(n+p)}, \quad p, x > 0 \end{aligned} \tag{2.102}$$

Generalized-Logistic of type II order statistics

Morais de Lemos (2009) introduced the beta-generalized logistic of type II distribution which is a special case of the beta-generalized logistic of type IV distribution with $p = 1$. She gave its pdf as,

$$f(x; a, b, q) = \frac{qe^{-bqx}}{B(a,b)(1+e^{-x})^{qb+1}}\Big[1 - \frac{e^{-qx}}{B(a,b)(1+e^{-x})^q}\Big], \quad a, b, q, x > 0 \tag{2.103}$$

We therefore, extend this work and obtain the r^{th} order statistic of the generalized logistic of type II distribution by replacing $a = r$ and $b = n - r + 1$.

$$\begin{aligned} f_{r:n}(x) &= \frac{qe^{-(n-r+1)qx}}{B(r, n-r+1)(1+e^{-x})^{q(n-r+1)+1}}\Big[1 - \frac{e^{-qx}}{B(r, n-r+1)(1+e^{-x})^q}\Big] \\ &= \frac{n!}{(r-1)!(n-r)!}\frac{qe^{-(n-r+1)qx}}{(1+e^{-x})^{q(n-r+1)+1}}\Big[1 - \frac{n!}{(r-1)!(n-r)!}\frac{e^{-qx}}{(1+e^{-x})^q}\Big] \\ &\text{where,} \quad q, x > 0 \end{aligned} \tag{2.104}$$

Subsequently, we get the minimum and the maximum order statistics for the generalized logistic of type II distribution as follows.

For $r = 1$ (minimum);

$$\begin{aligned} f_{1:n}(x) &= \frac{n!}{(1-1)!(n-1)!} \frac{qe^{-(n-1+1)qx}}{(1+e^{-x})^{q(n-1+1)+1}} \Big[1 - \frac{n!}{(1-1)!(n-1)!} \frac{e^{-qx}}{(1+e^{-x})^q}\Big] \\ &= \frac{nqe^{-nqx}}{(1+e^{-x})^{nq+1}} \Big[1 - \frac{ne^{-qx}}{(1+e^{-x})^q}\Big], \quad q, x > 0 \end{aligned} \tag{2.105}$$

For $r = n$ (maximum);

$$\begin{aligned} f_{n:n}(x) &= \frac{n!}{(n-1)!(n-n)!} \frac{qe^{-(n-n+1)qx}}{(1+e^{-x})^{q(n-n+1)+1}} \Big[1 - \frac{n!}{(n-1)!(n-n)!} \frac{e^{-qx}}{(1+e^{-x})^q}\Big] \\ &= \frac{nqe^{-nqx}}{(1+e^{-x})^{q+1}} \Big[1 - \frac{ne^{-qx}}{(1+e^{-x})^q}\Big], \quad q, x > 0 \end{aligned} \tag{2.106}$$

Generalized-Logistic of type III order statistics

Morais de Lemos (2009) introduced the beta-generalized logistic of type III distribution which is a special case of the beta-generalized logistic of type IV distribution with $p = q$. She gave its pdf as,

$$\begin{aligned} f(x; a, b, p) = \frac{B(p,p)^{1-a-b}}{B(a,b)} \frac{e^{-px}}{(1+e^{-x})^{2p}} \Big[B_{\frac{1}{1+e^{-x}}}(p,p)\Big]^{a-1} \Big[B_{\frac{e^{-x}}{1+e^{-x}}}(p,p)\Big]^{b-1} \\ \text{where,} \quad a, b, p, x > 0 \end{aligned} \tag{2.107}$$

We therefore, extend this work and obtain the r^{th} order statistic of the generalized logistic of type III distribution by replacing $a = r$ and $b = n - r + 1$.

$$\begin{aligned} f_{r:n}(x) &= \frac{B(p,p)^{1-r-(n-r+1)}}{B(r, n-r+1)} \frac{e^{-px}}{(1+e^{-x})^{2p}} \Big[B_{\frac{1}{1+e^{-x}}}(p,p)\Big]^{r-1} \Big[B_{\frac{e^{-x}}{1+e^{-x}}}(p,p)\Big]^{(n-r+1)-1} \\ &= \frac{n!}{(r-1)!(n-r)!} B(p,p)^{-n} \frac{e^{-px}}{(1+e^{-x})^{2p}} \Big[B_{\frac{1}{1+e^{-x}}}(p,p)\Big]^{r-1} \Big[B_{\frac{e^{-x}}{1+e^{-x}}}(p,p)\Big]^{n-r} \\ &\text{where,} \quad p, x > 0 \end{aligned} \tag{2.108}$$

Subsequently, we get the minimum and the maximum order statistics for the generalized logistic of type III distribution as follows.

For $r = 1$ (minimum);

$$\begin{aligned} f_{1:n}(x) &= \frac{n!}{(1-1)!(n-1)!} B(p,p)^{-n} \frac{e^{-px}}{(1+e^{-x})^{2p}} \left[B_{\frac{1}{1+e^{-x}}}(p,p) \right]^{1-1} \left[B_{\frac{e^{-x}}{1+e^{-x}}}(p,p) \right]^{n-1} \\ &= nB(p,p)^{-n} \frac{e^{-px}}{(1+e^{-x})^{2p}} \left[B_{\frac{e^{-x}}{1+e^{-x}}}(p,p) \right]^{n-1}, \quad p, x > 0 \end{aligned} \tag{2.109}$$

For $r = n$ (maximum);

$$\begin{aligned} f_{n:n}(x) &= \frac{n!}{(n-1)!(n-n)!} B(p,p)^{-n} \frac{e^{-px}}{(1+e^{-x})^{2p}} \left[B_{\frac{1}{1+e^{-x}}}(p,p) \right]^{n-1} \left[B_{\frac{e^{-x}}{1+e^{-x}}}(p,p) \right]^{n-n} \\ &= nB(p,p)^{-n} \frac{e^{-px}}{(1+e^{-x})^{2p}} \left[B_{\frac{1}{1+e^{-x}}}(p,p) \right]^{n-1}, \quad p, x > 0 \end{aligned} \tag{2.110}$$

Beta prime order statistics

The beta-beta prime distribution was introduced by Morais de Lemos (2009), and obtained from the beta-generalized logistic of type IV distribution using the transformation $x = e^{-y}$ where y is a random variable following the beta-generalized logistic of type IV distribution. She gave the pdf of the beta-beta prime distribution as,

$$\begin{aligned} f(x; a, b, p, q) = \frac{B(p,q)^{1-a-b}}{B(a,b)} \frac{x^{q-1}}{(1+x)^{p+q}} \left[B_{\frac{x}{1+x}}(q,p) \right]^{a-1} \left[B_{\frac{1}{1+x}}(p,q) \right]^{b-1} \\ \text{where,} \quad a, b, p, q, x > 0 \end{aligned} \tag{2.111}$$

We therefore, extend this work and obtain the r^{th} order statistic of the beta prime distribution by replacing $a = r$ and $b = n - r + 1$.

$$f_{r:n}(x) = \frac{B(p,q)^{1-r-(n-r+1)}}{B(r,n-r+1)} \frac{x^{q-1}}{(1+x)^{p+q}} \Big[B_{\frac{x}{1+x}}(q,p)\Big]^{r-1} \Big[B_{\frac{1}{1+x}}(p,q)\Big]^{(n-r+1)-1}$$
$$= \frac{n!}{(r-1)!(n-r)!} B(p,q)^{-n} \frac{x^{q-1}}{(1+x)^{p+q}} \Big[B_{\frac{x}{1+x}}(q,p)\Big]^{r-1} \Big[B_{\frac{1}{1+x}}(p,q)\Big]^{n-r} \quad (2.112)$$
$$\text{where,} \quad p,q,x > 0$$

Subsequently, we get the minimum and the maximum order statistics for the beta prime distribution as follows.

For $r = 1$ (minimum);

$$f_{1:n}(x) = \frac{n!}{(1-1)!(n-1)!} B(p,q)^{-n} \frac{x^{q-1}}{(1+x)^{p+q}} \Big[B_{\frac{x}{1+x}}(q,p)\Big]^{1-1} \Big[B_{\frac{1}{1+x}}(p,q)\Big]^{n-1}$$
$$= nB(p,q)^{-n} \frac{x^{q-1}}{(1+x)^{p+q}} \Big[B_{\frac{1}{1+x}}(p,q)\Big]^{n-1}, \quad p,q,x > 0 \quad (2.113)$$

For $r = n$ (maximum);

$$f_{n:n}(x) = \frac{n!}{(n-1)!(n-n)!} B(p,q)^{-n} \frac{x^{q-1}}{(1+x)^{p+q}} \Big[B_{\frac{x}{1+x}}(q,p)\Big]^{n-1} \Big[B_{\frac{1}{1+x}}(p,q)\Big]^{n-n}$$
$$= nB(p,q)^{-n} \frac{x^{q-1}}{(1+x)^{p+q}} \Big[B_{\frac{x}{1+x}}(q,p)\Big]^{n-1}, \quad p,q,x > 0 \quad (2.114)$$

F order statistics

The pdf of the beta-F distribution was obtained by Morais de Lemos (2009) as,

$$f(x;a,b,u,v) = \frac{B(a,b)^{-1}}{B(\frac{u}{2},\frac{v}{2})} \frac{(\frac{v}{u})^{\frac{v}{2}} x^{\frac{v}{2}-1}}{\Big[1+(\frac{v}{u})x\Big]^{(u+v)/2}} \Big[I_{\frac{(\frac{v}{u})x}{1+(\frac{v}{u})x}}(\frac{u}{2},\frac{v}{2})\Big]^{a-1} \Big[I_{\frac{1}{1+(\frac{v}{u})x}}(\frac{u}{2},\frac{v}{2})\Big]^{b-1} \quad (2.115)$$
$$\text{where,} \quad a,b,u,v,x > 0$$

We therefore, extend this work and obtain the r^{th} order statistic of the F distribution by replacing $a = r$ and $b = n - r + 1$.

$$f_{r:n}(x) = \frac{B(r, n-r+1)^{-1}}{B(\frac{u}{2}, \frac{v}{2})} \frac{(\frac{v}{u})^{\frac{v}{2}} x^{\frac{v}{2}-1}}{\left[1 + (\frac{v}{u})x\right]^{(u+v)/2}} \left[I_{\frac{(\frac{v}{u})x}{1+(\frac{v}{u})x}}(\frac{u}{2}, \frac{v}{2})\right]^{r-1} \left[I_{\frac{1}{1+(\frac{v}{u})x}}(\frac{u}{2}, \frac{v}{2})\right]^{n-r}$$
$$\text{where,} \quad u, v, x > 0 \tag{2.116}$$

Subsequently, we get the minimum and the maximum order statistics for the F distribution as follows.

For $r = 1$ (minimum);

$$\begin{aligned} f_{1:n}(x) &= \frac{n!}{(n-1)!} B(\frac{u}{2}, \frac{v}{2}) \frac{(\frac{v}{u})^{\frac{v}{2}} x^{\frac{v}{2}-1}}{\left[1 + (\frac{v}{u})x\right]^{(u+v)/2}} \left[I_{\frac{(\frac{v}{u})x}{1+(\frac{v}{u})x}}(\frac{u}{2}, \frac{v}{2})\right]^{1-1} \left[I_{\frac{1}{1+(\frac{v}{u})x}}(\frac{u}{2}, \frac{v}{2})\right]^{n-1} \\ &= nB(\frac{u}{2}, \frac{v}{2}) \frac{(\frac{v}{u})^{\frac{v}{2}} x^{\frac{v}{2}-1}}{\left[1 + (\frac{v}{u})x\right]^{(u+v)/2}} \left[I_{\frac{1}{1+(\frac{v}{u})x}}(\frac{u}{2}, \frac{v}{2})\right]^{n-1}, \quad u, v, x > 0 \end{aligned} \tag{2.117}$$

For $r = n$ (maximum);

$$\begin{aligned} f_{n:n}(x) &= \frac{n!}{(n-1)!} B(\frac{u}{2}, \frac{v}{2}) \frac{(\frac{v}{u})^{\frac{v}{2}} x^{\frac{v}{2}-1}}{\left[1 + (\frac{v}{u})x\right]^{(u+v)/2}} \left[I_{\frac{(\frac{v}{u})x}{1+(\frac{v}{u})x}}(\frac{u}{2}, \frac{v}{2})\right]^{n-1} \left[I_{\frac{1}{1+(\frac{v}{u})x}}(\frac{u}{2}, \frac{v}{2})\right]^{n-n} \\ &= nB(\frac{u}{2}, \frac{v}{2}) \frac{(\frac{v}{u})^{\frac{v}{2}} x^{\frac{v}{2}-1}}{\left[1 + (\frac{v}{u})x\right]^{(u+v)/2}} \left[I_{\frac{(\frac{v}{u})x}{1+(\frac{v}{u})x}}(\frac{u}{2}, \frac{v}{2})\right]^{n-1}, \quad u, v, x > 0 \end{aligned} \tag{2.118}$$

Burr XII order statistics

Paranaíba et al. (2010) introduced the beta-burr XII distribution and gave its pdf as,

$$\begin{aligned} f(x;a,b,p,q,k) &= \frac{1}{B(a,b)}\Big[1-\Big(1+(\frac{x}{q})^p\Big)^{-k}\Big]^{a-1}\Big[\Big(1+(\frac{x}{q})^p\Big)^{-k}\Big]^{b-1} pkq^{-p} \\ &\Big(1+(\frac{x}{q})^p\Big)^{-k-1} x^{p-1} \\ &= \frac{pkq^{-p}x^{p-1}}{B(a,b)}\Big[1-\Big(1+(\frac{x}{q})^p\Big)^{-k}\Big]^{a-1}\Big[1+(\frac{x}{q})^p\Big]^{-kb-1} \\ &\text{where,} \quad a,b,p,q,k,x>0 \end{aligned} \tag{2.119}$$

We therefore, extend this work and obtain the r^{th} order statistic of the burr XII distribution by replacing $a = r$ and $b = n - r + 1$.

$$\begin{aligned} f_{r:n}(x) &= \frac{pkq^{-p}x^{p-1}}{B(r,n-r+1)}\Big[1-\Big(1+(\frac{x}{q})^p\Big)^{-k}\Big]^{r-1}\Big[1+(\frac{x}{q})^p\Big]^{-k(n-r+1)-1} \\ &= \frac{n!}{(r-1)!(n-r)!} pkq^{-p}x^{p-1}\Big[1-\Big(1+(\frac{x}{q})^p\Big)^{-k}\Big]^{r-1}\Big[1+(\frac{x}{q})^p\Big]^{-k(n-r+1)-1} \\ &\text{where,} \quad p,q,k,x>0 \end{aligned} \tag{2.120}$$

Subsequently, we get the minimum and the maximum order statistics for the burr XII distribution as follows.

For $r = 1$ (minimum);

$$\begin{aligned} f_{1:n}(x) &= \frac{n!}{(1-1)!(n-1)!} pkq^{-p}x^{p-1}\Big[1-\Big(1+(\frac{x}{q})^p\Big)^{-k}\Big]^{1-1}\Big[1+(\frac{x}{q})^p\Big]^{-k(n-1+1)-1} \\ &= npkq^{-p}x^{p-1}\Big[1+(\frac{x}{q})^p\Big]^{-nk-1}, \quad p,q,k,x>0 \end{aligned} \tag{2.121}$$

For $r = n$ (maximum);

$$\begin{aligned} f_{n:n}(x) &= \frac{n!}{(n-1)!(n-n)!} pkq^{-p}x^{p-1}\Big[1-\Big(1+(\frac{x}{q})^p\Big)^{-k}\Big]^{n-1}\Big[1+(\frac{x}{q})^p\Big]^{-k(n-n+1)-1} \\ &= npkq^{-p}x^{p-1}\Big[1-\Big(1+(\frac{x}{q})^p\Big)^{-k}\Big]^{n-1}\Big[1+(\frac{x}{q})^p\Big]^{-k-1}, \quad p,q,k,x>0 \end{aligned}$$

(2.122)

Log-logistic order statistics

The beta - log logistic distribution was introduced by Paranaíba et al. (2010) as a special sub-model of the beta-burr XII distribution.
They gave its pdf as,

$$\begin{aligned} f(x;a,b,p,\lambda) &= \frac{\left[\frac{(\lambda x)^p}{1+(\lambda x)^p}\right]^{a-1}\left[1-\frac{(\lambda x)^p}{1+(\lambda x)^p}\right]^{b-1}\frac{\lambda p(\lambda x)^{p-1}}{(1+(\lambda x)^p)^2}}{B(a,b)} \\ &= \frac{\lambda p(\lambda x)^{ap-1}}{B(a,b)(1+(\lambda x)^p)^{a+b}} \\ &\text{where,} \quad a,b,p,\lambda,x>0 \end{aligned} \tag{2.123}$$

We therefore, extend this work and obtain the r^{th} order statistic of the log logistic distribution by replacing $a = r$ and $b = n - r + 1$.

$$\begin{aligned} f_{r:n}(x) &= \frac{\lambda p(\lambda x)^{rp-1}}{B(r,n-r+1)(1+(\lambda x)^p)^{r+n-r+1}} \\ &= \frac{n!}{(r-1)!(n-r)!}\frac{\lambda p(\lambda x)^{rp-1}}{(1+(\lambda x)^p)^{n+1}}, \quad p,\lambda,x>0 \end{aligned} \tag{2.124}$$

Subsequently, we get the minimum and the maximum order statistics for the log logistic distribution as follows.

For $r = 1$ (minimum);

$$\begin{aligned} f_{1:n}(x) &= \frac{n!}{(1-1)!(n-1)!}\frac{\lambda p(\lambda x)^{(1)p-1}}{(1+(\lambda x)^p)^{n+1}} \\ &= \frac{n\lambda p(\lambda x)^{p-1}}{(1+(\lambda x)^p)^{n+1}}, \quad p,\lambda,x>0 \end{aligned} \tag{2.125}$$

For $r = n$ (maximum);

$$f_{n:n}(x) = \frac{n!}{(n-1)!(n-n)!}\frac{\lambda p(\lambda x)^{(n)p-1}}{(1+(\lambda x)^p)^{n+1}} \\ = \frac{n\lambda p(\lambda x)^{np-1}}{(1+(\lambda x)^p)^{n+1}}, \quad p, \lambda, x > 0 \tag{2.126}$$

2.5 Distribution of the Median, Range and Other Statistics

2.5.1 Distribution of the sample median

The median is a measure of location that might be considered an alternative to the sample mean. It is less affected by etreme observations.

The sample median, which we will denote by M, is a number such that approximately one-half of the observations are less than M and one-half are greater. In terms of order statistics, M is defined as;

$$M = \begin{cases} X_{((n+1)/2)} & \text{if } n \text{ is odd} \\ [X_{(n/2)} + X_{(n/2+1)}]/2 & \text{if } n \text{ is even} \end{cases} \tag{2.127}$$

Consider when the **sample size n is odd**. Then from equation (2.11), the pdf of the sample median $\tilde{X_n} = X_{(n+1)/2:n}$ is

$$f_{\tilde{X_n}}(x) = \frac{n!}{(\frac{n+1}{2}-1)!(n-\frac{n+1}{2})!}[F(x)]^{\frac{n+1}{2}-1}[1-F(x)]^{n-\frac{n+1}{2}}f(x) \\ = \frac{n!}{[(\frac{n-1}{2})!]^2}[F(x)(1-F(x))]^{\frac{n-1}{2}}f(x), \ -\infty < x < \infty \tag{2.128}$$

Suppose now the **sample size n is even**, the sample median is given by $\tilde{X_n} = (X_{(n/2):n} + X_{(n/2)+1:n})/2$. To derive the distribution of $\tilde{X_n}$ in this case, we first from equation (2.16) get the joint density function of $X_{(n/2):n}$ and $X_{(n/2)+1:n}$ to be

$$\begin{aligned} f_{\frac{n}{2},\frac{n}{2}+1}(x_1,x_2) &= C_{x_1,x_2}[F(x_1)]^{\frac{n}{2}-1}[F(x_2)-F(x_1)]^{\frac{n}{2}+1-\frac{n}{2}-1}[1-F(x_2)]^{n-(\frac{n}{2}+1)}f(x_1)f(x_2) \\ &= \frac{n!}{[(\frac{n}{2}-1)!]^2}[F(x_1)(1-F(x_2))]^{\frac{n}{2}-1}f(x_1)f(x_2), \; -\infty < x_1 < x_2 < \infty \\ &\text{where } C_{x_1,x_2} = \frac{n!}{(\frac{n}{2}-1)!(\frac{n}{2}+1-\frac{n}{2}-1)!(n-(\frac{n}{2}+1))!} \end{aligned} \tag{2.129}$$

Secondly, from equation (2.129), we obtain the joint density function of $X_{(n/2):n}$ and $\tilde{X}_n$ to be

$$\begin{aligned} f_{X_{(n/2):n},\tilde{X}_n}(x_1,x) &= C_{x_1,x}[F(x_1)]^{\frac{n}{2}-1}[1-F(2x-x_1)]^{n-\frac{n}{2}-1}[F(2x-x_1)-F(x_1)]^{n-n} \\ &\quad f(x_1)f(2x-x_1) \\ &= \frac{2n!}{[(\frac{n}{2}-1)!]^2}[F(x_1)]^{\frac{n}{2}-1}[1-F(2x-x_1)]^{\frac{n}{2}-1}f(x_1)f(2x-x_1) \\ &\text{where } -\infty < x_1 < x < \infty, \quad C_{x_1,x} = \frac{2n!}{(\frac{n}{2}-1)!(n-\frac{n}{2}-1)!(n-n)!} \end{aligned} \tag{2.130}$$

Integrating out x_1 in equation (2.130), we obtain the pdf of the sample median $\tilde{X}_n$ as

$$f_{\tilde{X}_n}(x) = \frac{2n!}{[(\frac{n}{2}-1)!]^2}\int_{-\infty}^{x}[F(x_1)]^{\frac{n}{2}-1}[1-F(2x-x_1)]^{\frac{n}{2}-1}f(x_1)f(2x-x_1)\,dx_1, \; -\infty < x < \infty \tag{2.131}$$

and from equation (2.131), the cdf of the sample median $\tilde{X}_n$ can be given as

$$\begin{aligned} F_{\tilde{X}_n}(x_\alpha) &= Pr(\tilde{X}_n \le x_\alpha) \\ &= \frac{2n!}{[(\frac{n}{2}-1)!]^2}\int_{-\infty}^{x_\alpha}\int_{-\infty}^{x}[F(x_1)]^{\frac{n}{2}-1}[1-F(2x-x_1)]^{\frac{n}{2}-1}f(x_1)f(2x-x_1)\,dx_1dx \\ &\text{where } -\infty < x_\alpha < \infty \end{aligned} \tag{2.132}$$

2.5.2 Distribution of the sample range

The sample range, $R_n = X_{n:n} - X_{1:n}$, is the distance between the smallest and largest observations. It is a measure of the dispersion in the sample and should reflect the dispersion in the population.

Here we shall obtain the distribution of the sample range $R_n = X_{n:n} - X_{1:n}$, the i^{th} quasirange $R_{i:n} = X_{n-i+1:n} - X_{i:n}$ which is a special case of $R_{i,j:n}$ and the sample midrange $V_n = (X_{1:n} + X_{n:n})/2$.
From the joint density function of $X_{1:n}$ and $X_{n:n}$ in equation (2.18), we get the joint density function of $X_{1:n}$ and R_n to be

$$f_{X_{1:n},R_n}(x_1, \omega) = n(n-1)[F(x_1+\omega)-F(x_1)]^{n-2} f(x_1) f(x_1+\omega), \ -\infty < x_1 < \infty; \ 0 < \omega < \infty \tag{2.133}$$

The pdf of the **sample range** R_n can be therefore derived by integrating out x_1 in equation (2.133) and obtain

$$\begin{aligned} f_{R_n}(\omega) &= \int_{-\infty}^{\infty} f_{X_{1:n},R_n}(x_1, \omega)\, dx_1 \\ &= n(n-1) \int_{-\infty}^{\infty} [F(x_1+\omega) - F(x_1)]^{n-2} f(x_1) f(x_1+\omega)\, dx_1, \ 0 < \omega < \infty \end{aligned} \tag{2.134}$$

However, the cdf of R_n takes a simpler form and is derived as

$$\begin{aligned}
F_{R_n}(\omega_\alpha) &= Pr(R_n \leq \omega_\alpha) \\
&= \int_0^{\omega_\alpha} \int_{-\infty}^{\infty} n(n-1)[F(x_1+\omega) - F(x_1)]^{n-2} f(x_1) f(x_1+\omega)\, dx_1 d_\omega \\
&= n(n-1) \int_0^{\omega_\alpha} \int_{-\infty}^{\infty} [F(x_1+\omega) - F(x_1)]^{n-2} f(x_1) f(x_1+\omega)\, dx_1 d_\omega \\
&= n \int_{-\infty}^{\infty} f(x_1) \left[(n-1) \int_0^{\omega_\alpha} [F(x_1+\omega) - F(x_1)]^{n-2} f(x_1+\omega)\, d_\omega \right] dx_1 \\
&= n \int_{-\infty}^{\infty} [F(x_1+\omega_\alpha) - F(x_1)]^{n-1} f(x_1)\, dx_1,\ 0 < \omega_\alpha < \infty
\end{aligned} \tag{2.135}$$

In order to derive the distribution of the **quasirange** $R_{i,j:n}$, we start by obtaining the joint density function of $X_{i:n}$ and $R_{i,j:n}$ from equation (2.16).

$$\begin{aligned}
f_{X_{i:n},R_{i,j:n}}(x_i, \omega) = &\frac{n!}{(i-1)!(j-i-1)!(n-j)!} \\
&[F(x_i)]^{i-1}[F(x_i+\omega) - F(x_i)]^{j-i-1}[1 - F(x_i+\omega)]^{n-j} f(x_i) f(x_i+\omega), \\
&-\infty < x_i < \infty,\ 0 < \omega < \infty
\end{aligned} \tag{2.136}$$

we then derive the pdf of $R_{i,j:n}$ by integrating out x_i and obtain

$$\begin{aligned}
f_{R_{i,j:n}}(\omega) = &\frac{n!}{(i-1)!(j-i-1)!(n-j)!} \\
&\int_{-\infty}^{\infty} [F(x_i)]^{i-1}[F(x_i+\omega) - F(x_i)]^{j-i-1}[1 - F(x_i+\omega)]^{n-j} f(x_i) f(x_i+\omega)\, dx_i, \\
&0 < \omega < \infty
\end{aligned} \tag{2.137}$$

Next, we present the derivation of the **sample midrange** $V_n = (X_{1:n} + X_{n:n})/2$. From equation (2.18), we have the joint density function of $X_{1:n}$ and V_n as

$$\begin{aligned} f_{X_{1:n},V_n}(x_1,\gamma) &= \frac{2n!}{(n-2)!}[F(2\gamma - x_1) - F(x_1)]^{n-2} f(x_1) f(2\gamma - x_1) \\ &= 2n(n-1)[F(2\gamma - x_1) - F(x_1)]^{n-2} f(x_1) f(2\gamma - x_1), \; -\infty < x_1 < \gamma < \infty \end{aligned} \tag{2.138}$$

Hence, the pdf of the sample midrange V_n is given by integrating out x_1 in equation (2.138)

$$f_{V_n}(\gamma) = 2n(n-1) \int_{-\infty}^{\gamma} [F(2\gamma - x_1) - F(x_1)]^{n-2} f(x_1) f(2\gamma - x_1)\, dx_1, \; -\infty < \gamma < \infty \tag{2.139}$$

with the cdf of midrange given as

$$\begin{aligned} F_{V_n}(\gamma_\alpha) &= 2n(n-1) \int_{-\infty}^{\gamma_\alpha} \int_{-\infty}^{v} [F(2\gamma - x_1) - F(x_1)]^{n-2} f(x_1) f(2\gamma - x_1)\, dx_1 d\gamma \\ &= n \int_{-\infty}^{v} f(x_1) \left[2(n-1) \int_{x_1}^{\gamma_\alpha} [F(2\gamma - x_1) - F(x_1)]^{n-2} f(2\gamma - x_1)\, d\gamma \right] dx_1 \\ &= n \int_{-\infty}^{v} [F(2\gamma_\alpha - x_1) - F(x_1)]^{n-1} f(x_1)\, dx_1, \; -\infty < \gamma_\alpha < \infty \end{aligned} \tag{2.140}$$

Similarly, density function and the distribution function of the general *quasi-midrange* $V_{i,j:n} = (X_{i:n} + X_{j:n})/2;\; 1 \leq i < j \leq n$ can be derived.

Remark 6. *We note that the expressions for the pdf and cdf derived for the sample median and range are for the case when the population distribution has an infinite support.*

2.6 Order Statistics for a Discrete Case

We mention that, though there are noted similarities between order statistics from continuous and discrete distributions, some properties by order statistics from continuous distributions do not hold for discrete distributions.

More work on order statistics from discrete distributions has been done by Nagaraja

(986a,b); Rüschendorf (1985); Arnold et al. (1992) and a review article by Nagaraja (1992). Here we show order statistics from a discrete case for the r^{th} and joint order statistics.

Suppose that $X_1, X_2, \cdots, X_n$ are n independent variates, each with cdf $F(x)$.
Let $F_{r:n}(x)$ where $r = 1, 2, \cdots, n$ denote the cdf of the r^{th} order statistic $X_{r:n}$. Then the cdf of the r^{th} order statistic is given by

$$\begin{aligned} F_{r:n}(x) &= Pr(X_{r:n} \leq x) \\ &= Pr(\text{at least } r \text{ of the } X_i \leq x) \\ &= \sum_{i=r}^{n} \binom{n}{i} [F(x)]^i [1 - F(x)]^{n-i} \end{aligned} \tag{2.141}$$

which is the binomial probability that exactly $i \leq x$.
Alternatively, equation (2.141) can be written as

$$F_{r:n}(x) = [F(x)]^r \sum_{j=0}^{n-r} \binom{r+j-1}{r-1} [1 - F(x)]^j \tag{2.142}$$

where the RHS is the sum of probabilities that exactly $r \leq x$. This is a negative binomial version of equation (2.141).
Alternatively, from the relation between binomial sums and incomplete beta function, we have

$$F_{r:n}(x) = I_{F(x)}(r, n - r + 1) \tag{2.143}$$

where $I_p(a, b)$ is the incomplete beta function.

Now let $f_{r:n}(x) = Pr(X_{r:n} = x)$ be the probability function of $X_{r:n}$, where $f(x)$ is discrete over $x = 0, 1, 2, \cdots$
From equation (2.143), we have

$$\begin{aligned} f_{r:n}(x) &= F_{r:n}(x) - F_{r:n}(x - 1) \\ &= I_{F(x)}(r, n - r + 1) - I_{F(x-1)}(r, n - r + 1) \\ &= Pr[F(x - 1) < U_{r:n} \leq F(x)] \\ &= \frac{1}{\beta(r, n - r + 1)} \int_{F(x-1)}^{F(x)} \mu^{r-1} (1 - \mu)^{n-r} \, d\mu \end{aligned} \tag{2.144}$$

The joint cdf $F_{r,s:n}(x,y)$ of $X_{r:n}$ and $X_{s:n}$ is obtained as below for discrete case.
For $x < y$,

$$\begin{aligned} F_{r,s:n}(x,y) &= Pr(\text{at least } r \leq x, \text{at least } s \leq y) \\ &= \sum_{j=s}^{n}\sum_{i=r}^{j} Pr(\text{exactly } i \leq x, \text{exactly } j \leq y) \\ &= \sum_{j=s}^{n}\sum_{i=r}^{j} \frac{n!}{i!(j-i)!(n-j)!}[F(x)]^{i}[F(y)-F(x)]^{j-i}[1-F(y)]^{n-j} \end{aligned} \tag{2.145}$$

For $x \geq y$, the inequality $X_{s:n} \leq y$ implies $X_{r:n} \leq x$, so that

$$F_{r,s:n}(x,y) = F_{s:n}(y) \tag{2.146}$$

From equation (2.145), we have the probability function $f_{r,s:n}(x,y) = Pr(X_{r:n} = x, X_{s:n} = y)$ as

$$\begin{aligned} f_{r,s:n}(x,y) &= F_{r,s:n}(x,y) - F_{r,s:n}(x-1,y) - F_{r,s:n}(x,y-1) + F_{r,s:n}(x-1,y-1) \\ &= Pr\Big[[F(\mu_{r:n})]^{-1} = x, [F(\mu_{s:n})]^{-1} = y\Big] \\ &= Pr\Big[F(x-1) < \mu_{r:n} \leq F(x), F(y-1) < \mu_{s:n} \leq F(y)\Big] \\ &= C_{rs} \iint v^{r-1}(w-v)^{s-r-1}(1-w)^{n-s}\, dvdw,\ x \leq y \end{aligned} \tag{2.147}$$

where C_{rs} is as in equation (2.17) and the integration is over the region

$$(v,w) : v \leq w, F(x-1) \leq v \leq F(x), F(y-1) \leq w \leq F(y)$$

2.7 Expected Values and Moments of Order Statistics

A fundamental definition regarding order statistics, which can be critically important in the computation of L-moments and probability-weighted moments (though these are not

discussed here), is the expectation of an order statistic. The expectation is defined in terms of the **QDF**.
Here we find the single moments $\mu_{r,n}$ and product moments, $\mu_{r,n}$ and $\mu_{s,n}$, of order statistics in general case.
From equation (3.1), we get the following results

$$\begin{aligned}\mu_{r:n}^{(k)} &= E(X_{r:n})^k \\ &= \int_{-\infty}^{\infty} x^k \, dF_{r:n}(x) \\ &= \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} x^k [F(x)]^{r-1} [1-F(x)]^{n-r} \, dF(x)\end{aligned} \tag{2.148}$$

For continuous distribution functions F, equation (2.148) can be expressed as

$$\mu_{r:n}^{(k)} = \frac{n!}{(r-1)!(n-r)!} \int_0^1 [G(u)]^k u^{r-1} [1-u]^{n-r} \, du \tag{2.149}$$

where $G(u)$ is the inverse of F.
For absolutely continuous distributions with pdf f, equation (2.148) can be expressed as

$$\mu_{r:n}^{(k)} = \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} x^k [F(x)]^{r-1} [1-F(x)]^{n-r} f(x) \, dx \tag{2.150}$$

We note that for $r = 1$, $n = 1$ and $k = 1$

$$E(X_{1:1}) = \int_0^1 x(F) \, dF \;=\; \mu \;=\; \text{arithmetic mean} \tag{2.151}$$

Consider two order statistics $X_{r:n}$ and $X_{s:n}$, $1 \le r < s \le n$.
Hence,

$$\begin{aligned}\mu_{r,s:n}^{(k,j)} &= E[(X_{r:n})^k (X_{s:n})^j], \; 1 \le r < s \le n \\ &= C_{r,s} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^k y^j [F(x)]^{r-1} [F(y)-F(x)]^{s-r-1} [1-F(y)]^{n-s} \, dF(x) dF(y)\end{aligned} \tag{2.152}$$

where $C_{r,s} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$
For absolutely continuous case

$$\mu_{r,s:n}^{(k,j)} = C_{r,s} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^k y^j [F(x)]^{r-1} [F(y)-F(x)]^{s-r-1} [1-F(y)]^{n-s} f(x) f(y) \, dx dy \tag{2.153}$$

In general, denoting $\mu^{(1)}_{r:n}$ by $\mu_{r:n}$ and $\mu^{(1,1)}_{r,s:n}$ by $\mu_{r,s:n}$, for convenience. We have the; variance given by

$$\sigma_{r,r:n} = Var(X_{r:n}) = \mu^{(2)}_{r:n} - (\mu_{r:n})^2,\ 0 \leq r \leq n \tag{2.154}$$

and covariance between $X_{r:n}$ and $X_{s:n}$ given by

$$\sigma_{r,s:n} = Cov(X_{r:n}X_{s:n}) = \mu_{r,s:n} - \mu_{r:n}\mu_{s:n},\ 0 \leq r < s \leq n \tag{2.155}$$

2.8 Recurrence Relations and Identities

Here we aim to study the recurrence relations and identities between the single and product moments of order statistics, this results in reduction of the number of independent calculations required for evaluation of the moments.

2.8.1 Identities

In general,

$$\Big(\sum_{r=1}^{n} X^k_{r:n}\Big)^m = \Big(\sum_{r=1}^{n} X^k_r\Big)^m \tag{2.156}$$

Let μ and σ^2 be the population mean and variance respectively and taking expectations both sides from equation (2.156) and choosing $m = 1$, we have

$$\sum_{r=1}^{n} \mu^{(k)}_{r:n} = nE(X^k) = n\mu^{(k)}_{r:n} \tag{2.157}$$

Then, if $k = 1$, we have

$$\sum_{r=1}^{n} \mu_{r:n} = nE(X) = n\mu \tag{2.158}$$

and, if $k = 2$, we have

$$\sum_{r=1}^{n} \mu^2_{r:n} = nE(X^2) \tag{2.159}$$

Now if $k = 1$ and $m = 2$ and using binomial expansion, we have

$$\begin{aligned}\sum_{r=1}^{n}\sum_{s=1}^{n}(X_{r:n}X_{s:n}) &= \sum_{r=1}^{n} X_{r:n}^2 + 2\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} X_{r:n}X_{s:n} \\ &= \sum_{r=1}^{n} X_r^2 + 2\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} X_r X_s\end{aligned} \tag{2.160}$$

Now taking expectations both sides

$$\begin{aligned}\sum_{r=1}^{n}\sum_{s=1}^{n} E(X_{r:n}X_{s:n}) &= \sum_{r=1}^{n} E(X_r^2) + 2\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} E(X_r X_s) \\ &= \sum_{r=1}^{n} \mu_r^{(2)} + 2\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} \mu_{r,s:n} \\ &= nE(X^2) + \frac{2n(n-1)}{2}[E(X)]^2\end{aligned} \tag{2.161}$$

When we apply equation (2.156) and simplify equation (2.161), we get an identity for product moments of order statistics as

$$\begin{aligned}\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} \mu_{r,s:n} &= \binom{n}{2}[E(X)]^2 \\ &= \binom{n}{2}\mu^2 \\ &= \frac{1}{2}n(n-1)\mu^2\end{aligned} \tag{2.162}$$

In summary, we have the following identities

$$\sum_{r=1}^{n} \mu_{r:n} = n\mu$$

$$\sum_{r=1}^{n} E(X_{r:n}^2) = nE(X^2)$$

$$\sum_{r=1}^{n}\sum_{s=1}^{n} E(X_{r:n}X_{s:n}) = \frac{1}{2}n(n-1)\mu^2$$

2.8.2 Recurrence relations

By starting from equation (2.11), we can establish the *triangle rule* for single moments of order statistics from any arbitrary distribution given by

$$r\mu_{r+1:n}^{(k)} + (n-r)\mu_{r:n}^{(k)} = n\mu_{r:n-1}^{(k)} \tag{2.163}$$

Similarly, a recurrence relation for the product moments of order statistics from any arbitrary distribution is given by

$$(r-1)\mu_{r,s:n}^{(k,m)} + (s-r)\mu_{r-1,s:n}^{(k,m)} + (n-s+1)\mu_{r-1,s-1:n}^{(k,m)} = n\mu_{r-1,s-1:n}^{(k,m)} \tag{2.164}$$

Chapter 3

Order Statistics from Uniform Distribution

3.1 Introduction

In chapter 2, we discussed some basic distributional properties of order statistics from arbitrary continuous and discrete populations. In this chapter, we apply these distributional properties to the case of uniform order statistics on the unit interval $(0, 1)$ and specifically show that the r^{th} order statistic from a random sample of size n from the uniform population has a $Beta(r, n-r+1)$ distribution. Similarly, we show that the r^{th} and s^{th} order statistics jointly have a bivariate $Beta(r, s-r, n-s+1)$ distribution. We use these distributional results to derive the means, variances and covariances of uniform order statistics. Additionally, we discuss other interesting properties of order statistics from uniform population.

3.2 Notations and Definitions

Suppose that we have a random sample $X_1, X_2, \cdots, X_n$ of size n from a continuous distribution with common distribution function $F_X(x) = F(x)$ and common density function $f_X(x) = f(x)$. The order statistics $X_{(1)} \leq X_{(2)} \leq \cdots < X_{(n)}$ are obtained by ordering the sample $X_1, X_2, \cdots, X_n$ in ascending order. Since this is random sampling from a continuous distribution, we assume that the probability of a tie between two order statistics is zero.

We had already shown in the previous chapter that the probability density function of the i^{th} order statistic is given as:

$$f_{X_r}(x) = \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1} [1 - F(x)]^{n-r} f(x) \tag{3.1}$$

3.3 Basic Distributional Results and Properties

Here we give the cdf and pdf of $u_{r:n},\ r = 1, 2, \cdots, n$ for a standard uniform uniform distribution with $f(x) = 1$ and $F(x) = u$, where $0 \leq x \leq 1$.

We first obtain the cdf of $u_{r:n},\ (1 \leq r \leq n)$ to be

$$\begin{aligned} F_{r:n}(x) &= Pr(X_{r:n} \leq x) \\ &= \sum_{r=1}^{n} \binom{n}{r} [F(x)]^r [1 - F(x)]^{n-r},\ -\infty < x < \infty \end{aligned} \tag{3.2}$$

Furthermore, by using the identity that

$$\sum_{r=1}^{n} \binom{n}{r} p^r [1-p]^{n-r} = \int_0^p \frac{n!}{(r-1)!(n-r)!} t^{r-1} (1-t)^{n-r}\, dt, 0 < p < 1$$

we can write the cdf of $X_{r:n}$ from equation (3.2) as

$$\begin{aligned} F_{r:n}(x) &= \int_0^{F(x)} \frac{n!}{(r-1)!(n-r)!} t^{r-1} (1-t)^{n-r}\, dt \\ &= I_{F(x)}(r, n-r+1),\ -\infty < x < \infty \end{aligned} \tag{3.3}$$

which is incomplete beta function (see Pearson (1934)).
Accordingly, from equations (3.2) and (3.3), we have the cdf of $u_{r:n}$ as

$$\begin{aligned} F_{r:n}(u) &= \sum_{r=1}^{n} \binom{n}{r} u^r [1-u]^{n-r} \\ &= \int_0^{u} \frac{n!}{(r-1)!(n-r)!} t^{r-1} [1-t]^{n-r}\, dt,\ 0 < u < 1 \end{aligned} \tag{3.4}$$

Similarly, from equation (2.11) we obtain the pdf of $u_{r:n}$, $(1 \leq r \leq n)$ to be

$$f_{r:n}(u) = \frac{n!}{(r-1)!(n-r)!} u^{r-1}[1-u]^{n-r}, \; 0 \leq u \leq 1 \tag{3.5}$$

The above density function is from the family of beta distributions; $Beta(r, n-r+1)$.

Hence, the pdf of the minimum value $(i.e\ r = 1)$ is given as

$$f_{X_{(1)}}(u) = n(1-u)^{n-1} \tag{3.6}$$

and the pdf of the maximum value $(i.e\ r = n)$ is given by

$$f_{X_{(n)}}(u) = nu^{n-1} \tag{3.7}$$

In general, the pdf of beta distribution with mean and variance are,

$$f_w(w) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} w^{a-1}[1-w]^{b-1}$$

where $0 < w < 1$ and $\Gamma(\cdot)$ is the gamma function

$$E(w) = \frac{a}{a+b}$$

$$Var(w) = \frac{ab}{(a+b)^2(a+b+1)}$$

Hence, the pdf of the r^{th} order statistic of standard uniform distribution with mean and variance is given as,

$$f_{r:n}(u) = \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)} u^{r-1}[1-u]^{(n-r+1)-1}, \; 0 \leq u \leq 1 \tag{3.8}$$

with

$$E(u_{r:n}) = \frac{r}{r+(n-r+1)} = \frac{r}{n+1} \tag{3.9}$$

and

$$Var(u_{r:n}) = \frac{r(n-r+1)}{(n+1)^2(n+2)} \tag{3.10}$$

Note, the proofs for the mean (3.9) and variance (3.10) will be done in section (3.4).

From equation (2.20) and (2.22) we have the joint cdf of $u_{r:n}$ and $u_{s:n}$, $1 \leq r < s \leq n$ to be

$$\begin{aligned} F_{r,s:n}(u_r, u_s) &= \sum_{r=s}^{n} \sum_{s=r}^{r} \frac{n!}{s!(r-s)!(n-r)!} u_r^s (u_s - u_r)^{r-s} (1-u_s)^{n-r} \\ &= \int_0^{u_r} \int_1^{u_s} \frac{n!}{(r-1)!(s-r-1)!(n-s)!} t_1^{r-1} (t_2 - t_1)^{s-r-1} (1-t_2)^{n-s} \, dt_2 dt_1 \\ & 0 \leq u_r < u_s \leq 1 \end{aligned} \tag{3.11}$$

with the joint pdf of $u_{r:n}$ and $u_{s:n}$ from equation (2.16) obtained as

$$\begin{aligned} f_{r,s:n}(u_r, u_s) &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} u_r^{r-1} (u_s - u_r)^{s-r-1} (1-u_s)^{n-s} \\ & 0 \leq u_r < u_s \leq 1 \end{aligned} \tag{3.12}$$

which takes the form of joint bivariate $Beta(r, s-r, n-s+1)$ distribution.

From equation (2.28), we can see that the joint density function of all n order statistics based on standard uniform distribution with density function $f(u) = 1;\ 0 \leq u \leq 1$ is given by

$$\begin{aligned} f_{1,2,\cdots,n:n}(u_1, u_2, \cdots, u_n) &= n![f(u_1) \cdot f(u_2) \cdots f(u_n)] \\ &= n! \prod_{i=1}^{n} f(u_i) \\ &= n!,\ 0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq n \end{aligned} \tag{3.13}$$

Theorem 3.3.1. *For a standard uniform distribution, the random variables* $V_1 = u_{r:n}/u_{s:n}$ *and* $V_2 = u_{s:n}$; $1 \leq r < s \leq n$ *are statistically independent, with* V_1 *and* V_2 *having* $Beta(r, s-r)$ *and* $Beta(s, n-s+1)$ *distributions respectively.*

Proof: *From equation (3.12), we have the joint density function of* $u_{r:n}$ *and* $u_{s:n}$ $(1 \leq r < s \leq n)$ *to be*

$$f_{r,s:n}(u_r, u_s) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} u_r^{r-1}(u_s - u_r)^{s-r-1}(1-u_s)^{n-s}$$

Taking the transformation;

$$V_1 = u_{r:n}/u_{s:n} \; and \; V_2 = u_{s:n}$$

we note that the jacobian of this transformation is, v_2*, hence the joint pdf of* V_1 *and* V_2 *is*

$$\begin{aligned} f_{v_1,v_2}(v_1, v_2) = & \frac{(s-1)!}{(r-1)!(s-r-1)!} v_1^{r-1}(1-v_1)^{s-r-1} \cdot \frac{n!}{(s-1)!(n-s)!} v_2^{s-1}(1-v_2)^{n-s} \\ & 0 < v_1 < 1, \; 0 < v_2 < 1 \end{aligned} \tag{3.14}$$

Implying that the random variables V_1 *and* V_2 *are statistically independent and also distributed as* $Beta(r, s-r)$ *and* $Beta(s, n-s+1)$ *respectively.*

Theorem 3.3.2. *Consider a standard uniform (0,1) distribution, the random variables*

$$V_1 = \frac{u_{r_1:n}}{u_{r_2:n}}, \; V_2 = \frac{u_{r_2:n}}{u_{r_3:n}}, \cdots, V_{i-1} = \frac{u_{r_{i-1}:n}}{u_{r_i:n}} \; and \; V_i = u_{r_i:n}$$

$(1 \leq r_1 < r_2 < \cdots < r_i \leq n)$ *are all statistically independent, and have distributions* $Beta(r_1, r_2 - r_1), Beta(r_2, r_3 - r_2), \cdots, Beta(r_{i-1}, r_i - r_{i-1})$ *and* $Beta(r_i, n - r_i + 1)$ *respectively.*

Proof: *From the above theorem, we obtain*

$$\begin{aligned}
E(\prod_{j=1}^{i} u_{r_{j:n}}^{m_j}) &= E\Big(\prod_{j=1}^{i} v_j^{k_1+k_2+\cdots+k_j}\Big) \\
&= \prod_{j=1}^{i} E(v_j^{k_1+k_2+\cdots+k_j}) \\
&= \prod_{j=1}^{i} \left(\frac{n!}{n+\sum\limits_{j=1}^{i} k_j}\right)\left(\frac{(r_j+k_1+k_2+\cdots+k_j-1)!}{(r_j+k_1+k_2+\cdots+k_{j-1}-1)!}\right) \qquad (3.15)\\
&= \frac{n!}{n+\sum\limits_{j=1}^{i} k_j}\prod_{j=1}^{i}\left(\frac{(r_j+k_1+k_2+\cdots+k_j-1)!}{(r_j+k_1+k_2+\cdots+k_{j-1}-1)!}\right)
\end{aligned}$$

with $k_0 = 0$

Hence, from equation (3.15), specifically we obtain that for $1 \leq r_1 < r_2 < r_3 < r_4 \leq n$.

$$E(u_{r_{1:n}}^{k_1} u_{r_{2:n}}^{k_2} u_{r_{3:n}}^{k_3} u_{r_{4:n}}^{k_4}) = \frac{n!(k_1+r_1-1)!(k_1+k_2+r_2-1)!(k_1+k_2+k_3+r_3-1)!(k_1+k_2+k_3+k_4+r_4-1)!}{(r_1-1)!(k_1+r_2-1)!(k_1+k_2+r_3-1)!(k_1+k_2+k_3+r_4-1)!(n+k_1+k_2+k_3+k_4)!} \qquad (3.16)$$

These first four cummulants and cross-cummulants of uniform order statistics obtained in equation (3.16) above may be used to develop some approximations for the corresponding quantities of order statistics from arbitrary continuous distribution $F(x)$. *This method of approximation is discussed in detail by David and Johnson (1954).*

3.4 Expected Values and Moments of Uniform Order Statistics

Here we find the single moments $U_{r:n}$ and product moments of $U_{r:n}$ and $U_{s:n}$ of order statistics from the case of the standard uniform distribution, hence, showing the means, variances, covariances and correlations.

From equation (3.1) and for any $\alpha > -r$, we get the following results

$$
\begin{aligned}
E(U_{r:n})^{\alpha} &= \int_0^1 x^{\alpha} f(x)\, dx \\
&= \frac{n!}{(r-1)!(n-r)!} \int_0^1 x^{\alpha} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x)\, dx \\
&= \frac{n!}{(r-1)!(n-r)!} \int_0^1 x^{\alpha} x^{r-1} (1-x)^{n-r}\, dx \\
&= \frac{n!}{(r-1)!(n-r)!} \beta(\alpha+r, n-r+1) \\
&= \frac{n!\Gamma(\alpha+r)\Gamma(n-r+1)}{(r-1)!(n-r)!\Gamma(n+\alpha+1)} \\
&= \frac{n!\Gamma(\alpha+r)}{(r-1)!\Gamma(n+\alpha+1)} \\
&= \frac{n!(\alpha+r-1)!}{(r-1)!(n+\alpha)!}
\end{aligned}
\tag{3.17}
$$

where $\beta(a,b)$ and $\Gamma(s)$ denote the beta function and gamma function respectively, which are related as

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

We note also that

$$\Gamma(n) = (n-1)! \text{ for } n = 1, 2, \cdots$$

Implying that the α^{th} moment is $B(r+\alpha, n-r+1)/B(r, n-r+1)$

For $\alpha = 1$;

$$
\begin{aligned}
E(U_{r:n}) &= u_{r:n} \\
&= \frac{n!\Gamma(r+1)}{(r-1)!\Gamma(n+2)} \\
&= \frac{n!r!}{(r-1)!(n+1)!} \\
&= \frac{r}{n+1},\ 1 \le r \le n
\end{aligned}
\tag{3.18}
$$

and

$$E(1/U_{r:n}) = \frac{n!\Gamma(r-1)}{(r-1)!\Gamma(n)} = \frac{n}{r-1},\ 2 \le r \le n \tag{3.19}$$

For $\alpha = 2$;

$$\begin{aligned} E(U_{r:n})^2 &= \frac{n!\Gamma(r+2)}{(r-1)!\Gamma(n+3)} \\ &= \frac{n!(r+1)!}{(r-1)!(n+2)!} \\ &= \frac{n!(r+1)r(r-1)!}{(r-1)!(n+2)(n+1)n!} \\ &= \frac{r(r+1)}{(n+1)(n+2)},\ 1 \le r \le n \end{aligned} \tag{3.20}$$

and

$$E\Big(\frac{1}{(U_{r:n})^2}\Big) = \frac{n(n-1)}{(r-1)(r-2)},\ 3 \le r \le n \tag{3.21}$$

In general, for $k = 1, 2, \cdots$, we have

$$\begin{aligned} E(U_{r:n})^k &= \frac{n!\Gamma(r+k)}{(r-1)!\Gamma(n+k+1)} \\ &= \frac{n!(r+k)(r+k-1)\cdots(r+1)r(r-1)!}{(r-1)!(n+k+1)(n+k)\cdots(n+2)(n+1)n!} \\ &= \frac{r(r+1)\cdots(r+k-1)}{(n+1)(n+2)\cdots(n+k)},\ 1 \le r \le n \end{aligned} \tag{3.22}$$

and

$$E(\frac{1}{(U_{r:n})^k}) = \frac{n(n-1)\cdots(n-k+1)}{(r-1)(r-2)\cdots(r-k)},\ k+1 \le r \le n \tag{3.23}$$

It follows from equations (3.18) and (3.20) that

$$\begin{aligned} Var(U_{r:n}) &= E(U_{r:n})^2 - [E(U_{r:n})]^2 \\ &= \frac{r(r+1)}{(n+1)(n+2)} - [\frac{r}{(n+1)}]^2 \\ &= \frac{r(r+1)(n+1) - r^2(n+2)}{(n+1)^2(n+2)} \\ &= \frac{nr^2 + r^2 + nr + r - nr^2 - 2r^2}{(n+1)^2(n+2)} \\ &= \frac{r(n-r+1)}{(n+1)^2(n+2)}, 1 \le r \le n \end{aligned} \tag{3.24}$$

and from equations (3.19) and (3.21)

$$\begin{aligned} Var(1/U_{r:n}) &= \frac{n(n-1)}{(r-1)(r-2)} - [\frac{n}{r-1}]^2 \\ &= \frac{n(n-1)(r-1) - n^2(r-2)}{(r-1)^2(r-2)} \\ &= \frac{n^2r - n^2 - nr + n - n^2r + 2n^2}{(r-1)^2(r-2)} \\ &= \frac{n(n-r+1)}{(r-1)^2(r-2)},\ 3 \le r \le n \end{aligned} \tag{3.25}$$

Consider two uniform order statistics $U_{r:n}$ and $U_{s:n}$, $1 \le r < s \le n$.
Hence

$$\begin{aligned} E(U_{r:n}^{m_r} U_{s:n}^{m_s}) &= \int_0^1 \int_0^1 U_{r:n}^{m_r} U_{s:n}^{m_s} f(U_r U_s)\ dU_r dU_s \\ &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \beta(r+m_r, s-r)\beta(s+m_r+m_s, n-s+1) \\ &= \frac{n!}{(n+m_r+m_s)!} \cdot \frac{(r+m_r-1)!}{(r-1)!} \cdot \frac{(s+m_r+m_s-1)!}{(s+m_r-1)} \end{aligned} \tag{3.26}$$

setting $m_r = m_s = 1$, we obtain

$$
\begin{aligned}
E(U_{r:n}U_{s:n}) &= \frac{n!}{(n+2)!} \cdot \frac{r!}{(r-1)!} \cdot \frac{(s+1)!}{s!} \\
&= \frac{n!}{(n+2)(n+1)n!} \cdot \frac{r(r-1)!}{(r-1)!} \cdot \frac{(s+1)s!}{s!} \\
&= \frac{r(s+1)}{(n+1)(n+2)},\ 1 \le r < s \le n
\end{aligned} \tag{3.27}
$$

Hence,

$$
\begin{aligned}
\sigma_{r,s:n} &= Cov(U_{r:n}U_{s:n}) \\
&= E(U_{r:n}U_{s:n}) - E(U_{r:n})E(U_{s:n}) \\
&= \frac{r(s+1)}{(n+1)(n+2)} - \frac{r}{(n+1)} \cdot \frac{s}{(n+1)} \\
&= \frac{r(s+1)}{(n+1)(n+2)} - \frac{rs}{(n+1)^2} \\
&= \frac{(rs+r)(n+1) - rs(n+2)}{(n+1)^2(n+2)} \\
&= \frac{nr + r - rs}{(n+1)^2(n+2)} \\
&= \frac{r(n-s+1)}{(n+1)^2(n+2)}
\end{aligned} \tag{3.28}
$$

and

$$
\begin{aligned}
Corr(U_{r:n}U_{s:n}) &= \frac{Cov(U_{r:n}U_{s:n})}{\sqrt{Var(U_{r:n})Var(U_{s:n})}} \\
&= \frac{r(n-s+1)}{(n+1)^2(n+2)} \frac{(n+1)^2(n+2)}{\sqrt{r(n-r+1)+s(n-s+1)}} \\
&= \left\{ \frac{r(n-s+1)}{s(n-r+1)} \right\}^{1/2}
\end{aligned} \tag{3.29}
$$

Subsequently, the correlation coefficient between the minimum value $r = 1$ and maximum value $s = n$ is given as,

$$
\begin{aligned}
Corr(U_{1:n}, X_{n:n}) &= \left\{ \frac{1(n-n+1)}{n(n-1+1)} \right\}^{1/2} \\
&= \left\{ \frac{1}{n^2} \right\}^{1/2} \\
&= \frac{1}{n}
\end{aligned}
$$

3.5 Distribution of the Median, Range and Other Statistics

Here we give the distribution of the sample median and sample range of order statistics from the standard uniform distribution.

3.5.1 Distribution of the sample range

The sample range is given by

$$R_n = X_{n:n} - X_{1:n}$$

Now let

$X_{1:n}$ take the value x

$X_{n:n}$ take the value y

R_n take the value w

Therefore,

$$R_n = w = y - x$$

and

$$y = w + x$$

From equation (2.16) and letting $r = 1$ and $s = n$, we have the pdf of the sample range given as

$$\begin{aligned}
f_{X_{1:n}X_{n:n}}(x,y) &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \\
&\quad [F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y) \\
&= \frac{n!}{0!(n-2)!0!}[F(x)]^0 f(x)[F(w+x)-F(x)]^{n-2}f(w+x)[1-F(w+x)]^0 \\
&= n(n-1)f(x)f(w+x)[F(w+x)-F(x)]^{n-2}
\end{aligned} \tag{3.30}$$

Since $f(x) = 1$ and $F(x) = u$, we get the pdf of R_n as;

$$\begin{aligned}
f_{X_{1:n}X_{n:n}}(x,w+x) &= n(n-1)\cdot 1 \cdot 1 \cdot [u+w-u]^{n-2} \\
&= n(n-1)w^{n-2}
\end{aligned} \tag{3.31}$$

for a starting point $u \varepsilon [0, 1-w]$ with interval length w.

To find the probability of $X_{1:n}$ and $X_{n:n}$, within some interval of w, we integrate over all (permissible) starting points, u:

$$\begin{aligned}
f(w) &= \int_0^{1-w} f_{X_{1:n}X_{n:n}}(x,w+x)\,du \\
&= \int_0^{1-w} n(n-1)w^{n-2}\,du \\
&= n(n-1)w^{n-2}\int_0^{1-w} du \\
&= n(n-1)w^{n-2}(1-w)
\end{aligned} \tag{3.32}$$

which is the distribution of the sample range for standard uniform distribution.

Using equation (3.32), we get the cdf of the sample range R_n as

$$\begin{aligned}
F_{R_n}(r) &= \int_0^{r} n(n-1)w^{n-2}(1-w)\,dw \\
&= nr^{n-1} - (n-1)r^n, \ 0 < r < 1
\end{aligned} \tag{3.33}$$

Knowing that $F(x_1 + r) \equiv 1$ when $x > 1 - r$, we get the cdf of sample range R_n from equation (2.135) as

$$\begin{aligned} F_{R_n}(r) &= \int_0^{1-r} r^{n-1}\, dx_1 + n \int_{1-r}^{1} (1 - x_1)^{n-1}\, dx_1 \\ &= nr^{n-1}(1-r) + r^n,\ 0 < r < 1 \end{aligned} \tag{3.34}$$

which is as expressed in equation (3.33).
As seen in equations (3.32) and (3.33), the sample range R_n from a standard uniform population has a $Beta(n-1, 2)$ distribution.

From equation (2.137), we can see that the pdf of *quasirange* $R_{i,j:n}$ for standard uniform distribution is,

$$\begin{aligned} f_{R_{i,j:n}}(\omega) &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\omega^{j-i-1} \int_0^{1-\omega} x_i^{i-1}(1-\omega-x_i)^{n-j} \\ &= \frac{n!}{(j-i-1)!(n-j+i)!}\omega^{j-i-1}(1-\omega)^{n-j+i},\ 0 < \omega < 1 \end{aligned} \tag{3.35}$$

Similarly, from equation (2.139), the pdf of the *sample midrange* V_n for standard uniform distribution is obtained as,
for $0 \le t \le \frac{1}{2}$;

$$\begin{aligned} f_{V_n}(t) &= 2n(n-1) \int_0^{t} (2t - 2x_1)^{n-2}\, dx_1 \\ &= 2^{n-1} n t^{n-1},\ 0 \le t \le \frac{1}{2} \end{aligned} \tag{3.36}$$

and for $\frac{1}{2} \le t \le 1$;

$$\begin{aligned} f_{V_n}(t) &= 2n(n-1) \int_{2t-1}^{t} (2t - 2x_1)^{n-2}\, dx_1 \\ &= 2^{n-1} n (1-t)^{n-1},\ \frac{1}{2} \le t \le 1 \end{aligned} \tag{3.37}$$

Using equations (3.36) and (3.37), we can obtain the cdf of the sample midrange V_n for the standard uniform distribution as;

$$\begin{aligned} F_{V_n}(t_0) = Pr(V_n \leq t_0) &= 2^{n-1}t_0^n, \text{ if } 0 \leq t_0 \leq \frac{1}{2} \\ &= 1 - 2^{n-1}(1-t_0)^n, \text{ if } \frac{1}{2} \leq t_0 \leq 1 \end{aligned} \tag{3.38}$$

Expectation of the sample range

We can further show the expectation of the sample range $R_n = X_{n:n} - X_{1:n}$ by taking the expectation of the difference of the random variables $X_{n:n}$ with $X_{1:n}$. From equation (2.11), we had the pdf of the r^{th} order statistic as,

$$f_{X_{r:n}}(u) = \frac{n!}{(r-1)!(n-r)!}[F_X(u)]^{r-1}f_X(u)[1-F_X(u)]^{n-r}$$

Letting $r = 1$ and $r = n$ for the 1^{st} and n^{th} order statistic (of the uniform distribution) respectively, we get

$$f_{X_{1:n}}(u) = n(1-u)^{n-1}$$

$$f_{X_{n:n}}(u) = nu^{n-1}$$

Now, taking the expectation of their difference, we get

$$\begin{aligned} E(R_n) &= E(X_{n:n} - X_{1:n}) \\ &= \int_0^1 uf(X_{n:n} - X_{1:n})\, du \\ &= \int_0^1 u[nu^{n-1} - n(1-u)^{n-1}]\, du \\ &= \int_0^1 nu^n\, du - \int_0^1 nu(1-u)^{n-1}\, du \\ &= \frac{n}{n+1} - \frac{1}{n+1} \\ &= \frac{n-1}{n+1} \end{aligned} \tag{3.39}$$

We note that $X_{n:n}$ and $X_{1:n}$ are not independent but that the expectation of the sums of random variables is still the same regardless.

The joint pdf of sample range (R_n) and mid-range (m)

We show the joint pdf of range R_n and mid-range m in a random sample of size n from a uniform population over the interval $(0, 1)$.

From $R_n = X_{n:n} - X_{1:n}$ and $m = \frac{1}{2}[X_{1:n} + X_{n:n}]$, we find two equations, $X_{n:n} = 2M - X_{1:n}$ and $X_{n:n} = R + X_{1:n}$, solving them simultaneously gives;

$$X_{n:n} = M + \frac{R}{2}$$

and

$$X_{1:n} = M - \frac{R}{2}$$

$$\text{J} = \begin{vmatrix} \frac{\delta X_{1:n}}{\delta R} & \frac{\delta X_{1:n}}{\delta M} \\ \frac{\delta X_{n:n}}{\delta R} & \frac{\delta X_{n:n}}{\delta M} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{vmatrix} = |-1| = 1$$

$$g(r, m) = f_{X_{1:n}X_{n:n}}(r, m) \cdot |J|$$

but

$$\begin{aligned} f_{X_{1:n}X_{n:n}}(x, y) &= \frac{n!}{(1-1)!(n-1-1)!(n-n)!}[F(x)]^{-1}f(x)[F(y) - F(x)]^{n-1-1}f(y)[1 - F(y)]^{n-n} \\ &= \frac{n!}{(n-2)!}f(x)[F(y) - F(x)]^{n-2}f(y) \\ &= n(n-1)[y + \frac{1}{2} - (x + \frac{1}{2})]^{n-2} \\ &= n(n-1)[y - x]^{n-2} \end{aligned}$$

since $f(x) \sim U(0, 1)$

$$f(x) = \frac{1}{1-0} = \begin{cases} 1, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

and

$$F(x) = \int_0^x 1\ dt\ =\ t|_0^x\ =\ x$$

Therefore,

$$\begin{aligned} g(r,m) &= [f_{X_{1:n}X_{n:n}}(r,m)]\,|{-1}| \\ &= n(n-1)[r+m-m]^{n-2} \\ &= n(n-1)r^{n-2};\ 0 \le r \le 1-2\,|m| \le 1 \end{aligned}$$

Which is the joint pdf of range and mid-range.

Hence, the pdf of mid-range m is given by;

$$\begin{aligned} g(m) &= \int_0^{1-2|m|} g(r,m)\ dr \\ &= \int_0^{1-2|m|} n(n-1)r^{n-2}\ dr \\ &= n(n-1)\frac{r^{n-1}}{n-1}|_0^{1-2|m|} \end{aligned}$$

$$g(m) = n[1-2|m|]^{n-1};\ |m| \le 1$$

3.5.2 Distribution of the sample median

The sample median, which we will denote by M, is defined as;

$$M = \begin{cases} X_{(n+1)/2:n} & \text{if } n \text{ is odd} \\ [X_{n/2:n} + X_{n/2+1:n}]/2 & \text{if } n \text{ is even} \end{cases} \tag{3.40}$$

Consider when n is odd. Then from equation (2.11) we have the pdf of the sample median $\tilde{X_n} = X_{(n+1)/2:n}$ to be

$$\begin{aligned} f_{\tilde{X_n}}(x) &= \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x) \\ &= \frac{n!}{((n+1)/2-1)!(n-(n+1)/2)!}[F(x)]^{(n-1)/2}[1-F(x)]^{(n-1)/2}f(x) \quad (3.41) \\ &= \frac{n!}{([(n-1)/2]!)^2}[F(x)(1-F(x))]^{(n-1)/2}f(x),\ -\infty < x < \infty \end{aligned}$$

The above pdf is symmetric about 0 if the population distribution is symmetric about 0. For the case of standard uniform distribution, the pdf of the sample median given in equation (3.41) becomes

$$f_{\tilde{u_n}}(u) = \frac{n!}{([(n-1)/2]!)^2}u^{(n-1)/2}[1-u]^{(n-1)/2},\ 0 \le u \le 1 \quad (3.42)$$

We can further work out the moments of the sample median $\tilde{X_n}$ from equation (3.41). The m^{th} moment is obtained as below from equation (3.17) and $r = (n-1)/2$.

$$\begin{aligned} E(\tilde{u_n}^m) &= \frac{n!(m+r-1)!}{(r-1)!(n+m)!} \\ &= \frac{n!}{(n+m)!} \cdot \frac{(m+r-1)!}{(r-1)!} \\ &= \frac{n!}{(n+m)!} \cdot \frac{(m+(n+1)/2-1)!}{((n+1)/2-1)!} \\ &= \frac{n!}{(n+m)!} \cdot \frac{(m+(n-1)/2)!}{((n-1)/2)!} \text{ where } m = 1, 2, \cdots \end{aligned} \quad (3.43)$$

Specifically, we can find the mean and variance as follows;

For mean, we set $m = 1$ and obtain

$$\begin{aligned} E(\tilde{u_n}) &= \frac{n!}{(n+m)!} \cdot \frac{(m+(n-1)/2)!}{((n-1)/2)!} \\ &= \frac{n!}{(n+1)!} \cdot \frac{(1+(n-1)/2)!}{((n-1)/2)!} \\ &= \frac{1}{(n+1)} \cdot \frac{((n+1)/2)!}{((n-1)/2)!} \\ &= \frac{1}{(n+1)} \cdot \frac{(n+1)/2((n-1)/2)!}{((n-1)/2)!} \\ &= \frac{1}{2} \end{aligned} \tag{3.44}$$

Hence, $E(\tilde{u_n}) = \frac{1}{2}$

If now, $m = 2$;

$$\begin{aligned} E(\tilde{u_n}^2) &= \frac{n!}{(n+m)!} \cdot \frac{(m+(n-1)/2)!}{((n-1)/2)!} \\ &= \frac{n!}{(n+2)!} \cdot \frac{(2+(n-1)/2)!}{((n-1)/2)!} \\ &= \frac{1}{(n+2)(n+1)} \cdot \frac{n+3}{2} \cdot \frac{n+1}{2} \\ &= \frac{(n+3)}{4(n+2)} \end{aligned} \tag{3.45}$$

Therefore, the variance is given by

$$\begin{aligned} Var(\tilde{u_n}) &= E(\tilde{u_n}^2) - [E(\tilde{u_n})]^2 \\ &= \frac{n+3}{4(n+2)} - \frac{1}{4} \\ &= \frac{1}{4(n+2)} \end{aligned} \tag{3.46}$$

Hence, $Var(\tilde{u_n}) = \frac{1}{4(n+2)}$

Suppose now the sample size n is even. Then the sample median is given by $\tilde{X_n} = (X_{n/2:n} + X_{n/2+1:n})/2$.
We then derive the distribution of $\tilde{X_n}$ in this case by first from equation (2.16) have the

joint density function of $\tilde{X_{n/2:n}}$ and $\tilde{X}_{n/2+1:n}$ as;

$$f_{\frac{n}{2},\frac{n}{2}+1}(x_1,x_2) = \frac{n!}{(\frac{n}{2}-1)!(\frac{n}{2}+1-\frac{n}{2}-1)!(n-(\frac{n}{2}+1))!}$$
$$[F(x_1)]^{\frac{n}{2}-1}[F(x_2)-F(x_1)]^{\frac{n}{2}+1-\frac{n}{2}-1}[1-F(x_2)]^{n-(\frac{n}{2}+1)}f(x_1)f(x_2)$$
$$= \frac{n!}{[(\frac{n}{2}-1)!]^2}[F(x_1)(1-F(x_2))]^{\frac{n}{2}-1}f(x_1)f(x_2),\ -\infty < x_1 < x_2 < \infty \tag{3.47}$$

From equation (3.47) above, we obtain the joint density function of $\tilde{X_{n/2:n}}$ and $\tilde{X}_{n/2+1:n}$ as;

$$f_{X_{n/2:n},\tilde{X}_n}(x_1,x) = \frac{2n!}{(\frac{n}{2}-1)!(n-\frac{n}{2}-1)!(n-n)!}$$
$$[F(x_1)]^{\frac{n}{2}-1}[1-F(2x-x_1)]^{n-\frac{n}{2}-1}[F(2x-x_1)-F(x_1)]^{n-n}f(x_1)f(2x-x_1)$$
$$= \frac{2n!}{[(\frac{n}{2}-1)!]^2}[F(x_1)]^{\frac{n}{2}-1}[1-F(2x-x_1)]^{\frac{n}{2}-1}f(x_1)f(2x-x_1),\ -\infty < x_1 < x < \infty \tag{3.48}$$

If we integrate out x_1 in equation (3.48), we obtain the pdf of the sample median $\tilde{X}_n$ as below

$$\begin{aligned} f_{\tilde{X}_n}(x) &= \int_{-x}^{x} f_{X_{n/2:n},\tilde{X}_n}(x_1,x)\,dx_1 \\ &= \int_{-x}^{x} \frac{2n!}{[(\frac{n}{2}-1)!]^2}[F(x_1)]^{\frac{n}{2}-1}[1-F(2x-x_1)]^{\frac{n}{2}-1}f(x_1)f(2x-x_1)\,dx_1 \\ &= \frac{2n!}{[(\frac{n}{2}-1)!]^2}\int_{-x}^{x}[F(x_1)]^{\frac{n}{2}-1}[1-F(2x-x_1)]^{\frac{n}{2}-1}f(x_1)f(2x-x_1)\,dx_1 \end{aligned} \tag{3.49}$$

where $-\infty < x < \infty$

In particular, for a standard uniform distribution, we obtain the pdf of the sample median $\tilde{X_n}$ as;

$$\begin{aligned}
f_{\tilde{u_n}}(u) &= \frac{2n!}{[(\frac{n}{2}-1)!]^2} \int_0^1 u^{\frac{n}{2}-1}[1-t]^{\frac{n}{2}-1}\,du \\
&= \frac{2n!}{[(\frac{n}{2}-1)!]^2} \cdot [1-t]^{\frac{n}{2}-1} \cdot \frac{u^{\frac{n}{2}-1}}{\frac{n}{2}}\Bigg|_0^1 \\
&= \frac{2n!}{[(\frac{n}{2}-1)!]^2}\frac{2}{n}[1-t]^{\frac{n}{2}-1}
\end{aligned} \tag{3.50}$$

The cdf of the sample median $\tilde{X_n}$ can simply be written from equation (3.49) as;

$$\begin{aligned}
F_{\tilde{X_n}}(x_0) &= Pr(\tilde{X_n} \leq x_0) \\
&= \frac{2n!}{[(\frac{n}{2}-1)!]^2} \int_{-x}^{x_0}\int_{-x}^{x} [F(x_1)]^{\frac{n}{2}-1}[1-F(2x-x_1)]^{\frac{n}{2}-1} f(x_1) f(2x-x_1)\,dx_1 dx
\end{aligned} \tag{3.51}$$

where $-\infty < x_0 < \infty$

By changing the order of integration, we get the cdf of $\tilde{X_n}$ as;

$$\begin{aligned}
F_{\tilde{X_n}}(x_0) &= \frac{2n!}{[(\frac{n}{2}-1)!]^2} \int_{-x}^{x_0} [F(x_1)]^{\frac{n}{2}-1} f(x_1) \left[\int_{x_1}^{x} [1-F(2x-x_1)]^{\frac{n}{2}-1} f(2x-x_1)\,dx\right] dx_1 \\
&= \frac{n!}{(\frac{n}{2}-1)!(\frac{n}{2})!} \left[\int_{-\infty}^{x_0} [F(x_1)]^{\frac{n}{2}-1}[1-F(x_1)]^{\frac{n}{2}} f(x_1)\,dx_1 \right. \\
&\quad \left. - \int_{-\infty}^{x_0} [F(x_1)]^{\frac{n}{2}-1}[1-F(2x_0-x_1)]^{\frac{n}{2}} f(x_1)\,dx_1 \right]
\end{aligned} \tag{3.52}$$

Notably, we have assumed an infinite support population in deriving equation (3.52).

For the case of standard uniform distribution, where the population is finite, we take $1-F(2x_0-x_1) \equiv 0$ whenever $x_1 \leq 2x_0-1$ and obtain the cdf of $\tilde{X_n}$ from equation (3.52)

as below.
For $0 \leq x_0 \leq \frac{1}{2}$;

$$F_{\tilde{X}_n}(x_0) = \frac{n!}{(\frac{n}{2}-1)!(\frac{n}{2})!}\left[\int\limits_0^{x_0} x_1^{\frac{n}{2}-1}[1-x_1]^{\frac{n}{2}}\ dx_1 - \int\limits_0^{x_0} x_1^{\frac{n}{2}-1}[1+x_1-2x_0]^{\frac{n}{2}}\ dx_1\right]$$
$$= I_{x_0}\Big(\frac{n}{2},\frac{n}{2}+1\Big) - \frac{n!}{(\frac{n}{2}-1)!(\frac{n}{2})!}\sum_{r=0}^{\frac{n}{2}}(-1)^r\binom{\frac{n}{2}}{r}[1-x_0]^{\frac{n}{2}-r}\int\limits_0^{x_0} x_1^{\frac{n}{2}-1}[x_0-x_1]^r\ dx_1$$
$$= I_{x_0}\Big(\frac{n}{2},\frac{n}{2}+1\Big) - \sum_{r=0}^{\frac{n}{2}}(-1)^r\binom{n}{\frac{n}{2}-r}x_0^{\frac{n}{2}+r}[1-x_0]^{\frac{n}{2}-r},\ 0 \leq x_0 \leq \frac{1}{2} \tag{3.53}$$

and for $\frac{1}{2} \leq x_0 \leq 1$;

$$F_{\tilde{X}_n}(x_0) = \frac{n!}{(\frac{n}{2}-1)!(\frac{n}{2})!}\left[\int\limits_0^{x_0} x_1^{\frac{n}{2}-1}[1-x_1]^{\frac{n}{2}}\ dx_1 - \int\limits_{2x_0-1}^{x_0} x_1^{\frac{n}{2}-1}[1+x_1-2x_0]^{\frac{n}{2}}\ dx_1\right]$$
$$= I_{x_0}\Big(\frac{n}{2},\frac{n}{2}+1\Big) - \frac{n!}{(\frac{n}{2}-1)!(\frac{n}{2})!}\sum_{r=0}^{\frac{n}{2}-1}\binom{\frac{n}{2}-1}{r}[2x_0-1]^{\frac{n}{2}-1-r}[1-x_0]^{\frac{n}{2}+r+1}\int\limits_0^1 t^{\frac{n}{2}+r}\ dt$$
$$= I_{x_0}\Big(\frac{n}{2},\frac{n}{2}+1\Big)\sum_{r=0}^{\frac{n}{2}-1}\binom{\frac{n}{2}+r}{r}\binom{n}{\frac{n}{2}-1-r}[2x_0-1]^{\frac{n}{2}-1-r}[1-x_0]^{\frac{n}{2}+r+1},\ \frac{1}{2} \leq x_0 \leq 1 \tag{3.54}$$

In equations (3.53) and (3.54), $I_{x_0}\Big(\frac{n}{2},\frac{n}{2}+1\Big)$ denotes an incomplete beta function, as used in equation (2.14).

Specifically, from equations (3.53) and (3.54), we get that;

$$F_{\tilde{X}_n}\Big(\frac{1}{2}\Big) = I_{\frac{1}{2}}\Big(\frac{n}{2},\frac{n}{2}+1\Big) - \frac{1}{2^n}\binom{n-1}{\frac{n}{2}-1} \tag{3.55}$$

3.5.3 Estimation of percentiles

In descriptive statistics, we define the sample percentiles using the order statistics (even though the term order statistics may not be used in a non-calculus based introductory statistics course). For example, if sample size is an odd integer $n = 2m+1$, then the sample median is the order statistic X_{m+1}. The preceding discussion on the order statistics of the uniform distribution can show us that this approach is a sound one.

Suppose we have a random sample of size n from an arbitrary continuous distribution. The order statistics listed in ascending order are:

$$X_1 < X_2 < X_3 < \cdots < X_n$$

For each $i \le n$, consider $W_i = F(X_i)$. Since the distribution function $F(x)$ is a non-decreasing function, the W_i are also increasing:

$$W_1 < W_2 < W_3 < \cdots < W_n$$

It can be shown that if $F(x)$ is a distribution function of a continuous random variable X, then the transformation $F(X)$ follows the uniform distribution $U(0,1)$. Then the following transformed random sample:

$$F(X_1), F(X_2), \cdots, F(X_n)$$

are drawn from the uniform distribution $U(0,1)$. Furthermore, W_i are the order statistics for this random sample. By the preceding discussion,

$$E[W_i] = E[F(X_i)] = \frac{i}{n+1}$$

Note that $F(X_i)$ is the area under the density function $f(x)$ and to the left of X_i. Thus $F(X_i)$ is a random area and $E[W_i] = E[F(X_i)]$ is the expected area under the density curve $f(x)$ to the left of X_i.

Recall that $f(x)$ is the common density function of the original sample $X_1, X_2, \cdots, X_n$.

For example, suppose the sample size n is an odd integer where $n = 2m + 1$. Then the sample median is X_{m+1}. Note that

$$E[W_{m+1}] = \frac{m+1}{n+1} = \frac{1}{2}$$

Thus if we choose X_{m+1} as a point estimate for the population median, X_{m+1} is expected to be above the bottom 50% of the population and is expected to be below the upper 50% of the population.

Furthermore, $E[W_i - W_{i-1}]$ is the expected area under the density curve and between X_i and X_{i-1}. This expected area is:

$$E[W_i - W_{i-1}] = E[F(X_i)] - E[F(X_{i-1})] = \frac{i}{n+1} - \frac{i-1}{n+1} = \frac{1}{n+1}$$

The expected area under the density curve and above the maximum order statistic X_n is:

$$E[1 - F(X_n)] = 1 - \frac{n}{n+1} = \frac{1}{n+1}$$

Consequently here is an interesting observation about the order statistics $X_1 < X_2 < X_3 < \cdots < X_n$. The order statistics X_i divides the area under the density curve $f(x)$ and above the x-axis into $n + 1$ areas. On average each of these area is

$$\frac{1}{n+1}$$

As a result, it makes sense to use order statistics as estimator of percentiles.
For example, we can use X_i as the $(100p)^{th}$ percentile of the sample where

$$p = \frac{i}{n+1}$$

Then X_i is an estimator of the population percentile τ_p where the area under the density curve $f(x)$ and to the left of τ_p is p.
In the case that $(n+1)p$ is not an integer, then we interpolate between two order statistics. For example, if $(n+1)p = 5.7$, then we interpolate between X_5 and X_6.

Chapter 4

Order Statistics from Logistic Distribution

4.1 Introduction

In this chapter, we apply the distributions of order statistics and their functions as described in chapter 2 to the case of logistic distribution.

4.2 Notations and Definitions

A random variable Y has the logistic distribution with location parameter a and scale parameter b if its density function is

$$f(y) = \frac{\exp(\frac{y-a}{b})}{b\Big[1 + \exp(\frac{y-a}{b})\Big]^2}, \ -\infty < y < \infty \tag{4.1}$$

In this study, however, we will restrict ourselves to the standard logistic distribution, with the pdf given as,

$$f(y) = \frac{e^{-y}}{(1 + e^{-y})^2}, \ -\infty < y < \infty \tag{4.2}$$

and cdf given as

$$F(y) = \frac{1}{(1 + e^{-y})}, \ -\infty < y < \infty \tag{4.3}$$

Let $Y_1, Y_2, \cdots, Y_n$ denote a random sample of size n from a logistic distribution with pdf $f(y)$ and cdf $F(y)$ as given in equations (4.2) and (4.3) respectively.
We also let $Y_{1:n}, Y_{2:n}, \cdots, Y_{n:n}$ be the associated order statistics obtained by arranging the $Y_i^{'s}$ in increasing order of magnitude.

4.3 Basic Distributional Results and Properties

Here we give the density function of $Y_{i:n}$ $(1 \leq i \leq n)$ as,

$$\begin{aligned} f_{i:n}(y) &= \frac{n!}{(i-1)!(n-i)!}[F(y)]^{i-1}[1-F(y)]^{n-i}f(y) \\ &= \frac{n!}{(i-1)!(n-i)!}[(1+e^{-y})^{-1}]^{i-1}[1-(1+e^{-y})^{-1}]^{n-i}e^{-y}(1+e^{-y})^{-2} \\ &= \frac{n!}{(i-1)!(n-i)!}[(1+e^{-y})^{-1}]^{i-1}[e^{-y}(1+e^{-y})^{-1}]^{n-i}e^{-y}(1+e^{-y})^{-2},\ -\infty < y < \infty \end{aligned} \tag{4.4}$$

For the case $i = 1$ (minimum), the pdf of $Y_{1:n}$ is given by

$$\begin{aligned} f_{1:n}(y) &= \frac{n!}{(1-1)!(n-1)!}[(1+e^{-y})^{-1}]^{1-1}[e^{-y}(1+e^{-y})^{-1}]^{n-1}e^{-y}(1+e^{-y})^{-2} \\ &= \frac{n!}{(n-1)!}[e^{-y}(1+e^{-y})^{-1}]^{n-1}e^{-y}(1+e^{-y})^{-2} \\ &= n[e^{-y}(1+e^{-y})^{-1}]^{n-1}e^{-y}(1+e^{-y})^{-2},\ -\infty < y < \infty \end{aligned} \tag{4.5}$$

The cdf of $Y_{1:n}$ is given by

$$\begin{aligned} F_{1:n}(y) &= 1 - [1-F(y)]^n \\ &= 1 - [1-(1+e^{-y})^{-1}]^n \\ &= 1 - [e^{-y}(1+e^{-y})^{-1}]^n \\ &= 1 - [e^{-ny}(1+e^{-y})^{-n}] \\ &= \Big[\frac{1}{1+e^{-y}}\Big]^n,\ -\infty < y < \infty \end{aligned} \tag{4.6}$$

For the case $i = n$ (maximum), the pdf of $Y_{n:n}$ is given by

$$\begin{aligned} f_{n:n}(y) &= \frac{n!}{(n-1)!(n-n)!}[(1+e^{-y})^{-1}]^{n-1}[e^{-y}(1+e^{-y})^{-1}]^{n-n}e^{-y}(1+e^{-y})^{-2} \\ &= \frac{n!}{(n-1)!}[(1+e^{-y})^{-1}]^{n-1}e^{-y}(1+e^{-y})^{-2} \\ &= n[(1+e^{-y})^{-1}]^{n-1}e^{-y}(1+e^{-y})^{-2},\ -\infty < y < \infty \end{aligned} \tag{4.7}$$

The cdf of $Y_{n:n}$ is given by

$$\begin{aligned} F_{n:n}(y) &= [F(y)]^n \\ &= [(1+e^{-y})^{-1}]^n \\ &= (1+e^{-y})^{-n},\ -\infty < y < \infty \end{aligned} \tag{4.8}$$

The joint density function of $Y_{i:n}$ and $Y_{j:n}$ $(1 \leq i < j \leq n)$ is given by

$$\begin{aligned} f_{i,j:n}(y_i, y_j) &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}[F(y_i)]^{i-1}[F(y_j)-F(y_i)]^{j-i-1} \\ &\quad [1-F(y_j)]^{n-j}f(y_i)f(y_j) \\ &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}[(1+e^{-y_i})^{-1}]^{i-1} \\ &\quad [(1+e^{-y_j})^{-1}-(1+e^{-y_i})^{-1}]^{j-i-1}[1-(1+e^{-y_j})^{-1}]^{n-j} \\ &\quad e^{-y_i}(1+e^{-y_i})^{-2}\cdot e^{-y_j}(1+e^{-y_j})^{-2},\ 1 \leq i < j \leq n \end{aligned} \tag{4.9}$$

Also, the joint density function for all n order statistics is given by

$$\begin{aligned} f_{1,2,\cdots,n:n}(y_1, y_2, \cdots, y_n) &= n!f(y_1)f(y_2)\cdots f(y_n) \\ &= n!\prod_{i=1}^{n} f(y_i) \\ &= n!\prod_{i=1}^{n} \frac{e^{-y_i}}{(1+e^{-y_i})^2},\ 0 \leq y_1 < y_2 < \cdots < y_n < \infty \end{aligned} \tag{4.10}$$

4.4 Expected Values and Moments of Logistic Order Statistics

In this section, we find the single moment $E(Y_{i:n}^k)$ which we will denote by $\sigma_{i:n}^k$ for $1 \leq i \leq n,\ k = 1$ and the product moments $E(Y_{i:n}, Y_{j:n})$ which we will denote by $\sigma_{i,j:n}^k$ for $1 \leq i < j \leq n$ of order statistics from the case of the standard logistic distribution. We also derive the mean, variance and covariance.

We obtain the moments of logistic distribution using the moment generating function (mgf) and the cumulant-generating function (cgf) techniques (for definition of mgf and cgf, see section on Notations, Terminologies and Definitions).

From equation (4.4), we obtain the moment generating function of $Y_{i:n}$ $(1 \leq i \leq n)$ as,

$$\begin{aligned} M_{i:n}(t) &= E[e^{tY_{i:n}}] \\ &= \frac{1}{B(i, n-i+1)} \int_{-\infty}^{\infty} \frac{e^{-(n-i+1)y+ty}}{(1+e^{-y})^{n+1}}\, dy \\ &= \frac{B(i+t, n-i+1-t)}{B(i, n-i+1)} \\ &= \frac{\Gamma(i+t)}{\Gamma(i)} \frac{\Gamma(n-i+1-t)}{\Gamma(n-i+1)},\ 1 \leq i \leq n \end{aligned} \tag{4.11}$$

where $B(\cdot, \cdot)$ and $\Gamma(\cdot)$ are complete beta and gamma functions respectively defined in the definition section.

From equation (4.11), we obtain the cumulant-generating function of $Y_{i:n}$ as

$$\begin{aligned} K_{i:n}(t) &= log M_{i:n}(t) \\ &= log\Big[\frac{\Gamma(i+t)}{\Gamma(i)} \frac{\Gamma(n-i+1-t)}{\Gamma(n-i+1)}\Big] \\ &= log\Gamma(i+t) + log\Gamma(n-i+1-t) - log\Gamma(i) - log\Gamma(n-i+1),\ 1 \leq i \leq n \end{aligned} \tag{4.12}$$

From equation (4.12), we obtain the m^{th} cumulant of $Y_{i:n}$ to be

$$\begin{aligned}
K_{i:n}^{(m)} &= \frac{d^m}{dt^m} K_{i:n}(t)|_{t=0} \\
&= \frac{d^m}{dt^m} \Big[log\Gamma(i+t) + log\Gamma(n-i+1-t) - log\Gamma(i) - log\Gamma(n-i+1) \Big] \\
&= \frac{d^m}{dt^m} log\Gamma(i+t)|_{t=0} + \frac{d^m}{dt^m} log\Gamma(n-i+1-t)|_{t=0} \\
&= \frac{d^m}{dt^m} log\Gamma(i) + \frac{d^m}{dt^m} log\Gamma(n-i+1) \\
&= \phi^{(m-1)}(i) + (-1)^m \phi^{(m-1)}(n-i-1)
\end{aligned} \tag{4.13}$$

where $\phi^{(m)}(\cdot) = \frac{d^m}{dt^m} log\Gamma(\cdot) = \frac{\Gamma'(\cdot)}{\Gamma(\cdot)}$ and denotes a polygamma function of order m.

From equation (4.13) we obtain the mean as;

$$\begin{aligned}
\mu_{i:n} &= K_{i:n}^{(1)} \\
&= \phi^{(1-1)}(i) + (-1)^1 \phi^{(1-1)}(n-i-1) \\
&= \phi(i) - \phi(n-i-1)
\end{aligned} \tag{4.14}$$

Also, the variance is obtained as;

$$\begin{aligned}
\sigma_{i,i:n} &= K_{i:n}^{(2)} \\
&= \phi^{(2-1)}(i) + (-1)^2 \phi^{(2-1)}(n-i-1) \\
&= \phi^{(1)}(i) + \phi^{(1)}(n-i-1)
\end{aligned} \tag{4.15}$$

where ϕ and $\phi^{(1)}$ are the digamma and trigamma functions respectively.

From equation (4.9), we obtain the joint mgf of $Y_{i:n}$ and $Y_{j:n}$ as

$$\begin{aligned}
M_{i,j:n}(t_1, t_2) &= E[e^{t_1 Y_{i:n} + t_2 Y_{j:n}}] \\
&= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \int_{-\infty}^{\infty} \int_{-\infty}^{y_j} e^{t_1 y_i + t_2 y_j} \\
&\quad [F(y_i)]^{i-1} [F(y_j) - F(y_i)]^{j-i-1} [1 - F(y_j)]^{n-j} f(y_i) f(y_j) \; dy_i dy_j
\end{aligned} \tag{4.16}$$

We now make the following transformations

$$s = F(y_i) = \frac{1}{1 + e^{-y_i}}$$

and

$$h = F(y_j) = \frac{1}{1 + e^{-y_j}}$$

This would imply that;

$$e^{y_i} = \frac{s}{1 - s}$$

and

$$e^{y_j} = \frac{h}{1 - h}$$

Re-writing equation (4.16), we get

$$\begin{aligned} M_{i,j:n}(t_1, t_2) &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \int_0^1 \int_0^h \left(\frac{s}{1-s}\right)^{t_1} \left(\frac{h}{1-h}\right)^{t_2} \\ &\quad s^{i-1}[h-s]^{j-i-1}[1-h]^{n-j}\, dsdh \\ &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \int_0^1 \int_0^h \left(\frac{s^{t_1}}{(1-s)^{t_1}}\right) \left(\frac{h^{t_2}}{(1-h)^{t_2}}\right) \\ &\quad s^{i-1}[h-s]^{j-i-1}[1-h]^{n-j}\, dsdh \end{aligned} \tag{4.17}$$

We now expand $(1-s)^{-t_1}$ as an infinite series in powers of s, and get

$$\begin{aligned} M_{i,j:n}(t_1, t_2) &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \sum_{r=0}^{\infty} \frac{(t_1 + r - 1)^{(r)}}{r!} \\ &\quad \int_0^1 \int_0^h s^{t_1+i-1+r}[h-s]^{j-i-1} h^{t_2} [1-h]^{n-j-t_2}\, dsdh \end{aligned} \tag{4.18}$$

where

$$(t_1 + r - 1)^{(r)} = \begin{cases} 1 & \text{if } r = 0 \\ t_1(t_1+1)^{(2)} \cdots (t_1 + r - 1)^{(r)} & \text{if } r \geq 1 \end{cases}$$

But we note that

$$\int_0^h s^{t_1+i-1+r}[h-s]^{j-i-1}\,ds = h^{j+t_1+r-1}B(t_1+i+r, j-i)$$

Hence, re-writing equation (4.18) we get

$$\begin{aligned} M_{i,j:n}(t_1,t_2) &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\sum_{r=0}^{\infty}\frac{(t_1+r-1)^{(r)}}{r!} \\ &\quad B(t_1+i+r,j-i)\int_0^1 h^{j+t_1+t_2+r-1}[1-h]^{n-j-t_2}\,dh \\ &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\sum_{r=0}^{\infty}\frac{(t_1+r-1)^{(r)}}{r!} \\ &\quad B(t_1+i+r,j-i)B(j+t_1+t_2+r,n-j-t_2+1) \\ &= \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n-j+1)}\sum_{r=0}^{\infty}\frac{(t_1+r-1)^{(r)}}{r!} \\ &\quad \frac{\Gamma(t_1+i+r)}{\Gamma(t_1+j+r)}\frac{\Gamma(t_1+t_2+j+r)\Gamma(n-j+1-t_2)}{\Gamma(n-1+t_1+r)} \end{aligned} \tag{4.19}$$

which is the joint mgf of $Y_{i:n}$ and $Y_{j:n}$.

Therefore, from equation (4.19) above, we can obtain the product moments as;

$$\begin{aligned} \sigma_{i,j:n}^{k_1,k_2} &= E[Y_{i:n}^{k_1}, Y_{j:n}^{k_2}] \\ &= \frac{d^{k_1+k_2}}{dt_1^{k_1}dt_2^{k_2}}M_{i,j:n}(t_1,t_2)|_{t_1=t_2=0} \end{aligned} \tag{4.20}$$

Now, if $k_1 = k_2 = 1$, we get the covariance $\sigma_{i,j:n}$ as,

$$\begin{aligned}\sigma_{i,j:n} &= E[Y_{i:n}, Y_{j:n}] \\ &= \frac{d^2}{dt_1 dt_2} M_{i,j:n}(t_1, t_2)|_{t_1=t_2=0} \\ &= \phi^{'}(j) + [\phi(i) - \phi(n+1)][\phi(j) - \phi(n-j+1)] \\ &+ \sum_{r=1}^{\infty} \frac{1}{r} \frac{(i+r-1)^{(r)}}{(n+r)^{(r)}} [\phi(j+r) - \phi(n-j+1)]\end{aligned} \tag{4.21}$$

4.5 Distribution of the Median, Range and Other Statistics

Here we give the distribution of the sample median and the sample range of order statistics from the standard logistic distribution.

4.5.1 Distribution of the sample range

We denote the sample range $Y_{n:n} - Y_{1:n}$ by R_n.

We first obtain the cdf of R_n from equation (2.135) as

$$\begin{aligned}F_{R_n}(r) &= Pr(R_n \leq r) \\ &= n \int_{-\infty}^{\infty} [F(y+r) - F(y)]^{n-1} f(y)\, dy,\ 0 \leq r < \infty\end{aligned} \tag{4.22}$$

We now expand $[F(y+r) - F(y)]^{n-1}$, in equation (4.22), binomially,

$$[F(y+r) - F(y)]^{n-1} = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} [F(y+r)]^{n-1-i} [F(y)]^i \tag{4.23}$$

Hence,

$$\begin{aligned} F_{R_n}(r) &= Pr(R_n \leq r) \\ &= n\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\int_{-\infty}^{\infty}[F(y+r)]^{n-1-i}[F(y)]^i f(y)\,dy \\ &= n\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\int_{-\infty}^{\infty}\frac{e^{-y}}{(1+e^{-r}e^{-y})^{n-1-i}(1+e^{-y})^{i+2}}\,dy \end{aligned} \tag{4.24}$$

We then substitute $s = \frac{1}{(1+e^{-r}e^{-y})}$ in equation (4.24) and obtain,

$$\begin{aligned} F_{R_n}(r) &= Pr(R_n \leq r) \\ &= n\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}e^{-(i+1)r}\Delta_{i:n}(r),\ 0 \leq r < \infty \end{aligned} \tag{4.25}$$

where, if $m = e^{-r} - 1$;

$$\begin{aligned} \Delta_{i:n}(r) &= \int_0^1 s^{n-1}(1+ms)^{-i-2}\,ds \\ &= \frac{1}{(-m)^n}\Big[(-1)^{i+1}\binom{n-1}{i+1}kn(1+m) \\ &+ \sum_{i=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{1}{(k-i-1)}\{(1+m)^{k-i-1}-1\}\Big] \\ &= \frac{1}{(1-e^{-r})^n}\Big[(-1)^i\binom{n-1}{i+1}r + \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{1}{(k-i-1)}\{e^{-(k-i-1)r}-1\}\Big] \end{aligned} \tag{4.26}$$

we set $\binom{n-1}{i+1}$ to zero if $i > n-2$.

Therefore, we substitute $\Delta_{i:n}(r)$ in equation (4.26) into (4.25) and obtain the cdf of the sample range R_n as;

$$\begin{aligned} F_{R_n}(r) &= Pr(R_n \leq r) \\ &= \frac{n}{(1-e^{-r})^n}\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\Big[(-1)^i\binom{n-1}{i+1}re^{-(i+1)r} \\ &+ \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\frac{1}{(k-i-1)}\{e^{-kr}-e^{-(i+1)r}\}\Big],\ 0 \leq r < \infty \end{aligned} \tag{4.27}$$

We therefore, obtain the pdf of the sample range R_n by differentiating equation (4.27) with respect to r.

$$\begin{aligned}
f_{R_n}(r) = & \frac{n^2 e^{-r}}{(1-e^{-r})^{n+1}} \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \Big[(-1)^i \binom{n-1}{i+1} r e^{-(i+1)r} \\
& + \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{1}{(k-i-1)} \{ e^{-kr} - e^{-(i+1)r} \} \Big] \\
& + \frac{n}{(1-e^{-r})^n} \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \Big[(-1)^i \binom{n-1}{i+1} e^{-(i+1)r} \{ 1 - (i+1)r \} \\
& - \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{1}{(k-i-1)} \{ k e^{-kr} - (i+1) e^{-(i+1)r} \} \Big], \ 0 \le r < \infty
\end{aligned} \tag{4.28}$$

where also, $\binom{n-1}{i+1}$ is set to zero if $i > n-2$.

4.5.2 Distribution of the sample median

The density function of the median, $Y_{\frac{n+1}{2}}$, of a random sample of size n (where n is odd) for any density function, $f_M(y)$, is given as,

$$\begin{aligned}
f_M(y) &= \frac{n!}{(\frac{n+1}{2}-1)!(n-\frac{n+1}{2})!} [F(y)]^{\frac{n+1}{2}-1} [1-F(y)]^{n-\frac{n+1}{2}} f(y) \\
&= \frac{n!}{[(\frac{n-1}{2})!]^2} [F(y)(1-F(y))]^{\frac{n-1}{2}} f(y)
\end{aligned} \tag{4.29}$$

Considering a logistic distribution with $f(y)$ and $F(y)$ as in equations (4.2) and (4.3) respectively, then equation (4.29) becomes

$$\begin{aligned}
f_M(y) &= \frac{n!}{[(\frac{n-1}{2})!]^2} e^{-y}(1+e^{-y})^{-2} [(1+e^{-y})^{-1}\{1-(1+e^{-y})^{-1}\}]^{\frac{n-1}{2}} \\
&= \frac{n!}{[(\frac{n-1}{2})!]^2} e^{-y}(1+e^{-y})^{-2} [e^{-y}(1+e^{-y})^{-2}]^{\frac{n-1}{2}}
\end{aligned} \tag{4.30}$$

Which can be summarized as

$$f_M(y) = \frac{n!}{(m!)^2} k k^m \tag{4.31}$$

where $m = \frac{n-1}{2}$ and $k = e^{-y}(1 + e^{-y})^{-2}$.

Chapter 5

Order Statistics from Exponential Distribution

5.1 Introduction

In this chapter, we apply the distributions of order statistics and their functions as described in chapter 2 to the case of exponential distribution.

5.2 Notations and Definitions

A random variable X has the exponential distribution with parameter $\lambda > 0$ if its density function is

$$f(x, \lambda) = \lambda e^{-\lambda x},\ x > 0 \tag{5.1}$$

Then we denote $X \sim Exp(\lambda)$.
We will restrict our study to the standard exponential distribution, which is obtained when $\lambda = 1$. Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed standard exponential, $e(1)$, random variables with a density function

$$f(x) = e^{-x},\ x \geq 0 \tag{5.2}$$

Let $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ be the associated order statistics obtained by arranging the $X_i^{\prime s}$ in increasing order of magnitude.

5.3 Basic Distributional Results and Properties

Here we give the density function of $X_{i:n}, 1 \leq i \leq n$.

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x) \\ &= \frac{n!}{(i-1)!(n-i)!}(1-e^{-x})^{i-1}e^{-(n-i+1)x}, \ 0 \leq x < \infty \end{aligned} \tag{5.3}$$

For the case $i = 1$ (minimum), the pdf of $X_{1:n}$ is given by

$$\begin{aligned} f_{1:n}(x) &= \frac{n!}{(1-1)!(n-1)!}(1-e^{-x})^{1-1}e^{-(n-1+1)x} \\ &= \frac{n!}{(n-1)!}e^{-nx} \\ &= ne^{-nx}, \ 0 \leq x < \infty \end{aligned} \tag{5.4}$$

This shows a remarkable result that the *minimum of n independent standard exponentials is itself an exponential with mean* $\frac{1}{n}$, *i.e. it is distributed exactly as* $e(\theta = 1/n)$.

The cdf of $X_{1:n}$ is given by

$$\begin{aligned} F_{1:n}(x) &= 1 - [1-F(x)]^n \\ &= 1 - [1-(1-e^{-x})]^n \\ &= 1 - [e^{-x}]^n \\ &= 1 - e^{-nx} \end{aligned} \tag{5.5}$$

For the case $i = n$ (maximum), the pdf of $X_{n:n}$ is given by

$$\begin{aligned} f_{n:n}(x) &= \frac{n!}{(n-1)!(n-n)!}(1-e^{-x})^{n-1}e^{-(n-n+1)x} \\ &= \frac{n!}{(n-1)!}(1-e^{-x})^{n-1}e^{-x} \\ &= ne^{-x}(1-e^{-x})^{n-1} \\ &= n\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}e^{-(i+1)x}, \ 0 \leq x < \infty \end{aligned} \tag{5.6}$$

with the cdf of $X_{n:n}$ given by

$$\begin{aligned} F_{n:n}(x) &= F(x)^n \\ &= (1-e^{-x})^n \\ &= (1-\frac{ne^{-x}}{n})^n \\ &\approx exp(-ne^{-x}) \end{aligned} \tag{5.7}$$

$$\therefore \lim_{n\to\infty} F_{n:n}(x) = \lim_{n\to\infty} exp(-ne^{-x}) = 0$$

Similarly, the joint density of $X_{i:n}$ and $X_{j:n}$, $1 \le i < j \le n$, is given by [see David (1981) and Arnold et al. (1992)]

$$\begin{aligned} f_{i,j:n}(x,y) &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}[F(x)]^{i-1}f(x)[F(y)-F(x)]^{j-i-1}f(y)[1-F(y)]^{n-j} \\ &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}(1-e^{-x})^{i-1}(e^{-y}-e^{-x})^{j-i-1}e^{-x}e^{-(n-j+1)y},\ 0 \le x < y < \infty \end{aligned} \tag{5.8}$$

with the joint pdf of $X_{1:n}$ and $X_{n:n}$ given as

$$f_{1,n:n}(x,y) = n(n-1)e^{-(x+y)}[e^{-y}-e^{-x}]^{n-2} \tag{5.9}$$

The joint density function for all n order statistics is similarly given by

$$\begin{aligned} f_{1,2,\cdots,n:n}(x_1,x_2,\cdots,x_n) &= n!f(x_1)f(x_2)\cdots f(x_n) \\ &= n!\prod_{i=1}^{n} f(x_i) \\ &= n!\prod_{i=1}^{n} e^{-x_i} \\ &= n!e^{-\sum_{i=1}^{n} x_i};\ 0 \le x_1 < x_2 < \cdots < x_n < \infty \end{aligned} \tag{5.10}$$

5.4 Expected Values and Moments of Exponential Order Statistics

We now let $Z_{1:n} \leq Z_{2:n} \leq \cdots \leq Z_{n:n}$ denote order statistics corresponding to the standard exponential distribution with density function

$$F(x) = 1 - e^{-x},\ x > 0$$

From the density function of $X_{i:n}$ in equation(5.3), we calculate the general single moment $E(Z_{i:n})^k$ as follows

$$\begin{aligned}
E(Z_{i:n})^k &= E(X_{i:n}^k) \\
&= \int_0^\infty x^k f_{i:n}(x)\ dx \\
&= \frac{n!}{(i-1)!(n-i)!} \int_0^\infty x^k [F(x)]^{i-1}[1-F(x)]^{n-i} f(x)\ dx \\
&= \frac{n!}{(i-1)!(n-i)!} \int_0^\infty x^k [1-e^{-x}]^{i-1}[1-(1-e^{-x})]^{n-i} e^{-x}\ dx \\
&= \frac{n!}{(i-1)!(n-i)!} \int_0^\infty x^k [1-e^{-x}]^{i-1} e^{-(n-i+1)x}\ dx \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} \int_0^\infty x^k e^{-(n-i+j+1)x}\ dx,
\end{aligned}$$

But

$$\begin{aligned}
\int_0^\infty x^k e^{-(n-i+j+1)x}\ dx &= (n-i+j+1)^{-(k+1)} \int_0^\infty t^k e^{-t}\ dt \\
&= \frac{\Gamma(k+1)}{(n-i+j+1)^{(k+1)}}
\end{aligned}$$

Therefore;

$$E(Z_{i:n})^k = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} \frac{\Gamma(k+1)}{(n-i+j+1)^{(k+1)}},\ i = 1, 2, \cdots\ k \geq 1 \tag{5.11}$$

If we let $i = 1$, then

$$\begin{aligned} E(Z_{1:n})^k &= \frac{n!}{(1-1)!(n-1)!}\sum_{j=0}^{0}(-1)^j\binom{1-1}{j}\frac{\Gamma(k+1)}{(n-1+j+1)^{(k+1)}} \\ &= \frac{n!}{(n-1)!}\frac{\Gamma(k+1)}{n^{(k+1)}} \\ &= n\frac{\Gamma(k+1)}{n^{(k+1)}} \\ &= \frac{\Gamma(k+1)}{n^k},\ k > -1 \end{aligned} \tag{5.12}$$

Now, if $i = 2$ and $k > -1$, we have

$$\begin{aligned} E(Z_{2:n})^k &= \frac{n!}{(2-1)!(n-2)!}\sum_{j=0}^{2-1}(-1)^j\binom{2-1}{j}\frac{\Gamma(k+1)}{(n-2+j+1)^{(k+1)}} \\ &= \frac{n!}{(n-2)!}\Big[\frac{(-1)^0\binom{1}{0}\Gamma(k+1)}{(n-1)^{(k+1)}} + \frac{(-1)^1\binom{1}{1}\Gamma(k+1)}{n^{(k+1)}}\Big] \\ &= n(n-1)\Big[\frac{\Gamma(k+1)}{(n-1)^{(k+1)}} - \frac{\Gamma(k+1)}{n^{(k+1)}}\Big] \\ &= n(n-1)\Gamma(k+1)\Big[(n-1)^{-(k+1)} - n^{-(k+1)}\Big] \end{aligned} \tag{5.13}$$

Similarly, for $i = n$ and $k > -1$, we have

$$\begin{aligned} E(Z_{n:n})^k &= \frac{n!}{(n-1)!(n-n)!}\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\frac{\Gamma(k+1)}{(n-n+j+1)^{(k+1)}} \\ &= \frac{n!}{(n-1)!}\Big[\frac{(-1)^0\binom{n-1}{0}\Gamma(k+1)}{1^{(k+1)}} + \cdots + \frac{(-1)^{n-1}\binom{n-1}{n-1}\Gamma(k+1)}{n^{(k+1)}}\Big] \\ &= n\Big[\Gamma(k+1) - 2^{-(k+1)}(n-1)\Gamma(k+1) + \cdots + (-1)^{n-1}n^{-(k+1)}\Gamma(k+1)\Big] \\ &= n\Gamma(k+1)\Big[1 - 2^{-(k+1)}(n-1) + \cdots + (-1)^{n-1}n^{-(k+1)}\Big] \end{aligned} \tag{5.14}$$

We now introduce a theorem to enable us show the mean, variance and other expected values of $Z_{i:n}$.

Theorem 5.4.1. *Let* $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ *be the order statistics from the standard exponential distribution. Then, the random variables* $Z_1, Z_2, \cdots, Z_n$*, where*

$$Z_i = (n-i+1)(X_{(i)} - X_{(i-1)}) \tag{5.15}$$

with $X_{(0)} \equiv 0$*, are statistically independent and also have standard exponential distributions.*

Proof: *From equation (5.10), we saw that the joint density function of* $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ *is*

$$f_{1,2,\cdots,n:n}(x_1, x_2, \cdots, x_n) = n! e^{-\sum_{i=1}^{n} x_i}; \; 0 \leq x_1 < x_2 < \cdots < x_n < \infty$$

We then consider the transformation given in equation (5.15) and obtain;

$$Z_1 = nX_{(1)}, Z_2 = (n-1)(X_{(2)} - X_{(1)}), \cdots, Z_n = X_{(n)} - X_{(n-1)} \tag{5.16}$$

From equation (5.16), we obtain an equavalent transformation below

$$X_{(1)} = \frac{Z_1}{n}, X_{(2)} = \frac{Z_1}{n} + \frac{Z_2}{n-1}, \cdots, X_{(n)} = \frac{Z_1}{n} + \frac{Z_2}{n-1} + \cdots + Z_n \tag{5.17}$$

We then find the jacobian of this transformation as;

$$J(Z_1, Z_2, \cdots, Z_n) = \begin{vmatrix} \frac{\partial X_{(1)}}{\partial Z_1} & \frac{\partial X_{(1)}}{\partial Z_2} & \cdots & \frac{\partial X_{(1)}}{\partial Z_n} \\ \frac{\partial X_{(2)}}{\partial Z_1} & \frac{\partial X_{(2)}}{\partial Z_2} & \cdots & \frac{\partial X_{(2)}}{\partial Z_n} \\ \vdots & \vdots & \vdots & \\ \frac{\partial X_{(n)}}{\partial Z_1} & \frac{\partial X_{(n)}}{\partial Z_2} & \cdots & \frac{\partial X_{(n)}}{\partial Z_n} \end{vmatrix} = \begin{vmatrix} \frac{1}{n} & 0 & \cdots & 0 \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & 0 \end{vmatrix} = \frac{1}{n!}$$

We also let $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} z_i$, *and obtain the joint density function of* $Z_1, Z_2, \cdots, Z_n$ *as;*

$$\begin{aligned}
f_{Z_1,Z_2,\cdots,Z_n}(Z_1, Z_2, \cdots, Z_n) &= |J|\, f_{Z_1,Z_2,\cdots,Z_n}(Z_1, Z_2, \cdots, Z_n) \\
&= |J|\, n! f(Z_1) f(Z_2) \cdots f(Z_n) \\
&= |J|\, n! \prod_{i=1}^{n} e^{-Z_i} \\
&= |J|\, n! e^{-\sum\limits_{i=1}^{n} Z_i} \\
&= \frac{1}{n!} n! e^{-\sum\limits_{i=1}^{n} Z_i} \\
&= e^{-\sum\limits_{i=1}^{n} Z_i},\ 0 \le Z_1 < Z_2 < \cdots < Z_n < \infty
\end{aligned} \tag{5.18}$$

Hence, Z_i *are statistically independent and also have standard exponential distribution.*

From the proof of the theorem (5.4.1), we see that if $Z_{(1:n)}, Z_{(2:n)}, \cdots, Z_{(n:n)}$ is the order statistics corresponding to n $i.i.d$ random variables from a standard exponential distribution, then

$$(Z_{(1:n)}, Z_{(2:n)}, \cdots, Z_{(n:n)}) \stackrel{d}{=} (\frac{w_1}{n}, \frac{w_1}{n} + \frac{w_2}{n-1}, \cdots, \frac{w_1}{n} + \frac{w_2}{n-1} + \cdots + \frac{w_{n-1}}{2} + w_n) \tag{5.19}$$

where $w_1, w_2, \cdots, w_n$ are independent exponential $E(1)$ random variables.

From equation (5.19) and when $k = 1$, we obtain

$$\begin{aligned}
\mu_{i:n} &= E(Z_{i:n}) \\
&= E(Z_{(1:n)}, Z_{(2:n)}, \cdots, Z_{(n:n)}) \\
&= E\Big(\frac{w_1}{n} + \frac{w_2}{n-1} + \cdots + \frac{w_i}{n-i+1}\Big) \\
&= \frac{E(w_1)}{n} + \frac{E(w_2)}{n-1} + \cdots + \frac{E(w_i)}{n-i+1} \\
&= \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-i+1} \\
&= \sum_{j=1}^{i} \frac{1}{n-j+1},\ 1 \le i \le n
\end{aligned} \tag{5.20}$$

Since $E(w_i) = Var(w_i) = 1$, if w_i has a standard exponential distribution. Hence

$$\begin{aligned}
\sigma_{i,i:n} &= Var(Z_{i:n}) \\
&= Var\Big(\frac{w_1}{n} + \frac{w_2}{n-1} + \cdots + \frac{w_i}{n-i+1}\Big) \\
&= \Big(\frac{Var(w_1)}{n} + \frac{Var(w_2)}{n-1} + \cdots + \frac{Var(w_i)}{n-i+1}\Big) \\
&= \frac{1}{n^2} + \frac{1}{(n-1)^2} + \cdots + \frac{1}{(n-i+1)^2} \\
&= \sum_{j=1}^{i} \frac{1}{(n-j+1)^2}, \ 1 \le i \le n
\end{aligned} \tag{5.21}$$

From equations (5.20) and (5.21) we obtain that

$$\begin{aligned}
E(Z_{i:n})^2 &= Var(Z_{i:n}) + [E(Z_{i:n})]^2 \\
&= \sum_{j=1}^{i} \frac{1}{(n-j+1)^2} + \Big[\sum_{j=1}^{i} \frac{1}{n-j+1}\Big]^2
\end{aligned} \tag{5.22}$$

If we compare equations (5.20) and (5.22) with equation (5.11) taking $k = 1$ and $k = 2$, then we get the following identities;

when $k = 1$,

$$\begin{aligned}
E(Z_{i:n}) &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} \frac{\Gamma(1+1)}{(n-i+j+1)^{(1+1)}} \\
&= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^j \frac{\binom{i-1}{j}}{(n-i+j+1)^2}
\end{aligned}$$

Therefore,

$$\sum_{j=1}^{i} \frac{1}{n-j+1} = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^j \frac{\binom{i-1}{j}}{(n-i+j+1)^2} \tag{5.23}$$

when $k = 2$,

$$\begin{aligned} E(Z_{i:n})^2 &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^j \begin{pmatrix} i-1 \\ j \end{pmatrix} \frac{\Gamma(2+1)}{(n-i+j+1)^{(2+1)}} \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^j \begin{pmatrix} i-1 \\ j \end{pmatrix} \frac{2}{(n-i+j+1)^3} \\ &= \frac{2(n!)}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^j \frac{\binom{i-1}{j}}{(n-i+j+1)^3} \end{aligned}$$

Therefore,

$$\sum_{j=1}^{i} \frac{1}{(n-j+1)^2} + \Big[\sum_{j=1}^{i} \frac{1}{n-j+1} \Big]^2 = \frac{2(n!)}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^j \frac{\binom{i-1}{j}}{(n-i+j+1)^3} \tag{5.24}$$

Now, from equation (5.20), we see that

$$E(Z_{1:n}) = \frac{1}{n} \tag{5.25}$$

and from equation (5.21), we see that

$$Var(Z_{1:n}) = \frac{1}{n^2} \tag{5.26}$$

this imply that $Var(Z_{i:n}) \longrightarrow 0$ as $n \longrightarrow \infty$

Also, from equation (5.14), we see that

$$\begin{aligned} E(Z_{n:n}) &= \sum_{j=1}^{n} \frac{1}{(n-j+1)} \\ &= \sum_{j=1}^{n} \frac{1}{j} \sim log(n) \end{aligned} \tag{5.27}$$

this imply that $E(Z_{n:n}) \longrightarrow \infty$ as $n \longrightarrow \infty$

and

$$Var(Z_{n:n}) = \sum_{j=1}^{n} \frac{1}{j^2} \tag{5.28}$$

which tends to $\frac{\pi^2}{6}$ as $n \longrightarrow \infty$.

Since $Z_i;\ i = 1, 2, \cdots, n$ are $i.i.d$ standard exponential random variables, the covariance of $Z_{r:n}$ and $Z_{s:n}$ $(1 \leq r < s \leq n)$ is given by

$$\begin{aligned} \sigma_{r,s:n} &= Cov(Z_{r:n}, Z_{s:n}) \\ &= \sum_{j=1}^{r} Var\Big(\frac{w_i}{n-j+1}\Big) \\ &= \sum_{j=1}^{r} \frac{1}{(n-j+1)^2},\ r \leq s \end{aligned} \tag{5.29}$$

5.5 Distribution of the Median, Range and Other Statistics

Here we give the distribution of the sample range of order statistics from the standard exponential distribution.

5.5.1 Distribution of the sample range

Let a random sample of size n from an exponential distribution $X_i \sim \exp(1)$. We seek to find the pdf of the sample range $R_n = X_{(n)} - X_{(1)}$.

Using the memoryless property of the exponential distribution stated elsewhere Galambos and Kotz (1978) pg. 13, the difference between $X_{(n)}$ and $X_{(1)}$ is independent of the actual value of $X_{(1)}$.

Hence, we find the pdf of R_n by first assuming that $X_{(i)} = 0$.
Thus;

$$\begin{aligned} Pr(R_n < r) &= Pr(n-1 < r) \\ &= \Big(\int\limits_0^r f(x)\,dx\Big)^{n-1} \\ &= \Big(\int\limits_0^r e^{-x}\,dx\Big)^{n-1} \\ &= (1-e^{-r})^{n-1} \end{aligned} \tag{5.30}$$

where $Pr(R_n < r)$ is the probability that the remaining $n-1$ sample observations all fall in the range $(0, r)$.

Differentiating equation (5.30), we obtain the pdf of R_n as;

$$f_{R_n}(r) = (n-1)e^{-r}(1-e^{-r})^{n-2},\ 0 \leq r < \infty \tag{5.31}$$

Chapter 6

Characterizations of Exponential Distribution Based on Order Statistics

6.1 Introduction

In this chapter, we review characterization results related to exponential distribution based on order statistics and in particular by the distributional properties, independence, and moments assumption of order statistics.
There is abundance of characterizations of exponential distribution and among them a considerable part is based on properties of order statistics. Most of them could be found in Ahsanullah and Hamedani (2010), Balakrishnan (1996) Galambos and Kotz (1978), and Johnson et al. (1994).

6.2 Notations and Definitions

Let $X_1, X_2, \cdots, X_n$ denote independent and identically distributed random variables with common exponential distribution having the pdf

$$f(x, \lambda) = \lambda e^{-\lambda x}, \text{ for } x \geq 0 \text{ and } \lambda > 0 \tag{6.1}$$

with cdf given as

$$F(x, \lambda) = 1 - e^{-\lambda x}, \text{ for } x \geq 0 \text{ and } \lambda > 0 \tag{6.2}$$

Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ denote the order statistics of $X_1, X_2, \cdots, X_n$. A characterization of the exponential distribution is shown by considering the identical distribution of the random variables $nX_{1:n}$ and $(n-i+1)(X_{1:n} - X_{i-1:n})$ for one i and one n with $2 \leqq i \leqq n$.

6.3 Characterization Based on Distributional Properties of Order Statistics

In this section, we study various characterization results of exponential distribution by distributional properties of order statistics.

One of the basic characterizations of the exponential distribution states that among the non-degenerate distributions only the exponential distribution has the property that $nX_{1:n}$ for all $n \geq 1$ is distributed as the population.

Theorem 6.3.1. *Let F(x) be a non-degenerate distribution, we let*

$$G(x) = 1 - F(x) \tag{6.3}$$

We assume that

$$G(x_1, x_2, \cdots x_n) = G(x_1)G(x_2)\cdots G(x_n) \tag{6.4}$$

holds for two integral values of n, n_1 *and* n_2 *(say), such that* $(logn_1)/(logn_2)$ *is irrational, and in equation (6.4),* $x_1 = x_2 = \cdots = x_n = x \geq 0$ *is arbitrary. Then* $F(x) = 1 - e^{-\lambda x}$ *where* $x \geq 0, \lambda > 0$

Proof: The result is due to Sethuraman (1965).
Assuming

$$[G(x)]^n = G(nx),\ \forall n \geq 0 \text{ and for } n = n_1 \text{ or } n_2 \tag{6.5}$$

Also, equation (6.5) can be written as

$$G(\frac{x}{n}) = [G(x)]^{\frac{1}{n}} \tag{6.6}$$

By induction, for any integer $N = n_1^s n_2^t$ where s, t are arbitrary integers,

$$[G(x)]^N = G(Nx) \tag{6.7}$$

Using $N = n_1^s n_2^t$, and from elementary mathematics

$$\begin{aligned} y &= logN \\ &= log[n_1^s n_2^t] \\ &= slogn_1 + tlogn_2 \end{aligned} \tag{6.8}$$

Hence, we can obtain two sequences; $s = s(b)$ and $t = t(b)$ such that the corresponding $y = y(b) \to 0$.
Let $y = y(b)$ be such a sequence.
Then by putting

$$g(x) = log[-logG(e^x)]$$

$$z = logx$$

and

$$y = logN$$

we get from equation (6.7),

$$\begin{aligned} & g(y+z) - g(z) = y \\ & \text{or} \\ & \frac{g(y+z) - g(z)}{y} = 1 \end{aligned} \tag{6.9}$$

If $y \to 0$, we have $g^{'}(z) = 1$ (whenever the derivative exists, which in this case it doesn't since y is arbitrary).

Further, if $\{V\}$ is an arbitrary set such that $v \to 0$, then we can construct two sequences $y_1 < v \leq y_2$ such that $y_1 \to 0$ but y_1 is of the form of equation (6.8) and $y_i/v \to 1$ for $i = 1, 2$.

Therefore,

$$\frac{g(y_1+z) - g(z)}{y_1} \cdot \frac{y_1}{v} \leq \frac{g(v+z) - g(z)}{v} \leq \frac{g(y_2+z) - g(z)}{y_2} \cdot \frac{y_2}{v} \tag{6.10}$$

which by equation (6.9), reduce to

$$\frac{y_1}{v} \leq \frac{g(v+z) - g(z)}{v} \leq \frac{y_2}{v} \tag{6.11}$$

Since the two extremes tend to one as $v \to o$, $g^{'}(z)$ exists and equals one.

This now gives

$$G(e^z) = exp(-e^{z+c}) \tag{6.12}$$

Hence,

$$G(x) = e^{-\lambda x},\ \lambda > 0 \tag{6.13}$$

From equation (6.3), we find that

$$F(x) = 1 - e^{-\lambda x},\ x \geq 0, \lambda > 0 \tag{6.14}$$

which was to be proved.

Theorem 6.3.2. *Assume that for any $n \geq 2$, $nX_{1:n}$ has the same distribution $F(x)$ as the population. If $F(x)$ is such that, $x \to 0^+$, lim $F(x) = \lambda > 0$ finite, then $F(x) = 1 - e^{-\lambda x},\ x \geq 0$.*

Proof: Both result and the method of proof are due to Arnold (1971), Gupta (1973) and Galambos and Kotz (1978).

Let

$$n(k) = n^k,\ k \geq 1 \tag{6.15}$$

If $X_1, X_2, \cdots, X_{n(k)},\ \ k \geq 2$ are observations, then we can form the following blocks $(X_1, X_2, \cdots, X_{n(k-1)}),\ (X_{n(k-1)+1}, X_{n(k-1)+2}, \cdots, X_{2n(k-1)}),\ \cdots,$ $(X_{(n-1)n(k-1)+1}, X_{(n-1)n(k-1)+2}, \cdots, X_{n(k)})$ and if we denote the minima of these blocks by $X^{(1)}_{1:n(k-1)}, X^{(2)}_{1:n(k-1)}, \cdots, X^{(n)}_{1:n(k-1)}$ respectively.

Then evidently,

$$X_{1:n(k)} = min(X^{(1)}_{1:n(k-1)}, X^{(2)}_{1:n(k-1)}, \cdots, X^{(n)}_{1:n(k-1)}) \tag{6.16}$$

Since the $X_{i's}$ are independent and identically distributed, so are the $X^{(i)}_{1:n(k-1)}$.

Therefore, for $k = 2$, each $X^{(i)}_{1:n(k-1)}$ is distributed as $X_{1:n}$, which by assumption has distribution $F(nx)$.
By basic fact that (see cdf of $X_{1:n}$ on section (2.3.1)),

$$Pr(X_{1:n} \geq x) = [1 - F(x)]^n \tag{6.17}$$

we have

$$[1 - F(x)]^n = 1 - F(nx) \tag{6.18}$$

Thus by equation (6.16),

$$\begin{aligned} Pr(X_{1:n(2)} \geq x) &= [1 - F(nx)]^n \\ &= 1 - F(n^2 x) \end{aligned} \tag{6.19}$$

Equations (6.16) and (6.17) and induction over k yields

$$\begin{aligned} Pr(X_{1:n(k)} \geq x) &= [1 - F(x)]^{n(k)} \\ &= 1 - F(n(k)x) \end{aligned} \tag{6.20}$$

Therefore, on one hand,

$$Pr\Big(X_{1:n(k)} < \frac{x}{n(k)}\Big) = 1 - \Big[1 - F(\frac{x}{n(k)})\Big]^{n(k)} \tag{6.21}$$

while on the other hand,

$$Pr\Big(X_{1:n(k)} < \frac{x}{n(k)}\Big) = F(x) \tag{6.22}$$

If $n \geq 2$, and $n(k) = n^k$ with $k \geq 1$. We then obtain

$$\begin{aligned} F(x) &= 1 - \Big[1 - F(\frac{x}{n(k)})\Big]^{n(k)} \\ &= 1 - \Big[1 - F(\frac{x}{n^k})\Big]^{n^k} \end{aligned} \tag{6.23}$$

By assuming, $k \to +\infty$,

$$F(xn^{-k}) = axn^{-k} + o(n^{-k}) \tag{6.24}$$

Consequently, using a relation

$$\lim_{s = +\infty} \Big(1 + \frac{y}{s} + o(\frac{1}{s})\Big)^s = e^y \tag{6.25}$$

and equation (6.23), imply that, for any $x > 0$

$$\begin{aligned} F(x) &= 1 - \Big[\lim_{k = +\infty} \Big(1 + \frac{\lambda x}{n^k} + o(\frac{1}{n^k})\Big)^{n^k}\Big] \\ &= 1 - e^{-\lambda x},\ x \geq 0 \end{aligned} \tag{6.26}$$

which is the required proof.

We note that, equation (6.23) can be satisfied for equations other than the exponential if we don't assume that $F(x)/x \to a > 0$ as $x \to 0^+$.

6.4 Characterization Based on Independence of Functions of Order Statistics

In this section, we study various characterization results of exponential distribution by independence of functions of order statistics.

We start by the following result from Fisz (1958).

Theorem 6.4.1. *Let X_1 and X_2 be independent random variables with common cdf $F(x)$. Assume that $F(0) = 0$ and that $F(x)$ is strictly increasing for all $x > 0$. Then $X_{2:2} - X_{1:2}$ and $X_{1:2}$ are independent if and only if, $F(x) = 1 - e^{-\lambda x}$ with some $\lambda > 0$.*

Proof: Using the special case $n = 2$ of theorem (3.1.1) of Galambos and Kotz (1978) pg. 37, shows that $X_{2:2} - X_{1:2}$ and $X_{1:2}$ are independent for the exponential distribution $F(x) = 1 - e^{-\lambda x}$.

Here we now prove the converse, that $X_{1:2}$ and $X_{2:2} - X_{1:2}$ are independent.
Let

$$Pr(X_{2:2} - X_{1:2} < x/X_{1:2} = z) = Pr(X_{2:2} - X_{1:2} < x) \text{ for almost all } z > 0 \tag{6.27}$$

Note that "almost all" can refer to lebesque measure because of the assumptions on $F(x)$ that it is continuous and strictly increasing for all $x > 0$.

Hence,

$$\begin{aligned} Pr(X_{2:2} - X_{1:2} < x/X_{1:2} = z) &= Pr(X_{2:2} < x + z/X_{1:2} = z) \\ &= Pr(X^*_{1:1} < x + z) \end{aligned} \tag{6.28}$$

where $X^*_{1:1}$ is the indicated order statistic from a population with parent distribution given as

$$F^*(x) = \begin{cases} \frac{F(x)-F(z)}{1-F(z)} & \text{if } x \geq z \\ 0 & \text{otherwise} \end{cases} \tag{6.29}$$

If we denote the right hand side (RHS) of equation (6.27) by $H(x)$, hence

$$\frac{F(x+z) - F(z)}{1 - F(z)} = H(x) \tag{6.30}$$

for all $x \geq 0$ and almost all $z > 0$.
If we let $z \to 0$, it implies that $F(x) = H(x)$.

Therefore, if we write

$$\frac{F(x+z) - F(z)}{1 - F(z)} = 1 - \left[\frac{1 - F(x+z)}{1 - F(z)}\right] \tag{6.31}$$

then equation (6.17) becomes

$$1 - F(x+z) = [1 - F(x)][1 - F(z)] \tag{6.32}$$

for all $x \geq 0$ and almost all $z > 0$.
Simplifying equation (6.32) further,

$$\begin{aligned} 1 - F(x+z) &= [1 - F(x)][1 - F(z)] \\ &= 1 - F(z) - F(x) + F(x)F(z) \\ F(x+z) - F(z) &= F(x) - F(x)F(z) \\ &= F(x)[1 - F(z)] \end{aligned}$$

Hence,

$$F(x+z) - F(z) = F(x)[1 - F(z)] \tag{6.33}$$

Now dividing equation (6.33) by $x > 0$, we obtain

$$\frac{F(x+z) - F(z)}{x} = \frac{F(x)}{x}[1 - F(z)] \tag{6.34}$$

If $z \geq 0$ is such that $f(z)$ is defined, then limit of left hand side (LHS) of equation (6.34) is

$$\overset{\lim}{x = 0^+} \frac{F(x)}{x} = \lambda \geq 0 \tag{6.35}$$

Denoting the derivative of the limit of the RHS by $f^+(z)$, we get

$$f^+(z) = \lambda[1 - F(z)] \tag{6.36}$$

Hence, $f(z)$ exists for all $z \geq 0$ and is also given as

$$f(z) = \lambda[1 - F(z)] \tag{6.37}$$

Therefore, the solution of equation (6.37) is the required exponential distribution, hence the proof.

Theorem 6.4.2. *Let X_i $(i = 1, 2, \cdots, n)$ be i.i.d random variables with common cdf $F(x)$. Then $X_{n:n} - X_{n-1:n}$ and $X_{n-1:n}$ are independent if and only if, $F(x) = 1 - e^{-\lambda x}$; $\lambda > 0, x \geq 0$.*

Proof: The proof was provided by Lee et al. (2002).

The joint pdf of $X_{n:n} - X_{n-1:n}$ and $X_{n-1:n}$ is

$$
\begin{aligned}
f_{n-1,n}(x_{n-1:n}, x_{n:n}) &= \frac{n!}{(n-1-1)!(n-n+1-1)!(n-n)!} \\
&\quad [F(x)]^{n-1-1} f(x)[F(y)-F(x)]^{n-n+1-1} f(y)[1-F(y)]^{n-n} \\
&= \frac{n!}{(n-2)!}[1-e^{-\lambda(x_{n-1:n})}]^{n-2}\lambda e^{-\lambda x_{n-1:n}}\lambda e^{-\lambda x_{n:n}} \\
&= \lambda^2 n(n-1)e^{-\lambda(x_{n-1:n}+x_{n:n})}[1-e^{-\lambda x_{n-1:n}}]^{n-2}
\end{aligned}
\tag{6.38}
$$

We then consider the following transformations, $Z_1 = (X_{n:n} - X_{n-1:n})$, $Z_2 = X_{n-1:n}$ and their inverses $x_{n-1:n} = z_1$, $x_{n:n} = z_1 + z_2$.

Since this is a one-to-one transformation, its jacobian is $|J| = 1$.
Thus, from equation (6.38), the joint pdf of Z_1 and Z_2 is

$$g(z_1, z_2) = \lambda^2 n(n-1)e^{-\lambda(z_1+2z_2)}(1-e^{-\lambda z_2})^{n-2} \tag{6.39}$$

We then use beta function, to get the marginal pdf of Z_1 as,

$$g_1(z_1) = \lambda e^{-\lambda z_1} \tag{6.40}$$

Also, the marginal pdf of Z_2 is,

$$g_2(z_2) = \lambda n(n-1)e^{-2\lambda z_2}(1-e^{-\lambda z_2})^{n-2} \tag{6.41}$$

We note that, $g(z_1, z_2) = g_1(z_1) \cdot g_2(z_2)$, hence Z_1 and Z_2 are independent.
That is, $X_{n:n} - X_{n-1:n}$ and $X_{n-1:n}$ are independent for the exponential distribution $F(x) = 1 - e^{-\lambda x};\ \lambda > 0,\ x \geq 0$.

Ahsanullah (1976) showed characterization of the exponential distribution by considering the identical distribution of the random variables $nX_{1:n}$ and $(n-i+1)(X_{1:n} - X_{i-1:n})$ for one i and one n with $2 \leqq i \leqq n$.
We summarize this characterization in the following theorem.

Theorem 6.4.3. *Let X be a non-negative random variable with absolutely continuous distribution function $F(x)$ for all $x \geq 0$ and $F(x)$ for all x.*
Then the following properties are equivalent

1. *X has an exponential distribution with density as given in equation (6.1)*

2. *For one i and one n with $2 \leqq i \leqq n$, the statistics $(n-i+1)(X_{i:n} - X_{i-1:n})$ and $nX_{1:n}$ are identically distributed.*

Proof: For part one, here we show that (1) imply (2).
We let $g(x) = f(x)/(1 - F(x)) = \lambda$
We then consider the joint pdf of $X_{i:n}$ and $X_{i-1:n}$ as,

$$\begin{aligned} f_{i-1,i}(x_{i-1:n}, x_{i:n}) &= \frac{n!}{(i-1-1)!(i-i+1-1)!(n-i)!} \\ &[F(x)]^{i-1-1} f(x)[F(y) - F(x)]^{i-i+1-1} f(y)[1 - F(y)]^{n-i} \\ &= \frac{n!}{(i-2)!(n-i)!}[1 - e^{-\lambda x_{i-1:n}}]^{i-2} \cdot \lambda e^{-\lambda x_{i-1:n}} \cdot \lambda e^{-\lambda x_{i:n}} \\ &\cdot [1 - (1 - e^{-\lambda x_{i:n}})]^{n-i} \\ &= \frac{n!}{(i-2)!(n-i)!}\lambda^2 e^{-\lambda(x_{i:n} + x_{i-1:n})}[1 - e^{-\lambda x_{i-1:n}}]^{i-2} e^{-\lambda(n-i)x_{i:n}} \end{aligned} \tag{6.42}$$

Using the transformation $V_1 = X_{i:n}$ and $V_2 = (n-i+1)(X_{i:n} - X_{i-1:n})$ with their inverses $x_{i:n} = v_1$ and $x_{i-1:n} = v_1 - \frac{v_2}{(n-i-1)}$

It can therefore, be shown that, from equation (6.42)

$$\begin{aligned} g(v_1, v_2) &= \frac{n!}{(i-2)!(n-i)!}\lambda^2 e^{-\lambda(v_1 + v_1 - \frac{v_2}{(n-i-1)})}[1 - e^{-\lambda(v_1 - \frac{v_2}{(n-i-1)})}]^{i-2} e^{-\lambda(n-i)v_1} \\ &= \frac{n!}{(i-2)!(n-i)!}\lambda^2 e^{-2\lambda v_1} \cdot e^{\lambda(\frac{v_2}{(n-i-1)})} \cdot e^{-\lambda(n-i)v_1} \cdot [1 - e^{-\lambda(v_1 - \frac{v_2}{(n-i-1)})}]^{i-2} \\ &= \frac{n!}{(i-2)!(n-i)!}\lambda^2 e^{-2\lambda(n-i)v_1} \cdot e^{\lambda(\frac{v_2}{(n-i-1)})} \cdot [1 - e^{-\lambda(v_1 - \frac{v_2}{(n-i-1)})}]^{i-2} \end{aligned} \tag{6.43}$$

Therefore, $V_2 = (n-i+1)(X_{i:n} - X_{i-1:n})$ is identically distributed as $nX_{1:n}$.

For part two, here we show that (2) imply (1)
Let $Z_i = X_{i:n} - X_{i-1:n}$, the pdf of Z_i is

$$f_{Z_i}(z) = \frac{n!}{(i-2)!(n-i)!} \int_0^\infty [F(x)]^{i-2}[1-F(x+z)]^{n-i} f(x) f(x+z) dx \tag{6.44}$$

If we now let $Y = (n-i+1)z_i$ and substitute in equation (6.44) above, we get the pdf of Y as

$$f_Y(y) = \frac{n!}{(i-2)!(n-i+1)!} \int_0^\infty [F(x)]^{i-2}[1-F(x+\frac{y}{n-i+1})]^{n-i} f(x) f(x+\frac{y}{n-i+1}) dx \tag{6.45}$$

Further, letting $W = nX_{1:n}$, we get the pdf of W as

$$f_W(w) = [1-F(\frac{w}{n})]^{n-1} f(\frac{w}{n}) \tag{6.46}$$

But Y and W are identically distributed (proved in part one), and using the fact that

$$\frac{(i-2)!(n-i+1)!}{n!} = \int_0^\infty [F(x)]^{i-2}[1-F(x)]^{n-i+1} f(x) dx \tag{6.47}$$

Then from equations (6.45) and (6.47), we get

$$0 = \int_0^\infty [F(x)]^{i-2}[1-F(x)]^{n-i+1} f(x) g(x,y) dx, \ \forall y > 0 \tag{6.48}$$

where

$$g(x,y) = [1-F(y/n)]^{n-1} f(y/n) - \{[1-F(x+y/(n-i+1))]/[1-F(x)]\}^{n-i}$$
$$f(x+y/(n-i+1))/[1-F(x)]$$

We then integrate equation (6.48) with respect to y from 0 to y_1 and obtain,

$$0 = \int_0^\infty [F(x)]^{i-2}[1-F(x)]^{n-i+1} f(x) G(x,y_1) dx, \ \forall y_1 > 0 \tag{6.49}$$

where

$$G(x, y_1) = \Big[\frac{1 - F(x + y_1/(n-i+1))}{1 - F(x)}\Big]^{n-i+1} - [1 - F(y_1/n)]^n$$

but

$$G(0, y_1) = 0, \ \forall y_1 > 0$$

Hence,

$$[1 - F(y/(n-i+1))]^{n-i+1} = [1 - F(y/n)]^n, \ \forall y_1 > 0 \tag{6.50}$$

If we now substitute $H(y) = 1 - F(y)$, $\phi(y) = -logH(y)$ and $z = y/n$, then

$$\phi(z) = \Big[\frac{(n-i+1)}{n}\Big]\phi(nz/(n-i+1)), \ \forall z > 0; \text{for one } i \text{ and one } n \text{ with } 2 \leqq i \leqq n \tag{6.51}$$

The solution of equation (6.51) is provided by Aczél and Oser (2006), pg 32 as

$$\phi(z) = cz, \ \text{where } c \text{ is a constant} \tag{6.52}$$

and so

$$F(z) = 1 - e^{-cz} \tag{6.53}$$

Using the boundary condition $F(0) = 0$ and $F(\infty) = 1$, we have

$$F(x) = 1 - e^{-\lambda x}, \ \lambda > 0 \tag{6.54}$$

6.5 Characterization Based on Moment Assumption

In this section, we study various characterization results of exponential distribution through moment assumption.

Let $X_1, X_2, \cdots, X_n$ be independent random variables with cdf $F(x)$.
We use notation

$$E_{r:n} = E(X_{r:n}) \tag{6.55}$$

which is always assumed to be finite.

Using equation (2.14), we have that

$$\begin{aligned} E_{r:n} &= \int_{-\infty}^{\infty} x dF_{r:n}(x) \\ &= r\binom{n}{r} \int_0^1 F^{-1}(t) t^{r-1}(1-t)^{n-r} dt \end{aligned} \tag{6.56}$$

where

$$F^{-1}(t) = inf\{x : F(x) \geq t\}$$

From equation (6.56) and using the relation (see section (2.8.2))

$$(n-r)\binom{n}{r}(1-t) + (r+1)\binom{n}{r+1}t = n\binom{n-1}{r}$$

we have that, for any integers $0 < r < n,\ n \geq 2$

$$(n-r)E_{r:n} + rE_{r+1:n} = nE_{r:n-1} \tag{6.57}$$

Therefore, if $r = r(n)$, arbitrary function of n such that $1 \leq r(n) \leq n$, then the sequence $E_{r(n):n},\ n \geq 1$, where $E_{1:1} = E(X_1)$ uniquely determines all values $E_{r:n},\ 1 \leq r \leq n,\ n \geq 1$.

We thus obtain the following theorem which was pointed by Galambos and Kotz (1978).

Theorem 6.5.1. *The triangular array* $E_{r:n},\ 1 \leq r \leq n,\ n \geq 1,$ *of numbers uniquely determines the population distribution* $F(x)$.

Proof: Let $Z_1, Z_2, \cdots, Z_n$ be *i.i.d* random variables with distribution function $H(x)$. Assuming that, for all $1 \leq r \leq n,\ n \geq 1$, then

$$E(Z_{r:n}) = E_{r:n} \tag{6.58}$$

where $E_{r:n}$ is defined in equation (6.55).

Using equation (6.56), we get

$$\int_0^1 F^{-1}(t)t^{r-1}(1-t)^{n-r}dt = \int_0^1 H^{-1}(t)t^{r-1}(1-t)^{n-r}dt \tag{6.59}$$

substituting $z = 1 - t$ and putting

$$f_r(z) = F^{-1}(1-z)(1-z)^{r-1} \tag{6.60}$$

and

$$h_r(z) = H^{-1}(1-z)(1-z)^{r-1} \tag{6.61}$$

results in

$$\int_0^1 f_r(z)z^k dz = \int_0^1 h_r(z)z^k dz,\ k = n - r \geq 0 \tag{6.62}$$

Both $f_r(z)$ and $h_r(z)$ are non-negative and their integrals are equal, therefore, there exists a constant $c > 0$ such that $c\ f_r(z)$ and $c\ h_r(z)$ are densities over the finite interval $(0, 1)$.

Therefore, from equation (6.62), we see that all moments of two absolutely continuous distributions are equal, provided their distributions are supported by the finite interval $(0, 1)$.

Hence, $F(x) = H(x)$, as claimed.

From theorem (6.5.1) above, we see the following interesting corollaries.

Corollary 6.5.1. *If* $E_{1:n} = \frac{1}{n}$ *for all* $n \geq 1$, *then* $F(x) = 1 - e^{-x},\ x \geq 0$

Proof: If $F(x) = 1 - e^{-x},\ x \geq 0$ then $E_{1:n} = \frac{1}{n}$ for all $n \geq 1$.
Hence, by triangular array of numbers, $E_{r:n}$ is the same for all $1 \leq r \leq n,\ n \geq 1$, as for a population distribution $F(x) = 1 - e^{-x}$. Therefore, theorem (6.5.1) implies our statement.

Corollary 6.5.2. *If* $E_{1:n} = \frac{1}{(n+1)},\ n \geq 1$, *then* $F(x) = x$ *for* $0 \leq x \leq 1$.

Proof: The proof is similar to the preceding one.
Since, $F(x) = x,\ 0 \leq x \leq 1,\ E_{1:n} = \frac{1}{(n+1)}$, theorem (6.5.1) imply that no other population can have this property.

Note, we see from corollaries (6.5.1) and (6.5.2) that the limits of $nE_{1:n}$ as $n \to \infty$, cannot characterize the population distribution.

However, Galambos and Kotz (1978), gave an argument to show that asymptotic values $E_{k:n} \sim h(k, n),\ 1 \leq k \leq n,\ n \to \infty$ may characterize population distributions within some families.

Huang (1974), characterized exponential distribution by expected value $E(nX_{1:n})$, we summarize this result in the following theorem.

Theorem 6.5.2. *If* $F(x)$ *does not degenerate at the origin and if*

$$E(nX_{1:n}) = E(X_1) < \infty,\ \forall n = 2, 3, \cdots \tag{6.63}$$

then $F(x)$ *is an exponential distribution function.*

Proof: We let $\lambda = E(X_1) = E(nX_{1:n})$. Since $E(X_{1:n})$ is a decreasing function of n, then $\lambda > 0$.
We first show that, for some $n \geq 2$, $nX_{1:n}$ is identically distributed as X_1, then X_1 possesses finite moments of all order.
If $nX_{1:n}$ is identically distributed as X_1 for some $n \geq 2$. Then for $i = 0, 1, 2, \cdots$

$$1 - F(n^i x) = [1 - F(x)]^{n^i},\ \forall x \tag{6.64}$$

Specifically, if we let $p = 1 - F(1)$. Then

$$1 - F(n^i) = p^{n^i} \tag{6.65}$$

Suppose $nX_{1:n} \overset{d}{=} X_1$. Then

$$F(x) \geqslant 0, \text{ for } x \lesseqgtr 0 \tag{6.66}$$

We further, let k be a positive integer and from equation (6.66)

$$\begin{aligned} E(X_1^k) &= \int_0^\infty Pr(X_1^k > u)du \\ &= \int_0^\infty kt^{k-1}[1 - F(t)]dt \\ &= \int_0^1 kt^{k-1}[1 - F(t)]dt + \sum_{m=0}^{\infty} \int_{n^m}^{n^{m+1}} kt^{k-1}[1 - F(t)]dt \end{aligned} \tag{6.67}$$

This shows that X_1 possesses finite moments of all order.

We note that the result of the proof of X_1 possessing finite moments is a weakening of the theorem (6.5.2) above, hence, $E(nX_{1:n}) = E(X_1)$ for some n and therefore, $F(x)$ is exponential.

Theorem 6.5.3. *If $E(X_1)$ is finite and if $F(x)$ is continuous, then $E(X_{2:2} - X_{1:2}/X_{1:2} = y)$ is constant almost surely with respect to $F(x)$, if and only if, $F(x)$ is exponential.*

Proof: The result and the proof is due to Ferguson (1967).
Since for any y such that $0 < F(y) < 1$,

$$E(X_{2:2} - X_{1:2}/X_{1:2} = y) = E(X_{2:2}/X_{1:2} = y) - y \tag{6.68}$$

Therefore,

$$E(X_{2:2} - X_{1:2}/X_{1:2} = y) = \int_y^\infty x dF^*(x) - y \tag{6.69}$$

where

$$F^*(x) = \frac{F(x) - F(y)}{1 - F(y)}, \text{ for } x \geq y \tag{6.70}$$

If the LHS of equation (6.69) is constant almost surely then equation (6.69) is equivalent to

$$\int_{y}^{\infty} x dF^{*}(x) = c + y \quad \text{a.s. (with respect to) } F \tag{6.71}$$

or to

$$\int_{y}^{\infty} x dF(x) = (c + y)[1 - F(y)] \quad \text{a.s. (with respect to) } F \tag{6.72}$$

where $0 < F(y) < 1$, and equation (6.72) is valid for all y.

Therefore,

$$F(y) = 1 - exp\Big[- \frac{1}{c}(y - B)\Big] \tag{6.73}$$

where B is an arbitrary constant. Since $F(y)$ is continuous by assumption, equation (6.73) is valid for all $y \geq B$, hence the proof.

Theorem 6.5.4. *Let $F(x)$ be continuous and assume that $E(X_1)$ is finite. If $\overline{X}$ denotes the arithmetic mean $(\frac{1}{n})(X_1 + X_2 + \cdots + X_n)$ and if $E(\overline{X} - y/X_{1:n} = y)$ is constant almost surely (with respect to F), then $F(x)$ is exponential.*

Proof: First, for conditional distribution of X_i given $X_{1:n} = y$, we have that

$$\begin{aligned} Pr(X_i < x/X_{1:n} = y) &= \frac{Pr(X_i < x) - Pr(X_{1:n} = y)}{Pr(X_{1:n} = y)} \\ &= \frac{1}{n} + \frac{n-1}{n}\frac{F(x) - F(y)}{1 - F(y)}, \text{ if } x > y \end{aligned} \tag{6.74}$$

But the conditional distribution is zero for $x \leq y$.

Therefore,

$$E(X_i/X_{1:n} = y) = \frac{1}{n} + \frac{n-1}{n[1 - F(y)]} \int_{y}^{\infty} x dF(x) \quad \text{a.s. (with respect to) } F \tag{6.75}$$

We observe that the RHS of equation (6.75) doesn't depend on i.
Thus denoting the expectation by

$$E(X_i/X_{1:n} = y) = g_n(y) \tag{6.76}$$

Then

$$E(\overline{X}/X_{1:n} = y) = \frac{1}{n}\sum_{i=1}^{n} E(X_i/X_{1:n} = y) \tag{6.77}$$

equals $g_n(y)$.

However,

$$E(\overline{X}/X_{1:n} = y) = y + c_n \tag{6.78}$$

Therefore,

$$g_n(y) = y + c_n \quad \text{a.s } (F) \tag{6.79}$$

We note that from equation (6.75),

$$\frac{1}{n-1}E(nX_i - y/X_{1:n} = y) = \frac{1}{n-1}E(nX_j - X_{1:n}/X_{1:n} = y) \tag{6.80}$$

which doesn't depend on n. Hence, its value is the same for all n.

Specifically, for $n = 2$, we have from equations (6.76) and (6.79)

$$\begin{aligned} E(2X_i - X_{1:2}/X_{1:2} = y) &= 2E(X_i/X_{1:2} = y) - y \\ &= y + c_2 \end{aligned} \tag{6.81}$$

Since $E(X_1/X_{1:2} = y) = E(X_2/X_{1:2} = y)$, we have that,

$$\begin{aligned} E(2X_i - X_{1:2}/X_{1:2} = y) &= E(X_1 + X_2 - X_{1:2}/X_{1:2} = y) \\ &= E(X_{1:2} + X_{2:2} - X_{1:2}/X_{1:2} = y) \\ &= E(X_{2:2}/X_{1:2} = y) \end{aligned} \tag{6.82}$$

Combining equations (6.81) and (6.82), we get

$$E(X_{2:2}/X_{1:2} = y) = y + c_2 \quad \text{a.s } (F) \tag{6.83}$$

which is directly the assumption of theorem (6.5.3). Therefore, we get conclusion of theorem (6.5.4) from theorem (6.5.3).

Chapter 7

Conclusion and Recommendation

Two methods of constructing order statistics distribution have been highlighted. Traditionally, order statistics distributions have been constructed using the transformation method. However, this study examined also the use of beta generated distributions approach fronted by Jones in 2004 in constructing order statistics. We therefore, extended this technique and constructed order statistics distribution for various distributions.

For order statistics from standard uniform distribution, the r^{th} order statistic from a random sample of size n has a $Beta(r, n-r+1)$ distribution and the r^{th} and s^{th} order statistics jointly have a bivariate $Beta(r, s-r, n-s+1)$ distribution.

For order statistics from standard exponential distribution, the pdf of the first (i=1) order statistic shows a remarkable result that the minimum of n independent standard exponentials is itself an exponential with mean $\frac{1}{n}$.

Characterization of a probability distribution is an important tool during usage and application of the distribution. We have highlighted a number of characterization methods for the exponential distribution, particularly, the principles underlying the characterization by distributional properties, independence of functions and moment assumption of order statistics. For each method, we gave the characterization theorems and the associated proofs, corollaries and references.

We would like to recommend further research on the use of beta generated distributions approach in constructing the distributions of order statistics.

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