University of Nairobi

School of Mathematics

ISSN: 2410-1397

Master Project in Applied Mathematics

The Numerical Solution For Laminar Boundary Layer Flow Over A Flat Plate

Research Report in Applied Mathematics, Number 17, 2017

MAKANGA ROBERT AYIECHA
I56/83129/2015

August 2017

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Applied Mathematics

The Numerical Solution For Laminar Boundary Layer Flow Over A Flat Plate

Research Report in Applied Mathematics, Number 17, 2017

MAKANGA ROBERT AYIECHA
I56/83129/2015

School of Mathematics
College of Biological and Physical sciences
Chiromo, off Riverside Drive
30197-00100 Nairobi, Kenya

Master of Science Project

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Applied Mathematics

Prepared for The Director
Board Postgraduate Studies
University of Nairobi

Monitored by Director, School of Mathematics

Abstract

The goal of this project, is to present the momentum laminar boundary layers of an incompressible fluid flow induced over a flat plate by a uniform free stream (Blasius flow). Similarity transformation technique is adopted to obtain a self-similar ordinary differential equations called ***Blasius' equation*** and then the self-similar equations are solved numerically by finite difference method

Dissertations in Mathematics at the University of Nairobi, Kenya.
ISSN 2410-1397: Research Report in Mathematics

DISTRIBUTOR: School of Mathematics, University of Nairobi, Kenya

Declaration and Approval

I the undersigned declare that this project report is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

ROBERT AYIECHA MAKANGA
Reg No. I56/83129/2015

In my capacity as a supervisor of the candidate, I certify that this report has my approval for submission.

Signature Date

Prof. C.B. SINGH
School of Mathematics,
University of Nairobi,
Box 30197, 00100 Nairobi, Kenya.
E-mail: singh@uonbi.ac.ke

Dedication

This project is dedicated to my loved father Yuvinalis Makanga, my mother Mary Ombati and my dear sisters Dolvin and Franciscah and finally my dear brothers Innocent and Evans and my fiancee, Faith.

Contents

Acknowledgments

First, I thank God for being with me all along my post graduate studies. I would like to acknowledge and extend my most heartfelt gratitude to my lecture and my supervisor of this project Prof C.B Singh who has spared his time to give me valuable comments and suggestions leading to great improvement of this dissertation. My thanks goes to the following lectures who taught me in postgraduate; Prof Ogana, Prof Porkhariyal, Dr Nyandwi, Dr Were and Dr Moindi. I also appreciate for the golden moments spent with my classmates Mathias Muia, Patrick Kamuri,Pamella Orina, Erick Muriithi and Daniel Kariuki. My special thanks goes to my family members and friends who gave me all the support and prayers for me to succeed. May the Almight God bless them all . . . .

Robert Ayiecha Makanga

Nairobi, 2017.

1 Introduction

The study of boundary layer flow began some time after the study by the Germany scientist, Prandtl [1904] on the laminar flow over a flat plate with the effect of viscosity. Prandtl presented a paper with only eight pages long for 10 minutes in a seminar about boundary concepts. In the paper presented, he gave the description of the boundary layer concept. He theorized the effects of the friction was to cause the fluid adjacent to the solid body's surface to stick under the circumstance of no-slip condition, which is referred to as ***boundary layers***. The ***boundary layer flow*** is defined as the flow of that portion of a viscous fluid which is in the neighborhood of a body in contact with the fluid and in motion relative to the fluid.
This concept has been very important in the field of fluid dynamics which has led the interests from many researchers. As we will seen some of the scholars who has shown their interests on this concept hereafter.
In this project, We shall deal with the Blasius equation for a steady incompressible and two dimensional viscous flow. We shall derive the equation through the similarity reduction of the equation of continuity and Navier stoke equation to obtain third order nonlinear ordinary differential equation. Thereafter, we solve it numerically by finite difference method and compare the results with other researchers.
Since the laminar boundary layer equation is a nonlinear third order ODE, this makes it so difficult to solve it numerically by finite difference method. First, we will linearize by using the Jacobian transformation then solve it by an iteration method e.g Newton-Raphson method.

1.1 Literature Review

The boundary layer theory is applied the fields of aerodynamics (e.g in airplanes, rockets and projectiles), hydrodynamics (e.g in ships, submarines, torpedoes), transportation (like automobiles, trucks, cycles), wind engineering (e.g in building, bridges) and ocean engineering (e.g in buoys, breakwaters).This has lead to many an interest from many scholars advancing on the boundary layer concept.
Blasius [1908] who was Prandtl's student used the similarity transformation technique in the governing equation to reduce the Navier Stoke equation for the viscous incompressible steady laminar flow over a solid boundary from PDE to an ODE. He obtained a laminar boundary layer equation (also known as ***Blasius equation***) which is *a third order nonlinear* ODE. Blasius solved the laminar boundary layer equation by using a series of expanding together with the shooting method. This has led greatly attracted researchers to work on the study of the steady flows of viscous, incompressible fluid over

solid boundary layers.This type of flow is known as ***Blasius flow***
Blasius equation is one of the basic equations in the fluid dynamics which has been the focus of many studies. Various theoretical developments through the following years have been reviewed by the following scholars;
Toopfer (1912) solved the Blasius equation numerically by the application of the method of the Runga and Kutta.
L. Howarth [1938] solved the Blasius equation using a modified numerical method which involves a series of expansion.
K. Vadasz (1997) solved the Blasius equation by assuming a finite power seires where the objective was to determine the power series coefficients.
Lin J. (1999) used an analytical method to solve the same equation, i.e. He used the parameter iteration method.
Abdul-Majid Wazwaz (2007) solved the equation with the use of variational iteration method.
D.D. Ganji (2008) solved the boundary layer equation by using the homotopy pertubation method(HPM).
Rafael C. (2010) investigated the solution for the Blasius equation by using a single initial value problem via the fourth order Runge-Kutta method algorithm with the shooting procedure.
Saba G.(2015) developed an integral solution for the Blasius equation via the use of the Green function idea as well as approximating the nonlinear term of the Blasius equation by using a trigonometric expansion.

1.2 Boundary Layer

1.2.1 Prandtl's Boundary Layer

Early in the 20th century there has been the theory of mechanics of fluids in the fields.
Firstly, in hydrodynamics in which the theory described the flow over surfaces and bodies by assuming that the flow is inviscid, incompressible and rotational.
Secondly, in hydraulics which was a mainly experimental field concerning the behaviour of the fluids in machinery like pipes, pumps and ships. However, hydrodynamics seemed to be good theory for fluids flowing not close to the solid boundaries but it could not explain the concept of friction and drag.
In 1904, a physicist, Ludwig Prandtl, presented his paper on the boundary layer theory at the third congress of mathematicians in Heidelberg, Germany. It was his theory that connected these two fields, hydrodynamics and hydraulics. He described the boundary layer as a very thin region close to the solid body's surface, the layer in which the flow is influenced by friction between the solid surface and fluid flow.
The theory was based on some important observation. i.e the viscosity of the fluid cannot be neglected in all regions. This leads to a very important condition known as the no-slip

condition. The no-slip condition is the effect due to viscosity the causes the fluid adjacent to the solid boundary to stick, the fluid will have zero velocity relative to the boundary. At certain distance from the solid boundary, under the no-slip condition, the very thin layer is formed which is streamlined in shaped. This very thin layer formed is known as a laminar boundary layer. In such layers, the velocity of the fluid changes rapidly form zero to its main stream value,U as shown in figure 1 below. This may imply a steep gradient of shearing stress. The wall stress can be expressed as follows;

$$\text{Wall shear stress} = viscosity(\nu) \times \text{velocity gradient}$$

A more precise criterion for the existence of a well defined laminar boundary layer is that the Reynolds number should be large, though not so large to an extend that cause a breakdown of the laminar flow.

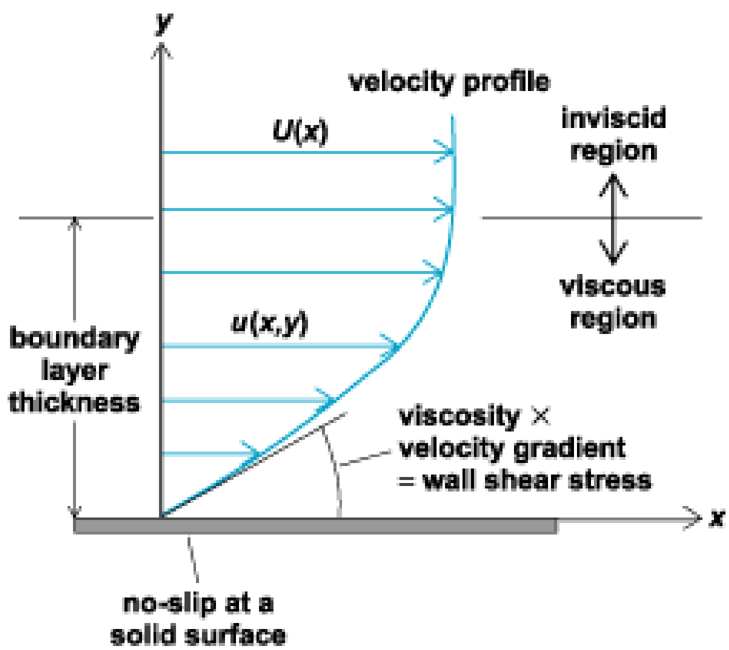


Figure 1. The laminar fluid flow over a flat plate under the no-slip condition

2 Derivation of The Governing Equation

2.1 Equation of Continuity

We derive the equation of continuity in Cartesian coordinate as follows;
We consider fluid flow in a parallelepiped as shown in figure below.

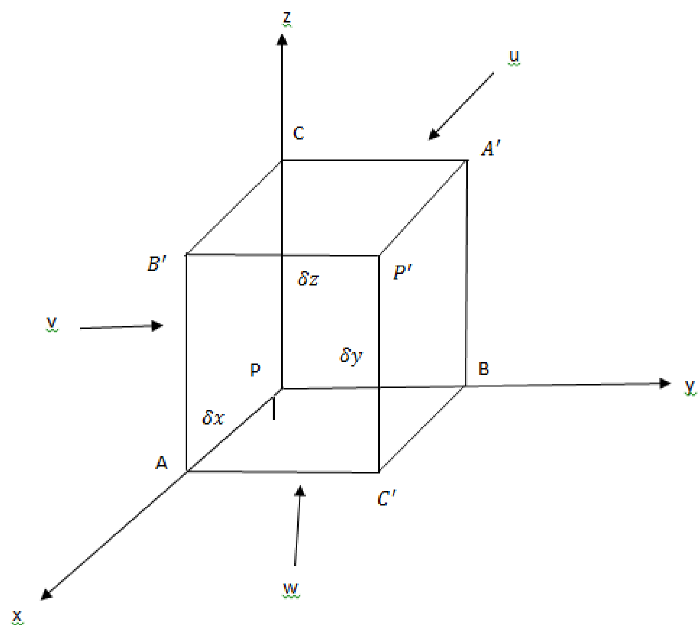


Figure 2. The fluid flow through a parallelepiped

Let there be a fluid particle at P whose coordinates are (x,y,z).
Let ρ be the fluid density at P at any time t.
Let u,v and w be the velocity components in the direction of x,y and z respectively.
Let δx, δy and δz be the edges of the parallelepiped drawn above along the axes as shown in the direction above.
Then, we have;
Mass of fluid flow inside parallelepiped per unit time through face $PCA^{'}B$;

$$\begin{aligned} &= \rho u \delta y \delta z \\ &= g(x,y,z) \qquad (say) \end{aligned} \tag{1}$$

and Mass of fluid flow out of the parallelepiped per unit time through face $AB^{'}P^{'}C^{'}$;

$$
\begin{aligned}
&= g(x+\delta x, y, z) \\
&= g(x,y,z) + \delta x \frac{\partial g(x,y,z)}{\partial x} + \text{.....terms containing higher powers of } \delta x. \text{ (Expand by Tarlor's series)}
\end{aligned} \tag{2}
$$

Thus, Mass of fluid within parallelepiped because of flow across the two faces per unit time;

$$
\begin{aligned}
&= g(x,y,z) - [g(x,y,z) + \delta x \frac{\partial g(x,y,z)}{\partial x} + ...] \\
&= -\delta x \frac{\partial g(x,y,z)}{\partial x} \qquad \text{to the first order of approximation.} \\
&= -\delta x \frac{\partial(\rho u \delta y \delta z)}{\partial x}
\end{aligned}
$$

$$
= -\delta x \delta y \delta z \frac{\partial(\rho u)}{\partial x} \tag{3}
$$

Similarly, the mass of fluid flow inside parallelepiped per unit time through face $PAB^{'}C$;

$$
\begin{aligned}
&= \rho v \delta x \delta z \\
&= g_1(x,y,z) \qquad (say)
\end{aligned} \tag{4}
$$

and mass of fluid flow out of the parallelepiped per unit time through face $BC^{'}P^{'}A^{'}$;

$$
\begin{aligned}
&= g_1(x, y+\delta y, z) \\
&= g_1(x,y,z) + \delta y \frac{\partial g_1(x,y,z)}{\partial y} + \text{....terms containing higher powers of } \delta y.
\end{aligned} \tag{5}
$$

Thus, mass of fluid within parallelepiped because of flow across the two faces per unit time is;

$$
\begin{aligned}
&= g_1(x,y,z) - [g_1(x,y,z) + \delta y \frac{\partial g_1(x,y,z)}{\partial y} + ...] \\
&= -\delta y \frac{\partial g_1(x,y,z)}{\partial y} + \text{...to first order of approximation} \\
&= -\delta y \frac{\partial(\rho v \delta x \delta z)}{\partial y} \\
&= -\delta x \delta y \delta y \frac{\partial(\rho v)}{\partial y}
\end{aligned} \tag{6}
$$

Finally, the mass of fluid flow inside the parallelepiped per unit time through face $PAC^{'}B$ is ;

$$
\begin{aligned}
&= \rho w \delta x \delta y \\
&= g_2(x,y,z) \qquad (say)
\end{aligned} \tag{7}
$$

and mass of fluid flow out of the parallelepiped per unit time through face $CB'P'A'$ is ;

$$\begin{aligned} &= g_2(x,y,z+\delta z) \\ &= g_2(x,y,z) + \delta z \frac{\partial g_2(x,y,z)}{\partial z} + \ldots \text{terms containing higher powers of } \delta z \end{aligned} \tag{8}$$

Thus, mass of fluid within the parallelepiped because of flow across the two faces per unit time;

$$\begin{aligned} &= g_2(x,y,z) - [g_2(x,y,z+\delta z \frac{\partial g_2(x,y,z)}{\partial z} + \ldots] \\ &= -\delta z \frac{\partial g_2(x,y,z)}{\partial z} + \ldots \qquad \text{to first order of approximation} \\ &= -\delta z \frac{\partial(\rho w \delta x \delta y)}{\partial z} \\ &= -\delta x \delta y \delta z \frac{\partial(\rho w)}{\partial z} \end{aligned} \tag{9}$$

Therefore, total mass of fluid within parallelepiped because of flow across all six faces;

$$\begin{aligned} &= -\delta x \delta y \delta z \frac{\partial(\rho u)}{\partial x} - \delta x \delta y \delta z \frac{\partial(\rho v)}{\partial y} - \delta x \delta y \delta z \frac{\partial(\rho w)}{\partial z} \\ &= -\delta x \delta y \delta z [\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}] \end{aligned} \tag{10}$$

Also mass of fluid inside parallelepiped at time t, is given as

$$= \rho \delta x \delta y \delta z$$

Therefore, rate of increase of mass of fluid inside parallelepiped per unit time,

$$\begin{aligned} &= \frac{\partial(\rho \delta x \delta y \delta z)}{\partial t} \\ &= \delta x \delta y \delta z \frac{\partial \rho}{\partial t} \end{aligned} \tag{11}$$

Using the principle of conservation of mass, from (10) and (11) We have;

$$\delta x \delta y \delta z \frac{\partial \rho}{\partial t} = -\delta x \delta y \delta z [\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}]$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial \rho w)}{\partial z} = 0 \tag{12}$$

Equation (12) is called ***equation of continuity*** in Cartesian coordinate system.
We may express (12) as ;

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial y} + v \frac{\partial \rho}{\partial y} + \rho \frac{\partial w}{\partial z} + w \frac{\partial \rho}{\partial z} \tag{13}$$

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} + \rho (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0$$

But, for the incompressible fluid where ρ = constant. Thus (13) becomes,

$$\rho(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z})=0 \tag{14}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \text{since} \qquad \rho\neq 0 \tag{15}$$

For the two dimension, incompressible fluid the equation (15) reduces to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{16}$$

Equation (16) is called ***the equation of continuity*** for the 2D, incompressible fluid. Where u, v are velocity components of the fluid in the x and y directions respectively.

2.2 Navier stokes equations

Let us consider a rectangular element of fluid with sides dx, dy and having a thickness b as depicted in figure 3 below. Using the Newton's second law of motion and applying;

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} \qquad \text{(divergence of vector V)}$$

where u, v and w are components in x, y and z direction respectively and this can be as well as expressed as div V or ∇V. Using equations (12) and (16) above, for the two-dimensional flow, i.e w=0, We have the following expressions;

$$\frac{\partial \rho}{\partial t}+div(\rho V)=0$$

$$\text{Or} \qquad \frac{\partial \rho}{\partial t}+\nabla(\rho V)=0$$

$$\text{Or} \qquad div(\rho V)=\nabla(\rho V)=0 \tag{17}$$

Flow of viscous Fluid

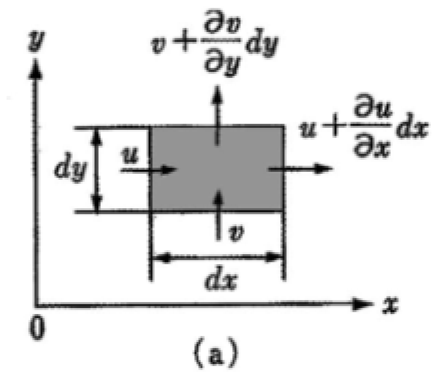


Figure 3. The velocity components on the fluid flow element in xy-plane

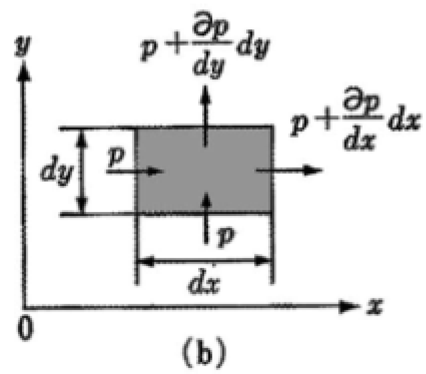


Figure 4. The pressure components on the fluid flow element in xy-plane

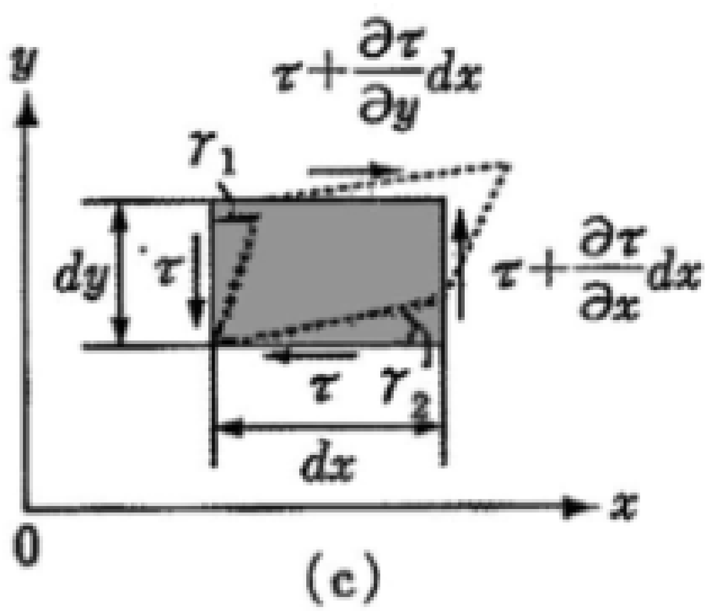


Figure 5. The figure of stress components on the fluid element in xy-plane

Let $F(F_x,F_y)$ be the forces acting on the fluid element, then in the x and y direction, we obtain the following equations respectively;

$$\rho bdxdy\frac{du}{dt} = F_x$$
$$\rho bdxdy\frac{dv}{dt} = F_y \tag{18}$$

where the left hand side of (18) represents the force of inertia given by the the mass of the fluid element times its acceleration.
Due to the change of position with time of the fluid element, we take its velocity to be given as;

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$
$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$
$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \tag{19}$$

In the similar way, we obtain the following;

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$$

substituting (19) into (18), we obtain;

$$\rho(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y})bdxdy=F_x$$

$$\rho(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y})bdxdy=F_y \tag{20}$$

Next, we let $F_B(B_x,B_y)$, $F_P(P_x,P_y)$ and $F_S(S_x,S_y)$ be the body, pressure and viscous forces respectively acting on the fluid elements. This implies that we can express F_x and F_y as follows;

$$\begin{aligned} F_x &= B_x+P_x+S_x \\ F_y &= B_y+P_y+S_y \end{aligned} \tag{21}$$

The body force, $F_B(B_x,B_y)$ is the force that act on the mass of the fluid element e.g the gravitational force. Let X , Y be components body forces in the x and y direction, we have the following;

$$\begin{aligned} B_x &= X\rho dxdy \\ B_y &= Y\rho bdydx \end{aligned} \tag{22}$$

The gravitational force in x direction is zero and $-g$ in y direction. The pressure force, $F_P(P_x,P_y)$ acting on the element in x and y direction is given by;

$$P_x = Pbdy-(P+\frac{\partial P}{\partial x})bdy=-\frac{\partial P}{\partial x}bdxdy$$

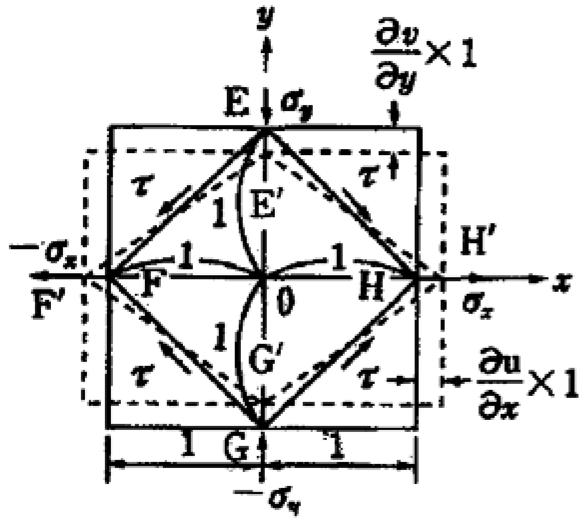
Similarly, we have;

$$P_y=-\frac{\partial P}{\partial y}bdxdy \tag{23}$$

The viscous force, $F_s(S_x,S_y)$ is the force in x-, y- direction cause by the angular deformation, S_x and S_y respectively.

Let the strain of the small element of the fluid be given as $\gamma=\gamma_1+\gamma_2$, thus the corresponding stress is expressed as;

$$\begin{aligned} \tau &= \mu\frac{\partial \gamma}{\partial t} \\ &= \mu(\frac{\partial \gamma_1}{\partial t}+\frac{\partial \gamma_2}{\partial t}) \\ &= \mu(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}) \end{aligned}$$



(d)

Figure 6. Tensile and shear stresses in x- and y- direction due elongation of the element

Then

$$S_{x_1} = \frac{\partial \tau}{\partial y} bdxdy$$
$$= \mu(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}) bdxdy$$

$$= \mu(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}) bdxdy \quad \textbf{(24)}$$

Suppose the elongation transformation causes a force S_{x_2} in the x direction and by considering a rhombus-shaped fluid element EFGH. Inscribing this fluid element in a cubic element ABCD of unit thickness, shows that an elongated flow in x-direction is contracted flow y-direction as shown in the figure 4 above.
The deformations results to an angular deformation on the face of the rhombus as seen above.
We can calculate the rate of deformation $\frac{\partial \gamma}{\partial t}$ as follows;

$$\frac{\partial \gamma}{\partial t} = \frac{\sqrt{2}\frac{\partial u}{\partial x}}{\sqrt{2}} = \frac{\partial u}{\partial x}$$

Therefore, shearing stress τ on the four faces rhombus is given as;

$$\tau = \mu \frac{\partial \gamma}{\partial t} = \mu \frac{\partial u}{\partial x}$$

For equalibrium, the stress force on face EG which is caused by tensile stress σ_x and stress forces on faces EH and HG caused by the shearing stress τ, will be given as;

$$\sigma_x = 2 \times \sqrt{2}\tau \cos 45 = 2\tau$$
$$= 2\mu \frac{\partial u}{\partial x}$$

We now consider the rectangular fluid element defined above, the tensile stress acting horizontally on a face of the area bdy, positioned at a distance dx is given as;

$$\sigma_x + \frac{\partial \sigma_x}{\partial x} dx.$$

and the stress force σ_{x_2} acting horizontally at the same position of the element is given as;

$$\begin{aligned} S_{x_2} &= -(\sigma_x)_x bdy + (\sigma_x)_{x+dx} bdy \\ &= \{-\sigma_x + (\sigma_x + \frac{\partial \sigma}{\partial x})\} bdy \\ &= \frac{\partial \sigma_x}{\partial x} bdxdy \\ &= 2\mu \frac{\partial^2 u}{\partial x^2} bdxdy \end{aligned} \tag{25}$$

Thus,

$$S_x = S_{x_1} + S_{x_2}$$
$$= \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) bdxdy \tag{26}$$

Similarly,

$$S_y = \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) bdxdy$$

Substituting equations (22), (23) and (26) into (20), We obtain;

$$\rho\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\} = \rho X - \frac{\partial P}{\partial x} + \mu\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\}$$
$$\rho \underbrace{\{\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\}}_{\text{Inertia force term}} = \underbrace{\rho Y}_{\text{body force term}} - \underbrace{\frac{\partial P}{\partial y}}_{\text{pressure term}} + \underbrace{\mu\{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\}\}}_{\text{Viscous term}} \tag{27}$$

Hence, the equation (27) is called the ***Navier stokes equation*** in Cartesian coordinate system.

2.3 Dimensional Analysis

This is a method for reducing the number and the complexity of experimental variables that affect a given physical phenomenon by using a sort of compacting technique. For example, if a physical phenomenon depends on n dimensional variables, then the dimensional analysis will the problem to only k dimensionless variables, where the reduction n-k=1,2,3 or 4, depending on the complexity of the problem. Generally, n-k equals the number of different dimensions which sometimes is called basic or fundamental dimensions.

Examples of Fundamental dimensions

Dimension	**symbol**	**units**
Mass	M	kg (Kilogram)
Lenght	L	m (meter)
Time	T	s (second)
Temperature	Θ	K(Kelvin)
Current	I	A(Amphere)

The dimension of any physical quantity can be expressed in terms of the fundamental dimensions.

Examples of derived dimensions

Quantity	**Dimension**
Area	L^2
Volume	L^3
Velocity	L/T
Acceleration	L/T^2
Mass density	$M/(LT^2)$
Pressure	$M/(LT^2)$
Force	ML/T^2

Non-dimensionalization

Nondimesionalizing a mathematical model is a constructive way to formulate the physical problem in terms of dimensionless quantity only. The significance of the dimensional analysis is that it provides the insight in the scaling relations of the system without using the knowledge of any governing equation. More importantly, is that the total number of the variable and parameters is minimal. Thus, the reduction of the number of parameter is also the purpose of scaling technique.
However, the dimensional analysis is more general the scaling. The only difference between the two is that, dimensional analysis is based on a transformation of both variables and parameters whereas in scaling only the variables are transformed.

3 Derivation of boundary layer equations for two-dimensional fluid flow

In general, the laminar boundary layer equations will form a partial differential equations which can be solved numerically. In this case, the very small viscosity or very large Reynolds number is considered. An important contribution to the science of fluid motion was made by L. Prandtl (1904).
Blasius (1908) reduced the Navier Stoke equation and the continuity equation to a single nonlinear ordinary differential equation through similarity transformation. This equation is called the laminar boundary layer equation (Blasius equation). In this chapter, We shall derive the Blasius equation through the simplification of the Navier Stoke and the equation of continuity.

3.1 The Governing Equations

We consider the two dimensional laminar boundary layer fluid flow over a semi-infinite (very thin) flat plate as shown in the figure 7 below. The full equations of motion for the 2D flow are;

1. Continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=0 \tag{28}$$

2. Momentum in x-direction:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{-1}{\rho}\frac{\partial p}{\partial x}+\nu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}) \tag{29}$$

3. Momentum in y-direction:

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{-1}{\rho}\frac{\partial p}{\partial y}+\nu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}) \tag{30}$$

where (x,y) are the Cartesian coordinates with the associated fluid velocities u, v and ρ is the fluid density, p is the fluid pressure and ν is the kinematic viscosity.

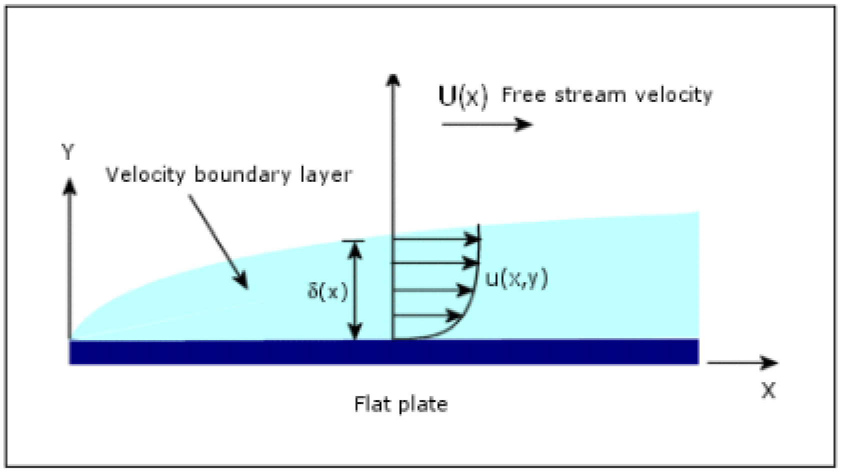


Figure 7. Laminar boundary layer over a flat plate

3.2 Non-dimensional Form of the equation

We start by considering non-dimensional variable;

$$x' = \frac{x}{L}, \qquad y' = \frac{y}{\delta}, \qquad u' = \frac{u}{U}, \qquad v' = \frac{v}{U}\frac{L}{\delta} \qquad p' = \frac{p}{\rho U^2} \qquad t' = t\frac{U}{L} \tag{31}$$

Where L is the horizontal length scale,δ is the boundary layer thickness at $x = L$, u is the fluid velocity in x-direction parallel to the solid boundary (flat plate). Thus, the non-dimensional form of the governing equations becomes;

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \tag{32}$$

$$\frac{U^2}{L}\frac{\partial u'}{\partial t'} + \frac{u'U^2}{L}\frac{\partial u'}{\partial x'} + \frac{U^2 v'}{L}\frac{\partial u'}{\partial y'} = \frac{-1}{\rho}\frac{\rho}{L}U^2\frac{\partial p'}{\partial x'} + \nu\frac{U}{L^2}\frac{\partial^2 u'}{\partial (x')^2} + \nu\frac{U}{\delta^2}\frac{\partial^2 u'}{\partial (y)'^2}$$

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{\nu}{UL}\frac{\partial^2 u'}{\partial (x')^2} + \frac{\nu L^2}{UL\delta^2}\frac{\partial^2 u'}{\partial (y')^2} \tag{33}$$

Similarly;

$$\frac{U^2\delta}{L^2}\frac{\partial v'}{\partial t'} + u'\frac{U^2\delta}{L^2}\frac{\partial v'}{\partial x'} + \frac{U\delta v'}{L}\frac{U\delta}{L\delta}\frac{\partial v'}{\partial y'} = -\frac{1}{\rho}\frac{\rho U^2}{\delta}\frac{\partial p'}{\partial y'} + \frac{\nu U\delta}{L^3}\frac{\partial^2 v'}{\partial (x')^2} + \frac{\nu U\delta}{L\delta^2}\frac{\partial^2 v'}{\partial (y')^2}$$

i.e $$\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} = -(\frac{L}{\delta})^2\frac{\partial p'}{\partial y'} + \frac{\nu}{UL}\frac{\partial^2 v'}{\partial (x')^2} + \frac{\nu}{UL}(\frac{L}{\delta})^2\frac{\partial^2 v'}{\partial (y')^2} \tag{34}$$

We consider a steady flow (flow at a particular position in space not changing with time). That is, $\frac{\partial ....}{\partial t} = 0$, equation (33) and (34) reduces to;

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{\nu}{UL}\frac{\partial^2 u'}{\partial (x')^2} + \frac{\nu L^2}{UL\delta^2}\frac{\partial^2 u'}{\partial (y')^2} \tag{35}$$

$$u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}=-(\frac{L}{\delta})^2\frac{\partial p'}{\partial y'}+\frac{\nu}{UL}\frac{\partial^2 v'}{\partial (x')^2}+\frac{\nu}{UL}(\frac{L}{\delta})^2\frac{\partial^2 v'}{\partial (y")^2} \quad \textbf{(36)}$$

Where the Reynolds number is given as;

$$Reynoldsnumber=\frac{Inertiaforce}{Viscousforce}$$

i.e.

$$R=\frac{UL}{\nu} \quad \textbf{(37)}$$

Next, We drop the primes from the non-dimensional governing eqautions (32) , (35) , (36) and using equation (37) we obtain;

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \textbf{(38)}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{R}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} \quad \textbf{(39)}$$

$$\frac{1}{R}(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y})=-\frac{\partial p}{\partial y}+\frac{1}{R^2}(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}) \quad \textbf{(40)}$$

This is due to inertia force and viscous force having the same order one

$$\text{i.}e \qquad \frac{\nu}{UL}(\frac{L}{\delta})^2=1 \qquad \Longrightarrow \frac{\delta}{L}=\sqrt{(\frac{\nu}{UL})}=\frac{1}{\sqrt{R}} \quad \textbf{(41)}$$

We make another assumptions that as the Reynolds number becomes large enough i.e $R\rightarrow\infty$, the equations above reduce to;

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0 \quad \textbf{(42)}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2} \quad \textbf{(43)}$$

$$-\frac{\partial p}{\partial y}=0 \quad \textbf{(44)}$$

The results for y- components of the momentum equation above shows the pressure gradient in y direction is zero. That implies that the pressure in y direction is constant. In terms of dimensional variables, the above system of equations resume the form;

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0 \quad \textbf{(45)}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \quad \textbf{(46)}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}=0 \tag{47}$$

We notice that the equation of the motion normal to the flat plate has been completely disappeared. Hence the number of the unknowns ones has been reduced by one. We therefore, remain with a system of two simultaneous equations. Again, We notice that one viscous term in the remaining equation of the motion has been dropped.
We take our assumptions for the boundary layer equations for a flat plate at angle of attack of zero incidence in two dimension, steady, incompressible flow at no-slip condition at the plate due to viscosity. The no-slip condition mean that $u=v=0$ at $y=0$
We now subject them to boundary conditions are follows;

$$At \qquad y=0; \qquad u=v=0 \tag{48}$$

$$At \qquad y\rightarrow\infty; \qquad u=U \tag{49}$$

Remember that we are not interested on how the flow outside the boundary layer reached the free stream velocity, U.
We assume no pressure gradient, i.e. $\frac{\partial p}{\partial x}=0$. Thus, the equation (46) reduces to;

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \tag{50}$$

We can further reduce equation (50) to an equation with a single dependent variable. We consider the stream function $\psi(x,y)$ which related to the velocities u(x,y) and v(x,y) defined as follows;

$$u=\frac{\partial\psi}{\partial y}, \qquad v=-\frac{\partial\psi}{\partial x} \tag{51}$$

We the substitute equation (51) into equation (50) to obtain;

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\nu\frac{\partial^3\psi}{\partial y^3} \tag{52}$$

In this case there is no length scale in the flow problem. Therefore, this suggests the possibility of similarity solution.
For a similarity solution, is the one that depends only on one variable. We therefore, only need ψ that depends only on a single variable η . We define the stream function, ψ and variable, η are related as indicated in the equation below;

$$\psi(x,y)=Ax^P f(\eta), \qquad \eta(x,y)=Bx^Q y. \tag{53}$$

Where we will determine P and Q as follows;
The terms in x-momentum equation are successively;

$$\frac{\partial\psi}{\partial y}=Ax^P f^{'}.Bx^Q=ABx^{P+Q}+f^{'} \tag{54}$$

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial x\partial y} &= (P+Q)ABx^{P+Q-1}f' + ABx^{P+Q}.f''QBx^{Q-1}y \\
&= AB(P+Q)x^{P+Q-1}f' + AB^2Qx^{P+2Q-1}yf'' \\
&= AB(P+Q)x^{P+Q-1}f' + ABQx^{P+Q-1}\eta f''
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
\frac{\partial\psi}{\partial x} &= APx^{P-1}f + Ax^{P}.f'QBx^{Q-1}y \\
&= APx^{P-1}f + AQx^{P-1}\eta f'
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial y^2} &= ABx^{P+Q}f''Bx^{Q} \\
&= AB^2x^{P+2Q}f''
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
\frac{\partial^3\psi}{\partial y^3} &= AB^2x^{P+2Q}f''' \cdot Bx^{Q} \\
&= AB^3x^{P+3Q}f'''
\end{aligned} \tag{58}
$$

Now , substituting equations (54), (55), (56), (57), (58) into equation (53), We obtain the following equation;

$$
\begin{aligned}
&ABx^{P+Q}f' \cdot [AB(P+Q)x^{P+Q-1}f' + ABQx^{P+Q-1}\eta f''] - \\
&[APx^{P-1}f + AQx^{P-1}\eta f'] \cdot [AB^2x^{P+2Q}f''] = \nu[AB^3x^{P+3Q}f''']
\end{aligned}
$$

or

$$
\begin{aligned}
&A^2B^2(P+Q)x^{2P+2Q-1}(f')^2 + A^2B^2Q\eta x^{2P+2Q-1}f'f'' - \\
&A^2B^2Px^{2P+2Q-1}ff'' - A^2B^2Qx^{2P+2Q-1}\eta f'f'' = \nu AB^3x^{P+3Q}f'''
\end{aligned}
$$

or

$$
(P+Q)(f')^2 - Pff'' = \nu\frac{B}{A}x^{-P+Q+1}f''' \tag{59}
$$

For a similarity solution, equation (59) should be independent of x. This implies that the power of x is zero.
Therefore,

$$
-P+Q+1=0 \tag{60}
$$

Again, We consider the boundary condition equation (48), $u(x,y)=0$. From equations (51) and (54), We have;

$$
u(x,y) = \frac{\partial\psi(x,y)}{\partial y} = ABx^{P+Q}f' \tag{61}
$$

Using the boundary condition, we get;

$$
u(x,0) = \frac{\partial\psi(x,0)}{\partial y} = ABx^{P+Q}f'(0) = 0 \tag{62}
$$

This implies $A, B \neq 0$. That is it has a trivial solution. This means that $f^{'} = 0$.
Similarly, if we consider another boundary condition at $y = 0$, $v(x,0) = 0$.
But,

$$v(x,0) = -\frac{\partial \psi}{\partial x} = -APx^{P-1}f(0) + ABQx^{P+Q-1}yf^{'} = 0 \tag{63}$$

Since $f^{'}(0) = 0$ already, it follows that $f(0) = 0$.
Finally, if we consider the last boundary condition $y \to \infty$, $u = U$.
But,

$$u(x,\infty) = \frac{\partial \psi(x,y)}{\partial y} = ABx^{P+Q}f^{'}(\infty) = U \tag{64}$$

with the same reason as (60), We find that;

$$P + Q = 0 \tag{65}$$

Solving equations (60) and (65) simultaneously, we obtain;

$$P = \frac{1}{2}, \qquad \text{and} \qquad Q = -\frac{1}{2} \tag{66}$$

We still need to determine A and B. By considering equation (59), we choose for convenience;

$$\nu \frac{B}{A} = 1 \tag{67}$$

Also from equation (64), We set

$$AB = U \qquad \text{so that} \qquad f^{'}(\infty) = 1 \tag{68}$$

Solving equation (67) and equation (68) simultaneously, we have;

$$\begin{aligned} A &= \nu B \\ &= \frac{\nu U}{A} \end{aligned}$$

Therefore,

$$A = \sqrt{\nu U} \tag{69}$$

Similarly,

$$B = \frac{A}{\nu}, \qquad \text{from equation (67).}$$

This implies that

$$B = \frac{U}{\nu B},$$

Substituting equation (68) into (67), we get the following;

$$B=\sqrt{\frac{U}{\nu}} \tag{70}$$

By substituting the equations (66), (69) and (70) into equation (59), we obtain;

$$-\frac{1}{2}ff^{''}=f^{'''}$$

$$f^{'''}+\frac{1}{2}ff^{''}=0$$

$$\text{Or} \qquad 2f^{'''}+ff^{''}=0 \tag{71}$$

The equation (71) is called ***Blasius' Equation*** (or the laminar boundary layer equation). The Blasius equation is basically obtained from the governing equation of the fluid flow (Navier Stokes equation) through similarity transformation to the third order nonlinear ordinary differential equation (ODE) as shown above.
As seen from equations (48) and (49) as well as (62) and (68), We obtain the boundary conditions as follows;

$$\text{At} \qquad \eta=0: \qquad f(0)=0, \qquad f^{'}(0)=0$$

$$\text{At} \qquad \eta\to\infty \qquad f^{'}(\infty)=1 \tag{72}$$

4 FINITE DIFFERENCE METHOD

4.1 Introduction

The finite difference methods are the explicit or implicit relations between the derivatives and the function values at the adjacent mesh points.
Let the interval [a,b] be divided into equal N+1 subintervals such that $x_j = a + jh$, for j=0,1,2.....N+1
Where $x_0 = a$, $x_{N+1} = b$ and $h = \frac{(b-a)}{N+1}$ where a and b are the boundary conditions at the nodal points while h is the distance between two adjacent mesh points.
The finite difference solution of a boundary value problem is obtained by replacing the differential equations at each nodal point by a difference equations.Thereafter, the boundary conditions are incorporated in the difference equation to obtain the system of algebraic equation. This system of algebraic equations are solved by a suitable method like Gaussian Elimination for the case of linear equation. If the algebraic equations are nonlinear, We must linearize it by using any iterative method e.g secant method, Newton-Raphson method.
The difference equations are derived from the Talyor's series.To obtain the appropriate the finite difference approximations to the derivatives, we proceed as follows;
If y(x) and its derivatives are single value continuous functions of x then Taylor's series expansion leads to;

$$y(x+h) = y(x) + hy^{'}(x) + \frac{h^2}{2}y^{''}(x) + \frac{h^3}{6}y^{'''}(x) + ... \tag{73}$$

which implies that;

$$y^{'}(x) = \frac{y(x+h) - y(x)}{h} - \frac{h}{2}y^{''}(x) - \frac{h^2}{6}y^{'''}(x) - ....$$

Thus, from above we obtain;

$$y^{'}(x) = \frac{1}{h}[y(x+h) - y(x)] + O(h) \tag{74}$$

Thus, the equation (74) is called the ***Forward difference*** approximation of $y^{'}(x)$ with an error of order h.
Similarly, We expand y(x-h) by the Taylor's series to obtain;

$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2}y^{''}(x) - \frac{h^3}{6}y^{'''}(x) + ... \tag{75}$$

On which we obtain the following;

$$y^{'}(x) = \frac{y(x) - y(x-h)}{h} + O(h) \tag{76}$$

The equation (76) is known as the ***backward difference*** approximation of $y^{'}(x)$ with an error of order h.
Subtracting equation (75) from equation (73), we obtain the following;

$$y(x+h) - y(x-h) = 2hy^{'}(x) + \frac{h^3}{3}y^{'''}(x) + ... \tag{77}$$

This leads to;

$$y^{'}(x) = \frac{y(x+h) - y(x-h)}{2h} + O(h^2) \tag{78}$$

which is the ***Central difference*** approximation of $y^{'}(x)$ with an error of order h^2.
Also by adding equations (73) and (75), We obtain the following;

$$y(x+h) + y(x-h)2y(x) + h^2y^{''}(x) + ... \tag{79}$$

This gives us;

$$y^{''}(x) = \frac{y(x+h) - y(x-h) - 2y(x)}{h^2} + O(h^2) \tag{80}$$

which is the ***Central Difference*** approximation to $y^{''}(x)$ with an error of order h^2 In the similar way, it is possible to derive the finite difference approximations of higher order derivatives.
Hence, the working expressions for the central difference approximates to the first three derivatives of y_i by ignoring the errors as;

$$y_i^{'} = \frac{y_{i+1} - y_{i+1}}{2h}$$
$$y_i^{''} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$
$$y_i^{'''} = \frac{y_{i+2} - 2y_{i+1}2y_{i-1} - y_{i-2}}{2h^3} \tag{81}$$

4.2 Nonlinear third order differential equation

For the instance, we consider the second order nonlinear differential equation;

$$u^{''} = f(x,u) \qquad a < x < b \tag{82}$$

We subject the equation (82) with the boundary conditions of the mixed type;

$$\begin{aligned} a_0 u(a) - a_1 u^{'}(a) &= \gamma_1 \\ b_0 u(b) + b_1 u^{'}(b) &= \gamma_2 \end{aligned} \tag{83}$$

The differential equation (82), subjected to the mixed boundary conditions (83), will have a unique solution iff it satisfies the following conditions;

(i) $f_u(x,u)$ is continuous and bounded.

(ii) $f_u(x,u) \geq 0$ for x $\in$ [a,b], u $\in (-\infty,\infty)$

Let x_j, be the nodal points where j=0,1,2....N+1. Using equation (81), we substitute into the differential equation (82) to obtain;

$$u_{j-1} - 2u_j + u_{j+1} = h^2 f(x_j,u_j) \qquad j = 1,2...N \tag{84}$$

The system (84) contains N equations in N+2 unknowns u_j, j=0,1,2,3......N+1. We then find the two more equations which are corresponding to the boundary conditions (83). We begin by writing as follows;

$$u_1 = u_0 + hu_0^{'} + h^2(\beta_0 u^{''} + \beta_1 u^{''}) \tag{85}$$

Where β_0 and β_1 are arbitrary constants to be determined. and the truncation error of (85) may be written as;

$$T_0 = U_1 - U_0 - hU_0^{'} - h_2(\beta_0 U_0^{''} + \beta_1 U_1^{''}) \tag{86}$$

Next, We expand the right hand side of (86) using Taylor's series about $x = x_0$ to obtain;

$$\begin{aligned} T_0 = (U_0 + hU_0^{'} + \frac{h^2}{2}U_0^{''} + \frac{h^3}{6}U_0^{'''} + \frac{h^4}{24}U_0^{(4)}....) - U_0 - hU_0^{'} \\ -h^2[\beta_0 U_0^{''} + \beta_1(U_0^{''} + hU_0^{''} + \frac{h^2}{2}U_0^{(4)} + ...)] \\ = (\frac{1}{2} - \beta_0 - \beta_1)h^2 U_0^{''} + (\frac{1}{6} - \beta_1)h^3 U_0^{'''} + (\frac{1}{24} - \frac{1}{2} - \beta_1)h^4 U_0^{(4)} + ... \end{aligned}$$

By setting the coefficient of $h^2U_0^{''}$ and $h^3U_0^{'''}$ to be zero, so that We determine β_0 and β_1 as follows ;

$$\beta_0+\beta_1=\frac{1}{2}, \qquad \beta_1=\frac{1}{6}$$

Thus,

$$\beta_0=\frac{1}{3}, \qquad \beta_1=\frac{1}{6}$$

From (85), we use the discretization at x_0 as

$$hu_0^{'}=u_1-u_0-\frac{h^2}{6}[2f(x_0,u_0)+f(x_1,u_1)] \tag{87}$$

where $T_0=O(h^4)$ and $a_0u_0-a_1u_0^{'}=\gamma_1$
Hence, we obtain ;

$$ha_0u_0-a_1[u_1-u_0-\frac{h^2}{6}\{2f(x_0,u_0)+f(x_1,u_1)\}]=h\gamma_1$$

or

$$(ha_0+a_1)u_0-a_1u_1+\frac{a_1h^2}{6}(2f_0+f_1)=h\gamma_1 \tag{88}$$

Thus, the equation (88) gives the difference approximation at the boundary point $x=x_0$, Similarly, at $x=x_{N+1}$, we use the discretization ;

$$hU_{N+1}=U_N+U_{N+1}+\frac{h^2}{6}[f(x_N,U_N)+2f(x_{N+1},U_{N+1})] \tag{89}$$

where $T_N=O(h^4)$ and $b_0U_{N+1}+b_1U_{N+1}=\gamma_2$
Thus, We have;

$$hb_0U_{N+1}+b_1[-U_N+U_{N+1}+\frac{h^2}{6}(f_N+2f_{N+1})]=h\gamma_2$$

$$-b_1U_N+(hb_0+b_1)U_{n+1}+\frac{b_1h^2}{6}(f_N+2f_{n+1})=h\gamma_2 \tag{90}$$

Hence, equation (90) gives the difference approximation at the boundary point at $x=x_{N+1}$. Thus, equations (84), (88) for j=1,2,3,...N and (90) are the required N+2 equations in N+2 unknowns u_j, j=0,1,2....N+1.
This gives the following systems of equations;

$$(1+c)u_0-u_1+\frac{h^2}{6}(2f_0+f_1)=\frac{h\gamma_1}{a_1}$$

$$-u_{j-1}+2u_j-u_{j+1}+h^2f_j=0 \qquad j=0,1,2....N$$

$$-U_N+(1+\alpha)U_{N+1}+\frac{h^2}{6}(f_N+2f_{N+1})=\frac{h\gamma_2}{b_1} \tag{91}$$

Where $c = \frac{ha_0}{a_1}$ and $\alpha = \frac{hb_0}{b_1}$

Hence, the sysytem of nonolinear equations can be solved by Newton's method or by any other iteration method.

4.3 Newton-Raphson method

We start by writing (91) in the following form;

$$F(u_0, u_1, u_2, ...u_{N+1}) = F(u) = 0 \tag{92}$$

where

$$u = [u_0, u_1, ...u_{n+1}]^\intercal, \qquad F = [F_0, F_1, ...F_{N+1}]^\intercal$$

$$\text{and} \qquad F_0 = (1+c)u_0 - u_1 + \frac{h^2}{6}(2f_0 + f_1) - \frac{(h\gamma_1)}{a_1}$$

$$F_j = -u_{j-1} + 2u_j - u_{j+1} + h^2 f_j, \qquad j = 1, 2....N$$

$$F_{N+1} = -U_N + (1+\alpha)U_{N+1} + \frac{h^2}{6}(f_N + 2f_{n+1}) - \frac{(h\gamma_2)}{b_1}$$

The Jacobian of $F(u_0, u_1, u_2, ...u_{N+1})$ is the tridiagonal matrix which is given as follows;

$$\begin{aligned} J(u_0, u_1, ...u_{N+1}) &= \frac{\partial f}{\partial u} \\ &= \begin{bmatrix} B_0 & C_0 & .. & 0 \\ A_1 & B_1 & C_1 & . \\ ... & ... & ... & ... \\ 0 & .. & ..A_{N+1} & B_{N+1} \end{bmatrix} \end{aligned} \tag{93}$$

5 METHODOLOGY

5.1 The Numerical Solution For Blasius' Equation

We solve the Blassius' equation

$$2f^{'''} + ff^{''} = 0 \tag{94}$$

with the boundary conditions;

$$f(0) = f^{'}(0) = 0, \qquad f^{'}(\infty) = 1 \tag{95}$$

by finite difference method.

Solution

Since the boundary conditions are at infinity and we are considering a thin boundary layer being induced over an semi-finite flat plate, we therefore let our infinity, $\infty = 3$. Using a step-size, h = 1, We divide the interval into equally four spaced mesh points such that;

$$\eta_0 = 0, \qquad \eta_1 = 1, \qquad \eta_2 = 2 \qquad and \qquad \eta_3 = 3$$

We note that our boundary conditions are of the mixed type. Therefore we proceed as follows;

$$f(0) - f^{'}(0) = 0 \tag{96}$$

$$f^{'}(0) = 1 \tag{97}$$

We start with equations (96) and (97) by replacing their derivatives by a central difference as follows;
$\frac{f_{n+1}-f_{n-1}}{2h} = f_n$

This yields, $\quad f_{n-1} = f_{n+1} - 2hf_n = f_{n+1} - 2f_n, \qquad n = 0,1,2,3 \qquad$ and **(98)**

$$\frac{f_{n+1} - f_{n-1}}{2h} = 1$$

$$f_{n+1} = f_{n-1} + 2 \qquad n = 0,1,2,3 \tag{99}$$

with the equations (98) and (99) at n=0, We have;

$$f_{-1} = f_1 - 2f_0 \qquad \text{and} \qquad f_4 = f_2 + 2 \tag{100}$$

We notice that f_{-1} and f_4 in equation (100) above are outside the domain. These mesh points are called the ***fictitious nodes or ghost nodes***. We shall see how to eliminate them later.
Next, we replace the derivatives for the equation (94) by a central difference to obtain;

$$\frac{f_{n+2} - 2f_{n+1} + 2f_{n-1} - f_{n-2}}{2h^3} = -\frac{1}{2}\left[\frac{f_{n+1} - 2f_n + f_{n-1}}{h^2}\right]f_n$$

$$f_{n-2} - 2f_{n-1} + 2f_{n+1} - f_{n+2} = f_n f_{n+1} - 2f_n^2 + f_n f_{n-1} \qquad \text{for n = 1,2} \tag{101}$$

at n=1

$$f_{-1} - 2f_0 + 2f_2 - f_3 = f_1 f_2 - 2f_1^2 + f_0 f_1 \tag{102}$$

we substitute the equation (100) into (102), we get;

$$f_1 - 4f_0 + 2f_2 - f_3 = f_1 f_2 - 2f_1^2 + f_0 f_1 \tag{103}$$

Given that $f_0 = 0$,equation (103) reduces to;

$$F_1(f_1, f_2) = f_1 + 2f_2 - f_1 f_2 + 2f_1^2 - 1.396842 = 0 \tag{104}$$

at n=2 the equation (101) becomes;

$$f_0 - 2f_1 + 2f_3 - f_4 = f_2 f_3 - 2f_2^2 + f_1 f_2 \tag{105}$$

substitute $f_0 = 0$ and $f_4 = f_2 + 2$ in (105), we get;

$$F_2(f_1, f_2) = 2f_1 + f_2 - 2f_3 + f_2 f_3 + f_1 f_2 - 2f_2^2 + 2 = 0 \tag{106}$$

Since we are supposed to determine the values of the function f 's for the internal mesh points only, We therefore use the shooting technique or by guessing. If We take our $f_3 = 1.396842$ then the equation (106) becomes;

$$F_2(f_1, f_2) = 2f_1 + 2.396842f_2 - 2f_2^2 + f_1 f_2 - 0.793684 = 0 \tag{107}$$

The equation (104) and (107) yield a system of;

$$F_1(f_1, f_2) = f_1 + 2f_2 - f_1 f_2 + 2f_1^2 - 1.396842 = 0$$

$$F_2(f_1, f_2) = 2f_1 + 2.396842f_2 - 2f_2^2 + f_1 f_2 - 0.793684 = 0$$

This system can be solved by any iterative method e.g Secant method, Newton-Raphson method etc. We will use the Newton-Raphson's method

to solve this as follows;
Therefore, We have;

$$\frac{\partial F_1}{\partial f_1} = 1 - f_2 + 4f_1 \qquad \frac{\partial F_1}{\partial f_2} = 2 - f_1$$

$$\frac{\partial F_2}{\partial f_1} = 2 + f_2 \qquad \frac{\partial F_2}{\partial f_2} = 2.396842 - 4f_2 + f_1$$

The Jacobian of $F(f_1, f_2)$ is given by;

$$J(f^{[s]}) \bigtriangleup f^{[s]} = -F(f_1, f_2)$$

where $J(f^{[s]}) = \begin{bmatrix} 1 - f_2^{[s]} + 4f_1^{[s]} & 2 - f_1^{[s]} \\ 2 + f_2^{[s]} & 2.396842 - 4f_2^{[s]} + f_1^{[s]} \end{bmatrix}$

Therefore we have the following;

$$\begin{bmatrix} 4f_1^{[s]} - f_2^{[s]} + 1 & 2 - f_1^{[s]} \\ 2 + f_2^{[s]} & f_1^{[s]} - 4f_2^{[s]} + 2.396842 \end{bmatrix} \begin{bmatrix} \bigtriangleup f_1^{[s]} \\ \bigtriangleup f_2^{[s]} \end{bmatrix} = - \begin{bmatrix} f_1^{[s]} + 2f_2^{[s]} - f_1^{[s]} f_2^{[s]} + 2(f_1^{[s]})^2 - 1.396842 \\ 2f_1^{[s]} + 2.396842 f_2^{[s]} - 2(f_2^{[s]})^2 + f_1^{[s]} f_2^{[s]} - 0.793684 \end{bmatrix} \tag{108}$$

where $\bigtriangleup f^{[s]}$ is the solution of ;

$$J(f^{[s]}) \bigtriangleup f^{[s]} = -F(f_1, f_2), \qquad s = 0, 1, 2....$$

Since J is a 2 $\times$ 2 matrix, We may invert it and solve the system. We obtain;

$$\begin{bmatrix} \bigtriangleup f_1^{[s]} \\ \bigtriangleup f_2^{[s]} \end{bmatrix} = -\frac{1}{D} \begin{bmatrix} f_1^{[s]} - 4f_2^{[s]} + 2.396842 & -2 + f_1^{[s]} \\ -2 - f_2^{[s]} & 4f_1^{[s]} - f_2^{[s]} + 1 \end{bmatrix} \times \begin{bmatrix} f_1^{[s]} + 2f_2^{[s]} - f_1^{[s]} f_2^{[s]} + 2(f_1^{[s]})^2 - 1.396842 \\ 2f_1^{[s]} + 2.396842 f_2^{[s]} - 2(f_2^{[s]})^2 + f_1^{[s]} f_2^{[s]} - 0.793684 \end{bmatrix}$$

Where $\qquad D = (4f_1^{[s]} - f_2^{[s]} + 1)(f_1^{[s]} - 4f_2^{[s]} + 2.43273) - (2 - f_2^{[s]})(2 + f_2^{[s]})$

and D is the determinant of the Jacobian matrix given above.
The next is to iterate which is obtained from ;

$$\begin{bmatrix} f_1^{[s+1]} \\ f_2^{[s+1]} \end{bmatrix} = \begin{bmatrix} f_1^{[s]} \\ f_2^{s} \end{bmatrix} + \begin{bmatrix} \triangle\, f_1^{[s]} \\ \triangle\, f_2^{[s]} \end{bmatrix}, \text{Where s=0,1,2...}$$

Taking $f_1^{[0]} = 0.5$ and $f_2^{[0]} = 1.0$, we get;

First iteration, s=0, $f_1^{[0]} = 0.5$ and $f_2^{[0]} = 1.0$

D=-6.706316

$$\begin{bmatrix} \triangle\, f_1^{[0]} \\ \triangle\, f_2^{[0]} \end{bmatrix} = \frac{1}{6.706316} \begin{bmatrix} -4.526431573 \\ 1.896842 \end{bmatrix}$$

Thus, this results to;

$$f_1^{[0]} = -0.1749505351 \qquad f_2^{[0]} = 1.282844113$$

Second iteration; s=1, $f_1^{[0]} = -0.1749505351 \qquad f_2^{[0]} = 1.282844113$

D= -4.301029038

$$\begin{bmatrix} \triangle\, f_1^{[1]} \\ \triangle\, f_2^{[1]} \end{bmatrix} = \frac{1}{4.301029038} \begin{bmatrix} -0.2763419055 \\ -2.643424288 \end{bmatrix}$$

Hence, we have;

$$f_1^{[2]} = f_1^{[1]} + \triangle\, f_1^{[1]} = -0.2392007187$$
$$f_2^{[2]} = f_2^{[1]} + \triangle\, f_2^{[1]} = 0.6682413599$$

Next iteration; s=2 $f_1^{[2]} = -0.2392007187$ and $f_2^{[2]} = 0.6682413599$

D=-5.652627577

$$\begin{bmatrix} \triangle\, f_1^{[2]} \\ \triangle\, f_2^{[2]} \end{bmatrix} = \frac{1}{5.652627577} \begin{bmatrix} 3.252495102 \\ 0.5195868835 \end{bmatrix}$$

$$f_1^{[3]} = f_1^{[2]} + \triangle\, f_1^{[2]} = 0.3382376184$$
$$f_2^{[3]} = f_2^{[2]} + \triangle\, f_2^{[2]} = 0.7604872747$$

Next iteration; s=3, $f_1^{[3]} = 0.3382376184$ and $f_2^{[3]} = 0.7604872747$

D= -4.245593845

$$\begin{bmatrix} \triangle f_1^{[3]} \\ \triangle f_2^{[3]} \end{bmatrix} = \frac{1}{4.245593845} \begin{bmatrix} -1.472700097 \\ 0.08581454691 \end{bmatrix}$$

$$f_1^{[4]} = f_1^{[3]} + \triangle f_1^{[3]} = -0.008639673755$$

$$f_2^{[4]} = f_2^{[3]} + \triangle f_2^{[3]} = 0.7806998881$$

Next iteration, s=4 , $f_1^{[4]} = -0.008639673755$ and $f_2^{[4]} = 0.7806998881$

D=-5.721134648

$$\begin{bmatrix} \triangle f_1^{[4]} \\ \triangle f_2^{[4]} \end{bmatrix} = \frac{1}{5.721134648} \begin{bmatrix} 0.2127856984 \\ -4.4833031419 \end{bmatrix}$$

$$f_1^{[5]} = f_1^{[4]} + \triangle f_1^{[4]} = 0.0285531816$$

$$f_2^{[5]} = f_2^{[4]} + \triangle f_2^{[4]} = 0.6962230891$$

Next iteration, s=5, $f_1^{[5]} = 0.0285531816$ and $f_2^{[5]} = 0.6962230891$

D=-5.465726524

$$\begin{bmatrix} \triangle f_1^{[5]} \\ \triangle f_2^{[5]} \end{bmatrix} = \frac{1}{5.465726524} \begin{bmatrix} 0.04798785962 \\ -0.01923195403 \end{bmatrix}$$

$$f_1^{[6]} = f_1^{[5]} + \triangle f_1^{[5]} = 0.03733295853$$

$$f_2^{[6]} = f_2^{[5]} + \triangle f_2^{[5]} = 0.6997417346$$

Next iteration, s=6 $f_1^{[6]} = 0.03733295853$ and $f_2^{[6]} = 0.6997417346$

D=-5.462700985

$$\begin{bmatrix} \triangle f_1^{[6]} \\ \triangle f_2^{[6]} \end{bmatrix} = \frac{1}{5.462700985} \begin{bmatrix} -0.004858955093 \\ -0.004267268505 \end{bmatrix}$$

$$f_1^{[7]} = f_1^{[6]} + \triangle f_1^{[6]} = 0.03644348003$$

$$f_2^{[7]} = f_2^{[6]} + \triangle f_2^{[6]} = 0.7075533804$$

We find the tolerance of the approximation as follows;

$$\text{Tolerance} = |\frac{\text{present approximation} - \text{previous approximation}}{\text{present approximation}}| \times 100\%$$

Next iteration, s=7 $f_1^{[7]} = 0.03644348003$ and $f_2^{[7]} = 0.7075533804$

D=-5.490376113

$$\begin{bmatrix} \triangle f_1^{[7]} \\ \triangle f_2^{[7]} \end{bmatrix} = \frac{1}{5.490376113} \begin{bmatrix} -0.012041766 \\ -0.085668322 \end{bmatrix}$$

$$f_1^{[8]} = f_1^{[7]} + \triangle f_1^{[7]} = 0.03425025; \text{Tolerance} = 6\%$$

$$f_2^{[8]} = f_2^{[7]} + \triangle f_2^{[7]} = 0.691950019 : \text{tolerance} = 2.25\%$$

Next iteration, s=8 $f_1^{[8]} = 0.03425025$ and $f_2^{[8]} = 0.691950019$

D=-5.441552225

$$\begin{bmatrix} \triangle f_1^{[8]} \\ \triangle f_2^{[8]} \end{bmatrix} = \frac{1}{5.441552225} \begin{bmatrix} 0.001152998272 \\ -0.0001364332767 \end{bmatrix}$$

$$f_1^{[9]} = f_1^{[8]} + \triangle f_1^{[8]} = 0.034462137; \text{Tolerance} = 0.6\%$$

$$f_2^{[9]} = f_2^{[8]} + \triangle f_2^{[8]} = 0.691924946; \text{tolerance} = 0.004\%$$

Since the **tolerance** $\leq 5\%$, then the roots becomes $f_1 = 0.034462137$ and $f_2 = 0.691924946$.

5.2 Results and Discussion

We summarize the results for the solution of Blassuis equation in the table below;

η	Numerical solution of $f(\eta)$	Analytical solution of $f(\eta)$	%Error
0	0.00000	0.0000	0.0000
1	0.034462137	0.16557	79.1858
2	0.691924946	0.65003	-6.4451
3	1.396842	1.39682	-0.001575

Comparison of the numerical values found by different methods

η	Current method	D.D.Ganji	Howarth	Rafael	S.Ghorbani	K.Parand
0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1	0.03446	0.165572	0.16557	0.16557	0.18498	0.16557
2	0.69192	0.650022	0.65003	0.65003	0.69365	0.65004
3	1.396842	1.396472	1.39682	1.39682	1.44689	1.39682

The resulting graph

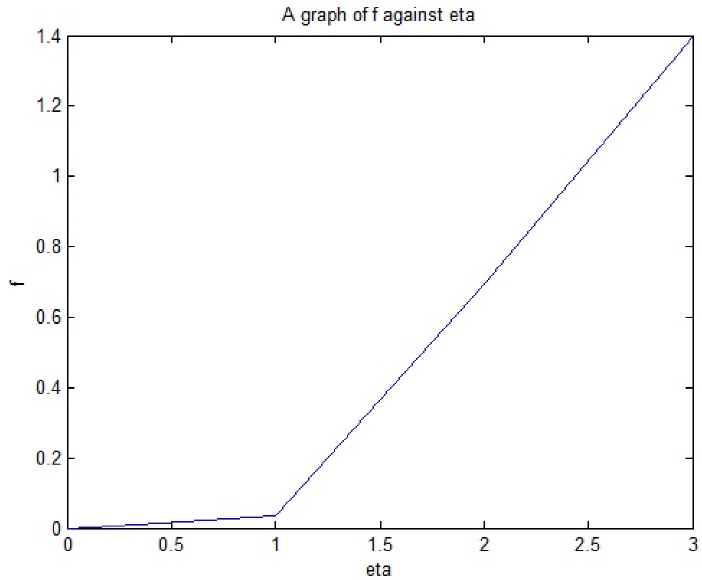


Figure 8. The result of the function of $f(\eta)$

5.3 Conclusion

We have considered the laminar boundary layer (Blasius) equation for an incompressible, laminar, viscous and two dimension flow over a flat plate. This equation occurs in the study of laminar boundary layer problems for the Newtonian fluids.
We have investigated the numerical solutions for the boundary layer equation(Blasius equation) by using the finite difference method. The difficult in this equations is due to the existence of its boundary condition at infinity. Since we are dealing with a very thin boundary layer, we consider the boundary condition at η= 3 rather than at infinity for easy approximation solution. We suppose $f(3) = 1.396842$ according to Rafael Cortell[20] so that we have the known boundary condition.
Even though the resulting was accurate as compared to the analytical solution obtained by L. Howarth [12], the solution at f(1) differed slightly with exact one.

5.4 Reference

1. Abbasbandy S.,(2007), "A numerical solution of Blasius equation by Adomian's decomposition method and comparison with homotopy perturbation method", *Chaos, Solitons and Fractals*,31: pp 257-260

2. Abdul - Majid Wazwaz,(2007), "The varational iteration method for solving two forms of Blasius equation on a half-infinite domain", *Applied Mathematics and computation*,188(1) : pp 485-491.

3. Ahmad F. and Al-Barakati W.H.,(2009), "An approximate analytic solution of the Blasius problem", *communications in Nonlinear science and Numerical Simulation*, Vol. 14, pp.1021-1024.

4. Allan F.M. and Syam M.I., (2005), "On the analytic solutions of the nonhomogenous Blasius problem", *Journal of computational and Applied Mathematics*, 182 : pp 362-371.

5. Anderson, J.D.(2007), "Fundamentals of aerodynamics", 4^{th} edition, *McGraw-Hill series in aeronautical and aerospace engineering*, Boston: McGrwa-Hill Higher Education. xxiv, 1008p. pp 78-86

6. Blasius H., (1908), "Grenzschichten in Flossigkeiten mit kleiner Reibung", *Zetschrift fur Mathematische Physik*, Vol. 56, pp 1-37.

7. Cortell R., (2005), "Numerical solutions of the classical Blasius equations", *Applied Math. Computation*, 140, pp. 217-222.

8. Datta B.K., (2003), "Analytic Solution for the Blasius equation", *Indian Journal of Pure Applied Mathematics*, Vol. 36, pp.237-240.

9. D.D.Ganji, H.Babazadeh, F.Noori, M.M.Pirouz, M.Janipour, (2009), "An Application of Homotopy Perturbation Method for Nonlinear Blasius Equation to Boundary Layer Flow Over a Flat plate", *International Journal of Nonlinaer Science*, Vol.7 : pp 399-404.

10. Fazio R., (2009), "Numerical transformation methods of Blasius problem and its variants", *Applied Mathematics and computation*, 215: pp 1513-1521.

11. Falkner V. M., Skan S.W., (1931), "Some approximate solutions of the boundary-layer equations", *Phil Mag*, 12: pp 865-896.

12. Howarth L., (1983), "On the solution of the Laminar Boundary Layer equations", *Poceedings of the Royal Society of London*, A.164: 547-549.

13. Liao S.J, (1999), An explicit totally analytic approximate solution for Blasius' viscous flow problems, *International Journal of Nonlinear Mechanics*, 34, pp 759-778.

14. Lin J., (1999), "A new approximate iteration solution of Blasius equation", *Communications in Nonlinear Science and Numerical Simulation*, 4 : pp 91-94

15. M.K. Jain, S.R.K Iyengar, R.K. Jain, (2007), *Numerical Methods for Scientific and Engineering Computation*, 5^{th} Ed. pp 45-56

16. Pahlavan A.A. and Boroujeni S.B., (2008), "On the analytical solution of viscous fluid flow past a flat plate", *Physcis Letters A*, vol. 372, pp. 3678-3682.

17. Parand K., Dehghan M., Baharifurd F., (2013), "Solving a Laminar boundary equation with the rational Geogenbauer Functions", *Applied Mathematical Modelling*, 37, pp 851-863.

18. Parand K., Dehghan M., Pirkhedri A., (2009), "Sinc-collocation method for solving the Blasius equation", *Physcis Letters A*, 373, pp 4060-4065.

19. Prandtl L., (1935), "The mechanics of viscous fluids", *In W. F. Durand*: Aerodynamics Theory III, pp 34-208.

20. Rafeal Cortell Bataller, (2010), "Numerical Comparisons of Blasius and Sakiadis Flows", *Matematika*, Vol 26(2), pp 187-196.

21. Saba Ghorbani, N. Amanifard, H.M.Deylami, (2015), " An Integral solution for the Blasius Equation", *Computational Research progress in Applied Science and Engineering*, Vol 01(03), pp 93-102.

22. Schlichting H. and K. Gersten, (2000), "Boundary Layer theory", 8^{th} rev. and enl. ed., Berlin ; Newyork: Springer . xxiii , 799p pp 91-97

23 Schlichting H.(1968), *Boundary-Layer Theory*, 6^{th} ed; McGraw-Hill Series in Mechanics Engineering, pp 127-149.

24. Toopfer C., (1912), "Bemerkungen zu dem Aufsatz von H. Blasius : Gronzachichten in Flussigkeiten mit kleiner Reibung", *Z. Math. Phys.*, 60, pp 397-398.

25. Wazwaz A. M., (2007), "The varation iteration method for solving two forms of Blasius equation on a half-infinite domain", *Applied Mathematics and Computation*, 188: pp 485-491.