THE RELIABILITY OF CAPITAL ASSET PRICING MODEL ON VALUATION OF LISTED FIRMS AT THE NAIROBI SECURITIES EXCHANGE

BY

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DECLARATION

This Research Project is my original work and has not been presented for an award of a degree in any other university or institute of learning.

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I56/61270/2013

This Research Project has been submitted for examination with my approval as the University supervisor.

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ABSTRACT

One of the most important modern financial theories that has dominated capital markets and influenced the field of finance and business for over the past five decades is the Capital Asset Pricing Model. However, due to its empirical shortcomings several researchers have attempted to improve CAPM and its assumptions by advancing various extensions of the model. The objective of this study was to conduct empirical tests of CAPM and its extensions on the Nairobi Securities Exchange and determine other applications of CAPM in insurance. Using decade-long data from forty seven listed companies we undertake the weekly study from January 2004 – December 2013.

We find that CAPM is not fully supported by the empirical data. Alphas for a majority of the NSE stocks are significantly different from zero whereas all the betas were significant as expected. The Security Market Line was linear and a positive correlation was observed between beta and returns but the SML intercept was significantly different from zero.

The study was extended to predict ex-ante beta using CAPM for year 2014 and compare with the realized ex-post beta during the year. The test assumes mean reversion of returns and reveals high forecast errors which negate CAPM as a true predictor of ex-ante betas. Zero-Beta CAPM indicated a much lower correlation and was forthwith rejected at the NSE. On the other hand, the Fama-French Three Factor Model reveals a higher correlation with NSE but also failed crucial compatibility tests.

The CAPM is therefore not a robust tool to predict the NSE and other model extensions need to be tested for conformity with the Kenyan market. Additional tests can help eliminate errors caused by unrealistic assumptions such as lack of wealth consumption, foreign investment, varying beta and risk premia. In case these extensions fail, then CAPM can be wholly rejected as a model of asset pricing at the NSE.

Key Words: *Capital Asset Pricing Model, Nairobi Securities Exchange, Systematic risk, Non-Systematic risk*

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CHAPTER 1: INTRODUCTION

William Sharpe (1964) and John Linter (1965) constructed the well-known Capital Asset Pricing Model (CAPM) showing that the true measure of risk for stocks is beta. The model's widespread popularity stems from its intuitive appeal, computational ease and was an extension of the earlier work of Harry Markowitz' (1959) mean-variance portfolio model. A Nobel Prize was later awarded to Sharpe as well as Harry Markowitz and Merton Miller in 1990 for their enormous contribution to financial theory. CAPM is a pricing theory that applies to a financial market under the condition of single period, perfect competition, and no conflict. The model itself is applied to obtain the linear relationship between risk and return of a financial asset based on numerous restricted assumptions.

The assumptions made were that all investors focus on a single holding period, and they seek to maximize the expected utility of their terminal wealth by choosing among alternative portfolios on the basis of each portfolio's expected return and standard deviation. All investors can borrow or lend an unlimited amount at a given risk-free rate of interest and there are no restrictions on short sales of any assets. All investors have identical estimated of the expected returns, variances, and covariance among all assets (that is, investors have homogeneous expectations). All assets are perfectly divisible and perfectly liquid (that is, marketable at the going price). There are no transaction costs and no taxes. All investors are price takers (that is, all investors assume that their own buying and selling activity will not affect stock prices).

The purpose of the assumptions related to financial asset market is to obtain a clear representation of the relationship between the expected return and the risk in the market, so that the inefficiency due to transaction costs, taxes, and the delay of information is avoided in the application of the model.

Although the assumptions do not perfectly match the realistic situation, they simplified the issues to generate the pricing model of the financial asset. Under the restricted assumptions, William Sharpe and Linter obtained the linear relationship between the expected return and the risk of the asset.

The model of CAPM can be presented as:

$E[R_i] = R_f + \beta_i E[(R_m) - R_f]$

R_i : The expected return of the stock i.

R_f: Risk free rate normally is the treasury bond rate with 90-day maturity.

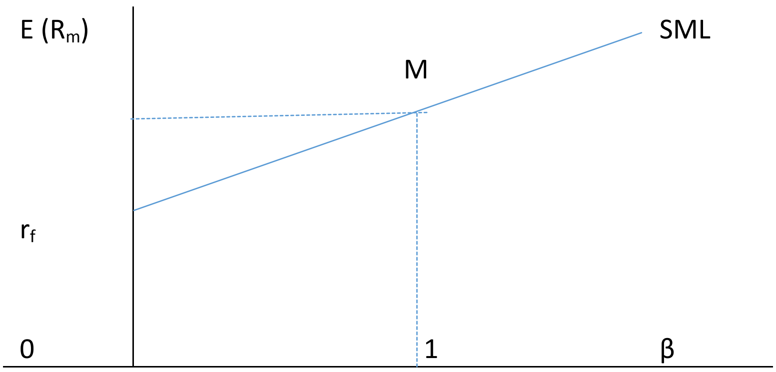
R_m: The expected return of the market portfolio.

β_i: β measures the systematic risk of stock i, or, it represents the reaction level of a financial asset's expected return towards the risk of it. Generally, it is defined as the volatility of an asset in relation to the volatility of the benchmark that said asset is being compared to.

The model of CAPM presents that the return of a risky assets consists of two parts, one is the return of risk free assets, represented by R_f; and the other part of the return of risky assets is the return of risky market, represented by $(R_m) - R_f$. β represents the level of the risk of the financial market, which indicates that a higher risk is always accompanied with a higher return.

CAPM suggests that not all the risk of the assets would need a remedy of the risk; the one with a remedy is systematic risk. Due to the reason that systematic risk cannot be reduced and eliminated through diversification, an accompanying return with the risk can attract investors; in contrast, unsystematic risks can be diversified; therefore, a remedy in order to attract investors is not needed. CAPM also indicates that the best portfolio is market portfolio due to the smallest risk it has among all, all the risky asset investors would prefer market portfolio.

The approach of defining the risk-return relationship of a single financial asset leads to the development of the security market line (SML). As stated in Graph 1, Stock Market Line indicates the risk-return relationship of a single stock under the condition of an efficient market. When efficient market applies, the relationship formula of a single stock risk and its return, the SML, can be obtained.



Graph 1: Security Market Line

The SML represents the investment's opportunity cost after investing in a combination of the market portfolio and the risk-free asset. The slope of the CAPM is equal to the market risk premium and displays the levels of risk against the expected rate of return of the entire market at a given point in time. It is obvious that, based on the model of CAPM, for any single stocks, the expected return increases with an increase of β, therefore, it is a positive linear relation between the value of β and the expected return of any single stocks.

CAPM can be used in classification of assets. According to the model of CAPM, an effective classification of assets can be done through the result of CAPM application. By using the risk factor β in CAPM to classify stocks, the classification of stocks can avoid risks and realize the returns for investors. As an example, when β>1, for instance, β=2, when the market portfolio return increased by 1%, such stocks would have an expected return increase 2%; when the market portfolio return decreased by 1, however, such stocks would have an expected return decrease 2%.

Therefore, it is obvious that the stocks in this category carries a relatively higher risk than that with a β=1. When β=1, such stocks will have a volatility same to the market volatility, and also they accurately reflect the market portfolio price change; when β<1, such stocks are more defensive compared to those discussed above. Apparently, different stocks carry different characteristics of return, based on which, an efficient financial asset management can be conducted according to the investors risk preference and their investment profile.

Further, CAPM is used in pricing financial assets, providing investment guidance to investors CAPM is a forecasting model for expected return of risky financial assets based on a balanced risk-return relationship; however, in the real life markets, returns of stocks are not balanced. Assuming that the calculated expected return is balanced, and a comparison can be conducted, so that the under-priced and overpriced financial assets can be easily explored. Furthermore, based on the rule of investing in low, shorting in high, such an application is guiding the investment behaviour of investors.

1.1 PROBLEM STATEMENT

Various asset pricing models have been postulated by scholars over the years but the challenge of choosing reliably profitable investments still continues to concern investors and financial institutions. How well do these economic theories fare in explaining asset prices and returns in actual markets and what other applications are derived from these theories.

The current research in this area seems to provide no conclusive support for the CAPM model. Several research studies have relied upon short NSE data periods and few stocks and have not studied the model extensions and its various applications.

1.2 OBJECTIVES OF STUDY

1. We carry out a mathematical and empirical study at how CAPM was derived, what difficulties are faced in its applications and what various extensions were put forward to address its shortcomings.
2. We shall examine whether CAPM is applicable in the Kenyan bourse - Nairobi Securities Exchange - and if it can be reasonably relied upon in explaining and predicting stock prices to aid in capital budget decision making.
3. In addition, we shall seek to find out how CAPM has been applied by actuaries in asset liability management of insurance firms and pricing of insurance premiums.

This study conducts the application and analysis of CAPM and its extensions at the Nairobi Securities Exchange over the decade commencing from year 2004 to year 2013. Chapter 1 mainly provides the background, objectives and introduction of the CAPM. Chapter 2 discusses the development of asset pricing theories, literature in existence for the CAPM model and research undertaken in different markets. Chapter 3 develops the mathematical derivation of CAPM, its critique and various extensions. In Chapter 4, the applications of the CAPM model in the insurance field will be discussed. Chapter 5 studies the data methodology, sources, test results and analysis of the results for the standard CAPM, Zero-Beta CAPM and Fama-French Three Factor Models. Further, an ex-ante beta forecast for year 2014 will be carried out using CAPM prediction and results compared with the actual beta realized in the NSE. Chapter 6 will provide the conclusion of the study and related recommendations. References used in the text shall be enumerated in the final section.

CHAPTER 2: LITERATURE REVIEW

The Capital Asset Pricing Model (CAPM) has been discovered and discussed since early 1960's through both theoretical and empirical aspects. Due to the long history of CAPM, the authors and researchers conducted numerical accomplishments related to the model itself and empirical cases, which is the main reason why CAPM has been developed into many directions as it is various depending on the application circumstances. Several authors have contributed to development of a model describing the pricing of capital assets under condition of market equilibrium including Eugene Fama, Michael Jensen, John Lintner, John Long, Robert Merton, Myron Scholes, William Sharpe, Jack Treynor and Fischer Black, some of whose findings will be discussed in this chapter of literature review. The CAPM was developed as a result of the mean-variance model which finds its basis on Bernoulli's utility theory.

2.1 HISTORICAL ORIGINS

Daniel Bernoulli (1738) wrote a remarkable paper, originally in Latin which was referenced widely in the fields of mathematics, logic and subsequently economics. This paper was available in English only after the 1950s.

At the beginning of the paper Bernoulli states, that all mathematicians who had studied the measurement of risk agree, that:

"Expected values are computed by multiplying each possible gain by the number of ways in which it can occur, and then dividing the sum of these products by the total number of cases where, in this theory, the consideration of cases which are all of the same probability is insisted upon."

Before Bernoulli turns to the famous Petersburg Paradox, he starts with an observation from the multiple publications on the topic, all claiming that: "since there is no reason to assume that of two persons encountering identical risks, either should expect to have his desires more closely fulfilled, the risks anticipated by each must be deemed equal in value."

This is a formulation of the principle of sufficient reason. (This states that nothing is without a ground or reason why it is the way it is). He further tries to formulate the problem in general terms by stripping individual characteristics of the persons themselves and banning them from the consideration; only those matters should be weighed carefully that pertain to the terms of the risk.

Bernoulli deduces from his concept of utility a fundamental rule: *If the utility of each possible profit expectation is multiplied by the number of ways in which it can occur, and we then divide the sum of these products by the total number of possible cases, a mean utility [moral expectation] will be obtained, and the profit which corresponds to this utility will equal the value of the risk in question.*

For Bernoulli it becomes evident that no valid measurement of the value of a risk can be obtained without consideration being given to its *utility.*

However, it hardly seems plausible to make any precise generalizations since the utility of an item may change with circumstances. The question if this kind of problem is accessible to mathematics was answered by Bernoulli with: yes. He claims "Now it is highly probable that *any increase in wealth, no matter how insignificant, will always result in an increase in utility which is inversely proportional to the quantity of goods already possessed.*"

This is the first published exposition of the *Principle of Decreasing Marginal utility*. This principle, later widely accepted in the theory of economic behaviour, states that marginal utility (the extra utility obtained from consuming a good) decreases as the quantity consumed increases; in other words, that each additional good consumed is less satisfying than the previous one.

Bernoulli's concept of decreasing marginal utility became central to economics, notably in the works of Jevons, Menger, Walras and Marshall. Mainstream economics originates from Jevons' and Menger's marginal utility and Walras' and Marshall's equilibrium approach. Menger and Jevins each managed to ground a theory of economic exchange on the fact that people are rational enough to choose which of two goods will provide them with the greatest benefit.

Bernoulli also introduced the concept of **maximization of expected utility**. However, despite endorsement by Laplace and others, Bernoulli's approach had little impact on the economics of decision making under risk until the formal development of the expected utility theory by Savage (1954) and Von Neuman and Oscar Morgenstern (1944, 1947) in their book Theory of Games and Economic Behaviour. The von Neumann-Morgenstern expected utility model is not without its limitations. One limitation is that it treats uncertainty as objective risk – that is, as a series of coin flips where the probabilities are objectively known. Savage's (1954) approach to choice under uncertainty, which rather than assuming the existence of objective probabilities attached to uncertain prospects makes assumptions about choice behaviour and argues that if these assumptions are satisfied, a decision-maker must act as if she is maximizing expected utility with respect to some subjectively held probabilities.

A point worth noting is that Savage's theory is about single person decision problems. If we imagine a situation where two Savage guys, A and B, are faced with uncertainty, there's no particular reason why when they form their subjective probabilities these probabilities should be the same (as would naturally be the case in a von Neumann-Morgenstern world where the probabilities are objectively specified). Nevertheless, in economic modelling, it is standard to assume that, if they have access to the same information, agents will form common subjective probabilities. This is called the common prior assumption, and is often identified with Harsanyi's (1968) development of Bayesian games. Essentially it holds that "differences in opinion are due to differences in information."

An old idea in the economics literature, dating back at least to Frank Knight (1921), is that a distinction should be drawn between risk (i.e. situations where it might be possible to assign probabilities) and uncertainty (i.e. situations where one is just clueless). According to this definition, the von Neumann-Morgenstern theory deals clearly with risk. Also, while Savage's theory does not assume known probabilities, it is nevertheless a model of risk in this sense – people at least behave as if they are assigning probabilities.

Arrow-Debreu model was developed as a model of general equilibrium that has been fundamental to economics and finance. Compared to earlier models, the Arrow-Debreu model basically generalized the notion of a commodity, differentiating commodities by time and place of delivery. For example, "apples in Malaysia in July" and "apples in Singapore in June" are considered as different commodities.

General equilibrium theory is concerned with the allocation of commodities (between nations, or individuals, across time, or under uncertainty, etc.). The Arrow-Debreu model studies those allocations which can be achieved through the exchange of commodities at one moment in time. When the descriptions are so precise that further refinements cannot yield imaginable allocations which increase the satisfaction of the agents in the economy, then the commodities are called Arrow-Debreu commodities.

Kenneth J. Arrow (1951) and Gerard Debreu (1951) work together to produce the first rigorous proof of the existence of a market clearing equilibrium, given certain restrictive assumptions. One of their key contributions is to introduce time and uncertainty into general equilibrium models.

First of all, it solves the long-standing problem of proving the existence of equilibrium in a Walrasian (competitive) system. This model analyses the exact situations of those markets that are very competitive. In economics, Arrow-Debreu model suggests that a set of prices such as aggregate supplies will equal to aggregate demands for every commodity under certain assumptions made about the economic conditions (i.e. perfect competition and demand independence).

With a general equilibrium structure, the model is applicable in evaluating the overall impact on resource allocation of policy changes in areas such as taxation, tariff, and price control. The functions of Arrow-Debreu model can be divided into six categories, asset pricing model, equity risk premium, corporate finance, Modigliani and Miller Theorem, Arrow-Debreu security and others.

An important feature of Arrow and Debreu (1954) is its conscious use of explicitly game-theoretic ideas, previously found in the related paper by Debreu (1952) on his own. However, a generalization of the usual notion of a game is involved. For there is an auctioneer whose strategy choice determines the price vector. Given this choice, agents are then constrained to choose net trades within their budget sets. Thus the strategic choice of the auctioneer limits the strategies that the other players are allowed to choose.

Markowitz (1952) in an article on Portfolio Selection postulates that an investor should maximize portfolio expected return $(\overline{r})$ while minimizing portfolio variance (σ_p^2). Markowitz paper is the first mathematical formalization of the idea of diversification of investments. It comes to a conclusion that through diversification, risk can be reduced (but not generally eliminated) without changing expected portfolio return.

One of the most important aspects of Markowitz's work was to show that it is not a security's own risk that is important to an investor, but rather the contribution the security makes to the variance of his entire portfolio - and that this was primarily a question of its covariance with all the other securities in his portfolio.

This school of thought formed the basis of the famous Markowitz problem which explicitly addresses the trade-off between expected rate of return and variance of the rate of return in a portfolio. A major set-back of the Markowitz model was the large data and computational requirements in analysis of a portfolio with many assets.

Tobin, on the other hand, realized that investors have a full range of liquidity preferences, and expanded potential investor choices to include low risk assets. He stated that investors should first determine their appetite for risk. This appetite should be satisfied from the one dominant equity portfolio determined from a Markowitz optimization (the single portfolio on the efficient frontier with the highest return per unit of risk). Then the liquidity and safety needs are satisfied with the local zero risk portfolio. In essence the investor has two buckets and need only choose how to divide his assets between them. So, each investor owns the same equity portfolio, but tempers the liquidity needs and risk-reward profile with different proportions of zero risk assets. His proposition was known as the “Separation Theorem”.

Treynor’s paper of 1961 had the intention to ‘lay the ground work for a theory of market value which incorporates risk’. In essence, the aims of the paper were to:

- Demonstrate that the overall behaviour of the agents leads to Proposition 1 of Modigliani and Miller (1958)
- Investigate the relation between risk and investment value
- Distinguish between insurable risk and uninsurable risk.

Treynor approached CAPM from the perspective of corporate cost-of-capital decision making hence striving to understand the relation between risk and the discount rate. Proposition 1 of Modigliani and Miller (1958) says that “In equilibrium, the market value of any firm is independent of its capital structure and is given by capitalizing its expected return at the rate that is appropriate for its class.”

The assumptions that Treynor gave included:

- No taxes
- No market frictions
- Trading does not affect prices
- Agents (investors) maximize utility in the essence of Markowitz
- Agents are risk averse
- A perfect lending market exists
- Agents have identical market knowledge and agree in their forecasts of future values

Sharpe, 1964, also set out to evaluate the relationship between the prices of assets and their risk attributes. Sharpe in 1964, noted that ‘through diversification, some of the risk inherent in an asset can be avoided so that its total risk is obviously not the relevant influence in its price; unfortunately little has been said concerning the particular risk component which is relevant.’

Sharpe (1964) and Lintner (1965) added some two key assumptions to the Markowitz model of the mean-variance portfolio.

- Given market clearing asset prices at t-1, investors agree on the joint distribution of asset returns from t-1 to t. And this is the distribution from which the returns we use to test the model are drawn from.
- Borrowing and lending is at a risk free rate, which is similar for all investors and does not depend on the amount borrowed or lent.

These assumptions meant that the market portfolio must be on the minimum variance frontier if the asset is to clear.

Michael Brennan (1970) derived the after tax CAPM where the before-tax return of stocks was positively related to the tax burden of equity securities. In his model, stocks paying higher dividend yields exhibit higher risk-adjusted returns than stocks paying lower or no dividends.

Fischer Black (1972) developed a version of the CAPM without risk free borrowing and lending; commonly known as the zero-beta CAPM. He showed that the CAPM result of market portfolio being mean-variance-efficient could also be achieved by allowing unrestricted short sales of risky assets. This implies that if there is no risk free asset, investors select portfolios from along the mean-variance-efficient frontier. The market portfolio thus becomes a portfolio of the efficient portfolios chosen by investors.

Mayers (1972) shows that when the market portfolio includes non-traded assets, the model also remains identical in structure to the CAPM. Solink (1974) and Black (1974) extended the model to encompass international investment.

Treynor and Black (1973) showed how best to construct a combination of active and passive portfolios. They do this by linking the CAPM with Sharpe (1963) index model. They explain when a portfolio should choose to run an almost perfectly diversified index (passively), and how the portfolio's diversification should vary with the prospects for the stocks in which the portfolio is invested; they also provide the first analysis to underpin market-neutral hedge funds. Modern portfolio optimisation and risk management systems are often extensions of the Treynor-Black model.

2.2 EMPIRICAL TESTS OF THE CAPM

Tests of the CAPM were based on three implications of the relation between expected return and market beta implied by the model:

- Expected returns on all assets are linearly related to their betas and no other variable has marginal explanatory power.
- The beta premium is positive; the expected return on the market portfolio exceeds the expected return on assets whose return is uncorrelated with the market return.
- In the Sharpe-Lintner version of the model, assets uncorrelated with the market have expected returns equal to the risk free interest rate, and the beta premium is the expected market return less the risk free rate.

In the case of Treynor-Sharpe-Lintner-Mossin CAPM, the slope of this line should be equal to the market risk premium, and the intercept should be equal to the risk free rate. For the zero-beta CAPM, the slope should be less than the market risk premium, while the intercept should be greater than the risk free rate. There should also be no systematic reward for non-market risk.

Black et al (1972) performed the earliest tests of the CAPM. The tests focused on the Treynor-Sharpe-Lintner model's predictions on the intercept and slope in the relation between expected return and market beta. The relationship between the mean excess return and the beta was linear and this was consistent with some type of CAPM. It however found that:

- Estimates of beta for individual assets were imprecise thus creating a measurement error problem when used to explain average returns
- The regression residuals had common sources of variations

To improve the precision of estimated betas, researchers such as Blume (1970), Friend and Blume (1970) and Black, Jensen and Scholes (1972) worked with portfolios rather than individual assets. Using portfolios in cross-section regressions of average returns on betas reduces the critical errors in variation problems. It however shrinks the range of betas and reduces statistical powers. To mitigate this problem, researchers sort securities on beta when forming portfolios; the first portfolio contains securities with the highest betas, up till the last one which contains securities with the lowest betas.

Roll (1977) created a major turning point in the empirical testing of the CAPM. He argued that previous tests of the CAPM had examined the relationship between equity returns and beta measured relative to a broad equity market index e.g. the S&P 500. However, the market defined in the CAPM was not a single equities market but an index of all wealth e.g. bonds, property, foreign assets, human capital etc. Thus the portfolio used by Black, Jensen and Scholes was not the true portfolio. Roll also shows that unless the market portfolio is known with certainty, CAPM could not be tested. Finally, he argues that the tests of CAPM are at best the tests of mean-variance efficiency of the portfolio taken as the market proxy.

More recent tests have been conducted on the CAPM. Gibbons (1982) proposed a methodology that directly tests the restriction on returns imposed by the CAPM. His method was based on Maximum Likelihood Estimation which avoids the need of separate steps. By estimating the beta and the risk premium simultaneously, errors-in-variable are avoided and there is an increase in precision of parameter estimates for the risk premium. His approach still rejected CAPM. Gibson, Ross and Shanken (1989) conducted statistical tests to confirm whether the market proxy is the tangency portfolio in the set of portfolios that can be constructed by combining the market portfolio with specific assets used as dependent variables in the time-series regression. Other authors who tried to handle the Roll critique include: Shanken (1987) and Kandel and Stambaugh (1987). They both argue that although the stock market is not the true market portfolio, it must nevertheless be highly correlated with the true market. Even with this insight, they still found evidence that CAPM did not hold. Stambaugh (1982) found that even when bonds and real estate are included into the market proxy, the CAPM is still rejected. Other risk factors were also found to influence stock prices. These include:

- Price to earnings ratio (Basu, 1977)
- Company Size (Banz, 1981)
- Book-to-Market equity (Fama and French, 1992)
- Other systematic influences (Dimson and Mussavian, 1998)

A key assumption of Markowitz portfolio optimization and the original CAPM was that investors only care about the mean and variance of one-period portfolio returns, which is an unrealistic assumption as investors can rebalance their portfolios frequently. Daily movements in prices of many assets cannot be explained by CAPM. Samuelson (1969), Hakansson (1970) and Fama (1970) worked on the intertemporal portfolio choice and asset pricing models, assuming that agents make portfolio and consumption decisions at discrete time intervals.

Merton's (1973) intertemporal capital asset pricing model (ICAPM) is a natural extension of the CAPM which assumes that time flows continuously. ICAPM has different assumptions about the investor objectives.

Merton summarized his result by saying "An intertemporal investor who currently faces a five per cent interest rate and a possible rate of either two or ten per cent next period will have portfolio demands different from a single-period maximize in the same environment or an intertemporal maximizer facing a constant interest rate of five per cent over time". The upshot is that a CAPM will hold at each point in time, but there will be multiple betas; the number of betas will be equal to the one plus number of state variables that drive the investment opportunity set through time.

Merton's analysis ran contrary to the basic assumption of CAPM, that an asset has greater value if its marginal contribution to wealth is greater.

Breeden (1989) came up with the Consumption Capital Asset Pricing Model. This allowed assets to be priced with a single beta like the CAPM; however, the beta was not measured with respect to aggregate market wealth, but with respect to aggregate consumption flow. He insisted that investor preferences must be defined over consumption and not wealth. One troubling feature though was the fact that the supply side of assets was being assumed away, yet the demand side of investors was adequately dealt with. One important insight of the ICAPM is that multiple risk factors are needed to explain asset prices.

Ross (1976) developed the arbitrage pricing theory (APT) as an alternative model that could potentially overcome the CAPM's problems; and retain the underlying message of the CAPM. The major idea of APT was that only a small number of systematic influences affect the long term average return of securities. Ross's APT was based on factor models which are multiple factors that represent the fundamental risks in the economy.

Multifactor models allow for many measures of systematic risk. Each measure captures the sensitivity of the asset to the corresponding pervasive factor. Ross said that the APT was more of an arbitrage relation than an equilibrium relation. If the factor model holds exactly and assets do not have specific risk, then the law of one price implies that the expected return of any asset is just a linear function of other assets' expected return. (If this was not the case, arbitrage would be taking place). When assets have no specific risk, all asset prices move in lockstep with one another and are therefore just leveraged 'copies' of each other. This result is much harder when assets have specific risk. It is possible to form portfolios where the specific risk is diversified away. To achieve full diversification, an infinite number of securities is required. With a finite set of securities, each of which has specific risk, the APT pricing restriction will only hold only approximately.

APT requires factor choice, number of factors, interpretation of factors etc which was a hotly contested debate. One of the earliest empirical studies was by Roll and Ross (1980), uses factor analysis. This was a statistical technique that allows the researcher to infer the factors from the data on security returns. The advantage of the factor analysis techniques is that the factors determined from the data explain a large proportion of the risks in that particular dataset over the period under consideration. The disadvantage is that factors usually have no economic interpretation. Roll and Ross concludes by arguing 'an effort should be directed at identifying a more meaningful set of sufficient statistics for the underlying factors.

An alternative to factor analysis is by using observed macroeconomic variables as the risk factors. Chen et al (1986) argued that at the most basic level, we can imagine some fundamental valuation model determines the price of assets i.e. the price of a stock will be the correctly discounted expected future dividends. Thus the choice of factors should include any systematic influences that impact future dividends, the way traders and investors form expectations and the rate at which investors discount future cash flows.

An example is the United States stock prices are significantly related to:

- Changes in industrial production
- The spread between the yield on short-term and long-term government bonds: This is interpreted as a proxy for business cycle.
- The spread between low-and-high grade bonds: Interpreted as a proxy for overall business risk in the economy.
- Changes in expected inflation
- Changes in unexpected inflation

Arguments against the APT include: Shanken (1982, 1985), he asserted that for individual securities the approximation implied by Ross was so imprecise, that it makes it impossible ever to test whether the APT is true or false. Shanken argues further that since the expected return for any security or portfolio is related only approximately to its factor sensitivities, to get an exact pricing relationship, additional assumptions are needed. He maintains that researchers who test the APT by assuming that the restriction holds, even for securities, are actually testing an equilibrium form of APT.

In conclusion, both CAPM and APT have fundamental limitations to any empirical verification.

Fama et. al (1993) explain the differences between the returns on the New York Stock Exchange (NYSE) and National Association of Security Dealers (NASD). Stocks on the NYSE have higher average returns than the stocks of similar size on the NASD during the test period. They use Fama and French three-factor model to explain the difference. Their analysis demonstrates that reason for this variation is the difference between the risk of the stocks, which is captured by Fama and French three-risk factor model. Fama et al. (1993, p.37) argue that stocks with high sensitivity tend to be firms with persistently poor earnings, which lead to low stock price and high book-to-market equity ratios. Stocks with low sensitivity to the book-to-market risk factor tend to have persistently high earnings which lead to low BE/ME. They conclude that book-to-market ratio is the most important risk factor that explains the difference in returns between NYSE stocks and NASD stocks.

Frazzini (2013) and Pedersen (2013) have presented a model named *Betting against Beta* that finds evidence that long leveraged low-beta assets and short high-beta assets produce significantly positive risk-adjusted returns.

CHAPTER 3: METHODOLOGY

3.1 MARKOWITZ MEAN-VARIANCE PORTFOLIO THEORY

Harry Markowitz (1952, 1959) developed his portfolio-selection technique, which came to be called modern portfolio theory (MPT). The mean-variance portfolio theory was designed to construct the optimal portfolio based on the idea that between risk and return there is a positive relation. Markowitz proved that investors should create their portfolio in order to offer them a maximum expected level of return for a given level of risk or, a minimum level of risk for a given expected level of return.

The **Markowitz model** is based on several assumptions concerning the behaviour of investors and financial markets namely:

A probability distribution of possible returns over some holding period can be estimated by investors. Investors have single-period utility functions in which they maximize utility within the framework of diminishing marginal utility of wealth. Variability about the possible values of return is used by investors to measure risk. Investors care only about the means and variance of the returns of their portfolios over a particular period.

Expected return and risk as used by investors are measured by the first two moments of the probability distribution of returns-expected value and variance. Return is desirable; risk is to be avoided[1]. Financial markets are frictionless.

Markowitz showed in his theory that stocks are related to each other and that the risk can be decreased through diversification. He was the first to clearly and rigorously show how the variance of a portfolio can be reduced through the impact of diversification. The proof is very simple: if one takes 2 stocks and he will calculate the correlation coefficient, the value of this coefficient would be less than one and if the respective stocks are included in a portfolio, the overall risk of this portfolio would decrease. Every possible asset combination can be plotted in risk-return space, and the collection of all such possible portfolios defines a region in this space. The line along the upper edge of this region is known as the efficient frontier. Combinations along this line represent portfolios (explicitly excluding the risk-free alternative) for which there is lowest risk for a given level of return. Conversely, for a given amount of risk, the portfolio lying on the efficient frontier represents the combination offering the best possible return. Mathematically the efficient frontier is the intersection of the set of portfolios with minimum variance and the set of portfolios with maximum return.

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[1] Markowitz model assumes that investors are risk averse. This means that given two assets that offer the same expected return, investors will prefer the less risky one. Thus, an investor will take on increased risk only if compensated by higher expected returns. Conversely, an investor who wants higher returns must accept more risk. The exact trade-off will differ by investor based on individual risk aversion characteristics. This implies a rational investor will not invest in a portfolio if a second portfolio exists with a more favorable risk-return profile - i.e., if for that level of risk an alternative portfolio exists which has better expected returns.

Using risk tolerance, we can classify investors into three types: risk-neutral, risk-averse, and risk-lover. Risk-neutral investor's do not require the risk premium for risk investments; they judge risky prospects solely by their expected rates of return. Risk-averse investors are willing to consider only risk-free or speculative prospects with positive premium; they make investment according to the risk-return trade-off. A risk-lover is willing to engage in fair games and gambles; and adjusts the expected return upward to account for the 'fun' of added risk.

A Markowitz portfolio model is one where no added diversification can lower the portfolio's risk for a given return expectation (alternately, no additional expected return can be gained without increasing the risk of the portfolio). The Markowitz Efficient Frontier is the set of all portfolios of which expected returns reach the maximum given a certain level of risk.

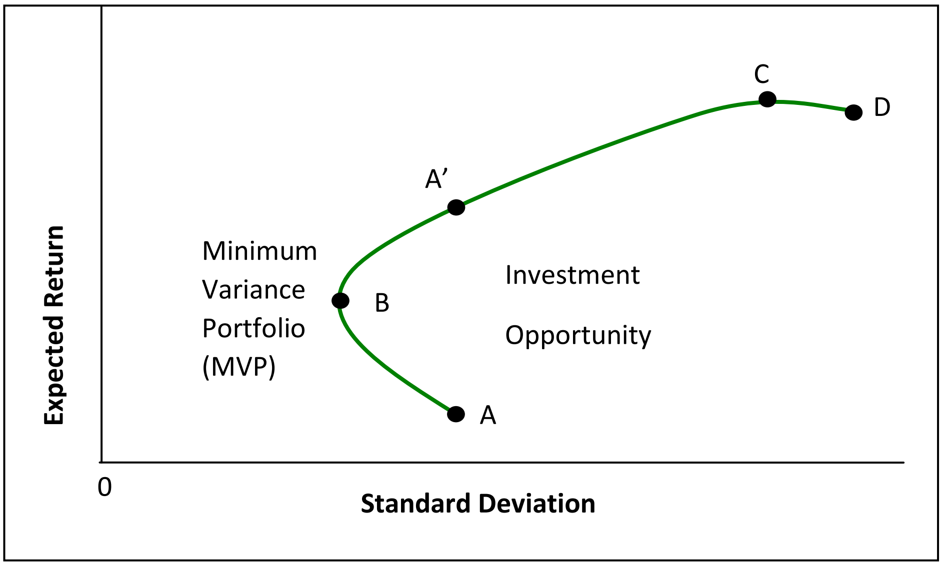

Figure 1: Investment opportunity set

Every possible asset combination can be plotted in risk-return space, and the collection of all such possible portfolios defines a region in this space. The line along the upper edge of this region is known as the efficient frontier. Combinations along this line represent portfolios (explicitly excluding the risk-free alternative) for which there is lowest risk for a given level of return. Conversely, for a given amount of risk, the portfolio lying on the efficient frontier represents the combination offering the best possible return. Mathematically the efficient frontier is the intersection of the set of portfolios with minimum variance and the set of portfolios with maximum return.

The area within curve ***ABCD*** is the feasible opportunity set representing all possible portfolio combinations. Portfolios that lie below the minimum-variance portfolio (point B) on the figure can therefore be rejected out of hand as inefficient. The portfolios that lie on the frontier ***BA*** in the Figure 1 would not be likely candidates for investors to hold. Because they do not meet the criteria of maximizing expected return for a given level of risk or minimizing risk for a given level of return. This is easily seen by comparing the portfolio represented by points A and A'. Since investors always prefer more expected return than less for a given level of risk, A' is always better than A. Using similar reasoning, investors would always prefer B to A because it has both a higher return and a lower level of risk. In fact, the portfolio at point B is identified as the **minimum-variance portfolio**; since no other portfolio exists that has a lower standard deviation. The curve ***BC*** represents all possible efficient portfolios and is the efficient frontier[2], which represents the set of portfolios that offers the highest possible expected rate of return for each level of portfolio standard deviation.

[2] The efficient frontier will be convex – this is because the risk-return characteristics of a portfolio change in a non-linear fashion as its component weightings are changed. (As described above, portfolio risk is a function of the correlation of the component assets, and thus changes in a non-linear fashion as the weighting of component assets changes.) The efficient frontier is a parabola (hyperbola) when expected return is plotted against variance (standard deviation).

The best choice among the portfolios on the upward sloping portion ***BC*** of the frontier curve is not as obvious, because in this region higher expected return is accompanied by higher risk. The best choice will depend on the investor's willingness to trade off risk against expected return.

Relaxing the assumption of **no short selling** in this development of the efficient frontier involves a modification of the analysis of the efficient frontier of constraint (not allowed short sales). If the number of short sales is unrestricted, then by a continuous short selling of the lowest-return asset *A* and reinvesting in highest-return asset *C* the investor could generate an infinite expected return. The efficient frontier of unconstraint portfolio is shown in Figure 2.

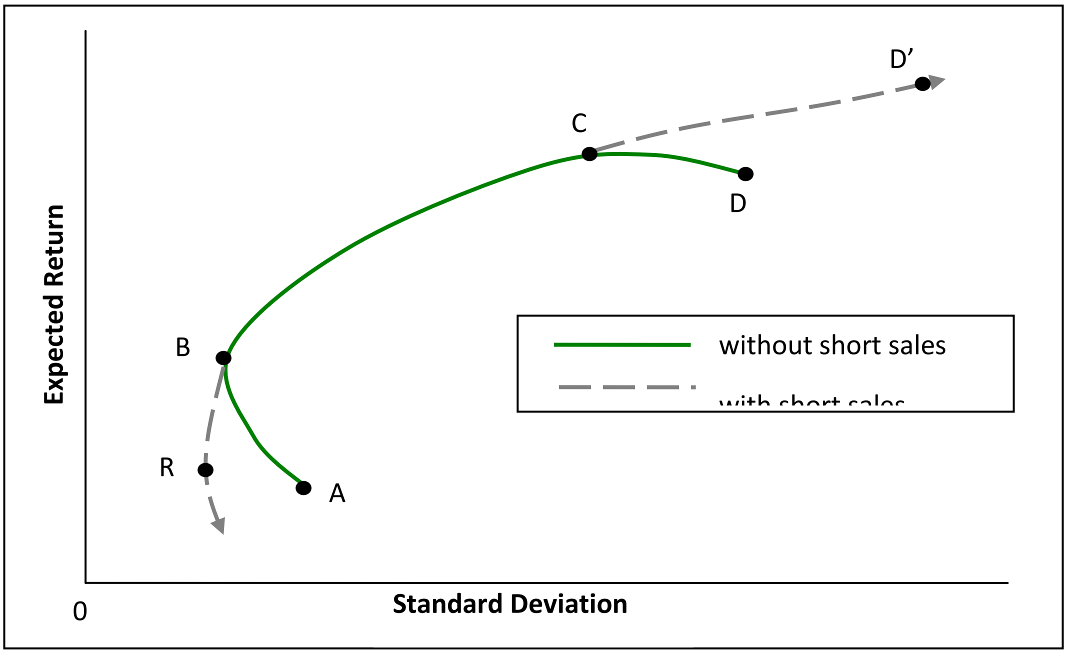

Figure 2: The efficient frontier of unrestricted/restricted portfolio

The upper bound of the highest-return portfolio would no longer be C but infinity (shown by the arrow on the top of the efficient frontier). Likewise the investor could short sell the highest-return security C and reinvest the proceeds into the lowest-yield security A[3], thereby generating a return less than the return on the lowest-return assets. Given no restriction on the amount of short selling, an infinitely negative return can be achieved, thereby removing the lower bound of B on the efficient frontier. Hence, short selling generally will increase the range of alternative investments from the minimum-variance portfolio to plus or minus infinity[4].

RETURN

Given any set of risky assets and a set of weights that describe how the portfolio investment is split, the general formulas of expected return for n assets is:

$$E(r_P) = \sum_{i=1}^{n} w_i E\left(r_i\right) \qquad \textbf{(F.1)}$$

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[3] Rational investor will not short sell a high-return asset and buy a low-return asset. This case is just for extreme assumption.

[4] Whether an investor engages in any of this short-selling activity depends on the investor's own unique set of indifference curves.

where:

$$\sum_{i=1}^{n} w_i = 1.0;$$

n = the number of securities;

w_i = the proportion of the funds invested in security i;

r_i, r_P = the return on ith security and portfolio p; and

$E(\ )$ = the expectation of the variable in the parentheses.

The return computation is nothing more than finding the weighted average return of the securities included in the portfolio.

RISK

The variance of a single security is the expected value of the sum of the squared deviations from the mean, and the standard deviation is the square root of the variance. The variance of a portfolio combination of securities is equal to the weighted average **covariance**[5] of the returns on its individual securities:

$$\operatorname{Var}\left(r_p\right) = \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \operatorname{Cov}\left(r_i, r_j\right) \qquad \text{(F.2)}$$

Covariance can also be expressed in terms of the correlation coefficient as follows:

$$\operatorname{Cov}\left(r_i, r_j\right) = \rho_{ij}\sigma_i\sigma_j = \sigma_{ij} \qquad \text{(F.3)}$$

where ρ_{ij} = correlation coefficient between the rates of return on security i, r_i, and the rates of return on security j, r_j, and σ_i, and σ_j represent standard deviations of r_i and r_j respectively. Therefore:

$$\operatorname{Var}\left(r_p\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \rho_{ij}\sigma_i\sigma_j \qquad \text{(F.4)}$$

Overall, the estimate of the mean return for each security is its average value in the sample period; the estimate of variance is the average value of the squared deviations around the sample average; the estimate of the covariance is the average value of the cross-product of deviations.

CALCULATING THE MINIMUM VARIANCE PORTFOLIO

In Markowitz portfolio model, we assume investors choose portfolios based on both expected return, $E(r_p)$, and the standard deviation of return as a measure of its risk, σ_p. So, the portfolio selection problem can be expressed as maximizing the return with respect to the risk of

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[5] High covariance indicates that an increase in one stock's return is likely to correspond to an increase in the other. A low covariance means the return rates are relatively independent and a negative covariance means that an increase in one stock's return is likely to correspond to a decrease in the other.

the investment (or, alternatively, minimizing the risk with respect to a given return, hold the return constant and solve for the weighting factors that minimize the variance). Mathematically, the portfolio selection problem can be formulated as quadratic program. For two risky assets A and B, the portfolio consists of w_A, w_B, the return of the portfolio is then, The weights should be chosen so that (for example) the risk is minimized, that is

$$\underset{w_A}{\text{Min}} \ \sigma_P^2 = w_A{}^2\sigma_A{}^2 + w_B{}^2\sigma_B{}^2 + 2w_A w_B \rho_{AB}\sigma_A\sigma_B$$

for each chosen return and subject to $w_A + w_B = 1, \ w_A \geq 0, \ w_B \geq 0$. The last two constraints simply imply that the assets cannot be in short positions.

Above, we simply use two-risky-assets portfolio to calculate the minimum variance portfolio weights. If we generalization to portfolios containing N assets, the minimum portfolio weights can then be obtained by minimizing the Lagrange function *C* for portfolio variance.

$$\text{Min } \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \rho_{ij}\sigma_i\sigma_j$$

$$\text{Subject to } \ w_1 + w_2 + ... + w_N = 1$$

$$C = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \text{Cov}\left(r_i r_j\right) + \lambda_1\left(1 - \sum_{i=1}^{n} W_i\right) \qquad \text{(X.8)}$$

in which λ_1 are the Lagrange multipliers, respectively, ρ_{ij} is the correlation coefficient between r_i and r_j, and other variables are as previously defined.

The efficient set that is generated by the aforementioned approach (equation X.8) is sometimes called the **minimum-variance set** because of the minimizing nature of the Lagrangian solution.

If we add a condition into the equation X.8*,* which is be subject to the portfolio's attaining some target expected rate of return, we can get the optimal risky portfolio.

$$\text{Min } \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j \rho_{ij}\sigma_i\sigma_j$$

Subject to

$\sum_{i=1}^{n} W_i E\left(R_i\right) = E^*$, where E^* is the target expected return and

$$\sum_{i=1}^{n} W_i = 1.0$$

The first constraint simply says that the expected return on the portfolio should equal the target return determined by the portfolio manager. The second constraint says that the weights of the securities invested in the portfolio must sum to one. The Lagrangian objective function can be written:

$$C=\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j\mathrm{Cov}\left(r_ir_j\right)+\lambda_1\left[E^*-\sum_{i=1}^{n}w_iE\left(r_i\right)\right]+\lambda_2\left(1-\sum_{i=1}^{n}w_i\right)$$

Taking the partial derivatives of this equation with respect to each of the variables, $w_1, w_2, ..., w_N$, λ_1, λ_2 and setting the resulting equations equal to zero yields the minimization of risk subject to the Lagrangian constraints. Then, we can solve the weights and these weights are represented optimal risky portfolio by using of matrix algebra.

If there no short selling constraint in the portfolio analysis, second constraint, $\sum_{i=1}^{n} w_i = 1.0$, should substitute to $\sum_{i=1}^{n}\left|w_i\right| = 1.0$, where the absolute value of the weights $\left|w_i\right|$ allows for a given w_i to be negative (sold short) but maintains the requirement that all funds are invested or their sum equals one.

The Lagrangian function is

$$C=\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j\mathrm{Cov}\left(r_ir_j\right)+\lambda_1\left[E^*-\sum_{i=1}^{n}w_iE\left(r_i\right)\right]+\lambda_2\left(1-\sum_{i=1}^{n}\left|w_i\right|\right) \qquad \text{(X.10)}$$

If the restriction of no short selling is in minimization variance problem, it needs to add a third constraint:

$$w_i \geq 0, \qquad i = 1, \ldots, N$$

The addition of this non-negativity constraint precludes negative values for the weights (that is, no short selling). The problem now is a quadratic programming problem similar to the ones solved so far, except that the optimal portfolio may fall in an unfeasible region. In this circumstance the next best optimal portfolio is elected that meets all of the constraints.

DEMERITS OF MARKOWITZ THEORY

Even though the theory of Markowitz was spectacular and useful in this field, it had some inconveniences. The calculation of portfolio return together with the weighted portfolio asset may perfectly accurate if it is just for two to three stocks. But when it comes to the addition of stocks D, E and so on up to a thousand or more, it becomes too long, time consuming, and almost impossible to calculate as the number of asset correlations becomes more and the calculation widens up into several algebraic numerations of every return variance being extended to three or more securities.

Further, it was done taking into account a very abstract concept in economics of expected utility model of Von Nuemann and Morgenstern (1953). The economical practice has shown that models constructed based on the idea of utility are very difficult or even impossible to apply. Also, the mathematics beyond of mean-variance is very sophisticated, which makes the application difficult when the portfolio consists of a great number of shares. Specifically, to estimate the benefits of diversification would require that practitioners calculate the covariance of returns between every pair of assets. Finally, critics of the model argued that it is a static one which made the results to be biased.

3.2 MATHEMATICAL DERIVATION OF CAPITAL ASSET PRICING MODEL

The basis of the CAPM is the portfolio theory with a riskless asset and unlimited short sales. We do not consider only the decision of a single investor, but aggregate them to determine a market equilibrium. Additionally to the assumptions for portfolio theory, we have to add that all investors have the same beliefs on the probability distribution of all assets, i.e. agree on the expected returns, variances and covariances. If all investors agree on the characteristics of an asset the optimal risky portfolio will be equal for all investors, even if they differ in their preferences (risk aversion). Because all assets have to be held by the investors the share each asset has in the optimal risky portfolio has to be equal to its share of the market value of all assets. The optimal risky portfolio has to be the market portfolio. Moreover all assets have to be marketable, i.e. all assets must be traded and there are no other investment opportunities not included into the model. Table 3.2.1 below summarizes the assumptions.

- No transaction costs and taxes
- Assets are indefinitely divisible
- Each investor can invest into every asset without restrictions
- Investors maximize expected utility by using the mean-variance criterion
- Prices are given and cannot be influenced by the investors (competitive prices)
- The model is static, i.e. only a single time period is considered
- Unlimited short sales
- Homogeneity of beliefs
- All assets are marketable

Table 3.2.1: Assumptions of the CAPM

Every investor j (j = 1, ... , M) maximizes his expected utility by choosing an optimal portfolio, i.e. choosing optimal weights for each asset. With the results of Arrow Pratt measure of risk aversion we get

$$\max_{\{x_i\}_{i=1}^N} E[U^j(R_p)] = \max_{\{x_i\}_{i=1}^N} U^j\left(\mu_p - \frac{1}{2}z_j\sigma_p^2\right)$$

$$= \max_{\{x_i\}_{i=1}^N}\left(\mu_p - \frac{1}{2}z_j\sigma_p^2\right)$$

$$= \max_{\{x_i\}_{i=1}^N}\left(\sum_{i=1}^N x_i\mu_i - \frac{1}{2}z_j\sum_{k=1}^N\sum_{i=1}^N x_i x_k\sigma_{ik}\right)$$

for all $j = 1,........,M$ with the restriction $\sum_{i=1}^N x_i = 1$.

The Lagrange function for solving this problem can easily be obtained as

$$L_j = \sum_{i=1}^N x_i\mu_i - \frac{1}{2}z_j\sum_{k=1}^N\sum_{i=1}^N x_i x_k\sigma_{ik} + \lambda\left(1 - \sum_{i=1}^N x_i\right)$$

The first order conditions for a maximum are given by

$$\frac{\partial L_j}{\partial x_i} = \mu_i - z_j\sum_{k=1}^N x_k\sigma_{ik} - \lambda = 0, \qquad i = 1,........,N,$$

$$\frac{\partial L_j}{\partial \lambda} = 1 - \sum_{i=1}^N x_i = 0$$

for all $j = 1,........,M$.

Solving the above equations for μ_i gives

$$\mu_i = \lambda + a_j\sum_{K=1}^N x_k\sigma_{ik} = \lambda + z_j Cov\left[R_i, \sum_{k=1}^N x_k R_k\right]$$

$$= \lambda + z_j Cov\left[R_i, R_p\right] = \lambda + z_j\sigma_{ip}$$

With $\sigma_{ip} = 0$ we find that $\mu_i = \lambda$ hence we can interpret that λ as the expected return of an asset which is uncorrelated with the market portfolio. As the riskless asset is uncorrelated with any portfolio, we can interpret λ as the risk free rate of return r:

$$\mu_i = r + z_j\sigma_{ip}$$

From (3.6) we see that the expected return depends linearly on the covariance of the asset with the market portfolio. The covariance can be interpreted as a measure of risk for an individual asset (*covariance risk*). Initially we used the variance as a measure of risk, but as has been shown in the last section the risk of an individual asset can be reduced by holding a portfolio. The risk that cannot be reduced further by diversification is called systematic risk, whereas the diversifiable risk is called *unsystematic risk*. The total risk of an asset consists of the variation of the market as a whole (systematic risk) and an asset specific risk (unsystematic risk). As the unsystematic risk can be avoided by diversification it is not compensated by the market, efficient portfolios therefore only have systematic and no unsystematic risk.

The covariance of an asset can be also interpreted as the part of the systematic risk that arises from an individual asset:

$$\sum_{i=1}^{N} x_i \sigma_{ip} = \sum_{i=1}^{N} x_i Cov\left[R_i, R_p\right] = Cov\left[\sum_{i=1}^{N} x_i R_i, R_p\right]$$

$$= Cov\left[R_p, R_p\right] = Var\left[R_p\right] = \sigma_p^2$$

Equation (3.6) is valid for all assets and hence for any portfolio, as the equation for a portfolio can be obtained by multiplying with the appropriate weights and then summing them up, so that we can apply this equation also to the market portfolio, which is also the optimal risky portfolio:

$$\mu_p = r + z_j \sigma_p^2$$

Solving for z_j and inserting into (3.6) gives us the usual formulation of the CAPM:

$$\mu_i = r + \left(\mu_p - r\right)\beta_i$$

Defining we can rewrite (3.8) as

$$\mu_i = r + \left(\mu_p - r\right)\beta_i$$

β_i represents the relative risk of the asset i (σ_{ip}) to the market risk (σ_p^2). The beta for the market portfolio is easily shown to be 1. We find a linear relation between the expected return and the relative risk of an asset. This relation is independent of the preferences of the investors (z_j), provided that the mean-variance criterion is applied and that the utility function is quadratic. This equilibrium line is called the *Capital Market Line* (*CML*). Figure 3.1 illustrates this relation.

For the risk an investor takes he is compensated by the amount of $\mu_p - r$ per unit of risk, the total amount $(\mu_p - r)\beta_i$ is called the *risk premium* or the *market price of risk*. The risk free rate of return r may be interpreted as the price for time.

It is the compensation for not consuming the amount in the current period, but wait until the next period.

Equation (4.9) presents a formula for the expected return given the interest rate, r, the beta and the expected return on the market portfolio, μ_p. However, the expected return of the market portfolio is not exogenous as it is a weighted average of the expected returns of the individual assets. In this formulation only relative expected returns can be determined, the level of expected returns, i.e. the market risk premium, is not determined by the CAPM. Although we can reasonably assume μ_p to be given when investigating a small capitalized asset, we have to determine μ_p endogenously.

Thus far we only considered the market portfolio, but an investor will in general not hold the market portfolio (optimal risky portfolio). As we saw in section 3 the optimal portfolio will be a combination of the market portfolio and the riskless asset, where the shares will vary among investors. We can now add another restriction to our model. The return on the optimal risky portfolio has to be such that the market for the riskless assets has to be in equilibrium. The amounts of riskless assets lent and borrowed have to be equal:

$$\sum_{j=1}^{M} x_{jr} = 0$$

where x_{jr} denotes the demand of the jth investor for the riskless asset. With this additional market to be in equilibrium it is possible to determine μ_p endogenously. We herewith have found an equilibrium in the expected returns. These expected returns can now be used to determine equilibrium prices according to section 1.

Nevertheless this result remains to determine only relative prices. The risk free rate r is not determined endogenously. Although it can reasonably be assumed that r can substantially be influenced by monetary policy, especially for longer time horizons it is not given.

CRITIQUE OF CAPM

The assumptions underlying the CAPM are very restrictive. The problems associated with the use of the mean-variance criterion and the quadratic utility function have already been mentioned in section 4.2. Some restrictions such as the absence of transaction costs and taxes, unlimited borrowing and lending at the risk free rate and short sales have been lifted by more recent contributions without changing the results significantly. One restriction however that mostly is not mentioned in the literature is that the assets have to be linearly dependent. This linearity, also used in portfolio theory is implied by the use of the covariance, which is only able to capture linear dependencies appropriately. This linearity of returns rules out the inclusion of derivatives that mostly have strong non-linearities in their pay-offs and have become an important tool for investment in recent years. By excluding such assets the need to include all assets is violated.

Beside these theoretical critiques empirical investigations show a mixed support for the CAPM. There exist a large number of empirical investigations of the CAPM using different econometric specifications. Early investigations mainly supported the CAPM, but more recent results show that the CAPM is not able to explain the observed returns. If other variables, such as book-market ratios, market value of a company or price-earnings ratios are included the beta has no significant influence on the observed returns.

Like the portfolio theory the CAPM is a static model, i.e. the investment horizon is assumed to be only a single period. As has been pointed out, after every period the portfolio has to be rebalanced even if the beliefs do not change. This rebalancing will affect the market equilibrium prices and hence the expected future returns and risk premia. Therefore a dynamic model will capture the nature of decision making more appropriate, such a modification is given with the Intertemporal CAPM to be presented later.

Another critical point in the CAPM is that unconditional beliefs (means, variances and covariances) are used, the investors are not able to condition their beliefs on information they receive. A direct implication of this is the assumption that beliefs are constant over time. Many empirical investigations give strong support that beliefs are varying over time, of special importance is the beta. But also the risk aversion and the risk premium have been found to vary significantly over time. These aspects are taken into account in the Conditional CAPM that will be presented in the next section.

The CAPM further takes the amount an investor wants to invest in the assets as exogenously given. However the amount to invest is a decision whether to consume today or to consume in the future. The influence of consumption is incorporated into the Consumption-based CAPM.

Finally the CAPM explains the expected returns only by a single variable, the risk of an asset relative to the market. It is reasonable to assume that other factors may as well influence the expected returns. We will therefore discuss the Arbitrage Pricing Theory as an alternative to the CAPM framework. The Intertemporal CAPM also allows for more risk factors.

Another assumption remains critical for the CAPM: the assumption that all assets are marketable. Some investment restrictions due to legislation in foreign countries are taken into account by the International CAPM, but assets such as human capital are not marketable. Therefore the market portfolio cannot be determined correctly. Roll (1977) showed that the determination of a correct market portfolio is important to achieve correct results. Only small deviations from the true market portfolio can bias the results significantly. Other investigations showed that small deviations do not have a large impact on the results. It remains an open question whether the assumption of all assets to be marketable is restrictive or not.

As a result of these critiques the CAPM has been modified and today a wide variety of extensions exist. Alternatives that use a different approach to asset pricing are not frequently found and had, with the exception of the Arbitrage Pricing Theory, no great impact as well in applications as the academic literature. The remaining sections will give an overview of the extensions of the CAPM as well as alternative approaches.

CAPM EXTENSIONS

3.3 THE ZERO-BETA CAPITAL ASSET PRICING MODEL:

This model was developed by Black (1972) and is applicable to markets where risk-free asset is partially or completely restricted. Zero Beta CAPM was generated for relaxing the assumption of a risk-free capital asset and the assumption that investors can borrow and lend on the basis of a risk-free interest rate. This is because even though most investors can make investments from an unlimited amount of risk-free assets; they however cannot borrow at the same amount limitlessly.

Zero beta model indicates that a risk free interest rate is not necessary in order for CAPM to be valid. Investors keep different risky portfolios; however all such portfolios take place on the efficient frontier. Union of the portfolios at the frontier at the same time takes place at the frontier. Linearity of the model is still valid and the beta coefficient continues to be the measure of systematic risk. However, one limitation of this model is the requirement that there is no restriction on short selling. Since the correlation coefficients of many of the assets in the market are positive, it is almost not possible to establish a Zero Beta portfolio without short selling.

For any frontier portfolio p, except the minimum variance portfolio, there exists a unique frontier portfolio with which p has zero covariance. Using the two-fund theorem, any portfolio can be written as a combination of 2 frontier portfolios, any portfolio the investors choose can be a combination of the market portfolio and its zero-beta counterpart. By the spanning property of the frontier portfolios, any portfolio on the frontier can be obtained as a linear combination of only two portfolios on the frontier. Therefore we have two-fund separation even without the risk-free asset.

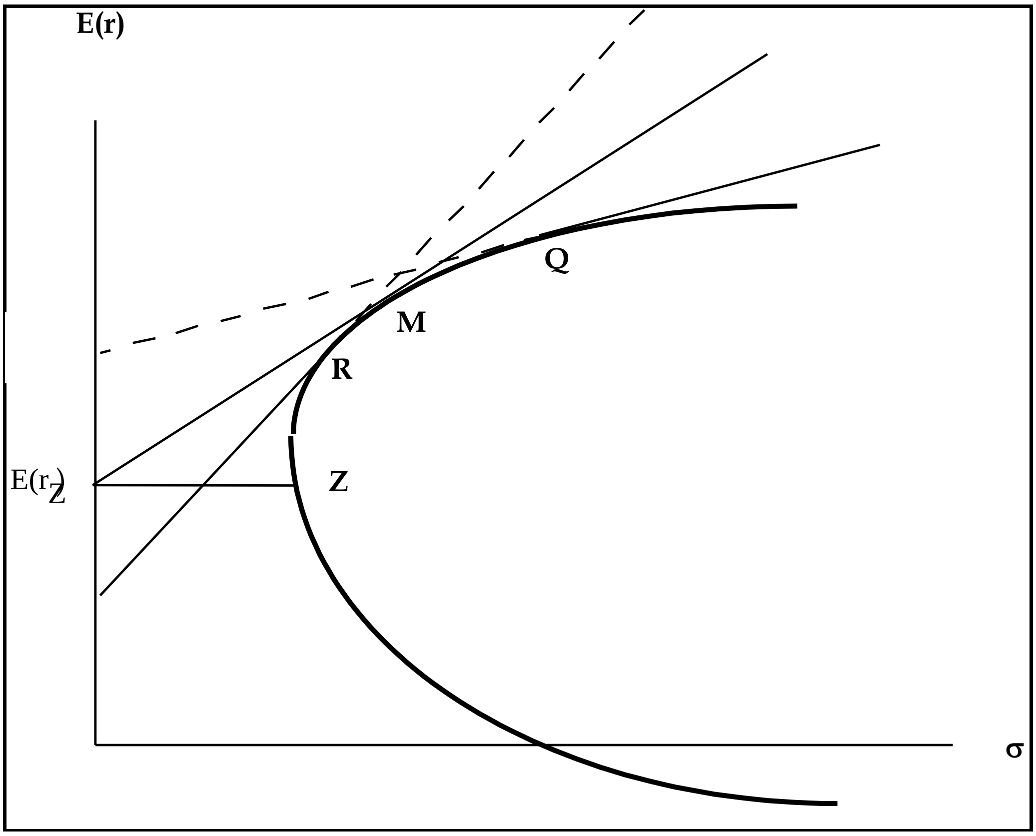
An important assumption behind the Zero-Beta CAPM is that short-sales are possible. To obtain zero-beta portfolios we typically would have to short sell some assets. If there are short-sales constraints the Zero-Beta CAPM fails to hold.

The stages in Zero Beta CAPM model analysis are the same; the only difference is the fact that a portfolio return, whose correlation with the market is zero takes place instead of a risk-free asset return. Since zero beta portfolio return is uncertain in this case, the second-pass regression equation is compared with CAPM, arranged as:

$$E(r_i) = E(r_z) + E[r_m - r_z]\beta_i$$

$$E(r_i) = E[r_z](1-\beta_i) + E(r_m)\beta_i$$

In this case, it is evident that γ_0 coefficient is an estimate of $r_z(1-\beta_i)$ value and γ_i is an estimate of $E(r_m)$ value. On the other hand, R_i and β_i values are estimates of uncertain real parameters of i capital asset. Therefore, differing from Standard CAPM, it has to be $\gamma_0 = r_z(1-\beta_i)$. Then, it has to be equal to $\gamma_0 / (1-\beta_i) = r_z$. Since R_z value is uncertain, the hypothesis, which has to be tested, is equality of all $\gamma_0 / (1-\beta_i)$. Chou (2002) used Wald test statistic in testing of this hypothesis and found that the International zero-beta CAPM was valid in at least 16 OECD over the period 1980-1997.



Since any combination of two efficient frontier portfolios is also efficient, the average (market) portfolio will also be efficient here, as depicted by point M. Moreover, the Zero Beta model must now apply, because the market portfolio is efficient and all investors choose risky portfolios that lie on the efficient frontier. As a result, the ray from the expected return on the efficient portfolio with zero correlation with M (and hence zero beta) to the efficient frontier, will be tangent at M. This can happen only if:

$r_f(1 - t) < E(r_Z) < r_f$

More generally, consider the case of any number of classes of investors with individual risk-free borrowing and lending rates. As long as the same efficient frontier of risky assets applies to all of them, the Zero-Beta model will apply, and the equilibrium zero-beta rate will be a weighted average of each individual's risk-free borrowing and lending rates. In the zero-beta CAPM the zero-beta portfolio replaces the risk-free rate.

Since the zero-beta portfolio is uncorrelated with the market portfolio, then we must have:

$$\beta_z = \frac{Cov(R_z, R_m)}{Var(R_m)}$$

$$Cov(R_z, R_m) = 0$$

hence $$\beta_z = 0$$

3.4 THE FAMA-FRENCH THREE-FACTOR MODEL

The Fama and French three-factor asset pricing model was developed as a response to poor performance of the CAPM in explaining realized returns. Eugene Fama and Ken French (1992) presented empirical arguments against the CAPM model showing that the cross section of average equity returns in the US market shows little statistical relation to the βs of the original CAPM model.

They discovered that anomalies relating to the CAPM are captured by the three-factor model. They base their model on the fact that average excess portfolio returns are sensible to three factors namely:

- excess market portfolio return;
- the difference between the excess return on a portfolio of small stocks and the excess return on a portfolio of big stocks (SMB, small minus big); and
- the difference between the excess return on a portfolio of high-book-to-market stocks and the excess return on a portfolio of low -book -to -market stocks (HML , high minus low) .

This is formulated as below:

$$E\left(R_i\right)-R_f=\beta_m^i E\left(r_m-r_f\right)+\beta_{SMB}^i E\left(SMB\right)+\beta_{HML}^i E\left(HML\right)$$

The coefficients in this model have similar interpretations to beta in the standard CAPM. β_M is a measure of the exposure an asset has to market risk (although this beta will have a different value from the beta in a CAPM model as a result of the added factors), β_{SMB} measures the level of exposure to size risk and β_{HML} measures the level of exposure to value risk.

In this equation, SMB (small minus big) is the difference between the returns on diversified portfolios of small and big market capitalization stocks, referred to as the size premium, and HML (high minus low) is the difference between the returns on diversified portfolios of high and low Book-to-Market stocks. This is the value placed by accountants on a company as a ratio relative to the book equity (BE) divided by its market equity (ME).

CAPM's assumption of a single risk factor explaining expected returns has been criticized. Fama and French (1992, 1993, 1995, 1996, 1998) proposed an alternative pricing model which incorporates these three factors as proxies for risk: the market, the size, and the value factors.

Although the Fama-French model has been adopted both by most practitioners and academics in financial issues pertaining to portfolio management, capital budgeting, and performance evaluation, the three-factor model suffers from many drawbacks.

In fact, while financial data exhibit multi scaling, i.e. are a combination of different multi-horizon dynamics, the Fama-French model is a single scale model that studies the relation between risk factors and expected returns on a global scale investment horizon.

Furthermore, most studies neglect the assets' long-term holding period focusing mainly on short-term analysis though it is crucial to take into account the tendency of some investors to hold stocks over the long-run.

The model fits two additional risk factors to the CAPM in order to explain the return variations better and cure the anomalies of the CAPM. Fama and French (1996) point out that the model captures many of the variations in the cross -section of average stock returns, and it absorbs most of the anomalies that have plagued the CAPM. In the same study they argue that the empirical success of their model suggests that it is an equilibrium pricing model, a three -factor version of Merton’s (1973) intertemporal CAPM or Ross’s arbitrage pricing theory.

This model is better than the CAPM to estimate expected returns, and captures in a better way the variation in average returns for portfolios formed on size, book to market, and the others factors, for which CAPM is not efficient.

From a theoretical perspective, the main shortcoming of the three factor model is its empirical motivation, as the SMB and HML explanatory returns are not motivated by predictions about state variables of concern to investors.

3.5 THE CONDITIONAL CAPITAL ASSET PRICING MODEL

As we saw in the last section the performance of the traditional CAPM is only poor. It has been proposed that the reason may be the static nature of the model and the therewith following assumptions of a fixed beta and fixed risk premia. Allowing the beta to vary over time can be justified by the reasonable assumption that the relative risk and the expected excess returns of an asset may vary with the business cycle.

It therefore is reasonable to use all available information on the business cycle and other relevant variables to form expectations, i.e. to use conditional moments. This gave rise to the *Conditional Capital Asset Pricing Model*.

DERIVATION OF THE MODEL

We assume that the CAPM as derived in the last section remains valid if we use conditional moments instead of unconditional moments:

$$E\left[R_i^t|\Omega_{t-1}\right]-E\left[r^t|\Omega_{t-1}\right]=\left(E\left[R_M^t|\Omega_{t-1}\right]-E\left[r^t|\Omega_{t-1}\right]\right)\beta_i^{t-1}$$

where

$$\beta_i^{t-1}=\frac{Cov\left[R_i^t,R_M^t|\Omega_{t-1}\right]}{Var\left[R_M^t|\Omega_{t-1}\right]}$$

R_i^t denotes the return of asset *i* at time *t*, r^t the risk free rate of return at time *t*, R_M^t the return of the market portfolio at time *t* and Ω_{t-1} the information available at time $t-1$. For notational simplicity we define

$$\gamma_0^{t-1}\equiv E\left[r^t|\Omega_{t-1}\right]$$

$$\gamma_1^{t-1}\equiv \mathrm{E}\left[R_M^t|\Omega_{t-1}\right]-E\left[r^t|\Omega_{t-1}\right]=E\left[R_M^t|\Omega_{t-1}\right]-\gamma_0^{t-1}$$

as the expected risk free rate and the expected risk premium, respectively. We can rewrite (5.1) as

$$E\left[R_i^t|\Omega_{t-1}\right]=\gamma_0^{t-1}+\gamma_1^{t-1}\beta_i^{t-1}$$

We want to explain unconditional returns because expectations cannot be observed. Taking expectations of (5.5) we get with the law of iterated expectations:

$$E\left[R_i^t\right]=E\left[\gamma_0^{t-1}\right]+E\left[\gamma_1^{t-1}\right]E\left[\beta_i^{t-1}\right]+Cov\left[\left[\gamma_1^{t-1},\beta_i^{t-1}\right]\right]$$

The risk premium, γ_1^{t-1}, and the risk, β_i^{t-1}, will in general be correlated. To see this imagine a recession in which future prospects of a company are very uncertain, then the beta will be relatively high as other companies, like consumer goods or utilities that form a part of the market portfolio, are affected much less. The risk premium will also be high to compensate the owners of the asset for the higher risk in a recession as the asset forms part of the market portfolio. In a boom relations are likely to change. Hence risk premium and risk will be correlated. Let the sensitivity of the conditional beta to a change in the market risk premium be denoted by θ_i:

$$\vartheta_i \equiv \frac{Cov\left[\beta_i^{t-1}, \gamma_1^{t-1}\right]}{Var\left[\gamma_1^{t-1}\right]}$$

We further define a residual beta as:

$$\eta_i^{t-1} = \beta_i^{t-1} - E\left[\beta_i^{t-1}\right] - \vartheta_i\left(\gamma_1^{t-1} - E\left[\gamma_1^{t-1}\right]\right)$$

The first two terms represent the difference of the conditional and the unconditional beta. The first term adjusts this difference for the deviation of the risk premium from its unconditional value. We find that

$$E\left[\eta_i^{t-1}\right] = E\left[\beta_i^{t-1}\right] - E\left[\beta_i^{t-1}\right] - \vartheta_i\left(E\left[\gamma_1^{t-1}\right] - E\left[\gamma_1^{t-1}\right]\right)$$

$$= 0$$

$$E\left[\eta_i^{t-1}\gamma_1^{t-1}\right] = E\left[\beta_i^{t-1}\gamma_1^{t-1}\right] - E\left[E\left[\beta_i^{t-1}\right]\gamma_1^{t-1}\right] - E\left[\vartheta_i\gamma_1^{t-1}\left(\gamma_1^{t-1} - E\left[\gamma_1^{t-1}\right]\right)\right]$$

$$E\left[\beta_i^{t-1}\right]E\left[\gamma_1^{t-1}\right] + Cov\left[\beta_i^{t-1}, \gamma_1^{t-1}\right] - E\left[\beta_i^{t-1}\right]E\left[\gamma_1^{t-1}\right] - \vartheta_i\left(E\left[\left(\gamma_1^{t-1}\right)^2\right] - E\left[\gamma_1^{t-1}\right]^2\right)$$

$$= Cov\left[\beta_i^{t-1}, \gamma_1^{t-1}\right] - \vartheta_i Var\left[\gamma_1^{t-1}\right]$$

$$= Cov\left[\beta_i^{t-1}, \gamma_1^{t-1}\right] - Cov\left[\beta_i^{t-1}, \gamma_1^{t-1}\right]$$

$$= 0.$$

We can now solve (3.8) for the conditional beta:

$$\beta_i^{t-1} = E\left[\beta_i^{t-1}\right] + \vartheta_i\left(\gamma_1^{t-1} - E\left[\gamma_1^{t-1}\right]\right) + \eta_i^{t-1}$$

We further can solve (3.7) for $Cov\left[\beta_i^{t-1}, \gamma_1^{t-1}\right]$ and insert the expression into (3.6) to obtain:

$$E\left[R_i^t\right] = E\left[\gamma_0^{t-1}\right] + E\left[\gamma_1^{t-1}\right]E\left[\beta_i^{t-1}\right] + Var\left[\gamma_1^{t-1}\right]\vartheta_i$$

The excess returns turn out to be a linear function of the expected beta and the sensitivity of the beta to a change in the risk premium. We want to express this relation with using the unconditional beta.

Equation (5.5) also remains valid for the market portfolio, so that we get with $\beta_M^{t-1} = 1$:

$$E\left[R_M^t | \Omega_{t-1}\right] = \gamma_0^{t-1} + \gamma_1^{t-1}$$

and from the definition of γ_0^{t-1}:

$$\gamma_1^{t-1} = E\left[R_M^t - \gamma_0^{t-1} | \Omega_{t-1}\right]$$

We define ε_i^t as the residual from relation (5.1):

with

$$\begin{aligned} E\left[\varepsilon_i^t | \Omega_{t-1}\right] &= E\left[R_i^t | \Omega_{t-1}\right] - E\left[\gamma_0^{t-1} | \Omega_{t-1}\right] - E\left[\left(R_M^t - \gamma_0^{t-1}\right)\beta_i^{t-1} | \Omega_{t-1}\right] \\ &= \gamma_0^{t-1} + \gamma_1^{t-1}\beta_i^{t-1} - \gamma_0^{t-1} - E\left[R_M^t - \gamma_0^{t-1}\right]\beta_i^{t-1} \\ &= \gamma_1^{t-1}\beta_i^{t-1} - \gamma_1^{t-1}\beta_i^{t-1} \\ &= 0 \end{aligned}$$

$$\begin{aligned} = E\left[\varepsilon_i^t R_M^t | \Omega_{t-1}\right] &= E\left[R_i^t R_M^t | \Omega_{t-1}\right] - E\left[\left(\gamma_0^{t-1} + \left(R_M^t - \gamma_0^{t-1}\right)\beta_i^{t-1}\right) R_M^t | \Omega_{t-1}\right] \\ & E\left[R_i^t R_M^t | \Omega_{t-1}\right] - E\left[E\left[r_i^t | \Omega_{t-1}\right] R_M^t | \Omega_{t-1}\right] \\ &= 0 \end{aligned}$$

By the law of iterated expectations we have:

$$E\left[\varepsilon_i^t\right] = E\left[E\left[\varepsilon_i^t | \Omega_{t-1}\right]\right] = 0$$

$$E\left[\varepsilon_i^t R_M^t\right] = E\left[E\left[\varepsilon_i^t R_M^t | \Omega_{t-1}\right]\right] = 0$$

$$\begin{aligned} E\left[\varepsilon_i^t \gamma_1^{t-1}\right] &= E\left[\varepsilon_i^t E\left[R_M^t - \gamma_0^{t-1} | \Omega_{t-1}\right]\right] \\ &= E\left[\mathrm{E}\left[\varepsilon_i^t E\left[R_M^t - \gamma_0^{t-1} | \Omega_{t-1}\right] | \Omega_{t-1}\right]\right] \\ &= E\left[\mathrm{E}\left[\varepsilon_i^t | \Omega_{t-1}\right] E\left[R_M^t - \gamma_0^{t-1} | \Omega_{t-1}\right]\right] \end{aligned}$$

$$= 0$$

By inserting (5.11) into (5.15) and solving for R_i^t we get

$$R_i^t = \gamma_0^{t-1} + \left(R_M^t - \gamma_0^{t-1}\right)E\left[\beta_i^{t-1}\right] + \left(R_M^t - \gamma_0^{t-1}\right)\left(\gamma_1^{t-1} - E\left[\gamma_1^{t-1}\right]\right)\vartheta_i + \left(R_M^t - \gamma_0^{t-1}\right)\eta_i^{t-1} + \varepsilon_i^t$$

From the definition and the properties of the covariance we get with (3.21):

$$Cov\left[R_i^t, R_M^t\right] = Cov\left[\gamma_0^{t-1}, R_M^t\right] + Cov\left[R_M^t - \gamma_0^{t-1}, R_M^t\right]E\left[\beta_i^{t-1}\right] + Cov\left[\left(R_M^t - \gamma_0^{t-1}\right)\left(\gamma_1^{t-1} - E\left[\gamma_1^{t-1}\right]\right)\vartheta_i, R_M^t\right]$$

$$+Cov\left[\left(R_M^t - \gamma_0^{t-1}\right)\eta_i^{t-1}, R_M^t\right]$$

$$Cov\left[R_i^t, \gamma_i^{t-1}\right] = Cov\left[\gamma_0^{t-1}, \gamma_i^{t-1}\right] + Cov\left[R_M^t - \gamma_0^{t-1}, \gamma_i^{t-1}\right]E\left[\beta_i^{t-1}\right] + Cov\left[\left(R_M^t - \gamma_0^{t-1}\right)\left(\gamma_1^{t-1} - E\left[\gamma_1^{t-1}\right]\right)\vartheta_i, \gamma_i^{t-1}\right]$$

$$+Cov\left[\left(R_M^t - \gamma_0^{t-1}\right)\eta_i^{t-1}, \gamma_i^{t-1}\right]$$

It has been shown by Jagannathan/Wang (1996, pp.38 ff.) that the last term in (5.22) and (5.23) becomes zero if we assume the residual betas, η_i^{t-1}, to be uncorrelated with market conditions.

We define

$$\beta_i = \frac{Cov\left[R_i^t, R_M^t\right]}{Var\left[R_M^t\right]}$$

as the traditional market beta which measures the unconditional risk and

$$\beta_i^\gamma = \frac{Cov\left[R_i^t, \gamma_1^{t-1}\right]}{Var\left[\gamma_1^{t-1}\right]}$$

As the premium beta $_i$which measures the risk from a varying beta. We can substitute (5.24) and (5.25) into (5.22) and (5.23), respectively, and obtain after solving for β_i and β^γ:

$$\beta_i = \frac{Cov\left[\gamma_0^{t-1}, R_M^t\right]}{Var\left[R_M^t\right]} + \frac{Cov\left[R_M^t - \gamma_0^{t-1}, R_M^t\right]}{Var\left[R_M^t\right]}E\left[\beta_i^{t-1}\right] + \frac{Cov\left[\left(R_M^t - \gamma_0^{t-1}\right)\left(\gamma_1^{t-1} - E\left[\gamma_1^{t-1}\right]\right), R_M^t\right]}{Var\left[R_M^t\right]}\vartheta_i$$

$$\beta_i = \frac{Cov\left[\gamma_0^{t-1},\gamma_1^{t-1}\right]}{Var\left[\gamma_1^{t-1}\right]} + \frac{Cov\left[R_M^t - \gamma_0^{t-1},\gamma_1^{t-1}\right]}{Var\left[\gamma_1^{t-1}\right]} E\left[\beta_i^{t-1}\right] + \frac{Cov\left[\left(R_M^t - \gamma_0^{t-1}\right)\left(\gamma_1^{t-1} - E\left[\gamma_1^{t-1}\right]\right),\gamma_1^{t-1}\right]}{Var\left[\gamma_1^{t-1}\right]} \vartheta_i$$

If we define

$$b_{10} \equiv \frac{Cov\left[\gamma_0^{t-1},R_M^t\right]}{Var\left[R_M^t\right]}\ ,\ b_{11} \equiv \frac{Cov\left[R_M^t - \gamma_0^{t-1},R_M^t\right]}{Var\left[R_M^t\right]}\ ,\ b_{12} \equiv \frac{Cov\left[\left(R_M^t - \gamma_0^{t-1}\right)\left(\gamma_1^{t-1} - E\left[\gamma_1^{t-1}\right]\right),R_M^t\right]}{Var\left[R_M^t\right]}\quad {}_i$$

$$b_{20} \equiv \frac{Cov\left[\gamma_0^{t-1},\gamma_1^{t-1}\right]}{Var\left[\gamma_1^{t-1}\right]}\ ,\ b_{21} \equiv \frac{Cov\left[R_M^t - \gamma_0^{t-1},\gamma_1^{t-1}\right]}{Var\left[\gamma_1^{t-1}\right]}\ ,\ b_{22} \equiv \frac{Cov\left[\left(R_M^t - \gamma_0^{t-1}\right)\left(\gamma_1^{t-1} - E\left[\gamma_1^{t-1}\right]\right),\gamma_1^{t-1}\right]}{Var\left[\gamma_1^{t-1}\right]}\quad {}_i$$

$$b = \begin{pmatrix} b_{10} \\ b_{20} \end{pmatrix}\ ,\ \ B = \begin{pmatrix} b_{11}b_{12} \\ b_{21}b_{22} \end{pmatrix}$$

we can write (5.26) and (5.27) in vector form as

$$\begin{pmatrix} \beta_i \\ \beta_i^\gamma \end{pmatrix} = b + B \begin{pmatrix} E\left[\beta_i^{t-1}\right] \\ \vartheta_i \end{pmatrix}$$

If $E\left[\beta_i^{t-1}\right]$ and ϑ_i are linear dependent, i.e. $E\left[\beta_i^{t-1}\right] = d_0 + d_1\vartheta_i$ we get from (5.28):

$$\beta_i = b_{10} + b_{11}(d_0 + d_1\vartheta_i) + b_{12}\vartheta_i$$

$$= (b_{10} + b_{11}d_0) + (b_{11}d_0 + b_{12})\vartheta_i$$

By inserting (5.29) into (5.12) we get

$$E\left[R_i^t\right] = E\left[\gamma_0^{t-1}\right] + E\left[\gamma_1^{t-1}\right]\left(d_0 + d_1\vartheta_i\right) + Var\left[\gamma_1^{t-1}\right]\vartheta_i$$

$$= E\left[\gamma_0^{t-1}\right] + E\left[\gamma_1^{t-1}\right]d_0 + \left(E\left[\gamma_1^{t-1}\right]d_1 + Var\left[\gamma_1^{t-1}\right]\right)\vartheta_i$$

$$= E\left[\gamma_0^{t-1}\right] + E\left[\gamma_1^{t-1}\right]d_0 - \frac{\left(E\left[\gamma_1^{t-1}\right]d_1 + Var\left[\gamma_1^{t-1}\right]\right)\left(b_{10} + b_{11}d_0\right)}{b_{12} + b_{11}d_0} + \frac{E\left[\gamma_1^{t-1}\right]d_1 + Var\left[\gamma_1^{t-1}\right]}{b_{12} + b_{11}d_0}\beta_i$$

$$= a_0 + a_1\beta_i$$

with $a_0 = E\left[\gamma_0^{t-1}\right] + E\left[\gamma_1^{t-1}\right]d_0 - \dfrac{\left(E\left[\gamma_1^{t-1}\right]d_1 + Var\left[\gamma_1^{t-1}\right]\right)\left(b_{10} + b_{11}d_0\right)}{b_{12} + b_{11}d_0}$

and $a_1 = \dfrac{E\left[\gamma_1^{t-1}\right]d_1 + Var\left[\gamma_1^{t-1}\right]}{b_{12} + b_{11}d_0}$.

In this case the conditional CAPM has the same form as the unconditional form presented in the last section. But if $E\left[\beta_i^{t-1}\right]$ is not a linear function of θ_i we are able to invert equation (5.28):

$$\begin{pmatrix} E\left[\beta_i^{t-1}\right] \\ \vartheta_i \end{pmatrix} = B^{-1}b - B^{-1}\begin{pmatrix} \beta_i \\ \beta_i^{\gamma} \end{pmatrix}$$

By rewriting (5.12) we get in vector form:

$$E\left[R_i^t\right] = E\left[\gamma_0^{t-1}\right] + \left[E\left[\gamma_1^{t-1}\right] Var\left[\gamma_1^{t-1}\right]\right]\begin{bmatrix} E\left[\beta_i^{t-1}\right] \\ \vartheta_i \end{bmatrix}$$

Inserting (5.31) becomes

$$E\left[R_i^t\right] = E\left[\gamma_0^{t-1}\right] + \left[E\left[\gamma_1^{t-1}\right] Var\left[\gamma_1^{t-1}\right]\right]B^{-1}b - \left[E\left[\gamma_1^{t-1}\right] Var\left[\gamma_1^{t-1}\right]\right]B^{-1}\begin{pmatrix} \beta_i \\ \beta_i^{\gamma} \end{pmatrix}$$

With $B^{-1} = \begin{pmatrix} c_{11}c_{12} \\ c_{21}c_{22} \end{pmatrix}$, $e_0 = E\left[\gamma_0^{t-1}\right] + \left[E\left[\gamma_1^{t-1}\right], Var\left[\gamma_1^{t-1}\right]\right]B^{-1}b$, $e_1 = E\left[\gamma_1^{t-1}\right]c_{11} + Var\left[\gamma_1^{t-1}\right]c_{21}$ and $e_2 = E\left[\gamma_1^{t-1}\right]c_{12} + Var\left[\gamma_1^{t-1}\right]c_{22}$ we can rewrite (5.33) as

$$E\left[R_i^t\right] = e_0 + e_1\beta_i + e_2\beta_i^{\gamma}$$

The expected return depends linearly on the market risk and the risk of a change in the market risk, i.e. it depends on two different beta. Although we have the dependence on two beta, this model is not a special case of other models that will be discussed later having multi-beta structures. The original form had a single (conditional) beta, only to derive the unconditional form this second beta turned out, no other source of risk than the market risk has been added. This second beta is due to the unobservability of expectations. Any risk factor that we can determine can change over time, i.e. we could derive the conditional version for various sources of risk, when determining the unconditional form for every risk factor such a second beta would turn out, hence the Conditional CAPM is a generalization of the unconditional form and not a generalization to include other risk factors.

EMPIRICAL RESULTS

Models with time varying betas and risk premia have attracted increased attention in recent years. The reason on one hand is the empirical evidence that covariances, variances and risk premia are not constant over time. On the other hand gives the poor performance of the traditional CAPM rise to modifications of this model.

By introducing time varying betas and risk premia the conditional CAPM remains to be a static model from its nature. Although a dynamic model would capture reality more appropriate, the conditional CAPM performs much better than its unconditional form. The model by Fama/French (1992), which adds other factors like the book-to-market ratio or firm size, does not perform approximately well and including other factors into the conditional CAPM does not improve the results much, suggesting that their factors are of no real importance.

Jagannathan/Wang (1996) also showed that by including other risk factors, different from those of Fama/French (1992), the performance can be increased significantly by using monthly instead of yearly data. This gives rise to other models allowing for more general risk factors and dynamic models.

Summarizing can be said that the Conditional CAPM fits the data much better than the traditional CAPM, but especially for explaining asset prices in the short run, it fails.

3.6 THE ARBITRAGE PRICING THEORY

Beside the problem of identifying the market portfolio and the critiques concerning the mean-variance criterion, a critical point in the concept of the CAPM is the aggregation of all risks into a single risk factor, the market risk. This aggregation is useful for optimal or at least well diversified portfolios, but for the explanation of returns of individual assets this aggregation may be problematic. It is well observable that assets are not only driven by general factors like the market movement, but that industry or country specific influences also have a large impact on returns.

This section presents an alternative to the CAPM, the *Arbitrage Pricing Theory* (*APT*) as first introduced by Ross (1976).

DERIVATION OF THE APT

Assume that the returns are generated by the following linear structure:

$$\tilde{R}_i = \mu_i + \beta_i \tilde{\delta} + \tilde{\varepsilon}_i$$

where $\tilde{R}_i$ denotes the realized return of asset *i*, μ_i the unconditional expected return, $\tilde{\delta}$ a vector of different risk factors, β_i a vector representing the influence each risk factor has on the asset return and $\tilde{\varepsilon}_i$ an error term summarizing the effects not covered by the model. We make the following assumptions on this structure:

$$E\left[\tilde{\delta}\right] = 0$$

$$E\left[\tilde{\varepsilon}_i\right] = 0$$

These assumptions state that the influence of effects not covered have on average no influence on the returns, i.e. there is no systematic bias. The factors having an influence on the returns are assumed to be normalized, i.e. only deviations from their average values are considered, the effect of the level of these factors are summarized in μ_i.

We do not have to assume that the ε_i are independent of each other, it is sufficient if they are not too dependent on each other, such that the law of large numbers can be applied. We need for all $i, j \left(i \neq j\right)$ that

$$E\left[\tilde{\varepsilon}_i \tilde{\varepsilon}_j\right] = Cov\left[\tilde{\varepsilon}_i, \tilde{\varepsilon}_j\right] \approx 0$$

By the law of large numbers we find a vector $x_n = \left(x_1, \ldots\ldots, x_n\right)'$ such that $Lim_{n\to\infty} x_n' \tilde{\varepsilon} = 0$ where $\tilde{\varepsilon} = \left(\tilde{\varepsilon}_1, \ldots\ldots., \tilde{\varepsilon}_n\right)$ ie. in a well-diversified portfolio the error terms have no influence on the return of a portfolio. We therefore interpret x_i as the weight asset *i* has in such a portfolio.

In general for any portfolio *x* we have

$$x'\tilde{R} = x'\mu + x'\beta\tilde{\delta} + x'\tilde{\varepsilon}$$

$$\approx x'\mu + x'\beta\tilde{\delta}$$

Where $\tilde{R} = \left(\tilde{R}_1,........,\tilde{R}_n\right)$, $\mu = \left(\mu_1,.......,\mu_n\right)$, $\beta = \left(\beta_1,........,\beta_n\right)$. The term $x'\beta$ denotes the risk from the common factors that cannot be eliminated by diversification, i.e. the systematic risk, while $x'\tilde{\varepsilon}$ can be eliminated by diversification. i.e. it is the unsystematic risk.

An *arbitrage portfolio* is defined as a portfolio with no risk, no net investment, but a positive certain return. Hence in an arbitrage portfolio neither systematic nor unsystematic risk must be present and the following restrictions apply:

$$x'\tilde{\varepsilon} = 0$$

$$x'\beta = 0$$

$$x'\iota = 0$$

where $\iota = \left(1,........,1\right)'$. Inserting these conditions into 7.4 gives us for an arbitrage portfolio:

$$x'\tilde{R} = x'\mu > 0$$

In equilibrium we must assure that no arbitrage is possible, i.e.

$$x'\mu > 0$$

If we continue to apply the mean-variance criterion, which is not implied by the APT but mostly done for conventional reasons, the condition of no arbitrage possibilities assures that with the introduction of more assets the efficient frontier does not converge to a vertical line in the $\left(\mu,\sigma\right)$ plane. i.e. $\frac{\partial\mu}{\partial\sigma} < \infty$. The assumptions underlying the APT as presented by Schneller (1990, pp. 2 ff.) and Ross (1976, pp. 347 and 351) are summarized in table 3.1.

- The returns are assumed by investors to follow equation (3.1)
- Investors are risk averse with a finite Arrow-Pratt measure of risk aversion
- No transaction costs or taxes exist
- No restrictions on short sales for any asset
- In equilibrium no arbitrage possibilities exist
- For at least one asset the possible loss from holding the asset is limited to $t < \infty$
- Every asset wants to be held by investors, i.e. the total demand for every asset is positive
- Homogeneity of beliefs, i.e. all investors expect the same $\mu_i < \infty$ and agree on β_i

Table 3.6.1: Assumptions of the APT

Let us consider an arbitrage portfolio fulfilling restrictions (3.5) and the expected return of this portfolio is $x'\mu = c > 0$

where

$$c = m + t$$

with $m = x_0\mu$ denoting the expected return of the optimal portfolio of the investor. From the limitation that $\mu_i < \infty$ for all assets we know that $m < \infty$. We will therefore consider an arbitrage portfolio whose certain return is larger than the expected return of the optimal portfolio plus the maximum loss associated to the asset with limited liability, i.e. the arbitrage portfolio should be preferred to all other portfolios. We do not assume this portfolio to have no unsystematic risk as it is not necessarily well diversified, we only assume it has no systematic risk.

Before deriving the optimal arbitrage portfolio we have to state some preliminaries from probability theory. Ross (1976, p. 359) states that if $\tilde{y}_n \rightarrow a$ in quadratic mean, then $E[U(\tilde{y}_n)] \rightarrow U(a)$ in quadratic mean.

Suppose that the unsystematic risk $x'Vx$, V denoting the covariance matrix, of the arbitrage portfolio converges to zero in quadratic mean i.e. $x'Vx \rightarrow 0$.

We then find that with the above result

$$E\left[U\left(x'\tilde{R}-t\right)\right] \to U\left(x'\mu - t\right) = U\left(c-t\right) > U\left(m\right)$$

By the ε-criterion of convergence there exists a number of assets n such that $E\left[U\left(x_n'\tilde{R}-t\right)\right] > U\left(m_n\right)$ hence the optimal portfolio x_{0n} with its return m_n would no longer be optimal. Hence $x'Vx$ does not converge to zero and there exists a number $a > 0$ such that

$$x'Vx \geq a$$

We now can determine the optimal arbitrage portfolio. From the theory of portfolio selection we know that the only efficient portfolio for a given expected return is the portfolio which has the lowest variance (risk). The risk of the arbitrage portfolio is only the unsystematic risk $x'Vx$. We have to apply the restrictions

$$x'\mu = 0$$

$$x'\beta = 0$$

$$x'\iota = 0$$

By defining $W = (\mu, \beta, \iota)'$ and $g = (c, 0, 0)$ we can rewrite these restrictions as

$$x'W = g$$

The Lagrange function shows up to be

$$L = x'Vx - 2\lambda\left(x'W - g\right)$$

The first order conditions for a minimum are

$$\frac{\partial L}{\partial x} = 2Vx - 2\lambda W = 0$$

$$\frac{\partial L}{\partial \lambda} = -2\left(x'W - g\right) = 0$$

which solves to

$$Vx = \lambda W$$

$$x'W = g$$

Solving (7.14) for x and inserting into (7.15) gives

$$\left(W'V^{-1}W\right)\lambda = g$$

We further find

$$x'Vx = x'W\lambda = g\lambda = g'\left[W'V^{-1}W\right]^{-1} g \geq a \geq 0$$

Ross (1976, pp. 357 f.) has proofed that there exists a vector a^* and a number $A > 0$ such that with $a^*g = 1$

$$\left(Wa^*\right)'\left(Wa^*\right) \leq A < \infty$$

If we define $ca^* = (1, -\gamma, -\rho)'$ we have $a^* g = 1$ and (7.18) becomes

$$\sum_{i=1}^{n}\left(\mu_i - \beta_i\gamma - \rho\right)^2 \leq c^2 A < \infty$$

As (7.19) has to hold for any n, also for $n = \infty$ this implies that on average

$$\left(\mu_i - \beta_i\gamma - \rho\right)^2 \to 0$$

A finite number of assets may not fulfill (7.20) but all of the remaining assets have to fulfil it, hence nearly all assets have expected returns according to

$$\mu_i \to \beta_i\gamma + \rho$$

what implies

$$\mu_i \approx \rho + \beta_i\gamma$$

If we assume asset 1 to be a zero-beta portfolio, i.e. $x_1\beta_1 = 0$ then the return of this asset becomes in the absence of systematic risk:

$$\begin{aligned} x_1\tilde{R}_1 &= x_1\mu_1 + x_1\beta_1\tilde{\delta} + x_1\tilde{\varepsilon}_1 \\ &= x_1\mu_1 \\ &= x_1\beta_1\gamma + x_1\rho \\ &= x_1\rho \end{aligned}$$

Hence we have found that

$$\tilde{R}_1 = \rho \equiv r$$

ρ is the return of the zero-beta portfolio. If there is a riskless asset it is the return of this asset.

Assume now that there exist portfolios which have only a risk on one factor, l, and that this risk is a unit, i.e. $x'\beta^l = 1$ and $x'\beta^s = 0$ for all s 6= l. In this case (7.22) becomes with inserting (7.24) and $x'\iota = 1$ as normalization:

$$x'\mu \approx x' r\iota + x'\beta\gamma = r + \gamma'$$

or

$$\gamma^{\iota} \approx x'\mu - r = \mu^{\iota} - r$$

where $\mu^{l} \equiv x'\mu$ is the expected return of the portfolio with only a unit risk in factor *l* and no risk in the other factors. Therefore we can interpret γ^{l} as the risk premium for having one unit of risk in factor *l*. β^{l}_{i} then represents the amount of risk asset *i* has from factor *l*.

By inserting (7.24) and (7.26) into (7.22) we get

$$\mu_i \approx r + \left(\mu^* - r\iota\right)\beta_i$$

Where $\mu^* = \left(\mu^1, \ldots\ldots\ldots, \mu^k\right)$

The exact relation only holds for an infinite number of assets, but as we find a very large number of assets the deviation will be very small.

At a first glance we could interpret the APT as a generalization of the CAPM to a multibeta model. The same structure of the result in equation (7.27) as in the CAPM seems to support this view. But it has clearly to be pointed out that the models differ substantially in their assumptions. The CAPM is concerned to find an equilibrium of the market by holding optimal portfolios as implied by portfolio theory, whereas the APT finds this equilibrium by ruling out arbitrage possibilities. We will address this distinction between the CAPM and the APT in the next section that provides a very similar extension of the CAPM.

The APT allows to include other sources of risk than only the market risk, e.g. industry specific factors. Furthermore a market portfolio has not to be determined, consequently we also do not face the problem of excluding assets from the considerations, with a sufficient number of asset this relation should hold. Also other measures of risk than variances and covariances could be used, how to determine the betas is not predetermined by this theory.

As noted in section 4.2, Fama/French (1992) and Fama/French (1993) finds evidence that other variables are able to explain the observed returns better than the market risk. The APT could be a framework to find a justification of their results on a sound theoretical basis. The next section will therefore give a short overview of the empirical findings concerning the APT.

EMPIRICAL EVIDENCE

The first problem to solve in applying the APT is to identify the risk factors $\tilde{\delta}$. Risk factors can either be constructed by finding a portfolio of assets that has a high correlation with a certain risk, this portfolio is called a *factor portfolio*, or by using other variables, such as macroeconomic data, e.g. GDP. The advantage of the former approach is that expected returns for the risk factor, μ^{l} can easily be determined from the market and the risk β_i can also be estimated from market data. The identification of these parameters for macroeconomic data imposes much more difficulties. This is the reason why in most cases factor portfolios are used.

To identify the systematic risk and hence the characteristics of the factor portfolios we can either use theoretical considerations or statistical methods to identify these risks. Widely used statistical methods are factor analysis and principal components method.

There exists a large number of surveys investigating the explanation of asset returns using APT. The factors mostly identified in these studies are related to dividends or earnings, book-market relations, the size of a company and the variance of asset returns. Most investigations show that three to five factors are sufficient to explain the observed returns, adding more factors does not improve the result substantially.

The investigations cited give evidence that the APT can explain the observed returns quite good for long and medium time horizons. For time horizons below one year they are not able to explain the data adequately. Compared to the present value model the time horizon can be reduced significantly from four to about one year, but as in the CAPM there remain many effects that cannot be explained sufficiently.

The assumption of a linear relation between the assets in the CAPM by the covariances is replaced by the assumption of a linear relationship with risk factors. Like in the CAPM this assumption limits the theory as nonlinear assets, like derivatives, cannot be modelled adequately. The advantage of the APT in this case is that it is not necessary to form a market portfolio and to include these assets, it enables to exclude human capital or real estate. It enables also to restrict the analysis to a certain group of assets, provided that the number of assets is sufficiently large that the approximation in (7.27) holds. The more assets are included the more precise the findings should be, with restricting to only a few assets the pricing relation does not break down as in the CAPM, it only becomes less precise, i.e. we should find more noise.

3.7 THE INTERTEMPORAL CAPITAL ASSET PRICING MODEL

The models of asset pricing considered so far had one feature in common: they were all static. The amount invested into assets was fixed for a given period of time and the amounts invested into each asset could not be changed. At the end of a time period it was assumed that the investors consume their wealth.

A more realistic setting would be to allow investors to change the amounts invested into each asset and also to withdraw a part of their investment for immediate consumption. Not only the amount invested into each asset but also the fraction of the wealth invested into assets will become an endogenous variable, while in the previous models this fraction was fixed.

The *Intertemporal Capital Asset Pricing Model* (*ICAPM*) as first developed by Merton (1973) takes these considerations in account.

THE MODEL

In the ICAPM we assume a perfect market, i.e. all assets have limited liability, we face no transaction costs or taxes, assets are infinitely divisible, each investor believes that his decision does not affect the market price, the market is always in equilibrium, hence we have no trades outside the equilibrium prices, investors can borrow and lend without any restrictions all assets at the same rate.

Further assumptions are that trading takes place continuously, i.e. all investors can trade at every point of time. All variables that can explain the prices and price changes of the assets (the *state variables*) follow a joint Markov process.

The state variables are further assumed to change continuously over time, i.e. no jumps are allowed.

If we let P_t^i denote the price of asset *i* at time *t*, Ω_t the information available at time *t* and *h* the number of time units, we have for the expected return of asset *i*, μ_i, and the variance of the returns, σ_i^2, per unit of time:

$$\mu_i = \frac{1}{h} E\left[\frac{P_{t+h}^i - P_t^h}{P_t^i} | \Omega_t \right]$$

$$\sigma_i^2 = \frac{1}{h} E\left[\left(\frac{P_{t+h}^i - P_t^i}{P_t^i} - h\mu_i \right)^2 | \Omega_t \right]$$

If we assume that μ_i and σ^2 exist and are finite and that $\lim_{h \to 0} \sigma_i^2 > 0$ i.e. by trading at every point of time (with a very short time horizon) the uncertainty cannot be eliminated.

- All assets have limited liability
- No transaction costs and taxes
- No dividends are paid
- All assets are infinitely divisible
- All investors believe that their decisions do not influence the market price
- All trades take place in equilibrium
- Unrestricted borrowing and lending of all assets at the same conditions
- Trading takes place continuously
- Uncertainty cannot be eliminated by a continuous revision of the portfolio
- The state variables follow a joint Markov process
- The state variables change continuously

Table 3.7.1 summarizes the main assumptions of the ICAPM.

Define $y_i(t)$ to be iid $N(0,1)$ distributed, we then can write the return dynamics implied by (8.2) with solving for $\frac{P_{t+h}^i - P_t^i}{P_t^i}$

$$\frac{P_{t+h}^i - P_t^i}{P_t^i} = h\mu_i + \sqrt{h}\sigma_i y_i(t)$$

Taking the limit of (8.3) with respect to h we get the differential equation of the return process:

$$\frac{dP^i}{P^i} = \lim_{h\to 0}\frac{p_{t+h}^i - p_t^i}{p_t^i} = \mu_i dt + \sqrt{dt}\sigma_i y_i(t)$$

Define dz_i to be a Wiener process:

$$dz_i = y_i(t)\sqrt{dt}$$

Inserting (3.5) into (3.4) gives us the result that returns follow an Ito process:

$$\frac{dP^i}{P^i} = \mu_i dt + \sigma_i dz_i$$

Expected returns and variances of returns are also assumed to follow an Ito process:

$$d\mu_i = a_i dt + b_i dq_i,$$

$$d\sigma_i = f_i dt + g_i dx_i$$

where dq_i and dx_i are two, not necessarily independent, Wiener processes. Equations (3.6)-(3.8) are assumed to form a joint Markov process.

For the further analysis we assume that we have n risky assets and one riskless asset, numbered $n+1$. The riskless rate of return can change over time, i.e. we have

$$\sigma_{n+1} = 0,$$

$$\mu_{n+1} = r,$$

$$b_{n+1} \neq 0.$$

Unlike in the CAPM the investor does not maximize his terminal utility at the end of a given time period (his time horizon), but the utility over the whole time period with length $T > 0$. The investor will maximize the function

$$E[\int_0^T u^k(C^k(t))e^{-p^k t}dt + U^k(W^k(T))e^{-p^k T}|\Omega_0],$$

where U^k denotes the utility of individual k, C^k its consumption at time t and ρ the discount factor for future utility. The first term denotes the present value of consumption from time 0 to time T and the second term denotes the present value of the utility from terminal wealth $W^k(T)$, or terminal consumption.

Merton (1990, pp. 124 ff.) derives the dynamics for the wealth $W^k(t)$ at any point of time. It consists of the number of assets that have been hold at a previous point of time, $N_i^k(t-h)$ multiplied by its present price, P_t^i:

$$W^k(t) = \sum_{i=1}^{n+1} N_i^k(t-h)P_t^i.$$

In the same moment of time he chooses a new portfolio, i.e. a new number of shares, $N_i^k(t)$ and the optimal consumption per unit of time, $C^k(t)h$. His wealth becomes

i

$$W^k(t) = \sum_{i=1}^{n+1} N_i^k(t)P_t^i + C^k(t)h.$$

Solving (8.13) and (8.14) for $C^k(t)h$ gives

$$C^k(t)h = \sum_{i=1}^{n+1}(N_i^k(t-h) - N_i^k(t))P_t^i$$

or with shifting the time period by h:

$$C^k(t+h)h = \sum_{i=1}^{n+1}(N_i^k(t) - N_i^k(t+h))P_{t+h}^i$$

$$= \sum_{i=1}^{n+1}(N_i^k(t) - N_i^k(t+h))(P_{t+h}^i - P_t^i)$$

$$+ \sum_{i=1}^{n+1}(N_i^k(t) - N_i^k(t+h))P_t^i$$

By letting $h \to 0$ (7.16) and (8.13) become

$$W^k(t) = \sum_{i=1}^{n+1} N_i^k(t)P_t^i,$$

$$-C^k(t)dt = \sum_{i=1}^{n+1} dN_i^k(t)dP_t^i + \sum_{i=1}^{n+1} dN_i^k(t)P_t^i.$$

Differentiating (8.18) by using Ito's lemma gives us with (8.19)

$$dW^k(t) = \sum_{i=1}^{n+1} N_i^k(t)dP_t^i + \sum_{i=1}^{n+1} dN_i^k(t)dP_t^i + \sum_{i=1}^{n+1} dN_i^k(t)P_t^i$$

$$= \sum_{i=1}^{n+1} N_i^k(t)dP_t^i - C^k(t)dt.$$

The first term represents the capital gains made from investing into assets and the last term are the losses in wealth due to consumption.

Define $w_i^k(t) = \dfrac{N_i^k(t)P_t^i}{W^k(t)}$ as the fraction of wealth invested into asset *i* after consumption. Inserting this definition into (3.20) we get:

$$dW^k(t) = \sum_{i=1}^{n+1} w_i^k(t)\frac{dP_t^i}{P_t^i}W^k(t) - C^k(t)dt.$$

Inserting (8.6) gives us the Ito process governing wealth:

$$dW^k(t) = \sum_{i=1}^{n+1} w_i^k(t)\mu_i W^k(t)dt$$

$$+ \sum_{i=1}^{n+1} w_i^k(t)\sigma_i W^k(t)dz_i - C^k(t)dt$$

$$= W^k(t)[\sum_{i=1}^{n+1} w_i^k(t)(\mu_i - r) + r]dt$$

$$-C^k(t)dt + W^k(t)\sum_{i=1}^{n+1} w_i^k(t)\sigma_i dz_i.$$

Define further $X = (X_1, \dots, X_m)$ to be the vector of state variables, which we assume to follow an Ito process:

$$dX = Fdt + GdQ$$

Where $dQ = (dq_q, ..., dq_m)$ is a Wiener process, with v_{ij} denoting the correlation between dq_i and dq_j, η_{ij} between dq_i and dz_j. It is further $F = (f_1, ..., f_m)$ and $G = (g_1, \dots, g_m)$.

After these preliminaries we now can solve the maximization problem of equation (8.12). For determining the maximum we have to find the optimal amount of consumption, $C^k(t)$ in every moment of time and the optimal share of the remaining wealth to invest into each asset, $\{w_i^k(t)\}_{i=1}^{n+1}$. We define a performance function $J^k(W^k, t, X)$

as

$$J^k(W^k, t, X) \equiv \max_{C^k(t), \{w_i^k(t)\}_{i=1}^n} E[\int_t^T U(C^k(\tau))e^{-p^k\tau}d\tau$$

$$+U(W^k(T))e^{-p^kT}|\Omega_t].$$

as

$$J^k(W^k, T, X) = U(W^k(T))e^{-p^kT}$$

we can rewrite (8.24) as

$$J^k(W^k, t, X) = \max_{C^k(t), (w_i^k(t))_{i=1}^n} E[\int_t^T U(C^k(\tau))e^{-p^k\tau}d\tau$$

$$+J^k(W^k, T, X)|\Omega_t].$$

with $v \equiv T - t$ applying the mean-value theorem states that there exists a $t^* \in [t, T]$ such that

$$\int_t^T U(C^k(\tau))e^{-p^k\tau}d\tau = U(C^k(t^*))e^{-p^kt^*}u.$$

We can expand $J^k(W^k, T, X)$ in a second order Taylor series around (W^k, t, X):

$$J^k(W^k, T, X) \approx J^k(W^k, t, X) + J_W^k\Delta W + J_t^k v + J_X^k \Delta X$$

$$+\frac{1}{2}J_{WW}^k\Delta W^2 + J_{WX}^k\Delta W\Delta X + \frac{1}{2}J_{tt}^k v^2$$

$$+J_{Wt}^k\Delta W v + J_{Xt}^k\Delta X v + \frac{1}{2}J_{XX}^k\Delta X^2,$$

where the sub-indices denote the derivative with respect to this variable evaluated at $(W^k(t), t, X(t)), \Delta W = W^k(T) - W^k(t)$ and $\Delta X = X(T) - X(t)$.

For simplicity we assume that the derivatives J_W^k and J_X^k do not vary with time i.e. $J_{Wt}^k = J_{Xt}^k = 0$ which is reasonable if we assume v to be not too large. Inserting (8.27) and (8.28) into (8.26), eliminating $J^k(W^k, t, X)$ and dividing by v gives us:

$$\max_{C^k(t),\{w_i^k(t)\}_{i=1}^n} E[U(C^k(t^*))e^{-p^k t^*} + J_W^k \frac{\Delta W}{v} + J_t^k$$

$$+J_X^k \frac{\Delta X}{v} + \frac{1}{2} J_{WW}^k \frac{\Delta W^2}{v} + J_{WX}^k \frac{\Delta W \Delta X}{v}$$

$$+\frac{1}{2} J_{tt}^k v + \frac{1}{2} J_{XX}^k \frac{\Delta X^2}{v} | \Omega_t] = 0.$$

Taking the limit as $v \to 0$, hence $t^* \to t$, we get as W^k and X follow a continuous Ito process:

$$\max_{C^k(t),\{w_i^k(t)\}_{i=1}^n} [U(C^k(t))e^{-p^k t} + J_W^k E\left[\frac{dW}{dt}\right] + J_t^k + J_X^k E\left[\frac{dX}{dt}\right] + \frac{1}{2} J_{WW}^k E\left[\frac{dW^2}{dt}\right]$$

$$+J_{WX}^k E\left[\frac{dWdX}{dt}\right] + \frac{1}{2} J_{XX}^k E\left[\frac{dX^2}{dt}\right]] = 0.$$

With the continuity of the Ito process we get $E[dW] = E[dX] = 0$ and therewith $E[dW^2]$ = $Var[dW]$ and $E[dX^2] = Var[dX]$. with the notations of (8.22) and (8.23) we find that

$$J_X^k E\left[\frac{dX}{dt}\right] = \sum_{i=1}^m J_i^k f_i,$$

$$J_{XX}^k E\left[\frac{dX^2}{dt}\right] = \sum_{i=1}^m \sum_{j=1}^m J_{ij} \vartheta_i \vartheta_j v_{ij},$$

$$J_{WX}^k E\left[\frac{dWdX}{dt}\right] = \sum_{i=1}^m \sum_{j=1}^n J_{iWw_j^k \vartheta_i \sigma_j \eta_{ij}} W^k v_{ij},$$

$$J_{WW}^k E\left[\frac{dW^2}{dt}\right] = J_{WW} \sum_{i=1}^m \sum_{j=1}^n w_i^k w_j^k \sigma_{ij} W^2,$$

where the subindices *i* and *j* are for the derivation with respect to the *i*'s and *j*'s state variable. We further have

$$\frac{dW^k}{dt} = W^k \left(\sum_{i=1}^{n+1} w_i^k (\mu_i - r) + r \right) - C^k.$$

Inserting these results into (8.30) gives

$$\max_{C^k(t),\{w_i^k(t)\}_{i=1}^n} [U(C^k)e^{-p^k t} + J_t^k$$

$$+J_W^k\left(W^k\left(\sum_{i=1}^{n+1} w_i^k\left(\mu_i-r\right)+r\right)-C^k\right)$$

$$+\sum_{i=1}^{m} j_i^k f_i+\frac{1}{2} J_{WW} \sum_{i=1}^{m} \sum_{j=1}^{n} w_i^k w_j^k \sigma_{ij} W^2$$

$$+\sum_{i=1}^{m} \sum_{j=1}^{n} J_{iWw_j^k \vartheta_i \sigma_j \eta_{ij}} W^k v_{ij}$$

$$+\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} J_{ij} \vartheta_i \vartheta_j v_{ij}]=0.$$

Defining the function in brackets for notational simplicity as Φ and denoting the derivative with respect to C^k as Φ_C and with respect to w_i^k as Φ_i we get the (n+1) first order conditions for a maximum:

$$\Phi_C=U_C-J_W^k=0,$$

$$\Phi_i=J_w^k\left(\mu_i-r\right) W^k+J_{WW}^k \sum_{j=1}^{n} w_j^k \sigma_{ij} W^2+\sum_{j=1}^{m} J_{jW}^k W^k \vartheta_j \sigma_i \eta_{ij}=0.$$

Condition (8.32) states the usual result that marginal utility of immediate consumption (U_C) has to equal marginal utility from deferred consumption (J_W).

If we define $V=\left\{\sigma_{ij}\right\}_{i,j=1}^{n}$ as the covariance matrix of the assets $d^k=\left\{w_j W^k\right\}_{j=1}^{n}$ as the demand vector for the assets of the investor k, $\mu=\left\{\mu_j\right\}_{j=1}^{n}$ as the vector of expected returns, $\sigma=\left\{\vartheta_j \sigma_i \eta_{ij}\right\}_{i,j=1}^{n,m}$ as the covariance matrix of the asset returns and the state variables $A^k=-\frac{J_W^k}{J_{WW}}, H_j^k=-\frac{J_j^k w}{J_{WW}^k}$ and $H^k=\left\{H_i^k\right\}_{j=1}^{n}$ we can rewrite (8.34) after dividing by W^k in vector form.

$$-A^k\left(\mu-r\iota\right)+V d^k-H^k \sigma=0.$$

Solving for the demand we have:

$$d^k=A^k V^{-1}\left(\mu-r\iota\right)+H^k V^{-1} \sigma$$

With v_{ij} as the (*i, j*)th element of V^{-1} and $\sigma_{ij}^*=g_j \sigma_i \eta_{ij}$ we can rewrite (8.35) for an individual asset as

$$d_i^k=A^k \sum_{j=1}^{n} \upsilon_{ij}\left(\mu_j-r\right)+\sum_{j=1}^{m} \sum_{\iota=1}^{n} H_\iota^k \sigma_{j\iota}^* \upsilon_{ij.}$$

The demand for an asset consists of two parts. The first term is the demand similar to that of the static CAPM, it is due to an effic ient investment by mean-variance maximizing. A^k includes a term for risk aversion of the investor and $\mu_j - r$ is the risk premium. The second term is an adjustment made for hedging against an unfavourable shift in the state variables. This adjustment can be either positive or negative, depending on the sign of $\frac{\partial C^k}{\partial x_i}$ which forms a part of H_i^k.

If the state variables do not vary over time, it turns out that $H_i^k = 0$ and the results are identical to the standard CAPM. In general, however, the state variables will change over time. For simplicity we now assume that only one state variable varies over time, e.g. the risk free interest rate r. In this case (8.36) becomes

$$d_i^k = A^k \sum_{j=1}^{n} \upsilon_{ij} \left(\mu_j - r\right) + H_\tau^k \sum_{j=1}^{m} \sigma_{j\tau}^* \upsilon_{jr}.$$

Assume there exists an asset, the nth asset, that is perfectly negative correlated with the state variable, i.e. $\rho_{nr} = -1$. Such an asset could be a bond, which is riskless in terms of default and whose value only depends on the interest rate.

Because of the varying interest rate the bond would no longer be riskless in its value. We find

$$p_{j\tau} = p_{jn} p_{n\tau} = -p_{jn},$$

such that

$$\sigma_{j\tau}^* = p_{j\tau} \sigma_j \vartheta_\tau = -\sigma_{jn} \sigma_j \vartheta_\tau = -\vartheta_\tau \frac{p_{jn} \sigma_j \sigma_n}{\sigma_n} = -\vartheta_\tau \frac{\sigma_{jn}}{\sigma_n}.$$

By using (3.39) the second term in (3.37) becomes $-H_r^k \frac{g_r}{\sigma_n} \sum_{j=1}^{n} v_{ij} \sigma_{jn}$. Because v_{rj} is an element of the inverse of σ we find that if $i = n$

$$\sum_{j=1}^{n} \upsilon_{ij} \sigma_{jn} = 1$$

and otherwise

$$\sum_{j=1}^{n} \upsilon_{ij} \sigma_{jn} = 0.$$

Therewith we can simplify (3.37) to

$$d_i^k = A^k \sum_{j=1}^{n} \upsilon_{ij} \left(\mu_j - \Upsilon\right) \text{ for } i = 1, \ldots\ldots, n-1,$$

$$d_n^k = A^k \sum_{j=1}^{n} \upsilon_{nj} \left(\mu_j - \Upsilon\right) - H_\tau^k \frac{\vartheta}{\sigma_n}.$$

We define now three portfolios:

Portfolio 1: Consists of all n risky assets with weights $\delta_i = \frac{\sum_{j=1}^{n} \upsilon_{ij}\left(\mu_j - \tau\right)}{\sum_{j=1}^{n}\sum \iota = 1^n \upsilon_{j\iota}\left(\mu_{j-\Upsilon}\right)}$, which are equal to the weights of the optimal risky portfolio in portfolio theory.

Portfolio 2: Consists only of the nth asset.

Portfolio 3: Consists only of the riskless asset.

By construction the compositions of the portfolios do not depend on the preferences of the investors. Let λ_i^k (i = 1, 2, 3) denote the fraction of his total wealth investor k invests into each of these portfolios. The total demand for the first $n-1$ assets is given by $\lambda_1^k \delta_i W^k$ and for the nth asset by $(\lambda_1^k \delta_n + \lambda_2^k) W^k$. These demands have to satisfy the conditions (8.42) if an investor has to be indifferent between investing into all n assets directly or investing only in the three portfolios:

$$\lambda_1^k \delta_i W^k = \lambda_1^k \frac{\sum_{j=1}^{n} \upsilon_{ij}\left(\mu_j - \Upsilon\right)}{\sum_{j=1}^{n}\sum \iota = 1^n \upsilon_{j\iota}\left(\mu_j - \Upsilon\right)} W^k = d_i^k = A^k \sum_{j=1}^{n} \upsilon_{ij}\left(\mu_j - \Upsilon\right),$$

$$\left(\lambda_1^k \delta_n + \lambda_2^k\right) W^k = d_n^k = A^k \sum_{j=1}^{n} \upsilon_{nj}\left(\mu_j - \Upsilon\right) - H_\tau^k \frac{\vartheta}{\sigma_n}.$$

Solving λ_1^k and λ_2^k gives

$$\lambda_1^k = \frac{A^k}{W^k} \sum_{j=1}^{n} \sum_{\iota=1}^{n} \upsilon_{ij}\left(\mu_j - \Upsilon\right),$$

$$\lambda_2^k = -\frac{H_\tau^k}{W^k} \frac{\vartheta_\tau}{\sigma_n}.$$

Instead of directly investing into all n risky assets, an investor will be indifferent to choosing a combination of the optimal risky portfolio and the nth asset that has the highest correlation with the changing state variable. The remaining wealth is invested into the third portfolio consisting only of the riskless asset. This result is known as the *Three-Fund Theorem*. It is the dynamic equivalent of Tobin's Separation Theorem.

Portfolios one and three ensure the investor to have a mean-variance efficient holding of assets, i.e. his holdings are on the efficient frontier of the CAPM. But as the state variable changes over time stochastically, the efficient frontier shifts over time, portfolio two hedges against an unfavourable shift.

In the general case where $m \geq 1$ state variables can change over time, it has been shown in Merton (1990, pp. 499 ff.) that (m + 2) portfolios are combined by the investors, hence it is known as the *(m+2)-Fund Theorem*. The properties in the general case do not change, one portfolio remains to be the optimal risky portfolio, one consists only of the riskless assets and the other *m* portfolios consist only of the asset having the highest correlation with one of the state variables.

We can now derive the equilibrium expected returns in the case of *m* = 1 state variables. Solving (8.34) for $\mu - r\iota$ gives

$$\mu - \Upsilon\iota = \frac{1}{A^k} V d^k - \frac{H_\tau^k}{A^k}\sigma.$$

As this relation holds for every investor individually, it also has to remain valid when aggregating the demands of all *K* investors.

Let $A = \sum_{K=1}^{K} A^k, H = \sum_{K=1}^{K} H^k, D_i = \sum_{K=1}^{K} d_i^k$ and $D = \{D_i\}_{i=1}^{n}$.

We can rewrite (8.45) as

$$\mu - \Upsilon\iota = \frac{1}{A} VD - \frac{H}{A}\sigma.$$

As we have assumed that trade only occurs in equilibrium, the total demand always has to equal the value of all assets, $M : \sum_{i=1}^{n+1} D_i = M$.

We define $w_i = \frac{D_i}{M}$ as the fraction of the *i*th asset in the total market value, which because of the equilibrium assumption has to equal relative demand. In scalar form (8.46) becomes for *i* = 1*, . . . , n*:

$$\mu_i - \Upsilon = \frac{M}{A}\sum_{j=1}^{n} w_j \sigma_{ij} - \frac{H}{A}\sigma_{i\Upsilon}$$

$$= \frac{M}{A}\sum_{j=1}^{n} w_j \sigma_{ij} + \frac{H}{A}\frac{\sigma_{in}\vartheta_r}{\sigma_n}$$

If we denote the market portfolio by the subindex M we have $\mu M - \Upsilon = \sum_{j=1}^{n} w_j\left(\mu j - \Upsilon\right), \sigma iM = \sum_{j=1}^{n} w_j \sigma_{ij}$ and $\sigma_M^2 = \sum_{j=1}^{n} w_j \sigma_j M$ and (8.47) becomes:

$$\mu i - \Upsilon = \frac{M}{A}\sigma_{iM} - \frac{H}{A}\sigma_{i\tau}.$$

Multiplying by *wj* and summing over all *i* = 1*, . . . , n* gives

$$\mu M - \Upsilon = \frac{M}{A}\sigma_M^2 + \frac{H}{A}\frac{\sigma_{Mn\vartheta\tau}}{\sigma n} = \frac{M}{A}\sigma_M^2 - \frac{H}{A}\sigma M\tau.$$

For *i* = *n* (8.47) becomes

$$\mu_n - \Upsilon = \frac{M}{A}\sigma_{nM} - \frac{H}{A}\sigma_{n\tau}.$$

By solving (8.49) and (8.50) for we get

$$\frac{M}{A}=\frac{\sigma_{n\tau}}{\sigma_{n\tau}\sigma_M^2-\sigma_{M\tau}\sigma_{nM}}\left(\mu M-\Upsilon\right)-\frac{\sigma M\tau}{\sigma_{n\tau}\sigma_M^2-M_\tau\sigma_{nM}}\left(\mu_n-\Upsilon\right),$$

$$\frac{H}{A}=\frac{\sigma_{nM}}{\sigma_{n\tau}\sigma_M^2-\sigma_{M\tau}\sigma_{nM}}\left(\mu_M-\Upsilon\right)-\frac{\sigma_M^2}{\sigma_{n\tau}\sigma_M^2-\sigma_{M\tau}\sigma_{nM}}\left(\mu_n-\Upsilon\right).$$

Inserting their results into (8.47) gives the equilibrium expected returns after rearranging terms:

$$\mu_i-\Upsilon=\frac{\sigma_{n\tau}\sigma_{iM}-\sigma_{nM}\sigma_{i\tau}}{\sigma_{n\tau}\sigma_M^2-\sigma_{M\tau}\sigma_{nM}}\left(\mu M-\Upsilon\right)$$

$$+\frac{\sigma_M^2\sigma_{i\tau}-\sigma_{M\tau}\sigma_{iM}}{\sigma_{n\tau}\sigma_M^2-\sigma_{M\tau}\sigma_{nM}}\left(\mu_n-\Upsilon\right)$$

$$=\beta_i^1\left(\mu_M-\Upsilon\right)+\beta_i^2\left(\mu_n-\Upsilon\right),$$

with $$\beta_i^1=\frac{\sigma_{n\tau}\sigma_{iM}-\sigma_{nM}\sigma_{i\tau}}{\sigma_{n\tau}\sigma_M^2-\sigma_{M\tau}\sigma_{nM}}\,and\,\beta_i^2=\frac{\sigma_M^2\sigma_{i\tau}-\sigma_{M\tau}\sigma_{iM}}{\sigma_{n\tau}\sigma_M^2-\sigma_{M\tau}\sigma_{nM}}.$$

The excess returns compensate investors for the systematic risk like in the CAPM and additionally for the risk of an unfavourable shift in the state variable

Even assets which have a beta of zero, i.e. are uncorrelated with the market portfolio, may have returns higher than the riskless rate of return because of the exposure to an unfavourable shift in the state variable. This relation can be observed in reality and could not be explained by the CAPM.

By allowing in general m≥ 1 state variable to change over time, Merton (1990, p. 510) has shown that (8.53) in general becomes

$$\mu_i-\Upsilon=\beta_i^1\left(\mu_M-\Upsilon\right)+\sum_{j=1}^{m}\beta_i^j\left(\mu_j-\Upsilon\right).$$

i.e. every risk of a state variable is compensated individually. The ICAPM extends the CAPM to a dynamic environment. The results are similar to those of the APT, but it has the advantage that the risks can be determined from characteristics of the assets. Cox et al. (1985) derive a similar result in a much more general framework than provided here.

EMPIRICAL EVIDENCE

Empirical investigations into the performance of the ICAPM are not frequently found. The reason is that at a first glance the ICAPM and the APT look alike from their results. The ICAPM only has a fixed risk factor, the market portfolio. In the APT the market portfolio is not a risk factors as it has been assumed that an investor is perfectly well diversified and hence the only source of risk are common factors. In the ICAPM portfolios have not to be perfectly diversified, nor the market portfolio neither the portfolios having the highest correlation with the state variables, hence we could interpret the risk from the market portfolio as arising from imperfect diversification. If all portfolios are perfectly diversified and the state variables equal the common factors, the ICAPM collapses to the APT. We could therefore view the APT as a special case of the ICAPM. Therefore the APT and ICAPM are often treated alike, despite their different theoretical foundations.

The investigation by Dokko/Edelstein (1991) explicitly uses the ICAPM as their reference model. They use the uncertainty about future inflation and real production as state variables. By using monthly data for the period 1960-1985 they show that excess returns are significantly affected by changes in these state variables.

They further find changes inflation uncertainty to be highly persistent, whereas changes in real production uncertainty are only temporary. Therefore they find that changes in inflation uncertainty influence asset prices significantly, whereas real production uncertainty has no important influence.

With these results Dokko/Edelstein (1991) can explain the behaviour of the stock market in their observed time period quite well. But as most models it fails to explain short-term movements of prices.

The ICAPM faces the same problem of identifying state variables as does one in the APT with identifying common factors. Fama (1998) addresses this problem of determining state variables.

3.8 THE CONSUMPTION-BASED CAPITAL ASSET PRICING MODEL

When we want to apply the ICAPM to explain the behaviour of asset prices we face the problem of identifying the relevant state variables, the theory does give no hint how to choose the relevant variables. The *Consumption-Based Capital Asset Pricing Model* (*CCAPM*) as first developed by Breeden (1979) develops the ICAPM further within the same theoretical framework to aggregate the risks from shifting state variables into a single variable, consumption.

DERIVATION OF THE MODEL

The assumptions underlying the CCAPM are identical to those of the ICAPM developed in the last section. The investors are also assumed to maximize the function already stated in equation (8.12)

$$E\left[\int_0^T U^k\left(C^k(t)\right)e^{-p^k t}dt + U^k\left(W^k(T)\right)e^{-p^k T}\middle|\Omega_0\right]$$

with respect to consumption $C^k(t)$ and the portfolio composition $\left\{w_i^k\right\}_{i=1}^n$.

The only additional assumption we have to make concerns consumption. We assume that there exists only a single consumption good and that the state variables influence also the consumption, i.e. consumption becomes a stochastic variable. This influence could e.g. be through influencing the price and therewith the number of goods that can be bought with a given wealth.

The first order conditions for maximizing (9.1) have already been derived in section 4.7. We restate them here for convenience slightly modified:

$$J_W^k = U_C,$$

$$d^k = w^k W^k = -\frac{J_W^k}{J_{WW}^k}V^{-1}\left(\mu - \Upsilon\iota\right) - \frac{J_{XW}^k}{J_{WW}^k}V^{-1}\sigma.$$

If we derive (9.2) again with respect to the state variables X and wealth W^k we get

$$J_{WX}^k = U_{CX} = U_{CC}C_X^k,$$

$$J_{WW}^k = U_{CW} = U_{CC}C_W^k.$$

Inserting these results into (9.3) we get

$$w^k W^k = -\frac{Uc}{U_{CC}C_W^k}V^{-1}\left(\mu - \Upsilon\iota\right) - \frac{U_{CC}C_X^k}{U_{CC}C_W^k}V^{-1}\sigma$$

$$= z^k\frac{1}{C_W^k}V^{-1}\left(\mu - \Upsilon\iota\right) - \frac{C_X^k}{C_W^k}V^{-1}\sigma,$$

With $z^k = -\frac{U_{CW}}{U_{CC}}$ as the Arrow Platt measure of risk aversion. Premultiplying (9.6) by $C_W^k V$ gives

$$C_W^k V w^k W^k = z^k \left(\mu - \Upsilon \iota\right) - C_X^k \sigma.$$

The term $V\ w^k\ W^k$ represents the vector of covariances of the asset returns with a change in the wealth. We will denote $V_{\mu Wk} \equiv V w^k W^k$. Rearranging (9.7) gives

$$z^k \left(\mu - \Upsilon \iota\right) = V_{\mu Wk} C_W^k + C_X^k \sigma.$$

The covariances of the asset returns with the investors consumption can be derived using Ito's lemma:

$$V_{\mu C^k} = V_{\mu W^K} C_W^k + C_X^k \sigma.$$

Inserting (9.9) into (9.8) gives:

$$z^k \left(\mu - \Upsilon \iota\right) = V_{\mu C^k}.$$

As (9.10) holds for every investor we can aggregate their decisions. Defining $z \equiv \sum_{k=1}^{K} z^k and V_{\mu C} \equiv \sum_{k=1}^{K} V_{\mu C^k} 4$ gives us

$$z\left(\mu - \Upsilon \iota\right) = V_{\mu C},$$

or for a single asset

$$z\left(\mu i - \Upsilon\right) = \sigma i C.$$

Equation (3.12) has also to be fulfilled for any portfolio. Let such a portfolio be denoted by the subindex *M*, it has not necessarily to be the market portfolio. With the weights w^M of this portfolio we can define the portfolio's expected return and covariance with consumption as $\mu_M \equiv \sum_{i=1}^{n} w_i^M \mu_i and \sigma_{MC} = \sum_{i=1}^{n} w_i^M \sigma i C.$

Multiplying (3.12) with and adding over all assets gives us

$$z\left(\mu_M - \Upsilon\right) = \sigma_{MC}.$$

Solving this equation for z and inserting into (3.12) gives after rearranging

$$\mu_i - \Upsilon = \frac{\sigma_{iC}}{\sigma_{MC}}\left(\mu M - \Upsilon\right).$$

By defining betas with respect to aggregate consumption in the conventional way as

$\beta_{iC} \equiv \frac{\sigma_{iC}}{\sigma_C^2} and \beta_{MC} \equiv \frac{\sigma_{MC}}{\sigma_C^2}$ we can rewrite (3.14) as

$$\mu_i - \Upsilon = \frac{\beta_{iC}}{\beta_{MC}}\left(\mu_M - \Upsilon\right).$$

Suppose there exists a portfolio M that is perfectly correlated with changes in aggregate consumption, then (9.15) reduces to

$$\mu_i - \Upsilon = \beta_{iC}\left(\mu_M - \Upsilon\right).$$

The risks of an unfavourable shift in the state variables that determined the excess returns in the ICAPM have now been aggregated into a general risk factor, aggregate consumption. The different risk factors all influence current and future consumption, hence they can be aggregated in this way. The term is also called the market price of consumption risk

In order to interpret this result we have to remember that the present utility, is, beside terminal wealth, affected by the amount of consumption and its timing. As we further can reasonably argue that i.e. the higher the wealth the higher consumption, and wealth is affected by these returns, consumption is also affected by the returns of the assets held.

Assume now that a state occurs that reduces current consumption and the expected return of an asset, i.e. $\beta_{iC} \gg 0$. The reduced current consumption reduces the investors present utility. Because expected future returns of the asset are also reduced, future expected wealth is reduced and hence future expected consumption, what reduces utility further. When holding the asset an investor faces this risk of an unfavorable shift in the state variables that influence his future consumption, hence he has to be compensated for this risk. The same argumentation can be used for a negative beta.

EMPIRICAL INVESTIGATIONS

An empirical investigation of the CCAPM faces several measurement problems concerning aggregate consumption:

The statistical data on aggregate consumption capture not consumption directly, but the expenditures for consumption goods and services. As the goods purchased are not necessarily consumed immediately (e.g. they can be stored for later consumption or can be consumed over time like consumer durables), the data will be biased, although as a result of aggregation the bias will be reduced, but it may lead or lag the business cycle.

The data available are not for an instant of time, but denote consumption for a certain period, at least a month or a quarter, whereas asset prices are available on a daily or even intraday basis.

The data are generated from samples and we face the problem of sampling errors.

Compared to the use of a market portfolio as in the CAPM the use of consumption data also has a virtue. While the market portfolio that is determined typically does not include important assets like real estate or human capital, consumption data cover a much larger fraction of the effective consumption.

Beside the data problem an empirical investigation faces many econometric problems, e.g. the determination of the portfolio with the highest correlation with a change in aggregate consumption (also called the *Maximum Correlation Portfolio*), which is the equivalent to the market portfolio in the CAPM. For an overview of these econometric considerations see Breeden et al. (1989).

Breeden et al. (1989) finds that excess returns of a zero beta portfolio are small, as predicted by the theory, whereas they are relatively large in the CAPM model. They report further that the market price of consumption risk is positive by observing data for the period 1929-1982 using quarterly and monthly data. The linearity of excess returns and consumption risk (β_{iC}) is only rejected for more recent sub periods (1947-1982) where the data quality is improved. This suggests that data quality is a crucial factor for interpreting the results.

Summarizing can be said that the results show a weak support for the CCAPM, a situation similar to the results concerning the CAPM. As other models presented thus far the CCAPM is especially not able to explain the behaviour of asset prices in the short run, but this may also be the consequence of missing consumption data for periods below a month.

Ahn/Cho (1991) show that with a time varying risk aversion, where the risk aversion depends on past returns, the CCAPM becomes much more consistent with the data.

3.9 THE INTERNATIONAL CAPITAL ASSET PRICING MODEL

The previous models of asset pricing implicitly assumed that all investors are located in the same country and consider only assets in their home country. In reality, however, we find investors in different countries and they also invest a part of their wealth abroad. In such a framework we therefore should consider the influence of exchange rates, different tastes for consumption across countries and barriers to foreign investment. These points were incorporated into the *International Capital Asset Pricing Model* (*International CAPM*) as developed by Stulz (1981b), Stulz (1981a) and Stulz (1995).

NO DIFFERENCES IN CONSUMPTION AND NO BARRIERS TO FOREIGN INVESTMENT

Stulz (1995) provides a model where neither the investors have different tastes across countries nor different costs of investing at home and abroad are faced. The assumptions underlying this model are very similar to those of the ICAPM, they are listed in table 10.1.

We assume that the price of asset *i*, denominated in the currency of country *j*, P_{ij} follows an Ito process:

$$\frac{dP_{ij}}{P_{ij}} = \mu_{ij}dt + \sigma_{ij}dz_{ij},$$

where μ_{ij} denotes the expected nominal return of asset *i* in currency *j*, σ_{ij} the standard deviation of this return and dz_{ij} a standard Wiener process. The price of the consumption good in country *j*, P_j^C, also follows an Ito process:

$$\frac{dP_j^C}{P_j^C} = \pi_j dt + \sigma_{\pi j}dz_{\pi j},$$

where π_j denotes the expected inflation rate in country j, $\sigma_{\pi j}$ the inflation rate's variance and $dz_{\pi j}$ a standard Wiener process. The real price of an asset in terms of the consumption good is given by:

$$P_{ij}^r = \frac{P_i j}{P_j^C}$$

and can be shown by Ito's lemma to follow

$$\frac{dP_{ij}^\tau}{P_{ij}^\tau} = \frac{d\,\frac{P_{ij}}{P_j^C}}{\frac{P_{ij}}{P_j^C}} = \left[\mu_{ij} - \pi_j - \sigma_{ij,\pi j} + \sigma_{\pi j}^2\right]dt + \sigma_{ij}dz_{ij} - \sigma_{\pi j}dz_{\pi j},$$

where $\sigma_{ij,\pi j}$ denotes the covariance between the nominal asset return and the inflation in this country. The first term denotes the expected real return of asset *i* in country $j, \mu_{ij}^{\tau} \equiv \mu_{ij} - \pi_j - \sigma_{ij,\pi j} + \sigma_{\pi j}^2$. As the sum of two Wiener processes is again a Wiener process we define a Wiener process $\sigma_{ij}^{\tau} dz_{ij}^{\tau} \equiv \sigma_{ij}^{\tau} dz_{ij}^{\tau} - \sigma_{\pi j} dz_{\pi j}$ and get an Ito process for the real price of the asset:

$$\frac{dP_{ij}^{\tau}}{P_{ij}^{\tau}} = \mu_{ij}^{\tau} dt + \sigma_{ij}^{\tau} dz_{ij}^{\tau}.$$

We have now formulated exactly the CAPM derived in section 4.3, only substituting nominal returns by real returns. With the law of one price we know that real prices and returns are equal in all countries and we can write (10.5) as

$$\frac{dP_i^{\tau}}{P_i^{\tau}} = \mu_i^{\tau} dt + \sigma_i^{\tau} dz_i^{\tau}.$$

From the CAPM we know that in this case

$$\mu_i^{\tau} - \Upsilon^{\tau} = \beta_i^{\tau} \left(\mu_W^{\tau} - \Upsilon^{\tau} \right),$$

where the superscript *r* denotes real variables and is the expected real return on the world market portfolio. In real terms there are no differences between countries and for the decisions it is of no importance where an investor and an asset is located.

It is more convenient to write the returns in nominal than in real returns. Assume therefore that there exists an asset that is riskless in nominal terms and that has a zero beta with the real world portfolio return. This asset also has to fulfil (10.7). By replacing μ^r by its original expression we get

$$\Upsilon_j - \pi_j - \sigma_{ij,\pi j} + \sigma_{\pi j}^2 - \Upsilon^{\tau} = 0.$$

Rearranging and inserting into (10.7) gives us

$$\mu_{ij} - \left(\sigma_{ij,\pi j} - \sigma_{\tau j,\pi j} \right) - \Upsilon_j = \beta_i^{\tau} \left(\mu_W^{\tau} - \Upsilon^{\tau} \right).$$

The last expression on the on the right side denotes the real excess return of the world market portfolio. As we know from the derivation of the real excess return this is given by

$$\mu_W^{\tau} - \Upsilon^{\tau} = \mu Wj - \pi j - \sigma W_{j,\pi j} + \sigma_{\pi j}^2 - \left(\Upsilon_j - \pi_j - \sigma_{rj,\pi j} + \sigma_{\pi j}^2 \right)$$

$$= \mu Wj - \left(\sigma W_{j,\pi j} - \sigma_{\tau j,\pi j} \right) - \Upsilon j.$$

Stulz (1995, p. 205) makes now the assumption that for all assets $\sigma_{ij,\pi j} = 0$,

i.e. the nominal asset returns are uncorrelated with the inflation rate. With this assumption we see from (10.9) and (10.10) that the real and nominal access returns coincide. The beta of the real returns also equals the beta of the nominal returns, Hence (10.9) becomes

$$\mu_{ij} - \Upsilon_j = \beta_{ij}\left(\mu W_j - \Upsilon_j\right).$$

We therewith found that with the assumption of no differences in tastes and no investment barriers the CAPM as derived in section 4.3 remains valid in an international setting. The market portfolio becomes the world market portfolio. We can view the world as an integrated market, different currencies and borders have no influence with the assumption of Purchasing Power Parity.

We will further on consider the influence different tastes and investment barriers have on the expected returns of assets.

Differences in consumption

Stulz (1981a) extends the CCAPM to the case that investors and assets are located in different countries and that the tastes of the investors differ between these countries but are identical within a country. For simplicity we assume that there exist only two countries, the home country and the foreign country, denoted by an asterisk at the variables. There exist no barriers to international investment.

To model the different tastes in the countries we assume that there exist k different consumption goods in the home country and k^* in the foreign country and at least one consumption good is consumed in both countries. For the goods consumed in both countries the law of one price applies as before, i.e. with an exchange rate of e between the two currencies we have for these consumption goods:

$$P_j^C = eP_j^{C*}.$$

The price for the jth domestic good in domestic currency is assumed to follow an Ito process:

$$\frac{dP_j^C}{P_j^C} = \pi_j dt + \sigma_{\pi j} dz_{\pi j}.$$

where μ^C_j denotes the expected price change, σ^C the variance of the price change and dz^C a standard Wiener process. The exchange rate also follows an Ito process:

$$\frac{de}{e} = \mu_e dt + \sigma_e dz_e,$$

with μ_e as the expected change in the exchange rate, σ_e its variance and dz_e a standard Wiener process. We assume that there are n risky assets that are traded among countries, hence the law of one price applies to all assets. Additionally each country has a riskless nominal asset in his currency with an interest rate of r and r^*, respectively. These riskless assets are also traded between the countries as the $(n+1)$th asset, but as the exchange rate may vary it is not riskless in the other country.

The risky foreign assets are assumed to follow an Ito process in foreign currency:

$$\frac{dP_{i*}^*}{P_{i*}^*} = \mu_{i*}^* dt + \sigma_{i*}^* dz_{i*},$$

Where is the expected return in foreign currency, the standard deviation of these returns and a standard wiener process. Using the law of one price for these assets the price in domestic currency is given by and by applying Ito's lemma the dynamics of the foreign assets in domestic currency are given by

$$\frac{dP_{i*}}{P_{i*}} = \frac{dP_{i*}^*}{P_{i*}^*} + \frac{de}{e} + \frac{de}{e}\frac{dP_{i*}^*}{P_{i*}^*}.$$

Analogous equations can be derived for the domestic assets. For the riskless assets in domestic currency these dynamics are given by

$$\frac{dP_{n+1}}{P_{n+1}} = rdt,$$

$$\frac{dP_{n+1*}}{P_{n+1*}} = \Upsilon^* dt + \frac{de}{e}.$$

Define the excess returns of a foreign asset in domestic currency, H_{i*}, as the return of this asset financed by borrowing abroad at the rate of r^*. We get by using (10.16) and (10.18):

$$\frac{dH_{i*}}{H_{i*}} = \frac{dP_{i*}}{P_{i*}} - \frac{dP_{n+1*}}{P_{n+1*}}$$

$$= \frac{dP_{i*}^*}{P_{i*}^*} + \frac{de}{e}\frac{dP_{i*}^*}{P_{i*}^*} - \Upsilon^* dt.$$

The dynamics of the exchange rate does not affect the return as the investor has fully hedged this risk by financing his investments abroad. Hence the exchange rate risk will not be compensated as it can be diversified, the only risk is the development of the return in foreign currency. Equation (10.19) for a foreign investor investing into a domestic asset becomes:

$$\frac{dH_i^*}{H_i^*} = \frac{dP_i}{P_i} - \frac{de}{e}\frac{dP_i}{P_i} - \Upsilon dt.$$

For a domestic investor investing at home we have:

$$\frac{dH_i}{H_i} = \frac{dP_i}{P_i} - \Upsilon dt.$$

The excess returns of foreign and domestic investors differ only in the term .i.e the covariance of the change in the exchange rate and the return of the asset in domestic currency. We further get the excess returns for a domestic investor for the foreign riskless asset by (10.17) and (10.18) as:

$$\frac{dH_{n+1^*}}{H_{n+1^*}} = \Upsilon^* dt + \frac{de}{e} - \Upsilon dt$$

$$= \left(\Upsilon^* - \Upsilon\right) dt + \frac{de}{e}$$

$$\frac{dH^*_{n+1}}{H^*_{n+1}} = \left(\Upsilon^* - \Upsilon\right) dt - \frac{de}{e} + \left(\frac{de}{e}\right)^2.$$

After these preliminaries we are now able to derive the model of international asset pricing. Unlike in the CCAPM presented above we now have more than one consumption good. Let q^k denote the quantities consumed by domestic investor *k* of consumption good *j*, then we have

$$C^k = \sum_{j=1}^{n} P_j^C \vartheta_j^k = P^C \vartheta^k,$$

where C^k denotes the consumption of the *k*th domestic investor, P^C the vector of domestic consumption good prices and q^k the vector of consumption for each consumption good. Besides the determination of the optimal amount spent for consumption, the investor now additionally has to find the optimal allocation the different consumption goods. We therefore define the utility function as

$$U^k\left(C^k\left(P^C\right), P^C\right) = \max_{q^k} u^k\left(q^k\right)$$

$$= \max_{q^k} \left(u^k\left(q^k\right) + \lambda(C^k - P^C q^k\right).$$

The first order conditions for this maximization are

$$u_q = \lambda P^C,$$

$$U_P = -\lambda q^k.$$

where $\lambda = U_C$ is the shadow price of increased consumption. Hence (10.25) and (10.26) become

$$u_q = U_C P^C,$$

$$U_P = -U_C q^k.$$

The optimal portfolio for a domestic investor has been derived in equations (9.2) and (9.3) to fulfil the following conditions:

$$J_W^k = U_C,$$

$$d^k = w^k W^k = -\frac{J_W^k}{J_{WW}^k} V^{-1}\left(\mu - \Upsilon \iota\right) - \frac{J_{XW}^k}{J_{WW}^k} V^{-1}\sigma.$$

We now assume that the first *K* state variables are the log-prices of consumption goods. Using (10.24) as the utility function and applying the theorem of implicit functions we get from (10.29) by differentiating:

$$J^k_{W,\iota nP} = U_{CC} C_{\iota nP} + U_{C,\iota nP}.$$

Let *Y* denote the vector of the state variables that are not prices of consumption goods, i.e. *X* = $(lnP, Y)'$, therewith we can write as

$$J^k_{XW} = \begin{bmatrix} J^k_{\iota nPW} \\ J^k_{YW} \end{bmatrix} = \begin{bmatrix} U_{CC} C_{\iota nP} + U_{C\iota nP} \\ U_{CC} C_Y \end{bmatrix} = U_{CC} C_X + \begin{bmatrix} U_{C\iota n P} \\ 0 \end{bmatrix}$$

by using the results and notations of the CCAPM. Inserting (10.32) into (10.30) and continuing as in 4.8 gives us

$$w^k W^k = -\frac{J^k_W}{J^k_{WW}} V^{-1}(\mu - \Upsilon\iota) - \frac{U_{CC} C_X}{J^k_{WW}} V^{-1}\sigma$$

$$-\frac{1}{J^k_{WW}} \begin{bmatrix} U_{C\iota nP} \\ 0 \end{bmatrix} V^{-1}\sigma$$

$$= \frac{z^k}{C_W} V^{-1}(\mu - \Upsilon\iota) - \frac{C_X}{C_W} V^{-1}\sigma - \begin{bmatrix} \frac{U_{C\iota nP}}{U_{CC} C_W} \\ 0 \end{bmatrix} V^{-1}\sigma.$$

We define α^k as the share of total consumption investor *k* spends for good *j*

$$\alpha^k_j = \frac{P^C_j q^k_j}{C^k}$$

and m^k as the marginal share from an increase in total consumption:

$$m^k_j = P^C_j \frac{\partial q^k_j}{\partial C^k} = P^k_j q^j_C.$$

By using these definitions we get with (10.28):

$$U_{C,\iota nP} = U_{CP} \frac{\partial P^C}{\partial \iota n P^C} = U_{PC} P^C = -P^C \left(Uccq^k + Ucqc \right)$$

$$= -U_{CC} P^C q^k - U_C P^C qc$$

$$= -U_{CC} \alpha^k C^k - U_C m^k,$$

where α^k and m^k are the vectors of the budget and marginal shares. Inserting into (10.33), multiplying by VC_W and rearranging gives

$$z^k \left(\mu - \Upsilon\iota - \begin{bmatrix} m^k \\ 0 \end{bmatrix} \sigma \right) = C_W w^k W^k V - \sigma \begin{bmatrix} \alpha^k C^k \\ 0 \end{bmatrix} + C_X \sigma$$

$$= V_{\mu C} - C^k \sigma \begin{bmatrix} \alpha^k \\ 0 \end{bmatrix}.$$

Define as the price of a basket of consumption goods that contain exactly one unit of asset g that is consumed in both countries and the expenditure for each asset is respectively. We therewith have in the first

Case units of asset j and in the second case units in these baskets.

Let be the vector of covariances of the asset returns with changes in

$$V_{\mu\alpha^k} = \sigma \begin{bmatrix} m^k \\ 0 \end{bmatrix},$$

And the vector of covariances of the returns with changes in

$$V_{\mu\alpha^k} = \sigma \begin{bmatrix} \alpha^k \\ 0 \end{bmatrix}.$$

Thus we can rewrite (10.37) as

$$z^k \left(\mu - \Upsilon \iota - V_{\mu m^k} \right) = V_{\mu C} - C^k V_{\mu\alpha^k}.$$

We can now aggregate (10.40) over all M domestic investors and get with $z = \sum_{k=1}^{M} z^k, C = \sum_{k=1}^{M} C^k, P_\alpha = \sum_{k=1}^{M} P_\alpha^k \frac{C^k}{C} andP_m = \sum_{k=1}^{M} P_m^k \frac{z^k}{z}$:

$$z\left(\mu - \Upsilon \iota - V_{\mu m} \right) = V_{\mu C} - CV_{\mu\alpha}.$$

The term $1/P_m$ denotes the real value of a marginal increase in domestic consumption as the result of a change of the value of the portfolio domestic investors hold, i.e. P_m depends on the preferences of the domestic investors.

For a foreign investor we get the equivalent equation in foreign currency:

$$z^* \left(\mu^* - \Upsilon^* \iota - V_{\mu^* m^*} \right) = V_{\mu^* C^*} - C^* V_{\mu^* \alpha^*}.$$

In order to aggregate domestic and foreign investors both equations have to be denoted in the same currency. Define $z^F = ez^*, C^F = eC^*, P_m^F = eP_m^*$ and $P_\alpha^F = eP_\alpha^*$

as the corresponding values in domestic currency. With Ito's lemma we get:

$$V_{\mu^*\alpha^*} = V_{\mu^* 1/e\alpha^F} = V_{\mu^*\alpha^F} - V_{\mu^* e},$$

$$V_{\mu^* m^*} = V_{\mu^*/em^F} = V_{\mu^* m^F} - V_{\mu^* e},$$

$$V_{\mu^* C^*} = V_{\mu^* 1/eC^F} = \frac{1}{e}\left(V_{\mu^* C^F} - V_{\mu^* e} C^F \right).$$

From (10.22) and (10.23) we see that for the excess returns of the riskless assets of the other country we have

$$\frac{dH_{n+1*}}{H_{n+1*}} = -\frac{dH_{n+1}^*}{H_{n+1}^*} + \left(\frac{de}{e}\right)^2$$

and from (10.20) and (10.21) for the risky assets:

$$\frac{dH_i}{H_i} = \frac{dH_i^*}{H_i^*} + \frac{de}{e}\frac{dP_i}{P_i}.$$

By defining $L = \begin{bmatrix} I_n & 0 \\ 0 & -1 \end{bmatrix}$ we get from (10.46) and (10.47):

$$L\left(\mu^* - \Upsilon^* \iota\right) = \left(\mu - \Upsilon \iota\right) + V_{\mu^* e}.$$

It can further be shown that for any stochastic variable y that

$$LV_{\mu y} = V_{\mu^* y}.$$

Multiplying (10.42) with *L* and inserting (10.43), (10.45), (10.48) and (10.49) we get

$$z^F \left(\mu - \Upsilon \iota - V_{\mu m^F}\right) = V_{\mu C^F} - C^F V_{\mu \alpha^F}.$$

This relation is now expressed in domestic currency and can therefore be aggregated with equation (10.41). By defining $z^W = z + z^F, P_m^W = P_m + P_m^F$ and $C^W = C + C^F$ we get

$$z^W \left(\mu - \Upsilon \iota - V_{\mu m^W}\right) = V_{\mu C^W} - C^W V_{\mu \alpha^W}.$$

By denoting the real world consumption we get with Ito's lemma:

$$P_\alpha^W V_{\mu c} = P_\alpha^W V_{\mu C^W / P_\alpha^W} = V_{\mu C^W} - C^W V_{\mu \alpha}.$$

Inserting into (10.51) and rearranging gives:

$$\mu - \Upsilon \iota - V_{\mu m^W} = \frac{P_\alpha^W}{z^W} V_{\mu c}.$$

This relation has to hold for any asset and any portfolio M (not necessarily the market portfolio):

$$\mu M - \Upsilon \iota - V_{\mu M m^W} = \frac{P_\alpha^W}{z^W} V_{\mu M^c}.$$

Solving for the preference parameter and inserting into (10.53) gives the final relationship

$$\mu - \Upsilon \iota - V_{\mu m^W} = \left(\mu M - \Upsilon \iota - V_{\mu M m^W}\right) \frac{V_{\mu c}}{V_{\mu M^c}}.$$

The portfolio *M* can freely be chosen, a useful approach in line with the traditional CAPM is to choose the world market portfolio of all jointly risky assets and the riskless asset of the other country.

The left hand side of equation (10.55) denotes the real excess return of the asset, on the right side the first term denotes the real excess return of the reference portfolio and the last term denotes the covariance of the nominal asset returns with world real consumption relative to this relation of the reference portfolio. The real expected excess returns are higher the higher this covariance is. A company that produces a product whose demand depends heavily on aggregate consumption would have a higher covariance. The excess returns depend on the product the company produces and the country in which their product is demanded.

This model showed that different tastes across countries do not affect the allocation of assets and their real expected returns, they only depend on the relation to world aggregate consumption, i.e. aggregated tastes. These findings are very similar to those of the CCAPM. By using a portfolio *M* as reference whose re- turns are perfectly correlated with changes in aggregate consumption, (10.55) equals the CCAPM in real terms.

To get differences in the allocation of assets, e.g. the observed bias towards domestic assets we have either to deviate from the law of one price for consumption goods and assets, e.g. by introducing transportation costs or to introduce barriers to international investment, what will be done in the next section.

BARRIERS FOR INTERNATIONAL INVESTMENT

Stulz (1981b) provides a model of international asset pricing in the presence of barriers to international investment. Like in the last section he assumes two countries, in each country there are n and n^* assets located, such that $n+n^* = N$. We assume that foreign investors are free to invest in their home country or abroad without facing any restrictions, whereas domestic investors can invest into domestic assets without any restrictions, but if they want to invest in foreign asset have to pay a tax of ϑ per time period and invested unit of wealth. This tax can either be a transaction tax, but it can also be interpreted as costs of obtaining additional information.

The expected return from holding a foreign asset is reduced to $\mu_i - \vartheta$ and from being short in an asset to $-\mu_i - \vartheta$. The tax of ϑ has not to be paid on the net position of a foreign asset, but on every position individually, i.e. holding one unit of a foreign asset long and one unit short results in a tax of 2ϑ.

With these conditions we assume every investor to optimize his portfolio holdings according to portfolio theory presented in section 4.2. If we denote the covariance matrix of all domestic and foreign assets by V and w^k the long and v^k the vector of short positions of the kth domestic investor the variance of a portfolio is given by

$$V_p = \left(w^k - \upsilon^k\right)' V \left(w^k - \upsilon^k\right).$$

The return of this portfolio consists of the expected returns from holding the assets, $(w^k - v^k)'\mu$, subtracting the taxes for holding foreign assets, $(w^k + v^k)'\iota_n\vartheta$, where ι_n denotes a vector with zeros in the first n and ones in the last n^* rows, and the return from holding the risk free asset

$$\mu_p = \left(w'k - \upsilon^k\right)'\mu - \left(w^k - \upsilon^k\right)'\iota_n\theta + \left(1 - \left(w^k - \upsilon^k\right)'\iota\right)\Upsilon.$$

From portfolio theory we know that to fi the efficient frontier we have to minimize (10.56) subject to the constraints

With λ^k denoting the Lagrangean multiplier for the first constraint we get the following first order conditions with L^k denoting the Lagrange function:

$$\frac{\partial L^k}{\partial w^k} = V\left(w^k - \upsilon^k\right) - \lambda^k\left(\mu - \Upsilon\iota - \theta\iota_n\right) \geq 0,$$

$$\frac{\partial L^k}{\partial \upsilon^k} = -V\left(w^k - \upsilon^k\right) + \lambda^k\left(\mu - \Upsilon\iota + \theta\iota_n\right) \geq 0,$$

$$\left(w^k\right)'\frac{\partial L^k}{\partial w^k} = 0,$$

$$\left(\upsilon^k\right)'\frac{\partial L^k}{\partial \upsilon^k} = 0.$$

By combining (10.58) and (10.59) we get

$$\lambda^k\left(\mu-\Upsilon\iota+\theta\iota_n\right)\geq V\left(w^k-\upsilon^k\right)\geq\lambda^k\left(\mu-\Upsilon\iota-\theta\iota_n\right),$$

Or with V_i denoting the ith column of V, i.e. $V_i(w^k - v^k)$ denoting the covariance of the ith asset with the portfolio $w^k - v^k$, and

$$1_F(i) = \begin{cases} 1 \text{ if the asset is foreign and the investor is domestic} \\ 0 \text{ if the asset is domestic or the investor is foreign} \end{cases}$$

We can rewrite (10.62) as

$$\lambda^k\left(\mu_i-\Upsilon+\theta 1_F\left(i\right)\right)\geq V_i'\left(w^k-\upsilon^k\right)\geq\lambda^k\left(\mu_i-\Upsilon-\theta 1_F\left(i\right)\right).$$

For domestic assets and foreign investors (10.64) reduces to

$$\lambda^k\left(\mu_j-\Upsilon\right)=V_j'\left(w^k-\upsilon^k\right).$$

If the second inequality in (10.64) holds strict we know from (10.58) and (10.60) that

$$\sum_{j=1}^{N} w_j^k\left(V_j'\left(w^k-\upsilon^k\right)-\lambda^k\left(\mu_j-\Upsilon-\theta 1_F\left(i\right)\right)\right)=0$$

holds only if Otherwise this sum must be positive by the restriction that and the second term being positive. Equivalently with (10.61) the strictness of the first inequality in (10.64) implies If both inequalities hold strictly the asset is not held by the investor. Hence one of the two inequalities has to be an equality. Defining for a domestic investor

$$\Pi^k(i) = \begin{cases} 1 & \text{for a foreign asset held long} \\ 0 & \text{for a domestic asset} \\ -1 & \text{for a foreign asset held short} \end{cases}$$

we can write

$$V_i'\left(w^k-\upsilon^k\right)=\lambda^k\left(\mu_i-\Upsilon-\theta\pi_k\left(i\right)\right).$$

Dividing (10.67) by (10.65) gives us

$$\frac{V_i'\left(w^k-\upsilon^k\right)}{V_j'\left(w^k-\upsilon^k\right)}=\frac{\mu_i-\Upsilon-\theta\pi_k\left(i\right)}{\mu_j-\Upsilon}$$

for $i \neq j$. The N net demands $w^k - v^k$ are the only unknown variables in these $N - 1$ equations and the relative demands of the assets can be determined. As no preferences enter equation (10.68) the relative demand for every domestic investor only depends on the properties of the assets and hence all investors have the same relative demand for risky assets, i.e. have the same optimal risky portfolio. But as we will see it has not to be the world market portfolio. The optimal risky portfolio for foreign investors will be different of that for domestic investors unless the term $\vartheta\pi^k(i)$ equals zero, i.e. no barriers for investment exist. Nevertheless all foreign investors will demand the same optimal risky portfolio.

Suppose that there exists a foreign asset whose return is uncorrelated with any other asset, domestic or foreign, i.e. $\sigma_{ij} = 0$ for $i \neq j$. Suppose further that the excess return is zero, as it is in the traditional CAPM, i.e. $\mu_i - r = 0$. Then (10.64) becomes

$$\lambda^k\theta \geq \sigma_i^2\left(w_i^k - \upsilon_i^k\right) \geq -\lambda^k\theta.$$

For holding the asset long we need As λ^k can be shown to be positive, this implies $w_i^k < 0$. In the same manner holding the asset short implies $_i < 0$. Both results violate the assumption of non-negativity of the positions. Therefore an asset with such a property is not traded by domestic investors. This result shows that not all assets are held by domestic investors.

Define now such that (10.64) becomes

$$\lambda^k\left(\mu_i - \Upsilon + \theta 1_F(i)\right) + \lambda^k Q_i^k = V_i'\left(w^k - \upsilon^k\right)$$

$$= \lambda^k\left(\mu_i - \Upsilon - \theta 1_F(i)\right) + \lambda^k q_i^k.$$

As for domestic assets $1_F(i) = 0$ we find that unless and for foreign assets $q^k + Q^k = 2\vartheta$.

Define w^S as the vector of the share the assets have on total world wealth W^W and let with W^k as the wealth of investor k define

$$T^k = \lambda^k W^k, T^D = \sum\nolimits_D T^k, T^F = \sum\nolimits_F T^k, \delta^k = \frac{T^k}{T^D}, \gamma^D = \frac{T^D}{T^D + T^F}, q^k = \left(q_1^k, \ldots, q_N^k\right)' \text{ and } q^D = \sum\nolimits_D \delta^k q^k,$$

Where D and F denote summing over all domestic and foreign investors, respectively.

Multiplying (10.62) by W^k and summarizing over all investors gives

$$V\sum_{D+F}\left(w^k - \upsilon^k\right)W^k = \left(T^D + T^F\right)\left(\mu - \Upsilon\iota + \theta\iota_n\right) + \sum_{D+F} T^k q^k.$$

Inserting the above definitions and using that $\vartheta\iota_n = q_i^k = 0$ for foreign investors we get

$$Vw^s W^w = \left(T^D + T^F\right)\left(\mu - \Upsilon\iota + \theta\iota_n \frac{T^D}{T^D + T^F} + \frac{1}{T^D + T^F}\sum_{D+F} T^k q^k\right)$$

$$= \left(T^D + T^F\right)\left(\mu - \Upsilon\iota + \gamma^D\theta\iota_n + \frac{T^D}{T^D + T^F}\sum_D \frac{T^k}{T^D} q^k\right)$$

$$=\left(T^D+T^F\right)\left(\mu-\Upsilon\iota+\gamma^D\theta\iota_n+\gamma^D\sum_D\delta^k q^k\right)$$

$$=\left(T^D+T^F\right)\left(\mu-\Upsilon\iota+\gamma^D\theta\iota_n+\gamma^D q^D\right).$$

Multiplying (10.71) by w^S and defining as the variance of the world market portfolio as the tax to be paid by a domestic investor for holding the world market portfolio and we get with as the expected return on the world market portfolio:

$$\sigma_m^2 W^W=\left(T^D+T^F\right)\left(\mu_m-\Upsilon-\theta_m\Upsilon^D+q_m\Upsilon^D\right).$$

Define $\beta^m=\dfrac{Vw^s}{\sigma_m^2}$ as the vector of betas of the assets with the world market portfolio. Solving (10.72) for and inserting into (10.71) we get

$$\mu-\Upsilon\iota-\Upsilon^D\theta 1_n+\Upsilon^D q^D=\beta^m\left(\mu_m-\Upsilon-\theta_m\Upsilon^D+q_m\Upsilon^D\right).$$

With ϑ = 0 we get the usual relationship of the CAPM as derived earlier. For domestic assets this reduces to

$$\mu_i-\Upsilon=\beta_i^m\left(\mu_m-\Upsilon-\theta_m\Upsilon^D+q_m\Upsilon^D\right).$$

We find a linear relationship between the beta of the asset with the world market portfolio and the excess return of the asset. The slope can be either smaller or larger than in the CAPM, where it is

For a foreign asset we get from (10.73):

$$\mu_i-\Upsilon=\beta^m\left(\mu_m-\Upsilon-\theta_m\Upsilon^D+q_m\Upsilon^D\right)+\Upsilon^D\theta-\Upsilon^D q_i^D.$$

We fit a similar Security Market Line as for domestic assets, it has the same slope, but it is shifted. If the asset is hold long, the second inequality in (10.64) becomes an equality and by inspection of (10.70) we see that q^k = 0 and hence

$q_i^D = \sum \pi^k q^k = 0$. The SML is shifted by γ^D ϑ upwards. If the asset is held short it follows in the same way that Q^D = 0 and hence from q^D + Q^D = 2ϑ that = 2ϑ and therefore the SML is shifted downwards by $\gamma\,\vartheta$. We fit to have three SML, one for domestic assets, one for foreign assets held long and one for foreign assets held short.

An asset not held by domestic investors fulfil both inequalities in (10.64) strictly, hence we find that for those assets 0 < q_i^k < 2ϑ and the assets plot between the two SML for foreign assets. Figure 10.1 illustrates these findings.

From (10.74) and (10.75) we see that with barriers to international investment an asset with a beta of zero with the world market portfolio may not have the same expected return as the risk free asset if it is a foreign asset. And a beta of one does not ensure to receive the market return.

At last we will show which properties an asset must have that it is held by domestic investors. Aggregating (10.64) over all domestic investors after having multiplied by W^k gives

$$T^D\left(\mu-\Upsilon\iota+\theta\iota_n\right)\geq V\sum_D\left(w^k-\upsilon^k\right)W^k\geq T^D\left(\mu-\Upsilon\iota-\theta\iota_n\right).$$

Define G^D as the fraction of total wealth invested into risky assets by domestic investors and G^F by foreign investors

$$G^D=\frac{1}{W^D}\sum_{i=1}^N\left(w_i^k-\upsilon_i^k\right)W^k,$$

$$G^F=\frac{1}{W^F}\sum_{i=1}^N\left(w_i^k-\upsilon_i^k\right)W^k$$

and w^D-v^D the vectors of fractions invested into each risky asset by domestic investors. The inequalities in (10.76) have to hold strictly for assets not hold by domestic investors. For foreign assets not held we get

$$T^D\left(\mu_i-\Upsilon+\theta\right)\succ G^DW^DV_i\left(w^D-\upsilon^D\right)W^k\succ T^D\left(\mu_i-\Upsilon-\theta\right).$$

From (10.65) we similarly get

$$V_i\left(w^F-\upsilon^F\right)=\frac{T^F}{G^FW^F}\left(\mu_i-\Upsilon\right),$$

$V_i^D=V_i\left(w^D-\upsilon^D\right) and V_i^F=V_i\left(w^F-\upsilon^F\right)$ denote the covariance of asset *i* with the portfolio held by domestic and foreign investors, respectively. Solving (10.78) for μ_i-r and inserting into (10.77) gives us

$$T^D\frac{G^FW^F}{T^F}V_i^F+T^D\theta\succ G^DW^DV_i^D\succ T^D\frac{G^FW^F}{T^F}V_i^F-T^D\theta,$$

or after rearranging

$$\theta\succ\frac{G^DW^D}{T^D}V_i^D-\frac{G^FW^F}{T^F}V_i^F\succ-\theta.$$

Defining $\xi^D=\dfrac{G^DW^D}{T^D} and \xi^F=\dfrac{G^FW^F}{T^F}$ enables us to rewrite this as

$\left|\xi^DV_i^D-\xi^FV_i^F\right|\prec\theta.$ Foreign assets that have a small covariance with the optimal risky portfolio of the domestic and foreign investors will not be held by domestic investors. These assets then will also have a small covariance and hence a small beta with the world market portfolio, which is a weighted average of these portfolios. The smaller the tax is the more assets will be traded, if ϑ = 0 all assets are traded as predicted by the CAPM

The reason that assets with a small beta with the world market portfolio are not held is that their expected benefits are too small to overcome the costs. The diversification effect can also be achieved by assigning a higher fraction to the riskless asset in the optimal portfolio.

Also foreign assets that have covariances close to those of domestic assets will not be traded, although their beta may differ significantly from zero. These foreign assets can easily be substituted by domestic assets without imposing the costs. These interpretations show that it is very likely to fi domestic investors to hold more domestic assets.

EMPIRICAL EVIDENCE

Most empirical investigations test the segmentation or integration of asset markets, i.e they want to fi or reject barriers to international investments. In general neither evidence for segmentation nor integration can be found at significant levels. The reason may be that these barriers in most cases affect investors from different countries or different types of investors of the same country not equally. Different tax structures, investment quotas and other factors differ widely between countries and types of investors. This will influence the behaviour of asset returns in a much more complicated way than modelled here. Additionally globalization and liberalizations change the rules permanently such that it is difficult to investigate asset returns over time. These difficulties make it very problematic to find support for or against the model.

The only substantial evidence can be derived from assets that differ only in their availability to foreign investors, e.g. the A- and B-shares of the Shenzen Stock Exchange in China. It can only be said that the more widely an asset is available to foreign investors the higher its price is.

- Asset prices and exchange rates are jointly log-normal distributed
- There exists a single consumption good or equivalently a common basket of consumption goods
- The consumption good is continuously and costless traded between the countries
- There are *n* risky assets that are traded continuously and costless
- There is an asset that is riskless in real terms, i.e. in terms of the consumption good
- There are *J* countries
- There exist no transaction costs, taxes, transportation costs or tariffs
- No restrictions on short sales
- No barriers to international investment
- Investors are price takers and have the same information
- Investors are risk averse
- For the consumption good the law of one price applies, i.e. the price is equal in all countries adjusted only by the exchange rate (Purchasing Power Parity, the exchange rate changes according to the difference of inflation in the countries.

Table 3.9.1: Assumptions of the International CAPM

3.10 THE PRODUCTION-BASED CAPITAL ASSET PRICING MODEL

The CCAPM and the ICAPM presented in the last sections took the productions side of the economy as given and only modelled the demand. Empirically strong evidence has been found that stock returns forecast GDP growth very well. By inverting this relation high expected GDP growth in the future should result in high asset returns now. By taking the demand side (consumption) as given and modelling the supply side (production) of the economy Cochrane (1991) developed the *Production-based Asset PricingModel* (*PAPM*).

THE MODEL

We assume a discrete time setting where a single good is produced in a finite number of states. We further have a single asset which pays a dividend in every period depending on the state. The discount factor also depends on the state.

From section 4.1 we know that the fundamental value of the asset with the current state as the only source of information is given by

$$P_t = E\left[\sum_{T=t}^{\infty}\left(\prod_{k=t}^{\tau} p_s^k\right) D_\tau \,|\, s^t\right],$$

where P_t denotes the price at time t, ρ^k the discount factor of investor k in state s^t, D_τ the dividend and s^t the current state. Defining the return as usual we get

$$R_{t+1} = \frac{P_{t+1} + D_{t+1} - P_t}{P_t}.$$

Using (11.2) it can easily be verified that

$$P_t = E\left[p_{t+1}^s (1 + R_{t+1}) P_t \,|\, s^t\right].$$

By inserting (11.2) into (11.3) and iterating by inserting (11.3) again, we get

$$\begin{aligned}
P_t &= E\left[p_{t+1}^s (1 + R_{t+1}) P_t \,|\, S^t\right] \\
&= E\left[p_{t+1}^s \left(p_{t+1} + D_{t+1}\right) | s^t\right] \\
&= E\left[p_{t+1}^s \left(p_{t+2}^s (1 + R_{t+2}) P_{t+1} + D_{t+1}\right) | S^t\right] \\
&= E\left[p_{t+1}^s \left(p_{t+2}^s \left(p_{t+2} + D_{t+2}\right) + D_{t+1}\right) | S^t\right] \\
&= \ldots\ldots\ldots\ldots \\
&= E\left[\sum_{\tau=t}^{\infty}\left(\prod_{k=t}^{\tau} p_s^k\right) D_T | S^t\right],
\end{aligned}$$

hence (11.3) and (11.1) are equivalent. By dividing (11.3) by P_t we get

$$E\left[p^s_{t+1}(1+R_{t+1})|S^t\right]=1.$$

For later convenience we rewrite in terms of the consumption good. Lets denote the price for the consumption good at time 0 for delivery at time $t+1$ if a certain state occurs (contingent contract). We assume that such a price exists for every state and date, i.e. we assume a complete market. The real price of this claim in terms of the good at time t in state s^t is given by

$$p^C\left(s^{t+1}\right)=\frac{P^C_0\left(s^{t+1}\right)}{P^C_0\left(s^t\right)}.$$

We also can express if we are in a certain state s^t by

$$P^C_0\left(s^{t+1}\right)=P^C_0\left(s^t\right)p^s_{t+1}\pi\left(s^{t+1}|s^t\right),$$

Where $\pi\left(s^{t+1}|s^t\right)$ denotes the probability that state s^{t+1} occurs given state s^t. Rearranging and using (11.6) gives

$$p^s_{t+1}=\frac{p^C(s^{t+1}}{\pi\left(s^{t+1}|s^t\right)}.$$

We now turn our attention to the production of the good. The total production, y_t, consists of the production of a single consumption good, c_t, and investments I_t

$$y_t\equiv c_t+I_t.$$

The total production is assumed to follow a certain production function

$$y_t=f\left(k_t,l_t,s^t\right),$$

where k_t denotes the capital stock and l_t the labour input, which we assume to be constant over time. The increase in the capital stock follows

$$k_{t+1}=\vartheta\left(k_t,I_t\right),$$

where in general $g(k_t,I_t)$ 6= k_t+I_t as we also account for adjustment costs of new investments, i.e. costs like education of employees.

Differentiating (11.10) and (11.11) totally with subscripts denoting the derivatives we have:

$$dk_{t+1} = \vartheta I(t) dI_t,$$

$$dk_{t+2} = \vartheta k(t+1) dk_{t+1} + \vartheta I(t+1) dI_{t+1},$$

$$dy_{t+1} = f_k(t+1) dk_{t+1}.$$

We can view the capital stock as fixed for a given period and labour input is fixed over the entire time by assumption, hence the marginal product of capital depends only on the production of the consumption good (sales) as the costs are fixed. It further depends on the adjustment costs of new investments in this period. This marginal product of capital has to equal the return on investment by standard neoclassical theory:

$$1 + R'_{t+1} = \frac{\partial c_{t+1}}{\partial k_{t+1}} \vartheta I(t).$$

We assume now that the capital stock for the periods $t+2$ and following is fixed and that the firm has to decide which amount to invest in period t and which amount to invest in period $t+1$. This implies the restriction

$$dk_{t+2} = 0.$$

By inserting (11.13) and rearranging we get

$$dI_{t+1} = -\frac{q_k(t+1)}{q_I(t+1)} dk_{t+1}.$$

By using (11.9),(11.12),(11.14) and (11.17) we get from (11.15)

$$1 + R'_{t+1} = \frac{\partial c_{t+1}}{\partial k_{t+1}} \vartheta I(t)$$

$$= \frac{dy_{t+1} - dI_{t+1}}{\vartheta I(t) dI_t} \vartheta I(t)$$

$$= \frac{f_k(t+1)\vartheta I(t) dI_t + \dfrac{\vartheta k(t+1)}{\vartheta I(t+1)} \vartheta I(t) dI_t}{dI_t}$$

$$= \left(f_k(t+1) + \frac{\vartheta k(t+1)}{\vartheta I(t+1)} \right) \vartheta I(t).$$

If we assume the company to maximize the expected profits by choosing the appropriate investment strategy, standard neoclassical theory suggests that marginal costs and benefits have to equal. By (11.10) the total production in a period is given by the capital stock as the only variable. Total production can only be divided between the production of the consumption good and new investments. An increase in investment by dI_t has to be accompanied by a decrease in sales of consumption goods with the same amount, hence marginal costs of increasing investments are $P^C(s^t)dI_t$. The marginal benefits are the increased production of the consumption good in the next period, due to a rise in total production and a reduction in future investment (the capital at $t+2$ is fixed to a certain amount), i.e. the marginal benefits are $E\left[\frac{P^C\left(s^{t+1}\right)}{\pi\left(s^{t+1}|s^t\right)}dc_{t+1}|s^t\right]$.

So the first order condition for a profit maximum is

$$P^C\left(s^t\right)dI_t = E\left[\frac{P^C\left(s^{t+1}\right)}{\pi\left(s^{t+1}|s^t\right)}dc_{t+1}|s^t\right].$$

Inserting (11.9), (11.14) and (11.17) gives

$$P^C\left(s^t\right)dI_t = E\left[\frac{P^C\left(s^{t+1}\right)}{\pi\left(s^{t+1}|s^t\right)}\left(f_k\left(t+1\right)+\frac{\vartheta k\left(t+1\right)}{\vartheta I\left(t+1\right)}\right)\vartheta I\left(t\right)dI_t|s^t\right].$$

Dividing by $P^C(s^t)dI_t$ and using (11.18), (11.6) and (11.8) we get

$$E\left[p_{t+1}\left(1+R_{t+1}\right)|s^t\right]=1.$$

By comparing this first order condition for a profit maximum on the return on investment with the asset pricing condition (11.5) we see that they both look alike. By writing these conditions in another form we get

$$\sum_{s^{t+1}}\pi\left(s^{t+1}|s^t\right)p_{t+1}\left(1+R'_{t+1}\right)=\sum_{s^{t+1}}\pi\left(s^{t+1}\mid s^t\right)p_{t+1}\left(1+R_{t+1}\right)=1.$$

It is obvious that the return on investment is a linear combination of asset returns and vice versa. In order to prevent arbitrage, i.e. the company may sell assets short and invest the proceedings to obtain a riskless profit or vice versa, the two expected returns must equal:

$$E\left[R'_{t+1}\right]=E\left[R_{t+1}\right].$$

Having shown the equality of expected asset returns and return on investment, we now can relate the asset returns easily to the growth rate of GDP, i.e. the business cycle. High expected growth of GDP will increase the return on investment and hence expected asset returns. The high expected returns for the near future will lead asset prices to increase as shown in section 4.1. This result will give the result of the stock market leading the business cycle, but this lead should not too far as the precision of expectation decrease with the forecast horizon.

We further have the rationale for asset pricing that the expected asset return should equal the return on investment.

EMPIRICAL EVIDENCE

An empirical investigation into the performance of the PAPM can use equation (11.23). Critical to the investigation is the determination of the return on investment, the return on assets can easily be derived from the market. This return can only be determined on the basis of the balance sheet, which is only published quarterly, in Europe often only semi-annual or annual, while stock returns are available daily. Further the data in the balance sheet may be biased due to poor accounting standards and actions taken for fiscal measures, i.e. the data quality may only be poor. We therefore face similar problems as in the CCAPM.

The investigation undertaken by Cochrane (1991, pp. 223 ff.) using quarterly data for the period 1947-1987 show nevertheless a high and significant correlation between asset returns and returns on investment. The returns on investment are more volatile than asset returns, suggesting that the arbitrage is not complete due to transaction costs or market incompleteness.

Cochrane (1991, pp. 232 ff.) further finds that the asset returns forecast GDP growth by about 9 months, a not too long period of time as predicted by theory.

These results suggest that in the longer run (for periods longer than a quarter) the stock market is driven by the business cycle, where for investigating single assets or industries the business cycle has to be decomposed into the parts relevant for this investigated assets. As other variables of interest to the expected returns, e.g. interest rate or consumption in the CCAPM, are also closely linked to the business cycle, its importance for explaining asset returns is further increased.

What remains an unsolved problem is the explanation of asset returns in the short run. Here only the Conditional CAPM, especially the ARCH specifications, give promising hints.

SUMMARY

This chapter gave an overview of the main models used in asset pricing, including some more recent developments. Their aim is to determine the fundamental value and/or an appropriate expected return. Most models relate expected returns to risks investors have to bear and have to be compensated for. They differ mainly in the risk factors they allow to enter into the model.

Although a large number of risk factors have been proposed in the literature and many models are used, none of the presented models or any other models used by practitioners are able to explain the observed asset prices or returns sufficiently. While most models work quite well for time horizons of more than a year, they fail in explaining short term movements. Only the Conditional CAPM, especially if covariances are modelled with a GARCH process, shows a satisfactory fit with the data, but the GARCH specification misses any economic reasoning.

The existence of many anomalies observed in asset markets, e.g. seasonal patterns, cannot be explained by any of these theories. They are subject to a large field of theories trying to explain a certain effect or a group of effects. To explain anomalies often even the assumption of rational acting investors, a central element in economic theories, is questioned in these efforts.

As there exists no generally accepted model of asset pricing the opinions of investors concerning the value of an asset diverge substantially. Access to *relevant* information and a superior model are central elements to make extraordinary profits in the market. Therefore not only academic research but also securities companies make huge efforts to improve the models.

Capital Asset Pricing Model: Sharpe-Lintner Version

In 1964 (Sharpe) and in 1965 (Lintner) continued the work of Markowitz and constructed the famous Capital Asset Pricing Model (CAPM). Basically, the model was developed to explain the differences in risk premium across assets. The CAPM shows clearly that these differences are generated by the differences in the riskiness of assets, that is, the higher the risk of an asset the higher the risk premium demanded by investors. With this extension, Sharpe-Lintner asset pricing model built the relationship between risk and expected return of financial assets based on the investors risk profile for the first time in the finance history.

Risk-free Rate

According to Damodaran (2006), the risk-free rate refers to the zero coupon government bond rate which matches the analysed cash flow time periods. For a long term analysis, a long term government rate is used as the risk-free rate of all cash flows, while in short term analysis, it is recommended to use entirely the short term government security rate as the risk-free rate. An investment should have no default risk in order to be risk-free.

Market Risk Premium

Damodaran (2006) stated that in an average risk investment, the market (risk) premium is the premium that investors demanded for investing with relativity to the risk-free rate. In other words, this premium demanded is the benchmark or the minimum amount of money needed to be exceeding by the expected return on investment. Moreover, there are three conditions for a market premium, which is; the premium should be greater than zero, there is an increase with the risk aversion of the market investors, and an increase in the riskiness of the average risk investment.

The CAPM Equation

The general equation of the model is:

$$\overline{r_i} = r_f + \beta_i \left(\overline{r_m} - r_f \right)$$

where:

$\overline{r_i}$ - expected return of stock i

β_i – relative risk of share i

$\overline{r_M}$ - expected return of the market portfolio

r_f – risk-free interest rate

This formula will be the main tool for testing the hypothesis H1: The market risk premium is positively related with the expected return. What CAPM says is that the equilibrium in the capital markets is characterized by only 2 numbers:-

(i) the return for waiting *i.e.* r_f;

(ii) the extra return *i.e.* $\overline{r_M} - r_f$.

A very important consequence of this model is the separation theorem, which says that in the capital markets the risk has two components: diversifiable (non-systematical) risk and non-diversifiable (systematical) risk. When pricing, the only significant risk is the systematic one, since investors can just get rid of the non-systematic risk through diversification. Sharpe & Lintner show that the true measure of risk is the well-known coefficient, *beta* computed as follows:

Beta (β)

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_m)}{\mathrm{Var}(R_m)}$$

Where:

Cov (Ri, Rm) represents the Covariance of the individual stock return with the market return,

And Var (Rm) represents the Variance of the market return.

Empirical evidence was in favour of CAPM and the model became extremely famous in the modern portfolio theory. Things were clear: stocks with beta lower than 1 were considered passive stocks and stocks with beta higher than 1 were considered aggressive and risky. Depending on their appetite towards risk, investors would choose the stocks in their portfolio according to the value of beta.

Though, some criticisms of CAPM emerged. One very known critic belonged to Fama and French. In 1992, they discovered a negative relationship between risk and return. Since then, a very important question was raised, '*is beta dead'*? The conclusion of these issues is that while academic debate still rages on, the CAPM may still be useful for those interested in the long run.

With this development, the accompanying assumptions of CAPM were stated as:

(a) all investors are risk-averse individuals, who maximise the expected utility of their end of period wealth,

(b) the investors are price takers and have homogenous expectations about asset returns that have joint normal distribution,

(c) there exist a risk-free asset such that investor may borrow or lend unlimited amounts at the risk-free rate,

(d) the quantities of asset are fixed, also all assets are marketable and perfectly divisible,

(e) asset markets are frictionless and information is costless and simultaneously available to all investors, and

(f) there are no market imperfections such as taxes, regulations, or restrictions on other selling (Attiya Y. Javed, Alternative Capital Asset Pricing Models: A Review of Theory and Evidence).

Concluding the Sharpe-Lintner model, the CAPM model can be written as:

$E(R_i) = R_f + \beta_i ((E(R_m) - R_f)$

The Equation of risk premium can be re-written as:

$E(R_i) - R_f = \beta_i ((E(R_m) - R_f)$

The version of CAPM by Sharpe and Linter has discovered a new field of finance study, which is the reason why the model has been widely used and tested based on both theoretical cases and empirical cases (Attiya Y. Javed, Alternative Capital Asset Pricing Models: A Review of Theory and Evidence)

CHAPTER 4: APPLICATION OF CAPM IN THE INSURANCE INDUSTRY

One of the main objectives of financial institutions is maximizing shareholder value and hence the main focus is on how the firm’s capital is allocated and utilized. Capital allocation in this context refers to the determination of the amount of a firm's equity capital that is assigned to each project or line of business undertaken by the firm. However, firms are usually concerned about capital allocation in the context of pricing and project selection, e.g., to determine the proportion of the firm’s overall cost of capital that must be contributed by each line of business in order to maximize firm value. This discussion of capital allocation is conducted in the context of the insurance industry. Various techniques have been suggested for allocating equity capital. However, most of the techniques discussed are perfectly general and can be applied in other industries as well.

Capital allocation is perhaps of special interest to financial firms such as insurers. For such firms, the principal providers of debt capital (insurance reserves) are also the firm’s principal customers. Unlike the holders of bonds and other (non-insurance) debt capital, insurance policyholders cannot protect themselves against the insolvency of specific debt issuers by holding a diversified portfolio. Unlike the diversified bond investor, the typical policyholder relies upon one insurer (or at most a few, in the case of life insurance) for each type of protection purchased (e.g., auto insurance, homeowners insurance, health insurance, etc.). Most insurance policies are purchased not as an investment but to protect against adverse financial contingencies. Thus, insolvency risk plays a special role in the insurance industry, and capital is held to assure policyholders that claims will be paid even if larger than expected.

In discussing capital allocation, it is important to keep in mind that the insurer’s entire capital is available to pay the claims arising from any specific policy or line of business. If the insurer becomes insolvent it is the entire company that enters bankruptcy – the firm does not go bankrupt line by line. Nevertheless, it is often useful to think of capital as being allocated by line of business for pricing, underwriting, and other types of decision making. Taking a close look at capital allocation in the context of the insurance industry is also useful to elucidate the interaction between financial decision making and the risk-based capital rules applied to the insurance industry by regulators. Capital allocation is also related to recently emerging concepts such as *risk adjusted return on capital* (RAROC) and *economic value added* (EVA), which have become important management decision making techniques for both financial and non-financial firms.

USING CAPITAL ALLOCATION TO MAXIMIZE VALUE

Why allocate capital? The motivation for anything a firm does should be to maximize shareholder value, that is, to increase the market value of equity capital. Unfortunately, this straightforward and powerful objective is often overlooked in practice. Through extensive discussions with people in the insurance industry, it appears that many firms are managing to their GAAP (generally accepted accounting principles) balance sheets and income statements, with the objective of showing healthy GAAP earnings and/or maximizing the value of GAAP equity. Of course, the firm needs to be cognizant of its GAAP performance because of the importance of accounting results to financial analysts and traders. However, it is a mistake to lose sight of the firm's true mission – the maximization of market value. Marking to market should play a critical role in the firm's internal decisions.

Capital allocation can be used to facilitate and improve the measurement of the economic profitability of businesses with different sources of risk and different capital requirements. In the insurance industry, it is customary to define businesses in terms of lines of insurance, for example, the commercial liability line or the auto liability line. Although this is the traditional approach, insurers need to step back and think carefully more about the issue of «what is a business?» in designing capital allocation and performance measurement systems.

Sometimes the banking literature talks about deposit accumulation or gaining demand deposits as one business, and making loans as another business. In this conceptualization, the economic concept of *transfer pricing* is used, whereby the bank's loan origination business will borrow money from the deposit accumulation business and pay an implicit rate of interest in return for having funds to invest in loans. This approach could also be used in insurance. The underwriting operation could be considered as a funds-generating business in which money is being borrowed from policyholders. The underwriting operation then would lend the funds to the investment business in return for a transfer price.

In either of the above lines of business concepts, the maturity and duration characteristics of the debt capital and the investments resulting from the insurer's different businesses must be recognized. Thus, funds generated by issuing long-tail liability policies are likely to lead to different investment objectives than funds raised by issuing short-tail property insurance policies in order to manage the risks of duration and convexity. One cannot assume that the long-term liability line has an asset portfolio that looks like the company; it has to be managed to meet the firm's overall objectives in terms of interest rate risk. Duration and convexity management is extremely technical. Therefore, special care must be exercised in giving a particular business credit for the money it generates, while, at the same time, charging it for the use of capital. This is the case for both the debt and equity capital needed to operate the business.

The primary link between capital allocation and value maximization is to enable the firm to measure performance by line of business to determine whether each business is contributing sufficiently to profits to cover its cost of capital and add value to the firm. To measure the cost of capital, it is necessary to determine the capital required to offer each type of insurance – relatively risky lines typically require more capital than less risky lines. For example, one might wonder whether the commercial liability insurance business is making an adequate profit, that is, whether the insurer should be charging higher or lower prices than at present, whether it should exit this business or perhaps devote more capital to it. Insurers can maximize value by shedding unprofitable businesses as well as by identifying profitable new projects. By withdrawing from unprofitable lines, the insurer may be able to increase the market value of equity, even as revenues decline. The ultimate objective should not be revenue growth, but maximizing net worth.

To provide a framework for the discussion of capital allocation methodologies, it is helpful to provide a simple mathematical statement of the capital allocation problem. Define x_i as the proportion of the firm's equity capital allocated to business i, where x_i is between 0 and 1. Thus, x_i indicates the proportion of capital which is allocated to business *i* and, therefore, the amount of capital allocated to business i is C_i, which is the total capital, C, multiplied by x_i. If the firm has N businesses, then

$$\sum_{i=1}^{N} x_i \leq 1 \quad \text{and} \quad \sum_{i=1}^{N} C_i \leq C$$

That is, the sum of the capital allocated to all of the firm's business will be less than or equal to the firm's total capital. While it may seem surprising that a firm may not assign all of its capital to its businesses, in fact some leading-edge researchers argue against allocating all of the capital and favor, instead, an allocation which results in less that 100 percent being assigned (Merton and Perold 1993). We return to this issue in the discussion below of the use of option models to allocate capital.

Once capital has been allocated by line, how can the resulting allocations be used to maximize firm value? One approach that has received considerable attention is to calculate the *risk-adjusted return on capital (RAROC).* RAROC is defined as the net income from a line, divided by the capital allocated to the line. That is,

$$RAROC_i = \frac{Net\ Income_i}{C_i}$$

where C_i is the capital allocated to line of business i.. The numerator of the RAROC formula, net income, also needs to be defined carefully. It may seem obvious, but it really is not once it is considered in economic terms. Basically, net income in the RAROC formula should be after taxes and interest expense. Even though interest expense is a banking term, it also applies to insurers in the form of underwriting loss. That is, the insurance market implicitly discounts the loss cash flows for the time value of money, meaning that the underwriting profit is negative in most cases. The negative underwriting return, which is analogous to interest expense, needs to be taken out when calculating the return from a line of business.

Once the RAROC for a line of business has been calculated, how does the firm know whether the line's current risk-adjusted return is adequate? The risk-adjusted return should be compared with the cost of capital for business i, where the cost of capital is obtained using an appropriate asset pricing model. If the risk-adjusted return equals or exceeds the cost of capital, then continuing to devote resources to this line of business is consistent with the goal value-maximization. However, if the risk-adjusted return is below the cost of capital, the line of business is reducing the firm's market value. In this circumstance, the firm should take some action to improve the situation such as re-pricing the insurance, tightening underwriting standards, or withdrawing from the line of business.

A slightly more formal way of determining whether a particular line of business is adding to firm value goes under the name of *economic value-added (EVA).*[1] Economic value-added measures the return on an investment in excess of its expected or required return. EVA seeks to identify lines that create value for the firm. EVA is net income minus the cost of capital, or hurdle rate, for a certain business, multiplied by the capital allocated to the business:

$$EVA_i = \text{Net Income}_i - r_i C_i$$

where ri = cost of capital (hurdle rate) for business i. Thus, if EVA ≥ 0, writing the line of business is consistent with value maximization, while if EVA < 0, the line is destroying firm value. The EVA formula can be changed slightly to put the results in rate of return format, creating a measure called the *economic value added on capital (EVAOC).* EVAOC is defined as EVA divided by the capital allocated to a line, i.e.,

$$EVAOC_i = (\text{Net Income}_i / C_i) - r_i$$

This is similar to RAROC, except that the line's cost of capital is subtracted. Again, if EVAOC is positive, the line is creating value for the firm.

An important detail is how to determine the cost of capital for business i. This too can present a problem in the insurance industry due to data limitations. One approach proposed by finance researchers to estimating the cost of capital for a line of business is the «pure play» technique. The pure play approach estimates the cost of capital by finding other firms that offer only one line of business. The cost of capital for a business in the multiple-line firm can then be based on the cost of capital of mono line firms writing only that specific type of business. This approach is problematical in the insurance industry because it is difficult to find firms that write only one line of business. Even if such a firm or firms could be found, the underwriting risk characteristics of the pure play firms could differ significantly from those of a given line of business written by the multiple line firm. An alternative to the pure play technique is the use of so-called "full-information betas" to determine the cost of capital (Kaplan and Peterson 1998). This estimation technique uses data on conglomerates (firms that write several lines of business) to conduct regressions that permit the estimation of the cost of capital by line.

Another problem in the insurance industry is the lack of quality data. An insurer thinking about implementing VaR, EVA, and the other economic methodologies discussed here needs to think about revising its data system to capture the data required to implement the methodologies. Insurers should be designing data systems that allow them to report underwriting results frequently, for example, on a monthly basis. Data quality is crucial. With inadequate data, even a perfect model will fail. Most insurers do not have the necessary data to implement VaR, RAROC, and EVA at the present time.

CAPITAL ALLOCATION TECHNIQUES

Various methodologies have been developed that could provide the foundation for a system of capital allocation. The following is by no means an exhaustive list. Many other proposals can be found throughout the related literature. I first provide an overview of the methods and then go into more detail in discussing each method separately.

OVERVIEW OF CAPITAL ALLOCATION TECHNIQUES

Regulatory Risk-Based Capital. In the United States, regulators have developed a formula to calculate the risk-based capital of insurers. A company's risk-based capital is used to define the minimum capital it must hold in order to avoid regulatory intervention. The regulatory thresholds or "action levels" are determined by the risk-based capital ratio, which is the ratio of the company's actual capital to its risk-based capital (see Cummins, Harrington, and Niehaus 1993). If the insurer's actual capital is greater than 200 percent of its risk-based capital, no regulatory action occurs. However, if actual capital falls below 200 percent of risk-based capital, various regulatory actions are taken, depending upon how far actual capital falls below 200 percent. An insurer's risk-based capital is computed by a formula designed to require more capital for riskier companies.

Some insurers actually use the regulatory risk-based capital formula in allocating capital for purposes of managing the firm. In my opinion, this is unwise, because the regulatory charges are of questionable accuracy and are based on book rather than market values. Furthermore, the regulatory charges ignore important sources of risk such as interest rate (duration and convexity) risk, as well an the insurer's transactions in the derivatives market. Even if the charges were accurate, they would be accurate only for the average firm in the industry. Consequently, in the case of firms with books of business having above or below-average risk, the regulatory charges would produce inappropriate allocations of capital. The result is that businesses may be charged for too much or too little capital, leading to sub-optimal decisions.

The Capital Asset Pricing Model. The second approach involves using one of the oldest of modern financial theory technologies, the capital asset pricing model (CAPM). This is not the best solution to the problem, but it may provide a helpful benchmark based on a familiar methodology. At the very least, the use of the CAPM allows managers to compare the preferred method to the results generated by a classic technique.

Value At Risk (VaR). This value at risk concept has become very popular in the banking and investment banking communities where there is a need to examine the risk exposure of the firm's trading book for foreign exchange, bonds, etc. Simply stated, the *value at risk (VAR)* is the amount the firm could lose with a specified small probability, such as 1 percent, in a specified period of time. The measurement of value at risk from currency and securities trading has advanced rapidly, thanks in part to daily and even more frequent data on exchange rates and asset prices which allow for very accurate and sophisticated calculations of VaR.[3]

VaR is also likely to be very useful to insurers and, in fact, is closely related to time honoured insurance and actuarial concepts such as the probability of ruin and the maximum probable loss. Unfortunately, the application of the most sophisticated VaR techniques requires very frequent data (monthly data is an absolute minimum), but insurance prices and losses are not observed with sufficient frequency nor in a market context. Most insurers do generate such data internally. Using sophisticated tools such as Value at Risk requires an integration of the capital allocation methodology with data processing and information systems to ensure that pertinent and useful data are generated to provide inputs for the Value at Risk models.

Marginal Capital Allocation. Marginal capital allocation is a term that can be applied to the capital allocation technique proposed by Robert Merton and Andre Perold (1993) and to a related technique developed by Stewart Myers and James Read (1999). Both techniques are based on the option pricing model of the firm. In the options view of the firm, the value of the policyholders' claim on the firm is equal to the present value of losses minus the value of the *insolvency put option*. The insolvency put option is the expected loss to policyholders due to the possibility that the firm will default. The simplest option interpretation of the firm involves a one-period, two-date model where the firm issues policies at time 0 and claim payments occur at time 1. If assets exceed liabilities at time 1, the firm pays the losses and the equity owners receive the residual value (the difference between assets and liabilities). However, if assets are less than liabilities, the insurer defaults and the policyholders receive the assets. The payoff to policyholders at time 1 is thus equal to: L - Max[L-A,0], where L = losses, A = assets, and Max[L-A,0] is the payoff on the insolvency put.

The Merton-Perold (M-P) and Myers-Read (M-R) capital allocation techniques take a different view of what is meant by a « marginal » approach to capital.allocation. Stated simply, the M-P approach views marginal capital allocation in terms of what happens to the insolvency put option if entire lines of business are added to or removed from the firm. The M-R approach is marginal in the instantaneous intepretation familiar from calculus, i.e., they allocate capital based on what happens to the firm's insolvency put option in response to very small changes in the liabilities of the lines of business written by the firm. The M-P method may leave some capital unallocated, while the M-R method makes a unique allocation of 100 percent of capital.

We now turn to a more detailed examination of how **The Capital Asset Pricing Model** is utilized in insurance.

THE CAPITAL ASSET PRICING MODEL

The CAPM states that the return on equity or cost of capital for a firm is determined by the following formula:

re = rf + ße [rm - rf] where re = cost of equity capital,

rf = default risk-free rate of interest,

rm = expected return on the "market", and

ße = the firm’s beta coefficient = Cov(re,rm)/Var(rm).

where Cov( ) = the covariance operator and Var( ) = the variance operator.

How can the CAPM be used by an insurance company to make pricing and investment decisions? Project decisions can be made by decomposing the beta coefficient to determine the betas by line of business. For example, let’s consider an insurer with two lines of business. Its net income would be:

I = rAA + r1P1

+r2P2

where I = net

income,

rA = return on assets,

r1, r2 = rates of return on underwriting from lines 1

and 2, A = assets, and

P1, P2 = premiums from lines 1 and 2.

Next introduce the balance sheet identity, which says that assets are equal to equity, plus the liabilities generated by the two lines. Then divide by equity to express the result as a rate of return:

rE = rA(E+L1+L2)/E+r1P1/E+r2P2/E

Then the Beta coefficient can be decomposed as

follows: ßE = ßA(1+k1+k2) + ß1s1 + ß2s2

where ßE, ßA, ß1, ß2 = betas for firm, assets, and insurance risk of lines 1

and 2, k1, k2 = liability leverage ratios for lines 1 and 2, = L_i/E, i = 1,

2, and s1, s2 = premium leverage ratios for lines 1 and 2, = P_i/E, i =

1, 2.

The calculation uses the property that the covariance is a linear operator.

The formula for the decomposition of ßE shows that the beta of the firm, which drives the required return on equity, is the beta of assets times 1, representing the investment of equity capital, plus the liability leverage ratios for lines one and two. Then the formula adds on the beta of each individual line's underwriting returns multiplied by the line-specific premium to surplus ratio. So we find a theoretical justification for the traditional rule of thumb leverage ratio that has been used for years in the insurance industry -- the premium to surplus ratio.

The model can be solved for the required rate of underwriting return on each line of business: $r_i = -k_i r_f + \beta_i (r_m - r_f)$

for lines of business i = 1 and 2. Thus, each line implicitly pays interest for the use of policyholder funds (the term $-k_i r_f$) and receives a rate of return based on the systematic risk of the line (the term $\beta_i (r_m - r_f)$).

The CAPM result has the following implication: It is not necessary to allocate capital by line using the CAPM, but rather to charge each line for at least the CAPM cost of capital, reflecting the line's beta coefficient and leverage ratio. Costs of capital based on other asset pricing models, such as the arbitrage pricing theory (APT), have similar implications.

Although the CAPM provides a useful way of conceptualizing the contributions of the firm's lines of business to the return on equity, there are at least three important problems with this model. (1) The CAPM only rewards the firm for bearing systematic underwriting risk, that is, underwriting risk that is correlated with the market portfolio. However, insurers also need to be concerned about extreme events, i.e., tail risk, that is simply not priced in the CAPM model. This is important in view of the role of insurers as financial intermediaries, where a firm's principal creditors are also its customers. (2) Line of business underwriting beta estimates are difficult to estimate given the data currently available (Cummins and Harrington 1988), although progress has been made in estimating costs of capital in the more recent literature (e.g., Lee and Cummins 1998). (3) Research has shown that rates of return are driven by other economic factors besides beta coefficients (e.g., Fama and French 1996). Thus, sole reliance on the CAPM would ignore important determinants of the cost of capital. The primary role for the CAPM is to serve as a benchmark to compare with the results of other estimation methodologies. If the two methods yield drastically different results, it would be appropriate to check the methodology and data carefully before proceeding.

ADDITIONAL ISSUES

Another important issue has to do with the *economic cost* of the firm's overall capital as well as the capital allocated to individual lines of business. The capital of financial institutions such as insurers is invested in marketable securities. If capital markets are efficient and frictionless, the invested funds will earn the equilibrium market rate of return and thus will be cost-less to the firm. However, the existence of various market and institutional imperfections lead to friction costs which imply that the capital invested will not earn the full fair market return required to avoid a loss to the insurer. Various types of friction costs are present that create costs for insurers that reduce the returns from the investment of their capital. The three most important sources of costly capital to insurers are:

- Agency and informational costs. It is well-known that managers of firms can behave opportunistically, and thus fail to realize the owners' objective of value maximization. In addition, adverse selection and moral hazard are endemic to insurance markets and will create costs to the extent they cannot be controlled through an insurer's pricing and underwriting decisions.
- The Federal income tax system leads to double taxation of investment income; and, as a result, investing in securities through an insurance company produces lower investment returns than investors could realize by investing directly in the market. And
- regulation, and especially the risk-based capital system, imposes costs on insurers in the form of a regulatory "option" on the insurer's assets. The option is created because the RBC system gives regulators the legal right to seize control of the insurer when its assets still exceed its liabilities. Other regulations such as investment restrictions also may lead insurers to hold inefficient portfolios, further reducing returns for any given level of risk.

The existence of market frictions means that a spread develops between the returns that could be earned by investing directly in capital markets and the returns actually earned on the capital held by insurers. It is this spread cost that must be taken into account in determining whether lines of business are earning the appropriate rates of return. Of course, the risk of individual lines also is important in determining their cost of capital. Usually, the type of risk recognized in the cost of capital context is systematic market risk, determined by an asset pricing model.

It is also interesting to consider the role of regulatory risk-based capital in the context of a marginal allocation system. With a well-designed marginal capital allocation system, is regulatory risk-based capital relevant? It can be, because there is a potential cost imposed on the firm by the regulatory risk-based capital system. For most insurers, and reasonably small EPD targets, no cost will be realized, since the insurer's total capital generally will be greater than the regulatory capital requirement. We also consider the following two cases:

- Regulatory capital for one or more lines exceeds the marginal capital allocated to these lines but the insurer's overall capital is greater than risk-based capital. In this case as well regulatory costs are not likely to be incurred because the risk-based capital test is applied to the entire firm and not by line of business. This conclusion would have to be modified if risk-based capital action level tests were to be applied by line.
- Regulatory risk-based capital exceeds the firm's overall capital, including both allocated and unallocated capital. In this situation, regulatory penalties will apply. Thus, the conclusion is that regulatory capital usually will not be a problem, even if one or more lines of business have allocated capital that is less than the by-line risk-based capital, as long as the firm's overall capital exceeds its overall risk-based capital.

The final point to be made is a caveat. Insurers should use caution in designing and implementing a capital allocation system. The use of an inappropriate system is likely to lead to rejection of some projects that should be accepted and acceptance of some projects that should be rejected. Systems that use stand-alone capital are likely to be particularly harmful. However, the adoption of an appropriate marginal capital allocation system can have significantly beneficial effects on the market value of the firm, by enabling the firm to identify projects and businesses that are creating value for shareholders as well as those that are destroying firm value

PAST RESEARCH OF CAPM

Numerous research on CAPM have been conducted in the developed world with a majority of studies revealing that the model fails the empirical test in most markets. Fama and French wrote a paper on ‘CAPM is wanted dead or alive’ where they criticized the model’s reliance on one factor model to explain risk. They added new variables to beta which revealed better results but still used beta as one of the variables. Therefore they showed that CAPM was not entirely wrong but needed to be strengthened.

CAPM has been tested in the Kenyan market by several research students in particular an MSc Finance student at JKUAT tested for the period 1999 – 2003. The test drew conclusions that CAPM was incompatible to the Kenyan stock market and recommended a CAPM test over a longer duration and using a larger sample.

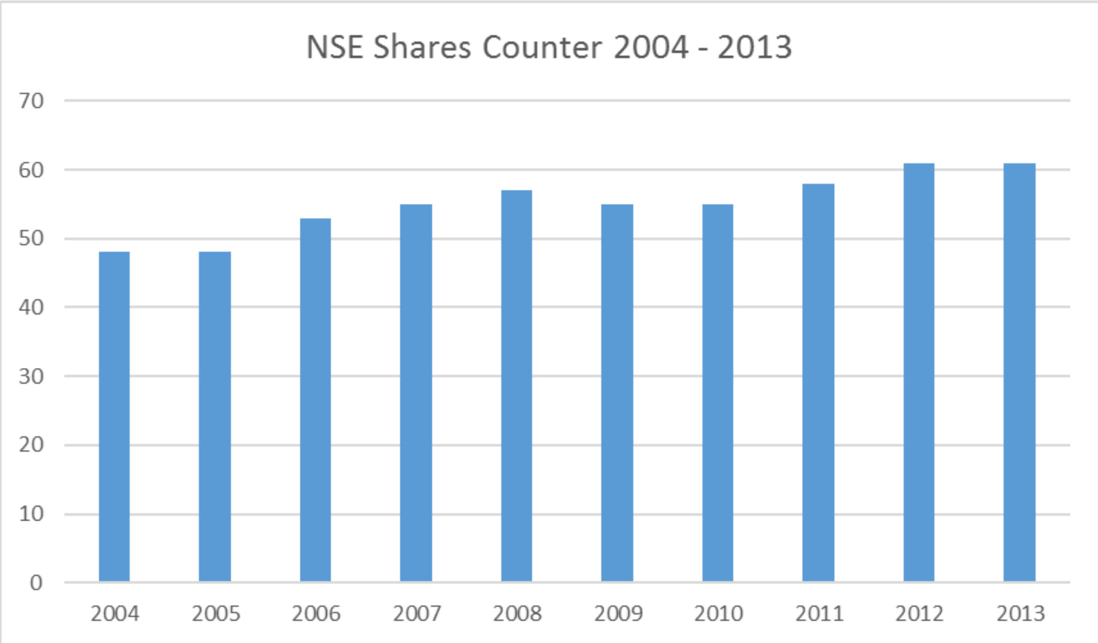
A recent test was also undertaken from January 2008 – December 2013 revealing inconsistencies in the model. The student recommended taking a larger sample of firms.

In this study, we analyse 47 equities spanning over a decade and further conduct a beta-prediction using CAPM for the year ending 31st December 2014.

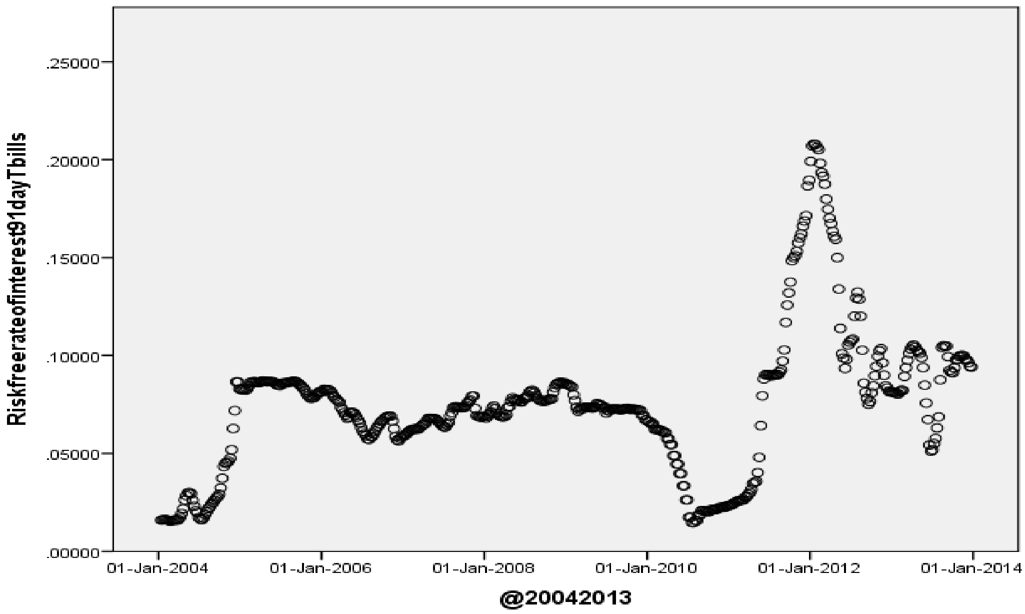
CHAPTER 5: DATA ANALYSIS & RESULTS

The study was conducted on the publicly listed stocks at the NSE over the ten years ranging from January 2004 – December 2013. The daily stock prices and stock market indices were obtained from the NSE whereas the weekly risk-free rates of return were acquired in the form of 90-day Treasury Bills rates from the Central Bank of Kenya.

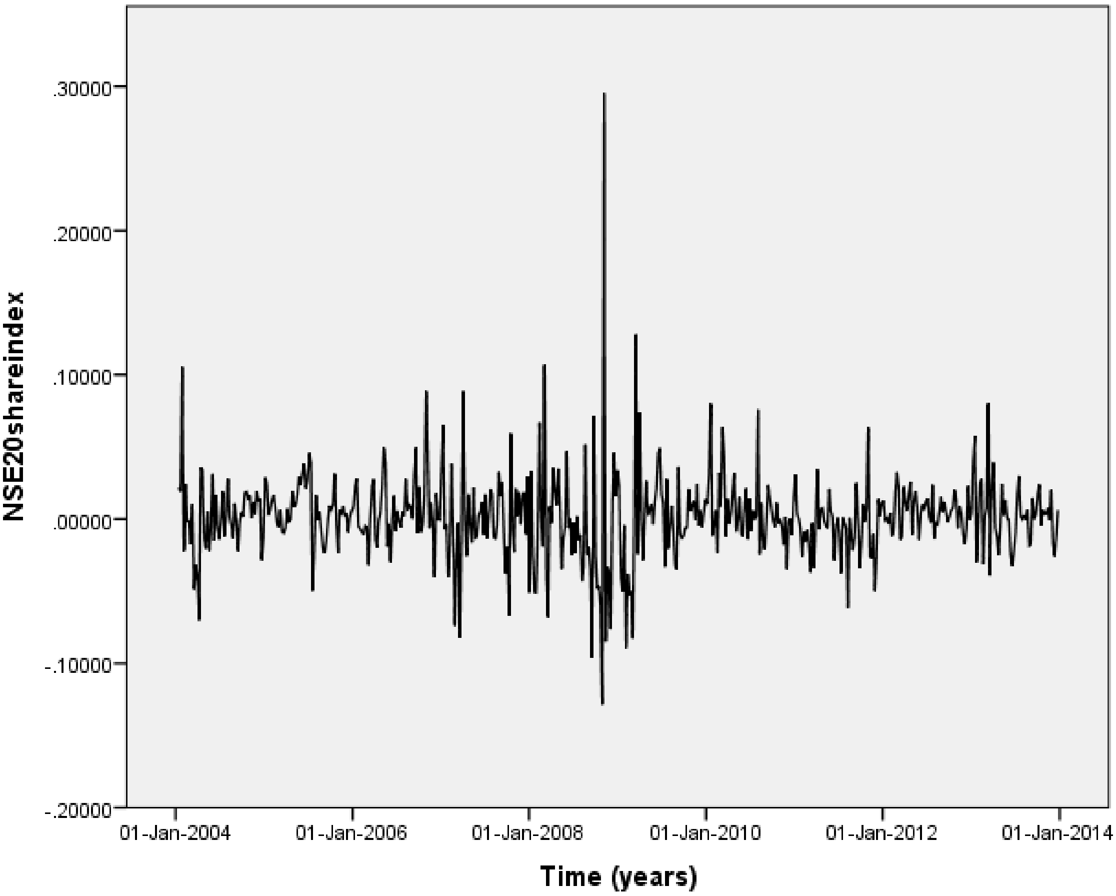
The number of shares listed over the period was as follows:-



The development of the risk-free rate of return over the decade:



Whereas the NSE 20-Share Index performed as follows:



Since the 91-day Treasury Bills were released by the government on a weekly basis, the NSE stocks have also been derived weekly for consistency.

5.1 NSE THROUGH THE YEARS

The Nairobi Stock Exchange has grown tremendously over the last decade from a market capitalization of Kshs 317.69 billion as at 1st January 2004 to Kshs 1.92 trillion as at 31st December 2013 representing a 604.59% increase through transactions such as initial public offers, rights issues, offers for sale, cross-listings, introductions, mergers, acquisitions and delistings. The equity turnover increased to Kshs 155 billion in 2013. The NSE 20-Share Index has improved from 2737.58 to 4926.97 – approximately 180% increase. The highest closing year index was 5200 in year 2008.

On 10th November 2004, the central depository system was commissioned which automated for the first time in Kenya's history the process of clearing and settlement of shares traded in the capital market. The NSE implemented live trading on its own automated trading systems (ATS) in September 2006. The NSE also implemented its Wide Area Network (WAN) platform in December 2007 facilitating remote trading which meant brokers and investment banks no longer required to trade from the floor of the house but through terminals in their offices linked to the NSE trading engine. In an effort to provide investors with a comprehensive measure of the performance of the stock market the NSE introduced the NSE All-Share Index (NASI) in February 2008 to complement the NSE 20-Share Index. The NASI's calculation was based on market capitalization, implying that the index level reflected the total market value of the constituent stocks.

The introduction of Safaricom through an Initial Public Offer (IPO) pushed the number of shares listed on the bourse to over 55 billion shares, from the previous 15 billion and market capitalization to reach Kshs 1.28 trillion. The IPO had a 532% subscription level and was the first IPO in which citizens of the East African Community were accorded the same treatment as domestic investors and it was the largest IPO in Sub-Saharan Africa at the time. In 2011, the equity settlement cycle moved from the previous T+4 cycle to the T+3 where investors were able to get their money three (3) days after the sale of their shares.

The Nairobi Stock Exchange Ltd also changed its name to the Nairobi Securities Exchange and re-classified equities under ten industry sectors in line with international best practise. The NSE later launched the FTSE NSE Kenya 15 and FTSE NSE Kenya 25 Equity Indices.

Growth Enterprise Market Segment (GEMS) was launched in 2013 and Home Afrika, a real estate company was the first company listed on the GEMS. As a result of its initiatives to increase company listings and diversify asset classes, the NSE was ranked the Most Innovative African Stock Exchange for 2013.

Further, the NSE was ranked the third most performing capital markets in Africa for year 2013 (at 48.6 percent rate of return) behind Ghana Stock Exchange (49.3 percent) and Malawi Stock Exchange (71 percent).

5.2 TESTING CAPM ASSUMPTIONS

The CAPM test was conducted using the SPSS statistical package which has capacity to handle large quantities of data and hence best suited for the NSE data of over 25,000 prices. To carry out the study, 47 shares were identified as being consistently traded on the NSE from years 2004 – 2013.

The CAPM assumption for **normality** was tested on SPSS and the following results obtained:

- Normality assumption – All weekly returns for the stocks, market and risk-free rates were evaluated using the Shapiro-Wilk test but found to be significant at a p-value of 0.05. The significance values were all at 0.00, hence we rejected the null hypothesis and assumed that the data did not follow a normal distribution.

5.3 TESTING CAPM

The idea behind CAPM is that investors need to be compensated in two ways: time value of money and risk. The time value of money is represented by the risk-free (R_f) rate and compensates the investors for placing money in any investment over a period of time. The other half of the formula represents risk and calculates the amount of compensation the investor requires for taking on additional risk. This is calculated by taking a risk measure (beta) that compares the returns of the asset to the market over a period of time and to the market premium (R_m-R_f). If a company has a beta of 3, then it is said to be three times more risky than the overall market.

Therefore, the CAPM is an equation that indicates the required rate of return one should demand for holding a risky asset as part of a diversified portfolio, based on the asset's beta. The higher the beta the higher the expected rate of return as the investor is rewarded for taking on risk that cannot be diversified away. If CAPM indicates a rate of return that is different from that predicted using other criteria (such as P/E ratios or stock charts), then one should, in theory, buy or sell the asset depending on the relationship of the different estimates.

The model argues that there is only one single source of systematic risk – its beta. However, the investor has the ability to diversify risk by collecting uncorrelated assets (implying a beta of zero) which lowers the expected rate of return due to the reduced risk.

The capital asset pricing model (CAPM) is expressed as follows:

$$E(R_i) = R_f + \beta_i [E(R_m) - R_f]$$

where:

- $E(R_i)$ is the expected return on the capital asset
- R_f is the risk-free rate of interest such as interest arising from government bonds
- β_i (the beta) is the sensitivity of the expected excess asset returns to the expected excess market returns,
- $E(R_m)$ is the expected return of the market
- $[E(R_m) - R_f]$ is sometimes known as the market premium (the difference between the expected market rate of return and the risk-free rate of return).

Expressed in terms of their risk premium, we find that:-

$$E(R_i) - R_f = \beta_i [E(R_m) - R_f]$$

Which states that the individual risk premium equals the market premium multiplied by β.

Combining the variables we get:

$E(Z_i) = B_i \, E(Z_m)$

Where:

$E(Z_i) = E(R_i) - R_f$

$E(Z_m) = E(R_m) - R_f$

Since $E(R_f) = R_f$

Removing expectations by the efficient market hypothesis:

$Z_i = B_i \; Z_m$

The regression equation will be:

$Z_{ti} = \alpha_i + \beta_i \, Z_{mt} + \varepsilon_{ti}$

The sample data of this study are gathered through NSE Stocks, which selected 47 stocks to be the samples of the analysis in the NSE. The time period is the weekly data set from January 4, 2004 to December 31, 2013.

In the examination part of this study and to investigate CAPM and its factors, a two-step regression will be applied to determine the variables.

In the first step, a calculation will be conducted to obtain the beta of the 47 stocks of the sample. This step is a time series analysis, which conducted a regression between the return of the stocks and the return of the market index in excess of riskless rate of return order to obtain the beta.

The second step is a cross-section regression, which targets every stock in the sample. In this step, the regression will be conducted between the beta obtained in the first step and the average return of the market, in order to obtain the Security Market Line of the studied market. This study attempts to identify whether the result of the second step is the same as the predicted result of CAPM.

A total of 47 ordinary least squares regressions of the CAPM equation were performed using the 10-year data for the NSE securities, the risk-free rate and the market return (NSE 20-Share) to determine the respective betas of each of the stocks.

The excess asset returns over the risk-free rate were computed for each stock over the testing period 2004 – 2013 on one hand, and the market premium values for each week derived on the other. The risk and market premiums were regressed and the following were the results:

Stocks	R squared	Constant (α_i)	Signif. Constant	Estimated Beta (β_i)	Significance Level
Eaagads Ltd Ord 1.25 AIMS	0.15100	-0.02600	0.00000	0.63400	0.00000
Kakuzi Ltd Ord.5.00	0.29700	-0.01000	0.03400	0.82100	0.00000
Rea Vipingo Plantations Ltd Ord 5.00	0.32200	-0.00100	0.80500	0.93400	0.00000
Sasini Ltd Ord 1.00	0.34000	0.01200	0.04600	1.13400	0.00000
Kapchorua Tea Co. Ltd Ord Ord 5.00 AIMS	0.23500	-0.02500	0.00000	0.64900	0.00000
Williamson Tea Kenya Ltd Ord 5.00 AIMS	0.16200	-0.02100	0.00000	0.67600	0.00000
The Limuru Tea Co. Ltd Ord 20.00 AIMS	0.33500	-0.02800	0.00000	0.59600	0.00000
Car & General (K) Ltd Ord 5.00	0.20500	-0.02000	0.00000	0.69200	0.00000
CMC Holdings Ltd Ord 0.50	0.40900	0.01300	0.01600	1.18000	0.00000
Marshalls (E.A.) Ltd Ord 5.00	0.25800	-0.02800	0.00000	0.62000	0.00000
Sameer Africa Ltd Ord 5.00	0.37800	0.00400	0.41700	1.06700	0.00000
Barclays Bank of Kenya Ltd Ord 0.50	0.36600	-0.00300	0.54100	0.99300	0.00000
CFC Stanbic of Kenya Holdings Ltd ord.5.00	0.14900	-0.00300	0.71700	0.90700	0.00000
Diamond Trust Bank Kenya Ltd Ord 4.00	0.49700	0.01000	0.02100	1.08300	0.00000
Housing Finance Co.Kenya Ltd Ord 5.00	0.39200	0.02400	0.00000	1.28100	0.00000
Kenya Commercial Bank Ltd Ord 1.00	0.41200	0.00700	0.15900	1.06800	0.00000
National Bank of Kenya Ltd Ord 5.00	0.45600	0.02300	0.00000	1.27700	0.00000
NIC Bank Ltd Ord 5.00	0.48600	0.00900	0.03700	1.10900	0.00000
Standard Chartered Bank Kenya Ltd Ord 5.00	0.57500	-0.00800	0.00700	0.89100	0.00000
Express Kenya Ltd Ord 5.00 AIMS	0.28500	-0.01400	0.00500	0.82500	0.00000
Hutchings Biemer Ltd Ord 5.00	0.44900	-0.03100	0.00000	0.58800	0.00000
Kenya Airways Ltd Ord 5.00	0.43400	0.00700	0.13300	1.07900	0.00000
Nation Media Group Ltd Ord. 2.50	0.56200	0.00000	0.93700	0.99300	0.00000
Standard Group Ltd Ord 5.00	0.32700	-0.00500	0.37700	0.94200	0.00000
TPS Eastern Africa Ltd Ord 1.00	0.41100	0.00100	0.85300	0.99500	0.00000
Uchumi Supermarket Ltd Ord 5.00	0.19200	-0.02700	0.00000	0.63000	0.00000
Athi River Mining Ord 1.00	0.33500	0.00100	0.90200	0.94700	0.00000
Bamburi Cement Ltd Ord 5.00	0.52100	-0.01800	0.00000	0.75900	0.00000
Crown Paints Kenya Ltd Ord 5.00	0.23100	-0.01200	0.03100	0.80500	0.00000
E.A.Cables Ltd Ord 0.50	0.29000	0.00900	0.16100	1.07300	0.00000
E.A.Portland Cement Co. Ltd Ord 5.00	0.18300	-0.02700	0.00000	0.62500	0.00000
KenolKobil Ltd Ord 0.05	0.21100	-0.01600	0.00900	0.81700	0.00000
Kenya Power & Lighting Co Ltd Ord 2.50	0.42700	0.00800	0.08600	1.12300	0.00000
Total Kenya Ltd Ord 5.00	0.50600	-0.00800	0.02100	0.90900	0.00000
Jubilee Holdings Ltd Ord 5.00	0.46700	0.00200	0.64700	0.97800	0.00000
Pan Africa Insurance Holdings Ltd Ord 5.00	0.22600	-0.01100	0.04200	0.80000	0.00000
Centum Investment Co Ltd Ord 0.50	0.35200	0.00600	0.27100	1.06500	0.00000

City Trust Ltd Ord 5.00 AIMS	0.01900	0.00600	0.80200	0.82600	0.00100
Olympia Capital Holdings Ltd Ord 5.00	0.20300	-0.01600	0.00800	0.80100	0.00000
A.Baumann & Co Ltd Ord 5.00 AIMS	0.07600	-0.02600	0.00100	0.62000	0.00000
B.O.C Kenya Ltd Ord 5.00	0.38400	-0.03300	0.00000	0.56900	0.00000
British American Tobacco Kenya Ltd Ord 10.00	0.44200	-0.02100	0.00000	0.69700	0.00000
Carbacid Investments Ltd Ord 5.00	0.13100	-0.02600	0.00000	0.63500	0.00000
East African Breweries Ltd Ord 2.00	0.42900	-0.00400	0.33800	0.94500	0.00000
Kenya Orchards Ltd Ord 5.00 AIMS	0.43200	-0.03400	0.00000	0.56600	0.00000
Mumias Sugar Co. Ltd Ord 2.00	0.35000	0.01500	0.01300	1.18700	0.00000
Unga Group Ltd Ord 5.00	0.28800	-0.00900	0.09400	0.87100	0.00000

The Betas ranged from 0.566 to 1.281. The estimated betas were all significant with p-values < than 0.05. The alphas (constant) ranged from -0.034 to 0.024 with 64% of the values being significant:

H_o: $\alpha_i = 0$

H_1: $\alpha_i \neq 0$

Hence we conclude that 64% of the alphas were significantly different from zero. CAPM requires that the intercepts of the estimated regression lines should be 0. Further, the R-squared ranged between 1.90% - 57.5%, indicating a weak relationship between the risk premium (response variable) and the regression (fitted) line.

5.4 THE SECURITY MARKET LINE

The next step will be to estimate the Security Market Line (SML) by regressing the average excess returns over the risk-free rate in 2004 – 2013 to their betas estimated above.

The Security Market Line is therefore:

$$E(R_i) = R_f + \beta_i [E(R_m) - R_f]$$

Re-arranging to calculate the risk premium:

$$E(R_i) - R_f = \beta_i [E(R_m) - R_f]$$

$$E(Z_i) = B_i \, E(Z_m)$$

Using estimated betas this time as the dependent variables we have:

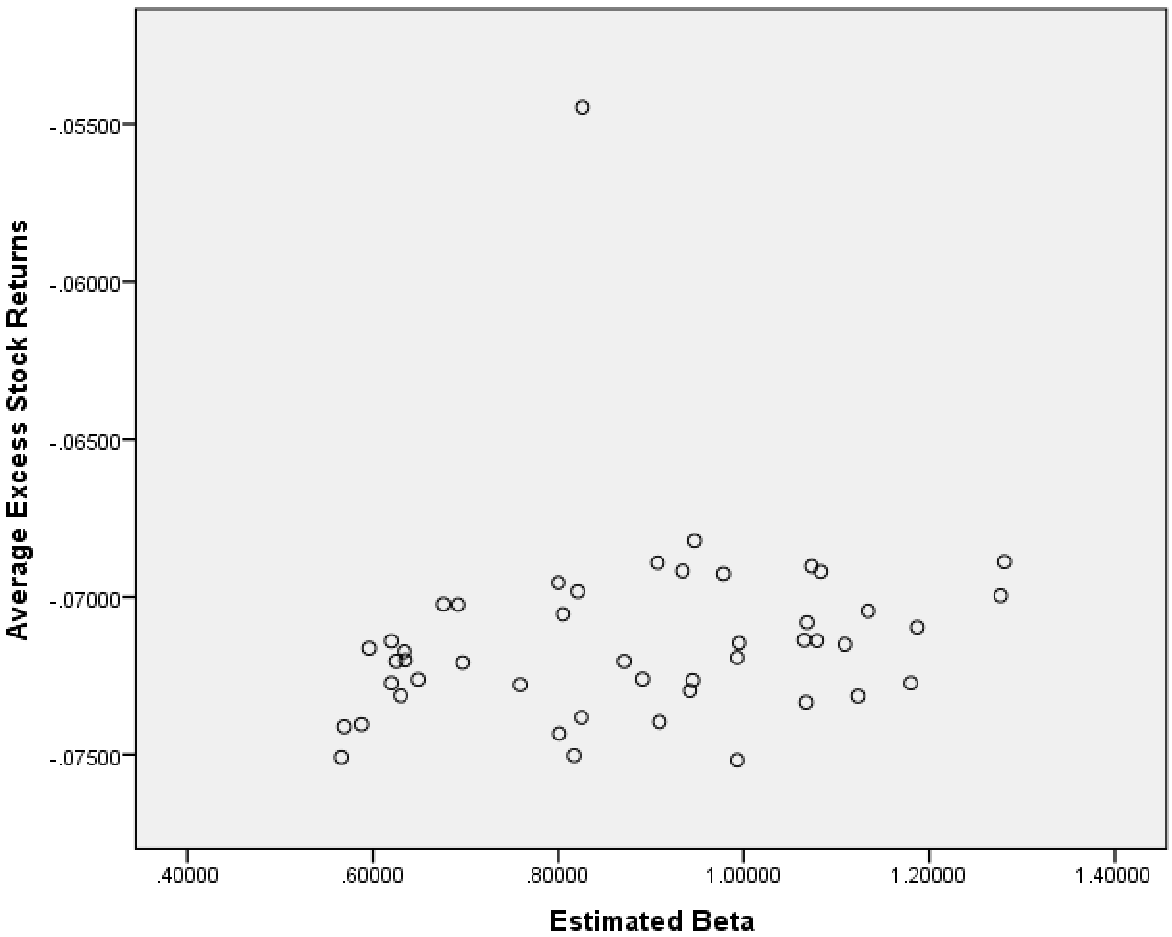
$$E(Z_i) = \gamma_0 + \gamma_1\beta_i + \varepsilon_i$$

Where:

γ_1 = market premium (slope)

If CAPM holds, then the y-intercept (constant) should be $\gamma_0 = 0$.

A graph of the average risk premium and estimated beta revealed a positive correlation:



The regression coefficients were as follows:

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	-.074	.002		-36.692	.000
	Estimated Beta	.003	.002	.176	1.199	.237

a. Dependent Variable: Average Excess returns

The y-intercept γ_0 was -0.074 with a market premium (slope) γ_1 of 0.003.

Despite the constant being negative, the value was significant at p-value < 0.05 hence we reject the null hypothesis that:

H_o: $\gamma_0 = 0$

H_1: $\gamma_0 \neq 0$

Subsequently, we shall perform a further non-linearity test. As predicted by CAPM, assets' returns are linearly related to their betas. We shall test this hypothesis by adding an term additional term to the previous equation:-

$$E(Z_i) = \gamma_0 + \gamma_1\beta_i + \gamma_2\beta^2_i + \varepsilon_i$$

Term β^2_i is simply the second power of our estimated beta. If the model holds and the linear relationship between beta and returns is strong, then adding the beta square should not influence the previous results.

Hence $\gamma_2 = 0$.

Additionally, a further check can be performed for the non-systematic risk. The CAPM argues that the only risk that matters to investors is measured by beta. Any other factors do not matter. To this end, we shall add on another parameter to the regression equation:

$$E(Z_i) = \gamma_0 + \gamma_1\beta_i + \gamma_2\beta^2_i + \gamma_3 RV_i + \varepsilon_i$$

The RV_i term stands for the variance of residuals of an asset i. We obtain the value from the first regression used to estimate betas. Residual variance will therefore be the measure of risk not accounted for by beta. If CAPM holds then:

$\gamma_3 = 0$

After performing the multiple regression above, the following results emerged:

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	-.075	.005		-15.853	.000
	Estimated Beta	.003	.011	.229	.314	.755
	Beta Squared	.000	.006	-.048	-.066	.948
	Variance of residuals	.000	.000	.838	10.556	.000

a. Dependent Variable: Average Excess returns

From the SPSS results, it was determined that the value of $\gamma_2 = 0$ with a p-value of 0.948. Hence we have no reason to reject the null hypothesis and conclude that the value is zero.

Further, $\gamma_3 = 0$ with a significant p-value of less than 0.05.

Therefore, we reject the null hypothesis and conclude that the value is significantly different from zero.

5.5 PREDICTING EX ANTE BETA BASED ON CAPM

We also set to find out if CAPM is a reliable predictor of ex ante beta. Using the formula:-

$$E(R_i) = R_f + \beta_i [E(R_m) – R_f]$$

$$E(R_i) - R_f = \beta_i [E(R_m) – R_f]$$

And making beta the subject:

$$\beta_i = [E(R_m) – R_f] / E(R_i) - R_f$$

We utilized a 'static/equilibrium' forecast of stock and market returns and assumed the stocks stay constant over time, based on their long-run historical average for the last 10 years. The risk-free rate of return is assumed to be the 1 year Treasury bond rate at the beginning of the year ie. 1st January 2014.

The ex ante beta results for year 2014 were as follows:

Stocks	Ex-Post Returns (2004-2013)	Annualised Historical Returns	Risk-Free 1 yr Treasury Bond	Ri-Rf	Ex Ante Beta, B(i)
Eaagads Ltd Ord 1.25 AIM	0.00250	0.139	0.105	0.034	**-1.617**
Kakuzi Ord.5.00	0.00440	0.257	0.105	0.152	**-7.249**
Kapchorua Tea Co. Ltd Ord Ord 5.00 AIM	0.00161	0.087	0.105	-0.018	**0.838**
Limuru Tea Co. Ltd Ord 20.00 AIM	0.00261	0.145	0.105	0.040	**-1.905**
Car & General (K) Ltd Ord 5.00	0.00399	0.230	0.105	0.125	**-5.972**
CMC Holdings Ltd Ord 0.50	0.00150	0.081	0.105	-0.024	**1.148**
Marshalls (E.A.) Ltd Ord 5.00	0.00150	0.081	0.105	-0.024	**1.147**
Sameer Africa Ltd Ord 5.00	0.00089	0.047	0.105	-0.058	**2.758**
Barclays Bank Ltd Ord 0.50	-0.00095	-0.048	0.105	-0.153	**7.313**
CFC Stanbic Holdings Ltd ord.5.00	0.00532	0.318	0.105	0.213	**-10.157**
Diamond Trust Bank Kenya Ltd Ord 4.00	0.00504	0.299	0.105	0.194	**-9.252**
Housing Finance Co Ltd Ord 5.00	0.00535	0.320	0.105	0.215	**-10.259**
Kenya Commercial Bank Ltd Ord 1.00	0.00342	0.194	0.105	0.089	**-4.275**
National Bank of Kenya Ltd Ord 5.00	0.00427	0.248	0.105	0.143	**-6.836**
NIC Bank Ltd Ord 5.00	0.00273	0.152	0.105	0.047	**-2.252**
Standard Chartered Bank Ltd Ord 5.00	0.00163	0.088	0.105	-0.017	**0.802**
Express Ltd Ord 5.00 AIM	0.00041	0.021	0.105	-0.084	**3.997**
Hutchings Biemer Ltd Ord 5.00	0.00019	0.010	0.105	-0.095	**4.551**
Kenya Airways Ltd Ord 5.00	0.00285	0.159	0.105	0.054	**-2.592**
Nation Media Group Ord. 2.50	0.00231	0.127	0.105	0.022	**-1.068**
Standard Group Ltd Ord 5.00	0.00126	0.067	0.105	-0.038	**1.792**
TPS Eastern Africa (Serena) Ltd Ord 1.00	0.00277	0.155	0.105	0.050	**-2.378**
Uchumi Supermarket Ltd Ord 5.00	0.00110	0.059	0.105	-0.046	**2.215**

ARM Cement Ltd Ord 5.00	0.00602	0.366	0.105	0.261	**-12.469**
Bamburi Cement Ltd Ord 5.00	0.00144	0.078	0.105	-0.027	**1.291**
Crown Berger Ltd Ord 5.00	0.00369	0.211	0.105	0.106	**-5.060**
E.A.Cables Ltd Ord 0.50	0.00521	0.310	0.105	0.205	**-9.801**
E.A.Portland Cement Ltd Ord 5.00	0.00219	0.121	0.105	0.016	**-0.743**
KenolKobil Ltd Ord 0.05	-0.00080	-0.041	0.105	-0.146	**6.961**
Kenya Power & Lighting Co Ltd Ord 2.50	0.00109	0.058	0.105	-0.047	**2.245**
Total Kenya Ltd Ord 5.00	0.00027	0.014	0.105	-0.091	**4.346**
Jubilee Holdings Ltd Ord 5.00	0.00497	0.294	0.105	0.189	**-9.024**
Pan Africa Insurance Holdings Ltd Ord 5.00	0.00468	0.275	0.105	0.170	**-8.129**
Centum Investment Co Ltd Ord 0.50	0.00286	0.160	0.105	0.055	**-2.634**
Olympia Capital Holdings ltd Ord 5.00	-0.00011	-0.005	0.105	-0.110	**5.279**
A.Baumann & Co Ltd Ord 5.00 AIM	0.00283	0.158	0.105	0.053	**-2.550**
B.O.C Kenya Ltd Ord 5.00	0.00011	0.006	0.105	-0.099	**4.733**
British American Tobacco Kenya Ltd Ord 10.00	0.00215	0.118	0.105	0.013	**-0.628**
Carbacid Investments Ltd Ord 1.00	0.00223	0.123	0.105	0.018	**-0.839**
East African Breweries Ltd Ord 2.00	0.00159	0.086	0.105	-0.019	**0.903**
Kenya Orchards Ltd Ord 5.00 AIM	-0.00086	-0.044	0.105	-0.149	**7.097**
Mumias Sugar Co. Ltd Ord 2.00	0.00327	0.185	0.105	0.080	**-3.824**
Unga Group Ltd Ord 5.00	0.00219	0.120	0.105	0.015	**-0.740**
NSE 20-SHARE INDEX - (1966 = 100)	0.00155	0.084	0.105	-0.021	

After computing the ex-ante betas, we collected the NSE data for the respective stocks over the year as well as the corresponding NSE 20-share indexes. The objective was to calculate ex post betas and determine if the beta forecasts matched with the actual stock betas in year 2014.

Tests of normality revealed the following:

Tests of Normality

	Kolmogorov-Smirnov[a]			Shapiro-Wilk		
	Statistic	df	Sig.	Statistic	df	Sig.
KapchoruaTeaCoLtdOrdOrd500AIM	.242	52	.000	.877	52	.000
LimuruTeaCoLtdOrd2000AIM	.306	52	.000	.786	52	.000
CarampGeneralKLtdOrd500	.181	52	.000	.894	52	.000
CMCHoldingsLtdOrd050	.170	52	.001	.763	52	.000
MarshallsEALtdOrd500	.196	52	.000	.916	52	.001
SameerAfricaLtdOrd500	.106	52	.200*	.971	52	.238
BarclaysBankLtdOrd050	.072	52	.200*	.979	52	.476
CFCStanbicHoldingsLtdord500	.107	52	.196	.952	52	.034
DiamondTrustBankKenyaLtdOrd400	.083	52	.200*	.986	52	.807
HousingFinanceCoLtdOrd500	.143	52	.010	.914	52	.001
KenyaCommercialBankLtdOrd100	.060	52	.200*	.985	52	.734
NationalBankofKenyaLtdOrd500	.160	52	.002	.926	52	.003
NICBankLtdOrd500	.135	52	.019	.915	52	.001
StandardCharteredBankLtdOrd500	.092	52	.200*	.968	52	.182
ExpressLtdOrd500AIM	.125	52	.042	.955	52	.047
HutchingsBiemerLtdOrd500	.464	52	.000	.149	52	.000
KenyaAirwaysLtdOrd500	.088	52	.200*	.966	52	.137
NationMediaGroupOrd250	.109	52	.175	.972	52	.267
StandardGroupLtdOrd500	.083	52	.200*	.955	52	.047
TPSEasternAfricaSerenaLtdOrd100	.086	52	.200*	.938	52	.010
UchumiSupermarketLtdOrd500	.113	52	.098	.939	52	.010
ARMCementLtdOrd500	.086	52	.200*	.961	52	.090
BamburiCementLtdOrd500	.157	52	.003	.942	52	.014
CrownBergerLtdOrd500	.152	52	.004	.928	52	.004
EACablesLtdOrd050	.097	52	.200*	.956	52	.052
EAPortlandCementLtdOrd500	.120	52	.058	.968	52	.177
KenolKobilLtdOrd005	.073	52	.200*	.983	52	.640
KenyaPowerampLightingCoLtdOrd250	.147	52	.007	.915	52	.001
TotalKenyaLtdOrd500	.115	52	.085	.959	52	.072
JubileeHoldingsLtdOrd500	.140	52	.012	.961	52	.085
PanAfricaInsuranceHoldingsLtdOrd500	.123	52	.049	.868	52	.000
CentumInvestmentCoLtdOrd050	.095	52	.200*	.970	52	.212
OlympiaCapitalHoldingsltdOrd500	.305	52	.000	.538	52	.000
ABaumannampCoLtdOrd500AIM	.170	52	.001	.763	52	.000
BOCKenyaLtdOrd500	.151	52	.005	.942	52	.014
BritishAmericanTobaccoKenyaLtdOrd1000	.137	52	.016	.941	52	.012
CarbacidInvestmentsLtdOrd100	.099	52	.200*	.963	52	.107
EastAfricanBreweriesLtdOrd200	.124	52	.046	.862	52	.000
KenyaOrchardsLtdOrd500AIM	.306	52	.000	.756	52	.000
MumiasSugarCoLtdOrd200	.131	52	.026	.891	52	.000
UngaGroupLtdOrd500	.467	52	.000	.174	52	.000
NSEALLSHAREINDEXNASI01stJan2008100	.089	52	.200*	.978	52	.433
NSE20SHAREINDEX1966100	.064	52	.200*	.982	52	.616
Riskfreerateofinterest91dayTbills	.170	52	.001	.763	52	.000
EaagadsLtdOrd125AIM	.113	52	.094	.972	52	.253
KakuziOrd500	.089	52	.200*	.987	52	.832

*. This is a lower bound of the true significance.
a. Lilliefors Significance Correction

Using the Shapiro-Wilk test results, about 16 of the 43 stocks lie above 5% level of significance which implies that only 37.2% of the data is normal. This can be explained by the low amount of data used in the study as only one year period of 52 weeks was considered.

The weekly excess stock and market returns were regressed and the following betas were computed:

Particulars	Ex Ante Beta, B(i)	Ex-Post Beta (2014)	Absolute Error	Mean Absolute Percent Error
Eaagads Ltd Ord 1.25 AIM	-1.617	-0.038	1.579	4155%
Kakuzi Ord.5.00	-7.249	-0.926	6.323	683%
Kapchorua Tea Co. Ltd Ord Ord 5.00 AIM	0.838	-0.123	0.961	781%
Limuru Tea Co. Ltd Ord 20.00 AIM	-1.905	-0.008	1.897	23713%
Car & General (K) Ltd Ord 5.00	-5.972	-0.891	5.081	570%
CMC Holdings Ltd Ord 0.50s	1.148	0.215	0.933	434%
Marshalls (E.A.) Ltd Ord 5.00	1.147	1.503	0.356	24%
Sameer Africa Ltd Ord 5.00	2.758	-0.179	2.937	1641%
Barclays Bank Ltd Ord 0.50	7.313	0.823	6.490	789%
CFC Stanbic Holdings Ltd ord.5.00	-10.157	0.877	11.034	1258%
Diamond Trust Bank Kenya Ltd Ord 4.00	-9.252	0.499	9.751	1954%
Housing Finance Co Ltd Ord 5.00	-10.259	0.808	11.067	1370%
Kenya Commercial Bank Ltd Ord 1.00	-4.275	0.632	4.907	776%
National Bank of Kenya Ltd Ord 5.00	-6.836	0.453	7.289	1609%
NIC Bank Ltd Ord 5.00	-2.252	1.066	3.318	311%
Standard Chartered Bank Ltd Ord 5.00	0.802	0.616	0.186	30%
Express Ltd Ord 5.00 AIM	3.997	-0.608	4.605	757%
Hutchings Biemer Ltd Ord 5.00	4.551	-0.808	5.359	663%
Kenya Airways Ltd Ord 5.00	-2.592	0.943	3.535	375%
Nation Media Group Ord. 2.50	-1.068	0.411	1.479	360%
Standard Group Ltd Ord 5.00	1.792	-0.110	1.902	1729%
TPS Eastern Africa (Serena) Ltd Ord 1.00	-2.378	0.787	3.165	402%
Uchumi Supermarket Ltd Ord 5.00	2.215	0.304	1.911	629%
ARM Cement Ltd Ord 5.00	-12.469	0.815	13.284	1630%
Bamburi Cement Ltd Ord 5.00	1.291	0.235	1.056	449%
Crown Berger Ltd Ord 5.00	-5.060	0.683	5.743	841%
E.A.Cables Ltd Ord 0.50	-9.801	0.386	10.187	2639%
E.A.Portland Cement Ltd Ord 5.00	-0.743	0.648	1.391	215%
KenolKobil Ltd Ord 0.05	6.961	0.721	6.240	865%
Kenya Power & Lighting Co Ltd Ord 2.50	2.245	0.989	1.256	127%
Total Kenya Ltd Ord 5.00	4.346	0.349	3.997	1145%
Jubilee Holdings Ltd Ord 5.00	-9.024	0.598	9.622	1609%
Pan Africa Insurance Holdings Ltd Ord 5.00	-8.129	0.834	8.963	1075%
Centum Investment Co Ltd Ord 0.50	-2.634	1.496	4.130	276%
Olympia Capital Holdings ltd Ord 5.00	5.279	-0.640	5.919	925%
A.Baumann & Co Ltd Ord 5.00 AIM	-2.550	0.215	2.765	1286%
B.O.C Kenya Ltd Ord 5.00	4.733	0.633	4.100	648%

British American Tobacco Kenya Ltd Ord 10.00	-0.628	1.463	2.091	143%
Carbacid Investments Ltd Ord 1.00	-0.839	0.936	1.775	190%
East African Breweries Ltd Ord 2.00	0.903	1.209	0.306	25%
Kenya Orchards Ltd Ord 5.00 AIM	7.097	2.825	4.272	151%
Mumias Sugar Co. Ltd Ord 2.00	-3.824	0.914	4.738	518%
Unga Group Ltd Ord 5.00	-0.740	-12.827	12.087	94%
AVERAGE FORECAST ERROR			4.651	1392%

Based on these results we computed error measurement statistics to measure the forecast error and determine whether a significant difference exists between the ex ante and ex post betas. The Mean Absolute Deviation (MAD) revealed an error of 4.651 whereas the Mean Absolute Percent Error (MAPE) shows a massive 1392%.

5.6 TESTING THE ZERO-BETA CAPM

A first-pass regression was conducted on the 47 stocks using raw returns instead of the excess returns tested in the standard CAPM. Beta estimates were obtained without deducting risk-free interest rate.

$$R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it}$$

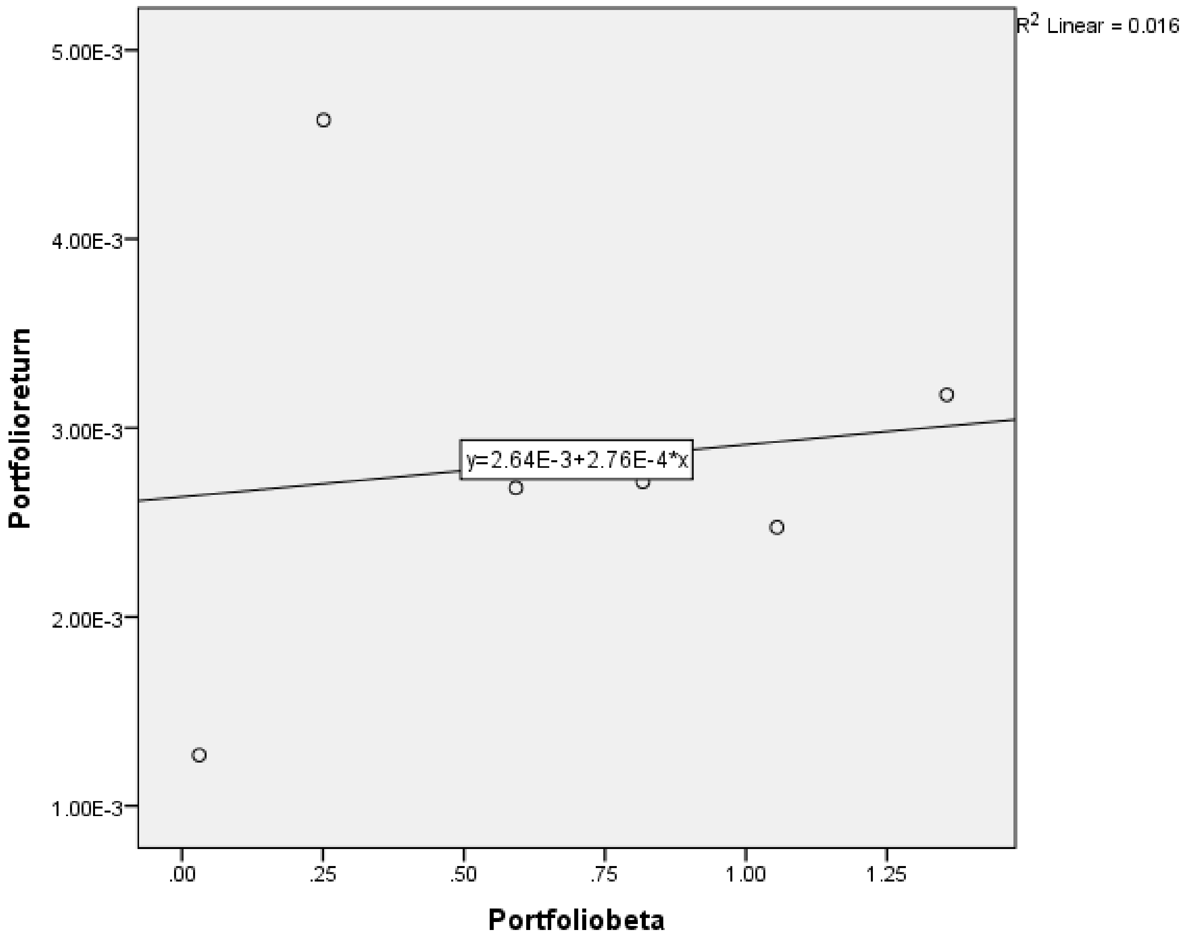
Stocks (2004-2013)	R squared	Constant (α_i)	Significance Level	Estimated Beta (Β)	Significance Level
Eaagads Ltd Ord 1.25 AIMS	0.00100	0.00200	0.40700	0.05800	0.54900
Kakuzi Ltd Ord.5.00	0.08200	0.00400	0.16900	0.57200	0.00000
Rea Vipingo Plantations Ltd Ord 5.00	0.12900	0.00400	0.16900	0.80000	0.00000
Sasini Ltd Ord 1.00	0.21600	0.00200	0.57500	1.27300	0.00000
Kapchorua Tea Co. Ltd Ord Ord 5.00 AIMS	0.01600	0.00100	0.57700	0.22000	0.00400
Williamson Tea Kenya Ltd Ord 5.00 AIMS	0.01100	0.00400	0.23500	0.24200	0.01700
The Limuru Tea Co. Ltd Ord 20.00 AIMS	0.00000	0.00300	0.05900	0.00500	0.92000
Car & General (K) Ltd Ord 5.00	0.01800	0.00400	0.18700	0.27700	0.00200
CMC Holdings Ltd Ord 0.50	0.30600	-0.00100	0.80000	1.43200	0.00000
Marshalls (E.A.) Ltd Ord 5.00	0.00200	0.00100	0.47200	0.07100	0.26700
Sameer Africa Ltd Ord 5.00	0.23800	-0.00100	0.73700	1.17600	0.00000
Barclays Bank of Kenya Ltd Ord 0.50	0.20000	-0.00300	0.34800	1.00700	0.00000
CFC Stanbic of Kenya Holdings Ltd ord.5.00	0.03800	0.00400	0.33100	0.66000	0.00000
Diamond Trust Bank Kenya Ltd Ord 4.00	0.31100	0.00300	0.14100	1.12600	0.00000
Housing Finance Co.Kenya Ltd Ord 5.00	0.30700	0.00300	0.37600	1.60900	0.00000
Kenya Commercial Bank Ltd Ord 1.00	0.27500	0.00200	0.55100	1.20500	0.00000
National Bank of Kenya Ltd Ord 5.00	0.35200	0.00200	0.51100	1.55800	0.00000
NIC Bank Ltd Ord 5.00	0.32500	0.00100	0.71900	1.21600	0.00000
Standard Chartered Bank Kenya Ltd Ord 5.00	0.27100	0.00100	0.72700	0.70300	0.00000
Express Kenya Ltd Ord 5.00 AIMS	0.09300	-0.00100	0.82500	0.64100	0.00000
Hutchings Biemer Ltd Ord 5.00	0.00000	0.00000	0.84400	0.00900	0.74200
Kenya Airways Ltd Ord 5.00	0.22500	0.00100	0.61900	1.02300	0.00000
Nation Media Group Ltd Ord. 2.50	0.35200	0.00100	0.67200	0.99500	0.00000
Standard Group Ltd Ord 5.00	0.11700	0.00009	0.97400	0.75200	0.00000
TPS Eastern Africa Ltd Ord 1.00	0.17900	0.00100	0.55100	0.85200	0.00000
Uchumi Supermarket Ltd Ord 5.00	0.02300	0.00100	0.80500	0.29500	0.00100
Athi River Mining Ord 1.00	0.15900	0.00500	0.09000	0.89200	0.00000
Bamburi Cement Ltd Ord 5.00	0.15700	0.00100	0.59300	0.45200	0.00000
Crown Paints Kenya Ltd Ord 5.00	0.07200	0.00300	0.36500	0.62500	0.00000
E.A.Cables Ltd Ord 0.50	0.14400	0.00400	0.30100	1.06000	0.00000
E.A.Portland Cement Co. Ltd Ord 5.00	0.00700	0.00200	0.45400	0.16400	0.05600
KenolKobil Ltd Ord 0.05	0.09300	-0.00200	0.53500	0.78100	0.00000
Kenya Power & Lighting Co Ltd Ord 2.50	0.29200	-0.00100	0.73300	1.28100	0.00000
Total Kenya Ltd Ord 5.00	0.25400	-0.00100	0.59300	0.80200	0.00000

Jubilee Holdings Ltd Ord 5.00	0.23800	0.00400	0.09400	0.89600	0.00000
Pan Africa Insurance Holdings Ltd Ord 5.00	0.05500	0.00400	0.20000	0.54400	0.00000
Centum Investment Co Ltd Ord 0.50	0.21300	0.00100	0.71800	1.15400	0.00000
City Trust Ltd Ord 5.00 AIMS	0.00100	0.01900	0.10900	0.34700	0.38200
Olympia Capital Holdings Ltd Ord 5.00	0.04700	-0.00100	0.77000	0.53800	0.00000
A.Baumann & Co Ltd Ord 5.00 AIMS	0.00000	0.00300	0.51800	0.03000	0.83500
B.O.C Kenya Ltd Ord 5.00	0.00300	0.00005	0.96900	0.04400	0.24700
British American Tobacco Kenya Ltd Ord 10.00	0.09000	0.00200	0.28200	0.35300	0.00000
Carbacid Investments Ltd Ord 5.00	0.00200	0.00200	0.52300	0.11000	0.29900
East African Breweries Ltd Ord 2.00	0.22300	0.00000	0.93100	0.90000	0.00000
Kenya Orchards Ltd Ord 5.00 AIMS	0.00000	-0.00100	0.33600	-0.00200	0.95200
Mumias Sugar Co. Ltd Ord 2.00	0.20700	0.00100	0.69700	1.27400	0.00000
Unga Group Ltd Ord 5.00	0.11600	0.00100	0.71900	0.76200	0.00000

The stocks were then arranged according to beta size and portfolios of 8 stocks each formed from the biggest to the smallest to minimize measurement errors.

The second-pass regression established with the results of the model is:

$$z_p = \gamma_0 + \gamma_1 \beta_p + \varepsilon_p$$



Rp = 0.00264 + 0.000276Bp

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.	95.0% Confidence Interval for B	
		B	Std. Error	Beta			Lower Bound	Upper Bound
1	(Constant)	.003	.001		2.945	.042	.000	.005
	Portfolio beta	.000	.001	.125	.253	.813	-.003	.003

a. Dependent Variable: Portfolio return

The alpha constant should be equal to the average risk-free rate of 0.074 in support of the CAPM since raw returns have been utilised but the value is much lower. This implies that risk-free is not accurate when taken as a basis for CAPM in the NSE. Further, an R-squared of 0.016 shows a very low coefficient of determination of the fitted line by the dependant variable.

This again shows that zero-beta CAPM is not valid at the NSE over the period of study.

One limitation of the two-factor model is that it relies rather heavily on the assumption that there are no short sales constraints. In other words, investors are assumed to be able to sell shares that they do not already own, then use the proceeds to purchase other shares. Empirically, almost all asset returns have positive correlations. This makes it virtually impossible to construct a zero-beta portfolio composed of only long positions in securities. Therefore the unconstrained use of short sales is a practical necessity to obtain zero-beta portfolios. With short positions the correlation between asset returns is reversed. For example, if you have sold a share short, you make positive returns when the asset price falls. In general, zero-beta portfolios would have to be composed of both long and short positions of risky assets. Ross [1977] has shown that in a world with short sales restrictions and no riskless asset the linear CAPM is invalid. In the NSE, short selling of stocks is prohibited by the Kenyan Capital Markets Authority hence rendering the zero-beta CAPM invalid.

5.7 FAMA FRENCH 3 FACTOR ANALYSIS

The NSE data for the decade was split into portfolios divided into the two additional Fama and French factors: market capitalization and book-to-market ratio. The stocks were first sorted into two groups based on their market capitalization with small and big cap stocks separated for each year. Subsequently, book-to-market (BE/ME) ratios were computed by dividing the share par value against the market share prices for each week. The shares were then ranked into low, medium or high pooled according to the bottom 30 percent (low), middle 40 percent (medium) and top 30 percent (high) BE/ME ratios.

6 portfolios were created namely: Small Low (S/L), Small Medium (S/M), Small High (S/H), Big Low (B/L), Big Medium (B/M) and Big High (B/H) for each year. These portfolios were re-constituted every year based on these factors. On forming the portfolios, their returns were averaged into one return for the portfolio.

Two portfolios for 'Small Minus Big' (SMB) and 'High Minus Low' (HML) were formulated using the formula:-

SMB = $\frac{1}{3}\left[\left(S/L+S/M+S/H\right)-\left(B/L+B/M+B/H\right)\right]$

which is the average return for the smallest 50% of stocks minus the average return of the largest 50% of stocks, and

$$HML=\frac{1}{2}\left[\left(S/H+B/H\right)-\left(S/L+B/L\right)\right]$$

which is average return for the 30% of stocks with the highest BE/ME (book-market) ratio minus the average return of 30% stocks with the lowest BE/ME ratio.

Excess returns were computed by deducting the risk-free interest rate from the new portfolios. With this, we are ready to regress the Fama and French 3 factor model as follows:

$$R_{it}-R_{ft}=\alpha_{it}+b_{it}\left(R_{mt}-R_{ft}\right)+s_{it}\left(SMB\right)+h_{it}\left(HML\right)+\varepsilon_{it}$$

where

R_{it} - Total return of portfolio i

R_{ft} - Risk-free rate of interest

R_{mt} - Total market portfolio return

b_{it} - Measure of exposure an asset has to excess market risk

s_{it} - Measures level of exposure to size risk

h_{it} - Measures level of exposure to value risk

We utilise this data in a multivariate regression to determine the value of alpha and the statistical likelihood that it is materially different from zero as measured by the relevant t-statistic. A multiple regression is carried out using SPSS for the 6 portfolios and the descriptive statistics were summarised as follows:

	Mean	Std. Deviation	Skewness		Kurtosis	
	Statistic	Statistic	Statistic	Std. Error	Statistic	Std. Error
S/L	-0.0716355	0.04829616	0.146	0.107	2.510	0.214
S/M	-0.0699920	0.05600663	3.276	0.107	33.012	0.214
S/H	-0.0730739	0.04698647	0.165	0.107	2.659	0.214
B/L	-0.0706900	0.04473459	-0.322	0.107	1.641	0.214
B/M	-0.0709455	0.04973337	0.377	0.107	3.962	0.214
B/H	-0.0727498	0.08169358	-1.866	0.107	26.308	0.214
R_m-R_f	-0.0726769	0.04656927	0.017	0.107	3.356	0.214
SMB	-0.0001054	0.03233147	1.974	0.107	21.432	0.214
HML	-0.0017491	0.04196458	-1.488	0.107	24.995	0.214

Looking at the three risk factors, excess-market returns and HML are more volatile than SMB but produce lower returns than SMB which is contrary to CAPM postulation that investors require high return for high risk. High BE/ME ratios earn the lowest returns as mentioned by Fama and French (1995). All portfolios produce negative returns with SMB having the highest returns followed by HML. In terms of the 6 portfolios, the returns were highest in the S/M, followed by B/L, B/M, S/L, B/H and finally S/H affirming that high BE/ME ratios earn the lowest returns. Further, on average, big cap stocks marginally outperform the small cap stocks.

Output	**R**	**R-Squared**
S/L	0.758	0.574
S/M	0.823	0.677
S/H	0.933	0.871
B/L	0.924	0.855
B/M	0.902	0.814
B/H	0.898	0.806

The Fama and French 3 factor model sees much improved results as the correlation ranges from 0.758 – 0.933. R-squared values are much higher ranging from 57.4% to 87.1% implying that the two additional risk factors of SMB and HML have significant impact in explaining the variation on excess portfolio returns for the NSE portfolios. We therefore note that the model is an improvement of the standard CAPM.

Multiple regression of the three Fama and French 3 Factors against the 6 portfolios gives the following alpha and beta values:

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.	95.0% Confidence Interval for B	
		B	Std. Error	Beta			Lower Bound	Upper Bound
1	(Constant)	-0.016	0.003		-5.999	0.000	-0.021	-0.011
	R_m-R_f	0.775	0.031	0.747	25.190	0.000	0.714	0.835
	SMB	0.320	0.059	0.215	5.405	0.000	0.204	0.437
	HML	-0.265	0.046	-0.230	-5.822	0.000	-0.355	-0.176

a. Dependent Variable: S/L

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.	95.0% Confidence Interval for B	
		B	Std. Error	Beta			Lower Bound	Upper Bound
1	(Constant)	-0.005	0.003		-1.786	0.075	-0.010	0.000
	R_m-R_f	0.883	0.031	0.735	28.445	0.000	0.822	0.944
	SMB	1.138	0.060	0.657	19.014	0.000	1.021	1.256
	HML	0.530	0.046	0.397	11.530	0.000	0.440	0.621

a. Dependent Variable: S/M

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.	95.0% Confidence Interval for B	
		B	Std. Error	Beta			Lower Bound	Upper Bound
1	(Constant)	-0.006	0.001		-4.529	0.000	-0.009	-0.004
	R_m-R_f	0.908	0.016	0.900	55.134	0.000	0.876	0.940
	SMB	0.518	0.032	0.356	16.307	0.000	0.455	0.580
	HML	0.376	0.024	0.336	15.401	0.000	0.328	0.424

a. Dependent Variable: S/H

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.	95.0% Confidence Interval for B	
		B	Std. Error	Beta			Lower Bound	Upper Bound
1	(Constant)	-0.008	0.001		-5.308	0.000	-0.010	-0.005
	R_m-R_f	0.876	0.017	0.912	52.649	0.000	0.844	0.909
	SMB	-0.304	0.032	-0.220	-9.467	0.000	-0.367	-0.241
	HML	-0.297	0.025	-0.279	-12.054	0.000	-0.346	-0.249

a. Dependent Variable: B/L

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.	95.0% Confidence Interval for B	
		B	Std. Error	Beta			Lower Bound	Upper Bound
1	(Constant)	-0.002	0.002		-1.336	0.182	-0.006	0.001
	R_m-R_f	0.947	0.021	0.886	45.248	0.000	0.906	0.988
	SMB	-0.219	0.040	-0.142	-5.433	0.000	-0.298	-0.140
	HML	-0.124	0.031	-0.104	-3.986	0.000	-0.184	-0.063

a. Dependent Variable: B/M

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.	95.0% Confidence Interval for B	
		B	Std. Error	Beta			Lower Bound	Upper Bound
1	(Constant)	-0.017	0.003		-5.649	0.000	-0.023	-0.011
	R_m-R_f	0.743	0.035	0.424	21.174	0.000	0.674	0.812
	SMB	-0.501	0.068	-0.198	-7.405	0.000	-0.634	-0.368
	HML	1.062	0.052	0.545	20.420	0.000	0.960	1.164

a. Dependent Variable: B/H

Summarized in the following table:-

Output	**Alpha**	**Significance**	**β(R_m-R_f)**	**Significance**	**β(SMB)**	**Significance**	**β(HML)**	**Significance**
S/L	-0.016	0.000	0.775	0.000	0.320	0.000	-0.265	0.000
S/M	-0.005	0.075	0.883	0.000	1.138	0.000	0.530	0.000
S/H	-0.006	0.000	0.908	0.000	0.518	0.000	0.376	0.000
B/L	-0.008	0.000	0.876	0.000	-0.304	0.000	-0.297	0.000
B/M	-0.002	0.182	0.947	0.000	-0.219	0.000	0.124	0.000
B/H	-0.017	0.000	0.743	0.000	-0.501	0.000	1.062	0.000

The alpha ranges between -0.002 to -0.017 and are all significant at 95% significance level except for the Small Medium (S/M) and Big Medium (B/M) portfolios implying that the hypothesis:

H_0: $\alpha_0 = 0$

vs H_1: $\alpha_0 \neq 0$

is rejected and the intercept is not zero for the four portfolios as stipulated by the Fama and French 3 Factor model.

The beta coefficients for the excess market return for the portfolios ranges from 0.743 – 0.947, while the level of exposure to size risk falls between –0.219 to 1.138 and the level of exposure to value risk is between -0.265 to 1.062.

Therefore, NSE still does not fully meet the requirements of the Fama French 3 factor model but shows improved results. This means that a further extension of the model such as a 4 factor model or the Arbitrage Pricing Theory (APT) would significantly improve the results.

5.8 SUMMARY OF RESULTS

The results of the above tests therefore indicate mixed results on the validity of CAPM in the Nairobi Securities Exchange.

(i) In the test for normality, the asset returns over the period 2004 – 2013 did not follow a normal distribution as per the CAPM assumption. Therefore, from the start we were curious to see whether the NSE data would follow CAPM.

(ii) Also a key assumption is that investors trade without transaction or taxation costs. However, the transactions at the Kenyan capital market are charged at a minimum of 2.1% of the trading value for the every transaction.

(iii) According to the first regression test, the Capital Asset Pricing Model was not fully supported by the empirical data. Alphas α_i for 64% of the data were significantly different from 0 and only 36% complied with the CAPM requirement for a zero intercept.

Further, R-squared ranged from 1.90% to 57.50% suggesting only a small portion of the variation in the risk premium is explained by the fitted regression line.

However, the beta values were quite promising with 100% of them significant with 95% confidence.

(iv) The Security Market Line portrayed a positive correlation between the estimated beta and the risk premium which supports the CAPM idea that expected returns are positively and proportionally related to beta.

However, the SML intercept was significantly different from zero and the alpha was negative which could be explained by the presence of transaction costs in the NSE.

(v) The non-linearity test showed an insignificant value of zero for $\gamma_{2,}$ supporting CAPM which postulates a linear relationship exists between asset returns and the beta.

(vi) The estimated value of the coefficient γ_3 showed a value of 0 but revealed a significant difference from 0 which goes against CAPM implying that some element of risk is not captured by beta and non-systematic risk could have some influence on the stocks' returns.

Nonetheless, an R-squared value of 73.40% was recorded which showed a good explanation of the variation of the excess returns by the fitted multiple regression line.

(vii) The ex ante beta forecast based on the assumption of mean reversion of stock and market returns from the past ten years presents a poor forecast of the ex post beta for year 2014. We can conclude that companies listed on the Nairobi Securities Exchange no longer have the same prospects as they once did over the past decade and new market considerations have come in to play which have changed the historical returns significantly.

(viii) The Zero-beta CAPM had R-squared ranging from 0% - 35% showing very low coefficient of determination of the fitted line by the dependant variable. Further, the alpha intercept was significant from zero at the 0.05 significance level leading to its rejection in the NSE.

(ix) The Fama-French 3 Factor Model produced an intercept of between -0.002 to -0.017 which was significant for 4 out of 6 portfolios. Therefore, the model was rejected despite the 3 risk factors showing high correlation and R-squared of above 80% for 4 out of 6 portfolios.

CHAPTER 6: CONCLUSION AND RECOMMENDATIONS

The main objective of this study is to test validity of the CAPM and its extensions in the Kenyan stock market. After performing the empirical analysis on the NSE data for the last 10 years from 2004 - 2013, the overall conclusion is that the Capital Asset Pricing Model is rejected despite some of its general conditions are true: linearity and a positive relationship between beta and returns.

Alphas for majority of the stocks were significantly different from zero implying that other factors are in play apart from the risk-free interest rate in arriving at the stock's expected rates of return.

Further, the true predictive power of CAPM is put to test. The ex-ante betas computed assuming mean reversion of returns indicate a high forecast error from the ex-post betas posted a year later.

Therefore, there is a weak compatibility of the Capital Asset Pricing Model to the NSE stocks.

These results are in line with other CAPM studies undertaken on emerging markets where beta is not a sufficient measure of risk as illustrated by the low correlation and R squares of beta to the risk premium from our study.

The zero-beta CAPM test conducted over the decade also indicated that this model extension is not valid at the NSE. This can be explained by the Capital Markets Authority stipulation that prohibits short-selling of stocks at the bourse. Zero-beta CAPM heavily relies on short sales to function.

The Fama-French Three Factor Model clearly improves on the standard CAPM with high R-squared values implying that the size and value premiums have a large effect on the expected stock returns at the NSE. However, once again, the intercepts are significantly different from zero and the model is also rejected.

We recommend that further research studies should be undertaken to determine which Asset Pricing Model is the best fit for the NSE and investigate what incremental market factors, other than beta, size and value premiums affect market prices at the Nairobi Securities Exchange.

Other model extensions discussed in our study need to be tested for conformity with the Kenyan market. These further tests can help improve the model by relaxing assumptions which have not been met by the NSE such as presence of transaction costs, wealth consumption, foreign investment, varying betas and risk premia. In the event that these extensions fail, then CAPM can be fully rejected as an unrealistic asset pricing model at the NSE.

In addition, further research needs to be undertaken to determine what other unique macroeconomic factors have the most influence on the NSE prices over the years. This could test effects of inflation, short term interest rates, currency exchange rates, oil and gold prices, political environment, investor behaviour among others. This can be evaluated using the APT model.

Also, a CAPM study needs to be undertaken in the Kenyan insurance sector to determine how well CAPM models the insurance data and if it can be relied upon to sufficiently aid decision making in asset liability management. Studies can also be carried out to determine whether CAPM is beneficial in deriving fair insurance premiums where the default risk has been put into consideration as postulated by Yueyuen Chen's theory.

This model however continues to serve as a useful benchmark for asset pricing and is the focus of numerous research as investors seek to make more informed and reliable capital budgeting decisions in financial markets and beyond.

CHAPTER 7: REFERENCES

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