ON WEYL SPECTRUM AND QUASI-SIMILARITY OF OPERATORS IN HILBERT SPACES

by

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Declaration and Approval

I the undersigned declare that this thesis is a presentation of my original research work. Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any other university or any other institution of learning.

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In our capacity as supervisors of the candidate's thesis, we certify that the above statements are true to the best of our knowledge and this thesis has our approval for submission.

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Chapter 1

Dedication

I dedicate this work to my lovely daughters, son and wife. I am very thankful for providing good working environment towards this success, I owe you alot.

Abstract

The task of looking for conditions under which two quasisimilar operators have equal spectra or essential spectra is quite evident in the extant literature of operator theory. However the case of equality of Weyl spectra for such operators is often overlooked. In this thesis we strive to fill this gap by considering mainly the intersection of classes of operators namely; dominant and biquasitriangular operators, ω-hyponormal and biquasitriangular operators and hypercyclic and biquasitriangular operators and possibly their subclasses. Many classes of operators defined on Hilbert spaces are defined by means of some inequalities. Which are obtained by relaxing the condition of normality. In this thesis we consider some of the spectral properties of classes of operators, and more precisely, our main interest concerns how Weyl spectrum of an operator say A, of dominant operator, ω-hyponormal operator and hypercyclic operators behaves with the structures of quasisimilarity. Together with properties of SVEP, Durnford‘s and Bishop‘s and Biquasitriangularity structure gives a general framework from which we explored the Weyl spectrum of the classes of operators above.

Contents

Chapter 2

Introduction and Preliminaries

2.1 General background

The concept of a Hilbert space as shown by P.Halmos in [1967], generalizes the notion of n-dimensional space in the sense of extending the methods of vector algebra and calculus from the two-dimensional planes and three-dimensional spaces to infinite dimensional spaces Polyanin [2008]. Recall that Hilbert spaces are distinguished among Banach spaces by being most closely linked to plane (and space). Euclidean geometry seems to be a correct description of our universe at many scales. But before the development of Hilbert spaces, other generalizations of Euclidean spaces were known to mathematicians and physicists. One of them was realized towards the end of the 19^{th} century. The idea of a space whose elements can be added together and multiplied by scalars known as an abstract linear space, were introduced in the first decade of the 20^{th} century. And the parallel developments led to the introduction of Hilbert spaces. P.Halmos [1967] showed that an Hilbert space is an abstract vector space possessing the structure of an inner product that allows length and angle to be measured. The spaces arise naturally and frequently in mathematics and physics, typically as infinite-dimensional function spaces. The significance of the concept of a Hilbert space offers one of the best mathematical formulations of quantum mechanics Lambert-Mogiliansky et al [2012], thus a basic mathematical object which is required for the description of particles in quantum mechanics. Since the quantum mechanical system are vectors in a Hilbert space Lambert-Mogiliansky et al [2012], with observables being self adjoint operators, the symmetries of the system are unitary operators and measurement are orthogonal projections. Hilbert through his study developed the spectral theorem and recognized the important work of Fredholm and others on integral operators, and thought to strengthen their work using L^2 space. He generalized what was then called the principal axis theorem to infinite dimensions and coined the term spectrum, which was later used to prove some version of the spectral theorem for self-adjoint operators E.B.Davies [2005]. Hilbert through his work discovered that the spectrum need not be discrete. The spectral theory broadly defined could be generalised as trying to classify all the structures of bounded operators on a Hilbert space. Broadly, history of spectral theory goes way back to the nineteenth century, when the objects of study used to be infinite systems of linear equations and integral equations. The subject was revolutionized in the late 1920‘s by Von Neumann, when he defined the notion of an abstract Hilbert space and considered bounded linear operators on it. In this modern sense a successful spectral theory was soon obtained by Riesz for all compact operators as a direct extension of the theory of finite square matrices. Operators are commonly used to perform a specific mathematical operation on another function. The operation can be to take the derivative or integrate with respect to a particular term, or to multiply, divide, add or substract a number or term with regards to the initial function. Operators are commonly used in physics, mathematics and chemistry, often to simplify complicated equations such as the Hamiltonian operator used to solve the Schrodinger equation in order to figure out the energy of a wave function. Also for two physical quantities to be simultaneously observable in quantum mechanics (that is, energy and time for example), their operator representations must commute. In 1909 a German mathematician, Hermann Weyl introduced a new concept in operator theory and developed a known classical Weyl theorem, and proved that given two self adjoint operators A and B Moslehian [2012], then spectrum of A and $A+B$, have same limit points with B being a compact operator. In 1910 Weyl proved that the essential spectra of a self adjoint operator is invariant under compact perturbations showed in S. Sunder [2010]. Also generalised that compact perturbation of an operator is in some sense small, and concluded that Fredholm index is a topological quantity. In 1935 von Neumann produced the converse that, if the spectra of self adjoint operators A and B have the same limit points, then given a compact operator K, it is a known fact that $A+K$ and B are unitarily equivalent Mc Guire et al [1988]. The study of Weyl spectrum was revealed and realised to be relevant for infinite dimensional Hilbert spaces. Intuitively, in finite dimensional Hilbert spaces all the operators are Fredholm operators of index zero, and the spectrum in this case consist of the set of eigenvalues, so the Weyl spectrum is empty, that is

$$\sigma_\omega(A) = \phi. \tag{2.1}$$

Hence the study of spectrum in general and Weyl‘s spectrum in particular is trivial in finite dimensional Hilbert spaces. The Weyl operators occurs in the theory of Fredholm operators, and has been singled out to be those Fredholm operators of zero index. The study of spectrum in general simplifies when the operators are decomposed into their simple forms.

It has been discovered by several reseachers that to investigate the structures of arbitrary operator in Hilbert spaces, we look into how the operator can be familiarised by decomposing it into its simple versions; namely direct sum decomposition (orthogonal decomposition), polar decomposition and other forms of decompositions, this is with respect to separable Hilbert spaces. A separable Hilbert space has its invariant subspaces, these are $H = H_1 \oplus H_2$ with H_1 closed subspace and H_2 orthogonal complement subspace. In Furuta [2001] a subspace H_1 of H is said to be invariant under $A \in B(H)$ if for every vector

$$\{x \in H_1 : Ax \in H_1.\}$$

A subspace H_1 of H is said to reduce A if H_1 and H_2 are both invariant under A with H_2 being orthogonal complement of H_1. Without loss of generality, the operator A has a decomposition given as follows,

$$A_1 = A \setminus H_1 \quad \text{and} \quad A_2 = A \setminus H_2 \tag{2.2}$$

thus A has a direct sum decomposition as

$$A = A_1 \oplus A_2. \tag{2.3}$$

The so-called direct sum decomposition is one of many known kinds of decomposition, which is largely motivated by the work of Nagy and Foias [1970]. Hence studying the properties of A is relaxed into studying the properties of its direct summands of A_1 and A_2 whose structures are now known to be less complicated than that of A . The study of the structures and properties of an arbitrary operator on Hilbert spaces is essentialy equivalent to study its complementary parts, its invariant and hyperinvariant lattices. In the literature many researchers have shown some tramendeous work. For instant L.R. William [1980] showed that every operator say A is always unitarily equivalent to direct sum $A_1 \oplus A_2$ where in B.P.Duggal [2005] A_1 is normal and A_2 is pure (completely non normal). Nagy and Foias in [1970] on their theory of contraction operators, proved that a contraction operator is a direct sum of a unitary part and completely non unitary part. In other words, in studying linear operators which are not normal, one of the major steps has been that of finding methods of decomposing such operators into various parts which are easier to handle. Finite dimensional Hilbert spaces suggest that two linear maps, which are linked by formula

$$AU_i = U_iB \tag{2.4}$$

for some invertible operator U_i mapping H_1 to H_2 share many similar properties, since U_i corresponds to change of basis in H_i. This fail in general in infinite dimensional case, where no good theory of basis exist. Similarly, one can redefine this idea if

$$H_1 = H_2 = H \tag{2.5}$$

by considering the operators A and B mapping H to H such that

$$AU = UB, \tag{2.6}$$

that is,

$$A = U^{-1}BU \tag{2.7}$$

which defines the notion of similarity between operators A and B. indeed an infinite dimensional Hilbert spaces seems not to generalised things as seen in the basic theory of the spectrum in $B(H)$. It is a known fact that direct sum decomposition splits a bounded linear operator into its normal and pure parts (completely non normal part) B.P.Duggal [2005]. The other form of decomposition which is majorly used in this study is the polar decomposition.

2.2 Research problem

With respect to what researchers have done, there are many notions of equivalence of Hilbert spaces operators. The most important are unitary equivalence, similarity and its weaker version of quasisimilarity, others are almost similarity and metric equivalence. But the first three are most important in the thesis. And inspite of what has been researched on, few have studied the quasisimilarity of operators in infinite dimensional Hilbert spaces with respect to Weyl spectrum. Thus, in this thesis we study the Weyl spectra of operators under the quasisimilarity relation.

2.3 Objectives

Overall objective

In this thesis we investigate the role played by quasisimilarity on the Weyl spectra of some classes of operators.

Specific objectives

Unlike in finite dimensional Hilbert spaces where unitary equivalence, similarity and quasisimilarity implies same thing, in infinite dimensional Hilbert spaces similarity and quasisimilarity are weaker versions of unitary equivalence. Almost all the equivalence relations have been shown to preserve the spectral pictures (that is spectrum, essential spectra and index function). Specifically in this thesis we study effects of quasisimilarity to the Weyl spectrum of some classes of operators as enumerated below;

1. Weyl spectrum and quasisimilarity of Dominant operators.

2. Weyl spectrum and quasisimilarity of ω-hyponormal operators.

3. Weyl spectrum and quasisimilarity of Hypercyclic operators.

2.4 Definitions, Notations and Terminologies

Definition 2.4.1. *Given H and K as two Hilbert spaces, an isomorphism is a $1 - to - 1$ correspondent that preserves the linear operator between linear spaces, and hence preserves the algebraic structure (see Kubrusly [2011]).*

Theorem 2.4.2. *Every separable Hilbert space of infinite dimension is isomorphic to ℓ^2.*

Remark 2.4.3. *Note that there exists* $\{e_n\}_{n=1}^{\infty}$, *for* $x \in H$, *with* $x_i = \langle x_i, e_i \rangle$ *and*

$$U(x) = (x_1, x_2, x_3, ...).$$

And by Parseval's identity, the series $\Sigma_{i=1}^{\infty} \mid x_i \mid^2$ *coverges, so the sequence* $(x_1, x_2, x_3, ...)$ *belongs to* ℓ^2.

Definition 2.4.4. *Suppose a Hilbert space* H *with* H_1 *being a closed subspace of* H. *The orthogonal complement subspace* H_2 *of* H_1 *is defined by*

$$H_2 = \{v \in H : \langle v, u \rangle = 0 \forall u \in H_1\}.$$

In this case any vector $w \in H$ *is expressed as* $w = u + v$, *where* $u \in H_1$ *and* $v \in H_2$, *so that* $H = H_1 \oplus H_2$ *(see Adamyan [2006]).*

Definition 2.4.5. *A transformation* $A : H \longrightarrow H$ *is called a linear operator if* A *satisfies*

$$A(u + v) = A(u) + A(v) \tag{2.8}$$

and

$$A(\lambda u) = \lambda A(u). \tag{2.9}$$

(see Furuta [2001]).

Definition 2.4.6. *Given a Hilbert space* H, *let* A *be any operator acting on* H, *the adjoint of* A *denoted by* A^* *is defined by*

$$\langle Au, v \rangle = \langle u, w \rangle = \langle u, A^* v \rangle \tag{2.10}$$

for all $u, v \in H$ *(see Furuta [2001]).*

Remark 2.4.7. *The concept of operator implies linear and bounded.*

Definition 2.4.8. *Given a Hilbert space* H, *any operator say* A *acting on* H *is bounded from below if for any* $c > 0$ *we have* $\parallel Au \parallel \leq c \parallel u \parallel$ *for all* $u \in H$ *and* c *a scalar. With operator* *norm of A denoted by $\| A \|$, is a number defined by $\| A \| = inf\{c > 0 : \| Au \| \leq c \| u \|\}$ for all $u \in H$ (see Furuta [2001]).*

Definition 2.4.9. *Let A be any operator acting on a Hilbert space, then A is left invertible if there exists an operator $L \in B(H)$ with $LA = I$, and right invertible if there exists $R \in B(H)$ with $AR = I$.*

Classes of Operators

Definition 2.4.10. *An operator $A \in B(H)$ is;*

- *self-adjoint if $A^* = A$.*
- *a projection if $A^2 = A$ and $A = A^*$.*
- *normal if $A^*A = AA^*$.*
- *If A commutes with the normal operator, that is $A(A^*A) = (A^*A)A$ then A is quasinormal operator.*
- *hyponormal if $(A^*A) \geq (AA^*)$.*
- *p-hyponormal if $(A^*A)^p \geq (AA^*)^p$, with $0 < p \leq 1$.*
- *quasihyponormal if $A^*\{A^*A - AA^*\}A \geq 0$.*
- *p-quasihyponormal if $A^*\{(A^*A)^p - (AA^*)^p\}A \geq 0$*
- *log-hypornormal if it is invertible and satisfies $log(A^*A) \geq log(AA^*)$.*
- *dominant if $R(A - \lambda I) \subset R(A - \lambda I)^*$, $\forall \lambda \in C$ (see Kubrusly [2012]).*
- *M-hyponormal if for $M > 0$ $\|(A - \lambda)^* x\| \leq M\|(A - \lambda)x\|$, $\forall \lambda \in C$.*
- *(p, k)-quasihyponormal if $A^{*k}(| A |^{2p} - | A^* |^{2p})A \geq 0,\ 0 < p \leq 1, k \in N$.*

Remark 2.4.11. *A 1-hyponormal operator is hyponormal and a $\frac{1}{2}$-hyponormal operator is said to be semi-hyponormal (see Duggal[2004]).*

- *is paranormal operator if* $\| Ax \|^2 \leq \| A^2x \|, \forall x \in H$.

- *is unitary if* $A^*A = AA^* = I$ *where* I *the identity operator, and in this case we have* $A^* = A^{-1}$.

- *is called an isometry if* $A^*A = I$.

- *is a partial isometry if* $A = AA^*A$.

Remark 2.4.12. *If a partial isometry is invertible, then it is unitary, and therefore its spectrum is a subset of the unit circle.*

Lemma 2.4.13. *If* A *is a contraction* ($\| A \| \leq 1$), *then* A *has a canonical decomposition* $A = A_o \oplus A_1$ *on* $H = H_o \oplus H_1$ *such that* A_o *on* H_o *is unitary and* A_1 *on* H_1 *is completely non unitary (see Jeon [2004]).*

Lemma 2.4.14. *If* A *is a normal operator, then* A *and* A^* *have the same kernel and range. Consequently, the range of* A *is dense if and only if* A *is injective.*

- *Is said to be compact operator if it maps bounded sets to relatively compact sets.*

- *Is Fredholm operator if* $dimker(A) < \infty$, $dimker(A^*) < \infty$ *and* $R(A)$ *is closed, where* A^* *is the adjoint of* A.

Remark 2.4.15. *The index of Fredholm operator is a continuous real-valued function given by* $ind(A) = dimker(A) - dimker(A^*)$.

Remark 2.4.16. *A bounded operator* $A : H \longrightarrow K$ *is Fredholm if and only if there exists orthogonal decomposition* $H = H_1 \oplus H_2$ *and* $K = K_1 \oplus K_2$ *such that* H_1 *and* K_1 *are closed subspaces and* H_2 *and* K_2 *are finite dimensional subspaces.*

- *Is Weyl operators if it is Fredholm operator of index zero, that is,* $dimker(A) = dimker(A^*)$ *(see M.Berkani [2002]).*

Definition 2.4.17. *The ascent* p *and the descent* q *of* $A \in B(H)$ *at* $\lambda \in C$ *are the extended integers given by* $p = inf\{n \geq 0 : N(A-\lambda)^n = N(A-\lambda)^{n+1}\}$ *and* $q = inf\{n \geq 0 : R(A-\lambda)^n = R(A-\lambda)^{n+1}\}$*, respectively. If the ascent and the descent of* A *at* $\lambda \in C$ *are both finite, then* $p = q$*.*

- *Is Browder if it is Fredholm of finite ascent and descent.*

Remark 2.4.18. *If* $A \in B(H)$ *is Weyl operator, then* A^* *is also Weyl operator where* A^* *is the adjoint operator. Also* $ind(A^*) = -ind(A)$*, since Weyl operators are Fredholm operators of index zero.*

- *Is semi-Fredholm if either* $kerA$ *or* $kerA^*$ *is finite dimensional and* $ranA$ *is closed (see H.Weyl [1950]).*

- *Is quasitriangular if and only if for each complex number* λ *with* $A - \lambda$ *semi-Fredholm,* $i(A-\lambda) \geq 0$ *(see X.Cao, [2006]).*

- *Is biquasitriangular if both* A *and* A^* *are quasitriangular (see Tanahashi [1999]).*

- *Equivalent* $A \in (BQT)$ *if and only if* $\sigma_{le}(A) = \sigma_{re}(A) = \sigma_e(A) = \sigma_\omega(A)$*.*

Remark 2.4.19. *The class of biquasitriangular operators is denoted by* BQT*.*

- *Is* ω*-hyponormal if* $\mid \tilde{A}| \geq \mid A \mid \geq |\tilde{A}|^*$*.*

Remark 2.4.20. *The classes of operator are related by, the following inclusions*

$$Self-Adjoint \subset Normal \subset Quasinormal \subset Hyponormal \subset Paranormal.$$

$$Hyponormal \subset p-hyponormal \subset quasi-hyponormal \subset (p)-quasihyponormal \subset (p,k)-quasihyponormal.$$

also

$$Unitary \subset Isometry \subset Partial - Isometry.$$

Also

$$Fredholm \subset Weyl \subset Browder.$$

Definition 2.4.21. *If there exist a pair of subspaces H_1 and H_2, of H that are invariant under A, with $H = H_1 \oplus H_2$. then the restriction map $A\backslash_{H_2}$ is quasinilpotent if $H_2 = H_1 = H$. Every quasinilpotent operator is nilpotent if H_2 is finite dimensional.*

Definition 2.4.22. *Two operators A and B on Hilbert space H are unitarily equivalent so long as we can find unitary operator U such that $A = U^*BU$. On the other hand A and B are similar whenever $A = U^{-1}BU$ for some invertible operator U.*

Definition 2.4.23. *$X \in B(H)$ is a quasi-affinity(quasi-invertible) if X is injective and has dense range (see Duggal [2002]). If given $A \in B(H)$, $B \in B(K)$ with quasi-affinities X and Y such that $AX = XB$ and $YA = BY$ then A and B are said to be quasi-similar.*

Remark 2.4.24. *The invertible operators are vector space isomorphisms and the only properties of A, which survive similarity conjugation are algebraic in nature, these are spectra, multiplicity of eigenvalues and so on.*

Examples of Weyl Operators

Example 2.4.25. *All linear operators in a finite dimensional Hilbert spaces are Weyl operators and identity operator in an infinite dimensional Hilbert spaces is also a Weyl operator.*

Remark 2.4.26. *Given a Fredholm operator F, then index of F is zero if and only if $F = A+K$ for some invertible operator A and compact operator K.*

Corollary 2.4.27. *Any Weyl operator $W \in B(H)$ can be decomposed canonically as $W = A+K$ for some invertible and compact operators A and K respectively.*

Definition 2.4.28. *$A \in B(H)$ on a Hilbert space H is invertible if,*

$$\| Ax \| \geq c \| x \|$$

for any $x \in H$ and $R(A)$ is dense in H, that is $\overline{R(A} = H$.

Definition 2.4.29. *We say that $A \in B(H)$ has the single valued extension property (SVEP) at $\lambda \in C$ if the following assertion is true: If D is an open neighborhood of λ and if $f : D \longrightarrow H$ is an H-valued analytic function such that $(\mu - A)f(\mu) = 0$, $\forall \mu \in D$, then f is identically zero on D. When A has SVEP at every $\lambda \in C$, we simply say that A has SVEP (see F.Kimura [2004]).*

Definition 2.4.30. *We say that A has Bishop property (property (β)) at $\lambda \in C$ if the following assertion is true: If D is an open neighborhood of λ and if $f_n : D \longrightarrow H$ $(n = 1, 2, ...)$ are H-valued analytic functions such that $(\mu - A)f_n(\mu) \longrightarrow 0$ uniformly on every compact subset of D (see F.Kimura [2004]).*

Definition 2.4.31. *We say that A has Dunford's property (C) if A has SVEP (see Duggal and Kubrusly [2014]), and also for every closed subset F of the complex plane, and corresponding local spectrum subspace $H_A(F) = \{x \in H; \sigma(A, x) \subset F\}$ is closed (see Duggal [2002]).*

Definition 2.4.32. *A bounded operator A is pure if A has no non-trivial reducing subspace in H with the restriction of A on the subspace is normal (see I.H.Jeon [2003]).*

Definition 2.4.33. *Let $A \in B(H)$, then the set $\{x, Ax, A^2x, ..., A^nx, ..\}$ is an orbit of x under A. If some orbit is dense in H then A is a hypercyclic operator and x a hypercyclic vector of A, (see Eungil Ko [2006]).*

Definition 2.4.34. *If A has a polar decomposition $A = U \mid A \mid$ with U an isometry operator, then the 1^{st} Aluthge transform is defined by $\tilde{A} = \mid A \mid^{\frac{1}{2}} U \mid A \mid^{\frac{1}{2}}$, and then $\tilde{A} \in H$ is a p-hyponormal with $p = \frac{1}{2}$, and defines semi-hyponormal operator (see Aluthge [1990]).*

Spectrum of an Operator

Definition 2.4.35. *Let $A \in B(H)$, then the set $\sigma(A) = \{\lambda \in C : A - \lambda I$ is not invertible$\}$ is called the spectrum of A, whereas, the complement of the spectrum of A is called the resolvent of A.*

Remark 2.4.36. *The set $\sigma(A)$ is a compact subset of the complex plane, whereas its complement in an open subset of the complex plane.*

Definition 2.4.37. *Let $A \in B(H)$, we say that Weyl's theorem holds for A if there is equality $\sigma(A) \backslash \sigma_\omega(A) = \pi_{00}(A)$. Where $\pi_{00}(A)$ is the set of isolated points of $\sigma(A)$ which are eigenvalues of finite multiplicity.*

Definition 2.4.38. *Let $A \in B(H)$, then the set $\sigma_p(A) = \{\lambda \in C : Ax = \lambda x$ for $x \neq 0\}$ is called the point spectrum of A.*

Definition 2.4.39. *Let $A \in B(H)$, then the set $\sigma_{ap}(A) = \{\lambda \in C : A - \lambda I$ is not bounded below$\}$ is called the approximate point spectrum of A.*

Definition 2.4.40. *The set of $\lambda \in C$ for which $A - \lambda I$ is injective but does not have dense range is known as the residual spectrum or compression spectrum of A and is denoted by $\sigma_r(A)$.*

Definition 2.4.41. *The set of all $\lambda \in (C)$ for which $A - \lambda I$ is injective and has dense range, but is not surjective is called continuous spectrum and is denoted by $\sigma_c(A)$.*

Remark 2.4.42. *Under the classical spectral theory, we have $\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$, where $\sigma_p(A)$, $\sigma_c(A)$ and $\sigma_r(A)$are mutually disjoint parts of $\sigma(A)$. Thus they partition the $\sigma(A)$.*

Remark 2.4.43. *The boundedness of the spectrum follows from the bound of $\| A \|$, which also shows the closedness of the spectrum. Thus, the spectrum of an unbounded operator is in general a closed, possibly empty subset of the complex plane (See Djordjevic [2002]).*

Definition 2.4.44. *Let $A \in B(H)$,then the spectral radius of A is given by the set*

$$r(A) = sup\{| \lambda |: \lambda \in \sigma(A)\}.$$

Definition 2.4.45. *Let $A \in B(H)$, the essential spectrum denoted by $\sigma_e(A)$ is the set given by $\sigma_e(A) = \{\lambda \in C : A - \lambda I$ is not Fredholm$\}$.*

Definition 2.4.46. *Let $A \in B(H)$, the Weyl spectrum of A is the set defined by $\sigma_\omega(A) = \{\lambda \in C : (\lambda I - A)$ is not Weyl$\}$.*

Definition 2.4.47. *let $A \in B(H)$, the upper semi-Fredholm spectrum of A is the set definen by $\sigma_{SF_+}(A) = \{\lambda \in C : (A - \lambda I)$ is not upper semi-Fredholm$\}$.*

Definition 2.4.48. *let $A \in B(H)$, the lower semi-Fredholm spectrum of A is the set definen by $\sigma_{SF_-}(A) = \{\lambda \in C : (A - \lambda I)$ is not lower semi-Fredholm$\}$.*

Definition 2.4.49. *Let $A \in B(H)$, the Browder spectrum is the set defined by $\sigma_b(A) = \{\lambda \in C : A - \lambda I$ is not Browder$\}$.*

Remark 2.4.50. *Under the modern spectral theory we have $\sigma_e(A) \subseteq \sigma_\omega(A) \subseteq \sigma_b(A) \subseteq \sigma(A)$. Where $\sigma_e(A)$, $\sigma_\omega(A)$, $\sigma_b(A)$ and $\sigma(A)$ forms the nested type of set (see Oyoo and Khalagai [2016]).*

Definition 2.4.51. *The set $SP(A)$ denote the spectral picture of $A \in B(H)$ consisting of the essential spectrum , hole in $\sigma_e(A)$ which is a non empty bounded component of $C \setminus \sigma_e(A)$ and a pseudohole in $\sigma_e(A)$ which is a non empty component of $\sigma_e(A) \setminus \sigma_{re}(A)$ or of $\sigma_e(A) \setminus \sigma_{le}(A)$.*

Theorem 2.4.52. *If A is a hypercyclic operator, then any operator in the uniformly closed, unital algebra generated by A is quasitriangular (see V.Matache [1993]).*

Definition 2.4.53. *The set $\partial\sigma(A)$ defines the boundary of spectrum of operator A.*

Chapter 3

Literature review

The spectra of operators have been studied by several authors with respect to properties of unitary equivalence, similarity and quasisimilarity. Similarly the essential spectra with relation to the same properties has been captured by several authors as well. We note that equality of spectra, essential spectra and Weyl spectra of similar operators is obvious but the spectra and essential spectra results under the quasisimilarity relation depends on the individual operators. And the Weyl spectra under quasisimilarity have been overlooked. The Weyl spectrum occurs in the theory of perturbation by compact operators and has the property of being invariant under perturbation. Coburn in [1996] used perturbation formula to define Weyl‘s spectrum, and proved that $\sigma_\omega(A) = \{0\}$ when A is compact and the space is infinite dimensional. Coburn extended a classical result of Weyl for normal operators, to hyponormal operator, then showed that

$$\sigma_\omega(A) = \sigma(A) - \pi_{00}(A)$$

where $\pi_{00}(A)$ denotes isolated points of $\sigma(A)$ and represent the eigenvalues of finite multiplicity (see Eungil Ko [2006]). It is a known fact that Weyl‘s theorem may or may not hold for direct sum of operators for which Weyl‘s theorem holds (see Duggal [2005] and W.Y.Lee [2001]) have both shown that if A and B are isoloid satisfying Weyl's theorem, then Weyl's theorem holds for

$$A = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Longleftrightarrow \omega \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \omega(A) \cup \omega(B).$$

Many researchers have studied the equality of spectrum and essential spectrum via quasi-similarities of operators. Douglas [1969,1966] showed that quasi-similar normal operators are unitarily equivalent, thus have equal spectra and essential spectra. Clary [1975] and I.H.Jeon [2008] proved that quassisimilar hyponormal have equal spectra and asked if it has equal essential spectra. William [1980] showed quasisimilar quasinormal operators and gave conditions under which quasisimilar hyponormal operators have equal essential spectra, that is if A and B are both hyponormal, partial isometry or are quasinormal. Gupta in [1985] and An-Hyun Kim [2006] showed that quasisimilar k-quasihyponormal and biquasitriangular operators have equal essential spectra. Yang [1993] proved that quasisimilar M-hyponormal operators have equal essential spectra. Zhang [2006] studied the invertibility and spectra of M_{A_0} where two operators A and B acting on $H \oplus K$ as having a matrix representation

$$M_{A_0} = \begin{pmatrix} A & A_0 \\ 0 & B \end{pmatrix}$$

with $A_0 \in B(K, H)$

Duggal [1996] showed that if A_i, $i = 1, 2$ are quasisimilar and U_i unitary in a representation $A_i = U_i \mid A_i \mid$ implies A_1 and A_2 have equal spectra and essential spectra, went further and showed that if $A \in HU(p)$, then $\sigma(A)$=$\sigma(\tilde{A})$ and $\sigma_\omega(A)$=$\sigma_\omega(\tilde{A})$ where $HU(p)$ is the class of p-hyponormal A with $U, A = U \mid A \mid$. Khalagai and Nyamai [1998], Fumihiko Kimura [2004] both proved that quasi-similar M-hyponormal operators have equal spectra. I.H.Jeon [2008] proved that quasisimilar p-hyponormal operators having equal spectra and essential spectra. I.H.Jeon [2008] proved that quasisimilar ω-hyponormal operators have equal spectra and essential spectra. Luketero et al [2015] and B.P.Duggal [1996] both showed equality of spectra and essential spectra for classes of operators that satisfy the Putnam-Fuglede property. Duggal and Kubrusly [2014] in their paper of biquasitriangularity and derivations it was shown that $A \in (BQT)$ if and only if $\sigma_e(A) = \sigma_\omega(A)$. Rashid et al [2008] and I.H.Jeon [2008] proved that quasisimilar (p,k)-quasihyponormal operators have the same Weyl spectra. Yang [1993] showed equality of spectra and essential spectra of quasisimilar M-hyponormal operators. extended further and use the result by Gupta [1985] who considered biquasitriangular and quasi-similar k-quasihyponormal operators having equal essential spectra. Yingbin and Zikun [2000], Fumihiko Kimura [2004] both showed that quasisimilar p-hyponormal having equal spectra, essential spectra. I.H. Kim [1998] showed that quasisimilar (p,k)-quasihyponormal operators have equal spectra and also equal essential spectra. Perez-Fernandez et al [2011] showed (as a corollary) that quasisimilar p-hyponormal or log-hyponormal or injective p-quasihyponormal operators have same spectra and essential spectra. I.H.Jeon et al [2004] proved that the normal parts of quasisimilar log-hyponormal operators are unitarily equivalent, a log-hyponormal operator compactly quasisimilar to an isometry is unitary, and a log-hyponormal spectral operator is normal. Apostol et al [1973] and William [1980] extended and showed that for A and B biquasitriangular, then $\sigma_e(A) = \sigma_e(B)$. Duggal and Kubrusly [2014] showed that $A \in BQT$ if and only if $\sigma_e(A)$ has no holes and pseudoholes, thus it is immediate from the definition that $\sigma_e(A) = \sigma_\omega(A)$ for every $A \in BQT$. Since BQT is the class of all biquasitriangular operators in $B(H)$, let $(BQT)_{qs}$ be the set of all operators $A \in B(H)$ such that A is quasisimilar to some biquasitriangular operator (see Duggal and Kubrusly [2014]). Thus by Nagy and Foias [1970], it was shown that the set $(BQT)_{qs}$ is a properly superset of BQT, and the set $(BQT)_{qs}$ is at least norm dense in $B(H)$, (which answers the problem whether $(BQT)_{qs} = B(H)$). Note that the hypothesis that the Hilbert space H is separable, is the hypothesis which quarantees SVEP for $A \in B(H)$ (see Duggal [2004]). The open problem in operator theory known as the invariant subspace problem asserts that, any Hilbert space operator acting on an infinite-dimensional, separable, complex Hilbert space operator has proper invariant subspaces in (Read C. [1988]), showed a negative answer to this problem in the case of a Banach space operators. As for Hilbert space operators, this is also an open problem. V.Matache [1993], proved that this problem had a negative answers if and only if there is some hypercyclic operator A on the Hilbert space H, such that any nonzero x in H is a hypercyclic vector of A. Hence interest in hypercyclic operators arises from the invariant subspace problem. C. Kitai [1982], proved that hyponormal operators are not hypercyclic. Duggal in 1996 gave a necessary and sufficient conditions for hypercyclic Banach spaces operator to satisfy a-Weyl's theorem. V.Matache in [1993], proved that if $A \in B(H)$ is hypercyclic and $\sigma(A)$ has no interior point, then A is biquasitriangular.

3.1 Knowledge Gap

From the extant literature it is clear that researchers have done almost exhaustively on essential spectra and spectra of operators under quasisimilarity, but very little on Weyl spectrum. And due to the fact that Weyl spectra under quasisimilarity have been overlooked, in this thesis we explore the Weyl spectra of Dominant, ω-hyponormal and hypercyclic operators under the quasisimilarity.

Chapter 4

Weyl Spectrum and quasisimilarity of dominant operators

4.1 Introduction

It is a known fact that every operator $A \in B(H)$ can be decomposed directly as $A = A_1 \oplus A_2$ (with respect to decomposition $H = H_1 \oplus H_2$) (see Duggal [1996]). In this chapter we look for conditions under which two quasisimilar operators have same Weyl spectrum. We do this by mainly intersecting quasisimilar dominant operators and biquasitriangularity. Recall $X \in B(H)$ is a quasi-affinity when X is both one-to-one and has dense range (see S.Clary [1975]). If given quasi-affinities X and Y with

$$AX = XB$$

and

$$BY = YA,$$

then A and B are said to be quasisimilar. It is well known in operator theory that two similar operators have same spectrum and essential spectrum (see S.Clary [1975] and William [1980]).

However, in the case of quasi-similarity it is not necessarily true.

Remark 4.1.1. *We note that from the flow of the extant literature above, authors are hardly considering equality of Weyl spectrum for quasisimilar operators. In this chapter we attempt to bridge this gap by mainly considering intersection of dominant and biquasitriangular operators.*

Remark 4.1.2. *Recall that $A \in B(H)$ is said to be dominant if for each $\lambda \in C$ there exists a positive number M_λ such that*

$$(A-\lambda)(A-\lambda)^* \leq M_\lambda (A-\lambda)^*(A-\lambda)(seeWilliam[1980]). \tag{4.1}$$

If constant M_λ are bounded by a positive number M, that is $M_\lambda < M$, then A is said to be M-hyponormal (see Khalagai and Nyamai [1998]).

We have the following set inclusions among the classes of operators mentioned above.

$$\{Normal\} \subset \{hyponormal\} \subset \{M - Hyponormal\} \subset \{Dominant\}$$

and

$$\{Normal\} \subset \{biquasitriangular\}.$$

From Mecheri, Salah [2016] it is a known fact that an operator A need not be hyponormal even though A and A^* are both M-hyponormal and from [Douglas [1966], Theorem 1] it is clear that every hyponormal operator is dominant. We require the following sequence of theorems;

Theorem 4.1.3. *L.R.William[1980]. Let A and B be dominant operators having Durnford's property (C) and are also quasisimilar with one of the intertwining quasi-affinities compact. Then $\sigma_e(A) = \sigma_e(B)$ and $\sigma(A) = \sigma(B)$.*

Theorem 4.1.4. *B.P.Duggal[2014]. Let $A \in B(H)$. Then $A \in BQT$ if and only if $\sigma_{SF_+}(A) = \sigma_{SF_-}(A) = \sigma_e(A) = \sigma_\omega(A)$.*

Theorem 4.1.5. *L.R.William[1980]. If A is a dominant operator, then A is biquasitriangular if and only if* $\sigma(A) = \sigma_{ap}(A)$.

Theorem 4.1.6. *L.R.William[1980]. Given A and B as quasisimilar pure dominant operators with one of the intertwining quasi-invertibles compact. Then we have* $\sigma_e(A) = \sigma(A)$ *and* $\sigma_e(B) = \sigma(B)$.

Remark 4.1.7. *We note that in Theorem 3.2.4 above the equality* $\sigma_e(A) = \sigma_\omega(A)$ *or* $\sigma_\omega(A) = \sigma(A)$ *follows trivially since* $\sigma_e(A) \subset \sigma_\omega(A) \subset \sigma(A)$. *In our results we strive to look at cases where equality involving Weyl spectrum is explicit.*

From the above sequence of theorems we obtain following results;

4.2 Main Results

Theorem 4.2.1. *Maina and Khalagai[2016]. Let* $A, B \in B(H)$ *be quasisimilar dominant operators, which are biquasitriangular and satisfying Dunford's property (C) with one of the intertwining quasiaffinities compact. Then* $\sigma_e(A) = \sigma_e(B)$, $\sigma_\omega(A) = \sigma_\omega(B)$ *and* $\sigma(A) = \sigma(B)$.

Proof. Since A and B are biquasitriangular, implies $\sigma_e(A) = \sigma_\omega(A)$ and $\sigma_e(B) = \sigma_\omega(B)$ (see B.P.Duggal [2005]). But A and B are dominant operators which satisfy Durnfod's property (C). Hence by Theorem 3.2.1, we have $\sigma_e(A) = \sigma_e(B)$ and $\sigma(A) = \sigma(B)$, (see B.P.Duggal [2011]), thus $\sigma_e(A) = \sigma_\omega(A) = \sigma_e(B) = \sigma_\omega(B)$, that is $\sigma_\omega(A) = \sigma_\omega(B)$. Hence $\sigma_e(A) = \sigma_e(B)$, $\sigma_\omega(A) = \sigma_\omega(B)$ and $\sigma(A) = \sigma(B)$. □

Corollary 4.2.2. *Maina and Khalagai[2016]. Let* $A, B \in B(H)$ *be quasisimilar dominant and biquasitriangular operators, such that AB and BA are also quasisimilar dominant and biquasitriangular operators satisfying Durnford's property (C), if A and B are quasiaffinities with one of them compact, then we have* $\sigma_e(AB) = \sigma_e(BA)$, $\sigma_\omega(AB) = \sigma_\omega(BA)$ *and* $\sigma(AB) = \sigma(BA)$.

Proof. It can easily be verified that in this case AB and BA are quasisimilar, since $(AB)A = A(BA)$ and $(BA)B = B(AB)$. Hence the result follows from the Theorem 3.2.6. □

Remark 4.2.3. *From B.P.Duggal [1996], note that if the operators A and B are restricted to the class of M-hyponormal, then in Theorem 3.2.1 we can drop the Dunford's property (C) as the following corollary shows.*

Corollary 4.2.4. *Maina and Khalagai[2016]. Let $A, B \in B(H)$ be quasisimilar M-hyponormal and biquasitriangular operators. Then we have $\sigma_\omega(A) = \sigma_\omega(B)$.*

Proof. Since A and B are quasisimilar M-hyponormal operators, we have by the result of (William [1980]) that $\sigma_e(A) = \sigma_e(B)$. But A and B are biquasitriangular implies $\sigma_e(A) = \sigma_\omega(A)$ and $\sigma_e(B) = \sigma_\omega(B)$ by Theorem 3.2.2. Thus $\sigma_\omega(A) = \sigma_e(A) = \sigma_\omega(B) = \sigma_e(B)$. □

Corollary 4.2.5. *Maina and Khalagai[2016]. Given $A, B \in B(H)$ a quasi-invertibles with AB and BA as M-hyponormal and biquasitriangular operators. Then we have $\sigma_\omega(AB) = \sigma_\omega(BA)$.*

Proof. In this case AB and BA are quasisimilar and result follows from Colollary 3.2.7. □

Remark 4.2.6. *Note that for an operator $A \in B(H)$ it has been said that A is consistent in invertibility (with respect to multiplication) or briefly that A is CI operator if for each $B \in B(H)$ AB and BA are invertible or non invertible together. Thus A is CI operator implies $\sigma(AB) = \sigma(BA)$. It is clearly shown that if $A \in B(H)$ is quasiinvertible (quasiaffinity), then A is a CI operator (see Luketero et al[2015]). In view of this result we have the following theorem;*

Theorem 4.2.7. *Maina and Khalagai[2016]. Given $A \in B(H)$ a quasiinvertible and $B \in B(H)$ is such that AB and BA are quasisimilar M-hyponormal operators, then $\sigma(AB) = \sigma(BA)$ and $\sigma_e(AB) = \sigma_e(BA)$.*

Proof. Since A CI operator (see Luketero et al[2015]), it follows that for any $B \in B(H)$, $\sigma(AB) = \sigma(BA)$ (see B.P.Duggal[2005]). The fact that $\sigma_e(AB) = \sigma_e(BA)$ follows from result by (Luketero et al[2015]). □

Corollary 4.2.8. *Maina and Khalagai[2016]. Given $A \in B(H)$ a quasiaffinity, $B \in B(H)$ be such that AB and BA are both M-hyponormal and biquasitriangular quasisimilar operators. Then we have $\sigma(AB) = \sigma(BA)$, $\sigma_e(AB) = \sigma_e(BA)$ and $\sigma_\omega(AB) = \sigma_\omega(BA)$.*

Proof. We note that $\sigma(AB) = \sigma(BA)$ follows from the fact that A is quasiaffinity and hence CI operator, while $\sigma_e(AB) = \sigma_e(BA)$ follows from result by (Luketero et al [2015]), and $\sigma_\omega(AB) = \sigma_\omega(BA)$ follows from the fact that AB and BA are biquasitriangular. □

Remark 4.2.9. *Duggal and Kubrusly [2014] proved the following result which shows that BQT operators are invariant under similarity.*

Theorem 4.2.10. *Duggal and Kubrusly [2014]. Given A, B and S such that $A \in BQT$, S is invertible and $AS = SB$. Then $B \in BQT$.*

The following result attempts to extend Theorem 3.2.15 to the property of quasisimilarity.

Theorem 4.2.11. *Maina and Khalagai[2016]. Let A and B be dominant operators which are quasisimilar with one of the intertwining quasi-invertibles compact and satisfy Dunford's condition (C). If $A \in BQT$ then $\sigma_a(B) = \sigma_a(A)$.*

Proof. In view of Theorem 3.2.1 we have $\sigma(A) = \sigma(B)$. Now by Theorem 3.2.3 we have that $\sigma(A) = \sigma_a(A)$, but A and B are dominant quasisimilar and satisfy Dunford's property (C) implies $\sigma_e(A) = \sigma_e(B)$, $\sigma(A) = \sigma(B)$ B.P.Duggal [2011], thus $\sigma(A) = \sigma_a(A)$ and $\sigma(B) = \sigma_a(B)$, that is $\sigma_a(B) = \sigma_a(A)$, since A and B are quasisimilar. □

Corollary 4.2.12. *Maina and Khalagai[2016]. Let $A, B \in B(H)$ be quasiaffinities with one of them compact. If AB and BA are dominant operators satisfying Dunford's condition (C), then we have that $AB \in BQT$ implies $\sigma_a(BA) \subset \sigma_a(AB)$.*

Proof. In this case AB and BA are quasisimilar since $(AB)A = A(BA)$ and $(BA)B = B(AB)$. Hence the result follows from Theorem 3.2.16. □

Remark 4.2.13. *Theorem 3.2.10 we do not know whether the condition* $\sigma_a(B) \subset \sigma_a(A)$ *for* $A \in BQT$ *would imply* $B \in BQT$*. If the answer to this question is affirmative then we can assert that the class of biquasitriangular intersection dominant operators which satisfy Durnford's condition (C) is invariant under quasisimilarity. Consequently this would generalise Theorem 3.2.10.*

Chapter 5

Weyl Spectrum and quasisimilarity of ω-Hyponormal operators

5.1 Introduction

In this chapter our task is to look at the classes of ω-hyponormal operators, by investigating the conditions under which two quasisimilar ω-hyponormal operators say A and B have equal Weyl spectrum. It is a known fact (see Aluthge and Wang[2000]), that every $A \in B(H)$ has decomposition $A = A_1 \oplus A_2$, and every $A \in B(H)$ has a decomposition given by $A = U \mid A \mid$, with U partial isometry and $\mid A \mid = (A^*A)^{\frac{1}{2}}$ (see Aluthge and Wang [2000]). Thus every ω-hyponormal A has a direct sum decomposition [20], given by $A = A_1 \oplus A_2$, where A_1 is also ω-hyponormal operator (see Rashid et al [2008]).

Remark 5.1.1. *Note that from the flow of the extant literature, authors are hardly considering equality of Weyl spectrum for quasisimilar operators. In this chapter we attempt to bridge this gap by mainly considering ω-hyponormal operators and possible subclasses.*

Remark 5.1.2. *Note that in Duggal [1996], if A is a p-quasihyponormal with dense range, then*

A is p-hyponormal, so p-hyponormal operators is p-quasihyponormal. Also if A has a polar decomposition $A = U \mid A \mid$, with U a partial isometry, then $\tilde{A} = \mid A \mid^{\frac{1}{2}} U \mid A \mid^{\frac{1}{2}}$, with $\tilde{A} \in H(p + \frac{1}{2})$, which defines semi-hyponormality (see Aluthge [1990]). And if we define the 2^{nd} Aluthge transforms of A, then we obtain hyponormal operator. In (Aluthge [1990]), the 1^{st} Aluthge transform denoted by operator $\tilde{A}$ is defined $\tilde{A} = \mid A \mid^{\frac{1}{2}} U \mid A \mid^{\frac{1}{2}}$. Also recall that A is ω-hyponormal if $\mid \tilde{A} \mid \geq \mid A \mid \geq \mid \tilde{A} \mid^*$.

We have the following set inclusions among the classes of operators under consideration.

$$Normal \subset Hyponormal \subset log - hyponormal \subset \omega - hyponormal,$$

$$Normal \subset hyponormal \subset p - hyponormal \subset p - quasihyponormal \subset \omega - hyponormal$$

and

$$Normal \subset Biquasitriangular$$

5.2 Log-hyponormal operators

Recall $A \in B(H)$ is log-hyponormal implies A is invertible and satisfies the following inequality

$$log(A^*A) \geq log(AA^*).$$

This section investigate conditions under which two quasisimilar log-hyponormal operators have equal Weyl spectrum.

Remark 5.2.1. *Note that every decomposable operator on H is biquasitriangular (see Funza [1971]). Also note that if A is a log-hyponormal operator, then the 1^{st} Aluthge transformation is semi-hyponormal and 2^{nd} Aluthge transform is hyponormal (see Aluthge [1990] and Tanahashi [1999]).*

In order to develope the main result, we first require the following sequence of results.

Theorem 5.2.2. *I.H.Jeon[2008]. Let $A \in B(H)$ be log-hyponormal. Then $A = A_1 \oplus A_2$ on $H = H_1 \oplus H_2$ where A_1 is normal and A_2 is pure and log-hyponormal.*

Theorem 5.2.3. *B.P.Duggal[2002]. If A is a log-hyponormal operator, then $\sigma_*(A) = \sigma_*(\tilde{A})$. Where σ_* denotes the $\sigma(A)$, $\sigma_p(A)$, $\sigma_{ap}(A)$, $\sigma_e(A)$ or $\sigma_\omega(A)$.*

Theorem 5.2.4. *B.P.Duggal[2002]. Let A be log-hyponormal, then Weyl's theorem holds for A; that is*

$$\sigma_\omega(A) = \sigma(A) - \pi_{00}(A)$$

. Where $\pi_{00}(A)$ is the set of isolated eigenvalues of finite multiplicity.

Theorem 5.2.5. *S.Jung[2011]. If log-hyponormal operators A and B are quasisimilar, then*

$$\sigma(A) = \sigma(B),$$

$$\sigma_e(A) = \sigma_e(B)$$

.

In this direction we now prove the following results.

Theorem 5.2.6. *Maina and Khalagai[2016]. Let $A, B \in B(H)$ be quasisimilar log-hyponormal operators which are also biquasitriangular, then $\sigma_e(A) = \sigma_e(B)$, $\sigma_\omega(A) = \sigma_\omega(B)$ and $\sigma(A) = \sigma(B)$.*

Proof. Since A and B are log-hyponormal, their polar decomposition are $A = U \mid A \mid$ and $B = V \mid B \mid$, let $\tilde{\mathrm{A}}$ and $\tilde{\mathrm{B}}$ be the 1^{st} Aluthge transforms of A and B. Since A and B are quasisimilar and from Theorem 4.2.3 $\sigma(A)$=$\sigma(\tilde{\mathrm{A}})$, $\sigma_e(A)$=$\sigma_e(\tilde{\mathrm{A}})$ and $\sigma_\omega(A)$=$\sigma_\omega(\tilde{\mathrm{A}})$. Similarly $\sigma(B)$=$\sigma(\tilde{\mathrm{B}})$, $\sigma_e(B)$=$\sigma_e(\tilde{\mathrm{B}})$ and $\sigma_\omega(B)$=$\sigma_\omega(\tilde{\mathrm{B}})$ (see Yingbin and Zikun [2000]), but quasisimilar log-hyponormal have equal spectra and essential spectra by Theorem 4.1.6, also A and B being biquasitriangular implies $\sigma_e(A) = \sigma_\omega(A)$ and $\sigma_e(B) = \sigma_\omega(B)$ (see A.H.Kim [2006]). Hence we have the result follows from the inclusion $\sigma_e(A) \subseteq \sigma_\omega(A) \subseteq \sigma(A)$ for $A \in B(H)$, thus $\sigma_e(A) = \sigma_e(B)$, $\sigma_\omega(A) = \sigma_\omega(B)$ and $\sigma(A) = \sigma(B)$. □

Lemma 5.2.7. *Maina and Khalagai[2016]. Given A and B as quasisimilar log-hyponormal operators and biquasitriangular with $\bar{A}$ and $\bar{B}$ also biquasitriangular then, $\sigma_\omega(\tilde{A})$=$\sigma_\omega(\tilde{B})$.*

Proof. Since A and B are quasisimilar and log-hyponormal implies their 1^{st} Aluthge transforms are semi-hyponormal. But it is a known fact from the literature review that quasisimilar p-hyponormal operators have equal spectra and essential spectra, also since Ã and B̃ are biquasitriangular, the result follows from Theorem 4.2.6. □

Corollary 5.2.8. *Maina and Khalagai[2016]. Given $A, B \in B(H)$ with AB and BA as log-hyponormal and biquasitriangular operators, if A and B are quasiaffinities with one of them compact, then we have $\sigma_e(AB) = \sigma_e(BA)$, $\sigma_\omega(AB) = \sigma_\omega(BA)$ and $\sigma(AB) = \sigma(BA)$*

Proof. It can easily be verified that in this case AB and BA are quasisimilar since $(AB)A = A(BA)$ and $(BA)B = B(AB)$, then the result follows from Theorem 4.2.6, that is $\sigma_e(AB) = \sigma_e(BA)$, $\sigma_\omega(AB) = \sigma_\omega(BA)$ and $\sigma(AB) = \sigma(BA)$. □

5.3 ω-hyponormal operators

Ahmed B.et al [2016] proved that if A is p-hyponormal and its Aluthge transform Ã=$| A |^{\frac{1}{2}} U | A |^{\frac{1}{2}}$ is normal, then A is normal and A=Ã. In (Duggal [2005]) it was proved that if A is ω-hyponormal, $ker(A) \subset ker(A^*)$ and its Aluthge transform Ã is normal, then A is normal and A=Ã. To construct our result we first consider the following results:

Theorem 5.3.1. *M.Putinar[1992]. Let A be invertible ω-hyponormal and completely non-normal operator. Then the point spectrum of A is empty.*

Theorem 5.3.2. *M.Putinar[1992]. If A is ω-hyponormal operator, then $\sigma_\omega(A) = \sigma(A) - \pi_{00}(A)$.*

Lemma 5.3.3. *Aluthge and Wang[2000]. For any operator A, $\sigma(A)=\sigma(\tilde{A})=\sigma(\tilde{\tilde{A}})$.*

Theorem 5.3.4. *C.Muneo[2005]. Let $A, B \in B(H)$ be injective ω-hyponormal operators, if A and B are quasisimilar, then they have same spectra and essential spectra.*

Theorem 5.3.5. *F.Kimura[2004]. If $A_i \in B(H_i)$, $(i = 1, 2)$ are quasi-similar ω-hyponormal operators, then $\sigma(A_1) = \sigma(A_2)$ and $\sigma_e(A_1) = \sigma_e(A_2)$.*

Theorem 5.3.6. *M.Putinar[1992]. Given A^* as p-hyponormal, B as ω-hyponormal and $XA = BX$ for $X \in B(H)$ quasi-affinity, then $XA^* = B^*X$.*

Theorem 5.3.7. *M.Putinar[1992]. Let A be ω-hyponormal and N normal operator. If $X \in B(H)$ has dense range with $AX = AN$, then A is also a normal operator.*

Corollary 5.3.8. *Let A as ω-hyponormal operator with $kerA \subset kerA^*$ and $B \in B(H)$ be a normal operator. If $AX = XB$ where X is a quasi-affinity, then A and B are unitarily equivalent normal operators.*

From the immediate results, we now prove the following results.

5.3.1 Main Results

Theorem 5.3.9. *Maina and Khalagai[2016]. Let A^* be p-hyponormal, $B \in B(H)$ be ω-hyponormal, where A and B are quasiaffinities. If $XA = BX$ for $X \in B(H)$ a quasiaffinity, then A and B are unitarily equivalent normal operators. Consequently $\sigma(A) = \sigma(B)$, $\sigma_e(A) = \sigma_e(B)$ and $\sigma_\omega(A) = \sigma_\omega(B)$.*

Proof. Since $XA = BX$ we have by Theorem 4.3.6 that $XA^* = B^*X$, thus $XA = BX$ and $XA^* = B^*X$ ....(1). Now by associativity property of operators we have

$$B(XA) = (BX)A.$$

Let $Y = XA = BX$. Then Y is a quasiaffinity and we have

$$YA = BY.$$

Now by Theorem 4.3.6 again $YA^* = B^*Y$ ....(2). By (2) we have

$$BXA^* = B^*XA.$$

By (1) we have $B^*XA = B^*BX$ and $BXA^* = BB^*X$. Since $B^*XA = BXA^*$ we have

$$BB^*X = B^*BX,$$

that is

$$(BB^* - B^*B)X = 0$$

and $BB^* = B^*B$ since X is a quasiaffinity. Similarly, by (1) we have $XAA^* = BXA^*$ and $XA^*A = B^*XA$, that is,

$$XAA^* = XA^*A,$$

implies

$$X(AA^* - A^*A) = 0$$

thus $AA^* = A^*A$ so that A and B are normal operators. Now taking adjoint on $XA^* = B^*X$ gives $AX^* = X^*B$. Thus $XA = BX$ and $AX^* = X^*B$ where X is quasiaffinity implies A and B are quasisimilar normal operators by Corollary 4.3.8, A and B are unitarily equivalent and $\sigma_\omega(A) = \sigma_\omega(B)$,$\sigma_e(A) = \sigma_e(B)$ and $\sigma(A) = \sigma(B)$. □

Corollary 5.3.10. *Maina and Khalagai[2016]. Given A^* p-hyponormal and B ω-hyponormal with A, B quasi-affinities. Then AB and BA are unitarily equivalent operators. Consequently $\sigma(AB) = \sigma(BA)$, $\sigma_e(AB) = \sigma_e(BA)$ and $\sigma_\omega(AB) = \sigma_\omega(BA)$.*

Proof. It can be easily verified that in this case AB and BA are quasisimilar, since $(AB)A = A(BA)$ and $(BA)B = B(AB)$, and the result follows from theorem above, that is $\sigma(AB) = \sigma(BA)$, $\sigma_e(AB) = \sigma_e(BA)$ and $\sigma_\omega(AB) = \sigma_\omega(BA)$. □

Lemma 5.3.11. *Maina and Khalagai[2016]. Given $A \in B(H)$ a ω-hyponormal operator with $\sigma(A) = \sigma(\tilde{A})$. Then, $\sigma_\omega(A) = \sigma_\omega(\tilde{A})$.*

Proof. Since A is ω-hyponormal operator, then by (Aluthge and Wang [2000]), $\tilde{A}$ is semi-hyponormal, thus $\sigma_\omega(\tilde{A})$=$\sigma(\tilde{A})$-$\pi_{00}(\tilde{A})$. Now by (Duggal and Kubrusly [2014]) $\sigma(A) = \sigma(\tilde{A})$. Thus, $\sigma_\omega(A) = \sigma(A) - \pi_{00}(A) = \sigma(\tilde{A})$-$\pi_{00}(\tilde{A})$=$\sigma_\omega(\tilde{A})$. That is, $\sigma_\omega(A) = \sigma_\omega(\tilde{A})$ and result follows immediately. □

Theorem 5.3.12. *Maina and Khalagai[2016]. Given $A, B \in B(H)$ as quasisimilar ω-hyponormal and biquasitriangular operators, we have $\sigma_e(A) = \sigma_e(B)$, $\sigma_\omega(A) = \sigma_\omega(B)$ and $\sigma(A) = \sigma(B)$.*

Proof. Since A and B are quasisimilar ω-hyponormal, we have $\sigma_e(A) = \sigma_e(B)$ and $\sigma(A) = \sigma(B)$ (see An-Hyun Kim [2006]). But A and B are biquasitriangular implies $\sigma_e(A) = \sigma_\omega(A)$ and $\sigma_e(B) = \sigma_\omega(B)$ (see Duggal and Kubrusly [2014]). Always $\sigma_e(A) \subset \sigma_\omega(A) \subset \sigma(A)$ and $\sigma_e(B) \subset \sigma_\omega(B) \subset \sigma(B)$ for any bounded operators A and B (see Duggal [2011]). From the above fact the result follows immediately. □

Remark 5.3.13. *Note any $A \in B(H)$ we say that A is consistent in invertibility (with respect to multiplication) if for any $B \in B(H)$ AB and BA are invertible or non invertible together. Thus A is completely invertible operator implies $\sigma(AB) = \sigma(BA)$. By (Luketero et al[2015]) if $A \in B(H)$ is quasiinvertible (quasiaffinity), then A is CI operator.*

In this direction we have the following result.

Theorem 5.3.14. *Maina and Khalagai[2016]. Let $A, B \in B(H)$ be ω-hyponormal and biquasitriangular operators such that AB and BA are quasi-affinities, with one of the quasi-affinity compact Then we have $\sigma(AB) = \sigma(BA)$, $\sigma_e(AB) = \sigma_e(BA)$ and $\sigma_\omega(AB) = \sigma_\omega(BA)$.*

Proof. Using the fact from Luketero et al [2015], it follows that $A, B \in B(H)$, $\sigma(AB) = \sigma(BA)$ (see Muneo Cho [2005]). The fact that $\sigma_e(AB) = \sigma_e(BA)$ follows from Luketero et al[2015] and $\sigma_\omega(AB) = \sigma_\omega(BA)$ follows from the biquasitriangularity property. □

F.Kimura [2004] showed that any operator $A \in B(H)$ has property (β) if $\tilde{A}$ has Bishop's property (β). Thus we obtain the following result;

Corollary 5.3.15. *Maina and Khalagai[2016]. Given $A, B \in B(H)$ quasisimilar ω-hyponormal operators with A and B satisfying Bishop's property (β), and $\sigma_e(\tilde{A})$=$\sigma_e(\tilde{B})$. Then, $\sigma_\omega(\tilde{A})$=$\sigma_\omega(\tilde{B})$.*

Proof. First recall for operator $A, B \in B(H)$ implies $\sigma(\tilde{A})$=$\sigma(A)$ and $\sigma(\tilde{B})$=$\sigma(B)$ (see Springer proceedings in mathematics [2015]). But A and B are ω-hyponormal and satisfy property (β) implies $\tilde{A}$ and $\tilde{B}$ also satisfies property (β) (see B.P.Duggal [2011]). But $\tilde{A}$ and $\tilde{B}$ are p-hyponormal operators (see An-Hyun Kim [2006]) and quasisimilar. With the above facts the result follows immediately and, $\sigma_e(\tilde{A})$=$\sigma_e(\tilde{B})$ and $\sigma_\omega(\tilde{A})$=$\sigma_\omega(\tilde{B})$. □

The restriction of a ω-hyponormal to its reducing subspace is also ω-hyponormal operator by (Rashid [2014]). We have seen that if A is ω-hyponormal then, it is of the form $A = A_1 \oplus \lambda I$ on

$$H = ker(A - \lambda I) \oplus ker(A - \lambda I)^{\perp},$$

where A_1 is ω-hyponormal operator with $ker(A_1 - \lambda I) = 0$ (see Rashid [2015]).

Chapter 6

Weyl Spectrum and quasisimilarity of hypercyclic Operators

6.1 Introduction

Halmos [1967] observed that the first proof of the existence of invariant subspaces for compact operators used the fact that such operators are very close to actually having an upper triangular matrix in some orthonormal basis. Thus introduced the concept of quasitriangular operator. This is simply an operator which can be written as an arbitrary small compact perturbation of an upper triangular matrix. So if A is reduced by subspaces of H then H can also be decomposed directly as

$$H = H_1 \oplus H_2$$

and relative to this A can have a matrix decomposition given by

$$A = \begin{pmatrix} A \setminus H_1 & A_0 \\ 0 & A_2 \end{pmatrix}$$

for operator

$$A_0 : H_2 \longrightarrow H_1 \tag{6.1}$$

and

$$A_2 : H_2 \longrightarrow H_2 \tag{6.2}$$

where $A \setminus H_1$ is the restriction of A to H_1. Conversely, if an operator A can be written as the triangulation

$$A = \begin{pmatrix} A_1 & A_0 \\ 0 & A_2 \end{pmatrix}$$

in terms of the decomposition

$$H = H_1 \oplus H_2,$$

then

$$A_1 = A \setminus H_1 : H_1 \longrightarrow H_1.$$

It follows that $A_0 = 0$ if and only if H_1 reduces A. This result is found in (W.Lee[2001]). Thus the direct summands of an operator A, are just the restrictions of A to reducing subspaces. Recall that given $x \in H$ and A operator on H. The set $\{x, Ax, A^2x, ..., A^nx, ..\}$ is the orbit of x under A (seeEungil Ko [2006]). If some orbit is dense in H then A is hypercyclic operator and x a hypecyclic vector of A (see Eungil Ko [2006]). Recall that $SP(A)$ denotes the spectral picture of $A \in B(H)$ consisting of the essential spectrum , hole in $\sigma_e(A)$ which is a non empty bounded component of $C \setminus \sigma_e(A)$ and a pseudohole in $\sigma_e(A)$ which is a non empty component of $\sigma_e(A) \setminus \sigma_{le}(A)$ or of $\sigma_e(A) \setminus \sigma_{re}(A)$. Recall that in our literature review we have already stated that Duggal and Kubrusly [23] showed that $A \in BQT$ if and only if $\sigma_e(A)$ has no holes and pseudoholes. Also (William [1980]) showed that for A and B biquasitriangular operators implies $\sigma_e(A) = \sigma_e(B)$.

Theorem 6.1.1. *S.Mecheri[2016] If A is not biquasitriangular then, either A or A^* has an eiqenvalue.*

Theorem 6.1.2. *S.Mecheri[2016]. Suppose that $A \in L(H)$ and $SP(A)$ contains no holes or pseudohole associated with a negative number, then A is quasitriangular.*

Theorem 6.1.3. *B.P.Duggal[2014]. Let $A \in B(H)$ with adjoint A^*, if both A and A^* have the SVEP, then A is biquasitriangular.*

Theorem 6.1.4. *[Cui, J. 2007]. Let $A \in B(H)$. Then the following statements are equivalent:*

$$A \in (BQT)$$

and

$$A = A_0 \oplus K,$$

where K is compact and A_0 is quasisimilar to a normal operator.

Remark 6.1.5. *It is clear from (Foias et al [1976]), that property of being biquasitriangular is not preserved under quasisimilarity by giving several examples, though there exists a biquasitriangular operator that is quasisimilar to non-quasitriangular operator. If we let $\theta_u(A) = \{UAU^* : U$ is unitary in B(H)$\}$ where the norm closure of $\theta_u(A)$ is given by $\bar{}\theta_u(A)$.*

Theorem 6.1.6. *C.Apostol[1973] Let A be an operator in $B(H)$, then there exist operators $B \in BQT$ and A' in $\bar{}\theta_u(A)$ (equivalently,A in $\bar{}\theta_u(A)$) with A' quasisimilar to B and $\sigma(A) = \sigma(B)$ and $\sigma_e(A) = \sigma_\omega(B)$.*

We note that in Theorem 5.1.6 the equality $\sigma_e(A) = \sigma_\omega(A) = \sigma(A)$, follows trivially since $\sigma_e(A) \subseteq \sigma_\omega(A) \subseteq \sigma(A)$. In our results we try to look at cases where equality involving Weyl spectrum is explicit.

6.1.1 Results

Theorem 6.1.7. *Let $A \in (BQT)$ and $B \in (BQT)_{qs}$, with A and B satisfying either Durnford's property (C) or property β, with $\sigma_e(A) = \sigma_e(B)$ by [0]. Then $\sigma_\omega(A) = \sigma_\omega(B)$ and $\sigma(A) = \sigma(B)$.*

Proof. Since A is biquasitriangular, then

$$\sigma_e(A) = \sigma_\omega(A) \tag{6.3}$$

and B being quasisimilar to some biquasitriangular operators implies that B is in $^{-}\theta_u(B)$ (see Duggal [1996]), thus A and B are quasisimilar and

$$\sigma_e(A) = \sigma_\omega(B) and \sigma(A) = \sigma(B) \tag{6.4}$$

Thus A and B are quasisimilar and satisfy either Durnfod's property or property (β). Indeed

$$\sigma_e(A) = \sigma_e(B) and \sigma(A) = \sigma(B). \tag{6.5}$$

Thus from equations 5.3, 5.4 and 5.5 and the fact that $\sigma_e(A) \subseteq \sigma_\omega(A) \subseteq \sigma(A)$ for any $A \in B(H)$ indeed $\sigma_\omega(A) = \sigma_\omega(B)$. Since $\sigma_e(A) = \sigma_e(B)$ (see B.P.Duggal [2005]), then $\sigma_\omega(A) = \sigma_\omega(B)$ and $\sigma(A) = \sigma(B)$. □

A normal operator is known to satisfy the SVEP, which is a useful property in the theory of spectral decomposition. Thus for any $A \in B(H)$ with adjoint A^*, if both A and A^* have the SVEP, then A is biquasitriangular. It follows that every decomposable operator on Hilbert space is biquasitriangular, since decomposable operators and their adjoints have the SVEP (see C.Apostol [1973]). It is well known (see Duggal and Kubrusly [2014]) that the operator $A = A_1 \oplus A_2 \in B(H_1 \oplus H_2)$. Thus we have the following;

Corollary 6.1.8. *Let $A, B \in (BQT)$, with $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ quasisimilar and satisfying Durnford's property and property (β), then*

$$\sigma_\omega(A_1) = \sigma_\omega(B_1).$$

Proof. From (S.Funza [1971]) we have that every decomposable operator on H is biquasitriangular, thus $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, where A_1 and B_1 are quasisimilar to normal operators. Thus A_1 and B_1 both have the SVEP and so are biquasitriangular. Implies both A_1 and B_1 are biquasitriangular normal operators and quasisimilar normal operators have equal spectra and essential spectra (see An-Hyun Kim [2006]), implying that $\sigma_e(A_1) = \sigma_e(B_1)$ and $\sigma(A_1) = \sigma(B_1)$.

But a normal operator $A \in B(H)$ is biquasitriangular if and only if $\sigma(A) = \sigma_{ap}(A)$, for any $A \in B(H)$, and biquasitriangularity implies $\sigma_e(A) = \sigma_\omega(B)$. Indeed the result follows from the inclusion $\sigma_e(A) \subset \sigma_\omega(A) \subset \sigma(A)$, that is

$$\sigma_\omega(A_1) = \sigma_\omega(B_1).$$

□

Remark 6.1.9. *We note that (Duggal and Kubrusly [2014]) were able to show that the class of biquasitriangular operators is stable under similarities and under perturbation by compact operators. This result has a natural extension to being invariant under perturbations by commuting (compact) operators (see Duggal [2011]). This lead to the following result*

Corollary 6.1.10. *Given $A, B \in B(H)$ with AB and BA are biquasitriangular operators satisfying Dunford's property (C), if A and B are quasiaffinities with one of them compact, then we have $\sigma_e(AB) = \sigma_e(BA)$, $\sigma_\omega(AB) = \sigma_\omega(BA)$ and $\sigma(AB) = \sigma(BA)$*

Proof. It can easily be verified that in this case AB and BA are quasisimilar and since they are biquasitriangular we have $\sigma_e(AB) = \sigma_\omega(AB)$, since $(AB)A = A(BA)$ and $(BA)B = B(AB)$ and satisfy property (C), we have $\sigma_e(AB) = \sigma_e(BA)$. Hence the result follows from the Theorem 5.1.6. □

6.2 On hypercyclic operator

Here we intend to investigate the conditions under which quasisimilar hypercyclic operators have equal Weyl spectrum.

Remark 6.2.1. *Note that in restricting operators A and B to the class of hypercyclic and biquasitriangular operators, then in Corollary 5.1.10 we can drop the Durnford's property (C) or*

Bishop's property (β). Also we note that (V.Matache [1993]) gave various theorems related to the class of hypercyclic and biquasitriangular operators as given below;

Theorem 6.2.2. *V.Matache[1993]. If A is a hypercyclic operator, such that there exist non constant polynomial p, $B = p(A)$ then B is quasitriangular.*

Theorem 6.2.3. *V.Matache[1993]. If A is hypercyclic and $\sigma(A)$ has no interior point, then A is biquasitriangular.*

Theorem 6.2.4. *V.Matache[1993]. Given H and K as Hilbert spaces, and X dense range operator acting from H into K. Let A be a hypercyclic operator on H and B an operator on K. If X intertwines the pair (A, B), that is if $XA = BX$, then B is hypercyclic.*

Remark 6.2.5. *From Theorem 5.2.5, if X and Y are intertwiners with dense ranges and are both one-to-one such that $XA = BX$ and $AY = YB$, then A and B are quasisimilar hypercyclic operators (see An-Hyun Kim [2006]). Also Katai [1982] observed that hypercyclic operators are quasitriangular and (Herrero [1991]), proved that a Hilbert space hypercyclic operator A has the property $\sigma(A) = \sigma_\omega(A)$. Thus we have*

Theorem 6.2.6. *V.Matache[1993]. Let $A \in B(H)$ be hypercyclic and $B \in \{A\}'$, where $\{A\}'$ the set of all operators commuting with A, then, $\partial\sigma(B) = \partial\sigma_e(B)$ and $\sigma(B) = \sigma_\omega(B)$.*

Theorem 6.2.7. *V.Matache[1993]. $A \in B(H)$ is hypercyclic implies $\sigma_e(A) = \sigma_{le}(A)$.*

6.2.1 Result

Theorem 6.2.8. *Let A, B be quasisimilar hypercyclic and biquasitriangular operators, with $\sigma_e(A) = \sigma_e(B)$. Then, $\sigma_\omega(A) = \sigma_\omega(B)$ and $\sigma(A) = \sigma(B)$.*

Proof. Recall that A and B are quasisimilar implies there exists quasi-affinities X and Y such that $AX = XB$ and $YA = BY$. Since A and B are hypercyclic, $\sigma_\omega(A) = \sigma(A)$ and $\sigma_\omega(B) = \sigma(B)$ by

Herrero [33]. But A and B are also biquasitriangular, thus $\sigma_e(A) = \sigma_\omega(A)$ and $\sigma_e(B) = \sigma_\omega(B)$, similarly, A and B are quasisimilar so $\sigma_e(A) = \sigma_e(B)$ and $\sigma(A) = \sigma(B)$ Cao, X. [64], hence result follows with inclussion that $\sigma_e(A) \subseteq \sigma_\omega(A) \subseteq \sigma(A)$. □

Corollary 6.2.9. *Given $A, B \in B(H)$ with AB and BA are hypercyclic and biquasitriangular operators. If A and B are quasiaffinities, then we have $\sigma_e(AB) = \sigma_e(BA)$, $\sigma_\omega(AB) = \sigma_\omega(BA)$ and $\sigma(AB) = \sigma(BA)$.*

Proof. It can be verified that in this case AB and BA are quasisimilar, since $(AB)A = A(BA)$ and $(BA)B = B(AB)$ Wei Chen [2009], and the result follows from theorem above. That is, $\sigma_e(AB) = \sigma_e(BA)$, $\sigma_\omega(AB) = \sigma_\omega(BA)$ and $\sigma(AB) = \sigma(BA)$. □

Theorem 6.2.10. *Let $A \in B(H)$ be hypercyclic and $B \in BQT$ with A quasisimilar to B, then $\sigma_\omega(A) = \sigma_\omega(B)$.*

Proof. Since A is hypercyclic, then $\sigma_e(A) = \sigma_{le}(A)$ and from Theorem 5.2.7 $\sigma(A) \subseteq \partial\sigma_e(A) \subseteq \sigma_{le}(A) \cap \sigma_{re}(A) \subseteq \sigma_e(A) \subseteq \sigma(A)$ (see B.P.Duggal [2005]). Similarly, by Theorem 5.2.3 together with Theorem 5.2.7 we obtain $\sigma(A) = \sigma_e(A) = \sigma_\omega(A) = \sigma_{le}(A)$, and in conclusion A is biquasitriangular. Since B is biquasitriangular, we have $\sigma_e(B) = \sigma_\omega(B) = \sigma_{le}(B)$ and by theorem 5.2.8 B is hypercyclic and by Theorem 5.2.5 A and B are quasisimilar, hence the result follows. □

Corollary 6.2.11. *Let $A \in B(H)$ be hypercyclic with $B \in BQT$, if A and B are quasi-affinities then, $\sigma_\omega(AB) = \sigma_\omega(BA)$.*

Proof. From chapter three it has been verified that AB and BA are quasisimilar and the result follows from theorem above. □

Chapter 7

Conclusion and Summary

7.1 Conclusion

Quasitriangular operators are easy to study since they have simple spectral structures. Operators which can be transform as a triangular operators, can as well be decomposed as a direct sum or as a polar decomposition. Fortunately, bounded linear operators acting on $B(H)$ are either triangularizable or reducible, such that every reducible operator can be expressed as a direct sum of a normal and completely non normal operator. Generally, dominant, ω-hyponormal and hypercyclic operators are not only non-normal but also quasitriangular. Using the concept of biquasitriangularity we have made several key contributions by exposing the Weyl spectrum for some classes of quasisimilar operators in Hilbert spaces by considering the equations $XA = BX$ and $AY = YB$ where X and Y are quasi-affinities then A and B are quasisimilar. Biquasitriangularity and dominant operators are dealt with in Chapter three, biquasitriangularity and ω-hyponormal have been given in Chapter four and biquasitriangularity and hypercyclic operators have been given in chapter five. We have extended some results by looking at CI operators. Most of the results in this thesis have shown the Weyl spectra of operators under the quasisimilarity relation.

7.2 Summary of main results

In this thesis, we have made tramendeous contribution especially, the behavior of Weyl spectrum under quasisimilarity of non-normal operators. In chapter one we considered some basic definitions and terminologies which form the basis of the entire work. In chapter two we considered the several contributions by other researchers from which we realise the knowledge gap. The major contributions are found in chapter three where a breakthrough is realised to this end, using the biquasitriangularity concept, we have proved independent result and deduced some valuable consequences. First, recall for instance that, if an operator A is dominant and biquasitriangular with Dunford property (C), then if they are quasisimilar then they have the same Weyl spectrum. Since under no condition, quasisimilar dominant operators fail to have same spectra and essential spectra. The concept is also extended to the product of two operators, Corollary 3.2.7. In chapter four we have tried to investigate the conditions under which two quasisimilar ω-hyponormal operators and its possible subclasses have same Weyl spectrum. In this case we make use of the 1^{st} Aluthge transform to develop the concept of ω-hyponormal operators. In Theorem 4.2.6 we have succeeded in looking at a subclass of ω-hyponormal operators, and found that given $A, B \in B(H)$, as quasisimilar log-hyponormal operators and also biquasitriangul, then $\sigma_e(A) = \sigma_e(B)$, $\sigma_\omega(A) = \sigma_\omega(B)$ and $\sigma(A) = \sigma(B)$. In Theorem 4.3.9 we succeeded by intersecting the ω-hyponormal and p-hyponormal operators, and obtained that if A^* is p-hyponormal and $B \in B(H)$ a ω-hyponormal with A and B quasi-affinities, if $XA = BX$ for $X \in B(H)$ a quasi-affinity, then A and B are unitarily equivalent normal operators. Consequently $\sigma(A) = \sigma(B)$, $\sigma_e(A) = \sigma_e(B)$ and $\sigma_\omega(A) = \sigma_\omega(B)$. In Theorem 4.3.12 we obtain the result by looking at the intersection of ω-hyponormal and biquasitriangular operators,that is, given $A, B \in B(H)$ a quasisimilar ω-hyponormal and biquasitriangular operators, then $\sigma_e(A) = \sigma_e(B)$, $\sigma_\omega(A) = \sigma_\omega(B)$ and $\sigma(A) = \sigma(B)$. Theorem 4.3.14 we looked at the consistent in invertibility operators of ω-hyponormal and biquasitriangular, such that AB and BA are quasi-affinities with one of the quasi-affinity compact. Then $\sigma(AB) = \sigma(BA)$, $\sigma_e(AB) = \sigma_e(BA)$ and $\sigma_\omega(AB) = \sigma_\omega(BA)$. In Chapter five, looked at the behaviour of Weyl spectrum under quasisimilarity of operators which are completely non-normal, of hypercyclic and biquasitriangular operators. And obtain the Weyl spectrum under quasisimilarity relation; that is in Theorem 5.2.9 we obtained that if A and B a quasisimilar hypercyclic and biquasitriangular operators, with $\sigma_e(A) = \sigma_e(B)$, then $\sigma_\omega(A) = \sigma_\omega(B)$ and $\sigma(A) = \sigma(B)$. In Theorem 5.2.11 looked at condition under which A hypercyclic and B biquasitriangular with A quasisimilar to B, then $\sigma_\omega(A) = \sigma_\omega(B)$.

7.3 Recommendation for further Research

Results in this thesis have great contributions. By investigating and looking into the conditions under which Weyl spectrum is equal to both the essential spectrum and spectrum of operators A and B with under quasisimilarity. It is clear that direct decomposition, polar decomposition and diagonalization of operators helps relax the complexities within operators, hence more familar one. When quasisimilarity cannot be instant, more analysis might be carried out to determine structures and properties of the spectra of operators under same relation. For instance, by using the biquasitriangularity property, Single Value Extension Property (SVEP), Durnford‘s property (C), Bishop property (β) as tools, we were able to link properties of Weyl spectrum with both essential spectrum and spectrum of classes of operators under quasisimilarity. This study has produced quite a number of results on quasisimilarity of operators. However, more are still in waiting as far as the research topic is conserned as given below on future research.

- It is well known from this thesis that, quasisimilar dominant operators which satisfy Durnford‘s property (C) and are biquasitriangular have equal Weyl spectrum. The nature of Weyl spectrum of quasisimilar dominant operators is not known when these properties are dropped.

- Under the same properties, it is not clear if Weyl spectrum of dominant operators are equal under almost similarity and metric equivalence.

- Also from this thesis it is clear that, quasisimilar and biquasitriangular log-hyponormal and ω-hyponormal operators have equal Weyl spectrum. Can similar results be obtained when the biquasitriangular property is dropped?. Or can we obtain the same results when the properties are retained but, under almost similarity and metric equivalence.

- Is browder spectrum of classes of operators equal under quasisimilarity relation?.

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Chapter 8

Appendix

8.1 The Publications