"OPERATOR EQUATIONS IN HILBERT SPACES"

A thesis submitted to the Faculty of Science, University of Nairobi in fulfillment for the award of the degree of Doctor of Philosophy in Mathematics.

By

Jairus Mutekhele Khalagai.

JULY 1983.

This Thesis is my original work and has not been presented for a degree in any other University.

J.M. KHALAGAI.

This Thesis has been submitted for examination with my approval as University Supervisor.

I.H. SHETH.

ACKNOWLEDGEMENTS.

My sincere thanks are to my supervisor, Dr. B.P. Duggal who started supervising me at M.Sc. level. He is the man who introduced me to real research work in Operator Theory. My thanks are also due to Dr. I.H. Sheth who not only took over and accomplished what was left by Dr. B.P. Duggal, but also widened my scope of research in Operator Theory by giving me relevant research advice which helped me in avoiding fallacious statements in this Thesis.

I am also indebted to my wife, Meave Miriam Nasimiyu, who gave me moral support during the protracted period of research work for this Thesis.

Finally, I thank the University of Nairobi for offering me the Tutorial Fellowship which enabled me to carry out the research, and Mrs. Mary Maina for her good job in typing this Thesis.

J.M. KHALAGAI

C O N T E N T S.

CHAPTER 0.

INTRODUCTION.

This Thesis is devoted to a study of three operator equations in a complex Hilbert space. The three operator equations are as follows:

1. $AH = KA$,
2. $AB + BA^* = A^*B + BA = I$,
3. $TST^* = S$.

We give below an account of the work done by several authors as far as these operator equations are concerned and also a brief chapterwise summary of the work done by us.

In the case of the operator equation $AH = KA$, a number of authors have based their work mainly on sufficient conditions under which H and K either belong to the same class of operators or happen to be equal. Some of the authors who have carried out this study are : M.R. Embry [12]*, B.P. Duggal [6], R. Nakamoto [20] and J.M. Patel [21] . We note that M.R. Embry [12] proved equality of H and K by requiring that H and K are commuting normal operators and zero does not belong to the interior of the numerical range of A. B.P. Duggal [6] generalised this result by only claiming normality of K and setting H to be of the form $H_n = A^*K^n A$, for all non-negative integer n. R. Nakamoto [20] generalised this result further by showing that it is enough to prove for $n = 0,1$ only.

* The number in the bracket denotes the number of the paper listed in the referance.

Introducing an extra condition that if also $AK = HA$, J.M. Patel [21] was able to prove equality of H and K under some conditions on A which we are going to improve in this Thesis.

The operator equation $AB + BA^* = A^*B + BA = I$, has been also studied by a number of authors. Firstly, the case in which the Hilbert space is finite-dimensional, the possibility of interest in those A and Hermitian B satisfying the equation was mentioned by Taussky in [30] . Barker [2] showed that: (i) If the number of eigen values of the matrix A with zero real part is zero, then A is normal if and only if there is a Hermitian matrix B such that the equation is satisfied and $AB = BA$, (ii) The solution A to the equation is normal if either $(A + A^*)$ and $-(A + A^*)$ have no common eigen-values, or B commutes with $(A + A^*)$. Kamei and Kato [14] extended Barker's result to the case in which the Hilbert space is infinite dimensional, and a number of other conditions guaranteeing the normality of solution A, have been given by Duggal and Khalagai in [10] and [11] . We carry on with this study in this Thesis.

The authors who have studied the operator equation $TST^*=S$ for unitary solutions in T, include S.K. Khasbardar and N.K. Thakare [18] who showed that if T and S satisfy the equation with T invertible and both T and T^{-1} are spectraloid then T is unitary provided zero does not belong to the closure of the numerical range of S. Singh and Mangla [29] proved that if T and S satisfy the equation with T invertible and S a cramped unitary operator, then T is unitary. Duggal [6] showed that for T and S satisfying the equation if T is an invertible normal operator and zero does not belong to the interior of the numerical range of S, then T is unitary. Patel and Sheth [23] proved that for T and S satisfying the equation, if T is left invertible and S is such that zero does not belong to the closure of its numerical range, then T is unitary provided it is either dominant or K-paranormal contraction. We improve this result and a few others in this Thesis.

CHAPTERWISE SUMMARY.

In the first chapter, we first improve one result due to J.M. Patel [21] and then exhibit a number of corollaries. Among the corollaries proved here, is one which apart from giving an alternative proof to the already known conditions, it also gives a few more conditions under which an operator B commutes with another operator A given that B commutes with A^2. We then prove an independent theorem which also gives sufficient conditions under which H = K. The chapter ends with a result which attempts to relax the commutativity condition on the operators H and K of M.R. Embry's result in [12] .

In the second chapter, we first show that for A and B satisfying the operator equation $AB + BA^* = A^*B + BA = I$, ReA and ReB are invertible. We then show that both A and A^* have no approximate proper values on the imaginary axis and zero cannot be a normal approximate propervalue for B. We deduce a number of corollaries from this result. We also prove a result to the effect that if λ is an approximate propervalue for A or A^*, then $\frac{1}{2Re\lambda}$ belongs to the closure of the numerical range of B, from which we also deduce a number of corollaries.

For sufficient conditions for existence of A or B satisfying the equation, we prove results similar to those of J.P. Williams [32] . For example, we show that if zero does not belong to the direct sum of the spectra of A and A^* or if the spectrum of A or equivalently that for some invertible operator T, the closure of the numerical range of $T^{-1}AT$, lies in the positive half complex plane, then there always exists a positive definite operator B such that $AB + BA^* = A^*B + BA = I$ provided B commutes with $A^* - A$. We also look at the uniqueness of solutions in which we are able to show that if either ReA and -ReA have no common spectra or zero does not belong to the interior of the numerical range of ReA, then the solution B to the equation is always unique.

The third chapter starts with application of our first result proved in chapter one. Here we give alternative proof for most of the already existing sufficient conditions for A satisfying the equation $AB + BA^* = A^*B + BA = I$, to be normal. We prove our second theorem which gives further conditions under which A is normal and deduce a number of corollaries. We also show that if B^n is normal for some even positive integer n, and B is invertible, then A is normal if and only if B^nA is normal. Giving an example, we are able to show that a similar result in which B^n is normal for some odd positive interger n is not possible. We are also able to answer the question raised by B.P. Duggal [9] by proving that a K-quasi-hyponormal operator A satisfying the equation is always normal. The chapter ends with a result giving sufficient conditions under which the solution B of the equation is self adjoint.

Here, we show among other conditions that if either ReA and -ReA have no common spectra or zero does not belong to the interior of the numerical range of ReA, then B is self adjoint.

In the fourth chapter, we first show that for a left invertible operator T satisfying the equation $TST^* = S$, T turns out to be invertible under very humble conditions on S, like S being invertible or right invertible. We then deduce a number of other results which give uniticity of T. We also improve on the result of Patel and Sheth [23], by proving that if T is a left invertible operator satisfying the equation and T is either dominant or K-paranormal, then T is unitary provided zero does not belong to the closure of the numerical range of either ...r ReS or ImS. The chapter ends with improvement of ...of J.M. Patel [22], S.M. Patel [24] and Singh ... [29] respectively.

...INITIONS.

... Thesis, G will denote a complex Hilbert space ...ner product function denoted by (. , .). B(G) ...ote the Banach algebra of all bounded linear ...tors on G. The elements of G will be denoted by ...ital letters such as A, B, H, K etc.

For $A \in B(G)$, $\rho(A)$ will denote the resolvent set of A i.e. the set of all λ for which $A - \lambda I$ is invertible. $\sigma(A)$ will denote the spectrum of A. i.e. the compliment of the resolvent set of A. $\sigma_{\pi}(A)$ will denote the approximate spectrum of A. i.e. the set of all λ such that for any $\varepsilon > 0$, there exists an $x \in G$ such that $||x|| = 1$ and $||(\lambda - A)x|| < \varepsilon$. Such a λ is called an approximate proper value

$\sigma_\delta(A)$ will denote the approximate defect spectrum of A. i.e. the set of all λ such that $A - \lambda$ is not onto.

The numerical range of A will be demoted by W(A). i.e. $W(A) = \{(Ax,x): ||x|| = 1\}$. The closure of the numerical range of A will be denoted by $\overline{W(A)}$, and the null space of A will be denoted by N(A), while R(A) will denote the range of A.

Let $[A,B] = AB - BA$ and $\{A\}' = \{B \epsilon B(G): [A,B] = 0\}$.

Let G_+ denote the right half complex plane and $\pi^+ = \{z: Rez>0, Imz>0\}$ where z is a complex number.

An operator A is said to be:

normal if $AA^* = A^*A$,

unitary if $A^*A = AA^* = I$,

isometric if $A^*A = I$,

hyponormal if $A^*A \geq AA^*$,

M-hyponormal if $(A - z)(A - z)^* \leq M(A - z)^*(A - z)$ for all complex numbers z, and M some positive number,

k-paranormal if $||Ax||^k \leq ||A^kx||.||x||^{k-1}$ for all $x \epsilon G$ and $K \geq 2$,

k-quasi-hyponormal if $A^{*k}(A^*A - AA^*)A^k \geq 0$,

dominant if $R(A - \lambda) \subset R(A^* - \lambda^*)$ for each $\lambda \epsilon \sigma(A)$,

contraction if $||A|| < 1$,

satisfying growth condition G_1 if $||(A - \lambda I)^{-1}|| \leq \frac{1}{d(\lambda,\sigma(A))}$, $\lambda \notin \sigma(A)$,

left invertible if $A_1A = I$ for some $A_1 \epsilon B(G)$.

CHAPTER ONE

ON THE OPERATOR EQUATION $AH = KA$.

In this Chapter, we consider sufficient conditions under which $H = K$ and its consequences. We start by improving one result of J.M. Patel [21] . He proved the following result:

THEOREM 1.A. If $AH = KA$ and $AK = HA$ with A unitary, then $H = K$ under any one of the following conditions:

(i) $\sigma(A) \cap \sigma(-A) = \emptyset$,

(ii) $0 \notin W(A)$.

The following result shows that the condition on A can be relaxed.

THEOREM 1.1. [16]. Let $A, H, K \in B(G)$.

If $AH = KA$ and $AK = HA$, then $H = K$ under any one of the following conditions:

(i) $\sigma(A) \cap \sigma(-A) = \emptyset$,

(ii) A is normal and either $\sigma(ReA) \cap \sigma(-ReA) = \emptyset$ or

$\sigma(ImA) \cap \sigma(-ImA) = \emptyset$,

(iii) $\{A\}' = \{A^{2m}\}'$, for some positive integer m and A is one-one or has dense range,

(iv) H and K are normal and $0 \notin W(A)$.

For the proof of the above theorem, we first prove the following lemmas:

LEMMA 1.2. [16] . If $AH = KA$ and $AK = HA$, then $[K, A^2] = 0 = [H, A^2]$.

Proof. From $AH = KA$, we have

$AHA = KA^2$, while from

$AK = HA$ we have

$A^2K = AHA$. Hence $[K, A^2] = 0$.

Similarly,

LEMMA 1.3. Let $A \in B(G)$. If $0 \notin W(A)$, then A is one-one and has dense range.

Proof. Since $0 \notin W(A)$, $(Ax,x) \neq 0$ for $x \neq 0$. i.e.

$Ax \neq 0$ for $x \neq 0$.

A being linear, this implies that A is one-one. If $\overline{R(A)} \neq G$, then $\left[\overline{R(A)}\right]^{\perp} = N(A^*) \neq G^{\perp} = \{0\}$, i.e. there exists a unit vector $x \in N(A^*)$ such that:

$(A^*x,x) = 0 = (x,Ax)$ which implies that $0 \in W(A)$, a contradiction. Hence $\overline{R(A)} = G$.

We note that the following results will also be required for the proof of our theorem 1.1. Firstly, M.R.Embry [12] proved the following corollaries:

COROLLARY 1.B. Let $A,H,K \in B(G)$.

Let $D = \{A: 0 \notin W(A)$ or $\sigma(A) \cap \sigma(-A) = \emptyset\}$.

If $AH = KA$ and $A^*H = KA^*$, where $A \in D$, then $H = K$ if either A is unitary or H and K are normal.

COROLLARY 1.C. Let $A \in B(G)$. If A is normal, then $\{A\}' = \{A^2\}'$ provided $0 \notin W(A)$.

M. Rosemblum [26] proved the following result:

THEOREM 1.D. Let $A,X \in B(G)$. If we have that $AX + XA = 0$ and $\sigma(A) \cap \sigma(-A) = \emptyset$ then $X = 0$ is the only solution.

C.C. Cowan [4] , proved the following result:

COROLLARY 1.E. Let $A \in B(G)$. If $\sigma(A) \subset \pi^+$, then $\{A\}' = \{A^4\}'$.

We also state the following result due to Putman and Fuglede.

PUTNAM-FUGLEDE THEOREM. Let $A_1, A_2, B \in B(G)$. If $BA_1 = A_2B$, A_1 and A_2 normal, then $BA_1^* = A_2^* B$.

Proof of theorem 1.1.

(i) $AH = KA$ and $AK = HA$ together imply that:

$A(H - K) = (K - H) A,$

or

$A(H - K) + (H - K) A = 0.$

Since $\sigma(A) \cap \sigma(-A) = \emptyset$, $H - K = 0$ is the only solution of the equation (by theorem 1.0. above) and hence $H = K$.

(ii) If A is normal, then $-A$ is also normal. Since

$A(H - K) = (H - K)(-A)$, by Putnam Fuglede theorem,

$A^*(H - K) = (H - K)(-A^*)$.

i.e. $(A + A^*)(H - K) + (H - K)(A + A^*) = 0,$

or

$(\mathrm{Re}A)(H - K) + (H - K)(\mathrm{Re}A) = 0.$

Since $\sigma(\mathrm{Re}A) \cap \sigma(-\mathrm{Re}A) = \emptyset$, $H - K = 0$ is the only solution of the equation and hence $H = K$.

The case in which $\sigma(\mathrm{Im}A) \cap \sigma(-\mathrm{Im}A) = \emptyset$ is proved similarly.

(iii) By lemma 1.2 above, $[K, A^2] = 0$, and so $[K, A^{2m}] = 0$.

Since by hypothesis $[K, A] = 0$ if and only if $[K, A^{2m}] = 0$, $[K, A] = 0$. Now, $AK = KA = AH$ and A being one-one, $K = H$. Also $AK = KA = HA$ and range of A being dense, $K = H$.

(iv) If H and K are normal, then by Putnam Fuglede theorem, we have also

$$AK^* = H^*A.$$

Taking adjoints gives:

$$A^*H = KA^*.$$

Since $0 \notin W(A)$, H and K are normal, $AH = KA$ and $A^*H = KA^*$, by corollary 1.8 above, $H = K$.

COROLLARY 1.4. [16]. If $AH = KA$ and $AK = HA$, then $H = K$ under any one of the following conditions:

(i) A is normal and $0 \notin W(A)$.

Proof. A is normal and $0 \notin W(A)$ together imply that $\{A\}' = \{A^2\}'$ by corollary 1.C. Hence by part (iii) of theorem 1.1. above, the result follows.

(ii) A is normal and $0 \notin W(ReA)$, with a similar statement holding when ReA is replaced by ImA.

Proof. We only need to show that $0 \notin W(ReA)$ implies $0 \notin W(A)$ and the result will follow from part (i) of this corollary. We do this by contradiction. Suppose $0 \notin W(ReA)$ and $0 \in W(A)$ Now let $A = J + iH$. By hypothesis $0 \notin W(J)$. If $0 \in W(A)$, then there exists a unit vector $x \in G$ such that $(Ax,x) = 0$.

i.e. $(Jx,x) + i(Hx,x) = 0$.

i.e. $(Hx,x) = 0 = (Jx,x)$ a contradiction to the hypothesis.

Hence $0 \notin W(A)$. The case in which $0 \notin W(ImA)$ is proved similarly.

(iii). $(Ax,x) > 0$ for $x \neq 0$.

Proof. If $(Ax,x) > 0$ for $x \neq 0$, then $0 \notin W(A)$. Also, since A is positive, it is normal. Hence result follows from part (i) of this corollary.

(iv) $\sigma(A) \subset \pi^+$.

Proof. If $\sigma(A) \subset \pi^+$, then A is invertible and by corollary 1.E above, $\{A\}' = \{A^4\}'$. Hence we can now apply part (iii) of theorem 1.1.

(v) $0 \notin \overline{W(A)}$.

Proof. We only need to show that if $0 \notin \overline{W(A)}$ then $\sigma(A) \cap \sigma(-A) = \emptyset$ and the result will follow from part (i) of theorem 1.1. We do this by contradiction. If $\sigma(A) \cap \sigma(-A) \neq \emptyset$ and $\lambda \in \sigma(A) \cap \sigma(-A)$, then $\lambda, -\lambda \in \sigma(A) \subset \overline{W(A)}$. $\overline{W(A)}$ being a convex set, $0 \in \overline{W(A)}$, a contradiction. Hence $\sigma(A) \cap \sigma(-A) = \emptyset$.

We note here that the conditions in theorem 1.1. and corollary 1.4. above, are an improvement of J.M. Patel's theorem 1.A. above.

COROLLARY 1.5. [28] . If A and T are operators such that $AT = T^{*-1}A$ and $AT^{*-1} = TA$, then T is unitary under any one of the conditions in theorem 1.1 with condition (iv) as T is normal and $0 \notin W(A)$ or under any one of those in corollary 1.4.

Proof. Set $H = T^{*-1}$ and $K = T$.

COROLLARY 1.6. [28] . If A and T are operators such that $AT = T^*A$ and $AT^* = TA$, then T is self adjoint under any one of the conditions in theorem 1.1. with condition (iv) as T is normal and $0 \notin W(A)$ or under any of those in corollary 1.4.

Proof. Set $H = T^*$ and $K = T$.

COROLLARY 1.7. [28] . Let A and B be any operators such that $[B, A^2] = 0$. Then $[B, A] = 0$ under any one of the following conditions:

(i) $\sigma(A) \cap \sigma(-A) = \emptyset$,

(ii) A is normal and either $\sigma(ReA) \cap \sigma(-ReA) = \emptyset$ or $\sigma(ImA) \cap \sigma(-ImA) = \emptyset$,

(iii) $\{A\}' = \{A^{2m}\}'$, for some positive integer m.

(iv) AB and BA are normal and $0 \notin W(A)$,

(v) A is normal and $0 \notin W(A)$.

(vi) A is normal and either $0 \notin W(\mathrm{Re}A)$ or $0 \notin W(\mathrm{Im}A)$,

(vii) $(Ax,x) > 0$ for $x \neq 0$,

(viii) $\sigma(A) \subset \pi^{+}$,

(ix) $0 \notin \overline{W(A)}$.

Proof. Since $[B,A^2] = 0$, we have:

$$A(AB) = (BA)A$$

and

$$A(BA) = (AB)A.$$

Now set $H = AB$ and $K = BA$, then each of the above conditions implies that $H = K$ by theorem 1.1. and corollary 1.4. above. Hence $[B,A] = 0$.

COROLLARY 1.8. [28] . If $AH = KA$ and $[K,A^2] = 0$, then $H = K$ under any one of the following conditions:

(i) $\sigma(A) \cap \sigma(-A) = \emptyset$,

(ii) $\{A\}' = \{A^{2m}\}'$, for some positive integer m, and A is one-one,

(iii) A is normal and $0 \notin W(A)$,

(iv) A is normal and either $0 \notin W(\mathrm{Re}A)$ or $0 \notin W(\mathrm{Im}A)$,

(v) A is normal and either $\sigma(\mathrm{Re}A) \cap \sigma(-\mathrm{Re}A) = \emptyset$ or $\sigma(\mathrm{Im}A) \cap \sigma(-\mathrm{Im}A) = \emptyset$,

(vi) $(Ax,x) > 0$ for $x \neq 0$,

(vii) $\sigma(A) \subset \pi^{+}$

(viii) $0 \notin \overline{W(A)}$.

Proof. Since $[K,A^2] = 0$, each of the above conditions implies $[K,A] = 0$. Now, $AH = KA = AK$ gives us $H = K$, since A is one-one by each of the above conditions. For the sake of completeness we give here proof of part (v). We give the proof here to show that $\sigma(\mathrm{Re}A) \cap \sigma(-\mathrm{Re}A) = \emptyset$ implies A is one-one. We do this by contradiction. If A is not one-one then $Ax = 0$ for some $x \neq 0$ in G.

A being normal, $A^*x = 0$. Hence $(A+A^*)x = 0$, which implies that $0 \in \sigma(\text{Re}A) \cap \sigma(-\text{Re}A)$, a contradiction. Hence A is one-one.

COROLLARY 1.9. [28] . If A and T are operators such that $AT = T^*A$ and $[T^*, A^2] = 0$, then T is self adjoint under any one of the conditions in corollary 1.8 above.

Proof. Set $H = T$ and $K = T^*$ and apply corollary 1.8.

COROLLARY 1.10. [28] . If A and T are operators such that $AT^{*-1} = TA$ and $[T, A^2] = 0$, then T is unitary under any one of the conditions in corollary 1.8 above.

Proof. Set $H = T^{*-1}$ and $K = T$ then apply corollary 1.8.

COROLLARY 1.11. Let $AH = KA$ and $AK = HA$, then $H = K$ if $0 \notin W(A)$ and $\dim G < \infty$.

Proof. If $\dim G < \infty$, then $W(A)$ is closed. Hence $W(A) = \overline{W(A)}$, and the result follows from part (v) of corollary 1.4.

COROLLARY 1.12 . [28] . If H and K are self adjoint operators such that $HA = AK$ with $A = UP$ invertible and $[H, U] = 0$, then $H = K$.

Proof. $HA = HUP = UHP = UPK$.

i.e. $HP = PK$.

Taking adjoints, we have:

$$PH = KP$$

i.e. $PH = KP$ and $HP = PK$.

Since $0 \notin \overline{W(P)}$, the result follows from part (v) of corollary 1.4 above.

We will require the following result of M.R. Embry [12] for the proof of our next result.

COROLLARY 1.F. Let A and E be operators such that $AE = -EA$, where either A or E is normal and either $0 \notin W(A)$ or $\sigma(A) \cap \sigma(-A) = \emptyset$, then $E = 0$.

In the following result, we assume normality of $H - K$ to get equality of H and K.

THEOREM 1.13. [28] Let A, H and K be operators such that $AH = KA$ and $AK = HA$ with $A = UP$. If $H - K$ is normal, then $H = K$ under any one of the following conditions:

(i) $0 \notin W(A)$,

(ii) $0 \notin W(U)$ and A is invertible,

(iii) $\sigma(U) \cap \sigma(-U) = \emptyset$ and A is invertible.

Proof. (ii) and (iii): A being invertible, U is unitary and P is an invertible positive operator. Now, $AH = KA$ gives us

$$UPH = KUP$$

or

$$PH = U^*KUP.$$

Also,

$$A^*AK = A^*HA \text{ gives us}$$

$$P^2K = PU^*HUP,$$

or

$$PK = U^*HUP$$

Hence, from $PH = U^*KUP$ and $PK = U^*HUP$, we have:

$$P(H - K) = U^*(K-H)\,UP.$$

$H - K$ being normal, $U^*(K - H)\,U$ is also normal, and so by Putnam Fuglede theorem, we have:

$$P(H - K)^* = U^*(K - H)^*\,UP.$$

Taking adjoints, we have:

$$(H-K)\,P = PU^*(K-H)\,U.$$

The equations:

$$(H-K)P = PU^*(K-H)U$$

and

$$P(H-K) = U^*(K-H)UP,$$

with $0 \notin \overline{W(F)}$, give us

$$H - K = U^*(K-H)\, U,$$

by part (v) of corollary 1.4.

i.e. $U(H - K) = -(H - K)\, U.$

Each of conditions (ii) and (iii) above now implies that $H - K = 0$ by corollary 1.F above. Hence $H = K$.

Part (i) is seen thus:

Since $AH = KA$ and $AK = HA$, we have:

$$A(H - K) = -(H - K)\, A.$$

Letting $E = H - K$; we get that $AE = -EA$.

Hence, by corollary 1.F again, $E = 0$, or $H - K = 0$. i.e. $H = K$.

In the sequel, we make an attempt to improve the following result of Embry [12] .

THEOREM 1.G. If H and K are commuting normal operators such that

$$AH = KA,$$

with $0 \notin W(A)$, then $H = K$.

We first prove the following result:

THEOREM 1.14. [28] . Let A, H and K be operators such that:

$$HA = AK \text{ and } H^*A = AK^*$$

with $0 \notin W(A)$, then $H = K$ if $[ReH, ReK] = 0$ and $[ImH, ImK] = 0$.

Proof. From $HA = AK$ and $H^*A = AK^*$, we have:

$$A(K + K^*) = (H+H^*)A$$

and

$$A(K - K^*) = (H - H^*)\, A.$$

i.e. $A(ReK) = (ReH)A$

and

$$A(ImK) = (ImH)A.$$

The proof of theorem 1.G can now be traced to give $ReK = ReH$ and $ImK = ImH$. i.e $H = K$.

REMARK. We note that in general, the conditions $[H,K] = 0$ $[ReH, ReK] = 0$ and $[ImH, ImK] = 0$ are independent. However, if H and K are normal, it can easily be shown that $[H,K] = 0$ implies $[ReH, ReK] = 0$ and $[ImH, ImK] = 0$. Hence the following corollary attempts to relax commutativity of H and K in theorem 1.6. above.

COROLLARY 1.15. [28]. Let H and K be normal operators such that

$$AH = KA,$$

then $H = K$ if $0 \notin W(A)$, $[ReH, ReK] = 0$ and $[ImH, ImK] = 0$.

Proof. Since H and K are normal, by Putnam-Fuglede theorem, we have:

$$AH^* = K^*A.$$

Hence result follows from theorem 1.14. above.

CHAPTER TWO

THE OPERATOR EQUATION $AB + BA^* = A^*B + BA = I$.

NECESSARY AND SUFFICIENT CONDITIONS FOR EXISTENCE OF A OR B.

In this chapter, we consider some necessary and sufficient conditions for the existence of A or B such that:

$$AB + BA^* = A^*B + BA = I \tag{1}$$

NECESSARY CONDITIONS.

We need the following result of C. Davis and P. Rosenthal [5], to prove our first result.

THEOREM 2.A. If $A \in B(G)$, then we have:

(i) $\sigma_\pi(A) = (\sigma_\delta(A^*))^*$,

(ii) $\sigma_\delta(A) = (\sigma_\pi(A^*))^*$

where $*$ denotes complex conjugation.

We now prove the following result:

THEOREM 2.1. [11] . Let $A,B \in B(G)$, then we have that:

(i) If $AB + BA^* = I$ has a solution B, then $0 \notin \sigma_\delta(A)$ and $0 \in \rho(\text{Re}B)$. Furthermore, $\|(\text{Re}B)^{-1}\| \leq 2\|A\|$.

(ii) If equation (1) has a solution B, then $0 \in \rho(\text{Re}A)$.

Proof. (i) Suppose that $0 \in \sigma_\delta(A)$. Then by theorem 2.A above, $0 \in \sigma_\pi(A^*)$ and so there exists a sequence of unit vectors $\{x_n\} \in G$, such that $A^*x_n \to 0$ as $n \to \infty$. Now $AB + BA^* = I$ gives us:

$$1 = (x_n, x_n) = (ABx_n, x_n) + (BA^*x_n, x_n) = (Bx_n, A^*x_n) + (A^*x_n, B^*x_n) \to 0.$$

This is a contradiction, hence $0 \notin \sigma_\delta(A)$. Since $AB + BA^* = I$, $AB^* + B^*A^* = I$ and so $A(\text{Re}B) + (\text{Re}B)A^* = I$. An argument similar to that above now shows that $0 \notin \sigma_\delta(\text{Re}B)$. But $\text{Re}B$ being self adjoin

$\sigma_\delta(\text{ReB}) = \sigma(\text{ReB})$. Hence $0 \in \rho(\text{ReB})$. i.e. ReB is invertible.

Now, to complete the proof, we note that for any $x \in G$,

$$||x||^2 = ((\text{ReB})x,\ A^*x) + (A^*x,\ (\text{ReB})x) \leq 2||(\text{ReB})x||\,||A^*x||.$$

Hence,

$$||(\text{ReB})^{-1}|| \leq 2||A||.$$

(ii) Proceeding as in (i), we have in this case that if $0 \in \sigma_\delta(\text{ReA})$, then

$$2 = ((AB + BA^* + A^*B + BA)x_n,\ x_n)$$

$$= 2\{((\text{ReA})Bx_n,\ x_n) + (B(\text{ReA})x_n,\ x_n)\} \to 0.$$

The contradiction implies that $0 \notin \sigma_\delta(\text{ReA})$, and so, since ReA is self adjoint, $0 \in \rho(\text{ReA})$.

We now prove the following result:

<u>THEOREM 2.2.</u> [28] . If there exist solutions A and B to (1), then

(i) A and A^* have no approximate proper value on the imaginary axis,

(ii) 0 cannot be a normal approximate proper value for B.

<u>Proof.</u> (i) Suppose $\lambda = i\mu, \mu$ real, is an approximate proper value for A. Then for a sequence $\{x_n\}$ of unit vectors, we have:

$$Ax_n - \lambda x_n \to 0.$$

Now, $(A - \lambda I)^* B + B(A - \lambda I) = (A^* + i\mu) B + B(A - i\mu) = A^*B + BA = I$.

Hence, we have:

$$1 = (x_n,\ x_n) = ((A - \lambda I)^* Bx_n,\ x_n) + (B(A - \lambda I)x_n,\ x_n)$$

$$= (Bx_n,\ (A - \lambda I)x_n) + (B(A - \lambda I)x_n,\ x_n) \to 0 \quad \text{as} \quad n \to \infty,$$

a contradiction. The case for A^* is proved similarly using $AB + BA^* = I$.

(ii) If 0 is a normal approximate proper value for B, then

$$Bx_n \to 0 \quad \text{and} \quad B^*x_n \to 0, \quad \text{for a sequence } \{x_n\} \text{ of unit vectors.}$$

Hence, we have that:

$$1 = (x_n,\ x_n) = (A^*Bx_n,\ x_n) + (BAx_n,\ x_n) = (A^*Bx_n,\ x_n) + (Ax_n, B^*x_n)$$

COROLLARY 2.3. [28] . If there exists a solution A to (1), then $\sigma(A)$ does not meet the imaginary axis. In particular $0 \notin \sigma(A)$.

Proof. Since for any operator A,

$\sigma(A) \subset \sigma_\pi(A) \cup (\sigma_\pi(A^*))^*$ (with * denoting the complex conjugate), and by theorem 2.2. above, $\sigma_\pi(A)$ and $\sigma_\pi(A^*)$ do not meet the imaginary axis, it follows that $(\sigma_\pi(A^*))^*$ does not meet the imaginary axis either. Hence $\sigma(A)$ does not meet the imaginary axis.

REMARK. Using the equation:

$||x||^2 = ((\mathrm{Re}B)x, A^*x) + (A^*x, \mathrm{Re}Bx)$ of theorem 2.1., we get

$||x||^2 \leq 2||\mathrm{Re}Bx||.||A^*x||.$

As A^* is invertible, this gives us $||A^{*-1}|| = ||A^{-1}|| \leq 2||\mathrm{Re}B||$.

COROLLARY 2.4. [28] . If there exists a normal solution B to (1), then $0 \notin \sigma(B)$.

Proof. For a normal operator B, every $\lambda \in \sigma(B)$ is a normal approximate proper value. Hence by part (ii) of theorem 2.2 above, $0 \notin \sigma(B)$.

THEOREM 2.5. [28]. If there exists a solution A to (1) with $\lambda = \alpha + i\beta$, $\alpha \neq 0$ as an approximate proper value of A or A^* then

$\frac{1}{2\alpha} \in \overline{W(B)}$.

Proof. Suppose $\lambda \in \sigma_\pi(A)$. Here $(A - \lambda I)x_n \to 0$ for some sequence $\{x_n\}$ of unit vectors.

Since $(A - \lambda I)^* B + B(A - \lambda I) = (A^* - \bar{\lambda})B + BA - \lambda B = A^*B + BA - (\lambda + \bar{\lambda})B$.

$= I - (\lambda + \bar{\lambda})B$

$= I - 2\alpha B$.

We have :

THEOREM 2.C. Suppose $0 \notin \sigma(A) + \sigma(A^*)$. Then we have:

(i) For every operator Y, there exists a unique operator X with

$$AX + XA^* = Y,$$

(ii) X is self adjoint if Y is self adjoint.

(iii) If Y is positive and invertible, then X is also invertible.

THEOREM 2.8. [10] . If $0 \notin \sigma(A) + \sigma(A^*)$, then there exists a positive definite operator B such that $AB + BA^* = I$. If also $[B, (A^* - A)] = 0$, then A and B satisfy equation (1).

Proof. Since I is positive and invertible, by theorem 2.C, there exists a positive invertible operator B such that $AB + BA^* = I$. Now, if $[B, (A^* - A)] = 0$, then $B(A^* - A) = (A^* - A)B$.
i.e $AB + BA^* = A^*B + BA$.

REMARK . We note here that in view of theorem 2.B above, the hypothesis $0 \notin \sigma(A) + \sigma(A^*)$ can be replaced by the hypothesis that $\sigma(A)$ or equivalently that for some invertible operator T, the closure of the numerical range of $T^{-1}AT$ lies in the positive half complex plane.

We now note that Phadke and Thakare [25] , proved the following result:

COROLLARY2.D. For an M-hyponormal operator A, $\sigma_{\pi}(A^*) = \sigma(A^*)$.
In view of corollary 2.D. above, the following result can be derived.

COROLLARY 2.9. [11] . If A is a M-hyponormal operator with $\sigma_{\pi}(A^*) \subset C_+$. then there exists a positive definite solution B such that $AB + BA^* = I$.

Proof. Since A is M-hyponormal, $\sigma_\pi(A^*) = \sigma(A^*)$ by corollary 2.D above. Since $\sigma(A^*) \subset G_+$ implies $\sigma(A) \subset G_+$, it follows from theorem 2.8. and the remark above that there exists a positive definite solution B such that $AB + BA^* = I$.

In view of corollary 2.6 and theorem 2.8 above, the following corollary is immediate.

COROLLARY 2.10. [28] . There exists a solution $B \geq 0$ to (1) if and only if $\sigma(A) \subset G_+$ and $[B, (A^* - A)] = 0$.

UNIQUENESS OF SOLUTION.

Assuming that solutions B to (1) exist, we consider now the problem of the uniqueness of these solutions. An important role here is played by the homogeneous form:

$$AY + YA^* = 0, \quad A^*Y + YA = 0 \tag{2}$$

of equation (1).

We first prove the following lemmas:

LEMMA 2.11. [11] . $Y = 0$ is the only solution of equations (2) if any one of the following conditions is satisfied.

(i) $0 \notin W(ReA)$,

(ii) $\sigma(ReA) \cap \sigma(-ReA) = \emptyset$.

Proof. Clearly, $Y = 0$ is a solution of (2). Let U be another solution of (2). Then we have:

$$AU + UA^* = 0 = A^*U + UA$$

and hence,

$$(ReA)U = -U(ReA).$$

Now, since ReA is normal, each of the conditions (i) and (ii) implies that $U = 0$ by corollary 1.r.

LEMMA 2.12. [11]. If $Y = 0$ is the only solution of (2), then the solution B of (1) is unique.

Proof. Suppose that B_1 and B_2 are two distinct solutions of (1). Then we have:

$$A(B_1 - B_2) + (B_1 - B_2)A^* = 0 = A^*(B_1 - B_2) + (B_1 - B_2)A.$$

But this implies that (2) has a non-zero solution $B_1 - B_2$, contrary to our hypothesis. Hence $B_1 = B_2$.

We note that combining lemmas 2.11 and 2.12, we have the following theorem for the unique solution B of (1).

THEOREM 2.13. [11]. The solution B of (1) is unique if any one of the following conditions is satisfied:

(i) $0 \notin W(\mathrm{Re}A)$,

(ii) $\sigma(\mathrm{Re}A) \cap \sigma(-\mathrm{Re}A) = \emptyset$.

We now show that if solutions Y to (2) are of a certain type, then the solutions B to (1) are unique in as much as they are self adjoint.

THEOREM 2.14. [11]. If for each solution Y of (2) the unique positive square root of Y^*Y is also a solution of (2), then the solutions B of (1) are self adjoint.

Proof. Let B be a solution of (1). Then we set $Y = B^* - B$. Since $A(B^* - B) + (B^* - B)A^* = 0 = (B^* - B)A + A^*(B^* - B)$, it is clear that A and Y satisfy equations (2). Hence we have:

$$AY^2 = -YA^*Y = Y^2A.$$

Since $Y^*Y = -Y^2 \geq 0$, there exists a unique $R \geq 0$ such that $R^2 = -Y^2$. Now since $[A, -Y^2] = 0$, $[A, R] = 0$. By hypothesi R is also a solution of (2). We now have:

$AR + RA^* = (A + A^*)R = 0$, and hence that range of $R \subseteq \text{Ker}(\text{Re}A)$. But $\text{Re}A$ being invertible, $\text{Ker}(\text{Re}A) = \{0\}$. Hence $R = 0$, and so, since $R^2 = Y^*Y$, $Y = 0$. This completes the proof.

CHAPTER THREE

OPERATOR EQUATION $AB + BA^* = A^*B + BA = I$. NORMAL SOLUTIONS.

In this chapter, we assume that there exist operators A and B such that the equation $AB + BA^* = A^*B + BA = I$ of which we shall still refer to as equation (1) is satisfied. We concern ourselves with the problem of finding sufficient conditions such that A or B is normal.

Firstly, let A and B satisfy equation (1), then we have:

$$BA^*A = (I - AB)A = A(I - BA) = AA^*B. \tag{3}$$

Similarly,

$$BAA^* = A^*AB. \tag{4}$$

We also have that if equation (1) is satisfied then:

$$B^2A = B(BA) = B(I - A^*B) = B - BA^*B = B - (I - AB)B = AB^2,$$

and

$$B^2A^* = B(BA^*) = B(I - AB) = B - BAB = B - (I - A^*B)B = A^*B^2.$$

Hence, we have:

$$[B^2, A] = 0, \quad [B^2, A^*] = 0. \tag{5}$$

Also from $AB + BA^* = I$ and $A^*B + BA = I$, we get

$$AB^* + B^*A^* = I \text{ and } A^*B^* + B^*A = I.$$

i.e. We have that:

$$A(\mathrm{Re}B) + (\mathrm{Re}B)A^* = I \text{ and } A^*(\mathrm{Re}B) + (\mathrm{Re}B)A = I.$$

In this case, letting $T = \mathrm{Re}B$, we see that T satisfies equation (1) hence as in (3) and (4) we have:

$$TA^*A = AA^*T \tag{6}$$

and

$$TAA^* = A^*AT \tag{7}$$

We also note that as already mentioned in the proof of theorem 2.13, a simple manipulation gives us:

$$A(B^* - B) + (B^* - B)A^* = 0 = (B^* - B) A + A^* (B^*-B).$$

Hence letting $W = \mathrm{Im}B$, we have:

$$AW = W(-A^*),\ A^*W = W(-A). \qquad (8).$$

Now, for the sake of convinience in this chapter, let

$A = X + iJ$ and $B = T + iW$ be the cartesian decomposition of A and B respectively. Also let $A = UP$, $B = VQ$ and $T = SR$ be polar decomposition of A,B and T respectively.

NORMALITY OF THE SOLUTION A TO (1).

E. Kamei and Y. Kato [14] proved the following result:

THEOREM 3.A. If A and B satisfy (1), then A is normal under any one of the following conditions:

(i) $[X,B] = 0$,

(ii) $[B,A] = 0$,

(iii) $\sigma(X) \cap \sigma(-X) = \emptyset$.

We also note that B.P. Duggal [7], proved the following result:

THEOREM 3.B. The solution A to (1) is normal if any one of the following conditions is satisfied.

(i) $[B, BA] = 0$ and $0 \notin W(B)$,

(ii) $[B, BC] = 0$ and $0 \notin W(B)$, where $C = A^* + A$,

(iii) $[B,A^*A] = 0$ and $0 \notin W(B)$.

We improve this result as follows:

THEOREM 3.1 [17]. Let A and B satisfy (1), then A is normal under any one of the following conditions:

(i) $0 \notin W(B)$,

(ii) $\{B\}' = \{B^{2m}\}'$, for some positive integer m,

(iii) $\sigma(B) \cap \sigma(-B) = \emptyset$.

Proof. (i) From equations (3) and (4), set $H=A^*A$ and $K=AA^*$ to give:

$$BH = KB$$

and

$$BK = HB.$$

Since H and K are normal, by part (iv) of theorem 1.1, $H = K$. i.e. $A^*A = AA^*$ or A is normal.

(ii) We note that by equation (5), $[B^2, A] = 0$, hence $[B^{2m}, A] = 0$. Thus if $\{B\}' = \{B^{2m}\}'$, then $[B,A] = 0$ and so by theorem 3.A. above, A is normal.

(iii) We also note that since $[B^2,A] = 0$, the condition $\sigma(B) \cap \sigma(-B) = \emptyset$ implies $[B,A] = 0$ by corllary 1.7. and hence A is normal by theorem 3.A again.

REMARK. In view of part (i) of theorem 3.1. above, some conditions in theorem 3.B. are redundant. In fact, these conditions carry through under the weaker hypothesis that range of B is dense in G.

COROLLARY 3.2. [11]. The solution A to (1) is normal under any one of the following conditions:

(i) $0 \notin W(T)$,

(ii) $\{T\}' = \{T^{2m}\}'$, m some positive integer,

(iii) $\sigma(T) \cap \sigma(-T) = \emptyset$.

Proof. We note that T satisfies equation (1). i.e. $AT + TA^* = A^*T + TA = I$. Hence the result is immediate from theorem 3.1.

We now note that Berberian [3] and Kato and Moriya [15] proved the following results respectively.

THEOREM 3.C. For any operator B, $Re\sigma(B) = \sigma(ReB)$ if B belongs to any of the following classes of operators:

(i) B is hyponormal,

(ii) B satisfies the growth condition G_1 and $\sigma(B)$ is connected.

THEOREM 3.D. For any operator B. $Re\sigma(B) = \sigma(ReB)$ if $[B^*B, ReB] = 0$

We now have the following corollary.

COROLLARY 3.3. [11] . The solution A to (1) is normal under any one of the following conditions:

(i) $\sigma(T) \subset G_+$,

(ii) B is normal and $\sigma(B)$ lies strictly on one side of the origin,

(iii) B is hyponormal and $Re\sigma(B) \subset G_+$

(iv) $[B^*B, T] = 0$ and $Re\sigma(B) \subset G_+$,

(v) B satisfies the growth condition G_1, $Re\sigma(B) \subset G_+$ and $\sigma(B)$ is connected.

Proof (i) If $\sigma(T) \subset G_+$, then T is an invertible positive operator. As such $0 \notin W(T)$ and result follows from part (i) of corollary 3.2. above.

(ii) We note that if B is normal, then it is convexiod. Hence $\sigma(B)$ lies strictly on one side of the origin implies $0 \notin Con\sigma(B) = \overline{W(B)}$, and result follows from part (i) of theorem 3.1.

If either of the conditions (iii), - (v) holds, then by theorems 3.C. and 3.D, $Re\sigma(B) = \sigma(ReB) = \sigma(T)$. Hence the proof follows from case (i).

We remark here that the example $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} + ix \\ \frac{1}{2} + it & -\frac{i}{2} \end{pmatrix}$, $x \neq t$, and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ shows that the condition $0 \notin W(B)$ in theorem 3.1. cannot be replaced by the condition $0 \notin \sigma(B)$, i.e. the condtion $0 \notin \sigma(B)$ is not sufficient to guarantee normality of A in (1). In a few results that follow we give further hypotheses under which $0 \notin \sigma(B)$ would give normality of A.

THEOREM 3.4. [28] . Let A and B satisfy (1), with $0 \notin \sigma(B)$. Then A is normal under any one of the following conditions:

(i) $0 \notin W(V)$,

(ii) $\sigma(V) \cap \sigma(-V) = \emptyset$,

(iii) $[P,V] = 0$.

Proof. (i) From equations (3) and (4) set $H=A^*A$ and $K=AA^*$. Then $H - K = A^*A - AA^*$ is normal. Since $0 \notin \sigma(B)$, $0 \notin W(V)$, and by part (ii) of theorem 1.13, $H = K$. i.e. A is normal.

(ii) Similarly, $\sigma(V) \cap \sigma(-V) = \emptyset$, by part (iii) of theorem 1.13, implies $A^*A = AA^*$.

(iii) Since $[P,V] = 0$, $[P^2, V] = 0$. i.e. $[A^*A, V] = 0$. Now from equation (4) set $H = A^*A$ and $K = AA^*$. Since H and K are self adjoint, by corollary 1.12, $H = K$ or $A^*A = AA^*$ and so A is normal.

COROLLARY 3.5. [28] . The solution A to (1) is normal under any one of the following conditions:

(i) $0 \notin W(S)$,

(ii) $\sigma(S) \cap \sigma(-S) = \emptyset$,

(iii) $[P, S] = 0$.

Proof. Since T is invertible by theorem 2.1., and satisfies (1), the result follows immediately from theorem 3.4. above.

THEOREM 3.6. - [10] . Let B^n be normal for some even positive integer n. If $0 \notin \sigma(B)$, then the solution A to (1) is normal if and only if B^nA is normal.

Proof. We first note that if A and B satisfy (1), then by equation (5), $[B^2, A] = 0$, $[B^2, A^*] = 0$. Suppose that B^nA is normal. Then by the fact that $[B^2, A] = 0$, $[B^2, A^*] = 0$ and the normality of B^n, we have:

$$B^nAA^*B^{*n} = AA^*B^nB^{*n} = AA^*B^{*n}B^n = B^{*n}AA^*B^n.$$

Also $A^*B^{*n}AB^n = B^{*n}A^*AB^n$. Thus :

$$B^{*n}(AA^* - A^*A)B^n = 0.$$

This implies that A is normal. If on the other hand, A is normal, then

$$B^nAA^*B^{*n} = B^nA^*AB^{*n} = A^*B^nB^{*n}A = A^*B^{*n}B^nA,$$ i.e. B^nA is normal.

COROLLARY 3.7. The solution A to (1) is normal if and only if T^nA is normal for some even positive integer n.

Proof. We first recall that T satisfies equation (1), and is invertible. Now, since $T = ReB$, T is self adjoint and hence T^n is normal. The result now follows easily from theorem 3.6 above.

REMARK. Notice that if B^nA, n some even positive integer, is normal, then so also is AB^n. A result similar to theorem 3.6 for the case in which B^n is normal for some odd positive integer n is not possible. Indeed suppose that the hypotheses that B is an invertible operator such that B^n and B^nA, n some odd positive integer, are normal, are sufficient conditions to guarantee the normality of A. Let B, $0 \notin \sigma(B)$, be self-adjoint. Then, using equations (3) and (5), we have:

$$BAA^*B^* = BAA^*B = A^*AB^2 = A^*B^2A = A^*B^*BA,$$

so that BA is normal. Hence if our proposition above were true, we would have that "If B is a self adjoint invertible operator satisfying (1), then A is normal." This, however, is not true, as the following example shows.

Let $A = \begin{pmatrix} -b + ix & a + ix \\ a - iy & b + iy \end{pmatrix}$, where

$x \neq y$ and $a \neq b$ are non-zero real numbers, and let

$B = \begin{pmatrix} c & c \\ c & -c \end{pmatrix}$, where c satisfies $2c\,(a - b) = 1$. Then B is an invertible self adjoint operator which along with A satisfies (1). A however, is not normal.

It is now clear that to obtain a result analogous to theorem 3.6 for the case in which n is odd, some additional hypotheses are required. Here are some partial results.

<u>THEOREM 3.6.</u> [10] . Let B^n be normal for some positive odd integer n. Suppose that $0 \notin \sigma(B)$.

(i) If B^nA and $B^{n+1}A$ are normal, then the solution A to (1) is normal,

(ii) If B^nA and AB^n are normal, and if either $\sigma(BA)$ is real or $BA \geq A^*B$, then the solution A to (1) is self adjoint. (Here, as usual, $BA \geq A^*B$ is to be taken as meaning that $BA - A^*B$ is a positive operator.)

<u>Proof.</u> (i) Let B^nA be normal. Then by (1) and (5) we have:

$$A^*B^{*n}B^nA = B^nAA^*B^{*n} = BAB^{n-1}A^*B^{*n} = B^{n-1}A^*B^{*n} - A^*B^nA^*B^{*n}$$

$$= B^{n-1}A^*B^{*n} - A^*B^nB^{*n-1}$$

$$+ A^*B^nB^{*n}A,$$

so that by the normality of B^n, $B^{n-1}(A^*B^* - A^*B)\,B^{*n-1} = 0$.

This implies that $A^*B^* = A^*B$, or what is the same that:

$$BA = B^*A . \tag{9}$$

We note that from equation (4) we have, by taking adjoints that:

$$B^*A^*A = AA^*B^* \tag{10}$$

Now, (3), (9) and (10) together imply that:

$$AA^*B^* = AA^*B = BA^*A = B^*A^*A.$$

Since $B^{n+1}A$ is normal, we have:

$$\begin{aligned} B^{n+1} AA^*B^{*n+1} &= A^*B^{*n+1} B^{n+1}A = A^*AB^{*n+1} B^{n+1} \\ &= A^*AB^*B^n B^{*n} B = BAA^* B^n B^{*n}B \\ &= B^{n+1}B^{*n} AA^* B = B^{n+1} B^{*n} AA^* B^* \\ &= B^{n+1} A^*AB^{*n+1} , \end{aligned}$$

and hence that A is normal.

(ii) As before, it is seen that if AB^n is normal then $AB=AB^*$. (11)

Now, (3), (9), (10) and (11) imply that both AB and BA are normal. Set $BA = T_1$ and $AB = T_2$, then (1), (10) and (11) imply that:

$$ReT_1 = \frac{1}{2}(T_1 + T_1^*) = \frac{1}{2} I, \quad ReT_2 = \frac{1}{2} (T_2 + T_2^*) = \frac{1}{2} I. \tag{12}$$

Clearly, if $\sigma(BA)$ is real, then BA is self adjoint. We show that BA is self adjoint in the case in which $BA \geq A^*B$. A simple calculation shows that:

$$0 \leq (BA - A^*B)^2 = BA + A^*B - 4A^*AB^2 = I - 4A^*AB^2 = 4(ReT_1)^2 - 4T_1^*T_1,$$

so that

$$T_1^*T_1 \lesssim (ReT_1)^2 .$$

But this is possible if and only if T_1 is self adjoint, since

$$(BA - A^*B)^* (BA - A^*B) = -(BA - A^*B)^2 \geq 0, \text{ and } (BA-A^*B)^2 \geq 0.$$

Hence we have:

$$BA = A^*B = A^*B^*, \tag{13}$$

and so by (10) that

$$BA = A^*B = \tfrac{1}{2} I. \tag{14}$$

We now show that (11) and (13) together show that T_2 is self adjoint, which in view of (11) and (12) would imply that $T_2 = T_2^* = \frac{1}{2} I$, and hence by (14) (Since $0 \notin \sigma(B)$) that $A = A^*$. Since by (13), $AA^* B^2 = T_2^2 = T_2^{*2}$, we have that:

$T_2^* T_2 = AA^* B^2 = \frac{1}{4} (T_2^* + T_2)^2 = (ReT_2)^2$, and hence that T_2 is self adjoint. This completes the proof.

<u>COROLLARY 3.9.</u> The solution A to (1) is normal under any one of the following conditions:

(i) T^nA and $T^{n+1}A$ are normal,

(ii) T^nA and AT^n are normal and either $\sigma(TA)$ is real or $TA \geq A^*T$, where n is some positive odd integer.

<u>Proof.</u> As in the proof of corollary 3.7, the result easily follows from theorem 3.8.

The following result provides an answer to a remark made by B.P. Duggal [9] in which he claims that "We do not know whether the solution A to (1) is normal whenever A is K-quasihyponormal." But we first exhibit the following result he proved.

<u>THEOREM 3.E.</u> The solution A to (1) is normal if A is hyponormal or co-hyponormal.

<u>THEOREM 3.10.</u> [28] . If for a given B, there exists a solution A to (1), then A is normal under any one of the following conditions:

(i) A is isometric,

(ii) A is quasi-hyponormal

(iii) A is K-quasi-hyponormal.

Proof. Since by corollary 2.3 A is invertible, isometric condition implies A is unitary. We also note that an invertible K-quasi-hyponormal operator A is hyponormal, hence by theorem 3.E above A is normal.

We now recall that in their dedication to the memory of the late Prof. T. Saito, A. Kobayashi and T.Okayasu [19] proved the following result:

THEOREM 3.F. Let W be an operator with dense range and $W = V|W|$ be its polar decomposition. Let also S and T be operators such that:

$$TW = WS \text{ and } T^*W = WS^*; \text{ then } \pi : A \to VAV^* \text{ is a}$$

-homomorphism of the C-algebra, $C^*(S)$ generated by S onto $C^*(T)$ carrying S to T.

The following result of T. Ando [1] is also required for the proof of our next result.

THEOREM 3.G. If both A and A^* are paranormal and $N(A) = N(A^*)$, then A is normal.

We now prove the following result:

THEOREM 3.11. [28] . Let A and B satisfy (1). If $W = \operatorname{Im}B$ has dense range, then A is normal whenever it is paranormal.

Proof. Since A is invertible, $N(A) = N(A^*)$ and from equations (8), we have:

$$AW = W(-A^*)$$

and

$$A^*W = W(-A).$$

Now, by theorem 3.F. if A is paranormal, then A^* is also paranormal. Hence result follows from theorem 3.G.

NORMALITY OF THE SOLUTION B to (1).

We now give sufficient conditions under which the solution B to (1) is normal.

THEOREM 3.12. [16] . The solution B to (1) is normal under any one of the following conditions:

(i) $\sigma(X) \cap \sigma(-X) = \emptyset$,

(ii) $0 \notin W(X)$,

(iii) $\sigma(X)$ is contained on one side of the origin,

(iv) $\{X\}' = \{X^{2m}\}'$, for some positive integer m.

Proof. (i) and (ii). We first note that as derived in equations (8), if A and B satisfy (1) then we have:

$$(A^* + A)(B^* - B) = -(B^* - B)(A^* + A)$$

or

$$X(B^* - B) = -(B^* - B)X.$$

Hence each of the conditions (i) and (ii) implies that $B^* - B = 0$ by corollary 1.F, and so $B = B^*$

(iii) If $\sigma(X)$ is contained on one side of the origin, then $0 \notin \overline{W(X)}$ and hence result follows from part (ii).

(iv) Since $X(B^* - B) = -(B^* - B)X$, $X^2(B^* - B) = -X(B^*-B)X = (B^*-B)X^2$ and hence,

$X^{2m}(B^* - B) = (B^* - B)X^{2m}$ for any positive integer m.

Under the given conditions, we have that:

$(B^*-B)X = X(B^*-B) = -(B^*-B)X$,

X being invertible, this gives

$B^* - B = -(B^* - B)$ or $B = B^*$.

CHAPTER FOUR

THE OPERATOR EQUATION $TST^* = S$. UNITARY SOLUTIONS.

In this chapter, we consider bounded linear operators T and S on a Hilbert space G such that:

$$TST^* = S. \tag{15}$$

We first note that many authors have considered equation (15) under the conditions that T is invertible and $0 \notin \overline{W(S)}$. In our first result, we derive the invertibility of T by merely assuming left invertibility of T and various conditions on S.

THEOREM 4.1. [27] . Let T and S be operators satisfying equation (15), with T left invertible. Then T is invertible under any one of the following conditions:

(i) S is right invertible,

(ii) Either $\text{Re}S$ or $\text{Im}S$ is right invertible,

(iii) S is invertible,

(iv) Either $\text{Re}S$ or $\text{Im}S$ is invertible,

(v) $\sigma(S) \cap \sigma(-S) = \emptyset$,

(vi) Either $\sigma(\text{Re}S) \cap \sigma(-\text{Re}S) = \emptyset$ or $\sigma(\text{Im}S) \cap \sigma(-\text{Im}S) = \emptyset$,

(vii) Either $0 \notin \overline{W(S)}$ or $0 \notin \overline{W(\text{Re}S)}$ or $0 \notin \overline{W(\text{Im}S)}$.

Proof. (i) Let T_1 be a left inverse of T and S_r be a right inverse of S. Since $TST^* = S$, we have:

$$ST^* = T_1S$$

and

$$T(ST^*)S_r = I$$

i.e. $$T(T_1S)S_r = I$$

or

$$TT_1 = I.$$

i.e. T is invertible.

(ii) Since $TST^* = S$, taking adjoints, $T S^* T^* = S^*$.

Hence,

$$T(S + S^*)T^* = S + S^*$$

i.e. $$T(ReS)T^* = ReS, \quad (16)$$

and

$$T(S - S^*) T^* = S - S^*$$

i.e. $$T(ImS)T^* = ImS. \quad (17)$$

Now, applying part (i), we get invertibility of T.

Trivially (iii) implies (i), and (iv) implies (ii). We also note that (v) implies (iii), (vi) implies (iv) and (vii) implies (iii) or (iv).

The following corollary is immediate.

COROLLARY 4.2. [27] . Let T and S satisfy (15). If T is an isometry, then T is unitary under any one of the conditions in theorem 4.1. above.

Proof. T being an isometry, T is left invertible. Hence, each of the conditions in theorem 4.1. above, implies T is invertible and so is unitary.

The following corollary due to B.P. Duggal [6] will be required for the proof of our next result.

COROLLARY 4.A. If E is an invertible normal operator and if there is an operator T such that $0 \notin W(T)$ and $TE^* = E^{-1}T$, then E is unitary.

THEOREM 4.3. [27] . Let T and S satisfy (15), with T a left invertible normal operator, then T is unitary under any one of the following conditions:

(i) $\sigma(\mathrm{Re}S) \cap \sigma(-\mathrm{Re}S) = \emptyset$,

(ii) $\sigma(\mathrm{Im}S) \cap \sigma(-\mathrm{Im}S) = \emptyset$,

(iii) Either $0 \notin W(\mathrm{Re}S)$ or $0 \notin W(\mathrm{Im}S)$.

Proof. (i) Since T is left invertible, the condition $\sigma(\mathrm{Re}S) \cap \sigma(-\mathrm{Re}S) = \emptyset$ implies that T is invertible by part (vi) of theorem 4.1 above. Hence we have:

$(\mathrm{Re}S)T^* = T^{-1}(\mathrm{Re}S)$ from equation (16). Now T being normal, by Putnam Fuglede theorem $(\mathrm{Re}S)T = T^{*-1}(\mathrm{Re}S)$. By taking adjoints, we have:

$$(\mathrm{Re}S)T^{-1} = T^*(\mathrm{Re}S).$$

Now, $(\mathrm{Re}S)T^* = T^{-1}(\mathrm{Re}S)$

and

$$(\mathrm{Re}S)T^{-1} = T^*(\mathrm{Re}S)$$

imply $(\mathrm{Re}S)(T^* - T^{-1}) + (T^* - T^{-1})(\mathrm{Re}S) = 0$.

Since $\sigma(\mathrm{Re}S) \cap \sigma(-\mathrm{Re}S) = \emptyset$, $T^* - T^{-1} = 0$. i.e. $T^* = T^{-1}$ or T is unitary.

Part (ii) is proved similarly, while (iii) follows easily from corollary 4.A above.

We note that Duggal [6] , proved the followeing result:

THEOREM 4.B. If $0 \notin W(A)$, and for a positive integer n, we have (i) $[A^{*n}A^n, A^{*n+1}A^{n+1}] = 0$, (ii) $[A^*A, A^{*n}A^n] = 0$, then A^n is normal. Also if (iii) $[AA^*, A^{*n+1}A^{*n+1}] = 0$, then A is normal.

We now use the above theorem to prove the following result:

THEOREM 4.4. [27] . Let T and S satisfy (15) with T invertible. If $0 \notin W(T)$ and for some positive integer n we have that $[T^{*n}T^n, T^{*n+1}T^{n+1}] = 0$ and $[T^*T, T^{*n}T^n] = 0$, then T^n is unitary under any one of the following conditions:

(i) $0 \notin W(S)$,

(ii) $\sigma(S) \cap \sigma(-S) = \emptyset$,

(iii) Either $\sigma(\mathrm{Re}S) \cap \sigma(-\mathrm{Re}S) = \emptyset$ or $\sigma(\mathrm{Im}S) \cap \sigma(-\mathrm{Im}S) = \emptyset$,

(iv) Either $0 \notin W(\mathrm{Re}S)$ or $0 \notin W(\mathrm{Im}S)$.

Proof. The conditions $0 \notin W(T)$, $[T^{*n}T^n, T^{*n+1}T^{n+1}] = 0$ and $[T^*T, T^{*n}T^n] = 0$ together imply that T^n is normal by theorem 4.B above. Since $TST^* = S$ implies $T^nST^{*n} = S$, we have that:

$$ST^{*n} = T^{-n}S$$

and

$$S^*T^{*n} = T^{-n}S^*.$$

This gives us:

$$(\mathrm{Re}S)T^{*n} = T^{-n}(\mathrm{Re}S)$$

and

$$(\mathrm{Im}S)T^{*n} = T^{-n}(\mathrm{Im}S).$$

Hence by corollary 4.B, each of the conditions implies that

$$T^{*n} = T^{-n}, \quad \text{or} \quad T^n \text{ is unitary.}$$

In order to prove our next result, we need the following result of R. Nakamoto [20].

<u>THEOREM 4.C.</u> Let $H_n = A^*B^nA$, where B is normal. If (i) $0 \notin W(A)$, and (ii) $[B, H_n] = 0$ for $n = 0, 1$, then $[B, A] = 0$.

<u>THEOREM 4.5.</u> [27]. Let T and S satisfy (15) with T left invertible and S has dense range such that $[S, TT^*] = 0$ and $0 \notin W(T)$, then T is unitary.

<u>Proof.</u> Since $TST^* = S$, Set $H_0 = TT^*$ and $H_1 = TST^*$. Now by theorem 4.C above, $0 \notin W(T)$, $[H_0, S] = 0$ and $[H_1, S] = 0$ together imply that $[S, T^*] = 0$. The relation $TST^* = S$ now becomes $TT^*S = S$. S having dense range, this gives us $TT^* = I$. Thus, T being both left and right invertible, T is invertible with $T^{-1} = T^*$. i.e. T is unitary.

We now consider the following results proved by Patel and Sheth [23].

<u>THEOREM 4.D.</u> If T is an invertible K - paranormal operator and S is an operator such that:

$$TST^* = S \quad \text{and} \quad 0 \notin \overline{W(S)}, \text{ then } T \text{ is unitary.}$$

THEOREM 4.E. If T is a left invertible operator and S is such that

$$TST^* = S,$$

with $0 \notin \overline{W(S)}$, then T is unitary provided it is either dominant or K-paranormal contraction.

In the sequel, we show that in theorem 4.E, once T is K-paranormal, the condition that it is a contraction can be dropped.

THEOREM 4.6. [27] . Let T be a left invertible operator which together with S satisfy (15). If T is dominant or K-paranormal, then T is unitary under any one of the following conditions:

(i) $0 \notin \overline{W(S)}$,

(ii) $0 \notin \overline{W(ReS)}$,

(iii) $0 \notin \overline{W(ImS)}$.

Proof. (i) Since T is left invertible and $TST^* = S$, with $0 \notin \overline{W(S)}$, by theorem 4.1, T is invertible. Now if T is dominant or K-paranormal, then by theorem 4.D, T is unitary.

We note that each of the conditions (ii) and (iii) implies (i).

COROLLARY 4.7. [27]. If S is an invertible operator such that either S^*S^{-1} or SS^{*-1} is dominant or K-paranormal, then S is normal under any one of the following conditions:

(i) $0 \notin \overline{W(S)}$,

(ii) $0 \notin \overline{W(ReS)}$,

(iii) $0 \notin \overline{W(ImS)}$.

Proof. Set $T = S^*S^{-1}$, then $TS = S^*$.

i.e. $TST^* = S^*T^* = S$.

Now, each of the above conditions implies that T is unitary by theorem 4.6. above. Hence we have:

$SS^* = S^*T^*TS = S^*S$, since $T^*T = I$.

i.e. S is normal. Similarly, we can consider the other case.

In our few results that follow, we improve some results due to J.M. Patel [22] , S.M. Patel [24] and Singh and Mangla [29] . Firstly, J.M. Patel [22] proved the following results:

<u>COROLLARY 4.F.</u> If T is a paranormal operator and $T^p = T^{*q}$ where p and q are integers, then T is normal.

<u>THEOREM 4.G.</u> If $T^{*p} = U^{-1}T^pU$, p any positive integer with T paranormal and U a cramped unitary operator, then T is normal.

We relax the condition on U in theorem 4.G above. But first we prove the following result:

<u>THEOREM 4.8.</u> [27] . Let T and S satisfy equation (15) with T left invertible. If T is paranormal and there exists an integer n such that T^n is normal, then T is unitary under any one of the following conditions:

(i) $0 \notin W(S)$ and S is right invertible,

(ii) $\sigma(S) \cap \sigma(-S) = \emptyset$.

<u>Proof.</u> $TST^* = S$ with S at least right invertible and T left invertible, imply T is invertible by theorem 4.1 above. Also, $TST^* = S$ implies $T^nST^{*n} = S$ for any positive integer n. Hence we have:

$$ST^{*n} = T^{-n}S$$

and

$$S^*T^{*n} = T^{-n}S^*.$$

Setting $H = T^{*n}$ and $K = T^{-n}$ by corollary 1.B above, each of the conditions (i) and (ii) implies $H = K$. i.e. $T^{*n} = T^{-n}$ or

T^n is unitary. This implies that $\sigma(T^n)$ and consequently $\sigma(T)$ lies on the unit circle. Since T is paranormal, this gives us uniticity of T.

THEOREM 4.9. [27] . If for any positive integer p, $T^{*p} = U^{-1}T^pU$, with T paranormal and U an invertible operator, then T is normal under any one of the following conditions:

(i) $0 \notin W(U)$ and U is unitary,

(ii) $\sigma(U) \cap \sigma(-U) = \emptyset$ and U is unitary,

(iii) T^p is normal and either $0 \notin W(U)$ or $\sigma(U) \cap \sigma(-U) = \emptyset$.

Proof. $T^{*p} = U^{-1}T^pU$ gives:

$$UT^{*p} = T^pU$$

and

$$U^*T^{*p} = T^pU^*$$

Now, by corollary 1.B again, each of the above conditions gives:

$$T^{*p} = T^p.$$

Since T is paranormal, by corollary 4.F above, T is normal ;

S. M. Patel [24] proved the following result:

THEOREM 4.H. If for a non-singular operator E, there exists an operator A with either $0 \notin W(A)$ or $\sigma(A) \cap \sigma(-A) = \emptyset$ such that $AE = E^{-1}A$, where either E is normal or A is a non-singular normal operator, then $E^2 = I$.

We improve the above theorem in the following way:

THEOREM 4.10. [27] . If for a non-sigular operator E, there exists an operator A such that:

$$AE = E^{-1}A,$$

then $E^2 = I$ under any one of the following conditions:

(i) $\sigma(A) \cap \sigma(-A) = \emptyset$,

(ii) $0 \notin W(A)$ and A is normal,

(iii) $0 \notin W(A)$ and $E - E^{-1}$ is normal,

(iv) Either $0 \notin W(\text{Re}A)$ or $0 \notin W(\text{Im}A)$ and A is normal,

(v) Either $\sigma(\text{Re}A) \cap \sigma(-\text{Re}A) = \emptyset$ or $\sigma(\text{Im}A) \cap \sigma(-\text{Im}A) = \emptyset$ and A is normal,

(vi) $\{A\}' = \{A^{2m}\}'$, m some positive integer and A has dense range or is one-one,

(vii) $\sigma(A) \subset \pi^+$.

<u>Proof.</u> We first note that $AE = E^{-1}A$ gives us $EA = AE^{-1}$. Thus we have:

$$AE = E^{-1}A$$

and

$$AE^{-1} = EA.$$

Now setting $H = E$ and $K = E^{-1}$, each of the conditions above, implies that $H = K$ by theorem 1.1 and corollary 1.4. i.e. $E = E^{-1}$ and hence $E^2 = I$.

We note that Singh and Mangla [29] proved the following result:

<u>THEOREM 4.I.</u> If T is an invertible operator such that $T^* = V^*T^{-1}V$ with V a cramped unitary operator, then T is unitary.

We improve this result as follows:

<u>THEOREM 4.11.</u> If T and S satisfy equation (15) with T left invertible and S invertible, then T is unitary under any one of the following conditions:

(i) $0 \notin W(S)$ and S is unitary,

(ii) $\sigma(S) \cap \sigma(S) = \emptyset$ and S is unitary.

<u>Proof.</u> Since S is invertible, T is invertible by theorem 4.1 above. Also, $TST^* = S$ gives $TS^*T^* = S^*$. Hence we have:

$$ST^* = T^{-1}S$$

and

$$S^*T^* = T^{-1}S^*.$$

Now, by corollary 1.8, each of the conditions (i) and (ii) implies that $T^* = T^{-1}$ or T is unitary.

REFERENCES.

[1] T. Ando, Operators with a norm condition, Acta. Sci. Math (Szeged), 33(1972), 169-178

[2] G.P. Barker, Normal matrices and Lyapunov Equation, SIAM J. Appl. Math. 26(1974), 1-4.

[3] S.K. Berberian, The numerical range of normal operator, Duke Math. J. 31(1964), 479-483.

[4] C.C. Cowan, Commutants and the operator equation $AX = \lambda XA$, Pacif. J. Math.80 No. 2., 337 - 339.-

[5] C. Davis and P Rosenthal, Solving linear equations, Canadian J. Math. 26 (1974), 1384-1389.

[6] B.P. Duggal, Some conditions on operators implying normality, Math. Japonica 22(1977), 357 - 360.

[7] ——————, The operator equation $AB+BA^* = A^*B+BA=I$, Math Japon. 25(1980), 33-35.

[8] ——————, Operator equations $AB+BA^*=I=A^*B+BA$, Normal solutions, Proc. Amer. Math Soc. To appear.

[9] ——————, Operator equation $AB+BA^*=I=A^*B+BA$, Existence and normality of solutions, Pre-print.

[10] B.P. Duggal and J.M. Khalagai, On the operator equation $AB+BA^*=A^*B+BA=I$, Math. Japon. 26(1981), 577-583.

[11] ——————, Operator equations $AB+BA^*=A^*B+BA=I$. Indian J.Pure. appl. Math Vol.13(11(1982),

[12] M.R. Embry, Similarities involving normal operators on Hilbert space, Pacif. J. Math. 35(1970), 331-336.

[13] P.R. Halmos, Hilbert space problem Book, Van Nostrand (1967), New York.

[14] E. Kamei and Y. Kato, On operator equation AB+BA*=A*B+BA=I, Math Japon. 23(1978), 129-130.

[15] Y.Kato and N. Moriya, On operators considered by Campbell, Math Japon. 22(1977), 233-238.

[16] J.M. Khalagai, Further ramarks on the operator equation AB+BA*=A*B+BA=I, Kenya J. Scie. Tech. Accepted for publication

[17] J.M. Khalagai and B.P. Duggal, The operator equation AB+BA*=A*B+BA=I, Journal of African Mathematical Union. To appear.

[18] S.K. Khasbardar and N.K. Thakare, Spectraloid operators, Similarity and related results, Indian J. Pure.appl. Math. 9 No.5 (1978), 494-500.

[19] A. Kobayashi and T. Okayasu, A remark on a theorem of Goya and Saito, Dedicated to the memory of the late Prof. T. Saito, Math. Japon. 27 No.4 (1982), 549 - 550.

[20] R. Nakamoto, On a problem of Duggal, Math. Japon. 22(1978), 441-442.

[21] J.M. Patel, Numerical range and similarity, Ph.D. Thesis, Gujarat University, Ahmadabad, (India).

[22] ——————, Operators having finite spectrum, Indian. J. appl. Math. 11(7) (1980), 897-903.

[23] J.M. Patel and I.H. Sheth, On intertwining by an operator having dense range, Pre-print.

[24] S.M. Patel, On Similarities of operators, The Yokohama Mathematical Journal Vol XXIII, Nos. 1-2, (1975), 85-90.

[25] S.V. Phadke and N.K. Thakare, M-hyponormal operators, invariant subspaces and spectral localization, Indian. J. Pure appl. Math. 8 (1977), 487-496.

[26] M. Rosemblum On operator equation BX - XA = Q, Duke Math. J. 23 (1956), 263 - 269.

[27] I.H. Sheth and J.M. Khalagai, The operator equation TST* = S. Unitary solutions, Kenya J. Scie. Tech. Submitted.

[28] ——————, On the operator equation AH = KA, Indian. J. pure appl. Math. Submitted.

[29] U.N. Singh and K. Mangla, Operators with inverses similar to their adjoints, Proc. A.M.S. 38 (1973), 258-260.

[30] O. Taussky, A generalisation of a theorem of Lyapunov, J. Soc. Indust. Appl. Math, 9(1961), 640-643.

[31] A.E. Taylor, Introduction to Functional Analysis, Wiley, New York, 1958.

[32] J.P. Williams, Similarity and the numerical range, J. Math. Anal. Appl. 26(1969(, 307-314.