EXTENDING THE NOTION OF RIEMANN INTEGRAL TO LEBESGUE INTEGRAL ON R^2 AND APPLICATIONS IN TIME SERIES ANALYSIS.

BY

FRANCIS NDONGA CHEGE

A project submitted to the school of mathematics in partial

fulfillment for a degree of master of science in pure mathematics

to School Of Mathematics. University Of Nairobi

2^{rd} September 2016

CONTENTS

DECLARATION

Declaration by the student

I the undersigned ,declare that this project is the original work

and has not been used as a basis for any degree in any other University

FRANCIS NDONGA CHEGE

REG.NO . I56/76350/2014

……………. …………….

This Project has been submitted with my approval as the university supervisor

SUPERVISOR

……………… ……………..

Signature Date

ACKNOWLEGEMENTS

Glory and Honor to the Almighty God to the highest .For giving me strength to Move on even when I am in the weakest spirit and moments.

I am very grateful to my supervisors Dr.J.N.Muriuki and Prof J. Khalaghai for their relentless support, direction and clarity while researching this project.

I would also not forget to reminisce the crucial background support from Msc. pure Math's lecturer's, colleagues, librarians, and authors .Finally much appreciation to University of Nairobi for the chance to undertake the postgraduate program.

DEDICATION

To my young , teenage and Adult friends, the selfless in doing together and sharing , that we extend often is a milestone .The non dangerous adventures ,cheerful spirit , conventional with understanding of the goals . The future is us, and we direct virtues , talents and hard work. You are uncountable and wish to be anonymous ,but request to Thank you all ,for this far. And great things in store are endless. We can do it again certainly.

ABSTRACT

This research work is intended for Senior undergraduate course in analysis ,'The 3^{rd} and 4^{th} year B.ed and B.sc mathematics options' and first year student mastering in mathematics. The project covers topics in calculus ,real analysis, measure theory and applications in time series. The beginning chapters lay the setting to Riemann integration in contrast with other earlier existing theories such us mid-ordinate rule and Trapezium method. Riemann defines partition of independent ordinate and take variation of the dependent ordinate then proceed to take the minimum and maximum sum of all the partitions possible and the integral is taken if the two Riemann sum are equal. Some examples of integration are also provided. The theory of Riemann stieltjes is an extension of Riemann theory that covers ;vector- valued functions and discontinuous functions such unit step functions and signum functions. It's bridge the gap of continuity and discontinuity by use of convergence of series and also extend the real line to R^n spaces. The final and most notable extension is the lebesgue integration. The construction of the lebesgue measure is done using countable base, whose members are open interval then the idea of measurable functions is extensively discussed ,before it's use in definition of measurable integral is important ,the we proceed to define monotone convergence theorems and lebesgue dominated convergence theorems. Finally the comparison of the two integration theories 'Riemann and lebesgue' is done by citing a number of similarity and loopholes in evaluation of integral in areas such as ;Bounded and Un bounded functions ,Complex and L_P-spaces and recovery of derivative functions. Finally application of the Fourier Series integrals in Time-Series Analysis is done by by smoothing time plot by regression and other methods which allow finding of auto correlation , wavelet and spectrum analysis.

CHAPTER ONE

1.0 INTRODUCTION

1.1 General Introduction

Integration means bringing parts together ,it is the process that is inverse to differentiation.

Thus the definite integration, "Let f be defined on the interval [a,b],the definite integral of f

Is given by $\int_a^b f(x)dx = \lim_{n\to\infty}\sum f(x_i)\Delta x$, ,provided the limits exists, where $\Delta x = \frac{(b-a)}{n}$ and x_i

is any value of x in i^{th} interval. This definite integral is a number (example of Riemann sum)

The fundamental Theorem of calculus ,'let f be continuous on the interval[a,b] and let F be any ant derivative of f .Then $\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b$ which shows connection of between ant derivatives and definite integrals.Other important theorems allied to Riemann includes the Archimedes ($287 - 213B.C$) ,First Principles and mean value theorem.

Riemann integral became inadequate and could not give solutions in discontinuity as well as Functions with increasing number of limits. Thus extensions such as Riemann Stieljes and Lebesgue integration theories allows us to integrate a much larger class of functions such as step-wise functions(discontinuous functions)and also many limits operations can be handled with a lot of ease.

1.2 PROBLEM STATEMENT

Many research studies has been done on the integration techniques ,but very few of their feedback narrow back to its development from reasonably well-behaved functions on sub-intervals of real line. As well as developed theories of integrations that can be applied to much large classes of functions whose domains are more or less arbitrary set ,including subsets of R^2 This research aim to put across different ways of approximating areas of the regions, the Riemann theory and extensions by Stieltjes and Lebesgue and also its applications in time series analysis

1.3 OBJECTIVES

The overall objectives is to survey the formulation (or derivation) of both Riemann integral and Lebesgue integral and make a brief comparison between theories.

1.4 Specific Objectives

1 .Investigate the fundamental concepts of Riemann and Riemann-Stieltjes theory of integration.

2. Construction of the lebesgue measure and integration and some of the main theorems of the theory.

3.Make a brief comparison stating where possible advantages of Lebesgue integral theory over the Riemann integral theory.

4.Exhibit examples to show applications in Time Series Analysis.

1.5 SIGNIFICANCE OF STUDY

Lebesgue integration have wide range of applications in statistics of expectations, Solutions to time series analysis and research methods. Furthermore integration and differentiation is very vital in applied and Engineering mathematics. It also occupy a central place in analysis , in the study of (L^2-Spaces and L^p-spaces).

CHAPTER 2

2.0 LITERATURE REVIEW

2.1 Motivation

Three Cambridge University Dons of mid -20^{th} Century in their three books,

'Cambridge Mathematics; Part I ,Part II ,and Part III ,classified the subject into

(i)Mathematics for pre-university/undergraduate mathematics

(ii)Applied mathematics of specialized courses and

(iii)Mathematics Analysis

Riemann and Lebesgue Theories Of Integration are some of earlier stage of analysis and extending the study of real line to R^n spaces just make it much involved .Furthermore application of orthogonal integral to time series analysis is crucial in Biostatistics ,geophysics and financial fields

2.2 Background Information.

The concepts of integration dates backs to ($(287-213B.C)$) where Archimedes and his contemporaries would apply the first principles to find area of planes figures even before the method of differentiation was discovered. Otherwise, the concepts of integration as a technique that both acts as a an inverse to the operation of differentiation and also compute area under curves ,goes back to the origin of calculus and the work of Isaac Newton $(1643-1727)$ and Leibnitz $(1646-1716)$

It was Leibnitz who introduced the $\int$ ...dx notation. The first rigorous attempt to understand integration as a limiting operation within the spirit of analysis was due to Bernard Riemann $(1826-1866)$. The approach of Riemann that is usually taught was however developed by Jean-Gaston Dar boux $(1842-1917)$ .at the time it was developed this theory seemed to be all that was needed but as the 19^{th} century drew closer, some problem appeared.

(i)One of the main tasks of integration is to recover a function f from it's derivative f'.

but some functions were discovered for which f' was bounded but not Riemann integrable.

(ii)Suppose (f_n) is a sequence of functions converging point wise to f. The Riemann integral could not be used to find conditions for which $\int f(x)dx = \lim_{n\to\infty} \int f_n\ (x)dx$

(iii)Riemann integration was limited to computing integrals over R^2 with respect to Lebesgue measure, although it is not yet apparent ,the emerging theory of probability would require the calculation of expectations of random variables $x; E(X) = \int_{\Omega} x(w)dp(w)$.The Lebesgue's technique allows us to investigate $\int_{s} f(x)dm(x)$ where $f; S \to R$ is a 'suitable' measurable function defined on a measure space $(S, \sum, M)$, If we take M to be the Lebesgue measure on $(R, B(R))$. we recover the familiar integral $\int_{R} f(x)dx$ but we will now be able to integrate many more functions (at least in principles)than Riemann and Darboux. If we take X to be a random variable on a probability space, we get it's expectation $E(x)$.

2.3 COMPARISON

Many authors such as have compared the two theories Riemann and Libesgue inform of integral theorem, but much of comparisons tools will depend on the calculus reader/student in identifying the key areas, applications and the successes or failure of each method. This article cite five such areas namely; Integration of discontinuous functions, Relation of differentiation and integration, complex functions and $L^2 - space$ s.

2.4 APPLICATION

There are wide range of stationary time series models methods for estimation of autocorrelation and spectrum as well as methods for multivariate stationary series, and those that forecasting future values . Authors who have written materials in this field includes

Priestly .M,' Spectral Analysis and Time Series'. Hannan. E.J,' Time Series Analysis.' etc..

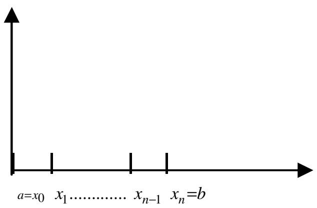
CHAPTER THREE

RIEMANN INTEGRATION

3.1.0 (Partition)

3.1.1 Definition; Let [a, b] be a compact interval. Then the set of points $p = \{x_o, x_1, ...x_n\}$

satisfying the inequality $a = x_0 < x_1 < x_2 .... < x_n = b$ is called a partition of $[a,b]$



3.1.2 Consequences

(a) $\Delta x_k = x_k - x_{k-1}$ such that $\sum_{k=1}^{n} \Delta x_k = b - a$

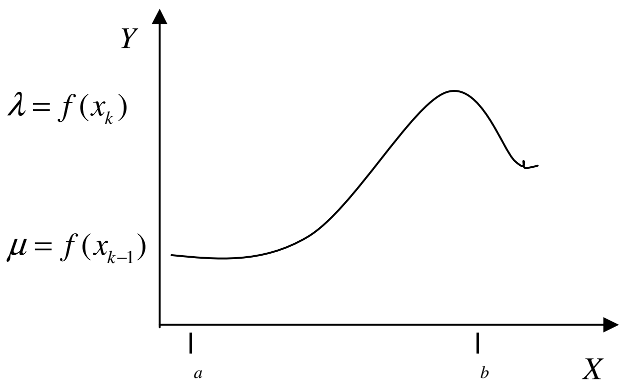
(b) collection of all possible partition on $[a,b]$ is denoted by $Q(a,b) \Rightarrow P \in Q[a,b]$

I .e P is a partition of $[a,b]$

3.2.0 Bounded Variation (Bounded Variation)

3.2.1 Definition ; Let f be a function on $[a,b]$ with $\Delta f(x_k) = f(x_k) - f(x_{k-1})$, if there exist a number M such that M>0 and $\sum | f(x_k) - f(x_{k-1}) | \leq M \quad \forall p \in Q[a,b]$.

Then the function f is said to be bounded variation on $[a,b]$ and is denoted by $f \in B.V[a,b]$.



3.2.2 Theorem

If f is monotonic on $[a,b]$ then $f \in B.V[a,b]$

Proof

A monotonic f is either an increasing $(\uparrow)$ or decreasing $(\downarrow)$ function on an interval $[a,b]$. (i)When f is increasing $(\uparrow)$ on $[a,b]$

Then for every partition of $[a,b]$ we have $\Delta f = f(x_k) - f(x_{k-1}) \geq 0$

Hence $$\sum_{i=1}^{n} f(x_k) - f(x_{k-1}) = \sum_{i=1}^{n} f(x_k) - \sum_{i=1}^{n} f(x_{k-1})$$

$$= f(b) - f(a)$$

Putting $f(b) - f(a) = M$, hence for all possible partitions ,

$f \in B.V[a,b]$ since $\sum_{k=1}^{n} |\Delta f x_k| \leq M$

(ii)If f is decreasing $(\downarrow)$ on $[a,b]$

Then for every partition of $[a,b]$

We have $\Delta f(x_k) = f(x_{k-1}) - f(x_k) \geq 0$

Hence $$\sum_{i=1}^{n} | f(x_k) - f(x_{k-1}) | = \sum_{i=1}^{n} f(x_{k-1}) - \sum_{i=1}^{n} f(x_k)$$

$$= f(b) - f(a)$$

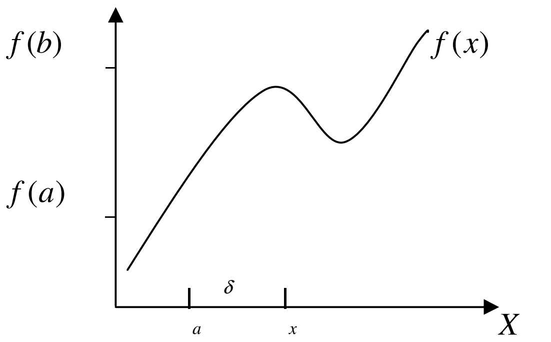
Putting $f(b) - f(a) = M$ implies that $\sum_{k=1}^{n} |\Delta f x_k| \leq M$

Hence for all partitions on $[a,b], f \in B.V[a,b]$

3.2.3 Def ($\varepsilon - \delta$,definition of continuity)

A function $f(x)$ is continuous at a point a if for every number $\varepsilon > 0$ their exist $\delta > 0$

Such that $|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon$

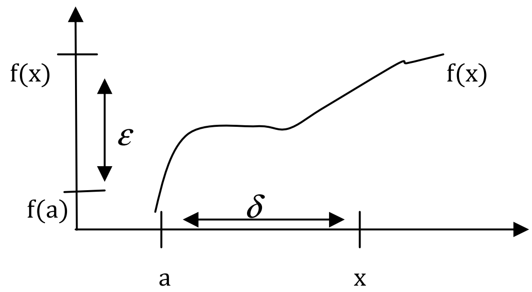


3.2.4 Example

The function $f(x) = \dfrac{x^2-5}{x-4}$ is continuous at $x=5$ since $\lim_{x\to 5} \dfrac{x^2-5}{x-4}$ has a value(exist).

On the contrary $f(x)$ is not continuous at $x=4$, because its limit has no value.

Proof



In this case $a=5$, $f(x) = \dfrac{x^2-5}{x-4}$

choose any $\varepsilon > 0$ and fix it such that $|f(x)-f(a)| < \varepsilon$

i.e $\left|\dfrac{x^2-5}{x-4} - 20\right| < \varepsilon$ or $\left|\dfrac{x^2-5-20x+80}{x-4}\right| < \varepsilon$

$$= \left|\frac{x^2-20x+75}{x-4}\right| < \varepsilon = \left|\frac{(x-5)(x-15)}{x-4}\right| < \varepsilon$$

$= |(x-5)||\frac{x-15}{x-4}|<\varepsilon$

$= |x-5|<\varepsilon|\frac{x-4}{x-15}|$ $\longrightarrow$ $\frac{1}{10}$ (for x close to 5)

i.e $|x-5|<\frac{\varepsilon}{10}=\delta$ Thus $\delta>0$ and $|x-5|<\delta$

whenever $|x-5|<\delta \Rightarrow |f(x)-f(5)|<\varepsilon$

3.2,5 . Theorem; Let f be continuous in $[a,b]$, if the derivative f' of the function f exist and is bounded on [a, b] such that for $\forall x\in(a,b)$, then f is of bounded variation.

Recall mean value theorem $f'(t_k)=\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}$ and $\Delta f(x_k)=f(x_k)-f(x_{k-1})$

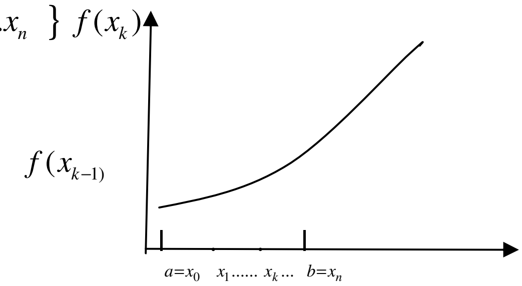
by mean value theorem $\Delta f(x_k)=f'(x_k)(x_k-x_{k-1}) = f'(t_k)\Delta x_k$ where $x_{k-1}\le t_k\le x_k$

And hence $\sum|\Delta f_k|=\sum|f'(t)\Delta x_k|\le A(b-a)$ Putting $A(b-a)=M$,

we have $\sum|\Delta f_k|\le M$ i. e f is a bounded variation.

3.3.0 Total Variation

3.3.1 Def ;Let $f \in B.V[a,b]$ and let $Sp = \sum | f(x_k) - f(x_{k-1}) |$ corresponding to the partition $p = \{x_0, x_1, x_2 \dots\dots\dots x_n \}$



Let $Q[a,b]$ be the set of all partition of $[a,b]$, the number

$V_f(a,b) = Sup\{s_p ; p \in Q(a,b) \}$

$$= Sup\{s_p = \sum | f(x_k) - f(x_{k-1}) | \mid P \in Q(a,b)\}$$ is called the total variation of f on $[a,b]$.

3.3.2 Theorem

Let $f \in B.V(a,b)$ and let $a < c < b$ then $f \in B.V[a,c]$ and $f \in B.V[c,b]$ furthermore $V_f[a,b] = V_f[a,c] + V_f[c,b]$

Proof

(I)Showing $V_f(a,c) + V_f(c,b) \le V_f(a,b)$

Let p_1 and p_2 be any arbitrary partitions of $[a,c]$ and $[c,b]$ respectively. Then $p_{0=} p_{1\cup} p_2$ is a partition of $[a,b]$.Let $Sp_i = \sum | f(x_k) - f(x_{k-1}) |$, corresponds to the partitions p_i (for arbitrary appropriate interval) then $\sum p_1 + \sum p_2 = Sp_0 \le V_f(a,b) \Rightarrow Sp_1$ and Sp_2 are bounded above by $V_f(a,b)$,Which implies that $Sp_1 = \sum | f(x_k) - f(x_{k-1}) | \le V_f(a,b)$ and $Sp_2 = \sum | f(x_k) - f(x_{k-1}) | \le V_f(a,b)$ hence f is of bounded on $[a,c]$ and $[c,b]$ and from above we have $V_f(a,c) + V_f(c,b) \le V_f(a,b)$.

(II)To show $V_f(a,b) \le V_f(a,c)+V_f(c,b)$

Let $p_0 = x_0, x_1, ...x_n$ be partition on $[a,b]$ and let $P' = P \cup \{c\ \}$ obtained by adjoining a point c in p_0.If $c \in (x_{k+1}, x_k)$ then $|f(x_k)-(x_{k-1})| \le |f(c)-f(x_{k-1})|+|f(c)+f(x_k)|$ so that $Sp_0 \le Sp'$ .The points P which belongs to $[a,c]$ and the points of P which belongs to $[c,b]$ determines the partitions p_1 and p_2 hence $Sp_0 \le Sp' = Sp_1 + Sp_2$ I .e $Sp_0 \le Sp_1 + Sp_2$

$$\le Vf(a,c)+Vf(c,b) \quad \Rightarrow V_f(a,b)=V_f(a,c)+V_f(c,b)$$

3.3.3 Theorem

Let $f \in BV[a,b]$ and consider the function F defined in [a, b] by $f(x)=\begin{cases} v_f(a,x); if ... a<x<b \\ 0; if .....x=a \end{cases}$ then F(↑) and F-f(↑)

Proof

For $a<x<y \le b$ we have $V_f(a,b)=V_f(a,x)+V_f(x,y)$ ....(i)

so that $F(y)=F(x)+V_f(x,y) \Rightarrow V_f(x,y)=F(y)-F(x)$

$$\Rightarrow F(y)-F(x) \ge 0$$

$$\Rightarrow F(x) \le F(y) \text{ but } x \le y \Rightarrow F \uparrow \text{ i. e non decreasing.}$$

Also for $a \le x \le y \le b$ we have $(F-f)y-(F-f)x = F(y)-f(y)-[F(x)-f(x)]$

$$=\{[F(y)-F(x)]-[f(y)-f(x)]$$

$$=V_f(a,y)-V_f(a,x)-[f(y)-f(x)]$$

$$=V_f(x,y)-[f(y)-f(x)] \ge 0$$

$$\Rightarrow (F-f)y-(F-f)x=0$$

$$\Rightarrow (F-f)x \le (F-f)y \text{ but } x \le y \text{ I .e } F-f \uparrow \text{ hence non-decreasing}$$

3.3.4 Theorem

A real valued function f defined on [a, b] is of bounded variation on [a, b] if and only if f can be expressed as a difference of two non-decreasing functions f_1 and f_2 i. e $f(x) = f_1(x) - f_2(x)$,

with f_1 and f_2 non-decreasing on [a ,b].

Proof

Let $f \in B.V[a,b]$ then $f = F - (F - f)$,

Let F be defined as $F(x) = \begin{cases} V_f = (a,x); a < x < b \\ 0; \ldots\ldots\ldots x = a \end{cases}$

Where both F and $F - f$ have been shown to be non-decreasing (by previous theorem)

Putting $F = f_1$ and $F - f_1 = f_2$ then f can be expressed as a difference of two non-decreasing functions.

Conversely

Let $f = f_1 - f_2$ when f_1 and f_2 are non-decreasing functions on [a, b]

f_1 and f_2 are monotonic on [a, b]

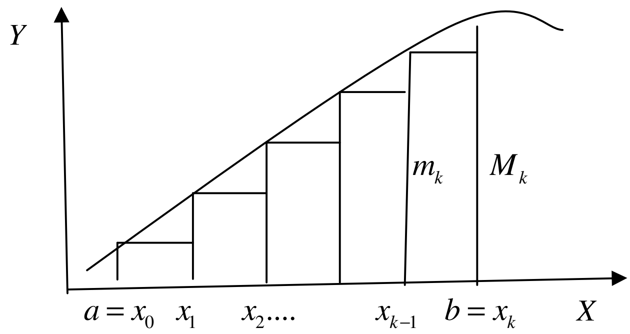
Thus f_1 and f_2 are of bounded variation on [a, b].

Hence the difference $f_1 - f_2$ is of bounded variation on [a, b]

I .e $f = f_1 - f_2$ is of bounded variation.

3.4.0 RIEMANN INTEGRATION

3.4.1. Definition; Let f be continuous and bounded on $[a,b]$, divide $[a,b]$ into n sub-divisions by points $x_0, x_1, ...x_n$



Thus partition $P = \{x_0, x_1, ..., x_n\}$ such that $a = x_0 < x_1 < .... < x_n = b$.

Let the largest sub-interval have value $\Delta x_k = x_k - x_{k-1}$

Let $M_k = \sup f(x) = \sup\{f(x); x \in (x_{k-1}, x_k)\}$ for $x_{k-1} < x < x_k$

$m_k = \inf(x) = \inf\{f(x); x \in (x_{k-1}, x_k)\}$, for $x_{k-1} < x < x_k$ and for each partition

form the sum $S_{(p)} = M_1(x_1 - x_0) + M_2(x_2 - x_1)............M_n(x_n - x_{n-1})$ = $\sum_{k=1}^{n} M_k \Delta x_k$

Similarly $s_{(p)} = m_1(x_1 - x_0) + m_2(x_2 - x_1) + ...m_n(x_n - x_{n-1})$ $= \sum_{k=1}^{n} m_k \Delta x_k$

S_p and $s_{(p)}$ are called the upper and lower sum respectively ,by varying the partition we obtain

set of $S_{(p)}$ and $s_{(p)}$,Let $U = \inf S_{(p)} = g.l.b$ of the values of $S_{(p)} \forall$ possible partition. Let $L = Sups_{(p)} = l.u.b$ of all values of $s_{(p)} \forall$ possible partition. These values which always exist

are called upper and lower Riemann integrals of f over [a ,b] denoted by $U = \int_a^b f(x)dx$ and

$L = \int_a^b f(x)dx$ If $L = U$ i. e If the lower and upper integrals are equal then f is said be

Riemann-integrable over $[a,b]$ and the common integral is denoted by $I=\int_a^b f(x)dx$

(i)if $U\neq L$, f is not integrable over the interval $[a,b]$

(ii) the expression $I=\int f(x)dx$ is called the Riemann integral.

3.4.2 Theorem

Let f be continuous on $[a,b]$ and $a<c<b$ then $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$

Proof

Let p_1 and p_2 be partition of $[a,c]$ and $[c,b]$ respectively and $P=p_1\cup p_2$

i.e P consists of at least one of the sets p_1 and p_2 ,where by $L=SupS(p)$

clearly $S(P)=S(p_1)+S(p_2)$ moreover $S(P)\leq L\leq\int_a^b f(x)dx$, then given any p_1 of $[a,b]$

and p_2 of $[a,b]$ $\Rightarrow S(p_1)+S(p_2)\leq\int_a^b f(x)dx\Rightarrow S(p_1)\leq\int_a^b f(x)dx-S(p_2)$..........$(i)$

For any part p_2 of (c,b) the right hand side of (i) forms an upper bound of $S(p_1)$,

$$\Rightarrow SupS(p_1)\leq\int_a^b f(x)dx-S(p_2)$$

$$\Rightarrow SupS(p_1)\leq\int_a^c f(x)dx\leq\int_a^b f(x)dx-S(p_2)\quad \text{i. e}\quad \int_a^c f(x)dx\leq\int_a^b f(x)dx-S(p_2)$$

$\Rightarrow S(p_2)\leq\int_a^b f(x)dx-\int_a^c f(x)dx$.....$(ii)$ $\forall$ partition p_2 in $[a,b]$, the right hand side of (ii)

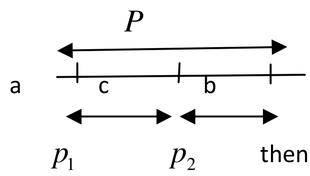
forms an upper bound $\Rightarrow SupS(p_1)\leq\int_a^b f(x)dx-\int_a^c f(x)dx$

$\Rightarrow SupS(p_2)\leq\int_c^b f(x)dx\leq\int_a^b f(x)dx-\int_a^c f(x)dx$ Thus $\int_a^c f(x)dx+\int_c^b f(x)dx\leq\int_a^b f(x)dx$ ....*

To show the reverse inequality

Let P be any partition of [a, b] and Q be the partition obtained from P

by adjoining a point C in [a, b]



then $s(p) \le s(Q)$

Let p_1 be the part of [a,b] consisting those points of Q which lie on [a, c] and p_2 be part of [a, b] consisting of those points of Q which lie on [c, b] then

$$s(p) \le s(Q) = s(p_1) + s(p_2) \quad \le \int_a^c f(x)dx + \int_c^b f(x)dx$$

i.e $s(p) \le \int_a^c f(x)dx + \int_c^b f(x)dx$ $\forall$, possible partition P on [a, b]

$$SupS(p) \le \int_a^c f(x)dx + \int_c^b f(x)dx \qquad \because SupS(p) \le \int_a^b f(x)dx \le \int_a^c f(x)dx + \int_c^b f(x)dx$$

then $\int_a^b f(x)dx \le \int_a^c f(x)dx + \int_c^b f(x)dx ......**$

By $*$ and $**$ equality is established i. e $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

*3.4.*3 Theorem

Let f be continuous on $[a,b]$ with $M = \max f(x)$ and $m = \min f(x)$ on $[a,b]$

Then $m(b-a) \le \int f(x)dx \le M(b-a)$

Proof

Let $S_{(p)} = \sum M_k \Delta x_k$, $s_{(p)} = \sum m_k \Delta x_k$ since $m \le m_k \le M_K \le M$,taking summation from $k=1$ to n $\sum m\Delta x_k \le \sum m_k \Delta x_k \le \sum M_K \Delta x_k \le \sum M\,\Delta x_k$ .For all possible partitions over

$[a,b]$ thus we have $m\sum \Delta x_k \le s_{(p)} \le S_{(p)} \le M\sum \Delta x_k \Rightarrow m(b-a) \le Sups_{(p)} \le \inf S_{(p)} \le M(b-a)$

But $Sups_{(p)} \le \int_a^b f(x)dx \le \inf S_{(p)}$ hence $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

3.4.4 Properties of Riemann integral

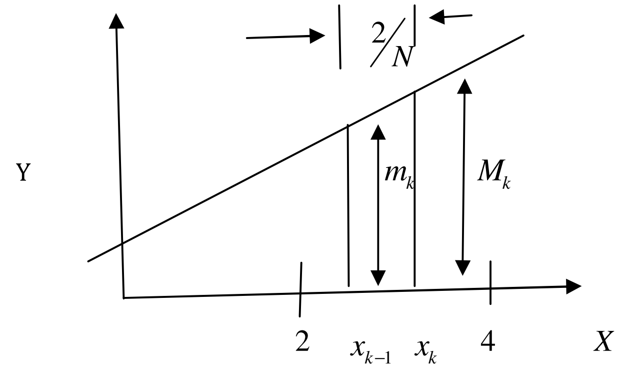
1.If $f(x)=c$ where c is constant then $\int_a^b f(x)dx = c(b-a)$.

2.Let f be continuous then $\int_a^b \{f(x)+c\} = \int_a^b f(x)dx + c(b-a)$

3.If f is continuous and integrable on $[a,b]$,then there exist a number c between a and b such that $\int_a^b f(x)dx = (b-a)f(c)$.

3.4.5 Example 1 Find the integral of $\int_2^4 (x+1)dx$ We need to decide on some partitions that would involve smaller and smaller segments, hoping that the corresponding upper and lower sums will get into N equal segments. $P_N; x_k = 2+\frac{k}{N}(4-2) = 2+\frac{2k}{N}, k=0,1,......N$

We determine the $\sup rema$ and $\inf ima$ for the sum, but this should be easy (see diag)



$$U(f,p_N)=\sum_{k=1}^{N} f(x_k)(x_k-x_{k-1})=\sum_{k=1}^{N}((\frac{2+2k}{N})+1).\frac{2}{n}$$

$$=\frac{6}{N}\sum_{k=1}^{N}1+\frac{4}{N^2}\sum_{k=1}^{N}k=\frac{6}{N}.N+\frac{4}{N^2}.\frac{N(N+1)}{2}$$

$$=6+\frac{2N+1}{N}$$

$$L(f,p_N)=\sum_{K=1}^{N} f(x_{k-1})(x_k-x_{k-1})=\sum_{k=1}^{N}((2+\frac{2(k-1)}{N}+1).\frac{2}{N}$$

$$=\quad \frac{6}{N}\sum_{K=1}^{N}1-\frac{4}{N^2}\sum_{K=1}^{N}1+\frac{4}{N}\sum_{K=1}^{N}K$$

$$=\frac{6}{N}.N-\frac{4}{N^2}.N+\frac{4}{N^2}.\frac{N(N+1)}{2}$$

$$=6-\frac{4}{N}+2(\frac{N+1}{N})$$

When we send N to infinity ,the sums approximate the area as well

$$Inf\{U(f,p)\leq \lim_{n\to\infty}(U(f,p_N)=\lim_{n\to\infty}(6+\frac{2N+1}{N})=8$$

$$Sup\{U(f,p)\}\geq \lim_{n\to\infty}(L(f,p_{_N}))=\lim_{n\to\infty}(6-\frac{4}{N}+2\frac{N+1}{N})=8$$

Thus

$$8\leq Sup\{U(f,p)\}=\inf\{U(f,p)\}\leq 8$$

$$Sup\{U(f,p)\}=\inf\{U(f,p)\}=8$$

Hence the function is Riemann integrable on and $\int_2^4 (x+1)dx=8$

3.4.6 Example 2

Show that a constant function k is integrable and $\int_a^b kdx = k(b-a)$

For any partition p of the interval $[a,b]$,

we have $L(p,f) = k\Delta x_1 + k\Delta x_2 + ... + k\Delta x_n$

$$= k(\Delta x_1 + \Delta x_2 + ... + \Delta x_n) = k(b-a)$$

$$\underline{\int_a^b} kdx = \sup L(p,f) = k(b-a)$$

$$\overline{\int_a^b} kdx = \inf U(p,f) = k(b-a)$$

Thus $\underline{\int_a^b} kdx = \overline{\int_a^b} kdx = k(b-a)$

3.4.7 Example3

Show that the function f defined by

$$f(x) = \quad f(x) = \begin{cases} 0; when..x..is..rational \\ 1; when..x..is..irrational \end{cases}$$ is not integrable on any interval

Let us consider a partition p of an interval [a,b]

$$U(p,f) = \sum_{i=1}^{n} M_1 \Delta x_1 = 1\Delta x_1 + 1\Delta x_2 + ... + 1\Delta x_n = b-a$$

$$\overline{\int_a^b} fdx = \inf U(p,f) = b-a$$

$$L(p,f) = \sup\{0\Delta x_1 + 0\Delta x_2 + ... + 0\Delta x_n\} = 0$$

$$\underline{\int_a^b} fdx = \sup L(p,f)$$

Thus $\overline{\int_a^b} fdx \neq \underline{\int_a^b} fdx$, hence, the function f is not integrable.

3.5.0. Some Calculus Theorems Allied to Riemann Integral

3.5.1 Definition

Let f be differentiable and defined on (a,b) and let f be continuous on $[a,b]$,

If f satisfies $F'(x) = f(x) \forall x \in (a,b)$, then F is called the anti derivative or primitive of f

3.5.2 Example

For $F(x) = x^2$ then anti derivative of $f(x)$ is defined by $F(x) = \frac{x^3}{3} + c$

3.5.3 Theorem

Let F be anti derivative for f and G be defined on $[a,b]$ .Then G is a primitive for f

on $[a,b]$ if and only if for some constants $c, G(x) = F(x) + c$

Proof

$F(x) + c$ is a primitive of f on $[a,b]$,suppose G is a primitive of f on $[a,b]$

then $F - G$ is continuous and differentiable on $[a,b]$

$$\Rightarrow D[F(x) - G(x)] = F'(x) - G'(x)$$

$$= f'(x) - f'(x)$$

$$= 0$$

$$\Rightarrow F(x) - G(x) = c$$

$$\Rightarrow G(x) = F(x) + c$$

3.5.4 Theorem(Fundamental theorem of integral calculus)

Any function f which is continuous on $[a,b]$ has a primitive on $[a,b]$.

If G is any primitive of f Then $\int_a^b f(x)dx = G(b) - G(a) \quad = [G(t)]_a^b$

Proof

Let F be defined on $[a,b]$ by $F(x) = \int_a^b f(t)dt \quad \forall \quad x \in [a,b]$,

then $\int_a^b f(t)dt = F(b) - F(a)$

$$= \{(G(b)+c) - (G(a)+c)\}$$

$$= G(b) - G(a) \ = [G(t)]_a^b$$

3.5.5 Theorem

Let f and g be continuous on $[a,b]$ and $\lambda, \mu \in R$,

Then $\int_a^b (\lambda f(x) + \mu g(x))dx = \lambda \int_a^b f(x)dx + \mu \int_a^b g(x)dx$

Proof

Let F and G be primitive of f and g on $[a,b]$,

then $h = \lambda F + \mu G,$ is a primitive of $\lambda f + \mu g$

and $\int_a^b \{\lambda f(t) + \mu g(t)\}dt = [\lambda F(t) + \mu G(t)]_a^b$ by $F.T.I.C$

$$= \lambda [F(t)]_a^b + \mu [G(t)]_a^b$$

$$= \lambda \int_a^b f(t)dt + \mu \int_a^b g(t)dt$$

3.5.6 Theorem(Integration by parts)

Suppose f and g are continuous on $[a,b]$ and have primitives F and G respectively on $[a,b]$

Then $\int_a^b f(t)G(t)dt = [F(t)G(t)]_a^b - \int_a^b F(t)dt$ where $F' = f(x)$ and $G' = g(x)$

Proof

$\Delta(FG) = G\Delta F + F\Delta G = Gf + Fg$

$\Rightarrow FG$ is a primitive of $fG + Fg$ on $[a,b]$,by previous theorem (fundamental theorem of integral calculus) $\Rightarrow \int_a^b (f(t)G(t) + F(t)g(t)dt = [F(t)G(t)]_a^b$

Distributing integration signs, we have

$$\int_a^b f(t)G(t)dt + \int_a^b F(t)g(t)dt = [F(t).G(t)]_a^b$$

$$\Rightarrow \int_a^b f(t)G(t)dt = [F(t)G(t)]_a^b - \int_a^b F(t)g(t)dt$$, hence integration by parts.

3.5.7 Theorem (Cauchy Criterion)

Let (f_n) be a sequence of functions defined on $S \subseteq R$

then their exist a function f ,such that f_n converges uniformly on S

iff the following is satisfied,

$\forall \varepsilon > 0 \quad \exists N$ such that $| f_n(x) - f(x) | < \varepsilon \quad \forall x \in s$ and $m,n > N$

3.5.8 Theorem (Cauchy –schwarz inequality)

Suppose f and g are continuous on $[a,b]$

then $\{\int_a^b f(t)g(t)dt\}^2 \leq \int_a^b \{f(t)\}^2 dt.\int_a^b \{g(t)\}^2 dt$

Proof,

For any $x \in [a,b]$, $0 \leq \int_a^b \{xf(t)+g(t)\}^2 dt = x^2\int_a^b \{f(t)\}^2 dt + 2x\int_a^b f(t).g(t)dt + \int_a^b \{g(t)\}^2 dt$

$$\equiv Ax^2 + Bx + C$$

i.e $Ax^2 + 2Bx + C = 0$, such a quadratic equation cannot have two different

Roots implies $\Rightarrow b^2 - 4ac \leq 0$ i. e $b^2 \leq 4ac$ Substituting $(2B)^2 \leq 4AC \Rightarrow B^2 \leq AC$

$$\Rightarrow \int_a^b \{f(t)g(t)dt\}^2 \leq \int_a^b \{f(t)\}^2 dt.\int_a^b \{g(t)\}^2 dt$$

3.5.9 Theorem ($M.V.T$ of Integral Calculus)

Let f be continuous on $[a,b]$,then $\exists$ $\xi \in (a,b)$ for which $\int_a^b f(x)dx = (b-a)f(\xi)$

where $f(\xi) = \dfrac{F(b)-F(a)}{b-a}$

Proof

Since f is continuous on $[a,b]$ then f is Riemann integrable $[m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)]$

thus $\exists$ μ between $\min$ and $\max$ such that $\int_a^b f(t)dt = \mu(b-a)$, but f is continuous

and takes all the values between $\min$ and $\max \Rightarrow \exists$ $\xi \in (a,b)$ such that $f(\xi) = \mu$

i.e $\int_a^b f(t)dt = f(\xi)(b-a)$

RIEMANN-STIELJES INTEGRAL

4.3.0 Review;

In Riemann integral $M_i = Sup\{f(x); x_{i-1} \le x \le x_i\}$ and $m_i = \inf\{f(x); x_{i-1} \le x \le x_i\}$, $\Delta x_i = x_i - x_{i-1}$

The upper and lower sums are defined by $U = \sum_{i=1}^{n} M_i \Delta x_i \equiv u(p,f)$ and $L = \sum_{i=1}^{n} m_i \Delta x_i \equiv L(p,f)$

And further $\bar{\int_a^b} f(x) = \inf \mu = \inf \mu(p,f)$ ...(i) $\underline{\int_a^b} f(x)dx = \sup L = \sup L(p,f)$ ......(ii)

Remark. Inf and Sup taken over all possible partition P of [a, b]. If (i) and (ii) are equal

i.e $u(p,f) = L(p,f)$ then f is said to be Riemann –Integrable on [a ,b].

4.3.1 Def (R.S integrals)

Let α be a real value on which f is monotonically $(\uparrow)$ on $[a,b]$,since $\alpha(a)$ and $\alpha(b)$ are finite .It follows that α is bounded on $[a,b]$,corresponding to each partition P of $[a,b]$

We write $\Delta\alpha = \alpha(x_i) - \alpha(x_{i-1})$ .Clearly , $\Delta\alpha \ge 0$,for any real valued function f which is bounded on $[a,b]$, We have $u(p,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$, $L(p,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$

We define $\bar{\int_a^b} f(x)d\alpha(x) = \bar{\int_a^b} fd(\alpha) = Inf(p,f,\alpha)$ and $\underline{\int_a^b} f(x)d\alpha(x) = \underline{\int_a^b} fd\alpha(x) = SupL(p,f,\alpha)$

If $\bar{\int_a^b} fd\alpha = \underline{\int_a} fd\alpha = \int_a^b fd\alpha$ ....(1)

Equation (1) is called the Riemann -Stieltjes integral of f with respect to α over $[a,b]$.

In this case f is said to be $R.S$ integral and is denoted by $f \in R(\alpha)$.

4.3.2 Remark

If $\alpha(x) = x$ then the $R.S$ integral reduces to Riemann integral

4.3.3 Theorem

If $P*$ is a refinement of P ,then $L(p, f, \alpha) \le L(P*, f, \alpha)..(i)$ $U(p^*, f, \alpha) \le U(p, f, \alpha)$ ...(ii)

Proof

To prove (i) ,suppose $P*$ contains only one point more than P and let $x*$ be the extra point

Such that $x_{i-1} < x^* < x_i$ where x_{i-1} and x_i are consecutive of P.

We put $W_1 = Inf\{f(x); x_{i-1} < x < x*\}$ and $W_2 = Inf\{f(x); x* < x < x_i\}$

Let $M_i = Inf\{f(x); x_{i-1} < x < x_i\}$,then clearly $w_1 \ge m_i$ and $w_2 \ge m_i$

And so $L(p*, f, x) - L(p, f, x) = w_1[\alpha(x*) - \alpha(x_{i-1})] + w_2[\alpha(x_i) - \alpha(x*)] - m_i[\alpha(x_i) - \alpha(x_{i-1})]$

$$= (w_1 - m_i)[\alpha(x*) - \alpha(x_{i-1})] + (w_2 - m_i)[\alpha(x_i) - \alpha(x*)] \ge 0$$

$$\Rightarrow L(p^*, f, \alpha) - L(p, f, \alpha) \ge 0 \quad \Rightarrow L(p, f, \alpha) \le L(p*, f, \alpha)$$

4.3.4 Corollary

$$\underline{\int_a^b} f(x)d\alpha(x) = \overline{\int_a^b} f(x)d\alpha(x)$$

Proof

Let $P*$ be the common refinement of two partition p_1 and p_2 $\Rightarrow P* = p_1 \cup p_2$ by theorem above $L(p_1, f, \alpha) \le L(p^*, f, \alpha) \le U(p*, f, \alpha) \le U(p_2, f, \alpha)$ Hence $L(p_1, f, \alpha) \le U(p_2, f, \alpha)$

and if p_2 is fixed and Sup taken over all possible partition p_1

$$Sup L(p, f, \alpha) = \underline{\int_a^b} f(x)dx \le U(p_2, f, \alpha)$$

Thus $\underline{\int}_{a}^{b} f(x)d\alpha(x)$ is a lower bound, taking $\inf imum$ over all possible partition p_2,

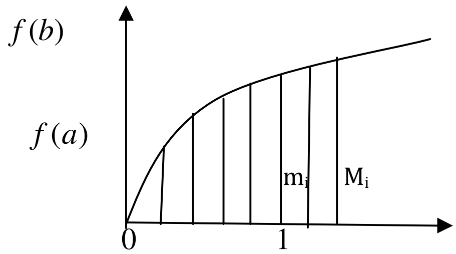
we obtain $\int_{a}^{b} f(x)d\alpha(x) \le InfU(p_2, f, \alpha)$

$$\underline{\int}_{a}^{b} fd\alpha \le InfU(p_2, f, \alpha) = \overline{\int}_{a}^{b} fd\alpha . \qquad \Rightarrow \underline{\int}_{a}^{b} fd\alpha \le \overline{\int}_{a}^{b} fd\alpha$$

4.3.5 Example

Let $\alpha(x) = x$ and define f on $[0,1]$ as $f(x) = \begin{cases} 1; if..rational \\ 0; if...irrational \end{cases}$

Show that $\int_{0}^{1} f(x)d\alpha(x) \le \int_{0}^{1} f(x)d(\alpha x)$



Solutions

For every partitions of $[0,1]$, $M_i = Sup\{f(x); x \in [0,1]\} = 1$ and $m_i = Inf\{f(x); x \in [0,1]\} = 0$

Since every sub-interval $[x_{i-1}, x_i]$ contains both rational and irrational and this holds to

each partitions hence $\forall P \quad u(p, f, \alpha) = u(p, f) = 1, \quad L(p, f, \alpha) = L(p, f) = 0$

Thus $\int_{0}^{1} f(x)d\alpha(x) \le \int_{0}^{1} f(x)d(\alpha x)$

Thus the $\int_{0}^{1} f(x)dx = \sup L(p, f) = 0$ and $\int_{0}^{1} f(x)dx = Inf(p, f) = 1$ .Then we compare the two

Since 0≠1 i.e. 0<1 and then $\int_{0}^{1} f(x)d\alpha(x) \le \int_{0}^{1} f(x)d(\alpha x)$

4.3.6 Theorem

$f \in R(\alpha)$ on $[a,b]$ if for every $\varepsilon > 0$ $\ni$ partition P s. t $U(p,f,\alpha) - L(p,f,\alpha) < \varepsilon$ ....*
(a criterion to show integral)

Proof

For every point P we have $L(p,f,\alpha) \le \underline{\int_a^b} fd\alpha \le \overline{\int_a^b} fd\alpha \le U(p,f,\alpha)$

$$\text{Thus } 0 \le \overline{\int_a^b} fd\alpha - \underline{\int_a^b} fd\alpha < \varepsilon$$

Since ε is arbitrary chosen

$\overline{\int_a^b} fd\alpha = \underline{\int_a^b} fd\alpha = \int fd\alpha$ i.e f is $R-S$ integral and $f \in R(\alpha)$

Conversely

Suppose $f \in R(\alpha)$ and let $\varepsilon > 0$, then there are partitions p_1 and p_2 of $[a,b]$

Such that, $u(p_2,f,\alpha) - \int_a^b fd\alpha < \frac{\varepsilon}{2}$ ......(i) and $\int_a^b fd\alpha - L(p_1,f,\alpha) < \frac{\varepsilon}{2}$ ....(ii)

Let P be common refinements of p_1 and p_2

$$\text{Then } U(p,f,\alpha) \le U(p_2,f,\alpha)\frac{\varepsilon}{2} + \int_a^b fd\alpha$$

Hence we have $u(p,f,\alpha) \le u(p_2,f,\alpha)\frac{\varepsilon}{2} < L(p_{1,}f,\alpha) + \varepsilon$

$$\Rightarrow u(p,f,\alpha) < \varepsilon + L(p_1,f,\alpha)$$

i.e. $u(p,f,\alpha) - L(p_1,f,\alpha) < \varepsilon$ where $f \in R(\alpha)$

4.3.7 Properties of R.S integration

(a)If $f_1 \in R(\alpha)$, $f_2 \in R(\alpha)$ on $[a,b]$ then $f_1 \pm f_2 \in R(\alpha)$

by linearity $c.f \in R(\alpha) \quad \forall c \in R$.

(b)If $f_1(x) \le f_2(x_o)$ then $\int_a^b f_1 d\alpha \le \int_a^b f_2 d\alpha$.

(d)If $f \in R(\alpha)$ on $[a,b]$, $f(x) \le M$, then $|\int_a^b fd\alpha| \le M[\alpha(b)-\alpha(a)]$

(e)Linearity, If $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$

Then $\int_a^b f(\alpha_1+\alpha_2) = \int_a^b fd\alpha_1 + \int_a^b fd\alpha_2$ And $f \in R(c\alpha) = c\int_a^b fd\alpha$

Proof (e)

If $f = f_1 + f_2$ and P is any partition of $[a,b]$

We have that $L(p,f_{1,}\alpha)+L(p,f_2,\alpha) \le L(p,f,\alpha) \le U(p,f,\alpha) \le U(p,f_{1,}\alpha)+U(p,f_2,\alpha)$.

If $f_1 \in R(\alpha)$ and $f_2 \in R(\alpha)$, let $\varepsilon > 0$ be given. There are partitions $p_j (j=1,2)$ such that $U(p_j,f_j,\alpha)-L(p_j,f_j,\alpha) < \varepsilon$ .These inequalities persists if p_1 and p_2 are replaced by their common refinement p .Thus $U(p,f,\alpha)-L(p,f,\alpha) < 2\varepsilon$ which proves that $f \in R(\alpha)$ and for this p we have $U(p,f_j,\alpha) < \int f_j d\alpha + \varepsilon \qquad (j=1,2)$

$\Rightarrow \int fd\alpha \le U(p,f,\alpha) < \int f_1 d\alpha + \int f_2 d\alpha + 2\varepsilon$,Since ε was arbitrary ,we have that

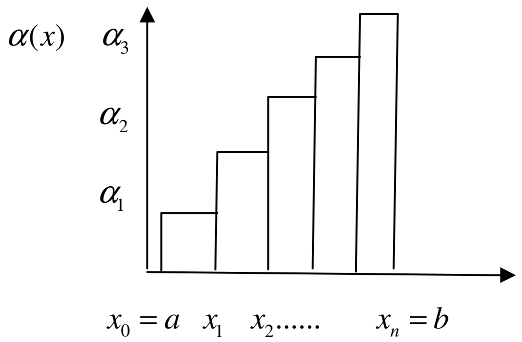
$\int fd\alpha \le \int f_1 d\alpha + \int f_2 d\alpha$ ....(a) If we replace f_1 and f_2 in (a) by $-f_1$ and $-f_2$,the inequality is reversed and equality is proved.

4.4.1 Definition; Unit Step function

A function α defined on $[a,b]$ is said to be a step function if $\exists$ a partition $P=\{x_0, x_1, ..., x_n\}$

With $a=x_0<x_1<...........<x_n=b$ such that α is constants on each interval.

The number $\alpha(x_k^+)-\alpha(x_k^-)$ is called the jump at x_k for $1<k<h$



4.4.2 Example

$I(x)=\begin{cases}0; x\le 0\\ 1; x>0\end{cases}$ and in general $I(x-\varepsilon)=\begin{cases}0; x\le \varepsilon\\ 1; x>\varepsilon\end{cases}$ the partition provides link

between R.S integral and finite series

4.4.3 Theorem

Let α be f_n on $[a,b]$ with $\alpha_k=\alpha(x_k^+)-\alpha(x_k^-)$ as in above.

Let f be defined such that both f and α are not discontinued from

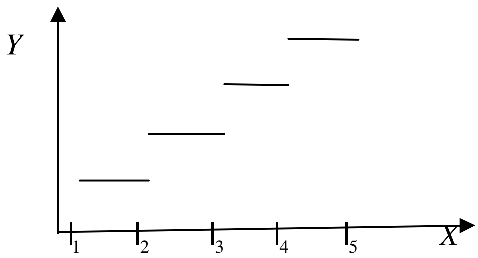
Right to left at each x_k then $\int_a^b f d\alpha$ exists

and $\int_a^b f(x) d\alpha(x)=\sum_{k=1}^{n} f(x_k)$.

4.4.4 Example (step function)

Let $[x]$ be the largest integer less than or equal to x,

referred to as greatest integer function,[x]≤x≤[x]+1 e.g. [π], [e]=2



Note $[\alpha]$ is continuous from the right with $\alpha_k = 1$. Thus If f is continuous on $[2,5]$ and

$\alpha(x) = [x]$ Then $\int_0^5 f(x)d\alpha(x) = \int_0^5 f(x)d[x]$ from theorem above

$$= \sum_{i=1}^{5} f(i) = 1+2+3+4+5 = 15$$

Now suppose f was x^2

$$\int_0^5 x^2 d[\alpha] = \sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

$$= 1+4+9+16+25 = 55$$

4.4.5 Example 2

$$\int_0^5 (x^2 d(x+[x]) = \int_0^5 x^2 dx + \int_0^5 x^2 d[x]$$

$$= \frac{x^2}{3} |_0^5 + \sum_{i=0}^{5} i^2$$

$$= \frac{125}{3} + 1+4+9+16+25 = 96\frac{2}{3}$$

4.5.0 Theorem (change of variable)

Suppose μ is a strictly increasing continuous function that maps an interval $[A,B]$ onto $[a,b]$

Suppose α is monotonically increasing on $[a,b]$ and $f \in R(\alpha)$ on $[a,b]$,

Define β and g on $[A,B]$ by $\beta(y)=\alpha(\mu(y))$ $g(y)=f(\mu(y))$ ………………..(I)

then $g \in R(\beta)$ and $\int_A^B g d\beta = \int_a^b f d\alpha$ ………….(II)

Proof

To each partition $P=\{x_0, x_1, ..., x_n\}$ of $[a,b]$ corresponds a partition $Q=\{y_0, y_1, ...y_n\}$ of $[A,B]$ such that $x_i=\varphi(y_i)$ and all partitions are obtained in this way .Since the values taken by f

on $[x_{i-1}, x_i]$ are exactly the same as those as those taken by g on $[y_{i-1}, y_i]$,we see that $U(Q,g,\beta)=U(p,f,\alpha)$, $L(Q,g,\beta)=L(P,f,\alpha)$ …….(III).Since $f \in R(\alpha)$,

can be chosen so that both $U(p,f,\alpha)$ and $L(p,f,\alpha)$ are close to $\int f d\alpha$ .and

$U(p,f,\alpha)-L(p,f,\alpha)<\varepsilon$, then $g \in R(\beta)$ and thus $\int_A^B g d\beta = \int_a^b f d\alpha$,if $\alpha(x)=x$ and

$\beta=\varphi$ and if $\varphi' \in R$ on $[A,B]$ then $\int_a^b f(x)dx = \int_A^B f(\varphi(y)\varphi'(y)dy$

4.5.1 Example

Evaluate $\int \sin^2 x \cos x dx$

solution

Let $u=\sin x$, then $\frac{du}{dx}=\cos x; du=\cos x dx$

Thus $\int \sin^2 x \cos x dx = \int u^2 d\mu = \frac{u^3}{3}+c = \frac{\sin^3 x}{3}+c$

4.6.0 Integration Of Vector-Valued Functions

Let $f_{1,} f_{2,}........f_k$ be real functions on [a, b] and $f = (f_1,........, f_k)$ be the corresponding mapping of [a, b] into R^k .If α increases monotonically on [a, b],to say $f \in R(\alpha)$,for $j = 1,......, k$.

in this case $\int_a^b fd\alpha = (\int_a^b f_1 d\alpha,........\int_a^b f_k d\alpha)$ i. e $\int fd\alpha$ is the point

in R^k whose j^{th} co-ordinates is $\int f_j d\alpha$

4.6.1 Theorem

If f maps $[a,b]$ into R^k and $f \in R(\alpha)$ for some monotonically increasing α on $[a,b]$

Then $|f| \in R(\alpha)$ and $|\int_a^b fd\alpha| \leq \int_a^b |f| d\alpha....(a)$

Proof

If $f_1.........f_k$ are components of f ,then $|f| = (f_1^2 + ... + f_n^2)^{\frac{1}{2}}$,each of $f_i^2 \in R(\alpha)$ and hence does their sum. Since square root function is continuous on [0,M] for every real M, $|f| \in R(\alpha)$,

To prove (a)Let $y = (y_1,......y_n)$ where $y_j = \int f_j d\alpha$ then we have that $y = \int fd\alpha$

$$\Rightarrow |y|^2 = \sum y^2 = \sum y_j \int f_j d\alpha = \int (\sum y_j f_j), \quad \text{by the Schwarz inequality}$$

$\sum y_j f_j(t) \leq |y||f(t)| \qquad (a \leq t \leq b)$ hence $|y^2| \leq |y| \int |f| d\alpha....(b)$

If $y = 0$ a is trivial, If $y \neq 0$, division of (b) by $|y|$ gives (a).

4.6.2 Example

If $A = (3x^2 + 6y)i - 14yzj + 20xz^2k$

Evaluate $\int_c A.dr$ from (0,0,0) to (1,1,1) along the following paths C

where $x = t \quad , y = t^2 \quad , z = t^3$

Solution

Points (0,0,0) and (1,1,1) corresponds to $t = 0$ and $t = 1$ respectively

$$dx = dt \quad , dy = 2dt \quad , dz = 3t^2 dt$$

$$\int_c A.dr = \int_{t=o}^{t=1} (3t^2 + 6t^2)dt \ -14(t^2)(t^3)2dt \ +20(t)(t^3)^2 3t^2 dt$$

$$= \int_0^1 9t^2 dt - 28t^2 dt + 60t^9 dt$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)dt \ = 3t^3 - 4t^7 + 6t^{10} |_0^1 = 5$$

4.6.3 Example2

Compute the length of the arc $x = (e^t \cos t)i + (e^t \sin t)j + e^t k \quad -\infty < t < \infty$

$$S = \int_0^t | \frac{dx}{dt} | dt = \int_0^t | e^t \cos t - e^t \sin t)i + (e^t \sin t + e^t \cos t)j + e^t k | dt$$

$$= \int_0^t [e^{2t}(-2\cos t \sin t) + e^{2t}(2\cos t \sin t + 1) + e^{2t}]^{\frac{1}{2}} dt$$

$$= \sqrt{3}\int_0^t e^t dt = \sqrt{3}(e^t - 1)$$

4.7.0 Rectifiable Curves

4.7.1 Definition ;For each curve γ in R^k there is associated a subset of R^k ,

i.e. the range of γ ,but different curves may have the same range.

We associate to each partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$ and to each Curve γ on $[a,b]$

the number $\wedge(P,\gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$ the i^{th} term in this sum is the distance (in R^k)

between the points $\gamma(x_{i-1})$ and $\gamma(x_i)$.

Hence $\wedge(p, y)$ is the length of a polygonal path with vertices at $\gamma(x_0), \gamma(x_1), ...\gamma(x_n)$

in this order. As our partitions becomes finer and finer this polygon approaches the range of γ more and more closely and is reasonable to define the length of γ as $\wedge(\gamma) = \sup \wedge (p,\gamma)$,

where the supre mum is taken over all partitions of $[a,b]$.

If $\wedge(\gamma) < \infty$,we say that γ is rectifiable.

In certain cases, $\wedge(\gamma)$ is given by a Riemann integral, this can be proved for

curves γ whose derivatives γ' is continuous.

4.7.2 Theorem

If γ' is continuous on $[a,b]$, then γ is rectifiable and $\wedge(\gamma) = \int_a^b |\gamma'(t)|\, dt$

Proof

(i)If $a \le x_{i-1} \le x_i \le b$ then $|\gamma(x_i) - \gamma(x_{i-1})| = |\int_{x_{i-1}}^{x_1} \gamma'(t)dt| \le \int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt$

Hence $\wedge(p,\gamma) \le \int_a^b |\gamma'(t)|\, dt$ for every partition P of $[a,b]$ thus $\wedge(\gamma) \le \int_a^b |\gamma'(t)|\, dt$ ....(i) (ii)To prove the reverse inequality let $\varepsilon > 0$ be given, Since γ' is uniformly continuous on $[a,b]$,

there exist $\delta > 0$ such that $|\gamma'(s) - \gamma(t)| < \varepsilon$ if $|s-t| < \delta$.

Let $P = \{x_0, x_1, ...x_n\}$ be a partition of $[a,b]$,with $\Delta x_i < \delta$ for all i,

If $x_{i-1} \le t \le x_i$ it now follows that $|\gamma'(t)| \le |\gamma'(x_i) + \varepsilon$

hence $\int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt \le |\gamma'(x_i)| \Delta x_i + \varepsilon \Delta x_i$

$$= \left| \int_{x_{i-1}}^{x_i} [\gamma'(t) + \gamma'(x_i) - \gamma'(t)] dt \right| + \varepsilon \Delta x_i$$

$$\le \left| \int_{x_{i-1}}^{x_i} \gamma'(t) dt \right| + \left| \int_{x_{i-1}}^{x_i} [\gamma'(x_i) - \gamma'(t)] dt \right| + \varepsilon \Delta x_i$$

$$\le |\gamma(x_i) - \gamma(x_{i-1})| + 2\varepsilon \Delta x_i$$

If we add these inequalities, we obtained

$$\int_a^b |\gamma'(t)|\, dt \le \wedge(p, y) + 2\varepsilon(b-a)$$

$\le \wedge(\gamma) + 2\varepsilon(b-a)$ and since ε was arbitrary

Thus $\int_a^b |\gamma'(t)|\, dt \le \wedge(\gamma)............(ii)$ From (i) and (ii) we have $\wedge(\gamma) = \int_a^b |\gamma'(t)|\, dt$

4.7.3 Example 1

If $x = f(t), a \le t \le b$ is a rectifiable arc, show that given an arbitrary $\delta > 0$ and $\varepsilon > 0$,

there exist a subdivision $a = t_o < t_1 < .....t_n = b$ with polygonal approximations P such

that (i) $t_i - t_{i-1} = 1,.....,n$ (ii) $|s - s(p)| < \varepsilon$,where s and $s(P)$ are the lengths

of $x = f(t)$ and P respectively.

Since s is the supremum of all possible $s(P)$, there exists subdivisions $a = t_o' < t_1' < .... < t_n' = b$

with polygonal approximations P' such that $s(P') > s-\varepsilon$.For otherwise, $s(P) \le s-\varepsilon$ for all $s(P)$, so that $s-\varepsilon$ is an upper bound of the $s(P)$ less than the supre mum s ,not impossible.

Now the above subdivision does not satisfy (i),a finer subdivision $a = t_o < t_1 < .... < t_n = b$ satisfying $(t_i - t_{i-1}) < \delta$ can be obtained by introducing additional points. But the new polygonal arc P' obtained this way satisfies $s(P) \le s(P') \le \varepsilon$ and therefore also $|s - s(P)| < \varepsilon$

4.7.4 Example 2

Show that a regular arc $x = f(t),\ a \le t \le b,$ is rectifiable .

let $a = t_o < t_1 < .... < t_n = b$ be arbitrary subdivision,

Then $s(P) = \sum_i |x_i - x_{i-1}| = \sum_i |f(t_i) - f(t_{i-1})|$

$$= \sum_n |(f_1(t_n) - f(t_{n-1}))i + (f_2(t_n) - f_2(t_{n-1}))j + (f_3(t_n) - (t_{n-1}))k|$$

$$\le \sum_n [|f_1(t_n) - f_1(t_{n-1})| + |f_2(t_n) - f_2(t_{n-1})| + |f_3(t_n) - f_3(t_{n-1})|]$$

$$\le \sum_n [|f_1^{'}(\theta_n)|(t_n - t_{n-1}) + |(f_2^{'}(\theta_n^{'})|(t_n - t_{n-1}) + |f_3^{'}(\theta_i^{"})|(t_i - t_{i-1})]$$

where we used the mean value theorem for the $f_i(t)$,and since $f_i^{'}(t)$ are continuous on closed interval $a \le t \le b$,they are bounded on $a \le t \le b$, say by M_n .Hence

$$s(P) \le (M_1 + M_2 + M_3)\sum_n (t_n - t_{n-1}) \le (M_1 + M_2 + M_3)(b-a)$$

Thus the $s(P)$ are all bounded by $(M_1 + M_2 + M_3)(b-a)$ and so the arc is rectifiable M_n .

CHAPTER FIVE

The LEBESGUE INTEGRATION

5.0 Introduction

5.1 Interval of a real line

Let I be an interval of real line and points (a,b), where $a<b$ i.e I is either of the following types $(a,b),[a,b],(a,b],[a,b)$ .Then the real number $b-a$ is called the length of either of these interval, we denote it by $\lambda(I)$, In this case I is bounded and is of the form $[a,b]$ .And the length taken as $+\infty$.

Remark

If $a=b$, then the length $\lambda(I)=0$, thus the void set $\varnothing$ has a length i.e $\mu(\varnothing)=0$.

5.2 .0 The Lebesgue Measure

5.2.1 Theorem

Consider R with the metric (Euclidean) then any open subsets E of the real line can be expressed as the union of at most countable family of mutually disjoint sub-interval of R.

Proof

Let A be any subsets of the real line R' then there is at least one open subset of R which Contains A (for instance R contains A),Let this open subset be expressed as a union of at most countable family of open sub-interval of R. Hence any subset A of R can be covered by at most countable family of open intervals denoted by $S(A)$ I. e the class of all such at most countable covers of A.

If γ is at most countable collection of open sub-interval's of R and thus $\gamma=(I_n)_1^{\infty}$, where each (I_n) is an open interval and $\bigcup_{n=1}^{\infty} I_n = S(A), \forall \gamma \in S(A)$

5.2.2 The Outer Lebesgue Measure

Let γ represent any at most countable collection of open sub-intervals of R'

We put $\gamma = \{I_n; n \in N\}$,each of I_n is an open sub-Interval of R' .such

that the non-negative extended real number $\lambda^*(\gamma) = \sum \lambda(I)$ i. e $\lambda^*(\gamma)$ -represent's sum

of the length's of all sub-interval in the collection γ . Let E be any subsets of R and let γ

be any at most countable collection of open sub-interval's that covers E which implies that

$\gamma \in (S(E))$. The extended real number $\inf\{\lambda^*(\gamma); \gamma \in (S(E))\}$ is called the outer lebesgue

measure of E denoted by $m^*(E)$.

Equivalently

Let $\gamma \in (S(E))$,at most countable sub-interval that covers E i. e $\gamma = (I_n)_{n=1}^{\infty}$, then the extended

real number $\lambda^*(\gamma) = \sum \lambda(I_n)$ i. e $\gamma \in S(E)$ is a set of real numbers $\lambda^*(\gamma_1), \lambda^*(\gamma_2)....$

Then we proceed to take the infimum, $\inf\{\lambda^*(\gamma); \gamma \in S(E)\}$

and $m^*(E) = \inf\{\lambda^*(\gamma); \gamma \in S(E)\}$,

Hence for each subset E of R' there corresponds a unique non-negative extended

number $m^*(E) \geq 0$ and it's infimum is such that $m^*; P(R') \rightarrow R_T^*$

extended real number is called the outer Lebesgue measure.

5.2.3 Remark ; Lebesgue measure is complete .For if E∈M and M(E)=0 and A⊆E

then A∈M and M*(A)=0

Proof ; Let M*(E)=0, and A⊆E, then by motone property M*(A) ≤M*(E)=0

$\Rightarrow o \leq M^*(A) \leq 0 ...thus..M^*(A)$

5.2.4 Theorem

Let m^* denote the outer lebesgue measure on R'

Then (i) $m^*(\phi)=0$

(ii) $m^*(E)\geq 0$,whenever $E\in F$ (non-negative)

(iii) If $A,B\in P(R)$ and $A\subset B$ then $m^*(A)\leq m^*(B)$

{monotone property of M*}

Proof

(i)We choose $\gamma=\phi \quad \Rightarrow \gamma\in (S(\phi))$ then $\lambda^*(\gamma)=0 \quad \forall\gamma\in (S(\phi))$

Now $m^*(\phi)=\inf\{\lambda^*(\gamma);S(\phi)\}=0$

(ii)Let $x\in R'$ consider $E=\{x\}$ then $\gamma=\{\frac{x-\varepsilon}{2},\frac{x+\varepsilon}{2}\}$ covers $\{x\}$ also

$\lambda^*(\gamma)=\sum\lambda(I_n)=(\frac{x+\varepsilon}{2}-\frac{x-\varepsilon}{2})$,The measure $m^*(\{x\})\leq\lambda^*(\gamma)=\varepsilon$

Implying the measure of infimum is positive i.e. $0\leq m^*(\{x\}\leq\lambda^*(\gamma)=\varepsilon$,

and $m^*(\{x\})=0$ if $\gamma=\varnothing$

(iii)Since $A\subseteq B$, $S(A)\subseteq S(B)$

Indeed if implying $\gamma\in S(B)$,

Then $\{\lambda^*(\gamma),\gamma\in S(A)\}\subseteq\{\lambda^*(\gamma);\gamma\in S(B)\}$

and hence $m^*(A)\leq m^*(B)$

5.2.5 Theorem

M^* is count ably sub-additive i. e if $(E_n)_{n-1}^{\infty}$ is a sequence of subsets of R'

then $m^*(\bigcup_{n=1}^{\infty}) \le \sum m^*(E_n)\ldots\ldots\ldots(i)$

Proof ;Suppose $m^*(E_{n_o}) = +\infty$ for some $n_o \in N$,then the right hand side of (i)

diverges, however since $E_{n_o} \subseteq \bigcup_{n=1}^{\infty} E_n$ introducing the measure $m^*(E_{n_o}) \le m^*(\bigcup_{n=1}^{\infty} E_n)$

thus $+\infty \le m^*(\bigcup E_n)$ hence(i) holds true for $m^*(E_{n_o}) = +\infty$

Assume $m^*(E_n) \le \infty$ by definition of m^* it follows that for each

$\varepsilon > 0 \quad \exists \gamma_n \in S(E)$ such that $\lambda^*(\gamma_n) \le m^*(E_n) + \frac{\varepsilon}{2^n}, n = 1, \ldots$

Let $\gamma = \bigcup_{n=1}^{\infty} \gamma_n$ then γ is atmost countable collection of open interval which covers $\bigcup_{n=1}^{\infty} E_n$

$\gamma \in S(\bigcup_{n=1}^{\infty} E_n)$ The measure of the union $m^*(\bigcup_{n=1}^{\infty} E_n) \le \lambda^*(\bigcup_{n=1}^{\infty} \gamma_n) = \lambda^*(\gamma)$

$$m^*(\bigcup_{n=1}^{\infty} E) \le \sum_{n=1}^{\infty} \lambda^*(\gamma_n) < \sum_{n=1}^{\infty} (m^*(E_n) + \frac{\varepsilon}{2^n}) = \sum_{n=1}^{\infty} m^*(E_n) + \varepsilon$$

$$m^*(\bigcup_{n=1}^{\infty} E_n) \le \sum_{n=1}^{\infty} m^*(E_n)$$

5.2.6 Thm; If $E \notin \mathrm{M}$ then there is a subset A of E with finite positive measure $(0 < m^*(A) < \infty)$

Proof

Since the measure $E \notin \mathrm{M}$ by definition $\exists x \subseteq R'$ such that $m^*(x) < m^*(x \cap E) + m^*(x \cap E^c)$

Suppose $m^*(x \cap E) = +\infty$,Since $x \supseteq x \cap E$ by monotone property $m^*(x) \geq m^*(x \cap E) = +\infty$

Thus $m^*(x) = +\infty$ and hence $m^*(x \cap E) < \infty$

Next suppose $m^*(x \cap E) = 0$ Thus $m^*(x) < m^*(x \cap E^c)$,

This is a contradiction since $x \supseteq x \cap E^c$ hence $m^*(x) \supseteq m^*(x \cap E^c)$ $\Rightarrow m^*(x \cap E) > 0$

i.e $0 < m^*(x \cap E) < \infty$ Putting $x \cap E = A$ we have $0 < m^*(A) < \infty$ where $A \subset E$

5.2.7 Theorem

If $A, B \in \mathrm{M}$,then so is $A \cup B$,Any finite union is measurable or M is closed under the union operation

Proof

Let $A \in \mathrm{M}$ by definition ,it follows that any $X \subseteq R'$ i. e $m^*(x) = m^*(x \cap A) + m^*(x \cap A^c) \ldots.(i)$

Similarly $B \in \mathrm{M} \Rightarrow \exists Y \subseteq R$ such that $m^*(Y) = m^*(Y \cap B) + m^*(Y \cap B^c) \ldots.(ii)$

In particular $Y = X \cap A^c$ .... (iii) ,using (iii) and (ii) we have

that $m^*(x \cap A^c) = m^*(x \cap A^c \cap B)$ $+ m^*(x \cap A^c \cap B^c) \ldots..(iv)$

Substituting (iv) and (i) gives $m^*(x) = m^*(x \cap A)$ $+ m^*(x \cap A^c \cap B) + m^*(x \cap A^c \cap B^c)$

or $m^*(x) = m^*(x \cap (A \cup B))$ $+ (A \cup B)^c$

Hence by finite sub-additivity of m*, $m^*(x \cap (A \cup B) \leq m^*(x \cap A) + m^*(x \cap (A^c \cap B^c))$

$\Rightarrow m^*(x) \geq m^*(x \cap (A \cup B))$ $+ m^*(x \cap (A \cap B)^c) \Rightarrow \exists x \subseteq R$ such that
$m^*(x) \geq m^*(x \cap (A \cup B)) + m^*(x \cap (A \cup B)^c)$ and from definition we have $A \cup B \in \mathrm{M}$

5.2.8 Theorem

If A and B are both L-measurable then $A \cap B \in \mathrm{M}$

Proof

$A, B \in \mathrm{M}$ from definition, $\Rightarrow A^c \in \mathrm{M}, B^c \in \mathrm{M}$

$$\Rightarrow A^c \cup B^c \in \mathrm{M}$$

$$\Rightarrow (A \cap B)^c \in \mathrm{M}$$

$$\Rightarrow A \cap B \in \mathrm{M}$$

5.2.9 Definition ($\Omega -$ Algebra or $\Omega -$ Field)

Let X be a non-void set and $\mathcal{F}$ be a class of subsets of X satisfying

the following (1) $\phi \in \mathcal{F}$

(2) If $E \in \mathcal{F}$ then $E^c \in \mathcal{F}$

(3) If $(E_n)_{n-1}^{\infty}$ is a sequence of members of $\mathcal{F}$ then $\bigcup_{n=1}^{\infty} E_n \in \bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$

Then $\mathcal{F}$ is called a $\Omega -$ algebra of subsets of X

5.2.10 Theorem (Disjoint Lemma)

Let X be a non-void set and Ω be an algebra of X

If $(E_n)_{n=1}^{\infty}$ is any sequence of sets in Ω such that

(i) $D_n \subseteq E_n$

(ii) $D_m \cap D_n = \varnothing$ whenever $m \neq n$ where $(D_n)_{n=1}^{\infty}$ is pair wise disjoint

(iii) $\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} E_n$,Then x belongs to at least one of the E_n's

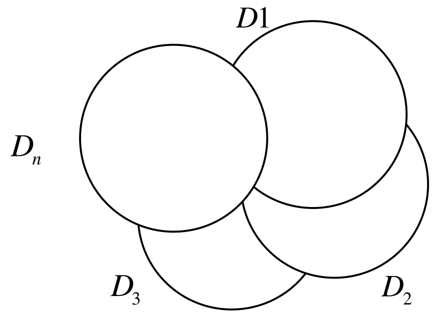
Proof

(i) $D_n \subseteq E_n \quad \forall n \in N$ since $E_n \in \Omega$ is an algebra

and D_n's are obtained from E_n's .Using operations of union of sets

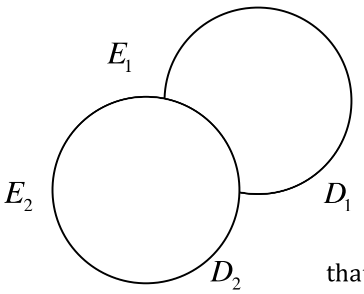
on finite number of sets i. e $D = E$ and ($E_1 \cup E_2 \cup .....E_n$) $n > 1$ and clearly that $D_n \subseteq E_n$

(ii) $D_m \cap D_n = \varnothing$, whenever $(D_n)_{n=1}^{\infty}$ is pair wise disjoint



From construction of D_n's it follows that $D_m \cap D_n = \varnothing$ for $n \neq m$

(iii) $\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} E_n$



From construction of D_n's it follows that $D_n \cap D_m = \varnothing$ for $n \neq m$ thus $D_n \subseteq E_n$

$\Rightarrow \bigcup_{n=1}^{\infty} D_n \subseteq \bigcup_{n=1}^{\infty} E_n$,the reverse inequality is clear from (i) and $\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} E_n$

Since $X \in \bigcup_{n=1}^{\infty} E_{n,}$ then x belongs to at least one of the E_n's

5.3.0 The Lebesgue Integral For Non-negative Simple Functions

5.3.1 Definition , Indicator or Characteristic Functions

Let (Ω, F) be a measurable space for a set $A \subseteq \Omega$ define

$X_A \to \{0,1\}$ by $\chi_A(x) = \begin{cases} 0; x \in A \\ 1; x \notin A \end{cases}$ this function is called the characteristic

or the indicator function of a set. If $f = I_A$ where i. e $I_A; \Omega \to R_e$

and $I_A(x) = \begin{cases} 1; x \in A \\ 0; x \notin A \end{cases}$ and $\int \chi_A(x) d\mu = 1.\mu(A) + 0.\mu(A^c)$

5.3.2 Defination ; Simple Functions

Suppose the range of S consists of the distinct numbers $a_1, a_2 .... a_n$

define simple non-negative function $S; \Omega \to R_e$ by $S(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$ where $a_i \geq 0, \forall A_i \in F$

and $\bigcup_{i=1}^{\infty} A_i \in \Omega$, with $A_i \cap A_j = 0 \quad i \neq j$.

5.3.3 Example

Consider $([0,1], \mathrm{M}, \mu)$,define $f(x) = \begin{cases} 1; if..x..is..rational \\ 0; if..x..is..irrational \end{cases}$.

This is a simple function with $A_1 = Q \cap [0,1]$ and $A_2 = A_1^c = Q^c \cap [0,1]$

Note that $f \in \mathrm{M}$ and $\int_{[0,1]} f d\mu = 1.\mu(Q \cap [0,1]) + 0.\mu(Q^c \cap [0,1]) = 0$

since rational s are countable then $\mu(Q \cap [0,1]) = 0$

5.4.0 Lebesgue Integration

5.4.1 Lebesgue Integral Of Non-negative Simple Functions

Integration is defined on a measure X in which F is the $\Omega - ring$ of measurable sets and μ

is the measure on it. Suppose $S(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$ where $\forall A_i \in F$, $\bigcup_{i=1}^{\infty} A_i = \Omega$ and

$a_i \geq 0 \in R$ is measurable and if S is measurable space (Ω, F, μ) and non-negative,

we define $\int S(x) d\mu = \sum_{i=1}^{n} a_i \mu(A_i) = \int S d\mu$ or $\int_E S d\mu = Sup I_E(s)$ ....(a)

The left side of (a) is the lebesgue integral of S ,with respect to μ over the set E

5.4.2 Properties Of The Integral

1. The integral is a non-negative extended real number $0 \leq \int S d\mu \leq +\infty$

2. If $s, s_1, s_2 \in L_0^+$ and $\alpha \in R_e$ such that $\alpha \geq 0$, the

(a) $\alpha s \in L_0^+$ and $\int (\alpha s) d\mu = \alpha \int s d\mu$

(b) $s_1 + s_2 \in L_0^+$ then $\int (s_1 + s_2) d\mu = \int s_1 d\mu + \int s_2 d\mu$

(c)If $s_1 \leq s_2$ then $\int s_1 d\mu \leq \int s_2 d\mu$

(d)If $\{s_n, n \geq 1\}$ is an increasing sequence functions in L_0^+ such that $\lim_{n\to\infty} S_n(x) = s(x)$

$\forall x \in R$ then $\int s(x) d\mu(x) = \lim_{n\to\infty} \int s_n(x) d\mu(x)$

5.5.0 The Integral Of a Non-Negative Measurable Functions

5.5.1 Definition

Let (Ω, F) be a measurable space ,the non-negative functions $f;\Omega \to R_e$ is said to be

$F-$measurable, If $\exists$ an increasing sequence $\{S_n; n \geq 1\}$ such that $\lim_{n\to\infty} S_n(x) = f(x)$

$\forall x \in \Omega$, we shall denote the class of all non-negative measurable function by L^+.

5.5.2 Theorem

(a)Suppose f is measurable and nonnegative on X.For $A \in \mathrm{M}$, define

$$\phi(A) = \int_A f d\mu \quad \text{, then } \phi \text{ is count ably additive on } \mathrm{M}$$

(b)The same conclusion holds if $f \in L(\mu)$ on X

Proof

To show $\phi(A) = \sum_{n=1}^{\infty} \phi(A_n)$, In general case, we have ,for every measurable simple

functions S such that $0 \leq s \leq f$ $\quad ,\int_A s d\mu = \sum_{n=1}^{\infty} \int_{A_n} s d\mu \leq \sum_{n=1}^{\infty} \phi(A_n) \quad \therefore \phi(A) \leq \sum_{n=1}^{\infty} \phi(A_n)$

Now if $\phi(A_n) = +\infty$ for some n, is trivial ,since $\phi(A) \geq \phi(A_n)$

suppose $\phi(A_n) < \infty$ for every n,such that $\phi(A_1 \cup A_2) \neq\geq \phi(A_1) + \phi(A_2)$

it now follows that for every n $\quad \phi(A \cup ...... \cup A_n) \geq \phi(A_1) + ... + \phi(A_n)$ since

$$A \supset A_1 \cup ...... \cup A_n \Rightarrow \phi(A) \geq \sum_{n=1}^{\infty} \phi(A_n)$$

5.5.3 Definition ;For a function $f \in L^+$, we define the integral of f with respect t o μ

by $\int f(x) d\mu(x) = \lim_{n\to\infty} \int S_n(x) d\mu x$

5.5.4 Properties Of the Integrals

Let f_1, f_2, f_3 then the following holds

1. $\int f d\mu \geq 0$ and for $f_1 \geq f_2 \quad \Rightarrow \int f_1 d\mu \geq \int f_2 d\mu$

2.For $\alpha, \beta \geq 0$, we have $\alpha f_1 + \alpha f_2 \in L^+$

and $\int (\alpha f_1 + \beta f_2) d\mu = \int \alpha f_1 d\mu + \int \beta f_2 d\mu \ = \alpha \int f_1 d\mu + \beta \int f_2 d\mu$

3.For every $E \in F$, we have $\chi_E f \in L^+$ and if $\upsilon\ (E) = \int \chi_E f d\mu$ is a measure on F

And $\upsilon(E) = 0$ iff $\mu(E) = 0$, the integral $\int \chi_E f d\mu = \int_E f d\mu$.

5.6.0 Monotone Convergence Theorem($M.C.T$ theorem)

Let $(X, \aleph, \mu)$ be a measure space , (f_n) be a sequence on $M^*(X, \aleph)$ s. t $f_n \leq f_{n+1} \quad \forall n \in N$

and $f_n \to f$ point wise on X , then $\int (f_n d\mu)_{n=1}^{\infty}$ converges to $\int f d\mu$ in R_e i. e

$$\lim_{n \to \infty} \int f_n d\mu = \int (\lim_{n \to \infty} f_n) d\mu = \int f d\mu$$

Proof

$f_n \in m^*(x,) \quad \forall n \in R$ and $f_n \to f$ point wise on $X \Rightarrow f \in m^*(X, \chi)$

since $f_n \leq f_{n+1} \leq f$ by monotone properties of S , we have that $\int f_n d\mu \leq \int f_{n+1} d\mu \leq \int f d\mu ...(i)$

Thus the sequence $(\int f_n d\mu)_{n=1}^{\infty}$ is increasing in R_e^* and hence

Conversely, If $A_n = \{x \in X; \alpha\phi x \le f_n(x)\}$ it can be shown that $A_n \in \aleph \quad \forall n \in N$

Moreover (i) $A_n \subseteq A_{n+1}$ (ii) $\bigcup_{n=1}^{\infty} A_n = X$

Since integral is a count ably additive set function $\alpha\phi(x) \le f_n(x)$ on $x \in A_n$,

by monotone property of $\int$ on m*(x, ℵ) , $\int \alpha\phi d\mu \le \int f_n d\mu$

i.e $\alpha\int \phi d\mu \le \int_{A_n} f_n d\mu \le \int_x f_n d\mu \le \int f d\mu$ …………(ii)

the two inequalities proof the theorem.

Remark; If we define $\lambda; \aleph \to R_e$ by $\lambda(E) = \int_E \phi d\mu \quad \forall E \in \aleph$

The $\lambda(E)$ is a measure and therefore λ is continuous from below.

Proof

$$\phi \in L_0^+ \Rightarrow \phi = \sum_{i=1}^{n} a_i \chi_{A_i} , \quad \bigcup_{i=1}^{n} A_i = \Omega$$

$$E \in F \Rightarrow \phi\chi_E = \sum_{i=1}^{n} a_i \chi_{A_i} \chi_E = \sum_{i=1}^{n} a_i \chi_{A_i \cap E}$$

where $\lambda(E) = \int \phi\chi_E d\mu = \sum_{i=1}^{n} a_i \mu(A_i \cap E)$ is it a measure or not

(i) $\lambda(\phi) = \sum_{i=1}^{n} a_i \mu(A \cap \varnothing) = \sum_{i=1}^{n} a_i \mu(\varnothing) = 0$

(ii)Since $a_i \ge 0$ and $\mu(A_i \cap E) \ge 0 \Rightarrow \sum_{i=1}^{n} a_i \mu(A_i \cap E) \ge 0$

(iii) λ is countable additive ,for let $E=\bigcup_{j=1}^{\infty}E_j ; E_j \in F$ for each j

then to show that $\lambda(E)=\sum_{j=1}^{\infty}\lambda(E_j)$ $\lambda(E)=\sum_{i=1}^{n}a_i\mu(A_i \cap E_j)=\sum_{j=1}^{n}a_n\mu(A_i \cap \bigcup_{j=1}^{\infty}E_j)$

$$=\sum_{i=1}^{n}a_i\mu(\bigcup_{j=1}^{\infty}(A_i \cap E_j))=\sum_{i=1}^{n}a_i\sum_{j=1}^{\infty}\mu(A_i \cap E_j)$$

$$=\sum_{j=1}^{\infty}\sum a_i\mu(A_i \cap E_j)=\sum_{j=1}^{\infty}\lambda(E_j) \quad \therefore \lambda(E) \text{ is a measure.}$$

5.6.1 Some Applications Of M.C.T.

Theorem; Let $(X,\aleph,\mu)$ be a measure space and $m^*(X,\aleph)$ and C non-negative real ,then (i) $\int cfd\mu = c\int fd\mu$ (ii) $\int (f+g)d\mu = \int fd\mu + \int gd\mu$

Proof

Let $(\phi_n),(\psi_n)$ be increasing $(\uparrow)$ sequence of simple $f_n(s)\in M^*(X,\aleph)$ such that $(\phi_n) increases(\uparrow)$ to f and $(\psi_n) increases(\uparrow) to\ g$.

$\Rightarrow c\phi_n$ is increasing sequence by M.C.T , $\lim_{n\to\infty}\int \phi_n d_n = \int (\lim_{n\to\infty}\phi_n)d_n = \int fd\mu$

$\lim_{n\to\infty}\int c\phi_n dn = \int cfd\mu$ ....* But $\int c\phi_n d\mu = c\int fd\mu$

$$\therefore \lim_{n\to\infty}\int c\phi_n d\mu = \lim_{n\to\infty}c\int \phi_n d\mu = c\lim_{n\to\infty}\int \phi_n d\mu = c\int fd\mu ......**$$

Thus from * and ** we have $\int cfd\mu = c\int fd\mu$.

(ii) by M.C.T $\lim_{n\to\infty}\int \psi_n d\mu = \int (\lim_{n\to\infty}\psi_n) d\mu = \int g d\mu$

Now $(\phi_n + \psi_n)$ increases $(\uparrow)$ to $f + g$ by M.C.T

$\lim_{n\to\infty}\int (\phi_n + \psi_n) d\mu = \int (f+g) d\mu .....*$

Since ϕ_n and ψ_n are simple $f_n 's \in M^*(X, \aleph)$

$\int (\phi_n + \psi_n) d\mu = \int \phi_n d\mu + \int \psi_n d\mu$

Thus $\lim_{n\to\infty}\int (\phi_n + \psi_n) d\mu = \int f d\mu + \int g d\mu ......**$

From * and ** we have $\int (f+g) d\mu = \int f d\mu + \int g d\mu$

5.6.2 Example

Let $(R, B(R), \mu)$ be a measurable space , where μ is the lebesgue measure on $B(R)$

Let $f_n = \chi_{(0,n)}$ $\forall n \in N$,where f_n is monotonic increasing to $f \in \chi_{[0,+\infty]}$

and f_n and f are $B(R)$ measurable functions

$$\int f_n d\mu = \int \chi_{[0,n]} d\mu = \mu[0,n] = n$$

and $\int f d\mu = \int \chi_{[0,n]} d\mu = \mu([0,+\infty]) = \infty$

Now $\int f d\mu = + \ \infty = \lim_{n\to\infty} n = +\infty$ and $\therefore$ M.C.T applies.

5.7.0 Fatou's Lemma

Let $(X, \aleph, \mu)$ be a measure space ,

and (f_n) be a sequence of elements of $M^*(x, \aleph)$,

Then $\int(\lim_{n\to\infty} f_n)d\mu \leq \lim_{n\to\infty}\int f_n d\mu$

Proof

For each $n \in N$,let $f_n = \inf\{f_n, f_{n+1}, ...\}$,

clearly $f_n \in M^*(x, \aleph)$ $\forall n \in N$ and $(f_n)\uparrow = \lim_{n\to\infty} f_n$

Hence by M.C.T $\lim_{n\to\infty}\int f_n d\mu = \int(\lim_{n\to\infty} f_n)d\mu$

i.e $\int(\lim_{n\to\infty} f_n)d\mu = \lim_{n\to\infty}\int f_n d\mu ....*$

now $f_m \leq f_n$ $\forall m \leq n$

By monotone property $\int f_n d\mu \leq \int f_m d\mu$

Taking the limits

$\lim_{m\to\infty}\int f_m d\mu \leq \lim_{n\to\infty}\int f_n d\mu ....**$

from * and **, we have $\int(\lim_{n\to\infty} f_n)d\mu \leq \lim_{n\to\infty}\int f_n$

5.7.1 Theorem

Let $(X, \aleph, \mu)$ be a measure space and $f, g \in M^{*}(x, \mu)$ and $f \leq g$

Let E and $F \in \aleph$ such that $E \subseteq F$ then(i) $\int f d\mu \leq \int g d\mu$ and (ii) $\int_{E} f d\mu \leq \int_{F} f d\mu$

Proof

(i)If $\phi \in M^{*}(x, \aleph)$ is simple and $\phi \leq f$ then $\phi \leq g$,further if $\Omega(f)$ is a set of all simple functions ,such that $\phi \leq f$ then $\phi \in \Omega(g)$ (simple functions s. t $\phi \leq g$) i. e $\Omega(f) \in \Omega(g)$

and hence $Sup \int_{\Omega(f)} \phi d\mu \leq Sup \int_{\Omega(g)} \phi d\mu$ i. e $\int f d\mu \leq \int g d\mu$

(ii)Consider $fX_{E}; fX_{F} \in M^{*}(x, \aleph))$ Since $E \subseteq F$, $\Rightarrow fX_{E} \leq fX_{F}$

By part (i) and monotony $\int fX_{E} d\mu \leq \int fX_{F} d\mu$ and $\int_{E} f d\mu \leq \int_{F} f d\mu$

5.7.2 Example

Consider $([0,1], F, \mu)$,and take $g_{n} = n\chi_{[\frac{1}{n},\frac{2}{n}]}$

Note that $g_{n} \rightarrow 0$ in [0,1], now $\int g_{n} dn = \int n.\chi_{[\frac{1}{n},\frac{2}{n}]} dn = n\mu([\frac{1}{n},\frac{2}{n}]) = n.\frac{1}{n} = 1$

$\Rightarrow \lim_{n\rightarrow\infty} \int g_{n} d\mu = \lim_{n\rightarrow\infty} 1 = 1$ Such that $\int g dn = 0 \neq \lim_{n\rightarrow\infty} \int g_{n} d_{n}$,M.C.T. does not apply

Now $g_{n} \rightarrow 0$ on [0,1], i.e $\int (\liminf_{n\rightarrow\infty} g_{n}) d\mu = \int 0 d\mu = 0$

And $\liminf_{n\rightarrow\infty} \int g_{n} d\mu = \liminf_{n\rightarrow\infty} 1 = 1 \therefore$ $\int (\liminf_{n\rightarrow\infty} g_{n}) = 0 \leq \liminf_{n\rightarrow\infty} \int g_{n} d\mu$,

fatou's lemma apply

5.8.0 Lebesgue Dominated Convergence Theorem(L.D.C.T)

Suppose $(f_n)_1^\infty$ is a sequence of measurable functions which converges $\mu.a,e$ to a function f.

Let g be an integrable functions such that $|f_n| \leq g$ Then f is integrable and $\lim_{n\to\infty}\int f_n d\mu = \int f d\mu$,

the function g is called a dominating function for the sequence $(f_n)_1^\infty$.

Proof.

Since $f_n + g \geq 0$, fatou's lemma shows that $\int (f+g)d\mu \leq \liminf_{n\to\infty} \int_E (f_n+g)d\mu$ e

$\int_E f d\mu \leq \liminf_{n\to\infty} \int f_n d\mu ...(i)$ Since $g - f_n \geq 0$ similarly

$$\int (g-f)d\mu \leq \liminf_{n\to\infty} \int_E (g-f_n)d\mu \quad -\int_E f d\mu \leq \liminf_{n\to\infty}[-\int_E f_n d\mu]$$

which is the same as $\int_E f d\mu \geq \lim_{n\to\infty} Sup \int_E f_n d\mu$ ...(ii) From (i) and (ii) we have $\lim_{n\to\infty}\int_E f_n d\mu = \int_E f d\mu$

5.8.1 Example . Let $f_n = n\chi_{[0,\frac{1}{n}]}$ for $n = 1,2,3,$,....,This functions $f_n(x) = \begin{cases} n; x\in (0; 1/n) \\ 0; otherwises \end{cases}$,

hence $f_n(x)$ cannot be dominated by a single integrable functions .Further at any point in $(0,1]$ the sequence contains only finite number of non-zero terms and indefinite number of zeros and at any point outside (0,1], each term of the sequence is zero Hence $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in n$,

Thus we have Further $\int_R f_n x dx = \int n\chi_{(0,\frac{1}{n})} dx = n\int_0^{\frac{1}{n}} dx = nm((0,\frac{1}{n}] = n.\frac{1}{n} = 1$

,Thus $\int_R f_n(x)dx = 1$ for all .Hence $\lim_{n\to\infty}\int_R f_n dx = 1 \neq 0 = \int_R \lim_{n\to\infty} f_n(x)dx$.

5.8.2 Example2

Show that $\lim_{n\to\infty}\int_0^1 f_n(x)=0$,where $f_n=\frac{nx}{1+n^2x^2}$

Sol

Let $nx=\frac{1+n^2x^2}{2}$ so that $\frac{nx}{1+n^2x^2}<\frac{1}{n}$

Let $g(x)=\frac{1}{2}$. since a constant is integrable, $g(x)$ is integrable

Hence $f_n(x)=\frac{nx}{1+n^2x^2}<g(x)$, $f_n(x)$ is dominated by an integrable function $g(x)$

Further $\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{nx}{1+n^2x^2}=0$, So that $f_n(x)\to 0$ as $n\to\infty$

Hence by lebesgue's dominated convergence theorem $\lim_{n\to\infty}\int_0^1\frac{nx}{1+n^2x^2}dx=\int_0^1 0dx=0$

5.8.3 Properties Of Lebesgue Integral For Bounded Measurable Functions

(a)If f is measurable and bounded on E , and $\mu(E)<\infty$, then $f\in\ell(\mu)$ on E

(b)If $a\le f(x)\le b$ for $x\in E$, and $\mu(E)<+\infty$, then $a\mu(E)\le\int_E fd\mu\le b\mu(E)$.

(c)If f and $g\in\ell(\mu)$ on E and if $f(x)\le g(x)$ for $x\in E$ then $\int_E fd\mu\le\int_E gd\mu$

(d)If $f\in\ell(\mu)$ on E , then $cf\in\ell(\mu)$ on E , and $\int_E cfd\mu=c\int_E fd\mu$

(e)If $f\in\ell(\mu)$ on E and $A\subset E$ then $f\in\ell(\mu)$ on A.

CHAPTER SIX

COMPARISON OF RIEMANN INTEGRAL AND LIBESGUE INTEGRAL THEORIES

6.1.0 Theorem(Equivalence of Riemann and Lebesgue)

(a) If $f \in R$ on [a, b], the $f \in L$ on [a, b] and $\int_a^b f dx = R\int_a^b f\,dx$.

(b) Suppose f is bounded on $[a,b]$, the $f \in R$ on $[a,b]$ if and only if f is continuous almost everywhere on $[a,b]$.

Proof ;(a)Suppose f is bounded , then there is a sequence $\{p_k\}$ of partitions of $[a,b]$ such that $\{p_{k+1}\}$ such that the distance between the adjacent points of P_k is less than $\frac{1}{k}$ and such that

$\lim_{n\to\infty} L(p_k, f) = R\underline{\int} fdx, \quad \lim_{n\to\infty} U(p_k, f) = R\overline{\int} fdx$, all the integrals are taken over $[a,b]$.

If $p_k = \{x_o, x_1, ...x_n\}$ with $x_o = a$ and $x_n = b$ define ,Putting $U_k(a) = M_i$ and $L_k(a) = m_i$ for $x_{i-1} < x < x_i, 1 \le i \le n$ and hence $L(p_k, f) = \int L_k dx, \quad U_k(p_k, f) = \int U_k dx$ so that $L_1(x) \le L_2(x) \le ....f(x)......U_2(x) \le U_1(x)$ for all $x \in [a,b]$, since p_{k+1} refines p_k.Thus there exist $L(x) = \lim_{k\to\infty} L_k(x)$, $U_k = \lim_{n\to\infty} U_k(x)$ and we observe that L and U are bounded and measurable functions on $[a,b]$ that $L(x) \le f(x) \le U(x)$ where $(a \le x \le b)$,and that $\int Ldx = R\underline{\int} fdx$, $\int Udx = R\overline{\int} fdx$, by the monotone convergence theorem, where the only assumption is that f is a bounded real function on $[a,b]$.We note that $f \in R$, if and only if its upper and lower Riemann integrals are equal. hence if and only if $\int Ldx = \int Udx$, since $L \le U$, $\int Ldx = \int Udx$ happens if and only if $L(x) = U(x)$ for all $x \in [a,b]$,

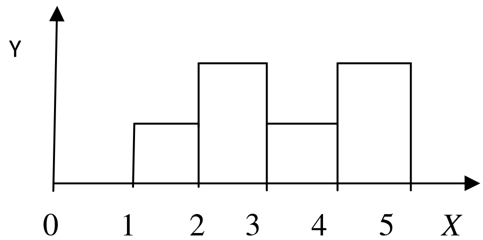
in this case $L(x) \le f(x) \le U(x) \Rightarrow L(x) = f(x) = U(x)$

almost everywhere on $[a,b]$, *so that* f is measurable, thus $\int_a^b fdx = R\int_a^b fdx$

(b)Furthermore ,if x belongs to number p_k ,it is quite easy to see that $U(x)=L(x)$ if and only if f is continuous at x .Since the union of sets P_k is countable, it's measure is 0,and we conclude that f is continuous almost everywhere on $[a,b]$ if and only if $L(x)=U(x)$ almost everywhere, Hence $\int_a^b fdx = R\int_a^b fdx$ if and only if $f \in R$.This completes the proof.

6.1.1 Example ; Evaluate $\int_0^5 f(x)dx = \begin{cases} 0; 0 \le x \le 1 \\ 1; \{1 \le x \le 2\} \cup \{3 \le x \le 4\} \\ 2; \{2 \le x \le 3\} \cup \{4 \le x \le 5\} \end{cases}$ by using the Riemann and Libesgue definitions of integrals.

(I)Using Riemann definition of the integrals(where the subdivisions is taken of the segments $[a,b]$) by the subdivisions points $x_0, x_1, x_2, ...x_n$ on $X-$axis.



the upper and lower Riemann sums tends to common value

$0(1-0)+1(2-1)+2(3-2)+1(4-3)+2(5-4)=6$ thus $R\int_a^b f(x)dx = 6$

(II)Evaluating the lebesgue integral where the sub-divisions is that of the interval $[0, 2+\delta], \delta \ge 0$

we get ,$0[1-0]+1[(2-1)+(4-3)]+2[(3-2)+(5-4)]=6$ thus $L\int_0^5 f(x)dx = 6$

6.1.2 Example 2

Let f be defined on $[a,b]$ as follows $f(x)=\begin{cases}0; if..x..rational\\1; if..x..is..irrational\end{cases}$, prove that f is lebesgue integrable but not Riemann integrable.

Solution

Consider a partition $P=\{a=x_o,<x_1<........<x=b\}$ of $[a,b]$.Then $M_i=1$ in $[x_{i-1},x_i]$ and $m_i=0$ in $[x_{i-1},x]$,Hence $S_p=\sum(x_i-x_{i-1})=b-a$ and $s_p=\sum 0(x_i-x_{i-1})=0$ so that $R\ R\int_a^{-b} f(x)dx=(b-a)$ and $R\int_{-a}^{b} f(x)dx=0$.This shows that f is not Riemann integrable. We prove that f is lebesgue integrable.

Let Q be the set of all rationales' in $[a,b]$,then CQ is the set of irrationals in $[a,b]$,where $[a,b]=Q\cup CQ$ and $Q\cap CQ=\varnothing$.Since Q is countable set it has a measure and hence it is measurable in $[a,b]$ and since the complement of a set is measurable, CQ is measurable.

By definitions f is the characteristic functions of CQ,Since CQ is measurable, f is measurable function. As f is bounded, it is integrable.

The lebesgue integral of f is $\int_a^b fdx=\int_{Q\cup CQ} fdx=\int_Q fdx+\int_{CQ} fdx$

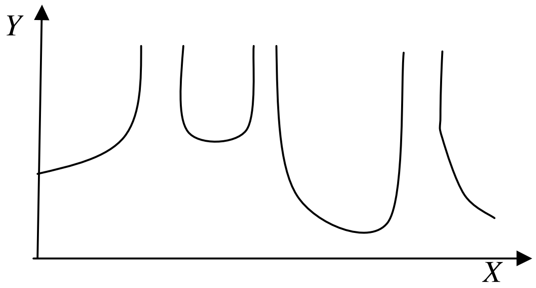
as $Q\cap CQ=0.m(Q)+1m(CQ)=m(CQ)$ .Next we find the measure CQ

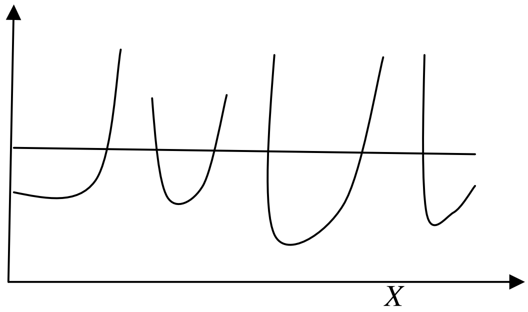
If E_1 and E_2 are disjoint measurable sets then $m(E_1)+m(E_2)=m(E_1\cup E_2)+m(E_1\cap E_2)$ where $E_1=Q$ and $E_2=CQ,$ taking $m(Q)+m(CQ)=m([a,b])+m(\varnothing)$, since $m(\varnothing)=0$ we have $m(CQ)=(b-a),$ thus $\int_a^b fdx=(b-a)$.Hence f is lebesgue integrable but not Riemann Integrable.

6.2.0 Comparison of Lebesgue and Riemann Integrals For Unbounded Functions.

Let f be a non-negative measurable functions on $[a,b]$. For each $x \in [a,b]$ and $n \in N$.

we define a function $F(x,n) = \begin{cases} f(x); 0 \le f(x) \le n \\ n; f(x) > 0 \end{cases}$





Thus $F(x,n) = \min(f(x),n)$, $F(x,n)$ being the minimum of $f(x)$ and hence measurable.

Which implies that for each $n \in N$, $F(x,n)$ is lebesgue integrable.

Now if $\lim_{n\to\infty} \int_a^b F(x,n)dx$ exist finitely then we say that the unbounded function f is lebesgue integrable and $\int_a^b f dx = \lim_{n\to\infty} \int_a^b F(x,n)dx$.

If the limit does not exist finitely then f is not lebesgue integrable The function $F(x,n)$ is called truncated function.

6.2.1 Example

Define $f(x) = \begin{cases} 1/x^{2/3}; 0 < x < 1 \\ 0; x = 0 \end{cases}$ show that f is lebesgue integrable on [0,1] and $\int 1/x^{2/3}dx=3$

Find also F(x,2), since $1/x^{2/3} \to \infty$, as $x \to 0$, so f is unbounded in [0,1]. In order to examine

Its lebesgue integral define d by $F(x, n)=1/x^{2/3}$, if $1/n^{3/2} \le x \le 1$

$= -1/3n^{-3/2}$ if $0<x<1/n^{3/2}$

$=0$ if x

For n=2 $F(x,2)=\begin{cases} 1/x^{2/3}, if\ 1/2^{3/2} \le x \le 1 \\ -\frac{1}{3}n^{-3/2}, if, 0<x< 1/n^{3/2} \\ 0, if, x=0 \end{cases}$

Now $\int_0^1 F(x,n)dx = \int_0^{(\frac{1}{n^{3/2}}} F(x,n)dx + \int_{\frac{1}{n^{3/2}}}^1 F(x,n)dx$

$$= \int_0^{\frac{1}{n^{1/2}}} -\frac{1}{3}n^{-3/2}dx + \int_{\frac{1}{n^{3/2}}}^1 \frac{1}{x^{2/3}}dx = \frac{1}{\sqrt{n}} + 3(1-(\frac{1}{n^{3/2}})^{1/3} \quad = 3 - \frac{2}{\sqrt{n}}, \forall n$$

Thus by the definition of lebesgue integral of unbounded functions ,we have

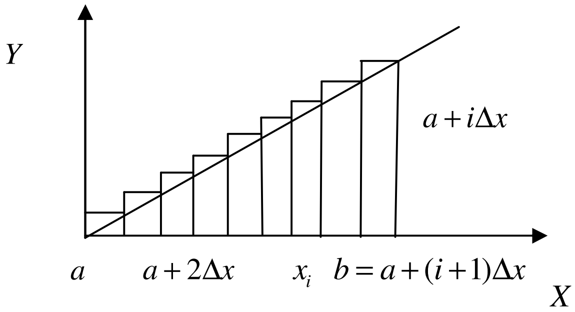
$$\int f(x)dx = \lim_{n\to\infty}\int F(x,n)dx = \lim_{n\to\infty}(3-\frac{2}{\sqrt{n}}) = 3$$

6.2.2 REMARK

The Riemann integral of f on unbounded set A can exist even though the Riemann integral of $|f|$ does not exist on A. For example, $R\int_0^\infty \frac{\sin x}{x}dx = \lim_{n\to\infty} R\int_a^b \frac{\sin x}{x}dx$ exists as an improper Riemann integral wheres the integral $\int_0^\infty |\frac{\sin x}{x}|dx$ does not exist. On the contrary the lebesgue integral of $L\int_0^\infty \frac{\sin x}{x}dx$ does not exist because $\int_0^\infty |\frac{\sin x}{x}|dx$ does not exist ,It shows that there exists improper Riemann integrals which are not integrable in lebesgue sense. This Indicates that nothing can be said about the equality of the two integrals when A is unbounded, Riemann integrals may exists when the lebesgue integral does not exists. Moreover if $|f|$ is Riemann integrable on A , then f is both Riemann and Lebesgue integrable on A and the two integrals are equal.

6.3.0 (III)Libesgue and Riemann Integrals and The Connection Between Integration And Different ion.

6.3.1 Definition, Let an interval of $R(a,b)$ be divided into N equal parts each of length $\Delta x = \frac{b-a}{N}$



Let $x \in [a+i\Delta x, a+(i+1)\Delta x]$ then $\lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\Delta x$ as $N \to \infty$ is called the definite

integral of f(x) in the interval (a,b) and is denoted by $\int_a^b f(x)dx$.

6.3.2 Theorem(fundamental theorem of differential calculus)

Let $f(x)$ have anti derivatives $F(x)$ in the interval $[a,b]$ Then $F(b)-F(a) = \int f(x)dx$.

proof, Let $F(x)$ be the anti derivatives of $f(x)$ the from mean value theorem

$$F(x_1)-F(x_0) = F'(c_0)\Delta x$$

$$F(x_2)-F(x_1) = F'(c_1)\Delta x$$

$$\frac{F(x_{n+1})-F(x_n) = F'(c_n)\Delta x}{F(x_{n+1})-F(x_n) = F'(c_i)\Delta x}$$ which implies $F(b)-F(a) = \int_a^b F(x)dx$.

6.3.3 Connection ;This familiar connection between integration and differentiation is carried over into lebesgue theory. For if $f \in \ell$ on $[a,b]$ and $F(x) = \int f(t)dt$ (a<x<b), then $F'(x) = f(x)$ almost everywhere on [a,b].Conversely, If F is differentiable at every point on $[a,b]$ {almost everywhere not good enough} And if $F' \in L[a,b]$ then $F(b)-F(a) = \int_a^b F'(t)$ $(a \le x \le b)$

6.3.4 Theorem

Let f be continuous function on [a, b], Then (i) f is integrable on [a b]

(ii)If F(x)= $\int_a^x f(t)dt$,where $a < x < b$, then $F(x)$ is differentiable and $F'(x) = f(x)$.

Proof

(i)Since f is continuous on [a,b],it is measurable on [a,b]

As a continuous functions is bounded on, let $|f| \leq M$,taking $g = M$ in the property, thus

f is integrable on [a,b].

(ii)Let $A = [a, x]$, $B = [x, x+h]$ so that $A \cup B = [a, x+h]$

Now we have $\int_{A \cup B} fdx = \int_a^x fdx + \int_x^{x+h} fdx$, using notation $F(x)$, we have

$F(x+h) = F(x) + \int_x^{x+h} fdx$, which gives $F(x+h) - F(x) = \int_x^{x+h} f(t), \ldots.(i)$

Since f is continuous function and the measure is the lebesgue measure,

we obtained earlier that $m(x, x+h) \leq \int_x^{x+h} f(t)dt \leq (x, x+h)M$ where $L \leq f(t) \leq M$

and $t \in [x, x+h]$, For L and M are bounds of continuous function f on [a,b].

Hence there is a point ε in $[x, x+h]$ such that $\int_x^{x+h} f(t)dt = hf\varepsilon \ldots.(2)$ where $\varepsilon = x + \theta$.

using (1) and (2) we have that $F(x+h) - F(x) = hf(\varepsilon)$, since $h \neq 0$ dividing by

h and taking the limits as $h \to 0$, we have $\lim_{h \to \infty} \frac{F(x+h) - F(x)}{h} = f(x)$

which proves that $F'(x) = f(x)$

In term of recovery of derivative functions the two integral are are effective.

6.4.0 (IV) Functions Of Class L^2

As an application of the lebesgue theory, perseval theorem and Bessels theorem already proved for Riemann integrable functions are extended to lebesgue functions.

Definitions

A trigonometric polynomials is a finite sum of the form

$$f(x) = a \;\; f(x) = a_o + \sum_{-N}^{N} (a_n \cos nx + b_n \sin nx) \qquad (x - real)$$

Where $a_o ....a_N , b_1 ....b_N$ are complete numbers, the sum can also be written in the form $f(x) = \sum_{-N}^{N} c_n e^{inx} \qquad (x - real)$

6.4.1 Definitions

We say a sequence of complex functions $\{\phi_n\}$ is an orthonormal set of functions on a measurable Space x if $\int_x \phi_n \phi_m d\mu = \begin{cases} 0; (n \neq m) \\ 1; (n = m) \end{cases}$,in particular ,we must have $\phi_n = \ell^2(\mu)$, If $f \in \ell^2(\mu)$ and If $c_n = \int_x f \phi_n d\mu$ (n=1,2,3,…………..),we write $f \sim \sum_{n=1}^{\infty} c_n \phi_n$.

The definitions of trigonometric Fourier series in L^2 (or even to L) on $(-\pi, \pi)$

6.4.2 Theorem(Bessel Inequality)

If $\{\phi_n\}$ is an ortho normal on $[a,b]$ and if $f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$

Then $\sum_{n=1}^{\infty} | c_n |^2 \leq \int_a^b | f(x) |^2 \, dx$,in particular $\lim_{n \to \infty} c_n = 0$,

The bessel inequality hold for any $f \in \ell^2(\mu)$.

6.4.3 Parseval's Theorem(Riemann version).

Suppose f and g are Riemann integrable functions with period 2π , and $f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$, ,

$g(x) \sim \sum_{-\infty}^{\infty} \gamma_n e^{inx}$,Then $\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} | f(x) - S_N(f;x)|^2 \, dx = 0$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, \bar{g}(x) dx = \sum_{-\infty}^{\infty} c_n \bar{\gamma}_n$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} | f(x)|^2 \, dx = \sum_{-\infty}^{\infty} | c_n |^2$$

Proof

Using the notation $\| h \|_2 = \{\frac{1}{2\pi}\int_{-\pi}^{\pi} | h(x)| \, dx\}^{\frac{1}{2}}$ let $\varepsilon > 0$ be given. Since $f \in R$ and $f(\pi) = f(-\pi)$,

by construction we obtain a continuous $2\pi -$periodic function h with $\| f - h \| < \varepsilon$ and we find a trigonometric polynomials P such that $| h(x) - p(x)| < \varepsilon$ for all x.

Hence $\| h - p \| < \varepsilon$ .If P has degree N_0. Thus $\| h - S_N(h) \|_2 \le \| h - p \| < \varepsilon$, for all $N \ge N_0$.

by bessel's inequality w ith $h - f$ in place of f , $\| S_N(h) - S_N(f) \|_2 = \| S_N(h - f) \|_2 \le \| h - f \|_2 < \varepsilon$

Now applying triangle inequality shows that $\| f - S_N(f) \|_2 < 3\varepsilon \quad N \ge N_0$

Thus $\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} | f(x) - S_N(f;x)|^2 \, dx = 0$

Next $\frac{1}{2\pi}\int_{-\pi}^{\pi} S_N(f)\, \bar{g} dx = \sum_{-N}^{N} c_n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\, \bar{g}(x)\, dx = \sum_{-N}^{N} c_n \bar{\gamma}_n$

And the Schwarz inequality shows that

$$|| \int f \bar{g} - \int S_N(f) \bar{g} | \le \int | f - S_N(f) || g | \le \{\int | f - S_N | \int | g |^2\}^{\frac{1}{2}} ,$$

which tends to zero as N→∞ ,if $g = f$

6.4.4 Parseval Theorem For $f \in \ell^2(\mu)$ {lebesgue version}

Suppose $f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$ .....(a) ,where $f \in \ell^2$ on $[-\pi, \pi]$

Let S_n be the partial sum of (a), Then $\lim_{n \to \infty} \| f - S_n \| = 0$

And $\sum_{-\infty}^{\infty} | c_n |^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} | f |^2 dx$

Proof

Let $\varepsilon > 0$ be given ,since $\| f - g \| = \{ \int_a^b (f-g)^2 dx \}^{\frac{1}{2}} < \varepsilon$, there is a continuous function g

such that $\| f - g \| < \frac{\varepsilon}{2}$ .Moreover, we can arrange it so that $g(\pi) = g(-\pi)$,then g

can be extended to a Periodic continuous function by Perseval Riemann version(earlier),

there is a trigonometric polynomial T ,of degree N ,say, such that $\| g - T \| < \frac{\varepsilon}{2}$.

Hence by Bessels inequality (extended to ℓ^2), $n \geq N$ implies $\| S_n - f \| \leq \| T - f \| < \varepsilon$

thus $\lim_{n \to \infty} \| f - S_n \| = 0$ and hence $\sum_{-\infty}^{\infty} | c_n |^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} | f |^2 dx$, as proved in perseval Riemann version.

6.4.5 Corollary

If $f \in \ell^2$ on $[-\pi, \pi]$ and if $\int_{-\pi}^{\pi} f(x) e^{-inx} dx = o$ $(n = 0, \pm 1, \pm 2, \pm ....)$ then $\| f \| = 0$,Thus if two functions in ℓ^2 have the same Fourier Series, they differ at most on a set of measure zero.

LIbesgue integral simplify the norm and working sums in ℓ^2 easier this is not the case with Riemann integral.

6.5.0 (V) Integration of Complex(Analytic)Expressions.

Complex expressions are well solved using U-substitution and Riemann improper integrals, we now extend this to lebesgue theory.

Suppose f is a complex-valued function defined on a measure space X and $f = u + iv$,

where u and v are real. We say f is measurable if and only if both u and v are measurable.

It is easy to verify that sums and products of complex measurable functions

are again measurable since $|f| = (u^2 + v^2)^{\frac{1}{2}}$.

Since $|f|$ is measurable for every complex measurable f. Suppose u is a measure on X,

and E is a complex function on X.We say that $f \in \ell(u)$ on E provided that f is

measurable and $\int |f| du < +\infty$ and we define $\int_E f du = \int_E u du + i \int_E v du$

Integral of $|f|$ is finite since $|u| \leq |f|$, $|v| \leq |f|$ and $|f| \leq |u| + |v|$ it is clear that finiteness of integral of $|f|$, holds if and only if $u \in \ell(u)$ and $v \in \ell(u)$ on E .

We know $|\int_E f du| \leq \int_E |f| du$. If $f \in \ell(u)$ on E, there is a complex number c, $|c| = 1$ Such

that $c \int_E f du \geq 0$ .If we put $g = cf = u + iv$, u and v real

then $|\int_E f du| = c \int_E f du = \int_E g du = \int_E u du \leq \int_E |f| du$,the third of the above

Equalities holds since the preceding one show that $\int g du$ is real.

6.6.0 The $L_p -$spaces.

Let $0 \le p \le \infty$ and $L_p(\mu)$ or $L_p(\Omega)$ or $L_p(\Omega, F, \mu)$ denote the space of all complex valued measurable functions on Ω such that $\int |f|^p \, d\mu < \infty$ .The space $L_p(\mu)$ is called the

P^{th} power integrable function of (Ω, F, μ)

A measurable function $f(x)$ defined on the segment $[a,b]$ is called the P^{th} power summable where $P \ge 1$, if $\int_a^b |f(x)|^p d\mu < \infty$, finite integrals.

The set of all such functions is denoted $L_p[a,b]$.

6.6.1 Example

$$f(x) = \frac{1}{\sqrt{x}} \in L_{P_1} \text{ i.e. } \int_0^1 f(x)dx = \int_0^1 \frac{dx}{\sqrt{x}}$$

$$= \int_0^1 x^{-\frac{1}{2}} dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2x^{-\frac{1}{2}} = 2\sqrt{x} = \int_0^1 f(x) = 2\sqrt{x} \;|_0^1 = 2$$

But $f \notin L_2(0,1)$ Since $\int_0^1 f(x) = \int_0^1 (\frac{1}{\sqrt{x}})^2 dx = \int_0^1 \frac{dx}{x}$

$$= \ln|x| \;|_0^1 = \infty \;,\; L_{p_2} \not\subset L_{p_1}$$

6.6.2 Example2

$$\int_0^1 \sqrt{(5-2x)dx} = \int_0^1 (5-2x)^{1/2} dx = -\frac{1}{3}\sqrt{(5-2x)^3} \;|_0^1 = 2 \quad f \in L_{p_1}$$

Now $f(x) = \int_0^1 5 - 2x dx \quad = 5x - x^2 \;|_0^1 = 4$

$$f \in L_{p_2} \quad \therefore L_{p_2} \subset L_{p_1}$$

The two examples shows that integration in ℓ^2 and of complex valued functions is not always guaranteed even though they were possible in R_1 .Continuity and finiteness of functions therefore must be considered when integrating.

6.6.3 Proposition

If $\mu(\Omega) \leq \infty$ and $1 \leq p_1 \leq \infty$ then $L_{p_2} \subset L_{p_1}$

Proof ;Take $f \in L_{p_2}$

$$|f|^{p_1} \leq |f|^{p_2} + 1 \qquad \forall x \in \Omega$$

$$\Rightarrow \int |f|^{p_1} \, d\mu \leq \int |f|^{p_2} \, d\mu + \int 1.d\mu < +\infty$$

Thus $\int |f|^{p_1} < +\infty \qquad \Rightarrow f \in L_{p_1} \;\; \therefore L_{p_2} \subset L_{p_1}$

6.6.4 Definition

For $f \in L_p(\mu)$, define $\| f \| = (\int | f | \, d\mu)^{\frac{1}{p}}$,called the P^{th} norm of $f \in L_p(\mu)$

6.6.5 Properties

(1)If $f, g \in L_P(\mu)$.The following hold $\| f \|_p = 0$ iff $f = 0$ a. e $x(\mu)$.

(2)The $\| \alpha f \|_p = |\alpha| \| f \|_p \qquad \forall \alpha \in C$

Proof

$$\| \alpha f \|_p = (\int | \alpha f |^p d\mu)^{\frac{1}{p}}$$

$$= (|\alpha|^p \int | f |^p \, d\mu)^{\frac{1}{p}}$$

$$= |\alpha| (\int | f |^p \, d\mu)^{\frac{1}{p}} = |\alpha| \| f \|_p$$

3.$\| fg \| \leq \| f \|_p + \| g \|_q$

Proof

Let p>1 and q>1 be such that $\frac{1}{p}+\frac{1}{q}=1$ (p and q are conjugate)

Let $f \in \ell_p(\mu)$ and $g \in \ell_q(\mu)$,Then $fg \in \ell_1(\mu)$ and $\int | fg | \leq (\int | f | d\mu)^{\frac{1}{p}} (\int | g |^q)^{\frac{1}{q}}$

Note that if $\| f \|_p = 0$ or $\| g \|_q = 0$ $\Rightarrow \int | f |^p \, d\mu = 0$ or $\int | g |^q \, d\mu = 0$

$\Rightarrow fg = 0$ a.e $x\mu$

Now assume $\| f \| \neq 0$ and $\| g \|_q \neq 0$

Apply the Holder's lemma by putting $t = \frac{1}{p}$

$a = (\frac{| f |}{\| f \|_p})^p$ $b = (\frac{| g |}{\| g \|_q})^q$

Substituting in the holders equalities $a^t b^{1-t} \leq ta + (1-t)b$ gives

$$\frac{| f |}{\| f \|_p} \cdot \frac{| g |}{\| g \|_q} \leq \frac{1}{p}(\frac{| f |}{\| f \|_p})^p + \frac{1}{q}(\frac{| g |}{\| g \|_q})^q \ldots\ldots\ldots\ldots(1)$$

Integrating both sides of (1) with respect to measure μ, we obtain

$$\frac{1}{\| f \|_p \| g \|_q} \int | fg | \, d\mu \leq \frac{1}{p \| f \|_p} \int | f |^p \, d\mu + \frac{1}{q \| g \|_q} \int | g |^q \, d\mu$$

$$\Rightarrow \frac{1}{\| f \|_p \| g \|_q} \int | fg | \, d\mu \leq \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \int | fg | \, d\mu \leq \| f \|_p \, \| g \|_q \quad \Rightarrow \| fg \| \leq \| f \|_p \| g \|_q$$

CHAPTER SEVEN

APPLICATION OF RIEMANN AND LIBESGUE INTEGRAL TO TIME SERIES ANALYSIS

REVIEW (I)

7.1.0 VARIATION ;The variation in observation can be due to;-

(i)Treatment effect's (ii)Random-error

The treatment model is an addition model of the form $y_{ij} = \mu + t_i + e_{ij}$

where (1) μ ;- is the grand mean i.e the mean yield if no treatment is applied.

(2) t_i ;- is effect of the i^{th} treatment .The i^{th} treatment will either increase or decrease of yield by t_i.

(3) e_{ij} is the randomization error effect.

7.1.1 REGRESSION MODEL.

7.1.2 Definition ;A regression model is a formal means of expressing the two essential ingredients of a statistical relation.

(a)The tendency of the dependent variable Y to vary both with the independent X in a systematic fashion.

(b)A scattering of points around the line of a statistical relationship.

7.1.3 Definition, First order model When there are two independent variable x_1 and x_2 the regression models becomes $Y_i = \beta_o + \beta_1 x_{i1} + \beta x_{i2} + \varepsilon_i$ is called a first order regression model with two independent variable. where Y_i is the dependent variable and the parameters of the model β_o , β_1 and β_2 and the error term is ε_i.The parameter β_1 indicates the change in the mean response per unit increase in x_1 when x_2 is held constant. Also β_2 indicates the change in mean response per unit increase in x_2 when x_1 is held constant.

7.1.4 Example

Suppose x_2 is held constant at level $x_2 = 20$, the regression function

$E(Y) = 20 + 0.95x_1 - 0.5(20)$ becomes $E(Y) = 10 + 0.95x_1$

7.1.5 General Linear Regression Model In Matrix Terms.

In matrix terms the general linear regression model is $\underset{\sim}{Y} = \underset{-}{x}\underset{-}{\beta} + \underset{-}{\varepsilon}$.....***

where $\underset{\sim}{Y}$;- is the vector of responses i.e $\underset{\sim}{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ $\underset{\sim}{x} = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n1} & \cdots & x_{np-1} \end{bmatrix}$

$\underset{\sim}{\beta}$ is the vector of parameters. For example if $Y_i = \beta + \beta_1 x_{i1} + \beta_2 x_{i2} + ... + \beta_{p-1} x_{ip-1}$

$\beta = \begin{bmatrix} \beta \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{bmatrix}$ and $\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$ is the vector of independent normal variables

with expectation $E(\varepsilon) = 0$.

7.1.6 LEAST SQUARES ESTIMATORS

Let us denote the vector of estimated regression coefficients $b_o, b_1, b_2, ...b_{p-1}$ as b

$\underset{-}{b} = \begin{bmatrix} b_o \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$ .The least squires normal equations for general regression model ***

are $(\underset{-}{x}'\underset{-}{x})b = \underset{-}{x}'\underset{-}{y}$ and the least squires estimators are $\underset{-}{b} = (x'x)^{-1}\underset{-}{x}\,\underset{-}{y}$.

7.1.7 FITTED VALUES AND RESIDUALS

Let the vectors of the fitted values Y_i be denoted by $\underline{\hat{Y}}$ and the vectors of the residual terms $e_i = y_i - \hat{y}_i$ be denoted by $\underline{e}$ $\hat{Y} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_n \end{bmatrix}$ and $\underline{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$

7.1.8 The fitted values are represented by $\hat{Y} = \underline{x}\underline{b}$ and residual terms by $\bar{e} = y - \hat{y} = \bar{y} - \underline{x}b$ The vectors of the fitted values $\hat{Y}$ can be expressed in terms of the matrix $\underline{H}$ as follows

$\hat{Y} = H\underline{Y}$ where $H = x(x'x)^{-1}x'$.

7.1.9 Similarly, the vector of the residuals can be expressed as follows $e = (\underline{I} - \underline{H})Y$.

The variance-covariance matrix of the residual is $\sigma^2(e) = \sigma^2(\underline{I} - \underline{H})$ which is estimated by $\sigma^2(\underline{e}) = MSE(\underline{I} - H)$.

7.2.0 FOURIER SERIES

7.2.1 Definition ,A trigonometric polynomial is a finite sum of the form $f(x) = a_o + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)....(a)$ where $a_o, ..., a_N$, are complex numbers .

Equation (a) can be written as $f(x) = \sum_{-N}^{N} c_n e^{inx}$ $(x - real)$ .Every trigonometric polynomial is periodic with period 2π .If n is a non zero integer , e^{inx} is the derivative of $\frac{e^{inx}}{in}$, which also has a period 2π . Hence $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}dx = \begin{cases} 1(if..n = 0) \\ 0(if..n = \pm 1, \pm 2....) \end{cases}$. $\sin x$ and $\cos x$ satisfy $f''(x) + f(x) = 0$, in general $f'(x)+\omega^2 f(x)=0$ is satisfied by $\sin\omega x$ and $\cos\omega x$.

7.2.2 $\sin x$ is an odd function and $\cos x$ is even $f(x)$ is said to be odd if

$f(-x)=-f(x)$ and even if $f(-x)=f(x)$.

e.g $\sin(\frac{-\pi}{2})=-1=-\sin(\frac{\pi}{2}).....odd \quad \cos(-\pi)=1=\cos\pi....even$.

7.2.3 $\sin\alpha\cos\beta=\dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2} \qquad \cos\alpha\cos\beta=\dfrac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$

$\sin\alpha\sin\beta=\dfrac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$

7.2.4 Then if m and n are non-negative integers then

(i) $$\int_{-\pi}^{\pi}\sin mx\cos nx dx=\frac{1}{2}\int_{-\pi}^{\pi}[\sin(m+n)x+\sin(m-n)x]dx$$

$$=\frac{1}{2}\int_{-\pi}^{\pi}\sin(m+n)xdx+\frac{1}{2}\int_{-\pi}^{\pi}\sin(m-n)xdx=0$$

Following the same arguments

$$\int_{-\pi}^{\pi}\sin mx\sin nx dx=\begin{cases}0; if ...m\neq n\\ \pi; if ...m=n>0\end{cases}$$ (iii) $$\int_{-\pi}^{\pi}\cos mx\cos nx dx\begin{cases}0; if ...m\neq n\\ \pi; if ...m=n>0\\ 2\pi; if ...m=n=0\end{cases}$$

(i),(ii) and (iii) are called the orthogonal formula.

7.2.5 Remark

Suppose the series

$\frac{a_o}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ converges then it's sum will be a function of x

i.e $f(x)=\frac{a_o}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

Suppose the convergence is uniform, then we can integrate term by term

$$\int_{-\pi}^{\pi} f(x)dx = \frac{a_o}{2}\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}[a_n \int_{-\pi}^{\pi} 1.\cos nxdx + b_n \int_{-\pi}^{\pi} 1.\sin nxdx .$$

For k=0. multiply by $\cos kx$

$$\int_{-\pi}^{\pi} f(x)\cos kxdx = \int_{-\pi}^{\pi} \frac{a_o}{2}\cos kxdx + \sum_{n=1}^{\infty}[a_n \int_{-\pi}^{\pi} 1.\cos nxdx + b_n \int 1.\sin nxdx$$

$$= \frac{a_o}{2}\int_{-\pi}^{\pi} \cos kxdx = \pi a_o \quad \text{i.e} \quad a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

For $k \geq 1$,multiply by $\cos kx$

$$\int_{-\pi}^{\pi} f(x)\cos kxdx = \int_{-\pi}^{\pi} \frac{a_o}{2}\cos kxdx + \sum[a_n \int_{-\pi}^{\pi} \cos nx\cos kxdx + b_n \int_{-\pi}^{\pi} \sin x\cos kxdx]$$

$$= \frac{a_o}{2}\int_{-\pi}^{\pi} \cos kxdx + \sum a_n \int_{-\pi}^{\pi} \cos nx\cos kxdx$$

$$= a_n \int_{-\pi}^{\pi} \pi dx = a_n\pi \quad \text{when} \quad n-k>0 \quad \text{Thus} \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$$

Similarly $\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{a_o}{2} + \sum[a_o \int_{-\pi}^{\pi} \cos nxdx + b_n \int_{-\pi}^{\pi} \sin nxdx]$

Multiply by $\sin kx$ for $k > 1$

$$\int_{-\pi}^{\pi} f(x)\sin kxdx = \frac{a_o}{2}\int_{-\pi}^{\pi} \sin kxdx + \sum[a_n \int_{-\pi}^{\pi} \cos nx\sin kxdx + b_n \int_{-\pi}^{\pi} \sin nx\sin kxdx \quad \text{where} \quad k = n$$

$$= a_n \int_{-\pi}^{\pi} \sin nx\sin kx + b_n \int_{-\pi}^{\pi} \sin nx\sin kxdx$$

$$\int_{-\pi}^{\pi} f(x)\sin kxdx = b_n\pi \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kxdx$$

7.2.6 Example 1

Compute the F series of $f(x)=x$ when $-\pi \le x \le \pi$

Solution

(i) $a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx = \frac{1}{\pi}\frac{x^2}{2}|_{-\pi}^{\pi} = 0$

(ii) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx = 0$ (since $x\cos x$ is odd function)

(iii) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx$ where $x\sin x$ is even

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx dx \quad = -\frac{2}{n\pi}x\cos nx|_0^{\pi} + \frac{1}{n\pi}\int_0^{\pi}\cos nx dx = \frac{2(-1)^{n+1}}{n}$$

7.2.7 Example 2

Compute the F series of f defined by $f(x)=\begin{cases} 0; if..-\pi \le x < 0 \\ 1; if..0 \le x \le \pi \end{cases}$

Solution

$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$ divides the integral to corresponds with the intervals

$= \int_{-\pi}^{\pi} f(x)dx + \frac{1}{\pi}\int_0^{\pi} f(x)dx \quad = \frac{1}{\pi}.\pi = 1$ For $n \ge 1$ $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(x)\cos nx dx + \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{\pi}\cos nx dx = 0$$

$$\frac{-1}{n\pi}(\cos n\pi - \cos 0) = \frac{-1}{n\pi}((-1)^n - 1)$$

$= \begin{cases} 0; if..n..is..even \\ \frac{2}{n\pi}; if..n..is..odd \end{cases}$ Thus $f(n) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\sin\frac{(2k+1)x}{2k+1}$

7.3.0 TIME SERIES

7.3.1 Definition; Time series is a set of data collected over time

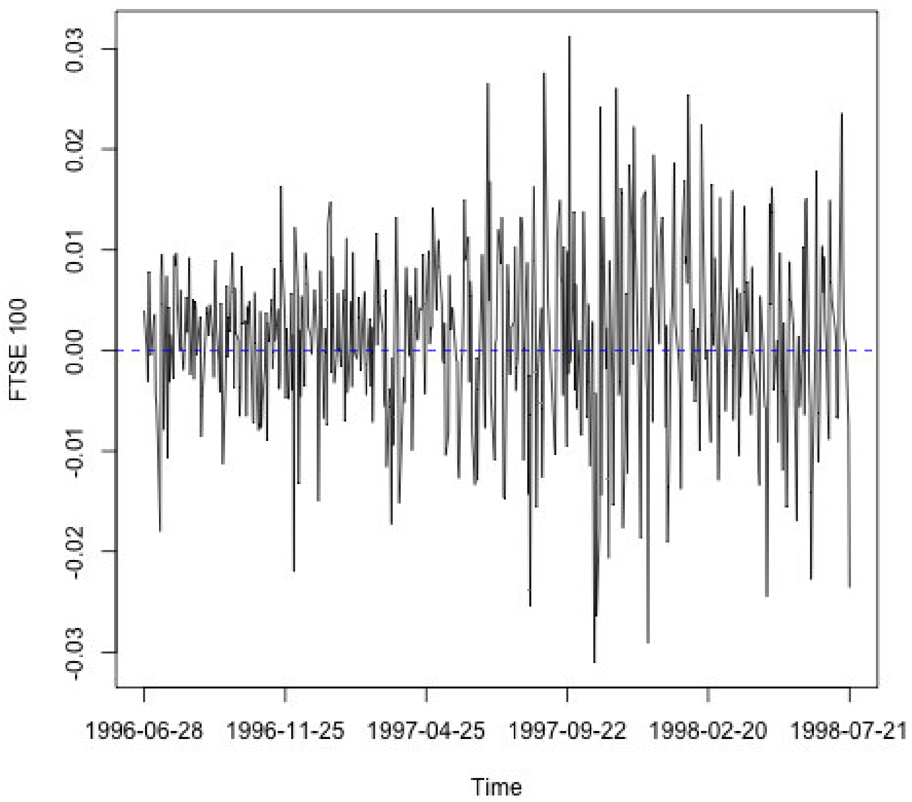
A time series can be expressed as a combination of cosine (or sine) waves with differing periods, amplitude .This properties can be utilized to examine the periodic (cyclical) behavior in a time series. Examples

(i).The prices of stocks and shares taken at regular intervals of time.

(ii)The temperature reading taken at regular interval in season at a place.

(iv)The values of brain activity measured every 2 seconds for 256 seconds

7.3.2 Example ;Picture of FTSE 100 share idex against time



7.3.3 Methods for time series analysis may be divided into two classes

(i)Frequency-domain methods;-which spectral analysis and wavelets analysis

(ii)And time-domain methods;-which includes auto-correlation and cross-correlation analysis.

7.3.4 Objectives Of Time Series Analysis

(i)Provide experiment and historic data.it may consist of graphical representation or a few summary statistic.

(ii)Monitoring of a time series to detect changes in behavior as they occur.

(iii)To fore-cast future values of a series.

(iv)Analysis of accommodate dependence in series and help in making inferences on parameters.

(v)Development of models with a view of understanding underlying mechanisms which generate the data.

7.4.0 Methods Of Analysis.

7.4.1 Time plot;-are pattern of plotted points or graphs of when the plotted and joined by straight lines.

7.4.2 Minimizing Randomness(Smoothing)

The process involves decomposing of independent variables y_t to trend estimate s_t and randomness r_t i.e $y_t = \hat{y}_t + e_t$ such that using simple linear regression model $Y_t = \mu(t) + u(t)$

implies that $\hat{y}$ is the estimate of the trend μt.

Ways of achieving stationary includes;- Moving averages, fitting polynomial regression, and spline regression.

7.4.3 (I)Moving Averages

A simple moving average is of the form $\hat{y}_t = \frac{(y_{t-1} + y_t + y_{t+1})}{3}$

and generally $\hat{y}_t = \sum_{-p}^{p} w_j y_{t+j} \quad ; t = p+1, ..., n-p$ where every increase positive integer p removes seasonal fluctuations but highlight more long-term trends.

7.4.4 Polynomial Regression

This is the matrix regression method ,where a polynomial represented by $\hat{\underset{\_}{Y}} = \underset{\_}{x}\underset{\_}{b}$ with residual terms

by $\bar{e} = y - \hat{y} = \bar{y} - x\underset{\_}{b}$ .The vectors of the fitted values $\hat{Y}$ can be expressed in terms of the

matrix $\underset{\_}{H}$ as follows $\hat{Y} = H\underset{\_}{Y}$ where $H = x(x'x)^{-1}x'$ ......(i) such that the polynomial is of the form $\hat{Y}_t = \sum_{j=0}^{p} H\underset{\_}{Y}$ and for large p, the values of $\underset{\_}{Y}$ can be adjusted to $\underset{\_}{Y} = Y - \bar{Y}$.

where $\bar{Y} = \sum_{i=1}^{n} Y_i \Big/ n$ .A further refinement can be done by replacing $\underset{\_}{Y}$ by orthogonal

polynomials $(a)..\int_{-\pi}^{\pi} \sin mx \cos nx dx = \frac{1}{2}\int_{-\pi}^{\pi} [\sin(m+n)x + \sin(m-n)x]dx$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin(m+n)x dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin(m-n)x dx = 0$$

(b) $\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0; if ...m \neq n \\ 1; if ...m = n > 0 \end{cases}$ or (c) $\int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0; if ...m \neq n \\ \pi; if ...m = n > 0 \\ 2\pi; if ...m = n = 0 \end{cases}$

where m and n are non-negative integers and the matrix $(X'X)$ in equation (i) is diagonal.

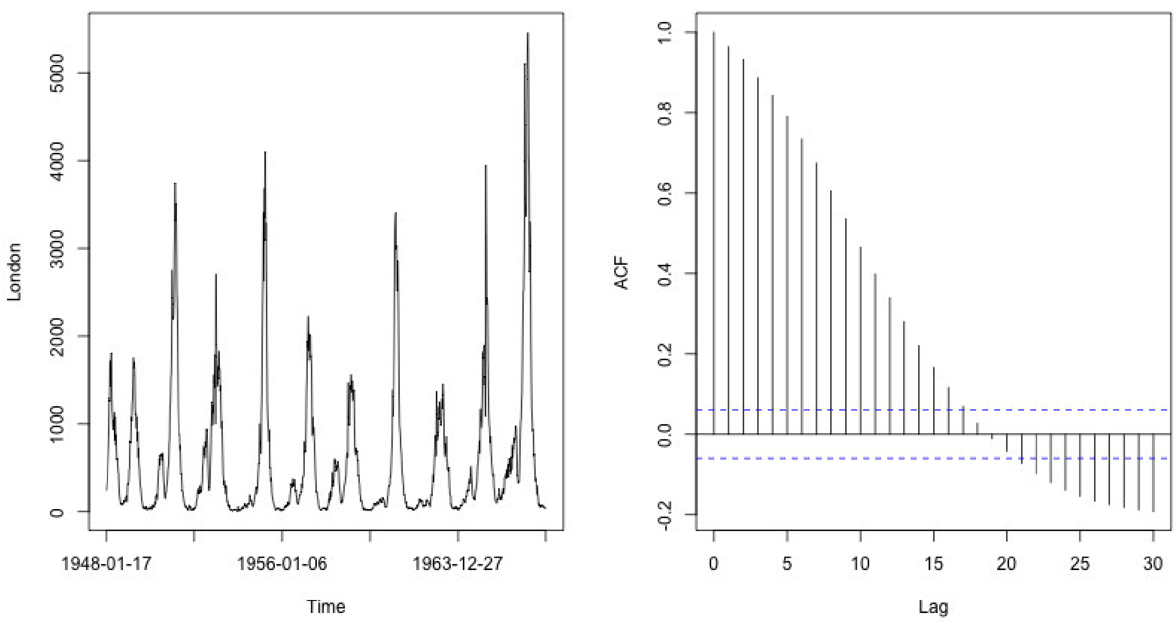
7.4.5 Spline Regression is a method of weighted moving averages applied to gain stationary which copes with arbitrary patterns of missing values in the data.

Equation $Q(\alpha) = \sum_{i=1}^{n} \{y_i - \mu(t_i)\} + \alpha \int_{-\infty}^{\infty} \{\mu''(t)\}^2 dt$. if α is close to zero, we tolerate a lot of roughness in μt to fit the data. if α is large we get smooth $\mu(t)$ and allow less close fit

7.5.0 Auto Correlation ,Sometimes known as a correlograms is a plot of the sample autocorrelations r_h versus h the time lag. It is a measure of internal correlation within a time series.

The variance-covariance matrix $\sigma^2(b) = \begin{bmatrix} \sigma^2(b_o) & \sigma^2(b_o b_1) & .... & \sigma^2(b_o, b_{p-1}) \\ \sigma^2(b_1, b_o) & \sigma^2(b_1) & .... & \sigma^2(b_1, b_{p-1}) \\ \vdots & \vdots & \vdots & \vdots \\ \sigma^2(b_{p-1} b_o) & ... & ... & \sigma^2(b_{p-1}) \end{bmatrix}$

where $\sigma^2(\underline{b}) = \sigma^2(x'x)^{-1}$ implying that the auto covariance function o f a stationary random function $Y(t)$ is $c_h = \mathrm{cov}(b_i, b_{j-h})$ and since $c(0)$ is the variance of Y_t, the auto correlation function becomes $\gamma_h = {c_h}/{c_0}$. The resulting values of r_h will be between -1 and $+1$ i.e $|r(k)| \leq 1$ and for independent variables $r_h = 0$.($+1$) implies there is a strong and positive association i.e the series values in two time interval are similar. whilst (-1) shows strong negative association(dissimilar) observation.

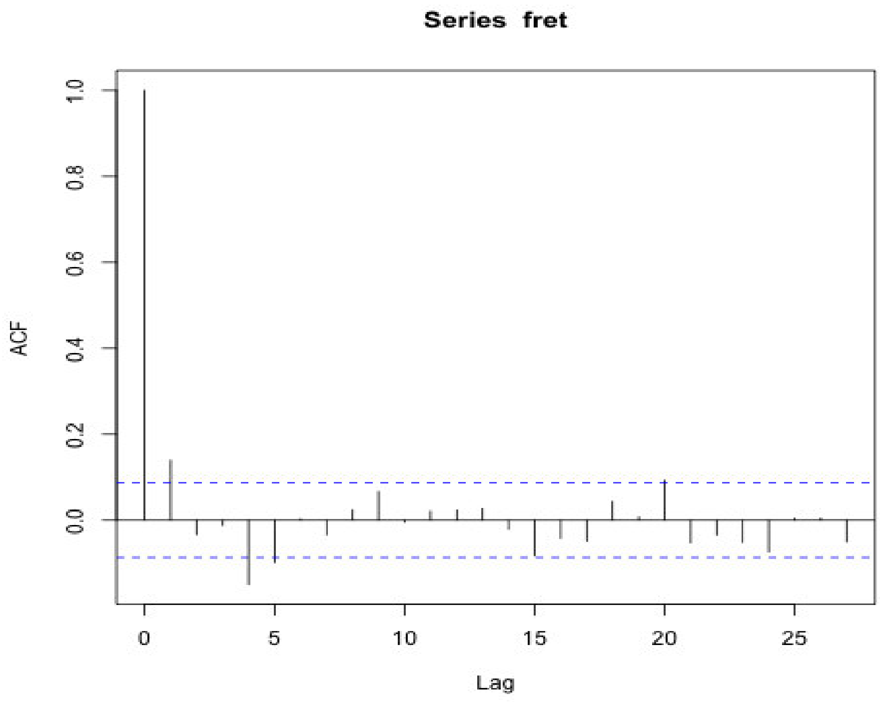


Equivalently , if we consider a random sequence $\{Y_t\}$ defined by $Y_t = \alpha Y_{t-1} + Z_t$ ....(c) , $\{Y_t\}$ is stationary in the range $-1 < \alpha < 1$. Taking expectations of both sides of eqn (c) and giventhat

$E(Z_t) = 0$,we deduce that $\mu = \alpha\mu$ therefore $\mu = 0$. Now multipliying both sides by Y_{t-k} taking expectations and dividing by $Var(Y_t)$ gives $\rho_k = \alpha\rho_{k-1}$. Finally , $\rho_o = 1$ gives the solution $\rho_k = \alpha^k ...; k = 0,1...$ then we proceed to plot ρ_k against k

7.5.1 Estimating The Autocorrelation Function For Equally Spaced Series(Correlograms)

For a series $\{Y_t, t = 1,......., n\}$ we use $\bar{y} = \frac{(\sum y_i}{n}$ and define the k^{th} sample auto covariance coefficient $g_k = \frac{\sum_{t=k+1}^{n} (y_t - \bar{y})(y_{t-k} - \bar{y})}{n}$ Then the k^{th} sample autocorrelation coefficient is $\gamma_k = \frac{g_k}{g_o}$,A plot of γ_k against K is called a correlogram of that data $\{y_t\}$ Each correlogram includes a pair of dashed horizontal lines representing the limits $\pm 2/\sqrt{n}$, which are used for informal assessment of departure from randomness

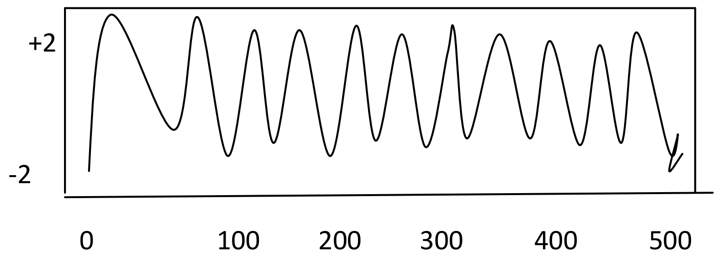


7.6.0 Wavelet;- Analysis is the analysis of the dominant frequencies in a time series

7.6.1 Introduction ;For the cosine function $X_t = 2\cos(2\pi \frac{1}{50} t + 0.6\pi)$ for $t = 1, 2, \ldots\ldots\ldots, 500$.

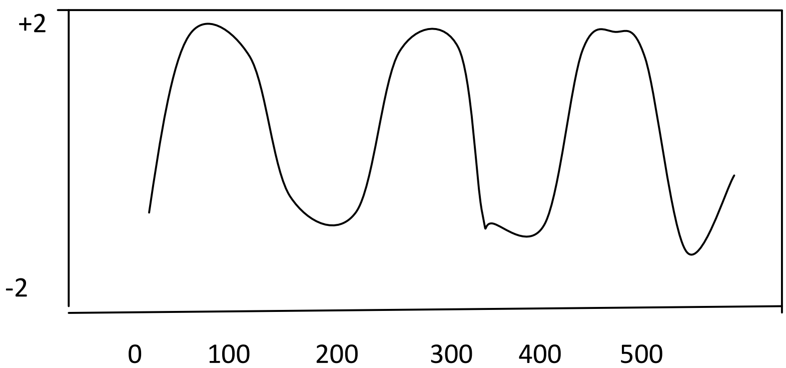
In addition normally distributed errors with mean 0 and variance 1

P=50 $\omega = \frac{1}{50}$, Thus it takes 50 times($\omega = \frac{1}{50}$) to cycle through the cosine function , before errors are added. The maximum and the minimum values are +2 and -2



If we change period to 250 and $\omega = \frac{1}{250} = 0.004$

then $X_t = 2\cos(2\pi \frac{1}{250} t + 0.6\pi)$ for $t = 1,500$



If the regression models becomes take s a cyclic shape $\sum_{k=1}^{m} y_t = \alpha\cos(\omega t) + \beta\sin(\omega t) + e_t ....(ii)$

where z_t is the randomness, $\omega = 2\pi/p$ the frequency and $\theta = (\alpha, \beta)$ parameters estimated by

least square i.e $\theta = (X'X)^{-1}XY$ and Suppose that we have observed at n distinct time points and for conviniences,we assume

that n is even.our goal is to identify important frequencies in the data.To pur sue the

investigation,we consider the set of possible frequencies $\omega_j = \frac{j}{n}$ for $j = 1, 2, ..., \frac{n}{2}$,This are

called the the harmonic frequencies.We will represent the time series as

$x_t = \sum_{j=1}^{\frac{n}{2}} [\beta_1(\frac{j}{n})\cos 2\pi(\omega_j t) + \beta_2(\frac{j}{n})\sin(2\pi(\omega_j t)$ .This is a sum of sine and cosine functions

at the harmonic frequencies.Think of the $\beta_1(\frac{j}{n})$ and $\beta_2(\frac{j}{n})$ as the regression parameters.

Then there are a total of n parameters because we let j move from 1 to $\frac{n}{2}$. This means that we have n data points and n parameters. So the fit of regressin model will be exact.The first step in the creation of the periodogram is the estimation of the $\beta_1(\frac{j}{n})$ and $\beta_2(\frac{j}{n})$ parameters It actually not necessary to carry out regression ($\theta = (X'X)^{-1}XY$) to estimate this parameters because Instead a mathematics device called the Fast Fourier Transform (FFT) is used.

After the parameters have been estimated we define $p(\frac{j}{n}) = \hat{\beta}_1^2(\frac{j}{n}) + \hat{\beta}_2^2(\frac{j}{n})$ .This is the sum of

squared “regression” coefficients at the frequencies $\frac{j}{n}$

7.6.2 Interpretation And Use

A relatively large value of $p(\frac{j}{n})$ indicates relatively more importance for the frequency $\frac{j}{n}$ (or near $\frac{j}{n}$) in explaining the oscillation in the observed series $p(\frac{j}{n})$ is proportional to the squared correlation between the observed series and cosine wave with frequencies $\frac{j}{n}$. The dorminant frequencies might be used to fit cosine(or sine) wave to the data or might be used simply to describe the important periodicities in the series.

7.6.3 Equivalently from Fourier the series $\frac{a_o}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ we

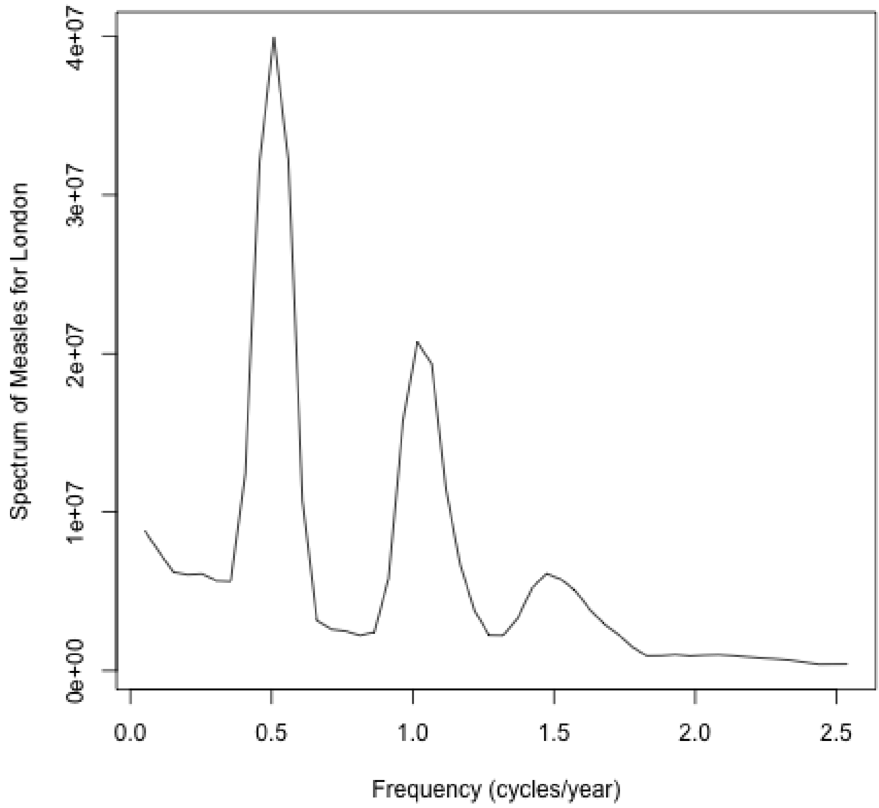
where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kxdx$ thus we can write

parameters as $\alpha = 2\left\{\sum_{t=1}^{n} y_t \cos(t\omega)\right\} \Big/ n$ and $\beta = 2\left\{\sum_{t=1}^{n} y_t \sin(t\omega)\right\} \Big/ n$ It can be shown that the Fourier series of $f(x)$ with $\omega = 0$ and n is odd take initial $y_t = \alpha + e_t$ $t = 1,\ldots\ldots..n$ where α is the sample mean $\alpha = \bar{y}$ Similarly for the even n, the Fourier series $f(x) = x$ is $y_t = \alpha(-1)^t + e^t$ $t = 1,\ldots\ldots,n$.

Equation (ii) show we can achieve an orthogonal partitioning of more variations by increasing m and since $\alpha = \left\{ \sum y_t(-1)^t \Big/ n \right\}$ and associated sum of squares is α^2

If $I(\omega)=\left[\left\{\sum_{t=1}^{n} y_t \cos(\omega t)\right\}^2 \left\{\sum_{t=1}^{n} y_t \sin(\omega t)\right\}^2\right] \Big/ n$ where $0 \leq \omega \leq \pi$ and the partitioning of the total variation in the series $\{y_t\}$ is $\sum_{t=1}^{n} y_t^2 = I(0) + 2\sum_{j=1}^{m} I(2\pi j/n) + I(\pi) \quad , j < n/2$

The graph of $I(\omega)$ against ω is called periodo gram.



The figure show the spectral analysis from the first of london measles time series.The largest peak occurs at the frequency of 0.5 cycles/year of biennial oscillation. There is also a large peak corresponding to annual oscillation and also a slightly smaller one at three cycles per two years

7.6.4 The Connection Between The Correlogram And The Period gram

Though the two have different rationales .The presented arguments ,show a connection between them .For Fourier frequency ω, we can write

$$I(\omega)=\frac{\left[\left\{\sum_{t=1}^{n} y_t\cos(\omega t)\right\}^2+\left\{\sum_{t=1}^{n} y_t\sin(\omega t)\right\}^2\right]}{n}=\frac{\left[\left\{\sum_{t=1}^{n}(y_t-\bar{y}\cos(\omega t)\right\}^2+\left\{\sum_{t=1}^{n}(y_t-\bar{y})\sin(\omega t)\right\}^2\right]}{n}$$

Since $\sum_{t=1}^{n}\cos(\omega t)=\sum_{t=1}^{n}\sin(\omega t)=0$. Expanding each squared term gives

$$nI(\omega)=\sum(y_t-\bar{y})^2\left\{\cos^2(wt)+\sin^2(\omega t)\right\}+\sum_{s\neq t}\sum(y_t-\bar{y})(y_s-\bar{y})\{\cos(\omega t)\cos(\omega s)+\sin(\omega t)\sin(\omega s)\}$$

$$=\sum(y_t-\bar{y})^2+2\sum_{k=1}^{n-1}\sum_{t=k+1}^{n}(y_t-\bar{y})(y_{t-k}-\bar{y})\cos(k\omega)$$

Now substituting the sample auto covariance coefficients we obtain

$\frac{I(\omega)}{g_o}=g_o+2\sum_{k=1}^{n-1}g_k\cos(k\omega)$ express Fourier transform as a sample of auto covariance

Finally dividing by g_o defines normalized period gram

$\frac{I(\omega)}{g_o}=1+2\sum_{k=1}^{n-1}\gamma_k\cos(k\omega)$ as the Fourier transform of the correlogram

7.7.0 The Spectrum Of A Stationary Random Process.

Consider a stationary random sequence $\gamma_t=\operatorname{cov}(Y_t,Y_{t-k})$ .The corresponding auto covariance generating function is $G(Z)=\sum_{k=-\infty}^{\infty}\gamma_k z^k ...(4)$ whose arguments z ,is a complex variable .If in equation (4). we now choose $z=e^{-iw}$ where ω is thereal variable ,we obtain the spectrum of $\{Y_t\}$,

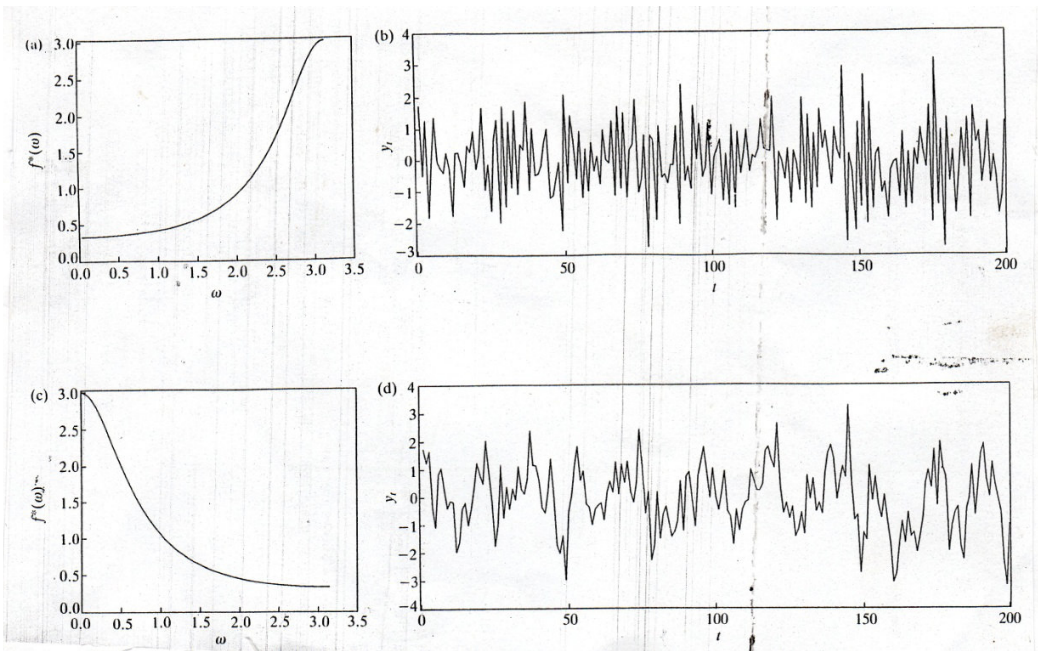
$f(\omega)=G(e^{-i\omega})=\sum_{k=-\infty}^{\infty}\gamma_k e^{-ik\omega}.......(5)$ because $\gamma_k=\gamma_{-k}$ and $e^{i\omega}+e^{-i\omega}=2\cos\omega$ we can write equation (5) as $f(\omega)=\gamma_0+2\sum_{k=1}^{\infty}\gamma_k\cos(k\omega)$, revealing that spectrum isa real-valued function. If σ^2 denotes the variance of Y_t.

we can similarly define a normalized spectrum $f^*(\omega)=\dfrac{f(\omega)}{\sigma^2}=1+2\sum_{k=1}^{\infty}\rho_k\cos(k\omega)$

Note; The normalized spectrum bears the same relationship to the autocorrelation function as does the spectrum to the auto covariance and any non-negative valued function $f(\omega)$ on ($(0,\pi)$ defines a legitimate spectrum.

7.7.1 Example

A first – order autoregressive process. Suppose that $\{Y_t\}$ is defined by $Y_t=\alpha Y_{t-1}+Z_t$ where $\{Z_t\}$ is a randomized sequence and $-1<\alpha<1$ we have already seen that the autocorrelation function $\{Y_t\}$ is $\rho_k=\alpha^k; k=0,1,.......,$ Thus the normalized spectrum of $\{Y_t\}$ is $f^*(\omega)=\sum_{k=-\infty}^{\infty}\rho_k e^{-ik\omega}$ It can be shown that $f^*(\omega)=(1-\alpha^2)\{1-2\alpha\cos(\omega)+\alpha^2\}^{-1}$ ....(6) Normalized spectrum for each of $\alpha=-0.5, 0.5$ and 0.9 Note For negative $\alpha, f^*(\omega)$ is an increasing function of ω



7.7.3 Discrete And Continuous Spectrum

Spectrum plots gives information about how power (or variance) in a series

is distributed according to frequencies.For auto covariance $c_h = \mathrm{cov}\{Y_t, Y_{t-h}\}$ and auto covariance

function is $\sum_{h=-\infty}^{\infty} c_h z^h$ and since $c_h = c_{-h}$ and $e^{i\omega} + e^{-i\omega} = 2\cos(\omega)$ we write a spectrum real

valued $f(\omega) = c_o + 2\sum_{h=1}^{\infty} \gamma_k \cos(h\omega)$ Conversion of time-indexed data into estimates of autocorrelation

or spectrum depends partly on Fourier transformation of $c(\tau)$ to obtain $F(A)$.If Continuous component is missing i.e $f(\lambda)$=0 for all λ. the time spectrum is said to have a discrete spectrum (point spectrum).

$C(\tau) = \sum_{k=-\infty}^{\infty} e^{i\lambda k\tau} p(\lambda_k)$ moreover $\sum_{k=-\infty}^{\infty} p(\lambda_k) = C(0) < \infty$

Thus since summable series are square summable $\sum_{k=-\infty}^{\infty} p^2(\lambda_k) < \infty$. It follows that the

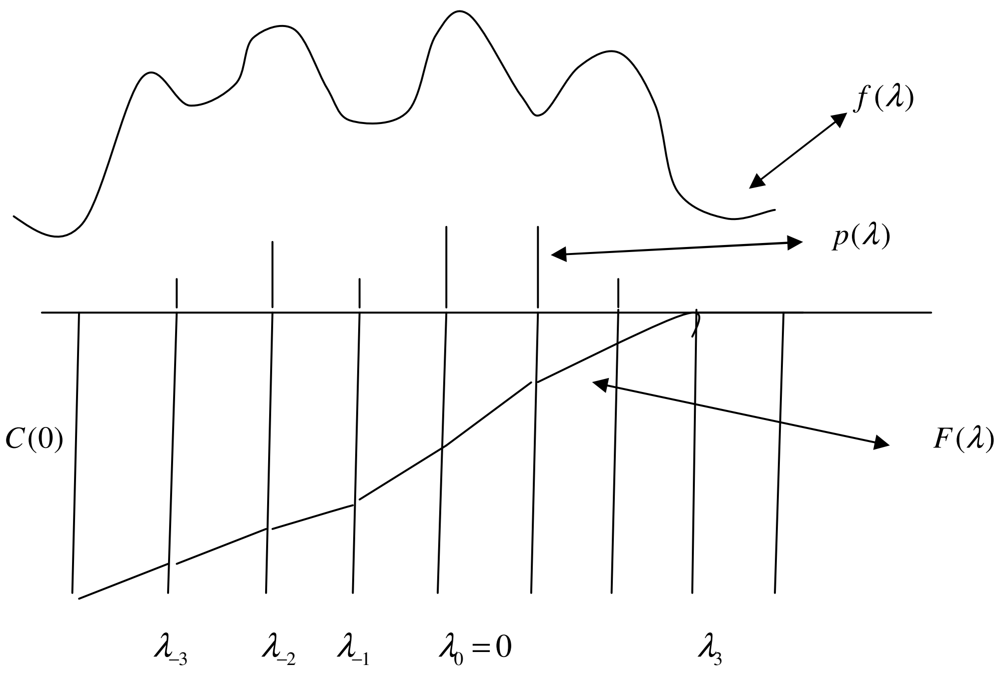
spectrum function can be obtained from auto covariance by the expression

$p(\lambda_k) = \lim_{t\to\infty} \frac{1}{2} \int_{-T}^{T} C(\tau) e^{-i\lambda_k \tau} d\tau$,expression yields $p(\lambda)$ for all λ and $F_d(A)$ can be

obtained. For continuous spectrum $C(\tau) = \int_{-\infty}^{\infty} e^{i\lambda\tau} f(\lambda) d\lambda$ is valid and $\int_{-\infty}^{\infty} f(\lambda) d\lambda = C(0) < \infty$

The auto covariance and spectrum of an almost periodic function

Let $Xt = \sum_{j=-\infty}^{\infty} C_j e^{i\lambda t}$ be an almost periodic function with $\sum_{j=-\infty}^{\infty} | C_j |^2 < \infty$



7.7.4 Univariate Spectral Models

Using the properties of inner product and orthonormality of functions $e^{i\lambda it}$. We can calculate the auto covariance functions for time series

$$C(\tau) = < x(t+\tau), x(t) > = < \sum_{j=-\infty}^{\infty} c_j e^{i\lambda_j \tau} e^{i\lambda_j t}, \sum_{k=-\infty}^{\infty} c_k e^{i\lambda_k \tau} >$$

$$= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_j \bar{c}_k e^{i\lambda_j k} < e^{i\lambda_j t}, e^{i\lambda_k t} > = \sum_{j=-\infty}^{\infty} | c_j |^2 e^{i\lambda_j t}$$

$$\Rightarrow P(\lambda) = \begin{cases} | c_j |^2 \; ....for...\lambda = \lambda_j ... j = 0, \pm 1, ... \\ 0; ....otherwise \end{cases} \quad \text{and} \quad C(0) = \sum_{j=-\infty}^{\infty} | c_j |^2$$

In practice spectral analysis imposes smoothing techniques on the period gram with certain assumptions .We can also create confidence interval to estimate the peak frequency regions. Spectral analysis can also be used to examine the association between two different time series.

RECOMMEDATION

To show further application of lebesgue integration in

(i) R^n -spaces and stokes and green theorems.

(ii) Statistical methods such discrete

and continuous solutions of expectations

(iii)In Time Series Analysis Solutions

CONCLUSION

This study describes the Extensions of Riemann theory of integrations, first to Riemann

Stieltjes integration, then to the most notable extensions, 'The Lebesgue Theory Of Integration.

As a result we are able to solve the discontinuous functions, such as step-functions, recover f(t) from

F'(t),and calculate areas covered by continuous functions with increased limits e.g R^n spaces.

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