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UNIVERSITY OF NAIROBI

COLLEGE OF BIOLOGICAL AND PHYSICAL SCIENCES

SCHOOL OF MATHEMATICS

ON APPLICATIONS OF OPERATORS AND GROUP - THEORETIC CONCEPTS IN SIGNAL PROCESSING AND TELECOMMUNICATION

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Declaration

Declaration by the candidate

This dissertation is my original work and has not been presented for a degree award in any other University

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Dedication

This project is dedicated to my beloved uncle Cyprian Oyugi and my beloved mother Dorcas Ogendi.

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I am grateful to the Almighty GOD who has seen me through my postgraduate studies. The bible has been my guide during my studies and project. The book of Mathew 7:7,"Ask and you shall be given, seek and you shall find, knock and it shall be opened unto you," has kept me in the spirit of working. Indeed GOD'S work is marvelous. My sincere gratitude goes to Dr. Bernard Mutuku Nzimbi for his encouragement, patience, perseverance, understanding and persistent assistance during the whole period of my study. Indeed he is the reason for my love of Pure Mathematics. I would like to acknowledge all my lecturers too in the school of Mathematics : Prof. J. M. Khalagai, Prof. G.P. Prokhariyal, Dr. J. N. Muriuki, Dr. J. H. Were, Dr. S. W. Luketero, Dr. N. Katende and Mr. C. Achola who tirelessly taught me through my postgraduate level. I will not forget to thank Prof. Weke and Dr. J. Ongaro for their tireless encouragement and support before and during my postgraduate studies. Indeed through ICTP- EAUMP, I have increased my knowledge in Pure Mathematics. Finally, I would like to sincerely thank my beloved uncle Cyprian Oyugi, my mother and the rest of my family members for their spiritual, mental and moral support without which I would not have gone this far. May GOD bless you all.

Abstract

Operator theory concepts such as frame operator, Gramian operator, synthesis operator and analysis operator are useful in signal processing and telecommunication. Finite dimensional Hilbert spaces and normalized (unit) vectors are rich in signals required in application. One of the most significant fields of applications and source of questions of group theory and operator theory is frames. The Mercedes- Benz frames are highly applicable in signal processing. In this thesis; we investigate frames, fourier analysis and wavelets and some of their applications in signal processing and telecommunication. We investigate why we need frames over bases, we study the dual frames, some operators: synthesis, analysis, frame and Gramian and finalize frames by looking at group frames. Here, we shall investigate the relationship between group symmetry and frames, representation theory and frames and also study group matrices and the Gramian of a group frame. We shall investigate properties of fourier analysis. We finalize by looking at the applications of frames and wavelets in signal processing and telecommunication.

List of Notations and Abbreviations

Ł$^2(R)$: The space of square integrable functions (with respect to Lebesgue measure).

dim(.) : Dimension of (.).

$Span(.)$: Linear span of (.).

FTT : Fourier Transform Theory.

$\mathbb{Z}$: The set of integers.

$\mathbb{N}$: The set of natural numbers.

$\mathbb{R}$: Denotes the set of all real numbers.

$\mathbb{C}$: The set of complex numbers.

$Ran(T)$: Rank of T.

$\| .\|$: $Norm.$

$\langle , \rangle$:scalar product.

ℓ^2 : set of square summable spaces.

$\mathcal{H}$: Hilbert space.

$T : \mathcal{H} \longrightarrow \mathcal{H}$: Linear operator.

$(e_j)_{j=1}^{M}$: Orthogonal normal basis for $\mathcal{H}$.

$(g_j)_{j=1}^{K}$: Orthonormal basis for $\mathcal{K}$.

T^* : Adjoint of T.

$B(X, Y)$: Banach Space of bounded linear operators.

det(.) : Determinant of(.).

I : Identity operator.

$C([a, b])$: Set of continuous functions on $[a, b]$.

$C_n([a,b])$: Set of functions on $[a,b]$ with continuous derivatives up to order n.

$B(\mathcal{H})$: Banach algebra of bounded linear operators on $\mathcal{H}$.
$G(\mathcal{H},\mathcal{K})$: Set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$.
$\| T \|$: The operator norm of T and invertible.
$\| x \|$: The norm of a vector x.
$\langle x, y \rangle$: The inner product of x and y on a Hilbert space $\mathcal{H}$.
$\mathrm{Ker}(T)$: The kernel of an operator T.
$M \bigoplus M^{\perp}$: The direct sum of the subspaces M and $M^{\perp}$

$\{T\}^{'}$: The commutant of T.

Contents

Chapter 1

PRELIMINARIES

1.1 Terminologies and Notations

We shall make use of the following $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \mathcal{K}, \mathcal{K}_1, \mathcal{K}_2$ to denote Hilbert spaces. A Hilbert space is a complete inner product space. $B(\mathcal{H})$ denotes the Banach algebra of bounded linear operators on $\mathcal{H}$. The Banach algebra of all bounded linear operators from a Hilbert space $\mathcal{H}_1$ into a Hilbert space $\mathcal{H}_2$ shall be denoted by $B(\mathcal{H}_1, \mathcal{H}_2)$ while T, T_1, T_2 etc denotes bounded linear operators. T^* denotes the adjoint of an operator T. $\langle x, y\rangle$ denotes the inner product of two vectors x and y. $\| x \|$ denotes the norm of vector x and $\| T \|$ the norm of operator T. We have a norm on $\mathbb{R}^n$ allowing us to measure distances between points. A norm on a linear space X is a real-valued function $\| . \|: X \to \mathbb{R}$ which satisfies the positivity, non- degenerate, absolute homogeneity and sub-additivity. $M^{\perp}$ denotes the orthogonal complement of M. $Ran(T)$ is the range of T while $Ker(T)$ denotes the kernel of T. We shall also use $M \bigoplus N$ to denote the orthogonal direct sum of M and N. $M_n(\mathbb{C})$ the algebra of n x n matrices over the field of complex numbers. We shall use $A \approx B$ to denote A is similar to B and $A \sim B$ to denote A and B as *injection-similar*. $A \backsimeq B$ where A and B are completely injection-similar. S_M is the symmetric group of permutations of $1, 2, \ldots M$. $GL(\mathcal{H})$ is the general linear group of linear maps a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{H}$.

An operator $T \in B(\mathcal{H})$ is said to be: An involution if $T^2 = I$, self-adjoint or Hermitian if $T^* = T$, unitary if $T^*T = TT^* = I$, normal if $T^*T = TT^*$, an isometry if $T^*T = I$, a co-isometry if $TT^* = I$, a partial isometry if $T = TT^*T$, quasinormal if $T(T^*T) = (T^*T)T$, or equivalently $[T^*T, T] = 0$, binormal if T^*T and TT^* commute, subnormal if it has a normal extension. That is, if there exists a normal operator N on a Hilbert space $\mathcal{K}$ such that H is a subspace of $\mathcal{K}$ and the subspace H is invariant under the operator N and the restriction of N to H coincides with T. That is, $T = N|_H$, hyponormal if $T^*T \geq TT^*$, seminormal if either T or T^* is hyponormal, a scalar if it is a scalar multiple of the identity operator (i.e. $T = \mu I, \mu \in \mathbb{C}$), left shift operator if $Tx = y$, where $x = (x_1, x_2, ...)$ and $y = (x_2, x_3, ...) \in \ell^2$, right shift operator if $Tx = y$, where $x = (x_1, x_2, ...)$ and $y = (0, x_1, x_2, ...) \in \ell^2$. An operator $T \in B(\mathcal{H})$ is said to be pure or completely non-normal (c.n.n.) if there exists no nontrivial reducing subspace $M \subset \mathcal{H}$ such that $T|_M$ (the restriction of T to M) is normal, that is, if T has no direct normal summand. When the subspace M is invariant under the operator T, then T induces a linear operator $T_M = T|_M$ on the space M. The linear operator T_M is defined by $T_M(x) = T(x)$, for $x \in M$. A part of an operator is a restriction of the operator to an invariant subspace.

If $\mathcal{K}$ is a Hilbert space, $H \subset \mathcal{K}$ is a subspace, $S \in B(\mathcal{K})$, and $T \in B(\mathcal{H})$, then S is a dilation of T (and T is a power-compression of S) provided that $T^n = P_H S^n|_H, n = 0, 1, 2, ...$

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. An operator $X \in B(\mathcal{H}, \mathcal{K})$ is invertible if it is injective (one-to-one) and surjective (onto or has dense range); equivalently if $\mathrm{Ker}(X) = 0$ and $\mathrm{Ran}(X) = \mathcal{K}$. We denote the class of invertible linear operators by $G(\mathcal{H}, \mathcal{K})$. The commutator of two operators A and B, denoted by $[A, B]$ is defined by $[A, B] = AB - BA$. The self-commutator of an operator A is $[A^*, A] = A^*A - AA^*$. Two operators $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ are similar (denoted $T \approx S$) if there exists an operator $X \in G(\mathcal{H}, \mathcal{K})$ such that $XT = SX$ (i.e. $T = X^{-1}SX$ or $S = XTX^{-1}$).

1.2 Introduction

A signal is a set of information or data. It is a quantitative description of a physical phenomenon, event or process e.g Electric current or voltage in a circuit, daily closing value of a share of stock last week and audio signal: continuous-time in its original form, or discrete-time when stored on a CD[30]. Any physical variable subject to variations represent a signal. Demand for study of signal and signal processing arises from the increase in demand in efficient communication and data storage e.g growth in internet communication that requires use of image and video signals.

In Chapter 1, we define the terms related to operators, group theory, Fourier analysis, wavelets and frame theory. A few theorems, lemma, propositions and relevant examples have been included. The definition of translation, modulation and dilation operators are discussed.

In Chapter 2, frames and frame operators have been discussed in details. The evolution of frames, basics from operators in Hilbert spaces, diagonalization of operators, analysis and synthesis operators have been discussed. Frame and Gramian operators with their difference in notation and application too have been looked into. The definition of frames and the concepts around frame theory too are included. The clear difference between Parseval frames and tight frames is well explained using examples. The Mercedes- Benz frame as a unit norm tight frame and its application in signal processing discussed. The chapter finalizes with excess of a Parseval frame as used in signal processing.

Chapter 3 contains the group frames. This chapter begins with the review of finite group representation theory then proceeds to the symmetries of a frame using group theory. The symmetries of the Mercedes-Benz frame with the diagrammatic representation showing the number of equiangular lines in $\mathbb{R}^2$ and $\mathbb{R}^3$ is discussed in details. The characterization of tight frames is studied then the discussion of The Harmonic frames. The chapter ends with central frames.

Chapter 4 is dedicated to Fourier analysis and wavelets with emphasis on application of Fourier series in music. The properties of Fourier analysis, Fourier transforms discussed. Major emphasis on wavelets regards the application of wavelets to signal processing and telecommunication. Chapter 5 looks at the application of frames in signal processing and telecommunication. Finally, the summary of the thesis and future research is done in Chapter 6.

1.3 Definition of terms

We begin by defining linear dependence and linear independence giving examples in each case.

Definition 1.3.1 *Let Y be a vector space over a field $\mathbb{K}$. Let X be a subset of Y. Then X is said to be linearly dependent if there exists a finite set of vectors $x_1, x_2, \cdots, x_k$ in X and scalars $b_1, b_2, \cdots, b_k$ not all zero such that*

$$\sum_{j=1}^{k} b_j x_j = 0.$$

Example 1.3.2 *Let* $x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, x_3 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ *be vectors.*

A simple computation shows that these vectors are linearly dependent.

The Example 1.2.2 above is then followed with the definition and example of linear Independence.

Definition 1.3.3 *Let Y be a vector space over a field $\mathbb{K}$. A subset X of Y is said to be linearly independent if there exists a finite set of vectors $x_1, x_2, \ldots, x_k \in X$ and $b_1, b_2, \ldots, b_k \in \mathbb{K}$ which are not all zero such that*

$$\sum_{j=1}^{k} b_j x_j \neq 0.$$

An example of linearly independent vectors is:

Example 1.3.4 *The set* $X = \{x_1, x_2, x_3\} = \{(1,1,0),(0,1,0),(0,0,1)\}$ *is linearly independent in* $\mathbb{R}^3$. *The set* $X \bigcup (1,1,1)$ *is linearly dependent in* $\mathbb{R}^3$, *since* $(1,0,0)+(0,1,0)+(0,0,1)-(1,1,1)=0$.

Having defined linear dependence and independence, we shall then define generating set or spanning set.

Definition 1.3.5 *Let* Y *be a vector space over a field* $\mathbb{K}$. *Let* $X \subseteq Y$ *be a finite or infinite set. Then Span* (X) *which is the set of all finite linear combinations of vectors in* X *denotes the span of* X *i.e*

$$Span(X) = \sum_{j=1}^{n} b_j x_j : n \in \mathbb{N}, b_j \in \mathbb{K}, x_j \in X, \forall j \in \{1,2,\ldots,n\}$$

X is said to span Y if span $(X) = Y$. X is called the spanning set for Y. We can also say that X generates Y or X is the generating set for Y.

Next is the definition and example of a basis of a vector space Y.

Definition 1.3.6 *Let* Y *be a vector space over a field* $\mathbb{K}$. *Let* $X \subseteq Y$. *Then* X *is said to be a basis of* Y *if* X *is linearly independent set such that span* $(X) = Y$.

Example 1.3.7 *Let* $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *be matrices over the vector space* $\mathbb{R}$. *Then the matrices* M_1, M_2, M_3 *and* M_4 *form a basis for* $M_{22}(\mathbb{R})$.

An inner product space is a vector space with an additional structure called an inner product. This also provide the means of defining orthogonality between vectors.

Definition 1.3.8 *Let* Y *be an inner product space. A basis* $\{x_1, x_2, \ldots, x_n\}$ *of* Y *is called orthogonal if* $\langle x_i, x_j \rangle = 0 \ \forall i \neq j$.

The definition 1.3.8 leads to the following proposition.

Proposition 1.3.9 *Let* X *be a subset of an inner product space* Y. *Then every vector of* $X^{\perp}$ *is orthogonal to every vector of span* (Y) *i.e* $\langle x, y \rangle = 0 \ \forall x \in span(X),\ y \in X^{\perp}$.

Proof

For any $x \in span(X)$, the vector x must be a linear combination of some vectors in X(by the definition of span (X)), say, $x = b_1x_1 + b_2x_2 + \ldots + b_mx_m$. Then for any $y \in X^{\perp}$, $\langle x, y \rangle = b_1\langle x_1, y\rangle + b_2\langle x_2, y\rangle + \ldots + b_m\langle x_m, y\rangle = 0$.

The following proposition shows that the orthogonal complement $X^{\perp}$ is a subspace of Y.

Proposition 1.3.10 *Let* X *is a non- empty subset of an inner product space* Y. *The orthogonal complement* $X^{\perp}$ *is also a subspace of an inner product space* Y.

Definition 1.3.11 *An orthogonal set of vectors is orthonormal set of vectors if:*
i) $\langle x_i, x_j \rangle = 0$ *for* $i \neq j$ *i.e vectors mutually and orthogonal*
ii) $\langle x_j, x_j \rangle = 1$ *for* $j = 1, 2, \ldots, k$ *i.e are all unit vectors.*

The following proposition shows that a vector space of non- zero elements which are mutually orthogonal are linearly independent elements.

Proposition 1.3.12 *Let*$\{y_1, y_2 \ldots, y_k\} \in Y$ *be non- zero, mutually orthogonal elements, so* $y_i \neq 0, \langle y_i, y_j \rangle = 0$ *for* $i \neq j$, *then they are linearly independent.*

We can then easily say that any collection of nonzero orthogonal vectors forms a basis for its span.

The proposition 1.3.13 leads to the following theorem which shows that a subspace X of an inner product space Y forms an orthogonal basis for Y.

THEOREM 1.3.13 *Let $y_1, \ldots, y_n \in Y$ be non - zero, mutually orthogonal elements of an inner product space Y . Then $y_1, \ldots, y_n$ form an orthogonal basis for their span $(X) =$ span $(y_1, \ldots, y_n) \subset Y$. Therefore it is a subspace of dimension $n = dim(X)$. Particularly, if dim $(Y) = n$, then $y_1, \ldots, y_n$ form an orthogonal basis for Y .*

THEOREM 1.3.14 *Let Y be an inner product space. Let $x_1, x_2, \ldots, x_n$ be an orthonormal basis of Y. Then an element $y \in Y$ can be written as a linear combination:*

$$y = b_1x_1 + \ldots + b_nx_n \tag{1.1}$$

in which its coordinates

$$b_j = yx_j, j = 1, ..., n, \tag{1.2}$$

are explicitly given as inner products. Moreover, its norm

$$\| y \| = \sqrt{b_1^2 + \ldots + b_n^2} = \sqrt{\sum y.x_j^2} \tag{1.3}$$

is the square root of the sum of squares of its orthonormal basis coordinates.

The above theorem is closely related to the theorem 1.3.15 below which gives a useful formula that appears in applications in signal processing.

THEOREM 1.3.15 *If $y_1, ..., y_n$ form an orthogonal basis, then the corresponding coordinates of a vector $y = a_1y_1 + a_2y_2 + \ldots + a_ny_n$ are given by*

$$a_j = \frac{\langle y, y_j \rangle}{\| y_j \|}^2 \tag{1.4}$$

In this case, its norm can be computed using the formula

$\| y \|^2 = \sum_{j=1}^{n} a_j^2 \| y_j \|^2 = \sum_{j=1}^{n} \frac{\langle y,y_j \rangle}{\|y_j\|}^2 (1.5)$

Equation 1.4, along with its orthonormal simplification, is one of the most useful formulas we shall establish and applications will appear repeatedly throughout the thesis.

Now, we shall look at a few definitions on the Hilbert space. We shall begin by defining an invariant subspace M in a Hilbert space $\mathcal{H}$.

Definition 1.3.16 *If $\mathcal{H}$ is a Hilbert space and T is a bounded linear operator in $\mathcal{H}$ then a subspace $M \subset \mathcal{H}$ is said to be invariant under T if $u \in M \Longrightarrow Tu \in M$ or $TM \subset M$.*

From Definition 1.2.14, we shall define a reducing subspace W of a Hilbert space $\mathcal{H}$.

Definition 1.3.17 *Let $\mathcal{H}$ be a Hilbert space and $A \in B(\mathcal{H})$. A closed subset M of $\mathcal{H}$ reduces A if and only if M is invariant under both A and A^*.*

Proof

Let M reduce A. Then both M and $M^\perp$ are invariant under A. $M^\perp$ invariant under A implies that $M^\perp$ is invariant under A^*. Hence M is invariant under both A and A^*.

Conversely, if M is invariant under both A and A^*, then M invariant under A^* implies that $M^\perp$ is invariant under A. Thus M and $M^\perp$ are invariant under A. Hence M reduces A.

Definition 1.3.18 *[24] Let X and Y be linear spaces over the same field $\mathbb{F}$. A linear operator from X into Y is a mapping $T : X \to Y$ such that $T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2$ for all $x_1, x_2 \in X$ and all $\alpha, \beta \in \mathbb{F}$.*

Recall that an operator T on a Hilbert space $\mathcal{H}$ is normal if it commutes with its adjoint.

The notion of reducing subspace enables us to define a pure (completely non-normal) operator.

Definition 1.3.19 *An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be pure (completely non-normal) if there exists no nontrivial reducing subspace W of $\mathcal{H}$ such that $T|_W$ is normal i.e if T has no direct normal summand.*

The Definition 1.2.19 leads to the definition of a linear operator T_W which is induced by the operator T.

Remark 1.3.20 *When the subspace W is invariant under the operator T , then T induces a linear operator $T_W = T|_W$ on the space W. The linear operator T_W is defined by $T_W(u) = T(u)$, for u in W.*

Definition 1.3.21 *[33]*

An operator $T \in B(\mathcal{H})$ is said to be positive if $\langle Tu, u\rangle = 0$, for all $u \in \mathcal{H}$ and T is self- adjoint.

The definition of an orthonormal basis makes it easier to define a Hilbert-Schmidt operator in a Hilbert space $\mathcal{H}$.

Definition 1.3.22 *An operator T on a Hilbert space $\mathcal{H}$ is Hilbert-Schmidt if $\| T \|_2 < \infty$, where*

$$\| T \|_2 = \{\sum_{n=1}^{\infty} \| T_{en} \|^2\}^{\frac{1}{2}}$$

and e_n is an orthonormal basis for $\mathcal{H}$.

Given $T \in B(\mathcal{H})$, we shall then define the Drazin inverse of T i.e T^D.

Definition 1.3.23 *Let $T \in B(\mathcal{H})$. If there exists an operator $T^D \in B(\mathcal{H})$ satisfying the following three operator equations*

$$(i) TT^D = T^D T.$$

$$(ii) T^D T T^D = T^D.$$

and

$$(iii) T^{k+1} T^D = T^k.$$

where $k = ind(T)$, the index of T, which is the smallest nonnegative integer for which

$$Ran(T^{k+1}) = Ran(T^k)$$

and

$$Ker(T^{k+1}) = Ker(T^k)$$

then T^D is called the Drazin inverse of T.

Definition 1.3.24 *Let $\mathcal{H}$ be a Hilbert space. A linear isometry which maps $\mathcal{H}$ onto $\mathcal{H}$ is called a unitary operator.*

Definition 1.3.25 *A contraction T in a Hilbert space $\mathcal{H}$ is said to be completely non-unitary (c.n.u) if there exists no nontrivial reducing subspace of T on which T acts unitarily, or equivalently if its unitary part acts on the zero space $\{0\}$.*

The next definition 1.2.36 is very important in this thesis since it forms the basis of our study. This is the definition of a frame of a Hilbert space $\mathcal{H}$.

Definition 1.3.26 *A **finite frame** is a finite sequence of vectors $\Theta = \{v_j\}_{j=1}^M \subset \mathcal{H}$ for which there exist constants $0 < \alpha \leq \beta < \infty$ such that for all vectors $v \in \mathcal{H}$*

$$\alpha \| v \|^2 \leq \sum_{j=1}^{M} | \langle v, v_j \rangle |^2 \leq \beta \| v \|^2$$

We call α and β the lower and upper frame bounds, respectively.

The completeness of a frame is defined as:

Definition 1.3.27 *Let X be a linear space. Any set of vectors Y in X is said to be a sequence of elements, say (x_n), then span Y = span $_n(x_n)$. (x_n) is said to be complete if the span of (x_n) is dense in X.*

The removal of a vector from from a frame leaves either a frame or an incomplete set.

The following gives an overview of a dual frame, analysis operator, synthesis operator and Gram matrix.

THEOREM 1.3.28 *Given any frame $\Theta = \{v_j\}_{j=1}^M$ for a Hilbert space $\mathcal{H}$, there exists another frame $\Phi = \{w_j\}_{j=1}^M$, called the **dual frame** such that any vector $v \in \mathcal{H}$ can be reconstructed by the given formula*

$$v = \sum_{j=1}^{M} \langle v, w_j \rangle v_j = \sum_{j=1}^{M} \langle v, v_j \rangle w_j.$$

The proof of the above theorem uses the concept of an analysis operator. Given a finite frame $\Theta = \{v_j\}_{j=1}^M \subset \mathcal{H}$, the analysis operator T is defined as a linear operator from the Hilbert space to the complex plane given by

$$Tv = [\ \langle v, v_1 \rangle \langle v, v_2 \rangle .. \langle v, v_n \rangle\ ] = \sum_{j=1}^{n} \langle v, v_j \rangle e_j,$$

where e_j is the standard orthonormal basis of $\mathbb{C}^n$. It has been shown in the literature that the analysis operator is one-to-one and has a synthesis operator T^* such that

$$T^*Tv = \sum_{j=1}^{n} \langle v, v_j \rangle v_j.$$

These operators can easily be written as matrices. Note that $G = TT^*$ is a Hermitian matrix whose entries are given as inner products, also known as a *Gram matrix.*

The definition of tight frames and Parseval frames which are so vital in application of frames in signal processing is as follows.

Definition 1.3.29 *A **tight frame** is a frame whose upper and lower bounds are equal. Such a frame is called a **Parseval frame** if upper and lower bounds are equal which equals to one.*

Example 1.3.30 *Let $\Theta = \{e_1, e_1, e_2, e_2, e_3, e_3, \ldots\}$. Θ is a tight frame with frame bounds $\alpha = 2$ and $\beta = 2$.*

Example 1.3.31 *Let $\Theta = \{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \ldots\}$.$\Theta$ is a tight frame with $\alpha = \beta = 1$ and hence a Parseval frame.*

We shall then look at a Parseval's identity for a Hilbert space $\mathcal{H}$.

THEOREM 1.3.32 ***(Parseval's Identity).***

Let $(h_j)_{j=1}^n$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Then for all vectors $v \in \mathcal{H}$,

$$\| v \|^2 = \sum_{j=1}^{n} | \langle v, h_j \rangle |^2$$

One of the reasons frames are interesting in signal processing and telecommunication is that they have many of the same properties as matrices. Frames can be a basis for a linear space. However, they are usually more than a basis and allow for redundant vectors while still spanning the space.

Two frames are said to be orthogonal if the product of their analysis operators is zero.

Definition 1.3.33 *Two frames $\mathbb{X}$ and $\mathbb{Y}$ are orthogonal if $T_{\mathbb{X}}^* T_{\mathbb{Y}} = 0$ where $T_{\mathbb{X}}$ and $T_{\mathbb{Y}}$ are the analysis operators for $\mathbb{X}$ and $\mathbb{Y}$ respectively.*

Related to the basis is the Riesz basis which is defined for a family of vetors v_j in an M- dimensional Hilbert space $\mathcal{H}^M$.

Definition 1.3.34 *A family of vectors $(v_j)_{j=1}^M$ in a Hilbert space $\mathcal{H}^M$ is a Riesz basis with lower (respectively, upper) Riesz bounds $\tau(resp.\mu)$, if, for all scalars $(a_j)_{j=1}^M$ we have:*

$$\tau \sum_{j=1}^{n} \mid b_j \mid^2 \leq \parallel \sum_{j=1}^{n} b_j v_j \parallel^2 \leq \mu \sum_{j=1}^{n} \mid b_j \mid^2$$

The following are equivalent statements for a Riesz basis.

Proposition 1.3.35 *If $(v_j)_{j=1}^M$ is a family of vectors then the following statements are equivalent.*
(i) $(v_j)_{j=1}^M$ is a Riesz basis for $\mathcal{H}^M$ with Riesz bounds τ and μ.
(ii) For any orthonormal basis $(e_j)_{j=1}^M$ for $\mathcal{H}^M$, the operator T on $\mathcal{H}^M$ given by $Te_j = v_j$ for all $j = 1, 2, ..., M$ is an invertible operator with $\parallel T \parallel^2 \leq \mu$ and $\parallel T^{-1} \parallel^{-2} \geq \tau$.

The notion of a dihedral group is so useful in the study of group frames.

Definition 1.3.36 *Let O and A be the center and vertex of a polygon respectively. Let a and b denote rotation about O and reflection mirror line OA. Then the dihedral group of order $2n$ is defined as $D_{2n} = \langle a, b : a^n = I, b^2 = I, b^{-1}ab = a^{-1} \rangle$.*

Definition 1.3.37 *[29]*
(i) Let $a \in \mathbb{R}$. We define the translation operator (or time shift) $T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by
$T_a f(x) := f(x - a)$.
(ii)Let $b \in \mathbb{R}$. The modulation operator (or frequency shift) $E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined by
$E_b f(x) := e^{2\Pi ibx} f(x)$.
iii) Let $c > 0$. We define the dilation operator $D_c : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $(D_c f)(x) := \frac{1}{\sqrt{c}} f(\frac{x}{c}), x \in \mathbb{R}$.

These Operators are bounded and linear.

Definition 1.3.38 *Let T be an operator in a Hilbert space $\mathcal{H}$. T^* is said to be the adjoint of T if $\langle Tf, g \rangle = \langle f, T^* g \rangle$ for all $f, g \in \mathcal{H}$.*

THEOREM 1.3.39 *The adjoint of a bounded linear operator of a Hilbert space always exists and is bounded linear.*

Definition 1.3.40 *Let f be a function whose domain is a set X. f is said to be injective if for all $x, y \in X$, if $f(a) = f(b)$, then $a = b$. Equivalently, if $a \neq b$ then $f(a) \neq f(b)$.*

THEOREM 1.3.41 *Let T be an operator which maps points in the domain to every points in the range . Let V be a vector space with $x, y \in V$. T defined on V is a surjection if there is an $x \in V$ such that $T(x) = y$ for all y.*

Chapter 2

FRAMES AND FRAME OPERATORS IN HILBERT SPACES

2.1 Introduction

Frames were introduced in 1952 by Duffin and Schaeffer [16] when they were studying some deep problems on nonharmonic fourier series for which they required a formal structure for working with highly overcomplete families of exponential functions in $L^2[0,1]$. They introduced the notion of a Hilbert space frame[21]. In 1986, Daubechies, Grossmann and Meyer [23] showed that frames can be used to find series expansions of functions in $L^2[\mathbb{R}]$. They intiated the role of frames in signal processing. Recently, Casazza and Kutyniok [10] introduced the notion of frames of subspaces or fusion frames to be applied in sensor networks and packet encoding. Li and Ogawa [27] introduced pseudo- frames in Hilbert spaces using two Bessel sequences. Another notion of frames known as oblique frames was introduced by Christensen and Eldar [13]. Generalizations of g- frames was introduced by Sun [36]. Adroubi [2] introduced two methods of generating frames for a Hilbert space $\mathcal{H}$. One method uses bounded linear operators on ℓ^2 while the other uses bounded operators on $\mathcal{H}$. Tight frames are of particular interest as they are closest to orthonormal bases in behaviour. Nairmark [1] and Han and Larson [22] proved that all tight frames are projections of orthonormal bases from a larger space. A uniform tight frame is one with all vectors having the same norm. In [20], it was shown that uniform tight frames optimize robustness to quantization noise . Also one erasure from a uniform tight frame cannot destroy the frame property was shown. Construction of frames from backgrounds of functional analysis [8],[11], operator theory[19],[26], number theory[26],[32] among others. Today, Frames is applied in pure and Applied Mathematics, Physics, Engineering, computer science among others. Frames are preferred due to their redundancy.

2.2 OPERATORS IN HILBERT SPACES

2.2.1 Some Basics from Operator Theory

We shall introduce some basic results from operator theory used in this thesis. Also we shall not forget that every linear operator has an associated matrix representation.

Definition 2.2.1 *Let $T : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{N}}$ be a linear operator. Let $(g_j)_{j=1}^{M}$ and $(h)_{k=1}^{N}$ be orthonormal bases for $\mathcal{H}^{\mathcal{M}}$ to $\mathcal{H}^{\mathcal{N}}$ respectively . Then the matrix representation of T (with respect to the orthonormal bases $(g_j)_{j=1}^{M}$ and $(h_k)_{k=1}^{N}$ is a matrix of size NxM and is given by $B = (b_{jk})_{j=1,k=1}^{N,M}$where*

$$(b)_{jk} = \langle Te_k, g_j \rangle$$

For all $x \in \mathcal{H}^{\mathcal{N}}$ with $c = (\langle x, g_j \rangle)_{j=1}^{M}$

$$Tx = Bc.$$

We shall begin by the following definition which gives some notion on a linear operator T.

Definition 2.2.2 *Let $T : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{N}}$ be a linear operator.*

(a) Then the kernel of T is defined as Ker $(T) := \{x \in \mathcal{H}^{\mathcal{M}} : Tx = 0\}$. The range of T defined by Ran(T) :=$\{Tx : x \in \mathcal{H}^{\mathcal{M}}\}$. The Rank of T is the dimension of the Range of T.

(b) T is injective (or one-to-one) if Ker (T): $= \{0\}$. T is surjective (or onto) if Ran $(T) := \mathcal{H}^{\mathcal{N}}$. T is bijective (or invertible) if T is both injective and surjective.

(c) The adjoint operator of T that is $T^* : \mathcal{H}^{\mathcal{N}} \longrightarrow \mathcal{H}^{\mathcal{M}}$ is defined by $\langle Tx, \tilde{x}\rangle = \langle x, T^*\tilde{x}\rangle$ for all $x \in \mathcal{H}^{\mathcal{M}}$ and $\tilde{x} \in \mathcal{H}^{\mathcal{N}}$.

(d) The norm of T is defined by $\| T \| := sup\{\| Tx \| : \| x \| = 1\}$.

The relationship between the notions in Definition 2.2.2 can be stated as in the result below.

Proposition 2.2.3 *(i) Let $T : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{N}}$ be a linear operator. Then dim $\mathcal{H}^{\mathcal{M}} = M =$ dim Ker (T) + Rank (T). Moreover, if T is injective, then T^*T is also injective.*

(ii) Let $T : \mathcal{H}^{\mathcal{M}} \longrightarrow H^M$ be a linear operator. Then T is injective if and only if it is surjective. Moreover, Ker(T) = (Ran $(T^*))^{\perp}$, and hence $\mathcal{H}^{\mathcal{M}}$ = Ker $(T) \bigoplus$ Ran(T^*).

Let $T : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{M}}$ be an injective operator. Then T is invertible. If $T : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{N}}$ is not injective, then by restricting T to (Ker $(T))^{\perp}$ makes T injective. Although $T|_{(Ker T)^{\perp}}$ may not be invertible. To make T surjective, we can consider the operator $T : (Ker(T))^{\perp} \longrightarrow Ran(T)$, which is now invertible.

The Pseudo inverse of an injective operator which provides a one-sided inverse for the operator can be defined as below.

Definition 2.2.4 *If $T : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{N}}$ is an injective linear operator then the Moore-Penrose inverse of T denoted by T^+, is defined by $T^+ = (T^*T)^{-1}T^*$.*

Moore- Penrose pseudoinverse is defined for any matrix and is unique. It brings great rotational and conceptual clarity to the study of solutions of arbitrary systems of linear equations and linear least square problems.

THEOREM 2.2.5 *Let T be in $\mathbb{R}^{mxn}$. Then*

$$T^+ = Lim(T^*T + \delta^2 I)^{-1}T^*$$

$\delta \to 0$

$= Lim\ T^*(TT^* + \delta^2 I)^{-1}$
$\delta \to 0$

$= T^*(TT^*)^{-1} = (T^*T)^{-1}T^*$.

Example 2.2.6 *Let $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then by limit definition of pseudoinverse, we find $T^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$*

Another example is:

Example 2.2.7 *Let $T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then by limit definition of the pseudoinverse, $T^+ = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$*

We can then show invertibility from the left using the result below.

Proposition 2.2.8 *Let $T : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{N}}$ be an injective, linear operator. Then $T^+T = I$.*

Here, T^+ projects a vector from $\mathcal{H}^{\mathcal{N}}$ onto $Ran(T)$. It plays the role of the inverse on $Ran(T)$ - not on all of $\mathcal{H}^{\mathcal{N}}$. Pseudo- inverse is a more general notion of this inverse. This is applicable to a non- injective operator. It restricts T^+ onto $(Ker(T))^{\perp}$. It enforces injectivity of the operator followed by application of the Moore -Penrose inverse of this new operator. By single value Decomposition, this pseudoinverse can be derived. An associated unique matrix representation can be derived if we fix orthonormal bases of the domain and range of a linear operator. We therefore will state this decomposition in terms of matrix.

THEOREM 2.2.9 *Let B be an $M \times N$ matrix. Then there exist an $M \times M$ matrix C with $C^*C = I$, and $N \times N$ matrix D with $D^*D = I$, and an $M \times N$ (where $M = N$)diagonal matrix E with nonnegative, decreasing real entries on the diagonal such that $B = CED^*$*

Definition 2.2.10 *Let B be an $M \times N$ matrix. Let C, E and D be chosen as in Theorem 2.2.9. Then $B = CED^*$ is called the singular value decomposition (SVD) of B. The column vectors of C are called the left singular vectors, and the column vectors of D are referred to as the right singular vectors of B.*

We shall then deduce the pseudo-inverse B^+ of B using Singular Value Decomposition as shown below.

THEOREM 2.2.11 *Let $B^+ = DE^+C^*$ be the Singular Value Decomposition of an $M \times N$ matrix B. Then $B^+ = DE^+C^*$ where E^+ is the $N \times M$ diagonal matrix obtained by inverting the nonzero(diagonal entries) of E^*.*

2.2.2 Diagonalization of Operators

Diagonalizations of operators are needed in deriving understanding of the action of an operator.

Definition 2.2.12 *Let $T : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{M}}$ be a linear operator. A nonzero vector $x \in \mathcal{H}^{\mathcal{M}}$ is an eigenvector of T with eigenvalue λ, if $Tx = \lambda x$. T is said to be orthogonally diagonalizable, if there exists an orthonormal basis $(e_j)_{j=1}^{M}$ of $\mathcal{H}^{\mathcal{M}}$ consisting of eigenvectors of T .*

Proposition 2.2.13 *Let $T : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{N}}$ be a linear operator. Then the nonzero eigenvalues of T^*T and TT^* are the same. If the operator is unitary, self-adjoint, or positive, the next result gives more information on the eigenvalues.*

Corollary 2.2.14 *Let $T : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{M}}$ be a linear operator.*
(i) If T is unitary, then its eigenvalues have modulus one.
(ii) If T is self-adjoint, then its eigenvalues are real.
(iii) If T is positive, then its eigenvalues are nonnegative.

Next is a fundamental result in operator theory which has its analogue in the infinite-dimensional setting called the spectral theorem.

THEOREM 2.2.15 *Let $\mathcal{H}^{\mathcal{M}}$ be complex and let $T : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{M}}$ be a linear operator. Then the following conditions are equivalent.*
(i) T is normal.
(ii) T is orthogonally diagonalizable.
(iii) There exists a diagonal matrix representation of T .
(iv) There exist an orthonormal basis $(e_j)_{j=1}^{M}$ of $\mathcal{H}^{\mathcal{M}}$ and values $\lambda_1, ..., \lambda_M$ such that

$$Tx = \sum_{j=1}^{M} \lambda_1 \langle x, e_j \rangle e_j$$

for all $x \in \mathcal{H}^{\mathcal{M}}$. In this case,

$$\parallel T \parallel = \max_{1 \leq j \leq M} |\lambda_j|.$$

Since every self-adjoint operator is normal, we obtain the following corollary (which is independent of whether $\mathcal{H}^{\mathcal{M}}$ is real or complex).

Corollary 2.2.16 *A self-adjoint operator is orthogonally diagonalizable.*

The following result, in particular allows the definition of the n-th root of a positive operator.

Corollary 2.2.17 *Let $T : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{M}}$ be an invertible positive operator with normalized eigenvectors $(\lambda_j)_{j=1}^M$ and respective eigenvalues $(\lambda_j)_{j=1}^M$ and let $b \in \mathbb{R}$. We can define an operator $T^b : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{M}}$ by*

$$T^b x = \sum_{j=1}^{M} \lambda_j^b \langle x, e_j \rangle e_j$$

for all $x \in \mathcal{H}^{\mathcal{M}}$. Then T^b is a positive operator and $T^b T^c = T^{b+c}$ for $b, c \in \mathbb{R}$. Particularly, T^{-1} and $T^{-1/2}$ are positive operators.

Using Theorem 2.2.15, we can define the trace of an operator by expressing it in terms of eigenvalues.

Definition 2.2.18 *Let $T : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{M}}$ be an operator. Then, the trace of T is defined by*

$$Tr(T) = \sum_{j=1}^{M} \langle Te_j, e_j \rangle$$

where $(e_j)_{j=1}^M$ is an arbitrary orthonormal basis for $\mathcal{H}^{\mathcal{M}}$.

The trace is well defined since the sum in the equation in Definition 2.2.17 is independent of the choice of the orthonormal basis.

Corollary 2.2.19 *Let T: $\mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{M}}$ be an orthogonally diagonalizable operator,and let $(\lambda_j)_{j=1}^M$ be its eigenvalues. Then*

$$Tr(T) = \sum_{j=1}^{M} \lambda_j.$$

2.2.3 Projection Operators

Projection operators map vectors onto the subspace either orthogonally or not. Orthogonal projections are more often used.

Definition 2.2.20 *Let $P : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{M}}$ be a linear operator. Then P is called a projection, if $P^2 = P$. In addition, P is orthogonal if $P = P^*$.*

We shall use the term projection to refer to orthogonal projections. A projection P has the crucial property that each given vector of Hilbert space $\mathcal{H}^{\mathcal{M}}$ is mapped to the closest vector in the range of P .

LEMMA 2.2.21 *Let W be a subspace of $\mathcal{H}^{\mathcal{M}}$. Let P be the orthogonal projection onto W and let $x \in \mathcal{H}^{\mathcal{M}}$. Then $\| x - Px \| \leq \| x - \tilde{x} \|$ for all $\tilde{x} \in W$. Moreover, if $\| x - Px \| = \| x - \tilde{x} \|$ for some $\tilde{x} \in W$, then $\tilde{x} = Px$.*

2.2.4 Redundancy of frames

Redundancy of a frame is the property that allows for excess vectors in a frame such that loss of some vectors in the frame does not make the collection of vectors to cease to be a frame.

Redundancy of a frame allows each vector in the space to have infinitely many representations with respect to the frame but it also has one natural representation given by the frame coefficients.

Redundancy allows for robustness i.e spreading information over a wider range of vectors hence avoid erasures and still have an accurate reconstruction e.g in internet coding, brain modelling among others.

Redundancy is also applicable in quantum tomography where classes of orthonormal bases are needed which have "constant" interactions with one another.

Redundancy gives information on orthogonality and tightness of a frame, about a number of spanning sets and minimal number of linearly independent sets a frame can be divided into.

Example 2.2.22 *Let $\Theta_1 = \{e_1, \ldots, e_1, e_2, e_3, \ldots, e_m\}$ where e_1 occurs p times. Let $\Theta_2 = \{e_1, e_1, e_2, e_2, \ldots, e_m, e_m\}$ with $e_1, \ldots, e_m$ being orthonormal basis for $\mathcal{H}$. The redundancy of Θ_1 seems localized while redundancy of Θ_2 seems quite uniform. Θ_2 is robust with respect to any erasure whereas Θ_1 is not.*

2.3 Analysis And Synthesis Operator

The analysis and synthesis operators are very important in frames. The analysis operator analyzes a signal in terms of the frame by computing its frame coefficients.

The following notions are related to a frame $\Theta = \{v_j\}_{j=1}^{M} \subset \mathcal{H}$.

(a) The constants α and β as in the Definition 1.3.26 are called the lower and upper frame bound for the frame, respectively. The largest lower frame bound and the smallest upper frame bound are denoted by α_{op}, β_{op} respectively are called the optimal frame bounds.

The lower frame bound ensures that a frame is complete (its closed span is the whole space). The upper frame bound, on the other hand, ensures that the Bessel map is well defined.

(b) Any family $\{v_j\}_{j=1}^{K}$ satisfying the upper frame inequality in Definition 1.3.26 is called a β-Bessel sequence.

(c) If $\alpha = \beta$ in Definition 1.3.26, then $\{v_j\}_{j=1}^{K}$ is called an α-tight frame.

(d) If $\alpha = \beta = 1$ in Definition 1.3.26, i.e if Parseval's identity holds, then $\{v_j\}_{j=1}^{M}$ is called a Parseval frame.

Example 2.3.1 *The example below shows that a Parseval frame can contain zero vectors. Let $(e_j)_{j=1}^{M}$ be an orthonormal basis for $\mathcal{H}^{\mathcal{M}}$. The system $(e_1, 0, e_2, 0, ..., e_m, 0)$ is a Parseval frame for $\mathcal{H}^{\mathcal{M}}$.*

(e) If there exists a constant b such that $| \langle v_j, v_k \rangle |= b$ for all $j = 1, 2, ..., M$, then $\{v_j\}_{j=1}^{M}$ is an equal norm frame. If $b = 1, \{v_j\}_{j=1}^{M}$ is a unit norm frame.

Example 2.3.2 *For $\mathbb{R}^2$, the Mercedes-Benz frame is the equal - norm tight frame for $\mathbb{R}^2$ given by:*

$$(\sqrt{\frac{2}{3}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \sqrt{\frac{2}{3}}\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{-1}{2} \end{pmatrix}, \sqrt{\frac{2}{3}}\begin{pmatrix} \frac{-\sqrt{3}}{2} \\ \frac{-1}{2} \end{pmatrix})$$

(f) If there exists a constant b such that $|\langle v_j, v_k \rangle| = b$ for all $j \neq k$, then $\{v_j\}_{j=1}^{M}$ is called an equiangular frame.

Remark 2.3.3 *The example 2.3.2 is an example of equiangular frame.*

(g) The values $(\langle x, v_j \rangle)_{j=1}^{K}$ are called the frame coefficients of the vector x with respect to the frame $\{\text{v}_j\}_{j=1}^{K}$.

(h) The frame $\{v_j\}_{j=1}^{K}$ is called exact, if $\{v_j\}_{j \in J}$ ceases to be a frame for $\mathcal{H}^{\mathcal{M}}$ for every $J = \{1, ..., K\}$.

The following are examples of frames.

Example 2.3.4 *A finite set of vectors in $\mathbb{C}^n$ is a frame for their (linear) span. In fact, any finite subset of a Hilbert space is a frame for its span.*

Example 2.3.5 *Let $\{e_j\}_{j=1}^{M}$ be an orthonormal basis for $\mathcal{H}^{\mathcal{M}}$. By repeating each element in $\{e_j\}_{j=1}^{M}$ twice, we obtain $\{v_j\}_{j=1}^{M} = \{e_1, e_1, e_2, e_2, ..\}$, which is a tight frame with frame bound $\alpha = 2$. If only e_1 is repeated, we obtain $\{v_j\}_{j=1}^{M} = \{e_1, e_1, e_2, e_3, ..\}$, which is a frame with bounds $\alpha = 1, \beta = 2$.*

Example 2.3.6 *Let $\{e_j\}_{j=1}^{M}$ be an orthonormal basis for $\mathcal{H}^{\mathcal{M}}$. $\{v_j\}_{j=1}^{M} := \{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \ldots\}$ that is, $\{v_j\}_{j=1}^{M}$ is the sequence where each vector $\frac{1}{\sqrt{l}}e_l, l \in N$, is repeated ℓ times. Then for each $v \in H$,*

$$\sum_{j=1}^{M} | \langle v, v_j \rangle |^2 = \sum_{j=1}^{M} l \mid \langle v, \frac{1}{\sqrt{l}} e_l \rangle |^2 = \| v \|^2 .$$

So $\{v_j\}_{j=1}^{M}$ is a tight frame for $\mathcal{H}^{\mathcal{M}}$ with frame bound $\alpha = 1$.

Example 2.3.7 *Let $\mathcal{H} = \mathbb{C}^2$ and analysis operator $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ then analysis operator is $T^* = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ are a frame of $\mathbb{C}^n$, and $T : \mathbb{C}\nvDash \rightarrow \mathbb{C}\nVdash$ is the associted frame operator $S = T^*T = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$. The frame bounds are $\alpha = 1, \beta = 3$.*

We can therefore define the analysis operator of a family of vectors.

Definition 2.3.8 *Let $\{v_j\}_{j=1}^K$ be a family of vectors in $\mathcal{H}^{\mathcal{M}}$. Then we can define the associated analysis operator $T : \mathcal{H}^{\mathcal{M}} \longrightarrow \ell_2^K$ by $Tv := (\langle v, v_j \rangle)_{j=1}^K$, $v \in \mathcal{H}^{\mathcal{M}}$.*

Two basic properties of the analysis operator are derived in the following lemma.
The adjoint to the analysis operator is known as synthesis operator. Synthesis operator is defined as:

Definition 2.3.9 *Let $(v_j)_{j=1}^K$ be a sequence of vectors in $\mathcal{H}^{\mathcal{M}}$ with associated analysis operator T . Then the associated synthesis operator is defined as the adjoint operator $T^* : \ell_2^K \longrightarrow \mathcal{H}^{\mathcal{M}}$ of T is given by*

$$T^*(b) = \sum_{j=1}^{K} b_j v_j$$

where $b = (b_j)_{j=1}^K \in \ell_2^K$.

LEMMA 2.3.10 *For a sequence of vectors $(v_j)_{j=1}^K$ in $\mathcal{H}^{\mathcal{M}}$ with associated analysis operator T, then:*
(i) Let $(e_j)_{j=1}^K$ denote the standard basis of ℓ_2^K.Then for all $j = 1, 2, ..., K$, we have $T^*e_j = v_j$, where $P : \ell_2^K \longrightarrow \ell_2^K$ denotes the orthogonal projection onto $Ran(T)$.
(ii) $(v_j)_{j=1}^K$ is a frame if and only if T^* is surjective.

Analysis operator of a frame analyzes the signal x in terms of the frame by computing its frame coefficients.

2.4 FRAME OPERATOR

Definition 2.4.1 *Let $(v_j)_{j=1}^K$ be a sequence of vectors in $\mathcal{H}^{\mathcal{M}}$ with associated analysis operator T. Then the associated frame operator $S : \mathcal{H}^{\mathcal{M}} \longrightarrow \mathcal{H}^{\mathcal{M}}$ is defined by*

$$Sv := T^*Tv = \sum_{j=1}^{K} \langle v, v_j \rangle v_j, v \in \mathcal{H}^{\mathcal{M}}.$$

Since $(v_j)_{j=1}^K$ is a Bessel sequence , then the series defining S converges unconditionally for all $\mathcal{H}^{\mathcal{M}}$. Lemma 2.4.1 below provides some important properties of S.

LEMMA 2.4.2 *Let $(v_j)_{j=1}^K$ be a frame with frame operator S and frame bounds α and β. Then :*
i) S is bounded, invertible, self- adjoint and positive.
ii) $S^{-1}(v_j)_{j=1}^K$ is a frame with frame operator S^{-1} and frame bounds α^{-1} and β^{-1}.
iii) Given that α and β are optimal frame bounds for $(v_j)_{j=1}^K$ then the optimal frame bounds for $S^{-1}(v_j)_{j=1}^K$ are α^{-1} and β^{-1}.

Proof
i) To show that S is bounded operator, we need to recall that S is composition of two bounded operators T^* and T, S is bounded.
$\| S \| = \| T^*T \| = \| T^* \| \| T \| = \| T \|^2 \leq \beta$.
Since $S^* = (T^*T)^* = T^*(T^*)^* = T^*T = S$. S is self- adjoint.
Using the frame inequality means that $\alpha \| v \|^2 \leq \langle Sv, v \rangle \leq \beta \| v \|^2$ for all $v \in \mathcal{H}^{\mathcal{M}}$ which is notated as $\alpha I \leq S \leq \beta I$ i.e S is positive.

Furthermore, $0 \leq I - \beta^{-1}S \leq \frac{\beta-\alpha}{\beta} I$, consequently,

$\| I - \beta^{-1}S \| = \sup_{\| v \| = 1} | \langle (I - \beta^{-1}S)v, v \rangle | \leq \frac{\beta-\alpha}{\beta} < 1$. Hence S is invertible.

ii) Since S is self- adjoint, S^{-1} is also self- adjoint. Then for all $v \in \mathcal{H}^{\mathcal{M}}$,

$$\sum_{j=1}^{K} | \langle v, S^{-1}v_j \rangle |^2 = \sum_{j=1}^{K} | \langle S^{-1}v, v_j \rangle |^2 \leq \beta \| S^{-1}v \|^2 \leq \beta \| S^{-1} \|^2 \| v \|^2$$

i.e $\{S^{-1}v_j\}_{j=1}^{K}$ is a Bessel sequence. Hence frame operator of $\{S^{-1}v_j\}_{j=1}^{K}$ is well defined. By definition,

$$\sum_{j=1}^{K} \langle v, S^{-1}v_j \rangle S^{-1}v_j = S^{-1} \sum_{j=1}^{K} \langle S^{-1}v, v_j \rangle v_j = S^{-1}SS^{-1}v = S^{-1}v.$$

Hence frame operator for $(S^{-1}v_j)_{j=1}^{K}$ equals S^{-1}. The operator S^{-1} commutes with both S and I.

Premultiplying with S^{-1}, we get $\beta^{-1}I \leq s^{-1} \leq \beta^{-1}I$ i.e $\beta^{-1} \| v \|^2 \leq \langle s^{-1}v, v \rangle \leq \alpha^{-1} \| v \|^2$ for all $v \in \mathcal{H}^{\mathcal{M}}$.

$$\beta^{-1} \| v \|^2 \leq \sum_{j=1}^{M} | \langle v, s^{-1}v_j \rangle |^2 \leq \alpha^{-1} \| v \|^2,$$

for all $v \in \mathcal{H}^{\mathcal{M}}$.

Hence , $\{S^{-1}v_j\}_{j=1}^{K}$ is a frame with frame bounds β^{-1} and α^{-1}.

iii) Let $\{v_j\}_{j=1}^{K}$ be a frame with an optimal lower bound α. Assuming that lower frame bound for $(S^{-1}v_j)_{j=1}^{K}$ is $c < \alpha^{-1}$. Using the frame $(S^{-1}v_j)_{j=1}^{K}$, we get $\{v_j\}_{j=1}^{K} = ((S^{-1})^{-1}S^{-1}v_j)_{j=1}^{K}$ has lower bound $c^{-1} > \alpha$ (contradiction). i.e $(S^{-1}v_j)_{j=1}^{K}$ has optimal upper bound α^{-1}.

Example 2.4.3 *Let $\mathcal{H} = \mathbb{C}^2$, and let $F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$. The columns of $F^* = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ are a frame of $\mathbb{C}^n$, and $F : \mathbb{C}^2 \longrightarrow \mathbb{C}^3$ is the associated frame operator. Its range in $\mathbb{C}^3$ is the span of its columns, i.e., all vectors of the form $\begin{pmatrix} a \\ b \\ a+b \end{pmatrix}$ and F is a bijection from $\mathbb{C}^2$ to this two-dimensional subspace of $\mathbb{C}^3$. Its frame bounds will be the squares of the singular values of F, which are the square roots of the eigenvalues of $F^*F = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ Thus, $\alpha = 1$ and $\beta = 3$.*

Example 2.4.4 *Let $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ be an analysis operator whose rows give the frame vectors. By singular value decomposition, the frame bounds are obtained i.e $\beta = 1, \alpha = \sqrt{3}$. This implies that $\| x \|^2 \leq | \langle x, v \rangle |^2 \leq 3 \| x \|^2$.*

We can then state without proof the following proposition.

Proposition 2.4.5 *If $(v_j)_{j=1}^{K}$ is a frame for $\mathcal{H}^{\mathcal{M}}$ with frame operator S, and F is an invertible operator on $\mathcal{H}^{\mathcal{M}}$ then $(Fv_j)_{j=1}^{K}$ is a frame with frame operator FSF^*.*

A frame operator is positive.

Example 2.4.6 *Let $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ then $T^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and frame operator $S = T^*T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$*

$\langle T^*Tx, x \rangle \geq 0$ then $\langle Tx, Tx \rangle \geq 0$. Hence $\| Tx \|^2 \geq 0$. Hence positive.

2.5 Gramian Operators

Let us begin by the definition of the Gramian operator.

Definition 2.5.1 *Let $(v_j)_{j=1}^K$ be a frame for $\mathcal{H}^{\mathcal{M}}$. Let T be the analysis operator. Then the operator $G : \ell_2^K \to \ell_2^K$ defined by*

$$G(b_j)_{j=1}^K = TT^*(b_j)_{j=1}^K = (\sum_{j=1}^K b_j\langle v_j, v_k\rangle)_{k=1}^K = \sum_{j=1}^K (b_j\langle v_j, v_k\rangle)_{k=1}^K$$

is called the Gramian (operator) for the frame $(v_j)_{j=1}^K$.

The (canonical) matrix representation of the Gramian of a frame $(v_j)_{j=1}^K$ for $\mathcal{H}^{\mathcal{M}}$ (also known as the Gramian matrix) is given by $\begin{pmatrix} \| v_1 \|^2 & \langle v_2, v_1\rangle & \cdots & \langle v_K, v_1\rangle \\ \langle v_1, v_2\rangle & \| v_2 \|^2 & \cdots & \langle v_K, v_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_1, v_K\rangle & \langle v_2, v_K\rangle & \cdots & \| v_K \|^2 \end{pmatrix}$

One property of the Gramian is immediate. For a unit norm frame, the entries of the Gramian matrix are exactly the cosines of the angles between the frame vectors e.g for an equiangular frame, all off-diagonal entries of the Gramian matrix have the same modulus. The result below contains the fundamental properties of the Gramian operator.

THEOREM 2.5.2 *Let $(v_j)_{j=1}^K$ be a frame for $\mathcal{H}^{\mathcal{M}}$ with analysis operator T, frame operator S, and Gramian operator G. Then:*

(i) An operator $(U_j)_{j=1}^K$ on $\mathcal{H}^{\mathcal{M}}$ is unitary if and only if the Gramian of $(U_j)_{j=1}^K$ coincides with G.

(ii) The nonzero eigenvalues of G and S coincide.

(iii) $(v_j)_{j=1}^K$ is a Parseval frame if and only if G is an orthogonal projection of Rank (M) (namely onto the Ran(T)).

(iv) G is invertible if and only if $K = M$.

Proof

(i) Follows from the entries of the Gramian matrix for $(U_j)_{j=1}^K$ which are of the form $\langle U_j, U_k\rangle$.

(ii) Since TT^* and T^*T have the same nonzero eigenvalues, so does G and S.

(iii) It is immediate to prove that G is self-adjoint and has Rank(M). Since T is injective, T^* is surjective, and $G^2 = (TT^*)(T^*T) = T(T^*T)T^*$ it follows that G is an orthogonal projection if and only if $T^*T = I$, which is equivalent to the frame being Parseval.

(iv) This is immediate by (ii).

The definition of Riesz basis enables us to define dual frame.

2.6 DUAL FRAME

An overcomplete frame is one which is not a Riesz basis.

Dual frames can be reconstructed in different ways. One way of reconstructing frames is from frame coefficients.

2.6.1 Reconstruction from frame coefficients

Reconstructing a frame from its coefficients i.e $(\langle v, v_j\rangle)_{j=1}^K$ is possible with respect to:

i)An orthonormal basis.

ii) A redundant system which requires a dual frame.

THEOREM 2.6.1 ***(Reconstruction Formula)***
Let $(v_j)_{j=1}^K$ be a frame for $\mathcal{H}^{\mathcal{M}}$ and et S be the frame operator. Then for all $v \in \mathcal{H}^{\mathcal{M}}$, we have

$$v = \sum_{j=1}^{K} \langle v, v_j \rangle S^{-1} v_j = \sum_{j=1}^{K} \langle v, S^{-1} v_j \rangle v_j.$$

Proposition 2.6.2 ***(Canonical Frame)*** *Let $(v_j)_{j=1}^K$ be a frame with frame bounds α and β and frame operator S. Then the sequence $(S^{-1}v_j)_{j=1}^K$ is a frame for $\mathcal{H}^{\mathcal{M}}$ with frame bounds $\frac{1}{\beta}$ and $\frac{1}{\alpha}$ and frame operator $\frac{1}{S}$ for an operator S.*

Proof
$(S^{-1}v_j)_{j=1}^K$ forms a frame for $\mathcal{H}^{\mathcal{M}}$ with associated frame operators $S^{-1}S(S^{-1})^* = S^{-1}$ giving rise to frame bounds $\frac{1}{\beta}$ and $\frac{1}{\alpha}$.

Proposition 2.6.1 calls for the definition of the canonical dual frame for a frame $(v_j)_{j=1}^K$.

Definition 2.6.3 *Let $(v_j)_{j=1}^K$ be a frame for $\mathcal{H}^{\mathcal{M}}$ with frame operator S. Then $(S^{-1}v_j)_{j=1}^K$ is called the canonical dual frame for frame $(v_j)_{j=1}^K$.*

Corollary 2.6.4 *Let $(v_j)_{j=1}^K$ be a Parseval frame for $\mathcal{H}^{\mathcal{M}}$. Then its canonical dual frame is the frame $(v_j)_{j=1}^K$ itself, and the reconstruction formula is*

$$v = \sum_{j=1}^{K} \langle v, v_j \rangle v_j, v \in \mathcal{H}^{\mathcal{M}}.$$

The result below shows the relationship between Parseval frames and orthonormal bases. It applies the above reconstruction formula for Parseval frames.

Proposition 2.6.5 ***(Trace Formula for Parseval Frames)***
Suppose $(v_j)_{j=1}^K$ is a Parseval frame for $\mathcal{H}^{\mathcal{M}}$ and T be a linear operator on $\mathcal{H}^{\mathcal{M}}$. Then

$$Tr(T) = \sum_{j=1}^{K} \langle Tv_j, v_j \rangle$$

.

Proof
Let $(e_k)_{k=1}^K$ be an orthonormal basis for $\mathcal{H}^{\mathcal{M}}$. Then

$$Tr(T) = \sum_{j=1}^{K} \langle Te_k, e_k \rangle$$

$$\Rightarrow Tr(T) = \sum_{j=1}^{M} (\sum_{j=1}^{K} \langle Te_k, v_j \rangle v_j, e_k)$$

$$= \sum_{k=1}^{M} \sum_{j=1}^{K} \langle e_k, T^* v_j \rangle \langle v_j, e_k \rangle$$

$$= \sum_{j=1}^{K} \langle \sum_{k=1}^{M} \langle v_j, e_k \rangle e_k, T^* v_j \rangle$$

$$= \sum_{j=1}^{K} \langle v_j, T^* v_j \rangle$$

$$= \sum_{j=1}^{K} \langle Tv_j, v_{\phi j} \rangle$$

The following example gives a frame and its dual frame.

Example 2.6.6 *Suppose* $T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ *is analysis operator with rows as the frame vectors. Then the dual frame operator of* $T = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$. *Then dual frame vectors are the rows of the matrix of the dual frame operator.*

We shall then discuss the properties of Dual Frames.

2.6.2 Properties of Dual Frames

Proposition 2.6.7 *Let* $(v_j)_{j=1}^K$ *be a frame for* $\mathcal{H}^{\mathcal{M}}$ *with analysis operator* T *and frame operator* S*. Then the following statements are equivalent:*
i) $(u_j)_{j=1}^K$ *is a dual frame for* $(v_j)_{j=1}^K$.
ii) The analysis operator T_1 *of sequence* $(v_j - S^{-1}v_j)_{j=1}^K$ *satisfy Ran(T)*$\perp$ *Ran(*T_1*).*

Proof
Let $\bar{v}_j = v_j - S^{-1}v_j$ for $j = 1, \dots, K$. Then

$$\sum_{j=1}^{K} \langle v, v_j \rangle v_j = \sum_{j=1}^{K} \langle v, v_k + S^{-1}v_j \rangle v_j = v + T^* T_1 v$$

Therefore, $(u_j)_{j=1}^K$ is a dual frame for $(v_j)_{j=1}^K$ if and only if $T^*T_1 = 0$ which is equivalent to (ii).

Corollary 2.6.8 *A frame* $(v_j)_{j=1}^K$ *for* $\mathcal{H}^{\mathcal{M}}$ *has a unique dual frame if and only if* $K = M$.

2.6.3 Converging sequence of Approximation

There are three iterative methods that can be used to derive a converging sequence of approximations of $v \in \mathcal{H}^{\mathcal{M}}$ from the knowledge of $(\langle v, v_j \rangle)_{j=1}^K$. These include the Frame Algorithm, Chebychev Algorithm and Conjugate Gradient Method. These may be required as the reconstruction formula of a frame $(v_j)_{j=1}^K$ using the canonical dual frame may not be utilized in practice as inversion is computationally expensive and numerically unstable.

Proposition 2.6.9 ***(Frame Algorithm)****[8]*
Let $(v_j)_{j=1}^K$ *be a frame for* $\mathcal{H}^{\mathcal{M}}$ *with frame bounds* α, β *and frame operator* S*. Given a signal* $v \in \mathcal{H}^{\mathcal{M}}$*, defining a sequence* $(w_k)_{k=0}^\infty \in \mathcal{H}^{\mathcal{M}}$ *by* $w_0 = 0, w_k = w_{k-1} + \frac{2}{\alpha+\beta} S(v - w_{k-1})$ *for all* $k \geq 1$*. Then* $(w_k)_{k=0}^\infty$ *converges to* $v \in \mathcal{H}^{\mathcal{M}}$ *and the rate of convergence is given by* $\| v - w_k \| \leq (\frac{\beta-\alpha}{\beta+\alpha})^k \| v \|, k \geq 0$.

One major drawback of the frame Algorithm is that the convergence rate depends on the ratio of frame bounds i.e a large ratio leads to a very slow convergence.

Therefore Chebyshev method and Conjugate Gradient method that lead to faster convergence than the Frame Algorithm were introduced to overcome the drawback.

Proposition 2.6.10 ***(Chebychev Algorithm)***
Let $(v_j)_{j=1}^K$ *be a frame for* $\mathcal{H}^{\mathcal{M}}$ *and* α, β *be frame bounds with frame operator* S*. Let* $\rho = \frac{\beta-\alpha}{\beta+\alpha}$ *and* $\delta = \frac{\sqrt{\beta - \sqrt{\alpha}}}{\sqrt{\beta - \sqrt{\alpha}}}$.

Given a signal $v \in \mathcal{H}^{\mathcal{M}}$, define a sequence $(w_j)_{j=0}^{\infty} \in \mathcal{H}^{\mathcal{M}}$ and corresponding scalars $(\lambda_k)_{k=1}^{\infty}$ by

$$w_0 = 0, w_1 = \frac{2}{\beta + \alpha} Sv$$

and $\lambda_1 = 2$ and for $k \geq 2$, $\lambda_k = \frac{1}{1 - \frac{\rho^2}{4}\lambda_{k-1}}$ and $w_k = \lambda_k(w_{k-1} - w_{k-2} + \frac{2}{\beta+\alpha}S(v - w_{k-1})) + w_{k-2}$.

Then $(w_k)_{k=0}^{\infty}$ converges to $v \in \mathcal{H}^{\mathcal{M}}$. Then the rate of convergence is $\| v - w_k \| \leq \frac{2\delta^2}{1+\delta 2k} \| x \|$.

The conjugate gradient method is advantageous as it does not require the knowledge of frame bounds.

Proposition 2.6.11 ***(Conjugate Gradient Method)***
If $(v_j)_{j=1}^{K}$ is a frame for $\mathcal{H}^{\mathcal{M}}$ with frame operator S. Let a signal $v \in \mathcal{H}^{\mathcal{M}}$, define sequences $(w_k)_{k=0}^{K}, (\gamma_k)_{k=0}^{\infty}$ and $(\wp_k)_{k=1}^{\infty}$ in $\mathcal{H}^{\mathcal{M}}$ and corresponding scalars $(\lambda_k)_{k=1}^{\infty}$ by $w_0 = 0$, $\gamma_0 = \wp_0 = Sv$ and $\wp_{-1} = 0$ and for $k \geq 0$ set $\lambda_k = \frac{\langle \gamma_k, \wp_k \rangle}{\langle \wp_k, S\wp_k \rangle}, w_{k+1} = w_k + \lambda_k \wp_k, \gamma_{k+1} = \gamma_k - \lambda_k S\wp_k$ and $\wp_{k+1} = S\wp_k - \frac{\langle S\wp_k, S\wp_k \rangle}{\langle \wp_k, S\wp_k \rangle}\wp_k - \frac{\langle S\wp_k, S\wp_{k-1} \rangle}{\langle \wp_{k-1}, S\wp_{k-1} \rangle}\wp_{k-1}$. Then $(w_k)_{k=0}^{\infty}$ converges to $v \in \mathcal{H}^{\mathcal{M}}$ and the rate of convergence is $||| v - w_K ||| \leq \frac{2\delta^K}{1+\delta^{2K}} ||| v |||$ where $\delta = \frac{\sqrt{\beta}-\sqrt{\alpha}}{\sqrt{\beta}+\sqrt{\alpha}}$ and $||| \cdot |||$ is the norm on $\mathcal{H}^{\mathcal{M}}$ given by $||| v ||| = \langle v, Sv \rangle = \| S^{\frac{1}{2}} v \|$, $v \in \mathcal{H}^{\mathcal{M}}$.

2.7 Construction of Frames

To find frames with certain desired properties, construction is necessary. This is very important for application of frames. There are many construction methods [9, 7] including Spectral Tetris, eigensteps among others. A prominent selection is used in this section.

2.7.1 Tight and Parseval Frames

In the construction of frames; tight frames are preferred. A frame reconstructed from tight frame coefficients is numerically optimally stable. In this section, most constructions discussed are geared towards modifying a given frame so that the result is a tight frame.

Let us begin with the generation of a tight frame. This applies $S^{\frac{-1}{2}}$ where S is a Parseval frame.

LEMMA 2.7.1 *Let $(v_j)_{j=1}^{K}$ be a frame for $\mathcal{H}^{\mathcal{M}}$ with a frame operator S. Then $(S^{\frac{-1}{2}} v_j)_{j=1}^{K}$ is Parseval frame.*

Proof
By Proposition 2.4.5, the frame operator for $(S^{\frac{-1}{2}} v_j)_{=1}^{K}$ is $(S^{\frac{-1}{2}})(S^{\frac{-1}{2}}) = I$. Hence the result.
It is only that this procedure requires inversion of the frame operator.
Invariance under orthogonal projections is the most basic invariance property of a frame can have. The orthogonal projection of a Parseval frame remains a Parseval frame. This operation maintains and may even improve frame bounds as shown below.

Proposition 2.7.2 *Let $(v_j)_{j=1}^{K}$ be frame for $\mathcal{H}^{\mathcal{M}}$ with frame bounds α and β. Let T be an orthogonal projection for $\mathcal{H}^{\mathcal{M}}$ onto a subspace W. Then $(Pv_j)_{j=1}^{K}$ is a frame for W with frame bounds α and β. Particularly, if $(v_j)_{j=1}^{K}$ is a Parseval frame for $\mathcal{H}^{\mathcal{M}}$ and P is an orthogonal projection on $\mathcal{H}^{\mathcal{M}}$ onto W then $(Pv_j)_{j=1}^{K}$ is a Parseval frame for W.*

Proof
For any

$$v \in W, \alpha \| v \|^2 = \alpha \| Pv \|^2 \leq \sum_{j=1}^{K} | \langle Pv, v_j \rangle |^2 = \sum_{j=1}^{K} | v, Pve_j \rangle |^2 \leq \beta \| Pv \|^2 = \beta \| v \|^2 .$$

Proposition 2.7.2 yields a corollary which can be interpreted as given an $M \times M$ unitary matrix, selecting N rows from the matrix then the column vectors of these N rows form a Parseval frame for $\mathcal{H}^{\mathcal{M}}$.

Corollary 2.7.3 *Let* $(e_j)_{j=1}^{M}$ *be an orthonormal basis for* $\mathcal{H}^{\mathcal{M}}$ *and let* T *be an orthogonal projection for* $\mathcal{H}^{M}$ *onto a subspace* W. *Then* $(Te_j)_{j=1}^{M}$ *is a Parseval frame for* W.

By scaling the frame vectors, a frame can easily be constructed from a given frame. Scalable frames are those can be scaled into a Parseval or tight frames. There is need to have a characterization of the class of frames which can be scaled.

Definition 2.7.4 *A frame* $(v_j)_{j=1}^{K}$ *for* $\mathcal{H}^{\mathcal{M}}$ *is said to be (strictly)scalable if there exists a nonnegative (respectively positive) numbers* $b_1, \ldots, b_K$ *such that* $(b_j v_j)_{j=1}^{K}$ *is a Parseval frame.*

Next is the Naimark's Theorem.

THEOREM 2.7.5 *(Naimark's Theorem)*
Let $(v_j)_{j=1}^{K}$ *be a frame for* $\mathcal{H}^{\mathcal{M}}$ *with analysis operator* T. *Let* $(e_j)_{j=1}^{M}$ *be a standard basis of* ℓ_2^K *and* $P : \ell_2^M \longrightarrow \ell_2^M$ *be the orthogonal projection onto* $Ran(T)$. *Then the following conditions are equivalent:*
i) $(v_j)_{j=1}^{K}$ *is a Parseval frame for* $\mathcal{H}^{\mathcal{M}}$.
ii) $\forall j = 1, \ldots K$ *we have* $Pe_j = T(v_j)_{j=1}^{K}$.
iii) There exists $u_1, u_2, \ldots u_M \in \mathcal{H}^{\mathcal{K}-\mathcal{M}}$ *such that* $(v_j \bigoplus u_j)_{j=1}^{K}$ *is an orthonormal basis for* $\mathcal{H}^{\mathcal{M}}$.
Moreover, if (iii) holds then $(u_j)_{j=1}^{K}$ *is a Parseval frame for* $\mathcal{H}^{\mathcal{K}-\mathcal{M}}$. *If* $(u_j)_{j=1}^{K}$ *is another Parseval frame as in (iii), then there exists a unique linear operator* T *on* $\mathcal{H}^{\mathcal{K}-\mathcal{M}}$ *such that* $(Tu_j)_{j=1}^{K} = u_j, j = 1, \ldots, K$ *and* T *is unitary.*

Proof
(i) $\Leftrightarrow$ (ii) By Theorem 2.5.2 (iii),$(v_j)_{j=1}^{K}$ is a Parseval frame if and only if $TT^* = P$. Since $\forall j = 1, 2, \ldots, K, T^* e_j = (v_j)_{j=1}^{K}$ then (i) is equivalent to (ii).
$(i) \Rightarrow (iii)$ Let $c_j := e_j - Tv_j$ for all $j = 1, 2, \ldots, K$ then by (ii), $c_j \in (Ran(T))^{\perp}$ for all j. Let $\Phi : (Ran(T))^{\perp} \longrightarrow \mathcal{H}^{\mathcal{K}-\mathcal{M}}$ be unitary and put $u_j = \Phi c_j, j = 1, 2, \ldots, K$ then since T is isometric, $\langle v_j \bigoplus u_j, v_k \bigoplus u_k \rangle = \langle v_j, v_k \rangle + \langle u_j, u_k \rangle = \langle Tv_j, Tv_k \rangle + \langle c_j, c_k \rangle = \delta_{jk}$ which proves (iii).
$iii) \Rightarrow i)$ follows from corollary 2.7.3
The moreover part follows from Corollary 2.7.3 that $(u_j)_{j=1}^{K}$ for $\mathcal{H}^{\mathcal{K}-\mathcal{M}}$. Let $\{u_j\}_{j=1}^{K}$ be another Parseval frame as in (iii). Let F and $F^{'}$ denote the analysis operators for $\{u_j\}_{j=1}^{K}$ and $\{u_j\}_{j=1}^{K}$ respectively. We make use of the decomposition $\mathcal{H}^{\mathcal{K}} = \mathcal{H}^{\mathcal{M}} \bigoplus \mathcal{H}^{\mathcal{K}-\mathcal{M}}$. Note that $U := (T, F)$ and $U^{'} := (T, F^{'})$ are unitary operators from $\mathcal{H}^{\mathcal{K}}$ to ℓ_2^K. Let P_{K-M} denote the projection operator from $\mathcal{H}^{\mathcal{K}}$ onto $\mathcal{H}^{\mathcal{K}-\mathcal{M}}$ and set

$$T := P_{K-M} U^{'*} U|_{\mathcal{H}^{\mathcal{K}-\mathcal{M}}} = F_{K-M} U^{'*} F.$$

Let $w \in \mathcal{H}^{\mathcal{M}}$. Then, since $U|_{\mathcal{H}^{\mathcal{M}}} = U^{'}|_{\mathcal{H}^{\mathcal{M}}=T}$, we have $P_{K-M} U^{'*} U w = F_{K-M} w = 0$. Hence $Tu_j = P_{K-M} U^{'*} U(v_j \bigoplus u_j) = P_{K-M} U^{'*} e_j = P_{K-M}(v_j \bigoplus \{u_j\} = \{u_j\}$. T is unique since both $(u_j)_{j=1}^{K}$ and $(u^{j})^{'}{}_{j=1}^{K}$ are spanning sets for $\mathcal{H}^{\mathcal{K}-\mathcal{M}}$.
If $\mathcal{H}^{\mathcal{M}}$ is real, then the following result applies, which can be utilized to derive a geometric interpretation of scalability.

THEOREM 2.7.6 *Let* $\mathcal{H}^{\mathcal{M}}$ *be real and let* $(v_j)_{j=1}^{K}$ *be a frame for* $\mathcal{H}^{\mathcal{M}}$ *without zero vectors. Then the following statements are equivalents:*
i) $(v_j)_{j=1}^{K}$ *is not scalable.*
ii) There exists a self-adjoint operator Y *on* $\mathcal{H}^{\mathcal{M}}$ *with* $Tr(Y) < 0$ *and* $\langle Yv_j, v_j \rangle \geq 0$ *for all* $j = 1, \ldots, K$
iii) There exists a self-adjoint operator Y *on* $\mathcal{H}^{\mathcal{M}}$ *with* $Tr(Y) = 0$ *and* $\langle Yv_j, v_j \rangle = 0$ *for all* $j = 1, \ldots, K$

2.8 Frames with Prescribed Frame Operator

To construct frames with prescribed frame operator, the eigenvalues of the frame operator are given with the assumption that the eigenvectors are the standard basis. An application of this is in noise reduction in case coloured noise is present.

The theorem below is so vital in frames with prescribed frame operator. It includes prescribing the eigenvalues of the frame operator.

THEOREM 2.8.1 *Let S be a positive self-adjoint operator on $\mathcal{H}^{\mathcal{M}}$, and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M > 0$ are the eigenvalues of S. Further, let $K \geq M$, and $c_1 \geq c_2 \geq c_3 \geq \ldots c_K$ are positive real numbers. Then the following conditions are equivalent:*

i) There exists a frame $(v_j)_{j=1}^K$ for $\mathcal{H}^{\mathcal{M}}$ with frame operator S satisfying $\| v_j \| = c_j \ \forall j = 1, \ldots, K$

ii) For every $1 \leq k \leq M$, we have

$$\sum_{k=1}^{K} c_k^2 \leq \sum_{k=1}^{K} \lambda_k$$

and

$$\sum_{j=1}^{K} c_j^2 = \sum_{j=1}^{M} \lambda_k$$

However, it is often preferable to utilize equal - norm frames, since then, roughly speaking, each vector provides the same coverage for the space.

There always exists an equal - norm frame with a prescribed frame operator. This is the content of the next result.

THEOREM 2.8.2 *For every $K \geq M$ and every invertible positive self-adjoint operator S on $\mathcal{H}^{\mathcal{M}}$ there exists an equal norm frame for $\mathcal{H}^{\mathcal{M}}$ with K elements and frame operator S. In particular, there exist equal norm Parseval frames with K elements in $\mathcal{H}^{\mathcal{M}}$ for every $M \leq K$.*

Proof

We define the norm of the to-be-constructed frame to be c, where

$$c^2 = \frac{1}{K} \sum_{k=1}^{M} \lambda_k$$

It is sufficient to prove that the conditions in Theorem 2.7.1(ii) are satisfied for $c_j = c$ for all $j = 1, 2, ..., K$. The definition of c immediately implies the second condition.

For the first condition, we observe that

$$c_1^2 = c^2 = \frac{1}{K} \sum_{k=1}^{M} \lambda_k \leq \lambda_1.$$

Hence this condition holds for $k = 1$. Now, toward a contradiction, assume that there exists some $m \in \{2, ..., M\}$ for which this condition fails for the first time by counting from 1 upward, i.e.,

$$\sum_{k=1}^{m-1} c_k^2 = (m-1)c^2 \leq \sum_{k=1}^{m-1} \lambda_k,$$

but

$$\sum_{k=1}^{m} c_k^2 = mc^2 > \sum_{k=1}^{m} \lambda_k \Rightarrow c^2 \geq \lambda_m$$

and thus $c^2 \geq \lambda_k$ for all $m+1 \leq k \leq M$.

Hence,

$$Kc^2 \geq mc^2 + (M-m)c^2 > \sum_{k=1}^{m} \lambda_k + \sum_{k=m+1}^{M} c_k^2 \geq \sum_{k=1}^{M} \lambda_k + \sum_{k=m+1}^{M} \lambda_k = \sum_{k=1}^{M} \lambda_k$$

which is a contradiction. Hence the result.

2.9 Excess of a Parseval Frame

The excess of a sequence in a Hilbert space $\mathcal{H}$ is the maximum number of elements that can be removed yet leave a set with the same closed span. Excess of a frame is the greatest integer n such that n elements can be deleted from the frame and still leave a complete set.

Duffin and Schaeffer [15] proved that if Θ is a frame for a Hilbert space $\mathcal{H}$ and $v \in \Theta$ is such that $\Theta\{v\}$ is complete in $\mathcal{H}$, then Θ $\{v\}$ is complete in $\mathcal{H}$, then Θ $\{v\}$ is a frame for $\mathcal{H}$. By iterating, it follows that in any overcomplete frame, at least finitely many elements can be removed yet still leave a frame.

Definition 2.9.1 *The Excess of a sequence $\Theta = \{v_j\}_{j=1}^M$ in a Hilbert space $\mathcal{H}$ denoted by $e(\Theta)$ is defined as $e(\Theta)= sup\{\mid G \mid : G \subset \Theta$ and $span(\Theta) = sp\bar{a}n(\Theta)\}$*

The excess to the dimension of the kernel to the synthesis operator and to certain inner products of frame elements with corresponding dual frame elements are connected as in lemma 2.9.2 below.

LEMMA 2.9.2 *Let $\Theta = \{v_j\}_{j=1}^K$ be a Bessel sequence in $\mathcal{H}$. Let $T : \mathcal{H} \to \ell^2(\mathcal{H})$ be the associated analysis operator.*

i) $e(\Theta) \geq dim(Ker\ T^*)$.

ii) If Θ is a frame then $e(\Theta) = dim\ (KerT^*)$.

Furthermore if $\bar{\Theta} = (v_j)_{j=1}^K$ is the canonical dual frame then

$$e(\Theta) = \sum_{j=1}^{M}(1 - \langle v, \bar{v}_j\rangle).\frac{e_n}{n}\}_{n\in N} \bigcup v$$

is not a frame. $e(\Theta) = 1$ and $(KerT^*) = 0$

Example 2.9.3 *Let $\Theta = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\} \in \mathcal{H}= R^2$. $\mid G \mid= 3$. $Scan(F)= R^2$.*

Example 2.9.4 *Let $\Theta = \{e_1, e_1, e_2, e_2, \ldots, e_m, e_m\}$. Then G can be $\{e_1, e_2, \ldots, e_m\}$. This implies that $e(\Theta) = m$.*

Chapter 3

GROUP FRAMES

3.1 Introduction

In this chapter, we shall investigate the relationship between group theory and frames. We shall investigate the symmetries of a group, representation theory and group frames, group matrices and the Gramian of a group, the characterization of all tight G- frames, Heisengerg and Harmonic group frames. The rotations of a triangle through $\frac{2\pi}{3}$ or the dihedral symmetries of a triangle as in definition 1.3.36 can be used to obtain the Mercedes- Benz frame as the orbit of a single vector. Most frames used in application are often obtained as the orbit of a single vector e.g Harmonic frames used in signal analysis and the equiangular Heisenberg frames used in quantum information theory. The basic theory of such group frames will be described, and some of the constructions that have been found so far discussed. However, before we describe this exploration, we shall provide some background information on these concepts.

3.2 Symmetries of a Frame

Formally, the symmetry group is a group of permutations (an abstract group) which acts as unitary transformations.

For a Mercedes- Benz frame, its symmetries are unitary maps i.e rotations and reflections that permute with its vectors.

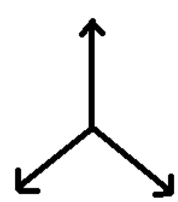

Figure 3.1: Symmetry of Mercedes Benz Frame

The following features of the symmetry group are well known (see [37])

i) For all finite frames, it is a group of the permutations on the index set.

ii) Similar frames have equal symmetry groups e.g a frame , its dual frame and its canonical tight frame.

iii) It is easily calculated from the Gramian of the canonical tight frame.

iv) The symmetry groups of tensor sums and direct sums of frames are related to those of the frames.

v) The symmetry group of a frame and its complementary frame are equal. We shall begin by defining the symmetry group of a finite frame $\Theta = (v_j)_{j=1}^{K}$ for a Hilbert space $\mathcal{H}$.

Definition 3.2.1 *Let $\mathcal{H} = \mathbb{F}^M$ and $\Theta = (v_j)_{j=1}^{K}$ be a finite frame for $\mathcal{H}$. The symmetry group of Θ denoted by sym(Θ) is defined as sym$(\Theta) := \{\sigma \in S_K$ such that there exists $U_\sigma \in GL(\mathcal{H})\}$ with $U_\sigma v_j = v_{\sigma j}, j = 1, \ldots, K$.*

We state and prove the following result on frames.

THEOREM 3.2.2 *Let Ω and Θ be similar frames i.e $\Omega = P\Theta$ where $P \in GL(\mathcal{H})$ or are complementary frames i.e $G_{\Omega^{can}} + G_{\Theta^{can}} = I$. Then sym$(\Omega)$= sym$(\Theta)$ where Ω^{can} denote the canonical tight frame $(\Omega\Omega^*)^{\frac{-1}{2}}\Omega$ for Ω.*

Particularly, a frame, its dual frame and its canonical tight frame have the same symmetry group.

Proof

Showing one inclusion: suppose $\sigma \in sym(\Omega)$ i.e $U_\sigma v_j = v_{\sigma j}$ for all j. Since $v_j = Pw_j$. This gives $P^{-1}U_\sigma P w_j = w_{\sigma j}$ for all j. Hence $\sigma \in$ sym (Θ).

An example of frames whose symmetric groups are equal is as shown below.

Example 3.2.3 *If Ω is a Mercedes- Benz frame, then the vectors of Ω add to zero. The complementary frame for Ω is $\Theta = ([1][1][1])$. Therefore sym$(\Theta) = S_3$ hence sym$(\Omega) = S_3$.*

Since a finite frame Θ is determined up to similarity by $G_{\Theta^{can}}$, the Gramian of the canonical tight frame, the proposition below shows that we can possibly compute Sym (Θ) from $G_{\Theta^{can}}$.

Proposition 3.2.4 *Let Θ be a finite frame. Then $\sigma \in sym(\Theta)$ if and only if $Q_\sigma^* G_{\Theta^{can}} Q_\sigma = G_{\Theta^{can}}$ where Q_σ is the permutation matrix given by $Q_\sigma e_j = e_{\sigma j}$.*

There are maximally symmetric frames of K vectors in $\mathbb{F}^M$, i.e., those with the largest possible symmetry groups as sym(Θ) is a subgroup of S_K.

Example 3.2.5 *If K is the number of equally spaced vectors in $\mathbb{R}^2$, then the vectors have the dihedral group (as defined in definition 1.3.36) of order $2K$ as symmetries. Although, in $\mathbb{C}^2$, this is not always the most symmetric frame of K vectors e.g if K is even the (Harmonic) tight frame given by the K distinct vectors*

$$\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} z \\ -z \end{pmatrix}, \begin{pmatrix} z^2 \\ z^2 \end{pmatrix}, \begin{pmatrix} z^3 \\ -z^3 \end{pmatrix}, \begin{pmatrix} z^4 \\ z^4 \end{pmatrix}, , \ldots \begin{pmatrix} z^{K-2} \\ z^{K-2} \end{pmatrix}, \begin{pmatrix} z^{-1} \\ z^{-1} \end{pmatrix}\} where z = e^{\frac{2\Pi j}{K}}$$

has a symmetry group of order $\frac{1}{2}K^2$.

Example 3.2.6 *The most symmetric tight frames of five vectors in $\mathbb{R}^3$ are as shown in Figure 3.2 below.*

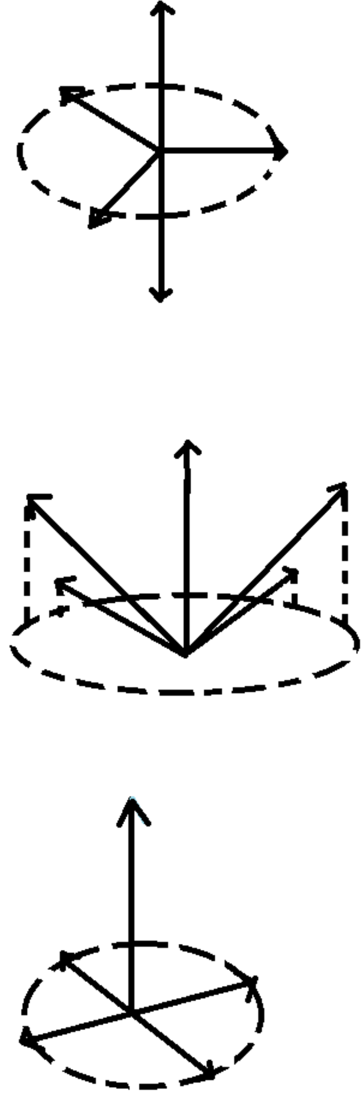

Figure 3.2: Symmetric Tight Frames.

The following proposition gives some of the conditions satisfied by symmetry groups of finite frames Θ and Ω.

Proposition 3.2.7 *Let Θ and Ω be finite frames. Then the symmetry groups of a finite frames satisfy*

i) Union of frames.

$sym(\Theta)\times sym(\Omega)\subset sym(\Theta \cup \Omega)$ *where* $\Theta \cup \Omega := (\begin{pmatrix} v_j \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \Omega_k \end{pmatrix})$

ii) Tensor product

$sym(\Theta)\times sym(\Omega)\subset sym(\Theta \bigotimes \Omega)$ *where* $\Theta \bigotimes \Omega := (v_j \bigotimes \Omega_k)$

iii) Direct sum

$sym(\Theta)\times sym(\Omega)\subset sym(\Theta \bigoplus \Omega)$ *where* $\Theta \bigoplus \Omega := (\begin{pmatrix} v_j \\ \Omega_k \end{pmatrix})$

where $\Sigma_j \langle f, v_j \rangle \Omega_k = 0, \forall f$.

3.3 Finite group Representation Theory.

Let G be a finite group, a representation of G is a homomorphism of groups $\rho : G \to GL_n(\mathbb{F})$,from G into the group of nxn invertible matrices over a field $\mathbb{F}$. A group action of the group is induced by the mapping on the vector space $\mathbb{F}^n$ by the product $v \to \rho(g)v$ where $g \in G$.

We can define the multiplication between the group element $g \in G$ and vector $v \in \mathbb{F}^n$ by the product $gv \to \rho(g)v$ thus producing another vector in $\mathbb{F}^n$. Since ρ is a homomorphism then $(hg)v = \rho(hg)v =$

$\rho(h)\rho(g)v = \rho(h)(\rho(g)v) = h(gv)$. The group identity 1 corresponds to the identity matrix. These properties give $\mathbb{F}^n$ the structure of an $\mathbb{F}G$- module. This module is also referred to as a representation of G.

Example 3.3.1 *Consider the action of the cyclic group $Z_3 = \{1, a, a^2\}$ on $\mathbb{C}^3$ by the permutation of the three.*

Let $\rho : Z_3 \to GL_3(\mathbb{C})$ be a map defined by $1 \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, a^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Then $\rho(1.a) = \rho(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \rho(1)\rho(a)$

and

$\rho(a.a^2) = \rho(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \rho(a)\rho(a^2)$

Similarly, $\rho(1.a^2) = \rho(1)\rho(a^2)$,

Since $\rho(gh) = \rho(g)\rho(h)$ for any $g, h \in Z_3, \rho$ is a representation of Z_3. Irreducible representations, which are representations whose associated $\mathbb{F}G$- modules have only trivial submodules i.e the only subspaces of V invariant under the action of G are the trivial subspace 0 and entire module are a primary method of understanding group representations and the groups they represent.

The following theorem 3.1.2 gives the relations between two representations.

THEOREM 3.3.2 *([25]Shur's Lemma)*

Let V and W be irreducible $\mathbb{C}G$- modules.

i) If $\Phi : V \to W$ is a $\mathbb{C}G$- homomorphism then either Φ is a $\mathbb{C}G$- isomorphism or $\Phi(v) = 0$ for all $v \in V$.

ii) If $\Phi : V \to V$ is a $\mathbb{C}G$- isomorphism then Φ is a scalar multiple of the identity endomorphism 1_v.

Proof

i) If $\Phi(v) \neq 0$ for some $v \in V$. Then Im $\Phi = 0$. As Im Φ is a $\mathbb{C}G$- submodule of W, and W is irreducible then Im $\Phi = W$. Ker Φ is a $\mathbb{C}G$- submodule of V; as Ker $\Phi \neq v$ and v is irreducible ; Ker $\Phi = 0$. Thus Φ is invertible , and hence is a $\mathbb{C}G$- isomorphism.

ii) The endomorphism Φ has an eigenvalue $\lambda \in \mathbb{C}$, and so $Ker(\Phi - \lambda 1_v) \neq \{0\}$. Thus Ker $(\Phi - \lambda 1_v)$ is a non- zero $\mathbb{C}G$- submodule of V. Since V is irreducible , Ker $(\Phi - \lambda 1_v) = v$. Therefore, $v(\Phi - \lambda 1_v) = 0$ for all $v \in V$ i.e $\Phi = \lambda 1_v$, as required.

Furthermore, we have that every representation of a finite group G is decomposable into a direct sum sum of irreducible submodules as stated and proved as in the Maschke's Theorem.

THEOREM 3.3.3 *(Maschke's Theorem) If G is a finite group and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If G is an $\mathbb{F}G$- module and U is an $\mathbb{F}G$- submodule of V, then there is an $\mathbb{F}G$- submodule W of V such that $V = U \bigoplus W$.*

From Schur's Lemma and Maschke's Theorem, we realise that every representation of a finite group G has a unique decomposition into irreducible subrepresentations and Schur's Lemma shows that the decomposition is unique [18].

3.4 Representation Theory and G-Frames

We shall begin by defining G-frames.

Definition 3.4.1 *Let Θ be a finite frame and sym (Θ) be its group symmetry. Let $\sigma \in sym(\Theta)$ then there is a unique $U_\sigma \in GL(\mathcal{H})$ with $U_\sigma f_j = f_{\sigma j}, \forall j$ and $sym(\Theta) \to GL(\mathcal{H}) : \sigma \to U_\sigma$ is a group homomorphism, i.e., a representation of $G = Sym(\Theta)$. If the action of the symmetry group is transitive on Θ under this action i.e Θ is the orbit of any one vector e.g the Mercedes- Benz frame then we have a $G-$frame.*

A representation of a group G provides a way of visualizing G as a group of matrices. To be precise, a representation is a homomorphism from G into a group of invertible matrices.

The definition of a G- frame is as below.

Definition 3.4.2 *Let G be a finite group. A group frame for a Hilbert space $\mathcal{H}$ denoted by G- frame is a frame $\Theta = (v_g)_{g\in G}$ for which there exists a unitary representation $\rho : G \to \mathcal{U}(\mathcal{H})$ with $gv_h := \rho(g)v_h = v_{gh}$, for all $g, h \in G$.*

From the definition 3.4.2, clearly a G- frame Θ is the orbit of a single vector $v \in H$ i.e $\Theta = (gv)_{g\in G}$. Hence Θ is an equal - norm frame.

Example 3.4.3 *The vertices of the regular K- gonal or Platonic solids are some of the examples of frames considered [6].*

We shall then in the next sections look at the basic properties and constructions for G- frames. Particularly, we shall realize that:

i) For abelian groups G, there is a finite number of G- frames of K vectors in $\mathbb{F}^M$. These frames are known as Harmonic frames.

ii) For nonabelian groups G, there is an infinite number of G- frames of K vectors in $\mathbb{F}^M$ of $K = M^2$ vectors in $\mathbb{C}^M$. These frames are known as the Heisenberg frames. This provide equiangular tight frames with the maximal number of vectors.

3.5 Group Matrices and the Gramian of a Group Frame

The representation defining a G- frame is unitary implying that $\rho(g)^* = \rho(g)^{-1} = \rho(g^{-1})$ such that $g^{-1}v = g^*v$, the Gramian of a G-Matrix $\Theta = (v_g)_{g\in G} = (gv)_{g\in G}$ is of the form: $\langle vg, vh\rangle = \langle gv, hv\rangle = \langle v, g^*hv\rangle = \langle v, g^{-1}hv\rangle = \eta(g^{-1}h)$, where $\eta : G \to \mathbb{F}$.

Clearly, we observe that the Gramian of a G- frame is a group matrix or G- matrix that is a matrix M, with entries indexed by elements of a group G with the form $M = [\eta(g^{-1}h)]_{g,h\in G}$.

The importance of Gramian of a G- frame being a group matrix is that it has a small number of angles i.e $\eta(g)$ such that $g \in G$ which make them useful for equiangular tight frames.

THEOREM 3.5.1 *If G is a finite group then $\Theta = (v_g)_{g\in G}$ is a G- frame (for its span $\mathcal{H}$) if and only if its Gramian G_Θ is a G-matrix.*

3.6 Characterization of all Tight Group Frames

If there exists a G- frame $\Theta = (gv)_{g\in G}$ i.e span $\{gv : g \in G\} = \mathcal{H}$ for a given representation, then the canonical tight frame is a tight G- frame. The description is as in the definition 3.6.1 below.

Definition 3.6.1 *If G is a finite group then we say that $\mathcal{H}$ is an $\mathbb{F}G$- module if there is a unitary action $(g, \nu) \to g\nu$ of G on $\mathcal{H}$ i.e a representation $G \to U(\mathcal{H})$.*

We say that a linear map $\sigma : V_j \to V_k$ between $\mathbb{F}G$- modules is an $\mathbb{F}G$- homomorphism if $\sigma g = g\sigma$ for all $g \in G$. If σ is a bijection then σ is an $\mathbb{F}G$- isomorphism. An $\mathbb{F}G$-module is irreducible if the corresponding representation is, and it is absolutely irreducible if it is irreducible when thought of as a $\mathbb{C}G$-module in the natural way.

A complete characterisation of which G-frames are tight, i.e which orbits $(gv)_{g\in G}$ under a unitary action of G gives a tight frame. A special case can be proved as below.

THEOREM 3.6.2 *If $\rho : G \to \mathcal{U}(\mathcal{H})$ is an irreducible unitary representation i.e span $\{gv : g \in G\} = \mathcal{H}$ then every orbit $\Theta = (gv)_{g\in G}, v \neq 0$ is a tight frame.*

Proof

Let $v \neq 0$, so that $\Theta = (gv)_{g\in G}$ is a frame. The frame operator S_Θ is positive definite, so there is an eigenvalue $\lambda > 0$ with corresponding eigenvector w. Since the action is unitary, we can calculate

$$S_\Theta(gw) = \sum_{h\in G}\langle gw, hv\rangle hv = g\sum_{h\in G}\langle w, g^{-1}hv\rangle g^{-1}hv = gS_\Theta(w) = \lambda(gw)$$

such that $S_\Theta = \lambda(I)$ on span $\{gw : g \in G\} = \mathcal{H}$. Hence Θ is a tight frame.

Example 3.6.3 *The symmetry groups of the five platonic solids acting on $\mathbb{R}^3$ as unitary transformations give irreducible representations, as do the dihedral groups acting on $\mathbb{R}^2$. Thus the vertices of the platonic solids and the K equally spaced vectors in $\mathbb{R}^2$ are tight G-frames.*

Theorem 3.6.2 can be generalized.

THEOREM 3.6.4 *Let G be a finite group which acts on $\mathcal{H}$ as unitary transformations,and let $\mathcal{H} = V_1 \bigoplus V_2 \bigoplus V_3 \ldots \bigoplus V_m$ be an orthogonal direct sum of irreducible $\mathbb{F}G$- modules for which repeated summands are absolutely irreducible. Then $\Theta = (gv)_{g\in G}, v = v_1 + v_2 + v_3 + \ldots + v_m, v_k \in V_k$ is a tight frame if and only if $\frac{\|v_k\|^2}{\|v_m\|^2} = \frac{dim(V_k)}{dim(V_m)}, \forall k, m$ and $\langle \sigma v_k, v_m\rangle = 0$ when V_k is $\mathbb{F}G$- isomorphic to V_m through $\sigma : V_k \to V_m$.*

Proposition 3.6.5 *If G is a finite group acting on $\mathcal{H}$ as unitary transformations then the associated canonical tight frame is a tight G-frame for $\mathcal{H}$ if there is a $v \in \mathcal{H}$ for which there is a $(gv)_{g\in G}$ is a frame i.e that spans $\mathcal{H}$.*

This is an alternative way to construct tight G-frames, but requires calculation of the square root of the frame operator.

3.7 Harmonic Frames

Harmonic or geometrically uniform tight frames are tight frames most commonly used in application due to their simplicity of construction and flexibility. They are an equal norm tight frame for $\mathbb{C}^M$ that can be obtained as the columns of any submatrix obtained by taking K rows of the Fourier transform matrix. The $K \times K$ Fourier matrix as below is a unitary matrix hence its rows or columns form an orthonormal basis for $\mathbb{C}^M$. Its projection is a tight frame. Below is the $K \times K$ Fourier transform

$$\frac{1}{\sqrt{K}}\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & z & z^2 & \ldots & z^{K-1} \\ 1 & z^2 & z^4 & \ldots & z^{2(K-1)} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & z^{K-1} & z^{2(K-1)} & \ldots & z^{(K-1)(K-1)} \end{bmatrix} \text{ where } z = e^{\frac{2\Pi j}{K}}$$

The irreducible representations for an abelian group G are one dimensional usually known as (linear) characters $\pi : G \to \mathbb{C}$. For cyclic group of order K i.e $G = Z_K$, the K characters are $\pi_k : K \to (z^k)^m, k \in Z_K$. These are the rows or columns of the Fourier transform.

For G a finite abelian group of order M, and $\tilde{G}$ is the character group, i.e., the set of K characters of G which forms a group under pointwise multiplication then the group $G \approxeq \tilde{G}$.

Definition 3.7.1 *If G is abelian group of order K then G- frame for $\mathbb{C}^K$ obtained by taking K rows or columns of the character table of G i.e $\Theta = ((\pi_k(g))_{k=1}^K)_{g\in G}, \pi_1, \pi_2, \ldots, \pi_K \in \tilde{G}$ or $\Theta = ((\pi(g_K))_{k=1}^K)_{\pi\in\tilde{G}}, \pi_1, \pi_2, \ldots, \pi_K \in G$ is called a Harmonic frame.*

We can characterise the G- frames for G abelian.

Since the number of abelian groups of order K are finite, we have the following result.

Corollary 3.7.2 *If $K \geq M$ (where K is the order of abelian group G while M is the dimension of the Hilbert space) then there is a finite number of tight frames of K vectors for $\mathbb{C}^M$ (upto unitary equivalence) given by the orbit of an abelian group of $M \times M$ matrices. This is the Harmonic frame.*

The following example illustrates the above Corollary 3.7.2.

Example 3.7.3 *The following Harmonic frames for $\mathbb{C}^2$ is obtained from the second and last rows of the Fourier Transform Matrix.*

$$\Theta = (\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} z \\ \bar{z} \end{pmatrix}, \begin{pmatrix} z^2 \\ \bar{z}^2 \end{pmatrix}, \begin{pmatrix} z^3 \\ \bar{z}^3 \end{pmatrix}, \ldots, \begin{pmatrix} z^{k-1} \\ \bar{z}^{k-1} \end{pmatrix})$$

An equal - norm finite tight frame for $\mathbb{C}^M$ is a group frame for an abelian group G and is also Harmonic.

THEOREM 3.7.4 *Let Θ be an equal - norm finite tight frame for $\mathbb{C}^M$. Then the following statements are equivalent:*

i) Θ is a G- frame where G is an abelian group.

ii) Θ is Harmonic.

Corollary 3.7.5 *There exists a real Harmonic frame of $K \geq M$ vectors for $\mathbb{R}^M$.*

Example 3.7.6 *The smallest noncyclic abelian group is $\mathbb{Z}_2 \times \mathbb{Z}_2$. Its character table can be calculated as the Kronecker product of that $\mathbb{Z}_2$ with itself , giving*

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \bigotimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

To get the Harmonic frame, we can take any pair of the last three rows.

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ *of four equally spaced vectors in $\mathbb{R}^2$, which is also given by $\mathbb{Z}_4$.*

Taking the first row and any other gives two copies of an orthogonal basis.

Therefore, Harmonic frames may be given by the character tables of different abelian groups; frames which arise from cyclic groups are known as cyclic Harmonic frames. There exists Harmonic frames of M vectors which are not cyclic.

The equivalence of subsets of a group G is defined through automorphism as follows.

Definition 3.7.7 *Two subsets X and Y of a finite group G are multiplicatively equivalent if there is an automorphism $\sigma : G \to G$ for which $Y = \sigma(X)$.*

Definition 3.7.8 *Two G- frames Θ and Ω are unitarily equivalent through an automorphism if $v_g = cU\Omega_{\sigma g}$, for all $g \in G$ where c is a constant.*

THEOREM 3.7.9 *Let G be a finite abelian group, $X, Y \subset G$. Then the following statements are equivalent:*

i) The subsets X and Y are multiplicatively equivalent.

ii) The Harmonic frames given by X and Y are unitarily equivalent through an automorphism.

The result below effectively illustrates Theorem 3.7.9:

THEOREM 3.7.10 *[12] Let G be an abelian group of order K, and let $\Theta = \Theta_X = (\pi \mid X)_{\pi \in G}$ be the Harmonic frame of K vectors for $\mathbb{C}^M$ given by $X \subset G$ where $\mid \times \mid = K$. Then*

- *Θ has distinct vectors if and only if X generates G.*
- *Θ is a real frame if and only if X is closed under taking inverses.*

Example 3.7.11 *There are representatives for the multiplicative equivalent classes of subsets of size three for $G = \mathbb{Z}_7$. Some vectors in $\mathbb{C}^3$.*

$\{1,2,6\}, \{1,2,3\}, \{0,1,2\}, \{0,1,3\}, \{1,2,5\}$, *(class size 6),* $\{0,1,6\}$, *(class size 3),*$\{1,2,4\}$,*(class size 2)*

Each gives an harmonic frame of distinct vectors(nonzero elements generate G). None of these are unitarily equivalent since their angles are different.

Example 3.7.12 *For $G = \mathbb{Z}_8$, there are 17 multiplicative equivalence classes of subsets of 3 elements. Two of these give frames with the same angles, namely $\{\{1,2,5\}\{3,6,7\}\}, \{\{1,5,6\}\{2,3,7\}\}$. The common angle multiset is $\{-1, \imath, \imath, -\imath, -\imath, -2\imath - 1, 2\imath - 1\}$*

These are unitarily equivalent frames but not through an automorphism.

3.8 Equiangular Harmonic Frames and Difference Sets

There exists Harmonic frames which are equiangular characterized by the existence of a difference set for an abelian group, which leads to some infinite families of equiangular tight frames.

Definition 3.8.1 *Let F be a subset of a finite group G of order K. If F is of order M then it is said to be (K, M, λ)- difference set if every nonidentity element of G can be written as a difference $a - b$ where $a, b \in F$ in exactly λ ways.*

Equiangular Harmonic frames are in $1 - 1$ correspondence with difference sets.

THEOREM 3.8.2 *If G is an abelian group of order K, then the frame of K vectors for $\mathbb{C}^M$ obtained by restricting the characters of G to $F \subset G, \mid F \mid= M$ is an equiangular tight frame if and only if F is (K, M, λ)-difference set for G. The parameters of a difference set satisfy $1 \leq \lambda = \frac{M^2-M}{K-1}$, and so equiangular Harmonic frame for K vectors for $\mathbb{C}^M$ satisfies $K \leq M^2 - M + 1$.*

Example 3.8.3 *Let $G = \mathbb{Z}_7$ then three of the seven Harmonic frames are equiangular i.e the ones given by the (multiplicatively inequivalent) difference sets $(1, 2, 4), (1, 2, 6), (0, 1, 3)$.*

3.9 Highly Symmetric Tight Frames and Finite Reflection Groups

We have already seen that there are finitely many G- frames for an abelian group G and infinitely many G-frames for a non - abelian group G.

Example 3.9.1 *Let $G = D_3$ be the dihedral group of symmetries of the triangle, $(\mid G \mid= 6)$, acting on $\mathbb{R}^2$, so as to express the Mercedes- Benz frame as the orbit of a vector v which is fixed by a reflection. If v is not fixed by a reflection then its orbit is a tight frame and it is easily seen that infinitely many unitarily inequivalent tight D_3 - frames of 6 distinct vectors for $\mathbb{R}^2$ can be obtained similarly.*

Definition 3.9.2 *Let Θ be a finite frame of distinct vectors. Then Θ is said to be highly symmetric if the action of its symmetry group, Sym (Θ) is irreducible, transitive and the stabilizer of any one vector (and hence all) is a notrivial subgroup which fixes a space of dimension exactly one.*

Example 3.9.3 *The vertices of platonic solids in $\mathbb{R}^3$, and the K equally spaced unit vectors in $\mathbb{R}^2$ are highly symmetric tight frames.*

THEOREM 3.9.4 *Fix $K \geq M$. Then there is a finite number of highly symmetric Parseval frames of K vectors for $\mathbb{F}^M$ (upto unitary equivalence).*

Proof

Let Θ be a highly symmetric Parseval frame of K vectors for $\mathbb{F}^M$. Then it is determined, upto unitary equivalence, by the representation induced by sym(Θ), and a subgroup $\mathcal{H}$ which fixes only the one- dimensional subspace spanned by some vector in Θ. There is a finite number of choices for sym(Θ) as its order is $\leq\mid S_K \mid= K!$, and hence (by Maschke's theorem) a finite number of possible representations. This class of frames is finite since there is a finite number of choices for $\mathcal{H}$.

Example 3.9.5 *Let $G = G(1, 1, 8) \cong S_8$, a member of one of the three infinite families of imprimitive irreducible complex reflection groups acting as permutations of indices of a vector $x \in \mathbb{C}^3$ in the subspace consisting of vectors with $x_1 + \ldots + x_8 = 0$. The orbit of the vector $\nu = 3z_2 = (3, 3, -1, -1, -1, -1, -1, -1)$ gives an equiangular tight frames of 28 vectors for a 7- dimensional space.*

The classification of all highly symmetric tight frames is in its earliest stages.

3.10 Central Group Frames

For an infinite nonabelian group G, the class of unitarily inequivalent G- frames may be narrowed down by imposing an additional symmetry condition.

Definition 3.10.1 *A G-frame $\Theta = (v_g)_{g\in G}$ is said to be central if $\Phi : G \to \mathbb{C}$ defined by $\Phi(g) := \langle v_1, v_g\rangle = \langle v_1, gv_1\rangle$ is a class function, i.e is constant on the conjugacy classes of G. This clearly shows that being central is equivalent to the symmetry condition $\langle gv, hv\rangle = \langle gu, hu\rangle, \forall g, h \in G, \forall v, u \in \Theta$.*

Example 3.10.2 *Let G be an abelian group then all G- frames are central. This is because all the conjugacy classes of an abelian group are singletons. Hence central G- frames generalise Harmonic frames to G nonabelian.*

Definition 3.10.3 *Let $\rho : G \to v(H)$ be a representation of a finite group G. $\tau := \tau_P : G \to \mathbb{C}$ defined by $\tau(g) := Trace(\rho(g))$ is the character of ρ.*

All central Parseval G-frames can be categorised in terms of Gramian. Hence the class of central G-frames is finite.

THEOREM 3.10.4 *If G is a finite group with irreducible characters $\tau_1, \tau_2, \ldots, \tau_r$ then $\Theta = (v_g)_{g\in G}$ is a Parseval G- frame if and only if its Gramian is given by Gram $(\Theta)_{g,h} = \sum_{j\in J} \frac{\tau_j(1)}{|G|}\bar{\tau}_j(g^{-1}h)$ for some $I \subset 1, \ldots, r$.*

The same way Harmonic frames were constructed from irreducible characters of G, the central G-frames can be constructed.

Corollary 3.10.5 *If G is a finite group with irreducible characters $\tau_1, \tau_2, \ldots, \tau_r$ then choosing Parseval G-frames for Θ_j for $\mathcal{H}_j$, $I = 1, \ldots, r$ with $Gram(\Theta_j) = \frac{\tau_j(1)}{|G|}K(\bar{\tau_j})$, $dim(\mathcal{H}_j) = \tau_j(1)^2$.*

Example 3.10.6 *Let $G = D_3 \cong S_3$ be the dihedral (or symmetric) group of order 6. $G = D_3 = \langle a, b : a^3 = b^2 = I, b^{-1}ab = a^{-1}\rangle$, and write class functions and G- matrices with respect to the order I, a, a^2, b, ab and a^2b. The conjugacy classes are $I, \{a, a^2\}, \{b, ab, a^2b\}$. The irreducible characters are*

$$\tau_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \tau_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \tau_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

There exists a central Parseval G-frame Θ_j for a space of dimension $\tau_j(1)^2$ corresponding to each of these. From the fact that τ_1 and τ_2 are one dimensional, $\tau_1 = \frac{1}{\sqrt{6}}(1,1,1,1,1,1)$ and $\tau_2 = \frac{1}{\sqrt{6}}(1,1,1,-1,-1,-1)$

A representation $\rho : D_3 \to u(\mathbb{C}^2) \subset \mathbb{C}^{2X2} \approx \mathbb{C}^4$ with trace

$$(\rho) = \tau_3$$

is given by $\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \rho(a) = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} \approx \begin{pmatrix} z \\ 0 \\ 0 \\ z^2 \end{pmatrix}, \rho(a^2) = \begin{pmatrix} z^2 & 0 \\ 0 & z \end{pmatrix} \approx \begin{pmatrix} z^2 \\ 0 \\ 0 \\ z \end{pmatrix}, \rho(b) =$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \approx \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \rho(ab) = \begin{pmatrix} 0 & z \\ z^2 & 0 \end{pmatrix} \approx \begin{pmatrix} 0 \\ z \\ z^2 \\ 0 \end{pmatrix}, \rho(a^2b) = \begin{pmatrix} 0 & z^2 \\ z & 0 \end{pmatrix} \approx \begin{pmatrix} 0 \\ z^2 \\ z \\ 0 \end{pmatrix}$

hence we obtain

$$\frac{1}{\sqrt{3}}(\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\begin{pmatrix}z\\0\\0\\z^2\end{pmatrix},\begin{pmatrix}z^2\\0\\0\\z\end{pmatrix},\begin{pmatrix}0\\1\\1\\0\end{pmatrix},\begin{pmatrix}0\\z\\z^2\\0\end{pmatrix},\begin{pmatrix}0\\z^2\\z\\0\end{pmatrix}).$$

Thus there are seven central Parseval D_3-frames, namely

$$\Theta_1, \Theta_2 \subset \mathbb{C}, \Theta_1 \bigoplus \Theta_2 \subset \mathbb{C}_2, \Theta_3 \subset \mathbb{C}^4 \Theta_1 \bigoplus \Theta_3, \Theta_2 \bigoplus \Theta_3 \subset \mathbb{C}_5, \Theta_1.\Theta_2 \bigoplus \Theta_3 \subset \mathbb{C}_6.$$

Below is an example of a representation from G to $\mathrm{GL}(n, F)$.

Definition 3.10.7 *A representation of G over $\mathbb{F}$ is a homomorphism ρ from G to $GL(n, \mathbb{F})$, for some n. The degree of ρ is the integer n i.e $\rho : G \to GL(n, \mathbb{F})$.*

Example 3.10.8 *Let Θ be a finite frame then the action of sym (Θ) on $\mathcal{H}$ given by the Definition 3.2.1 above in G-frame is a representation. If Θ is tight then the action is unitary.*

Chapter 4

FOURIER ANALYSIS AND WAVELETS

4.1 Introduction

The usefulness of Fourier theory to mathematicians, engineers and scientists is ever increasing. It is used by engineers and practical phycists to treat experimental data, extract information from noisy signals, design electrical filters, 'clean' television pictures e.t.c. These transforms are done digitally and there is minimum mathematics involved. Wavelets are mathematical functions that cut up data into different frequency components, and then study each component with a resolution matched to its scale. It is advantageous over Fourier methods in analyzing physical situations where the signal contains discontinuities and sharp spikes. Wavelets are applied in image compression, turbulence, human vision, radar and earthquake prediction. We shall study Fourier analysis and how the idea leads to wavelet analysis. We shall also compare Fourier transforms and Wavelets, state properties of Fourier transform and wavelets, their applications e.g of wavelets in image compression and denoising data.

In this chapter, we investigate the Fourier series and its application to music. We shall also investigate the Fourier analysis and integrals, reasons for Fourier analysis, Windowed Fourier analysis, properties of periodic sequences. We shall also discuss the Fast Wavelet Theorem.

4.2 FOURIER SERIES

French Mathematician Joseph Fourier (1768- 1830) while trying to solve a problem on heat conduction, he needed to express a function as an infinite series of sine and cosine functions.

$$f(x) = a_o + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx) = a_o + a_1 cosx + a_2 cos2x + a_3 cos3x + \ldots + b_1 sinx + b_2 sin2x + b_3 sin3x + \ldots$$

This series is known as Fourier Series or trigonometric series.

We start by assuming that the Trigonometric series converges and has a continuous function $f(x)$ as its sum on the interval $[-\pi, \pi]$ i.e

$$f(x) = a_o + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx) for - \pi \leq x \leq \pi$$

We shall then define Fourier Series.

Definition 4.2.1 *Let f be a piecewise continuous function on $[-\pi, \pi]$. Then the Fourier series of f is the series*

$$a_o + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx)$$

where the coefficients a_n and b_n in the series are defined by

$$a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdxandb_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx$$

are called the Fourier series coefficients of f.

Given a function f with period other than 2π, we can find find its Fourier series by making a change of variable.

THEOREM 4.2.2 *Let f be a piecewise continuous function $[-L, L]$. Then the Fourier series of f is:*

$$a_o + \sum_{n=1}^{\infty}[a_n cos(\frac{n\Pi x}{L}) + b_n sin(\frac{n\Pi x}{L})]$$

where $a_o = \frac{1}{2L}\int_{-L}^{L} f(x)dx$
and for $n \geq 1$,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos(\frac{n\Pi x}{L})$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin(\frac{n\Pi x}{L})$$

.

Below is the Fourier convergence Theorem.

THEOREM 4.2.3 *If f is a periodic function with period 2π, f and $f^{'}$ are piecewise continuous on $[-\pi, \pi]$ then the Fourier series is convergent. The sum of the Fourier series is equal to $f(x)$ for all number x where f is continuous . At the number x where f is discontinuous , the sum of the Fourier series is the average of the right and left limits i.e $\frac{1}{2}[f(x^+) + f(x^-)]$.*

The Fourier convergence Theorem is valid for functions with period $2L$.

Example 4.2.4 *The Fourier series of the triangular wave defined by $f(x) =\mid x \mid$ for $-1 \leq x \leq 1$ and $f(x+2) = f(x)$ for all x can be determined.*

Solution
Putting $L = 1$, we find the Fourier coefficients

$$a_o = \frac{1}{2}\int_{-1}^{1} \mid x \mid dx = \frac{1}{2}\int_{-1}^{0} \mid -x \mid dx + \frac{1}{2}\int_{0}^{1} \mid x \mid dx = [\frac{-1}{4}x^2]_0^1 + [\frac{1}{4}x^2]_0^1 = \frac{1}{2}$$

and for $n \geq 1$
$a_n = \int_{-1}^{1} \mid x \mid$cos $(n\pi x)$ dx = 2 $\int_0^1$ cos $(n\pi x)$dx.
Since $y =\mid x \mid cos(n\pi x)$ is an even function. Integrating by parts with $u = x$ and $dv = cos(n\pi x)dx$ i.e $a_n = 2[\frac{x}{n\pi}sin(n\pi x)]_0^1 - \frac{2}{n\pi}\int_0^1$ sin $(n\pi x)dx = 0 - \frac{2}{n\pi}[\frac{-cos(n\pi x)}{n\pi}]_0^1 = \frac{2}{n^2\pi^2}$ cos $(n\pi - 1)$.
Since $y =\mid x \mid$ sin $(n\pi x)$ is an odd function, we get $b_n = \int_{-1}^{1} \mid x \mid$ sin $(n\pi x)dx = 0$.
The series can be written as

$$\frac{1}{2} + \sum_{n=1}^{\infty}\frac{2(cosn\pi - 1)}{n^2\pi^2}cos(n\pi x).$$

But cos $n\pi = 1$ if n is even and cos $n\pi = -1$ if n is odd.

Therefore, $a_n = \frac{2}{n^2\pi^2}$ (cos $(n\pi x - 1)) = 0$ if n is even $\frac{-4}{n^2\pi^2}$ if n is odd.

Therefore the Fourier series is

$\frac{1}{2} - \frac{4}{\pi^2}$ cos $(\pi x) - \frac{4}{9\pi^2} cos(3\pi x) - \frac{4}{25\pi^2} cos(5\pi x) - \dots$

$$= \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4}{(2k-1)^2\pi^2} cos(2k-1)\pi x).$$

The Fourier Convergence Theorem gives

$$f(x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4}{(2k-1)^2\pi^2} cos(2k-1)\pi x.$$

4.2.1 Fourier series and Music

Heat equation and Wave equation use Fourier series to solve. We shall in this sub- section study the analysis and synthesis of musical sounds.

To hear sound from a musical instrument, air pressure fluctuations arrive at our eardrums and converted into electrical impulses processed by the brain. Our concern is: How do we distinguish between a note of a given pitch produced by two different instruments?

Difference between wavefronts can be expressed as sums of Fourier series

$Q(t) = a_o + a_1$ cos $(\frac{t\Pi}{L}) + b_1$ sin $(\frac{t\pi}{L}) + a_2\cos(\frac{2t\pi}{L}) + b_2\sin(\frac{2t\pi}{L}) + \dots$

Doing this enables us to express the sound as a sum of simple pure sounds. Difference in sounds between two instruments can be attributed to the relative sizes of the Fourier coefficients of respective wavefronts i.e the n^{th} term Fourier series i.e a_n cos $(\frac{nt\pi}{L}) + b_n(\frac{nt\pi}{L})$ is the n^{th} Harmonic of Q. The amplitude of n^{th} Harmonic is $A_n = \sqrt{a_n^2 + b_n^2}$ and its square $A_n^2 = a_n^2 + b_n^2$ is sometimes known as Energy of the n^{th} Harmonic for the Fourier series.

Fourier series enables us to synthesize sounds. For music synthesizers, various pure tones (Harmonics) can be combined to create a higher sound through emphasizing certain Harmonics by assigning a larger Fourier coefficient thus higher corresponding energies.

4.3 FOURIER ANALYSIS

4.3.1 Fourier Analysis and Integrals

The Fourier Transform of a function $f : \mathbb{R} \to \mathbb{C}$ can be defined in various ways. We shall define it using integration.

Definition 4.3.1 *Let $f : \mathbb{R} \to \mathbb{R}$. The Fourier transform of $f \in L^1(\mathbb{R})$ denoted by $\Upsilon[f](.)$ is given by the integral*

$$\Upsilon[f](x) := \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{+\infty} f(t)e^{(-ixt)}dt$$

for $x \in \mathbb{R}$ for which the integral exists.

4.3.2 The Dirichlet condition for inversion of Fourier Integrals

THEOREM 4.3.2 *Let $f : \mathbb{R} \to \mathbb{R}$. Let*

i) $\int_{-\infty}^{+\infty} \mid f \mid dt$ converge and

ii) in any finite interval, $f, f^{'}$ are piecewise continuous with at most finitely many maximal or minimal or discontinuities.

Let $F = \Upsilon[f]$. Then if f is continuous at $t \in \mathbb{R}$ then

$$f(t) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{+\infty} F(x)e^{itx}dx.$$

If f is discontinuous at $t \in \mathbb{R}$ and $f(t+0)$ and $f(t-0)$ denote the right and left limits of f at t then $\frac{1}{2}[f(t+0)+f(t-0)] = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{+\infty} F(x)e^{itx}dx.$

THEOREM 4.3.3 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous, $f^{'}$, $g^{'}$ piecewise continuous. If $\Upsilon[f](x) = \Upsilon[g](x)$ for all x then $f(t) = g(t)$, for all t.*

Proof

$f(t) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{+\infty} \Upsilon[f](x)e^{itx}dx = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{+\infty} \Upsilon[g](x)e^{itx}dx = g(t).$

Example 4.3.4 *It can be shown that the Fourier Transform of $f(t) = e^{-|t|}$. Hence deduce that* $\int_0^{+\infty} \frac{dx}{1+x^2} = \frac{\Pi}{2}$ *and* $\int_0^{+\infty} \frac{xsin(xt)}{1+x^2}dx = \frac{\Pi e^{-t}}{2}, t > 0.$

Solution

$F(x) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{+\infty} f(t)\ e^{-itx}dt = \frac{1}{\sqrt{2\Pi}}[\int_{-\infty}^{0} e^{t(1-ix)}dt + \int_0^{+\infty} e^{-t(1+ix)}dt] = \sqrt{\frac{2}{\Pi}}\frac{1}{1+x^2}.$

By inversion formulas

$e^{(-|t|)} = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{+\infty} F(x)e^{itx}dx = \frac{1}{\Pi}[\int_0^{+\infty} \frac{e^{ixt}+e^{-itx}}{1+x^2}] = \frac{2}{\Pi}\int_0^{+\infty} \frac{cos(xt)}{1+x^2}dx.$

If we substitute for $t = 0$ in the above equation gives first required identity. Differentiating with respect to $t > 0$ gives the second required identity.

4.3.3 L^p and Convergence

Consider the convergence of $\int_{-\infty}^{+\infty} f(x)e^{-ixt}dx$ for various values of t. Since $| e^{-itx} |= 1$. Looking at $\int_{-\infty}^{+\infty} | f(x) | dx$ known as L^1 norm of f and written as $\| f \|_1$. On the set of f where $\| f \|_1$ is finite (this is called $L^1(\mathbb{R})$), this $\| \, . \, \|_1$ is norm in the usual normed vector space. In $L^1(\mathbb{R})$, Fourier Transformation is quite well behaved.

Generally, let X be a set with a positive measure μ. Then we can define $L^p(x, \mu)$ to be the set $f : X \to \mathbb{C}$ such that f is measurable and $\int_x | f |^p d\mu < \infty$.

4.3.4 Some Reasons for Fourier Analysis

In Differential equations: Looking at $v(x) = e^{itx}$ then v is an eigenvector for differential operator i.e $\frac{d}{dx}v = \frac{d}{dx}(e^{itx}) = ite^{itx} = itv$. Fourier analysis is useful in solving physical problems which involves differential equations e.g wave equation $\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = 0$ which describes transmission of electromagnetic waves(light) , sound etc. The wave equation is useful for engineers working with light or sound signals. Fourier Transform of a signal shows them the frequency in each signal and $\| \, . \|_2^2$ corresponds to the energy of the signal hence results easy to interpret.

Utility of Fourier Transforms

The utility of a Fourier Transform lies on its ability to analyze a signal in time domain for its frequency domain content. First a function in time domain is translated into a function in the frequency domain. The signal is then analyzed for its frequency content as the Fourier coefficients of the transformed function represents the contribution of each sine and cosine function at each frequency.

4.3.5 The windowed Fourier Transform

For a non- periodic signal f(t), the summation of periodic functions , sine and cosine does not accurately represent the signal. To make the signal periodic, we may extend it though this requires additional continuity at the end points. The Windowed Fourier Transform is one solution to the problem of better reproducing non-periodic signals. The Windowed Fourier Transform equips one with information on signals simultaneously on the time domain and the frequency domain. The input signal f(t) is chopped up into sections and each section analyzed for its frequency content separately. If the signal has sharp transitions, we window the input data for sections to converge to zero at endpoints. The window's effect is to localize signals in time. To get more localized information about the frequency in a signal, we examine the signal through a "window". To know what frequencies are present in some region of a signal, we somehow cut out part of the signal around the region and look at the Fourier Transform of this new signal.

The window signal has some drawbacks. The window is of fixed width, so we need to choose how wide to make it i.e we must know the length of 'notes' of interest within the signal or must experiment to get a good width. For too narrow window, we will not be capable of looking at signals with a wavelength much longer than this width. For too wide window, the adjacent notes are blurred together.

4.3.6 The continuous Wavelet Transform

Since the windowed Transform has a weakness of the window being of fixed width, the weakness can be corrected by varying width of window. Choosing a window width of about half the wavelength helps in picking up the frequency. We can choose a window (say $e^{-x^2)}$ and a wave (say $sinx$) and find the product to get a wavelength $\psi(x) = e^{-x^2}sinx$. Centering the wavelet over the signal, $\psi(x-x_0) = e^{-(x-x_0)^2}sin\omega(x-x_0)$ where ω is the frequency. By scaling the whole thing by ω, we get $sin\omega$ that help get the frequency ω.

$\psi(\omega(x-x_0)) = e^{-\omega^2(x-x_0)^2}sin\omega(x-x_0)$.

The wavelet ψ is known as "mother wavelet" and the changes in scale and position are often written as $\psi_{a,b}$ where $\psi_{a,b}(x) =\mid a \mid^{-\frac{1}{2}} \psi(a^{-1}(x-b))$ where $a = \omega^{-1}, b = x_0$ and factor $\mid a \mid^{\frac{-1}{2}}$ is to normalize the wavelets in $L^2(\mathbb{R})$. This leads to continuous Wavelet Transform of some signal f i.e $F(a,b) = \int f(x)\overline{\psi_{a,b}}(x)dx$.

4.3.7 Properties of Fourier Transform

Let $\Upsilon[f](x) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{+\infty} f(t)e^{-ixt}dt = \frac{1}{\sqrt{2\Pi}}\hat{f}(x)$.

$\Upsilon^*[g](t) = \frac{1}{\sqrt{2\Pi}}\int_{-\infty}^{+\infty} g(x)e^{-ixt}dx$.

Let $f, g \in L^1[-\infty,\infty] i.e \int_{-\infty}^{\infty} \mid f(t) \mid dt < \infty$. Then

i) Υ and Υ^* are linear operators. This clearly means that $\Upsilon[af+bg] = a\Upsilon[f]+b\Upsilon[g]$ and $\Upsilon^*[af+bg]$ $= a\Upsilon^*[f]+b\Upsilon^*[g]$ for all $a, b \in \mathbb{C}$.

ii) For $t^nf(t) \in L^1[-\infty,\infty]$ and integer $n > 0$, $\Upsilon[t^nf(t)](x) = i^n\frac{d^n}{dx^n}\Upsilon[f](x)$.

iii) For $x^nf(x) \in L^1[-\infty,\infty]$ and integer $n > 0$, $\Upsilon^*[x^nf(x)](t) = i^n\frac{d^n}{dt^n}\Upsilon^*[f](t)$.

iv) Let the n^{th} derivative $f^{(n)}(t) \in L^1[-\infty,\infty]$ be piecewise continuous functions for some integer $n > 0$. f and lower derivatives are all continuous in $(-\infty,\infty)$. Then $\Upsilon[f^n](x) = (ix)^n\Upsilon[f](x)$.

v) Let n^{th} derivative for some positive integer n i.e $f^n(x) \in L^1[-\infty,\infty]$ and piecewise continuous for some positive integer n_1. f and the lower derivatives are all continuous in $(-\infty,\infty)$ then $\Upsilon^*[f^(n)](t) = (-it)^n\Upsilon^*[f](t)$.

vi) $\Upsilon[f(t-b)](x) = e^{-ixb}\Upsilon[f](x)$ gives the Fourier Transform of a translation by a real number b.

vii) $\Upsilon[f(at)](x) = \frac{1}{a}\Upsilon[f](\frac{x}{a})$ gives the Fourier Transform of a scaling by a positive number a.

viii)$\Upsilon[f(at-b)](x) = \frac{1}{a}e^{-ix\frac{b}{a}}\Upsilon[f](\frac{x}{a})$.

Example 4.3.5 *Given the rectangular box function with support in $[c,d]$ i.e* $\mathbb{R}(t) = \begin{cases} 1 & if c < t < d \\ \frac{1}{2} & t = c, d \\ 0 & otherwise \end{cases}$.

The box function $\prod(t) = \begin{cases} 1 & if -\pi < t < \Pi \\ \frac{1}{2} & t = pm\pi \\ 0 & otherwise \end{cases}$.

R(t)= $\begin{cases} 1 & if c < t < d \\ \frac{1}{2} & t = c, d \\ 0 & otherwise \end{cases}$

has the Fourier Transform $\hat{\Pi}(x) = 2\Pi$ *sinc* x, *but we can obtain* R *from* Π *by first translating* $t \to s = t - \frac{(c+d)}{2}$ *and then rescaling* $s \to \frac{2\Pi}{d-c}s$.

R(t)= $\pi(\frac{2\pi}{d-c}t - \pi\frac{c+d}{d-c})$

$\hat{R}(x)= \frac{4\pi^2}{d-c}e^{\frac{j\pi x(c+d)}{d-c}}$ *sinc* $(\frac{2\pi x}{d-c})$.

The inverse of the Fourier Transform of $\hat{R}$ *is* R *i.e* $\Upsilon^*(\Upsilon)R(t)$ *=R(t)*.

Another Example is as shown below.

Example 4.3.6 *We shall consider the truncated sine wave* $f(t) = \begin{cases} sin3t & if -\pi \leq t \leq \pi \\ 0 & otherwise \end{cases}$
with $\hat{f}(x) = \frac{-6isin(x\pi)}{9-x^2}$.
The derivative of $f(t)$ *i.e* f' *is* $3g(t)$ *(except at 2 points) where* $g(t)$ *is the truncated sine wave* $g(t)$= $\begin{cases} cos3t & if -\pi < t < \pi \\ \frac{-1}{2} & t = \pm\pi \\ 0 & otherwise \end{cases}$.
So already $\hat{g}(x)$= $\frac{2xsin(x\pi)}{9-x^2}$ *is computed. Hence* $3\hat{g}(x)$= $(ix)\hat{f}(x)$ *as was predicted.*

4.3.8 The Periodic Sequence

Definition 4.3.7 *Let* N *be the period of a periodic sequence* $\tilde{x}(n)$. *Then the periodic sequence with period* N *denoted by* $\tilde{x}(n)$ *is defined as* $\tilde{x}(n) = \tilde{x}(n + tN)$ *where* t *is an integer.*

Example 4.3.8 $W_N^{tn} = e^{-i\frac{2\Pi}{N}tm}$ *is called the Twiddle factor. It has the following properties.*
Period: $W_N^{tm} = W_N^{t+N}n = W_N^{t(n+N)}$
Symmetric: $W_N^{-tm} = (W_N^{tm}) = W_N^{N-t}n = W_N^{t(N-n)}$
Orthogonal:

$$\sum_{t=0}^{N-1} W_N^{tm}$$

$=\begin{cases} N & if n= rN \\ 0 & otherwise \end{cases}$.

For periodic sequence $\tilde{x}(n)$ with period N, only N samples are independent i.e N sample in one period is enough to represent the whole sequence.

4.3.9 The Discrete Fourier Transform

Let $x[n]_{n=0}^{N-1}$ be a signal. The Discrete Fourier Transform of the signal is a sequence $X[t] = f$ or $t = 0, 1, ..., N - 1$

$$X[t] = \sum_{n=0}^{N-1} X[n]e^{\frac{-2}{N}}$$

The Discrete Fourier Transform estimates the Fourier Transform of a function from a finite number of its sampled points. The sample points are supposed to be typical of what the signals look like at all other times. The Discrete Fourier Transform has symmetric properties almost exactly same as the Continuous Fourier Transform.

Discrete Fourier Transform is widely used in spectral analysis, acoustics, medical imaging and telecommunication.

4.3.10 Fast Fourier Transform

This refers to a very efficient algorithm for computing Discrete Fourier Transform. Approximating a function by samples and approximating the Fourier Integral by the Discrete Fourier Transform requires applying a matrix whose order is the number of samples points N. The number of multiplications involve determines the time taken to evaluate Discrete Fourier Transform in a computer. Multiplying $N \times N$ matrix by a vector costs on the order of N^2 arithmetic operations.The problem worsens as the number of sample points increases. If sample points are uniformly spaced, then Fourier Matrix can be factored into a produced just a few sparse matrices and the resulting factors applied to a vector in a total of order $NlogN$ arithmetic operation. This is the so called Fast Fourier Transform. Discrete Fourier Transform needs N^2 multiplications while Fast

Fourier Transform needs $Nlog_2N$ multiplications. This algorithm arises from the fact that Discrete Fourier Transform of a sequence of N points can possibly be written in terms of two Discrete Fourier Transforms of length $\frac{N}{2}$ i.e if N is a power of 2. This can recurrently be done until we get a Discrete Fourier Transform of single points.

For instance

$$X[t] = \sum_{n=0}^{N-1} x[n]e^{\frac{-2}{N}}$$

which can be re-written as

$$X[t] = \sum_{n=0}^{N-1} x[n]W_N^{nt}.$$

You can clearly notice that the same values of W_N^{nt} are calculated repeatedly using symmetric property of the Twiddle factor lots of computing can be saved.

$$X[t] = \sum_{n=0}^{N-1} x[n]W_N^{nt}$$

=

$$\sum_{r=0}^{\frac{N}{2}-1} x(2r)W_N^{2tr} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1)W_N^{t(2r+1)}$$

=

$$\sum_{r=0}^{\frac{N}{2}-1} x_1(r)W_{\frac{N}{2}} + W_N^t \sum_{r=0}^{\frac{N}{2}-1} x_2(r)W_{\frac{N}{2}}^{tr}$$

=

$$X_1(t) + W_N^t X_2(t).$$

This clearly shows that N point Discrete Fourier Transform can be obtained from two $\frac{N}{2}$- point transforms, one on even input data and other on odd input data.

4.4 WAVELETS

4.4.1 Overview of Wavelets

Wavelets are used in representation of data and other functions. In 1800s, Joseph Fourier discovered a way of superposing sines and cosines to represent other functions. In wavelet analysis, the scale that we use to look at data plays a special role. Wavelet algorithms processes data at different resolutions(or scales). Wavelet analysis help approximate functions that are contained neatly in finite domains. They are well- suited for data approximation with data discontinuities. Sines and Cosine functions compromise the bases of Fourier analysis. They are not effective in approximating sharp spikes.

In data compression, wavelets is an effective tool due to spark coding. The wavelet analysis procedure is to adapt a wavelet prototype function called an analyzing wavelet or mother wavelet. Frequency analysis is performed with a dilated low- frequency version of the prototype wavelet. Data operations can be done using just the corresponding wavelet coefficients. Further choosing the best wavelets adapted to your data or truncating the coefficients below a threshold leads to sparse data representation.

4.4.2 The Evolution of Wavelets

Wavelets by Joseph Fourier (1807) in his theories of Frequency analysis. According to Fourier, any 2π periodic function $f(x)$ is the sum

$$a_o + \sum_{t=1}^{\infty}(a_t cos(tx) + b_t sin(tx))$$

of its Fourier series where a_o, a_t and b_t are coefficients. These coefficients are calculated using the formula $a_o = \frac{1}{2\Pi}\int_0^{2\Pi} f(x)dx, a_t = \frac{1}{\Pi}\int_0^{2\Pi} f(x)cos(tx)dx$ and $b_t = \frac{1}{\Pi}\int_0^{2\Pi} f(x)sin(tx)dx$.

The analysis of a function $f(x)$ from frequency analysis to scale analysis by creating mathematical structures that vary in scale is done by constructing a function and shifting it by by some amount and changing its scale. The structure is applied in approximating a signal. The procedure is repeated. The basis structure is taken, shifted and scaled again. This is applied to the same signal to get a new approximation. This scale analysis is less sensitive to noise as it measures average fluctuations of the signal at different scales.

In 1909, A. Haar mentioned wavelet in an appendix . The characteristic of Haar wavelet is: It had a compact support i.e it vanishes outside a finite interval. The challenge of Haar Wavelet is that Haar wavelets are not continuously differentiable.

In 1930s , different groups worked on representation of functions using scale- varying basis functions. Paul Levy, a phycist investigated Brownian motion (a type of random signal) using a scale- varying basis function (called Haar Basis function). This basis function was more superior to the Fourier basis functions. Littlewood, Paley and Stein also researched on computing the energy of a function $f(x)$ i.e $E = \frac{1}{2}\int_0^{2\Pi} \mid f(x) \mid^2 dx$. Scientists were disturbed as the computation produced inconsistent results. Then a function that can vary in scale and can conserve energy when computing functional energy was discovered.

Between 1960- 1980 there were more developments on Wavelets. Guido Weiss and Ronald R. Coifman studied the simplest elements of a function space, called atoms, with goals of finding the atoms for a common function and finding the "assembly rules" that allow the reconstruction of all the elements of the function space using these atoms. In 1980, Grossman and Morlet, a phycist and an Engineer respectively, broadly defined wavelets in the context of quantum physics.

In 1985, Stephane Mallat while working on signal processing discovered the relationship between quadrature mirror filters, pyramid algorithms and orthonormal wavelet bases. From part of these results, Y. Meyer constructed the first non- trivial wavelets. Meyer wavelets are continuously differentiable though they have no compact support. Later Ingrid Daubechies used Mallat's work to construct a set of wavelet orthonormal basis functions; most elegant and have become greatly used in wavelet applications today.

4.4.3 Wavelet Transform and Fourier Transform

There are similarities and differences between Wavelet Transform and Fourier Transform.

4.4.4 Similarities between Wavelet Transform and Fourier Transform

One similarity of both Discrete Wavelet Transform and Fast Fourier Transform is that both are linear operators that generate a data structure that contains log_2N segments of various lengths , usually filling and transforming it into a difference data vector of length 2^N.

Secondly, mathematical properties of matrices involved in the Fast Fourier Transform and Discrete Wavelet Transform are similar. The Fourier matrix for both Fast Fourier Transform and Discrete Wavelet Transform is the transpose of the original. Both transforms can be viewed as a rotation in function space to a different domain.

Third, basis functions are localized making mathematical tools such a power spectra (how much power is contained in a frequency interval) and scalegrams useful at picking out frequency and calculating power distribution.

4.4.5 Differences between Wavelet Transform and Fourier Transform

One difference is that Fast Fourier Transform has Fourier Sine and Cosine which are not localized in space while Discrete Wavelet Transform has individual wavelet Transforms localized in space. This make many functions and operators using wavelets 'sparse' when transformed into wavelet domain. This sparseness , in turn, results in a number of useful applications such as Data compression, detecting features in image.

Another difference is that Fast Fourier Transform has a single set of basis functions as it just utilizes sine and cosine functions while Discrete Wavelet Transform do not have a single set of basis functions. It has an infinite set of possible basis functions i.e provide immediate access to information that can be obsecured by other time frequency methods such as Fourier Analysis.

4.4.6 Wavelet Analysis

The wavelet functions are used to analyze wavelet functions.

4.4.7 The Discrete Wavelet Transform

The wavelet basis is defined by $\Theta_{s,t}(x) = 2^{\frac{-s}{2}}\Theta(2^{-s}x - t)$ where s and t are integers that scale and dilate the mother function Θ to generate wavelets such as Daubechies wavelet families. s provides the wavelet's width while t provides its position . Mother functions are dilated by powers of 2 and translated by integers.

One of the useful features of wavelets is the ease with which a scientist being capable of choosing the defining coefficients for a given wavelet system to be adapted for a given problem eg The Haar Wavelet is often used in educational purposes. The coefficients $c_0, ..., c_n$ is thought of as filters. They can be placed in a transformation matrix applied to raw data vector. They are ordered using two dominant patterns i.e one that works on a smoothing filter and the other working to bring out the data information. These two ordering of coefficients are known as quadrature mirror filter pair in signal processing parlance. The wavelet coefficient matrix is applied to data vector in a hierarchical algorithm sometimes known as pyramidal algorithm. They are arranged so that odd rows contain an ordering of wavelet coefficients that act as the smoothering filter and the even rows contain an ordering of wavelet coefficient with different signs that act to bring out the details of the data. The matrix is first applied to the original, full- length vector then vector smoothed and decimated by half and matrix applied once more. This is continued until a trivial number of "smooth.......smooth.........smooo.....th......" data remain. Each matrix application leads to a higher resolution of the data while at the same time smoothening the remaining data. The output of the Discrete Wavelet Transform consists of the remaining (smooth etc) components and all of the accompanying "data" component.

4.4.8 Fast Wavelet Transform

Discrete Wavelet Transform decomposes into a product of a few sparse matrices using self- similarity properties to solve problems of the Discrete Wavelet Transform matrix is not sparse in general. This results in an algorithm that requires only order N operators to Transform an N- sample vector. This is the "fast" Discrete Wavelet Transform of Mallat and Daubechies.

4.4.9 Wavelet Packets

Wavelet Transform is a subset of a wavelet packet Transform. Wavelet Packets are particular linear combinations of wavelets. They form bases which retain many of the orthogonality, smoothness and localization properties of the parent wavelets. The coefficients in the linear combination are computed by a recursive algorithm making each newly computed wavelet packet coefficient sequence the root of its own analysis.

4.4.10 Adapted Waveform

A basis of adapted waveform is the best basis function for a given signal representation . The basis carries substantial information about the signal and if the basis description is efficient then the signal informationhas been compressed.

4.4.11 Properties of Adapted Wavelet Bases

Some of the properties of Adapted Wavelet Bases are:

i) Speedy computation of inner products with the other basis functions.

ii) Speedy superposition of the basis functions.

iii) Good frequency localization so researchers can identify signal oscillations.

iv) Independence.

v) Good spatial localization, so researchers can identify the position of a signal that is contributing a large component.

4.5 Applications of wavelets

1. **Musical Tones**

Victor Wickerhauses suggested that wavelet packets could be useful in sound synthesis. A single wavelet Packet generally could replace a large number of oscillators. He feels that sound synthesis is a natural use of wavelets. To approximate the sound of a musical instrument, a sample of the note produced by the instrument could be decomposed into its wavelet packet coefficient reproducing the notes required reloading those coefficients into a wavelet packet generally and playing back the result. Characteristics such as intensity of variations of how the sound starts and ends could be controlled separately using longer wave packets and encoding those properties as well into each note.

2.**Computer and Human Vision**

Maar's theory states that image processing in the human visual system has a complicated hierarchical structure that involves several layers of processing. At each processing level the retina system provides a visual representation that scales progressively in a geometrical manner. Maar further states that intensity changes occur at different scales in an image, so that their optional detection requires the use of operators of different sizes. He further suggests that sudden intensity changes produce a peak or trough in the first derivative of the image. These two hypothesis require that a vision filter has two characteristics i.e it should be a differential operator and its should be capable of being tuned to act at any desired scale Maar's operator was a wavelet referred today as Maar Wavelet.

3. **Federal Bureau of Investigation (FBI) Fingerprints compression**

Since 1924 to date, the United States FBI has approximately 30 million sets of fingerprints. There is demand in the Criminal Jutice Community for a digitization and a compression standard. In 1993, FBI Criminal Justice Information Service Division developed standard for fingerprint digitization and compression. Fingerprints images are digitized at a resolution of 500 pixels per inch with 256 levels of gray- scale information per pixel. Therefore, it is expensive to store an uncompressed image. Hence need for data compression.

4.**Denoising Noisy Data**

By methods of Wavelet shrinkage and thresholding, worked on by David Denoho, recieving a true signal from incomplete indirect or noisy data is possible. This thechnique works by decomposing data set using wavelets by using filters that act as averaging filters and others that produce details. Some of the resulting wavelet coefficients correspond to details in the set of data. If there are small details, they may be ommitted. Therefore, the idea of thresholding is to set to zero all the coefficients less than a particular threshold. These coefficients are used in an inverse wavelet transformation to reconstruct the set of data. The signal is transformed, thresholded and inverse transformed. Denoising is done without smoothing out the sharp structures. The result is cleaned- up signal that still shows important details. This is applied in Nuclear Magnetic Resonance (NMR).

5. **Detecting self- similar behaviour in a Time - Series.**

Wavelets characterizes behaviour especially self- similar behaviour over a wide range of time- scales. One great strength of wavelets is that they can process information effectively at different scales. Scargle used a wavelet tool known as scalegram to investigate time series. He defines scalegram of a time series as the average of the wavelet coefficients at a given scale. When plotted as a function of scale, it depicts much of the same information as does the Fourier power spectrum plotted as a function of frequency. To implement scalegram, you need to sum the product of the data with a wavelet function while to implement the Fourier power spectrum requires summing the data over sine or cosine function. The formulation of scalegram makes it more efficient that Fourier Transform as certain relationships between the different time scales become easier to see and correct such that seeing and correcting for photon noise.

Chapter 5

APPLICATION OF FRAMES IN SIGNAL PROCESSING AND TELECOMMUNICATION

Introduction

Finite frames are applied in cases where application require redundant, yet stable, decomposition e.g for analysis or transmission of signals, but surprisingly also for more theoretically oriented positions. Redundancy is often used to gain stability, robustness, and resilience to noise. Some of the applications include:

1. **Application of Frames to Discretized Data**

Our focus will be on Discrete Fourier Transform Frame as it is a Parseval Frame meaning it has a direct implementation resulting from the fact that the frame operator is just the identity function. The frame coefficients can easily be calculated and relatively computationally cheap way to retrieve the original vector from those coefficients. This arises from the fact that the frame operator is self- adjoint and has an inverse.

The challenge of implementing Discrete Fourier Transform Frame is that it needs a fast and reliable framework capable of performing complex vector Algebra. C++ has been designed to be able to work with speed for large files quickly and reliably.

Through frame simulation software, we are able to compare directly the data redundancy capabilities of the Discrete Fourier Frame with a traditional direct transmission where no other data redundancy or parity scheme is in use.The simulation software accepts a loss percentage as a parameter at run time and randomly will erase coefficients to simulate loss in transmission.

In cases where there are data loss expectation , a frame may be applied and its coefficients transmitted. This is applied in most digital wireless transmission e.g cellular phones, radio modems etc. Reduction of data loss is important in image,audio and video processing the receiving end when slight variance is acceptable in place of precision. Since the computation for the frame coefficients for the Discrete Fourier Transform frame is fairly expensive and as they are complex values the transmission of them does require a fair amount of bandwidth overhead in addition to the Central Processing Unit resources utilized on both the sender and receiver machines. A more specialized tight frame does this better than the frame discussed.

Frames can be applied in data encryption e.g If one was to take the Discrete Fourier Frame as applied within the simulation software and populate the imaginary components of the data vectors with random data seeded by the system time. We can possibly create an encryption algorithm that always generates slightly different output data but could always be decrypted to original data.

2. **Resilience to Noise and Erasures**.

One of the most common problems in signal transmission is Noise and Erasures [34, 35]. Frame redundancy can reduce and compensate for such disturbances.

There are different strategies for reconstruction depending on whether the reciever is aware of noise and erasures special types of erasures [4] or selection of dual frames for reconstruction [28, 31] are taken into account recently.

We shall then illustrate resilience to noise by using a Mercedes- Benz frame as below.

Considering the Unit- Norm Tight frame version of a Mercedes- Benz frame given as $\begin{pmatrix} 0 & 1 \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}$

Frames are considered robust under additive noise. While modeling additive quantization noise as white is technically false and can be misleading. Classical oversampled A/D conversion directly uses a Harmonic tight frame and - when quantizer resolution is held fixed- attains Mean Square Error (MSE) inversely proportional to the oversampled factor. Frames have been successfully used in the analysis and optimization of quantization and oversample A/D conversion where redundancy is used to gain robustness. An oversampled A/D conversion essentially corresponds to a tight frame expansion. An oversampled analog signal can be reconstructed by filters that have flat transfer characteristics in the frequency band occupied by the signal and arbitrary characteristics outside that band, leading to nonuniqueness of the reconstruction . This allows for noise reduction upon reconstruction with the minimum norm dual as the noise outside the signal's band is discarded, thus leading to the fact that the Mean Square Error(MSE) behaves inversely proportional to the oversampling factor.

3.**Application of Equiangular frames to Coding theory**

A unit norm frame with the property that there is a constant c such that $| \langle v_j, v_k \rangle |= c \forall j \neq k$ is called equiangular frame at an angle c.

Equiangular tight frames have applications in signal processing, communications, coding theory etc. A detailed study of this class of frames was introduced by Strohmer and Heath and Holmes and Paulsen. Holmes and Paulsen showed that equiangular tight frames give error correction codes that are robust against two erasures. Recently, Bodmann, Casazza, Edidin and Balan showed that equiangular tight frames are useful for signal reconstruction when all phase information is lost. Construction of capacity achieving signature sequences for multiuser communication system in wireless communication theory. The tightness condition allows equiangular tight frames to achieve the capacity of a Gaussian channel and their eqiuangularity allows them to satisfy an interface invariance property.

4. **Compressive sensing**

Finite frames play an important role in sparsifying systems and in designing the measurement matrix. The connection of this to frames can be found in [14, 3] for the connection to structured frames such as fusion frames [17, 5]

Compressive sensing is a method for solving underdetermined systems if we have some form of sparsity of the incoming signal. It is one of the most active areas of research today.

Given a vector $v = (b_1, b_2, \dots, b_M) \in \mathcal{H}^{\mathcal{M}}$ is k- sparse if $| \{1 \leq j \leq N : b_j \neq 0\} | \leq k$

The fundamental tool in compressive sensing is the class of Restricted Isometry Property (RIP) matrices.

Definition 5.0.1 *A matrix ϕ has the (k, ϕ) - RIP if $(1 \sim \sigma) \parallel x \parallel^2 \leq \parallel \phi v \|^2 \leq (1 + \sigma) \parallel v \parallel^2$, for every k-sparse vector v. The smallest σ for which ϕ is (k, σ)- RIP is the restricted isometry constant (RIP) σ_k*

5. **Phase Retrieval**

Phase retrieval is the problem of recovering a signal from the absolute values of linear measurement coeffficients called intensity measurements.There are two major approaches to the problem of Phase Retrieval i.e

i) Restricting the problem to a subclass of signals on which the intensity measurements become injective.

ii) Using a larger family of measurements so that the intensity measurements map any signal injectively.

This is applied in X- ray crystallography, electron microscopy, coherence theory, difractive imaging, astronomical imaging, X- ray tomography, optics, digital holography and speech recognition.

Definition 5.0.2 *A family of vectors $\Theta = \{v_j\}_{j=1}^{K}$ does phase retrieval on $\mathcal{H}^{\mathcal{M}}$ if whenever $v, w \in \mathcal{H}^{\mathcal{M}}$ satisfy*

$| \langle v, v_j \rangle |=| \langle w, v_j \rangle | \forall j = 1, 2, \dots, M$, then $v = cw$ where $| c |= 1$

6. **Denoising**

The authors of Fletcher et al. analyze denoising by sparse approximation with frames. The known approximation information about the signal x is that it has known sparsity k i.e it can be represented through k nonzero frame coefficients (with respect to a given frame Θ). After having been corrupted by noise yielding $\hat{v}$.

7. **Robust Transmission**

In communication, pioneered by Goyal and Kelner;the problem of creating multiple descriptions of the source so that when transmitted and in the presence of losses, the source could be reconstructed based on recieved material. This clearly means that some amount of redundancy needs to be present in the system , since , if not, the ;loss of even one description would be irreversible.

8. **Quantization Robustness**

Quantization is typically applied to the Transform coefficients which in this case are (redundant) frame coefficients [20, **?**]. Sigma- Delta algorithms and noncanonical dual frames reconstruction play a role in exploring the redundancy of a frame.

Frames show resilience to additive noise as well as numerical stability of reconstruction. Given some noise distributed over the (inner products), then we can argue that when there are k^m coefficients (frame as opposed to $n(m > n)$ bases), it is easier to deal with that lower level of noise per coefficient.

An Example is the Mercedes - Benz Frame: Considering Mercedes- Benz Frame and its unit norm tight frames. Suppose we perturb our frame coefficients by adding white noise w_j to the channel j, where $E[w_j] = 0, E[wjwk] = \sigma^2 \sigma_{jk}$ for $j, k = 1, 2, 3$.

To find the error of reconstruction; we can use:

$$v - \hat{v}$$

$$= \frac{2}{3} \sum_{j=1}^{3} \langle v_j, v \rangle v_j - (\sum_{j=1}^{3} \langle v_j, v \rangle + w_j) v_j$$

$$= -\frac{2}{3} \sum_{j=1}^{3} w_j v_j$$

Then the averaged Mean - Squared Error (MSE) per component is MSE=

$$\frac{1}{2} E \parallel v - \hat{v} \parallel^2$$

$$= \frac{1}{2} E \parallel \frac{2}{3} \sum_{j=1}^{3} w_j v_j \parallel^2$$

$$= \frac{1}{2} \sigma^2 \frac{4}{9} \sum_{j=1}^{3} \parallel v_j \parallel^2$$

$$= \frac{2}{3} \sigma^2$$

.

In the context of sigma- Delta quantization, frames have been used with success.

Chapter 6

SUMMARY AND FUTURE RESEARCH

6.1 SUMMARY

In Chapter 2, we have discussed the evolution of frames todate. More insights on frames and frame operators has been given. The difference between synthesis and analysis operators, Gramian and frame operators was provided using an operaor and its adjoint. The convergence of frames using Algorithm was discussed and its importance to signal processing. In Chapter 3 we have shown how Representation Theory is used in analyzing Parseval Frames. Group - Theoretic concepts i.e symmetric and dihedral groups with respect to Parseval Frames was shown elaborately. In Chapter 4, we discussed Fourier series and showed using an example that a function with period other than 2Π has a Fourier series. The application of Fourier series in music was discussed. Wavelets application e.g in data compression is explained well. In Chapter 5, The application of frames due to frame redundancy was discussed especially the Mercedes- Benz Frame.

Wavelets have been used longer than frames in signal processing and have been used this long. Frames too are here to stay. Frames are becoming a standard tool in the signal processing toolbox, spurred by a host of recent applications requiring some level of redundancy. Despite the mathematical simplicity of finite-dimensional theory, it is extremely useful for practical digital signal processing applications, which invariably involve a finite number of samples. It also enlightens as it exhibits many of the algebraic aspects that subsit in the more abstract and mathematically interesting settings, without the analytic subtleties that occur eg when dealing with limit processes and their interchange.The frame algorithm performs well under certain conditions though the convergence rate fall too very low whenever $\frac{\beta}{\alpha} >> 1$. These problems have been recognized and there are several well understood ways of circumventing them.

6.2 FUTURE RESEARCH

This thesis touches on the analysis operator, synthesis operator, frame operator and Gramian operator. Not much work has been done on Gramian operator. I would like to do my future research on Gramian operator and its application to signal processing and telecommunication.

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