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Research Report in Mathematics, Number 24, 2017

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December 2017

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Actuarial Science

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Master Thesis

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Abstract

The most important variable or parameter in the markets for financial options is the volatility. In practice, when taking practical measures to determine volatility based on observed market prices, it turns out to be variable. In particular, its curve is in the form of U with respect to the strike price. This is called colloquially "The Smile of Volatility." Many researchers believe that the problem of volatility smile is a complex question which, together with all its ramifications, is one of the most important problems of quantitative approaches. It has also been found that the underlying process seems to be correlated with the volatility process. In this project, the approach to the problem of the smile of the volatility is given in a descriptive and mathematical way. Also, the problem that is encountered when trying to fit the two stochastic processes on a binomial lattice is addressed by using the change of variable technique.

In this project, a bivariate binomial model for pricing options that take into consideration the correlation between the underlying process and the stochastic volatility is developed. The results obtained are then compared against the developed models in the market such as Binomial Lattice, Monte Carlo Simulation, Hull and White, and the Black-Scholes model for pricing options.

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Declaration and Approval

I the undersigned declare that this dissertation is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

KIGUTA JOHN GIKONYO
Reg No. I56/82155/2015

In my capacity as a supervisor of the candidate's dissertation, I certify that this dissertation has my approval for submission.

Signature Date

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Dedication

I dedicate this thesis to my Dad, my family, friends, and relatives. They have given me all the required support and motivation to handle any task or challenge with certainty and determination. With all of their love and respect, this project has been made possible.

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I thank God for everything. I would also like to thank my supervisor Dr. Mwaniki for his support, technical advice and also immeasurable value addition to the achievement of the goal of this study. A lot of thanks also go to my colleagues who have been so supportive and provided guidance whenever needed.

John Gikonyo Kiguta

Nairobi, 2017.

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1 Chapter one: Introduction

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1.1 Background of the Study

Prior to the economic crash of the 1987 global market, the evaluation model of [Black and Scholes, 1973] seemed to describe the option markets reasonably well [Alexander and Kaeck, 2012]. After the crash and ever since much has been done to improve on the model that assumed that there exists an effective state in the market where the volatility is constant, and the stock price follows the geometric Brownian motion. There have been several attempts to approach this assumption of constant volatility by Black-Scholes model. The stochastic volatility model by [Hull and White, 1987] was the first contribution in the option valuation literature that incorporated stochastic volatility. Unfortunately, these models require an estimate of the market price of volatility risk. In other words, with volatility stochastic, a second factor is introduced since the option is needed to satisfy a bivariate stochastic differential equation. This factor is the market price of the risk associated with volatility.

The BSM model paved the way for the development of the financial derivative market. However, the limitation lies in the fact that its strict presumed assumptions and actual financial markets do not match. This undermines the efficiency, accuracy and applicability of its pricing. The theory of option pricing has made a lot of improvements from the BSM model. One of the ways is relaxing the assumptions to meet the actual conditions and to promote more complex derivatives pricing.

[Cox et al., 1979] presented the simple method to design an option pricing model called the Binomial Model. This was primarily used to calculate the value of American options. The model assumes that there is only upward and downward stock price movement and that the magnitude of the stock price fluctuating upward (or downward) each time is the same throughout the study period. The model strictly assumes that volatility is a function of stock price and that the stock price directly determines the volatility.

The binomial option pricing model and the Black-Scholes option pricing model are two complementary approaches. The binomial option pricing model is relatively simple to derive, which is more suitable for explaining the basic concept of option pricing. The binomial option pricing model is based on the assumption that there are two possible directions for the price movement of securities in a given time interval: up or down. Although this assumption can be visible, the binomial option pricing model is suitable for dealing with more complex options, since it is possible to subdivide a given time segment into smaller units of time.

As the number of price changes to be considered increases, the distribution function of the binomial option pricing model tends more and more towards a normal distribution. The binomial option pricing model is consistent with the Black-Scholes option pricing model. The advantage of the binomial option pricing model is that it simplifies the calculation of the option pricing and adds to its intuitiveness, it has now become one of the major pricing standards for major stock exchanges around the world.

[Harrison and Kreps, 1979] proposed the martingale method of option pricing, using martingale measures to describe the non-arbitrage market and the incomplete market. They prove that the necessary and sufficient condition for the market to have no arbitrage is the existence of an equivalent martingale measure. The condition is that the equivalent martingale measure exists and is unique; when the market is complete, any or all of the required rights and interests are available and may be copied by the arbitrage-free method from the underlying securities on the market.

With the Black-Scholes theory, the equivalent martingale measure for the underlying process is constructed using the most direct technique. The change of the probability measure is promising in the pricing of the option. Tests from various studies conclude that the implied volatility is stochastic and that it seems to have a U-shaped function of the ratio between the price of the underlying asset and the option's strike price [Perelló et al., 2008]. This is known as the moneyness ratio. The fitting of the implied volatility against the ratio between the price of the underlying asset and the strike price of the option uses the terms "smile" or "smirk" effect because the function is usually asymmetric [Lee and Lee, 2010]. This phenomenon shows that the geometric Brownian model is not adequate. However, the continuous-time assumption provides other alternative models that try to explain this effect comprehensively. One of these alternative models is the Stochastic Volatility (SV) models that assume two-dimensional diffusion processes. One of the dimensions of the SV describes the dynamics of the asset price, whereas the other captures the volatility.

Many studies to check whether the SV classes of models can capture the dynamics of the underlying assets have been done. Many professionals acknowledge the significance of volatility in determining the dynamism of the financial market. Among the essential statistical properties of volatility in the financial markets seem to cause the clustering in the changes of price. That is, large changes tend to be followed by large changes, similar to the small changes [Bollen and Whaley, 2004]. Another feature is that regarding the prices changes that show low autocorrelations but there is significant volatility correlation for considerable time lags longer than twelve months. Also, there exists the leverage effect.

Volatility smile is a common phenomenon in option prices, despite the fact that there should be no smile in a real Black-Scholes world because the volatility should remain unchanged across the strike price and time [Lee and Lee, 2010]. The stochastic volatility models and option pricing are both exciting themes that are as boundless. The entire financial industry operates from the fair pricing of the financial instruments and modeling the market behaviors correctly. As such, the knowledge and the ability to evaluate complex derivatives and understand the underlying processes and concepts, are vital to the stakeholders of the financial markets. The statistical and mathematical theories that provide the essential basis of these tools are some of the methods developed in the past years and still are not efficient in providing reliable solutions. The efforts to improve more efficient models and concepts build on the establishment of the Black-Scholes-Merton model in 1970's.

1.2 Statement of the problem

There are many risks involved in the financial market. As such, a decision to get into the foreign exchange market makes the investors vulnerable to these risks. One of the objectives of business is to maximize the profit while minimizing the risks. This makes the investors prefer to accrue maximum expected return and minimal risk. The success of an investor is dependable on the ability to mitigate the risks involved. One of the ways of managing risks in the foreign exchange business is by investing in the currency options. Although option pricing has not gained popularity in Kenya, the trading of currency option over-the-counter has been occurring at a considerable rate. It is easy to price the currency options by considering only the final value. The convenience of this pricing activity occurs because of the existence of the closed-form formulas. However, there is a significant question of what happens when a person considers stochastic processes of both the underlying processes.

The solutions to pricing problems have been provided in the literature review. These solutions range from analytical methods to lattice approaches and the Monte-Carlo simulations. The price of hedging is less subject to the manipulation of the price. As such, options are useful in hedging. There are advanced closed-form solutions to European options. However, it is essential to develop a method that illustrates the correlation between the underlying asset and volatility. As such, this research will explore the lattice methods featured in the literature and utilize the binomial lattices to price the options. The decision to use the lattices is because they are more efficient and direct than the Monte-Carlo simulations [Li, 1992].

The complexity of option valuation makes it very complicated for mathematicians, even for researchers to approach the subject. Many of the methods of valuation of options have been published in magazines and international journals and in them the level of explanation is not always to a level at which gives fair price values. That is why I have decided to investigate the lattice method.

1.3 Research Objectives

1.3.1 General Objectives

To value options under stochastic volatility when the price of the underlying process has a negative or positive correlation with the stochastic volatility.

1.3.2 Specific Objectives

1. To apply the bivariate binomial lattice to determine the value of options.
2. To compare the output values from the model against the already developed models.
3. To estimating the Greeks from the bivariate binomial model.

2 Chapter Two: Literature Review

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2.1 Introduction

This chapter is divided into three sections, introducing recent literature related to this work and briefly describing it. The first part is a brief description of lattice models and the Black-Scholes model; the second section will introduce other related work on options with stochastic volatility.

2.2 Methods of valuation of options

To a large extent, market option prices arise from the laws of supply and demand of the market. However, this does not prevent it from being evaluated to find patterns to extract current prices.

2.2.1 Binomial lattice

While it is true that you can find closed-form solutions for the evaluation of options, the numerical methods have an important role as long as its computational implementation is simple than the development of analytical solutions. Using a set of fixed input parameters such as stock price, correlation, variance of volatility, volatility, moneyness ratio, binomial models have shown to provide accurate approximations of the stochastic volatility price for both European and American cases.

[Cox et al., 1979] presented the first implementation of the binomial model for the evaluation of options assuming the log-normal process. This method of option pricing is undoubtedly one of the most popular and useful among other things because of its simplicity, which on the other hand makes it not always recommended. It consists of building what is known as a binomial tree, which is a diagram that represents the possible paths that could be followed by the asset underlying the option. At every moment or step of time the price has a probability of going up by a percentage amount and a probability of going down by a certain percentage amount. This method can help assess options using the basic principles of non-arbitration and risk-neutral assessment. The binomial model is based on the possibility of forming a hedge by combining a long position in stock with a call sold on them. They propose to approach the problem through a simulation strategy in which the price of the underlying can suffer only one of the following two changes: increase in a rate x, or reduce in a rate y. This method, however, is not effective since the aspect of market perfection is not always the case. Another negative aspect of this technique is that only two changes in prices are taken into account. It is for this reason that is not the ideal method for valuing an option.

A more realistic approach to determine the price of the options, is presented by [Rubinstein, 1994] model. It is based on the calculation of the probability distribution used by market agents to evaluate the options with a common maturity and the same underlying asset. This results in a multiplicative binomial scheme, where the volatility at each node depends on the price of the asset. Prices of the options are calculated in the same way as in the conventional 'binomial trees. This proposal has some drawbacks, as it does not take into account options with shorter maturities and, also, it is assumed that all trajectories leading to the same final value have the equal probability of neutral risk.

[Derman et al., 1996] developed the technique presented by [Rubinstein, 1994], by taking into account options with different maturities and considering a set of options with exercise prices equal to the prices of the underlying asset in the previous node and with expiration in the immediate subsequent node. The drawback of this model is that it presents negative probabilities. Although this failure can be corrected, the method is more unstable numerically as the number of time steps becomes bigger. Another limitation of this model is that only options of European type can be assessed, but in [Chriss, 1996] the possibility of valuing Americans options is introduced.

To improve the stability of the model, [Derman et al., 1996] proposes a realignment of the central nodes of the tree as a function of term prices instead of the cash prices of the underlying process. At each stage of the options the exercise price equal to the term price of the underlying asset instead of its spot price. The lattice formed represents the different possible trajectories that can be followed by the price of an asset over the life of the option. This provides an excellent approximation using relatively simple mathematical results.

Finally, the proposal presented in Hilliard and Schwarz (1996) overcomes the limitations observed through the implementation of a model in which fits both the volatility and the underlying process in a binomial lattice. This scheme can be used to assess European options with a maturity before the corresponding time to the 'last node' of the tree, and can even be adjusted to assess Americans options.

2.2.2 Black- Scholes

In 1973, American scholars Black and Scholes proposed the Black-Scholes model for stock options. This model is a major breakthrough in option pricing, therefore, Black Scholes and Scholes are recognized as outstanding representatives of the theory of option pricing. This method however can be complicated much depending on the option, given the complex mathematical and scientific basis of the stochastic models. The main thing is to understand that it is a method valid for valuing options, but is not commonly used because of its difficulty. Admittedly, Black-Scholes option pricing formula has laid the groundwork for option pricing. However, a large number of studies later show that the assumptions of the model tend to be strict. For example, the stock price follows the log-normal distribution, continuous trading and non-existent trading restrictions, these are in fact some not very realistic assumptions.

Relaxing the assumptions of the Black-Scholes model and amending them has become a hot topic in the field of option pricing that has attracted a large number of scholars to study it. Throughout the decades of research, we can see a large number of studies that focus on transaction costs, transaction limits and the distribution of underlying asset prices. To relax the Black-Scholes option pricing model, more models are derived as briefly described below.

2.2.3 Stochastic –Volatility Model

Stochastic volatility models have gained popularity over the past years as a way of deriving the pricing and hedging. The increase in the use of the Stochastic Volatility models has occurred because of the existence of the non-flat implied volatility surface that has existed from the 1987 crash. As such, it is vital to look into the factors that have contributed to the popularity of the stochastic volatility model and its longevity.

There are many supporting views for modeling volatility as a random process. One of these is that it could represent estimation uncertainty. SV models specify the volatility algorithm as a linear stochastic process, where volatility is considered an unobservable component of the modeled series through a linear, self-regressing process.

2.2.4 Hull-White (1987)

In 1987, Hull-White proposed a two-factor model of stock price movements. The change of price is the process of random diffusion, and the instant standard deviation of the price change is another random diffusion Process can be expressed by the following formula.

$$dS = m_s dt + f(S)h(V)dZ_s \tag{2.1}$$

$$dV = m_v dt + bV dZ_v \tag{2.2}$$

Where M_s and M_v are expected returns of the stock price and variance at the moment of change, f(S)h(V) is the standard deviation of the moment of stock price change. Under this model, Hull-White assumes that the correlation coefficient between stock price and variation is zero use the Monte Carlo Simulation Method to calculate the European option price, and the calculated result is similar to that of Black-Scholes.

2.2.5 Nelson-Ramaswamy (1990)

In a 1990 study, Nelson-Ramaswamy using the binary tree approach proposed to approximate the stochastic model described above. In order to simplify the calculation of two trees, [Nelson and Ramaswamy, 1990] uses some assumptions and transformations on the variable into a simple binomial tree and then use the inverse function to deduce the target values. The major mathematical and practical challenge here is the estimation of parameter and stability of the estimates in time. In some stochastic volatility models, estimating the risk neutral parameters is not easy especially where there is no formula for the price of the option. As such, a person has to run simulations on binomial lattice at each step in the procedure of the iterative search. Various models aim to enhance a closed-form solution. This implies assuming the volatility is independent of the Brownian motion that drives the stock price, whereas the common and empirical evidence suggest a negative correlation [Fouque et al., 2000].

The above-mentioned effects of non-constant volatility would be suggesting that market participants are implicitly attributing a distribution different from that assumed by Black-Scholes. We will now consider two deviations from the distribution assumed by the Black-Scholes formula of special importance in practice. Given the overwhelming empirical evidence existing against the assumptions of the Black-Scholes formula, researchers have tried to propose alternative models that try to incorporate the smile effect of volatility.

2.2.6 Hilliard-Schwartz (1996)

Although Hull-White proposed a stochastic volatility model in 1987 to calculate the European option where the correlation coefficient between the stock price and the variation is zero, the Monte Carlo simulation method used to calculate at that time still could not solve calculations of American option. The constraints encountered when choosing the right time to exercise on American option proved to be both complicated and time-consuming, and there would be situations where convergence would not be possible, so Hilliard-Schwartz suggested the concept of Nelson-Ramaswamy of variable transformation to make it possible to fit on a bivariate tree. The underlying asset and standard deviation have their own diffusion process, and can form two simple trees, then combine the two simple binomial trees into a binary binomial tree under the binary binomial tree

Each node has its corresponding transformed price of stock and standard deviation, calculated by the inverse function. From this you can get the corresponding stock price of each node, and then use the derived probability formula for option price. In this model, not only can the price of the American option be calculated, but it can be used to calculate the option price in the presence of a non-zero correlation coefficient between stock price and variance.

2.2.7 Heston's Stochastic Volatility Process

In 1993 Steve L. Heston in his article "A Closed-Form Solution for Options with Stochastic Volatility" published in the Review of Financial Studies evaluates an option on an action with stochastic volatility. A relevant feature in Heston's article is that it obtains the characteristic functions of risk-neutral probabilities as solutions of a second order partial differential equation. By means of these risk-neutral probabilities, a formula similar to that of Black and Scholes is obtained to value a European option of purchase price; the put option can be obtained with put-call1 parity.

One notable feature of Heston's (1993) model is that it presents a closed formula for the price of an option with the assumption of a correlation between the price of the asset and its volatility. The option price is obtained by calculating the probability that a call option expires within-the-money, although such a probability cannot be calculated directly, it can be obtained by reversing the characteristic function of the price logarithm of the sub-asset. The stochastic dynamics driving volatility in the Heston model is defined below. Assume that the current price S_t of a stock is driven by:

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{v_t} dW_{1,t} \tag{2.3}$$

Where μ is the trend parameter, and $W_{1,t}$ is a Wiener process. The volatility $\sqrt{v_t}$ is driven by the process:

$$d\sqrt{v_t} = \beta\sqrt{v_t}dt + \sigma dW_{2,t} \tag{2.4}$$

Where $W_{2,t}$ is a Wiener process correlated with $W_{1,t}$ i.e., $Cov(dW_{1,t}, dW_{1,t}) = \rho dt$. To simplify the model, we apply the itô lemma to obtain the process for variance v_t which is expressed as a Cox, Ingersoll and Ross (1985) process:

$$dv_t = k(\theta - v_t)dt + \sigma\sqrt{v_t}dW_{2,t} \tag{2.5}$$

In the context of stochastic volatility models, the parameters θ, κ and σ are interpreted as the long-term variance, the long-run variance rate and the volatility of the variance (often referred to as volatility of volatility), respectively.

In the Heston model (1993) the price of the option is driven by a diffusion process analogous to that of the Black model and Scholes, except that Heston assumes that volatility depends on time and is driven by an independent diffusion process. There are two parameters of the prices of the purchase options obtained with the Heston model (1993) that deserve special attention: the correlation between the Brownians who drive the price of the stock and the variance denoted by: ρ i.e., $Cov(dW_{1,t}, dW_{1,t}) = \rho dt$ and the volatility of the variance σ . Regarding the correlation, a negative value of ρ will induce a negative bias in the distribution of so that negative shocks to the stock price will lead to positive shocks to the variance.

[Merton, 1973] proposes that volatility is a deterministic function of time, manages to explain the different levels that are reached in implied volatility, for different periods in time, but fails to explain the form of smile for different exercise prices. [Dupire, 1994] and [Rubinstein, 1994], suggest that not only time is indexed in volatility; they propose dependence with a volatility coefficient, just as they cannot explain the form of volatility. Smile for different periods in time.

On the other hand, Heston's (1993) model is considered one of the most representative, since the process for volatility is not negative and has a reversion to the average (which is observed in the markets) and has a closed solution for plain vanilla options. It also explains the smile of volatility. This pricing measure is seen in traded at-the-money European options prices, and as a common practice, this smile data is used for calibration.

2.2.8 Longstaff-Schwartz Algorithm

After the binomial tree and Black-Scholes methods, [Longstaff and Schwartz, 2001] proposed a method known as Least Square Monte Carlo (LSM). It is used for finding the optimal exercise strategy with American-style options. They grant freedom when choosing the moment of exercising the option. This method unlike Black-Scholes can be applied to exotic instruments as Asian options.

[Stein and Stein, 1991] assume that the volatility and the underlying return on assets has nothing to do with different volatility path used by the Black-Scholes model to calculate the option price. Based on the deficiencies of the above model, Heston (1993) uses the Eigenvalue method to address the problem. In the case of assets related to foreign exchange options and bond options pricing formula empirical study of Heston (1993) modelshows that the underlying asset return and volatility process explains the negative bias of the yield and the BS model period Price bias phenomenon.

Other stochastic volatility options pricing models similar to [Hull and White, 1987] including [Andersen et al., 2002] who validated Hull and White (1987) model using the warrants on the Stock Exchange. Warrants have better pricing performance, which means that considering the stochastic volatility can improve the pricing of warrants results. The empirical analysis shows that the stochastic volatility model can better fit the market data. However, there is an important flaw in the stochastic volatility model where the parameters cannot be directly observed, which gives the actual estimation great difficulties.

[Bollerslev, 1986] proposed GARCH model based on the ARCH class model, [Engle, 1982]. GARCH model can well explain the volatility of financial time series clustering and fat tail phenomenon. The volatility of the yield of a bond obeys the GARCH model, and [Duan, 1995] gives the options in physical measure and risk transition between the measures, and thus establish the basis of the European option pricing. In numerical calculation, [Duan and Simonato, 2001] presented a Markov chain technique to approximate the price of options. [Lehar et al., 2002] and other empirical analysis also shows that compared with the Black-Scholes model, the GARCH model the pricing of the option is more in line with the market price.

Li (1992) constructed a Possion Jumping Option Pricing Model unlike the other volatility models, the Possion jump process does not have the components of continuous martingales it is purely a jump process. There are other models that use implied volatility for pricing, but much of this literature focuses on how to explain the "implied volatility smile" effect, such as [Canina and Figlewski, 1993].

3 Chapter Three: Methodology

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3.1 Introduction

This chapter introduces the algorithm presented by Hilliard and Schwarz (1996) and explains the various aspects of the model.

Definition of Options
An option is a non biding contract where the holder has the right but not the obligation to exercise it. In an option contract, five elements are specified:

(i) Option type: call or put option (American or European),

(ii) Underlying asset: assets (stocks, currencies, interest rates, oil, gold, etc.),

(iii) Amount of traded asset: is the amount, in units, of the underlying asset that is stipulated that can be bought or sold for each option contract,

(iv) Expiration date: is the date on which the contract expires,

(v) Exercise price: is the price at which the contract can be exercised, that is, the price at which the underlying asset may be bought or sold, according to the option to be bought or sold.

Options are of two types, the call and the put option. A call option gives the buyer the right to buy a particular asset at a certain future date and at a particular price whereas a put option gives the right to trade a particular asset by a certain date and at a particular price [MacKenzie, 2006].

There are various styles of options in today's stock market. However, the major options are the European and American options. These two types of option differ in date at which they are exercised. The European options can only be exercised at the maturity date whereas the American options can be exercised at any time before the expiration date [Chance et al., 2000].

3.2 Price an of option

The option price is the premium that the buyer/seller of an option pay/receive for an option. Later on we will go deeper into some of their most important valuation methods. However, there are a number of relationships between the premium and the different components of the options that allow us to know the behavior of this one. We will also see how each one of us influences of these in the option price assuming that the other factors remain constant.

The price of the underlying asset is one of the factors influencing the price. For call options, the premium is what you must pay for having the right to buy. For this reason, if the price of the asset we intend to buy (at a price fixed through the option) increases, it has to cost us more money to acquire this right. Conversely, the right to sell (put option) has a lower cost in the event of an increase in the price of the underlying asset, as the higher the put option comes, the more difficult it will be to exercise the put option.

The strike price is the price agreed today to buy or sell a product. Sell the underlying asset in the call or put options at maturity. It has a negative relationship with the premium for purchase options. The higher the strike price, the lower the premium. For put options, a positive relationship is given, i. e. it will cost more money that which allows you to sell more expensive.

Time to maturity is another element that influences the premium of directly, this means that the longer term the higher the premium. As the time to maturity progresses, options lose their temporary value.

When working with stock options, you should consider the treatment of dividends embodying the underlying shares of the option. These dividends are paid to the owners of the shares and not to the option holder. When the payment of the dividend is made, the share price is reduced since it deducts this last flow, so when the price is reduced, the price of the call option premium is also reduced, on the contrary, it goes with the put option, if the option price decreases due to the payment of a dividend, the premium for buying the put option will increase in value.

Interest rates have a double effect on the price of the option. In the case of call, an increase in the interest rate causes the strike to be discounted to a higher premium. In addition to this relationship, it should also be noted which effect is more predominant in view of the rise in the interest rate, since if it has greater influence on the share price (raises in interest rates lead to falls in the price of a share), the price of the call option may decrease. For the PUT option, we see that interest rate rises decrease the premium, although a strong impact on the decline in share price could offset this drop in the premium, causing a rise.

Volatility is a measure of uncertainty about changes in the future price of shares in the future. It is the most influential factor in the price of options. When volatility increases, the likelihood of stock prices rising or falling is much greater. Greater volatility leads to higher premiums for both call options and put options, as the buyer of the option can earn very high or unlimited profits with losses always limited to the premium.

Knowledge of the future volatility of a share allows strategies to be carried out, regardless of the direction of possible declines or rises in the price of the underlying.

3.3 The basic Pricing of options

As a derivative security, the value of an option is determined by the value of the underlying asset. The element of risk in the underlying asset causes its random changes in price. An example of risk involved in the underlying asset is the uncertainty of the future price. Similar to the fluctuations in the price of the underlying asset, the value of then corresponding option is not constant. Formulating the option could lead to easy comprehension of the fundamental of option pricing. This thesis will use notions to formulate the options for better understanding.

We denote V_t as option value at a predetermined time t and S_t as the underlying asset value at a particular time t. Using the notations V_t and S_t,and denoting the underlying asset volatility by σ, the interest rate by r, the date of maturity of the option by T and the underlying asset's exercise price by K, then the payoff of the option is:

$$Payoff = \begin{cases} (S_T - K)^+, \text{(Call option)} \\ (K - S_T)^+, \text{(Put option)} \end{cases}$$

The fluctuations of the underlying asset price and the uncertainty in such changes make it significant to find the underlying asset properties. The properties of the underlying asset are significant in modeling its price process. The variable whose price changes frequently in an unpredictable way is said to follow a stochastic process. We will assume that the price process of the underlying asset will satisfy the Markov Property. A stochastic process is said to have the Markov Property only if the future conditional probability depends on the present state and not the sequence of events that preceded it. A process that has this property is referred to as a Markov process. In a Markov process, predicting the future value depends on only the present value of the variable [Gamerman and Lopes, 2006]. To model this process, a person has to find the process that satisfies the property.

3.4 Concepts of Volatility

Volatility is the level of uncertainty attached to the size of changes in the value of a security. As such, volatility is the key to understanding the cause of fluctuation of the option prices and why they act the way they do. Volatility is one of the most significant concepts applied when valuing options. In addition to valuation of options, volatility is a key factor that affects the pricing of option. For example, the probability that the option will be exercised for a profit is affected by the level of volatility while considering the changes of the price of the underlying instruments [Angeletos et al., 2001]. The anticipation of high profit from an option increases its value. High volatility means that the value of the particular security can be potentially be distributed over a wide range of values. This implies that there is a probability of dramatic change of the security price over a short time in either direction.

A lower volatility implies that the value of a security does not change at a significant rate but fluctuates steadily over a span of time [Angeletos et al., 2001]. In the case of options, volatility is good because the price of volatility generates greater value for a particular option. The explanation for the increase in the value of an option is that the higher the underlying asset volatility, the higher the option value. However, volatility is not good. The purchases of options enjoy only the upside potential and not the downside risk, unlike in the case of other financial assets that have both risks. Volatility affects binomial model in that volatility level determine the value of binomial model.

The two related, but distinct concepts of volatility that one should distinguish include the volatility of a financial instrument and the implied volatility of an option that is written on such an instrument.

Underlying Volatility and Implied Volatility
One of the key determinants when calculating the price of an option is the assessment of implied volatility. The magnitude of the expected fluctuations has a direct impact on the high cost of an option. Unlike other determinants such as the risk-free interest rate or the current price of the underlying, implied volatility is not directly observable.

To evaluate the implied volatility of an asset, operators rely primarily on past prices, ie on historical volatility while underlying volatility is estimated from asset returns [Satchell and Knight, 2011]. Implied volatility is often considered the best forecast of future volatility, regardless of the underlying asset. This assertion is generally tested on the basis of the joint assumption of informational efficiency of the options market - the forecasts must be unbiased and the forecasting errors must be orthogonal to the set of available information - and the validity of the model option used to infer implied volatility. Indeed, when empirical studies use options traded on organized markets, the use of an option valuation formula is necessary to extract the expected volatility from quoted option prices.

Foreign exchange options on the over-the-counter market are quoted in implied volatility. This is in particular a function of the exercise price and the maturity chosen by the client. The operators introduce this anticipated volatility in the formula of Black and Scholes (1987) for exchange options - in order to calculate the price of the corresponding option.

3.4.1 Brownian motion

Brownian motion, also called Brownian movement is used to describe the physical phenomena in which molecules undergo small, random fluctuations or movement. When using the Brownian motion, if a number of particles are present in a given medium and there is no preferred direction for the random movement, the particles will tend to spread evenly throughout the medium over a period of time. A process W that occurs in Brownian motion has two significant properties:

The first property is that for a small time change Δt, the change of ΔW is given by $N\sqrt{\Delta t}$ Where $N \sim N(0,1)$. The second property is that the values of ΔW for any different intervals, are independent and have stationary increments. This leads to the following:

$$E(\Delta W) = 0$$

$$Var(\Delta W) = \Delta t$$

Consider $W(T) - W(0)$ and $n = \frac{T}{\Delta t}$ then

$$W(T) - W(0) = \sum_{i=1}^{n} N_i \sqrt{\sigma t}$$

, $N_i \sim N(0,1)$ where i =1,2,3 ... n
The distribution of $W(T) - W(0)$ is normal with mean 0 and variance $n\Delta t = T$

3.5 Modeling stochastic volatility

As mentioned in the previous section, the volatility of an asset is not constant, nor is it observable. Therefore, an adequate treatment is required in the valuation of options. The alternative is to model it as a stochastic process.

Stochastic volatility models for valuing options are characterized by describing the dynamics of the underlying price by the following stochastic process.

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t \tag{3.1}$$

W_t is a geometric Brownian motion and σt is a process of volatility. This process must meet certain conditions for S_t to have a solution. For example, the volatility process must remain positive all the time, for this to happen σt is changed by a positive function $f(Y_t)$, where Y_t is a specific stochastic process. The most common stochastic differential equations for Y_t are:

1. Log-Normal (LN) which is given by $dY_t = \mu Y_t dt + \sigma_t Y_t dU_t$
2. Ornstein-Uhlenbeck which is given by $dY_t = \alpha(m - Y_t)dt + \beta dU_t$
3. Cox-Ingersoll-Ross which is given by $dY_t = k(\theta - Y_t)dt + \eta\sqrt{Y_t}dU_t$

U_t is a geometric Brownian motion; which in general is correlated with the Brownian motion W_t; that is, $Cov(W_t, U_t) = \rho dt$, $\rho \in [-1, 1]$.

3.5.1 Hull-White Model

The model proposed by Hull and White in 1987 for valuing options was based on the fact that the volatility of the underlying asset is guided by a geometric Brownian movement. The resulting formula is an approximation that considers a series of Taylor terms to be third order.

In this model the underlying asset price follows a Log-Normal distribution, and is given by the following equation,

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t \tag{3.2}$$

Where the parameter r represents the risk-free interest rate and we add the assumption that the underlying asset variance follows a Brownian motion, similar to a Log-Normal

$$dV_t = \mu V_t dt + \sigma_t V_t dU_t \tag{3.3}$$

The value of a European option is given by the Black-Scholes valuation formula, when it is integrated over the probability distribution of the average stochastic variance. One of the disadvantages of this model is that the dynamics of volatility do not represent a reversion to the mean. This means that volatility does not tend to take the value of the average over time. One of the models that solve this problem is the Hilliard and Schwarz (1996) model.

3.5.2 Bivariate Binomial Model

A stochastic volatility model is a model that allows the volatility and any involved variables involved to randomly fluctuate over time rather than remaining constant [Barndorff-Nielsen and Shephar

In the Hilliard-Schwartz(1996) model , a continuous-time risk-neutral diffusion is considered. This process is in the form:

$$dS = m_s dt + f(S)h(V)dZ_s \tag{3.3}$$

$$dV = m_v dt + bVdZ_v \tag{3.3}$$

Where S represents the underlying asset value, dZ_v and dZ_s are Wiener processes that have a correlation $Corr(dZ_s, dZ_v) = \rho_{sv}$,V denotes the stochastic volatility (SV), and $f(S)h(V)$ have the form $S^{\theta}V^{\alpha}$. Θ is the constant elasticity of variation.

3.6 Constructing the Lattice

Hilliard and Schwarz (1996) developed stochastic differential equations using a transformation of the S and V variables to recombine in a two-dimensional tree. Since the variation of the diffusion equation of the asset price is itself a diffusion process, the resulting lattice does not recombine resulting in an exponential explosive tree. In order to construct a path independent of the two trees, we must use the change of variable method to come up with a constant volatility. To fit this in a tree method, we consider the transformation of both processes. To transform Equation 3.3 we use,

$$Y = \frac{ln(V)}{b} \tag{3.4}$$

Using to Itô's lemma, we can get the diffusion process of Y, and the variation becomes constant: Itô's lemma states that for a drift-diffusion process of the form

$$dX_t = \mu_t dt + \sigma_t dB_t \tag{3.5}$$

and for any twice differentiable scalar function $f(t,x)$ of two real variables t and x, one has

$$df(t,X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma_t \frac{\partial f}{\partial x} dB_t \tag{3.6}$$

Therefor we have

$$dY = Y_v dV + \frac{1}{2} Y_{vv} dV^2 \tag{3.7}$$

Where $Y_v = \frac{1}{bV}$ and $Y_{vv} = -\frac{1}{bV^2}$ From the equation $dV = m_v dt + bVdZ_v$ so we can get

$$dV^2 = b^2V^2dt \tag{3.8}$$

Therefore

$$dY = \frac{1}{bV}(m_v dt + bVdZ_v) + \frac{1}{2}(-\frac{1}{bV^2})(b^2V^2dt) \tag{3.9}$$

This yields a process with unit volatility

$$dY = (\frac{m_v}{bV} - \frac{b}{2})dt + dZ_v \quad \textbf{(3.10)}$$

$$dY = m_y dt + dZ_v \quad \textbf{(3.11)}$$

Where the coefficient of dt, m_y is the drift term for Y.
The lattice in Y recombines as required because the coefficient dZ_v does not change, that is, it is a constant. The transformation of S, however, to constant volatility is not obvious due to the fact that the volatility of S from the equation has both the random variables V and S. In this case, a two-step transformation is used by first considering a transformation $H(S,V)$ of the form

$$H(S,V) = h^{-1}(V)\int \frac{dS}{f(S)} \quad \textbf{(3.12)}$$

Taking $h(V) = V^{\alpha}$ we can derive it as follows

$$H_s = \frac{\partial H}{\partial s} = \frac{1}{f(S)V^{\alpha}}$$

$$H_{ss} = \frac{\partial^2 H}{\partial s^2} = -\frac{f_s}{f^2 V^{\alpha}}$$

$$H_v = \frac{\partial H}{\partial v} = \frac{\alpha H}{V}$$

$$H_{vv} = \frac{\partial^2 H}{\partial v^2} = \frac{\alpha H(1+\alpha)}{V^2}$$

$$H_{sv} = \frac{\partial H^2}{\partial s \partial v} = -\frac{\alpha}{f(S)V^(\alpha+1)}$$

From this the diffusion process dH is given by

$$\begin{aligned} dH &= H_s dS + H_v dV + \frac{1}{2}[H_{ss}dS^2 + 2H_{sv}dSdV + H_{vv}dV^2] \\ &= H_s f(S)h(V)dZ_s + H_v bV dZ_v + m_h dt \end{aligned} \quad \textbf{(3.13)}$$

Where drift term of H is m_h is and which depends on m_s, m_V and second-order partials.
We then transform H to Q where

$$Q = (\alpha b)^{-1} ln(\alpha bH - \rho_{sv} + \sigma_h) \quad \textbf{(3.14)}$$

Where

$$\sigma_h = \sqrt{1 - 2\alpha bH\rho_{sv} + \alpha^2 b^2 H^2} \quad \textbf{(3.15)}$$

The diffusion in this case is in the form

$$dQ = m_q dt + dZ_h \tag{3.16}$$

At this point, the diffusion for Q has a constant volatility which is 1 as required. Using Ito's formula, the drift terms for H and Q can be derived as follows:

$$m_h = \frac{m_s}{f(S)V^{\alpha}} - \frac{m_v \alpha H}{V} - \frac{1}{2} f_s V^{\alpha} + \frac{1}{2}\alpha H(1+\alpha)b^2 - \alpha b \rho_{sv} \tag{3.17}$$

And

$$m_q = \frac{m_h}{\sigma_h} + \frac{1}{2}\frac{\alpha b \rho_{sv} - \alpha^2 b^2 H}{\rho_h} \tag{3.18}$$

Both Y and Q have unit volatility, which makes it easy to construct the bivariate binomial grid on the Y×Q space. Inverse transformation is used to give the values of the variables V and S. From Equation 3.4 we can get V as follows:

$$V = \exp(bY) \tag{3.19}$$

We can also get S using the inverse transforming from the two processes H and Q
Define $Q = Q(H)$

$$\begin{aligned} Q &= (\alpha b)^{-1} ln(\alpha b H - \rho_{sv} + \sigma_h) \\ &= (\alpha b)^{-1} ln(\alpha b H - \rho_{sv} + \sqrt{1 - 2\alpha b H \rho_{sv} + \alpha^2 b^2 H^2}) \end{aligned} \tag{3.20}$$

We can simplify the expression to solve for H and S

$$\alpha b Q = ln(\alpha b H - \rho_{sv} + \sqrt{1 - 2\alpha b H \rho_{sv} + \alpha^2 b^2 H^2})$$

$$\begin{aligned} \exp(\alpha b Q) &= \alpha b H - \rho_{sv} + \sqrt{1 - 2\alpha b H \rho_{sv} + \alpha^2 b^2 H^2} \\ \exp(\alpha b Q) + \rho_{sv} - \alpha b H &= \sqrt{1 - 2\alpha b H \rho_{sv} + \alpha^2 b^2 H^2} \\ [\exp(\alpha b Q) + \rho_{sv} - \alpha b H]^2 &= 1 - 2\alpha b H \rho_{sv} + \alpha^2 b^2 H^2 \\ [\exp(\alpha b Q) + \rho_{sv}]^2 - 2[\exp(\alpha b Q) + \rho_{sv}]\alpha b H &= 1 - 2\alpha b H \rho_{sv} \\ [\exp(\alpha b Q) + \rho_{sv}]^2 - 1 &= 2\alpha b H \exp(\alpha b Q) \end{aligned}$$

Therefore

$$H = \frac{2\rho - (1 - \rho_{sv}^2)\exp(-\alpha b Q) + \exp(\alpha b Q)}{2\alpha b} \tag{3.21}$$

$$S = \begin{cases} [V^{\alpha}(1-q)H]^{\frac{1}{1-q}}, & \text{when } q \neq 1 \\ \exp(h(V)H), & \text{when } q = 1 \end{cases} \tag{3.22}$$

The increments dZ_v and dZ_h under these transformations have correlation

$$(dZ_v, dZ_h) = \frac{\rho_{sv} - \alpha bH}{\sigma_h} \tag{3.23}$$

And

$$Corr(dY, dQ) = Corr(dZ_V, dZ_H)$$

3.6.1 Joint Probabilities and Binomial Jumps

Just as in the case of the standard univariate model that has unit volatility, the process for deriving the binomial jumps for the two transformation processes Q and Y are:

$$Y_1^{\pm} = Y_0 \pm \sqrt{\Delta t} \quad \text{and} \quad Q_1^{\pm} = Q_0 \pm \sqrt{\Delta t}$$

Where Δt is the size of the time step and $\sqrt{\Delta t}$ is the magnitude of the binomial jumps. In this case the Y and Q probabilities for the upward jumps are:

$$p = 0.5(1 + m_y\sqrt{\Delta t}) \quad \text{and} \quad q = 0.5(1 + m_q\sqrt{\Delta t}) \quad \text{respectively}$$

The joint probabilities are given by:

$$P_{11} = Prob(Q_1^+, Y_1^-) \qquad P_{12} = Prob(Q_1^+, Y_1^+)$$
$$P_{21} = Prob(Q_1^-, Y_1^-) \qquad P_{22} = Prob(Q_1^-, Y_1^+)$$

It is easy to derive the joint probabilities when dZ_V and dZ_h are independent. For example,

$$P_{11} = q(1-p) \qquad P_{12} = pq$$
$$P_{21} = (1-q)(1-p) \qquad P_{22} = p(1-q)$$

But in our case dZ_V and dZ_h are dependent. According to the diffusion process of Y and Q, $Corr(dY, dQ) = Corr(dZ_V, dZ_H)$.These correlations will be taken into account in the joint probability of constructing a bivariate tree change, so we need to set up marginal probabilities constraints and a cross-product moment constraints so that we can have the desired joint probabilities.

1. $\Delta Q = Q_1 - Q_0$ will have a constraint on its Marginal probability of

$$P_{11} + P_{12} = q \tag{3.24}$$

2. $\Delta Y = Y_1 - Y_0$ will have a constraint on its Marginal probability of

$$P_{11} + P_{21} = 1 - p \tag{3.25}$$

3. $E(\Delta Y \Delta Q)$ will have a constraint on its cross product moments of

$$\Delta t(2P_{12}+2P_{21}-1) \tag{3.26}$$

From Equation 3.30 we can make the necessary adjustments for non zero correlation on the joint probabilities. We can then denote the covariance between ΔY and ΔQ together with their correlation as follows:

$$Cov(\Delta Y,\Delta Q)=2\delta t(P_{12}+P_{21}+p+q-2pq-1)$$
$$Corr(\Delta Y,\Delta Q)=\frac{Cov(\Delta Y,\Delta Q)}{4\Delta t\sqrt{p(1-p)q(1-q)}} \tag{3.27}$$

Solving the 4 Equations 3.24 to 3.27 we will obtain the following joint probabilities:

$$P_{11}=q(1-p)-Corr(dY,dQ)\sqrt{p(1-p)q(1-q)}$$
$$P_{12}=pq+Corr(dY,dQ)\sqrt{p(1-p)q(1-q)}$$
$$P_{21}=(1-q)(1-p)+Corr(dY,dQ)\sqrt{p(1-p)q(1-q)}$$
$$P_{22}=p(1-q)-Corr(dY,dQ)\sqrt{p(1-p)q(1-q)}$$

3.7 Tree Presentation

Considering the transformations derived it is now possible to construct an additive binomial tree for the transformed process S and V where the two trees recombines. They can be be diagrammatically represented as:

Figure 1: The recombined Tree for Q

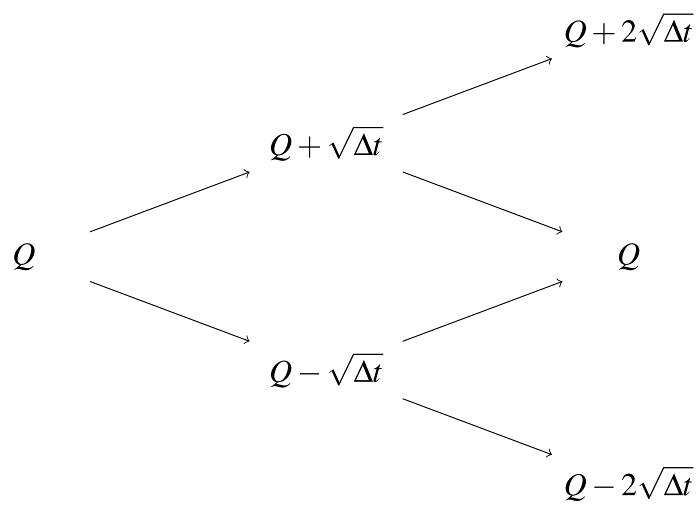
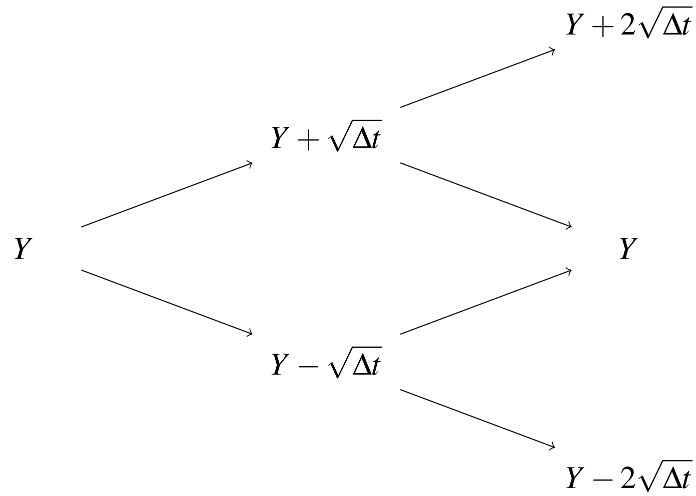
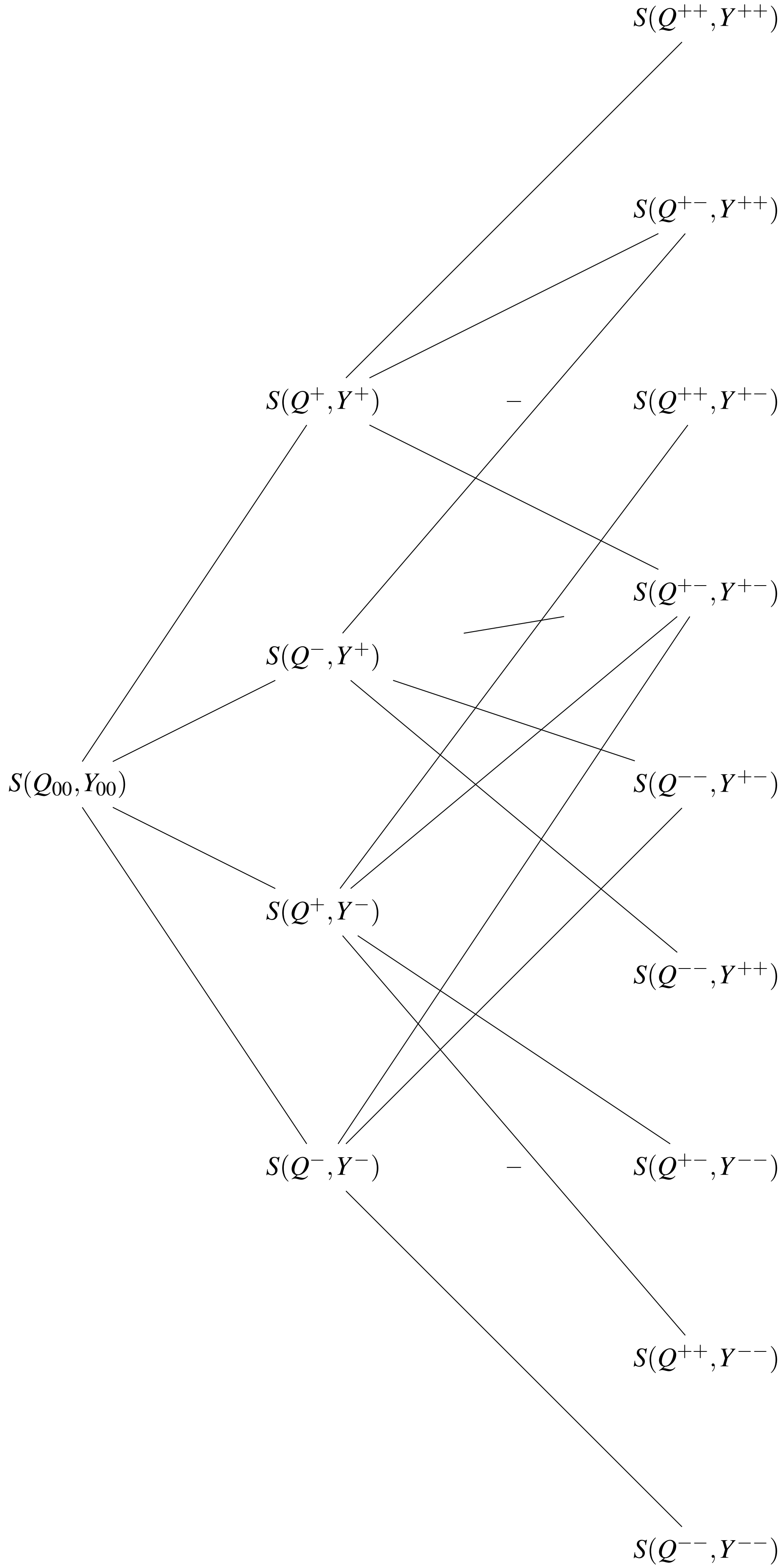


Figure 2: The recombined Tree for Y



To arrive at the original process S the inverse transformations are performed. Since the above trees recombine now we can construct the bivariate binomial tree in the $Y \times Q$ space. This results in a path-independent tree that has $(n+1)^2$ nodes. The binomial tree for S is shown in the figure below

Figure 3: The Tree for S under Transformation



3.8 Monte-Carlo Simulation

A Monte-Carlo simulation method simulates different paths for the price of the underlying asset under Q. This method also sets the value of the option equal to the mean of the payoff generated by each path and discounted to time zero. Although Monte-Carlo simulation method is time consuming, it is easy to apply it in different options including those whose price depends on the entire path of the underlying asset. In addition, this method is applicable for the options that involve more than one stochastic process.

This section shows an example of Monte-Carlo simulation whereby this method is applied to the Black-Scholes model that has time-dependent interest rate. The function for this differential equation is:

$$dS_t = r(t)S_t dt + \sigma S_t dW_t, \quad S_0 = s \tag{3.27}$$

The simulation of a path of S under this model can be done by using the discrete time evolution to approximate the SDE

$$S_{t+\Delta t} = S_t + r(t)S_t\Delta t + \sigma S_t Z(t)\sqrt{\Delta t}S(0) = s \tag{3.28}$$

With $Z(t) \sim N(0,1)$

When Monte-Carlo method is applied to this model, it yields n random values with standard normal distribution. The values have a period of time $(i\Delta t, (i+1)\Delta t]$ to obtain one value of S_T. The max payoff given by a call option is $(S_T - K, 0) = (S_T - K)^+$ and the payoff of a put option is $(K - S_T)^+$. These payoffs occur at the end of the European option's life time. The final step of the Monte-Carlo method involves taking the mean of these discounted payoffs to obtain the approximate price of the option. The fair value of an option in the Black-Scholes world is given by the present value of the anticipated payoff at expiry date. This payoff occurs under a risk-neutral random walk for underlying.

The following simple steps are used in Monte-Carlo simulation:

- A random walk model in risk-neutral world is used to generate the path of stock
- Obtains payoff
- Perform similar realizations over time horizon
- Get the expected payoff by calculating the average payoff for all realization
- The approximation of the option price is the present value of this average payoff

3.9 Estimating the Greeks from the Tree Model

The Greeks also known as hedge ratios measures the sensitivity of the option price to the underlying risk factor. The Black-Scholes model provides an easy way to estimate the price sensitivity to various parameters in the model. This is because the price is a function of the time to maturity T, the spot price S, the volatility σ, the strike price X and the risk free rate r. Using Taylor series expansion any sensitivity of price to changes in these parameters can easily be estimated.

From a binomial tree the Greeks can only be estimated using finite difference. The key function to help us estimate these sensitivities are:

$$\frac{df}{dx}(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{3.29}$$

$$\begin{aligned} \frac{d^2 f}{dx^2}(x) &\approx \frac{\frac{f(x+\Delta x)-f(x)}{\Delta x} - \frac{f(x)-f(x-\Delta x)}{\Delta x}}{\Delta X} \\ &\approx \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2} \end{aligned} \tag{3.30}$$

Delta Δ
This is the change in call (put) prices for a given change in the spot asset S. An estimation of the delta can be computed directly from the tree at any node as shown. This parameter is specified in underlying or premium asset units depending on the position we hold in options (bought or sold). As they are changing parameters, the delta is only significant in a certain period of time, having to be updated periodically.

$$\Delta_{0,0} = \frac{C_{1,1} - C_{1,0}}{S_{1,1} - S_{1,0}}$$

For purchase options, this parameter is positive, since a increase in the price of the underlying asset increases the call price. Conversely, an increase in the underlying asset leads to a reduction in the put price, which means that the call option maintains a negative delta position. The underlying asset, by definition, has a positive delta.

A delta hedging means obtaining a neutral delta in our portfolio of assets. Then, the combination of both call and put options (both bought and sold) and the position in the underlying (both bought and sold) must provide us with a neutral delta position.

Gamma (Γ)
This measures the rate of change in Delta with respect to the rate of change in the price

of the underlying asset. We could say that gamma is a protection against large changes in the price of the underlying asset, while delta is a protection against small changes.

$$\Gamma = \frac{\partial^2 C}{\partial S^2} \approx \frac{\left[\frac{C_{2,2}-C_{2,1}}{S_{2,2}-S_{2,1}}\right] - \left[\frac{C_{2,1}-C_{2,0}}{S_{2,1}-S_{2,0}}\right]}{\frac{1}{2}(S_{2,2}-S_{2,0})}$$

Theta (Θ)

This is the change in option pricing with respect to time. Since the bivariate binomial tree recombines we can approximate Theta as follows.

$$\Theta = \frac{C_{2,1} - C_{0,0}}{2\Delta_t}$$

Vega (ν)

This is the rate of change of the price of an option with respect to the changes in the volatility of the underlying asset. Vega try to measure portfolio price developments in terms of volatility. Like gamma, it then tries to protect the portfolio against major changes in the price of the underlying asset.

$$\nu \approx \frac{C(\sigma+\Delta\sigma) - C(\sigma-\Delta\sigma)}{2\Delta_\sigma}$$

Rho (ρ)

Rho is the derivative with respect to the rate of risk-free interest.

$$\rho \approx \frac{C(r+\Delta r) - C(r-\Delta r)}{2\Delta_r}$$

4 Chapter Four: Data Analysis and Results

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4.1 Introduction

In this chapter I present the results obtained after fitting the bivariate binomial tree model. On the basis of the assumption that there is a non-changing probability over time and a discrete time frame the results are compared against the Black-Scholes model, Hull and White model and the Monte Carlo Simulation.

4.2 Binomial Solution

To ensure that the price converges to a fair price value, the numbers of time steps are increased. The main steps to follow in this method is

i Transform the initial processes S and V to the required form Q and Y respectively.

ii Generate the trees from the transformed processes.

iii At each node transform back to S and V using the inverse transformation

iv Evaluate the payoff and get the highest at each node and then discount using backward induction to get the present value of the option.

4.3 Numeric example

To fit the model I have considered the parameters used in the Hilliard-Schwartz model. For various money-ness ratio the price is computed assuming a risk free rate of 5%, 0.5 years time to maturity, 15% stock volatility and 25% volatility parameter of the volatility diffusion process.

The bivariate binomial tree method is used in this thesis to obtain the values of European puts. If are fully observed, a person could realize that the values arrived at are close to the values of Hull-White, Monte Carlo simulation, and those of Black-Scholes. From the values, bivariate binomial tree method differs from other binomial methods in a value less than 0.1. The Monte Carlo simulation values match the above values and there is a little

Table 1. European Put Prices

S/X X=100	Black-Scholes Model	Hull-White Model	Bivariate Binomial	Monte-Carlo N=10^5	Monte-Carlo N=10^6	Monte-Carlo N=10^7
0.8	17.6431	17.6451	17.6783	17.6432	17.6476	17.6475
0.84	13.8763	13.8783	13.9315	13.8945	13.8799	13.8786
0.88	10.3983	10.3971	10.4591	10.3862	10.4004	10.3993
0.92	7.3656	7.3623	7.4092	7.3654	7.3524	7.3570
0.96	4.9036	4.8985	4.9049	4.8937	4.8864	4.8898
1	3.0585	3.0531	3.0153	3.0529	3.0409	3.0401
1.04	1.7841	1.7825	1.7404	1.7698	1.7754	1.7746
1.08	0.9753	0.9757	0.9504	0.9753	0.9726	0.9743
1.12	0.5341	0.5012	0.4775	0.5008	0.5044	0.5038
1.16	0.2411	0.2443	0.229	0.2485	0.2484	0.2488
1.2	0.1112	0.1123	0.1091	0.1182	0.1172	0.1168

difference between Monte Carlo simulation, Hull- White, and Black-Scholes values. This difference is less than ± 0.01.

The values are obtained with 300 time steps. The volatility diffusion parameter (b) is 25%. The correlation between price and volatility is zero. The parameters used in pricing of the puts prices are: risk-free is 5% rate, 0.5 year maturity time, 15% stock volatility, and the value of exercise price is $100.

4.4 Effects of changes in correlation

European put options with long maturity can be affected by stochastic volatility as illustrated in the table that follows. Here the volatility is correlated with the underlying price of asset. The value of the European put option shown has a constant stochastic volatility parameter. At various correlation levels the put values are calculated which is compared against various models. The values obtained at various correlations have slight deviation from the values obtained by the Black-Scholes model. This shows that correlation has an effect on the obtained prices. The values from the Black-Scholes model when the correlation is negative show that the model overprices in-the-money puts options. Also when the correlation is positive there is overpricing in the out-of-money.

Table 2. $\rho_{sv} = -0.50$

S/X	Black Scholes Model	Correlation = -0.50	
X=100		Hilliard and Schwartz	Bivariate Binomial Model
0.8	15.7161	14.267	15.7476
0.85	12.8482	11.465	12.5868
0.9	10.3933	9.2173	9.955
0.95	8.3253	7.4512	7.9368
1	6.6112	6.0734	6.3011
1.05	5.2083	4.9993	4.9053
1.1	4.0744	4.1594	3.8004
1.15	3.1673	3.4956	2.987
1.2	2.449	2.965	2.376

Table 3. $\rho_{sv} = -0.25$

S/X	Black- Scholes Model	Correlation = -0.25	
X=100		Hilliard and Schwartz	Bivariate Binomial Model
0.8	15.716	14.734	15.9532
0.85	12.848	11.815	13.1263
0.9	10.393	9.423	10.4334
0.95	8.325	7.511	8.0619
1	6.611	6.013	6.4324
1.05	5.208	4.848	5.1443
1.1	4.074	3.944	3.9978
1.15	3.167	3.241	2.9661
1.2	2.449	2.688	2.2605

Table 4. $\rho_{sv} = 0$

S/X	Black-Scholes Model	Correlation = 0	
X=100		Hilliard and Schwartz	Bivariate Binomial Model
0.8	15.716	15.1452	15.8232
0.85	12.848	12.114	13.0572
0.9	10.393	9.581	10.6938
0.95	8.325	7.518	8.319
1	6.611	5.893	6.056
1.05	5.208	4.626	5.0036
1.1	4.074	3.655	4.0464
1.15	3.167	2.914	3.1141
1.2	2.449	2.341	2.1989

Table 5. $\rho_{sv} = 0.25$

S/X	Black-Scholes Model	Correlation = 0.25	
X=100		Hilliard and Schwartz	Bivariate Binomial Model
0.8	15.716	15.4991	15.897
0.85	12.848	12.3512	12.7746
0.9	10.393	9.6835	10.4087
0.95	8.325	7.4646	8.2785
1	6.611	5.6943	6.2736
1.05	5.208	4.3191	4.6571
1.1	4.074	3.2748	3.7123
1.15	3.167	2.4925	2.9328
1.2	2.449	1.9122	2.2014

Table 6. $\rho_{sv} = 0.50$

S/X	Black-Scholes Model	Correlation = 0.50	
X=100		Hilliard and Schwartz	Bivariate Binomial Model
0.8	15.716	15.796	15.7668
0.85	12.848	12.538	12.7522
0.9	10.393	9.721	10.0662
0.95	8.325	7.338	7.8827
1	6.611	5.409	6.1161
1.05	5.208	3.905	4.5511
1.1	4.074	2.775	3.2651
1.15	3.167	1.961	2.3614
1.2	2.449	1.385	1.8019

4.5 American put option

Among the factors that generalizes the bivariate binomial model as compared to other methods is it can evaluate American options when considering a stochastic volatility. It is possible to price the American put options when the underlying asset and volatility correlate. Since it is not possible to evaluate American options using the Black –Scholes formula, and the Monte-Carlo method has difficulty as a result of the embedded forward simulation algorithm to capture the early exercise premium that is exhibited by the American options, then we consider using CRR binomial tree as a benchmark for our model. In this case, a comparison of the early exercise value and the end value of the option at maturity is made at each node and then picking the highest value from the two.

The figures below illustrate the value of an American put option that has a parameter of stochastic volatility of 1. In addition, the correlation between the volatility and price at various points is considered. 500 time steps are used as an estimate of the bivariate binomial values. In this exhibit, it is seen that there is a close relation between the values calculated by the bivariate binomial tree and those given by Hilliard and Schwartz (1996). When these values are compared with those obtained by CRR model, it is seen that the CRR model has a higher price than in-the-money puts. This case happens when the correlation is negative and the out-of-money when the correlation is positive.

Table 7. $\rho_{sv} = -0.50$

S/X	CRR Binomial Tree	correlation = -0.50
X=100		Bivariate Model
0.80	20.000	20.0000
0.85	15.020	15.0000
0.90	10.668	10.3857
0.95	7.2260	6.8253
1.00	4.649	4.3683
1.05	2.864	2.7325
1.10	1.675	1.6576
1.15	0.927	0.9961
1.20	0.500	0.6010

Table 8. $\rho_{sv} = -0.25$

S/X	CRR Binomial Tree	Correlation = -0.25
X=100		Bivariate Model
0.80	20.000	20.0000
0.85	15.020	15.0000
0.90	10.668	10.4973
0.95	7.226	6.8686
1.00	4.649	4.2006
1.05	2.864	2.6736
1.10	1.675	1.6107
1.15	0.927	0.8882
1.20	0.500	0.5251

Table 9. $\rho_{sv} = 0$

S/X	CRR Binomial Tree	Correlation = 0
X=100		Bivariate Model
0.80	20.000	20.0000
0.85	15.020	15.0000
0.90	10.668	10.4443
0.95	7.226	7.1015
1.00	4.649	3.8561
1.05	2.864	2.6229
1.10	1.675	1.6896
1.15	0.927	0.7824
1.20	0.501	0.4713

Table 10. $\rho_{sv} = 0.25$

S/X	CRR Binomial Tree	Correlation = 0.25
X=100		Bivariate Model
0.80	20.000	20.0000
0.85	15.021	15.0000
0.90	10.658	10.4751
0.95	7.236	7.0674
1.00	4.648	4.1368
1.05	2.863	2.4744
1.10	1.678	1.5925
1.15	0.928	0.8502
1.20	0.501	0.4234

Table 11. $\rho_{sv} = 0.50$

S/X	CRR Binomial Tree	Correlation = 0.50
X=100		Bivariate Model
0.80	20.000	20.0000
0.85	15.021	15.0244
0.90	10.668	10.6129
0.95	7.226	7.0567
1.00	4.649	4.3218
1.05	2.864	2.4137
1.10	1.675	1.3549
1.15	0.927	0.7844
1.20	0.501	0.4078

4.6 Sensitivities on European Options

A comparison on the bivariate binomial calculation of delta, Vega, and gamma with Black-Scholes calculations is shown below.

The computations of the three sensitivities are done using the volatilities of 15%, 20%, and 25%. In addition, the option has an interest rate of 5%, moneyness ratio of the range 0.8 to 1.2, and a small volatility parameter value of volatility diffusion of 0.0001. The reason for choosing a small volatility parameter is to enable a comparison of the results with the corresponding parameters from the Black-Scholes model. As shown in the table, bivariate binomial estimates correspond with the Black-Scholes parameters to the second or third decimal place.

Table 12. Delta

Moneyness Ratio	Volatility					
	Volatility = 0.15		Volatility = 0.20		Volatility = 0.25	
	Black-Scholes	Bivariate-Model	Black-Scholes	Bivariate-Model	Black-Scholes	Bivariate-Model
0.8	0.035	0.034	0.092	0.091	0.151	0.151
0.9	0.241	0.24	0.309	0.309	0.357	0.357
1.0	0.614	0.613	0.598	0.598	0.591	0.591
1.1	0.883	0.882	0.822	0.822	0.779	0.779
1.2	0.978	0.978	0.938	0.938	0.896	0.896

Table 13. Gamma

Moneyness Ratio	Volatility					
	Volatility = 0.15		Volatility = 0.20		Volatility = 0.25	
	Black-Scholes	Bivariate-Model	Black-Scholes	Bivariate-Model	Black-Scholes	Bivariate-Model
0.8	0.00905	0.00896	0.01455	0.0145	0.01655	0.01655
0.9	0.03261	0.03261	0.0277	0.02772	0.02345	0.02348
1.0	0.03608	0.03612	0.02736	0.02739	0.02198	0.02201
1.1	0.01691	0.01693	0.01677	0.01681	0.01527	0.01529
1.2	0.00418	0.00416	0.00722	0.00723	0.00849	0.00850

Table 14. Vega

Moneyness Ratio	Volatility					
	Volatility = 0.15		Volatility = 0.20		Volatility = 0.25	
	Black-Scholes	Bivariate-Model	Black-Scholes	Bivariate-Model	Black-Scholes	Bivariate-Model
0.8	0.044	0.043	0.093	0.093	0.132	0.133
0.9	0.198	0.198	0.224	0.225	0.237	0.238
1.0	0.271	0.272	0.274	0.274	0.275	0.279
1.1	0.153	0.148	0.203	0.193	0.231	0.225
1.2	0.045	0.048	0.104	0.105	0.153	0.153

Regular problems arise when the price of the asset is high (above 100), the volatility process has a large volatility (b greater than 50%), or when the maturity time of the options is long. However, it is possible to avoid these problems through scaling the underlying asset price appropriately. The asset and is adjusted and the price stricken to a new value while maintaining a well-behaved values in the tree. The option price obtained from the tree is rescaled in order to obtain the values of the option.

5 Chapter Five: Summary conclusion and recommendation

The bivariate binomial model presented has shown flexible possibility, compared to the Black-Scholes and Simulation approaches. This is because input parameters, such as exercise price and volatility, can be adjusted easily throughout the life of the option. Jumps can also be adjusted without major complexities.

In the proposed model, a generalized additive recombination and variable transition probabilities are obtained that determine the dynamic behavior of stochastic processes associated to the two processes, subject to appropriate constraints for the parameters. In addition, the generalized binomial tree scheme is presented as an alternative numerical method to evaluate asset options that can be modeled using the general linear stochastic differential equation with constant parameters or by any of its derivative processes. The rapid convergence of the the method with its absolute error rates and the results of the valuation of European put options for different processes with different expiration times in the "real world" and in a world of neutral risk are graphically illustrated.

A later work will allow us to study the convergence speed of the method for American put options. Likewise, this procedure can be extended to more general processes in which stochastic deterministic parameters are considered, for systems of two or three factors and even more complex stochastic models that include jumps.

I recommend more work to be done on American options to find a fair price for the put options and also investigation on sensitivities.

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