University of Nairobi

School of Mathematics

ISSN: 2410-1397

Master Project in Mathematics

Generating Probability Distributions Based on Burr Differential Equation

Research Report in Mathematics, Number 23, 2017

MOMANYI REINPETER ONDEYO
I56/74629/2014

December 2017

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Statistics

Master Project in Mathematics **University of Nairobi**
December 2017

Generating Probability Distributions Based on Burr Differential Equation

Research Report in Mathematics, Number 23, 2017

MOMANYI REINPETER ONDEYO
I56/74629/2014

School of Mathematics
College of Biological and Physical sciences
Chiromo, off Riverside Drive
30197-00100 Nairobi, Kenya

Master of Science Project
Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Statistics

Prepared for The Director
Graduate School
University of Nairobi

Monitored by School of Mathematics

Abstract

The differential equation approach was a significant method in the study of generating statistical distributions. It was first applied by Pearson in late 19th century when he constructed twelve density functions. Burr had in mind the Pearson family of distributions that was the only popular system in existence, when he constructed the twelve types of distribution functions.

In recent decades, there has been an increased interest in developing flexible statistical distributions. The common new method used is the exponentiated approach, where the cdf of an existing distribution is raised to a power of an additional parameter.

Traditionally, distributions of order statistics have been constructed using the transformation method. Here we have used new techniques of beta generated and beta exponentiated generated approach to construct distributions of order statistics.

Finally, we hope that the knowledge summarized in this study will help in the construction of more distributions based on Burr differential equation.

Dissertations in Mathematics at the University of Nairobi, Kenya.
ISSN 2410-1397: Research Report in Mathematics

DISTRIBUTOR: School of Mathematics, University of Nairobi, Kenya

Declaration and Approval

I the undersigned declare that this project report is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

MOMANYI REINPETER ONDEYO
Reg No. I56/74629/2014

In my capacity as a supervisor of the candidate, I certify that this report has my approval for submission.

Signature Date

Prof J.A.M Ottieno
School of Mathematics,
University of Nairobi,
Box 30197, 00100 Nairobi, Kenya.
E-mail: Joseph.otieno@uonbi.ac.ke

Dedication

I dedicate this project to my loving parents Yuvenalis and Violet Momanyi. A special feeling of gratitude to my siblings Joshua and John for their overwhelming social support and encouragement.

Contents

List of Tables

Acknowledgments

I wish to express my profound gratitude to my supervisor, Professor J.A.M Ottieno for his guidance, persistent encouragement, and support throughout the development and writing of this project.

The co-operation and ever-present assistance rendered by all staff in the School of Mathematics is also thankfully acknowledged.

My classmates of two years at the School of Mathematics made it all a fun journey, over and above the continuous learning from each other; and for that, I am immensely thankful.

I also wish to recognize the social support of my colleagues, friends and all those who helped and supported me in writing this project.

I am particularly grateful to the Directors of Phillips Healthcare Technologies Limited for study time extended to me throughout my years of study and through the process of writing this project.

Finally, i would like to thank the Almighty God for giving me strength and energy to finish this project.

MOMANYI REINPETER ONDEYO

Nairobi, 2017.

1 Introduction

1.1 Background Information

1.1.1 Families of Distributions

During the late nineteenth century all distributions were regarded as normal. The incompatibility of the normal distribution to explain theoretically and empirically many data situations forced the development of generalized frequency curves. Families of distributions functionally provide simple approximations to observe distributions in situations where it is difficult to derive a model. Since a trial and error approach to find out the appropriate model for the data is undesirable and time consuming, flexible systems of distributions must be evolved, which should incorporate, if not all, atleast the most common shapes that arise in practice.

The best way to understand the relevance of a particular system is through theoretical arguments which lead to the model. Their value also needs to be judged primarily on practical requirements such as; ease of computation, willingness to algebraic manipulation, richness in members, flexibility exhibited through the number of parameters in the system, easy methods of inferring the parameters and easy interpretation of the system through a defining equation.

With this aim in mind many families of distributions have been constructed in literature based on a differential equations. Such differential equations generate what is called **systems of frequency functions**. The oldest and most broadly studied system is the **Pearson system**, developed in 1895, 1901 and 1916 by Karl Pearson where he constructed twelve density functions based on a differential equation. The introduction of the Pearson system was a significant development because it provided a theoretical framework for various families of sampling distributions discovered subsequently by Pearson and others. After the early 1900s, the introduction of additional families of distributions and systems of frequency curves progressed sporadically. There was a continuing interest in methods of curve fitting and various attempts were made to extend the available systems to systems of frequency surfaces for fitting bivariate frequency data.

During the 1940s, three of the most common curve systems introduced were Burr system, Symmetric Tukey lambda distributions and Johnson (Translation) system. Since the 1950s, interest in systems of distributions for curve fitting has diminished due to the development of nonparametric density estimation methods which provide alternatives.

1.1.2 The Burr System

The Burr system was introduced by Irving W. Burr (1942), in an attempt to generate frequency functions that could be used in determining theoretical probabilities and expected frequencies. He developed the system with the aid of a differential equation involving distribution function $F(x)$, as in his opinion distribution functions are theoretically much better than density functions (employed in the Pearson system) as this has the advantage of closed forms, sometimes even for the quantile function.

Burr gave the following twelve solutions in table 1.1 (usually referred to by number). The Roman numeral description for the 12 types was first used by Johnson and Kotz (1970).

Table 1.1. The Burr System of Distributions

Type	$F(x)$	Support
I	x	$0<x<1$
II	$[e^{-x}+1]^{-r}$	$-\infty<x<\infty$
III	$[x^{-c}+1]^{-r}$	$0<x<\infty$
IV	$\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-r}$	$0<x<c$
V	$[ke^{-\tan x}+1]^{-r}$	$-\frac{\pi}{2}<x<\frac{\pi}{2}$
VI	$\left[ke^{-c\sinh x}+1\right]^{-r}$	$-\infty<x<\infty$
VII	$2^{-r}[1+\tanh x]^{r}$	$-\infty<x<\infty$
VIII	$\left[\frac{2}{\pi}\arctan e^{x}\right]^{r}$	$-\infty<x<\infty$
IX	$1-\frac{2}{c\left[(1+e^{x})^{k}-1\right]+2}$	$-\infty<x<\infty$
X	$\left[1-e^{-x^{2}}\right]^{r}$	$0<x<\infty$
XI	$\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^{r}$	$0<x<1$
XII	$1-[1+x^{c}]^{-k}$	$0<x<\infty$

where c, k and r are positive real numbers.

1.2 Definitions, Notations, Terminologies and Abbreviations

cdf cumulative distribution function, $F(x)$

pdf probability density function, $f(x)$

F^{-1} quantile function

$GB1$ generalized beta distribution of the first kind, $f(x) = \dfrac{ax^{ap-1}\left[1-\left(\frac{x}{b}\right)^{a}\right]^{q-1}}{b^{ap}B(p,q)}$

$GB2$ generalized beta distribution of the second kind, $f(x) = \dfrac{ax^{ap-1}}{b^{ap}B(p,q)\left[1+\left(\frac{x}{b}\right)^{a}\right]^{p+q}}$

$P(I)$ Pareto Type I distribution, $F(x) = 1-\left[\dfrac{x}{\sigma}\right]^{-\alpha}$

$P(II)$ Pareto Type II distribution, $F(x) = 1-\left[\dfrac{x+\sigma}{\sigma}\right]^{-\alpha}$

$P(III)$ Pareto Type III distribution, $F(x) = 1-\dfrac{\sigma e^{-\beta x}}{[\sigma+x]^{\alpha}}$

$P(IV)$ Pareto Type IV distribution, $F(x) = 1-\left[1+\left(\dfrac{x-\mu}{\sigma}\right)^{\frac{1}{\beta}}\right]^{-\alpha}$

$\Gamma(x)$ gamma function, $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$

$B(a,b)$ beta function, $B(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \int_0^\infty \dfrac{t^{a-1}}{(1+t)^{a+b}}dt$

$B(x;a,b)$ incomplete beta function, $B(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt$

$I_x(a,b)$ regularized incomplete beta function, $I_x(a,b) = \dfrac{B(x;a,b)}{B(a,b)}$

$\sinh x$ hyperbolic sine, $\sinh x = \dfrac{e^x-e^{-x}}{2} = \dfrac{e^{2x}-1}{2e^x} = \dfrac{1-e^{-2x}}{2e^{-x}}$

$\cosh x$ hyperbolic cosine, $\cosh x = \dfrac{e^x+e^{-x}}{2} = \dfrac{e^{2x}+1}{2e^x} = \dfrac{1+e^{-2x}}{2e^{-x}}$

$\tanh x$ hyperbolic tangent, $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x-e^{-x}}{e^x+e^{-x}} = \dfrac{e^{2x}-1}{e^{2x}+1} = \dfrac{1-e^{-2x}}{1+e^{-2x}}$

beta distribution of the first kind/Beta, $f(x) = \dfrac{x^{p-1}[1-x]^{q-1}}{B(p,q)}$

beta distribution of the second kind/Beta prime/Inverted Beta, $f(x) = \dfrac{x^{p-1}[1+x]^{-p-q}}{B(p,q)}$

1.3 Research Problem

The focus of this study is on re-examination of Burr's system. Solutions of Burr differential equation are expressed in terms of cumulative distribution functions.

Burr (1942) considered only twelve distribution functions known in literature as the Burr system of distributions, yet there are more than that in number.

Re-examining the Burr system we have realised that nine out of the twelve Burr distributions are powers of $cdf's$, popularly now known as exponentiated distributions. The other three are direct solutions in terms of $cdf's$.

This motivated us to generalize solutions of Burr differential equation by generator approach.

1.4 Study Objective

The main objective of this project is to construct distribution functions based on Burr differential equation.

1.4.1 Specific Objectives

1. To solve the Burr differential equation to obtain the twelve Burr distribution functions and more distribution functions.
2. To obtain Beta generated distributions.
3. To obtain Exponentiated generated distributions.
4. To obtain Beta-Exponentiated generated distributions.
5. To obtain minimum and maximum order statistic distributions.
6. To obtain moments of these distributions.

1.5 Literature Review

1.5.1 Solutions to ODE

The study of Ordinary Differential Equations began in 1675 with Gottfried Wilhelm von Leibniz (1646-1716). The theoretical development for general methods of integrating differential equations began when Isaac Newton (1642-1727) classified first order differential equations into three:

1. $\dfrac{dy}{dx} = f(x)$
2. $\dfrac{dy}{dx} = f(x,y)$
3. $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$

Class 1 and 2 contain only ordinary derivatives of one or more dependent variables with respect to a single independent variable and are today called **ordinary differential equations**. Class 3 involved the partial derivatives of one dependent variable and today are called **partial differential equations**.

Practically all original discoveries of almost all known methods of solving first order differential equations had been found by the end of the seventeenth century during the Bernoulli dynasty by James Bernoulli (1654-1705) and John Bernoulli (1667-1748).

1.5.2 Methods for Generating Families of Distributions

There has been a renewed interest in developing more flexible statistical distributions in recent decades. Kotz and Vicari (2005) highlighted milestones in early development of statistical distributions.

Systems of Frequency Functions

Methods developed before 1980s may be summarized into three categories:

1. **Differential equation Approach:** This approach was a major milestone in the methods for generating statistical distributions. Pearson's system was the most significant development. The Burr system was another important development.
   - **Pearson system** constructed to fit theoretical curves with widely varying shapes for obtaining improved approximations frequency data.
   - **Burr system** constructed to fit cumulative distributions rather than density functions to frequency data, to avoid numerical integration problems encountered when probabilities are evaluated from Pearson curves.
2. **Transformation Approach:** The **Johnson (Translation) system** developed by Norman L. Johnson (1949) was the milestone in this method. The system utilized four distinct monotone transformations which yield approximate normality when applied to skewed frequency data.

3. **Quantile Approach:** Hastings et al. (1947) proposed the use of quantile to generate the **Symmetric Tukey lambda distributions** to advance a study of order statistics. Tukey (1977) applied transformation with quantile to develop **Tukey's GH-system**.

Methods of Combination

After 1980, methods of generating new distributions shifted to adding parameters to an existing distribution or combining existing distributions into new distributions. The five general methods of combination and their variations are summarized as follows:

1. **Method of Generating Skew Distributions:** This is the most studied among the methods of combination in the modern era. The developments along this method continue to be very active. Azzalini (1985) was the first to propose the method of combining two symmetric distributions to form a skewed distribution. He introduced the skew normal family of distributions.

2. **Method of Adding Parameters To An Existing Distribution:** The popular method in this category is to raise the cdf of an existing distribution to a power of an additional parameter. This method of adding parameters can be applied to any family of distributions generated through the methods of combination to generate the **exponentiated family**.

3. **Beta-generated Method:** This method was pioneered by Eugene et al (2002). It uses the property that the cdf of a beta distribution and the beta random variable are between 0 and 1. By integrating the beta density from 0 to $F(x)$, the cdf of another random variable X, the method generates a new distribution for any $F(x)$.

4. **Transformed-transformer Method (T-X family):** This method was introduced by Alzaatreh et al (2013). It is a generalization of the beta generated method. Instead of using the beta distribution as the 'generator' to be 'transformed' through the 'transformer' $F(x)$, the method allows for any continuous pdf as the 'generator' to be 'transformed' through $W(F(x)) \in [a,b], -\infty \leq a < b \leq \infty$, a properly defined function of $F(x)$. The $W(.)$ satisfies the conditions for the cdf of the 'transformed' random variable T.

5. **Composite Method:** Cooray and Ananda (2005) proposed this method by combining two truncated $pdf's$ f_1 and f_2 defined on two domains overlapping at a point θ. The new distribution is determined by imposing the continuity and differentiability conditions $f_1(\theta) = f_2(\theta)$ and $f_1^{'}(\theta) = f_2^{'}(\theta)$ at θ to ensure that the combined distribution is a smooth pdf.

In this study, we focus on beta generator method, exponentiated generator method and a new beta-exponentiated generator method (a combination of beta and exponentiated generator methods) to construct new distributions .

1.5.3 Order Statistics

Developments in the field of order statistics from the early 1960's are summarized by Sarhan and Greenberg (1962). David (1981) gave an exciting encyclopedic representation of order statistics. An introductory level of order statistics was prepared by Ahsanullah et al. (2013).

In this study, we give a description of order statistics presenting distribution functions and exponentiated distribution functions with their moments.

1.6 Significance of the Study

1.6.1 Review of Results

The most widely known of the constructed distribution functions is the Burr System of distributions.

The Burr I consists of the uniform distribution whose density has a rectangular shape. The Burr III, IV, V, IX, and XII yield a variety of density shapes and involve four parameters, the most that can be efficiently estimated with the method of moments. Out of the twelve Burr distributions, the most researched members are Burr type III and type XII distributions. The two distributions are the simplest functionally and therefore most desirable for statistical modelling. Few papers have been written about the remaining types of the Burr distributions. However over the years there has been some interest in Burr type II, V and X distributions.

In addition to fitting frequency data, Burr system of distributions are useful for dealing with a number of statistical problems in which a class of distributions with functional simplicity and a variety of density shapes is required.

Burr XII Distribution

The Burr XII distribution is a versatile and flexible probability distribution, has a non-monotone hazard function and yields a useful range of values of skewness and kurtois as discussed in Hatke (1949), Burr (1968) and Burr and Cislak (1968). It has appeared in various literature under different names such as the Burr distribution in actuarial literature and Singh-Maddala distribution in economic literature.

Rodriguez (1977) and Tadikamalla (1980) showed that if the parameters are chosen appropriately, the Burr XII distribution contains the shape characteristics of the Burr II, Weibull, Normal, log-normal, Gamma (Pearson Type III), Logistic and Exponential (Pearson Type X) distributions, as well as a significant portion of the Pearson Types I (Beta), II, V, VII, IX and XII families.

The Burr XII distribution is related to various other distributions, namely, Lomax (Pareto Type II), Exponential-Gamma, Weibull-Gamma, Weibull-Exponential, Log-Logistic and GB2.

The distribution can be used to fit almost any given set of unimodal data by matching their mean and variances. It is very popular in modeling lifetime data and events with monotone failure rates. Other than the application to analysis of life time data, the Burr XII distribution is used in various fields such as survival(life table) analysis, finance, quality control, acceptance sampling and hydrology.

Examples of data modeled by the distribution are household income, crop prices, insurance risk, travel time and flood levels. McDonald and Ransom (1979a); Dagum (1983); Majumder and Chakravarty (1990); McDonald and Mantrala (1993, 1995); McDonald and Xu (1995); Bordley, McDonald, and Mantrala (1996); Henniger and Schmitz (1989); Bell, Klonner, and Moos (1999) fit a Singh–Maddala(Burr XII) distribution to income data. Hogg and Klugman (1983) fit the Burr XII distribution to 35 observations on hurricane losses in the United States; Cummins et al. (1990) fit the Burr XII distribution to aggregate fire losses using the data are provided in Cummins and Freifelder (1978).

The distribution function, inverse of the cumulative distribution function and the survival(reliability) function of the Burr XII distribution exist in simple closed form, thus simplifying the computation of the percentiles and the likelihood function for censored data. This fact enhances its applicability in simulation studies, quantal response and approximation of theoretical distributions whose moments are known, but whose functional forms cannot be expressed directly. Drane et al (1978) discussed the Burr XII distributions as response functions in analyzing quantal response experiments. Burr (1967) used the distribution to examine the effect of nonnormality on constants used in computing sample averages and ranges for plotting control charts.

The flexibility of the Burr XII distribution makes it a useful model in reliability studies. Cook and Johnson (1986) used the Burr XII model to obtain better fit to a uranium survey data set. Zimmer, Bert Keats and Wang (1998) discussed the statistical properties of the Burr XII distribution in reliability analysis. Woo and Ali (1998) calculated the moments and established some simple properties of hazard rate while Gupta et al (1996) obtained location of critical points for failure rate and mean residual life function.

The versatility of the Burr XII distribution turns it quite attractive as a tentative model for data whose underlying distribution is unknown. Wu et al (2007) studied the estimation problems for this distribution based on progressive type II censoring with random removals, where the number of units removed at each failure time has a discrete uniform distribution.

The Burr XII distribution has algebraic tails which are effective for modeling failure data that occur with less frequency. Dubey (1972, 1973); Evans and Simons (1975); Wingo (1983, 1993); Al-Hussaini and Jaheen (1992, 1994, 1995); Al-Hussaini, Jaheen and Mousa (1992) discussed the role, usefulness and properties of the Burr XII distribution as a failure model.

Burr III Distribution

Until the 1990s, the Burr III distribution was unknown in the economic journals. It has appeared in various literature under different names. In the actuarial literature as the inverse Burr distribution and in the economic literature as the Dagum distribution.

The Burr III distribution is related to various other distributions, namely, the Log-Logistic, the two parameter kappa, the three parameter kappa and the GB2. It contains the shape characteristics of the Weibull, reciprocal Weibull, Normal, Uniform, Logistic and Exponential (Pearson Type X) distributions.

The distribution function and inverse of the cumulative distribution function of the Burr III distribution exist in simple closed form. This fact plays an important role in the selection of a particular family of distributions as a stochastic model in simulation studies.

The Burr III distribution has been used to model a variety of data such as in forestry by Gove et al. (2008) and Lindsay et al. (1996), in fracture roughness by Nadarajah and Kotz (2006 and 2007), in life testing by Wingo (1983 and 1993), Operational risk Chernobai et al. (2007), in option market price distributions by Sherrick et al. (1996), in meteorology by Mielke (1973), in modeling crop prices by Tejeda and Goodwin (2008), in reliability quality control by Abdel-Ghaly et al. (1997), in geophysics by Clark, Cox and Laslett (1999), in simulation studies by Hasegawa and Kozumi (2003); Cowell and Victoria-Feser (2006) and in incomes.

Dagum and Lemmi (1989), Majumder and Chakravarty (1990), McDonald and Mantrala (1993, 1995), Victoria-Feser (1995, 2000), Bantilan et al. (1995), Bordley, McDonald, and Mantrala (1996), Botargues and Petrecolla (1997, 1999a,b), Pocock, McDonald and Pope (2003) fit the Dagum distribution to various income data.

The Dagum distribution provides a better fit to income data than the Singh-Maddala distribution. According to Kleiber (1996) this is so because the Dagum distibution has one extra parameter in the region where the majority of the data are.

In the actuarial literature, Cummins et al. (1990) fit the Dagum (Burr III) distribution to aggregate fire losses from Cummins and Freifelder (1978).

Benjamin et al. (2013) used Dagum distribution to model the maximum daily levels of tropospheric ozone.

Biewen and Jenkins (2005); Quintano and D'Agostino (2006) have used the Burr III distribution to model conditional distributions in a regression framework.

Burr II, V and X Distributions

The Burr II distribution has been used by Piorer (1980), Fry and orme (1998), Smith (1989) in binary choice model.

The Burr Type V distribution is still waiting for the interest of researchers. Many properties of the parameters of the distribution under different estimation procedures are still to be revealed. Recent study has shown it can be used to model lifetime data. Feroze and Aslam (2013) considered the maximum likelihood estimation of the Burr type V parameters under left censored samples.

The Burr X distribution has relations with the gamma, Weibull, exponentiated exponential and exponentiated Weibull distributions. This skewed distribution can be used quite effectively in analyzing lifetime data. Surles and Padjett (1998, 2001) considered the inference on reliability in stress strength model of Burr type X. Jaheen (1996), Sarwati and Abusalih (1991) considered the Bayesian estimation of Burr X model.

1.6.2 Present Study

A detailed study of using generator approach techniques to generate Burr distributions has not been undertaken in literature. With this aim in mind, the use of beta, exponentiated and beta exponentiated generator approach techniques has been proposed for generating distributions. It is interesting to know that over 75 distribution functions have been constructed out of which 69 are unique. This is more than the 12 proposed by Burr (1942).

In the following chapter, 5 cases of solving the Burr differential equation has been highlighted. In chapter 3, we obtain the Standard Uniform (Burr 1) distribution through a special case of $g(x)$. In Chapter 4, we concentrate on the Burr's assumption case which yields half of the 12 Burr System of distributions. In Chapter 5, we focus on the concept of income elasticity proposed by Stoppa (1990a). Chapter 6 highlights the case of reverse hazard function. Chapter 7 introduces the case of hazard function to generate distributions.

Finally, in Chapter 8, suggestions for further research on construction of more distribution functions are made.

2 Mathematical Formulation of the Problem

2.1 Introduction

A brief description of mathematical formulas which are going to be used in this research are given along with cases of Burr differential equation to be considered.

The solutions $F(x)$ obtained from the Burr differential equation are used to construct beta generated, exponentiated generated and beta-exponentiated distributions.

Order statistic distributions are special cases of beta generated and beta-exponentiated generated distributions.

2.2 Burr Differential Equation

Burr (1942) introduced the Burr system of distributions based on the differential equation of the form

$$y' = y(1-y)g(x,y) \tag{2.1}$$

where

$$y' = \frac{dy}{dx} = \frac{dF(x)}{dx} = f(x)$$

$$y = F(x)$$

and

$g(x,y)$ is a non negative function for $0 \leq y \leq 1$ and x in the range over which the solution is to be used.

The first problem is to solve the Burr differential equation for different cases of $g(x,y)$ to obtain the various distribution functions $F(x)$.

The cases to be considered are:

Case I: $g(x,y)=\dfrac{g(x)}{y(1-y)}$

Case II: $g(x,y)=g(x)$

Case III: $g(x,y)=\dfrac{g(x)}{xy}$

Case IV: $g(x,y)=\dfrac{r(x)}{(1-y)}$

Case V: $g(x,y)=\dfrac{\mu(x)}{y}$

2.3 Distributions Based on Generator Approach

Generator approach is one method of constructing probability distributions. In this research we are going to consider beta generator, exponentiated generator and beta-exponentiated generator approaches.

2.3.1 Beta Generator Approach

Eugene et al (2002) was the first to introduce the beta generated distribution through its cdf defined by,

$$W(y)=\int_0^y \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad a>0,\ b>0 \tag{2.2}$$

where $0\leq y\leq 1$ and $0\leq t\leq 1$

Since $0\leq F(x)\leq 1$ for $-\infty<x<\infty$, (2.2) can be rewritten by repalcing y with $F(x)$.

Thus

$$W[F(x)]=\int_0^{F(x)} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad a>0,\ b>0$$

which is a cdf of a cdf and a function of x.

Let

$$\begin{aligned} G(x) &= W[F(x)] \\ &= \int_0^{F(x)} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt \end{aligned} \quad (2.3)$$

Therefore

$$\begin{aligned} g(x) &= \frac{d}{dx} G(x) \\ &= \frac{d}{dx} W[F(x)] \\ &= \left\{ W'[F(x)] \right\} F'(x) \\ &= \left\{ W[F(x)] \right\} f(x) \\ &= \frac{[F(x)]^{a-1}[1-F(x)]^{b-1}}{B(a,b)} f(x) \end{aligned} \quad (2.4)$$

Alternatively,

$$g(x) = \frac{d}{dx} \int_0^{F(x)} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt$$

By Leibinitz principle of differentiation, we have

$$\begin{aligned} g(x) &= d \int_0^{F(x)} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dF(x) \\ &= \frac{1}{B(a,b)} \left[t^{a-1}(1-t)^{b-1} \right]_0^{F(x)} dF(x) \\ &= \frac{[F(x)]^{a-1}[1-F(x)]^{b-1}}{B(a,b)} f(x) \end{aligned}$$

which is a beta generated distribution.

i^{th} order statistic of $F(x)$

Jones (2004) derived a special case of (2.4) by putting $a = i$ and $b = n - i + 1$ and called it a **generalized i^{th} order statistic distribution** with following pdf

$$g_i(x) = \frac{[F(x)]^{i-1}[1-F(x)]^{n-i}}{B(i, n-i+1)} f(x) \tag{2.5}$$

The minimum order statistic is obtained by putting $i = 1$

$$g_1(x) = n[1-F(x)]^{n-1} f(x) \tag{2.6}$$

and the maximum order statistic is obtained by putting $i = n$

$$g_n(x) = n[F(x)]^{n-1} f(x) \tag{2.7}$$

2.3.2 Exponentiated Generator Approach

Let

$$G(x) = [F(x)]^r, \quad r > 0 \tag{2.8}$$

where $F(x)$ is the old/parent cdf and $G(x)$ is the new cdf.

Then $G(x)$ is an exponentiated distribution and its corresponding pdf is

$$g(x) = r[F(x)]^{r-1} f(x), \quad -\infty < x < \infty,\ r > 0 \tag{2.9}$$

2.3.3 Beta Exponentiated Generator Approach

A combination of beta generator and exponentiated generator gives a beta-exponentiated generator approach whose cdf is

$$
\begin{aligned}
G(x) &= W\left[F(x)\right]^{r} \\
&= \int\limits_{0}^{[F(x)]^{r}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt
\end{aligned}
\tag{2.10}
$$

Therefore

$$
\begin{aligned}
g(x) &= \frac{d}{dx} G(x) \\
&= \frac{d}{dx} W\left[F(x)\right]^{r} \\
&= \left\{W'\left[F(x)\right]^{r}\right\}\left\{r\left[F(x)\right]^{r-1}\right\} F'(x) \\
&= \left\{W\left[F(x)\right]^{r}\right\}\left\{r\left[F(x)\right]^{r-1}\right\} f(x) \\
&= \left\{\left[F(x)\right]^{r(a-1)}\right\}\left\{1-\left[F(x)\right]^{r}\right\}^{b-1} \frac{r\left[F(x)\right]^{r-1} f(x)}{B(a,b)} \\
&= \frac{r}{B(a,b)}\left[F(x)\right]^{ra-r+r-1}\left\{1-\left[F(x)\right]^{r}\right\}^{b-1} f(x) \\
&= \frac{r}{B(a,b)}\left[F(x)\right]^{ra-1}\left\{1-\left[F(x)\right]^{r}\right\}^{b-1} f(x)
\end{aligned}
\tag{2.11}
$$

Alternatively,

$$
\begin{aligned}
g(x) &= \frac{d}{dx} \int\limits_{0}^{[F(x)]^{r}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt \\
&= d \int\limits_{0}^{[F(x)]^{r}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} r\left[F(x)\right]^{r-1} dF(x) \\
&= \frac{r}{B(a,b)}\left[t^{a-1}(1-t)^{b-1}\right]_{0}^{[F(x)]^{r}}\left[F(x)\right]^{r-1} dF(x) \\
&= \frac{r}{B(a,b)}\left[F(x)\right]^{r(a-1)}\left\{1-\left[F(x)\right]^{r}\right\}^{b-1}\left[F(x)\right]^{r-1} f(x) \\
&= \frac{r}{B(a,b)}\left[F(x)\right]^{ra-r+r-1}\left\{1-\left[F(x)\right]^{r}\right\}^{b-1} f(x) \\
&= \frac{r}{B(a,b)}\left[F(x)\right]^{ra-1}\left\{1-\left[F(x)\right]^{r}\right\}^{b-1} f(x)
\end{aligned}
$$

i^{th} **order statistic of** $[F(x)]^r$

The i^{th} order statistic of the exponentiated distribution is given by the following pdf

$$g_i(x) = \frac{r[F(x)]^{ri-1}\{1-[F(x)]^r\}^{n-i}}{B(i,n-i+1)} f(x) \tag{2.12}$$

The minimum order statistic is obtained by putting $i = 1$

$$g_1(x) = rn[F(x)]^{r-1}\{1-[F(x)]^r\}^{n-1} f(x) \tag{2.13}$$

and the maximum order statistic is obtained by putting $i = n$

$$g_n(x) = rn[F(x)]^{rn-1} f(x) \tag{2.14}$$

2.4 Moments of the Distributions Based on Generator Approach

The m^{th} moment about the origin of a continous random variable X is given by

$$E(X^m) = \int x^m f(x)dx, \quad m = 1, 2, ... \tag{2.15}$$

2.4.1 m^{th} Moment of a Beta Generated Distribution

Using (2.4) as the pdf of a beta generated distribution,

$$E(X^m) = \int x^m \frac{[F(x)]^{a-1}[1-F(x)]^{b-1}}{B(a,b)} f(x)dx, \quad m = 1, 2, ... \tag{2.16}$$

m^{th} **Moment of the** i^{th} **order statistic of** $F(x)$

Using (2.5) as the pdf of the i^{th} order statistic of $F(x)$,

$$E(X_i^m) = \int x^m \frac{[F(x)]^{i-1}[1-F(x)]^{n-i}}{B(i,n-i+1)} f(x)dx, \quad m = 1, 2, ... \tag{2.17}$$

Putting $i = 1$ in (2.17) gives the m^{th} moment of the minimum order statistic of $F(x)$ as

$$E(X_1^m) = n\int x^m [1 - F(x)]^{n-1} f(x)dx, \quad m = 1, 2, ... \tag{2.18}$$

Putting $i = n$ in (2.17) gives the m^{th} moment of the maximum order statistic of $F(x)$ as

$$E(X_n^m) = n\int x^m [F(x)]^{n-1} f(x)dx, \quad m = 1, 2, ... \tag{2.19}$$

2.4.2 m^{th} Moment of an Exponentiated Generated Distribution

Using (2.9) as the pdf of an exponentiated generated distribution,

$$E(X^m) = r\int x^m [F(x)]^{r-1} f(x)dx, \quad m = 1, 2, ... \tag{2.20}$$

2.4.3 m^{th} Moment of a Beta-Exponentiated Generated Distribution

Using (2.11) as the pdf of a beta exponentiated generated distribution,

$$E(X^m) = r\int \frac{x^m [F(x)]^{ra-1} \{1 - [F(x)]^r\}^{b-1}}{B(a,b)} f(x)dx, \quad m = 1, 2, ... \tag{2.21}$$

m^{th} Moment of the i^{th} order statistic of $[F(x)]^r$

Using (2.12) as the pdf of the i^{th} order statistic of an exponentiated distribution,

$$E(X_i^m) = r\int x^m \frac{[F(x)]^{ri-1} \{1 - [F(x)]^r\}^{n-i}}{B(i, n-i+1)} f(x)dx, \quad m = 1, 2, ... \tag{2.22}$$

Putting $i = 1$ in (2.22) gives the m^{th} moment of the minimum order statistic of a an exponentiated distribution.

$$E(X_1^m) = nr\int x^m [F(x)]^{r-1} \{1 - [F(x)]^r\}^{n-1} f(x)dx, \quad m = 1, 2, ... \tag{2.23}$$

Putting $i = n$ in (2.22) gives the m^{th} moment of the maximum order statistic of an exponentiated distribution.

$$E(X_n^m) = nr\int x^m\,[F(x)]^{rn-1} f(x)dx, \quad m = 1,2,.... \tag{2.24}$$

3 Distributions based on Case I of Burr Differential Equation

This is the case when $g(x,y) = \dfrac{g(x)}{y(1-y)}$.

Therefore (2.1) becomes

$$y' = g(x)$$

implying that

$$\int dy = \int g(x)dx$$

Hence

$$y = \int g(x)dx$$

That is

$$F(x) = \int g(x)dx \tag{3.1}$$

3.1 $g(x) = 1$

Then (3.1) becomes

$$F(x) = \int dx$$

Substituting

$$\int dx = x$$

We get

$$F(x) = x$$

Now,

$$F(x) = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow x = 1$$

Therefore

$$F(x) = x, \quad 0 < x < 1 \tag{3.2}$$

and its corresponding pdf is

$$f(x) = 1, \quad 0 < x < 1 \tag{3.3}$$

which is the **Standard Uniform distribution** referred to as **Burr I distribution**.

The m^{th} moment of (3.3) is

$$\begin{aligned} E(X^m) &= \int_0^1 x^m dx \\ &= \left[\frac{x^{m+1}}{m+1}\right]_0^1 \\ &= \frac{1}{m+1} \\ &= B(m+1,1) \end{aligned} \tag{3.4}$$

3.1.1 Beta Standard Uniform Distribution

Putting (3.2) in (2.3) gives

$$G(x) = \int_0^x \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt \tag{3.5}$$

and its corresponding pdf is

$$g(x) = \frac{x^{a-1}\left[1-x\right]^{b-1}}{B(a,b)}, \quad 0 < x < 1, \ a > 0, \ b > 0 \tag{3.6}$$

which is the **Beta distribution**.

The m^{th} moment of (3.6) is

$$\begin{aligned} E(X^m) &= \int_0^1 x^m \frac{x^{a-1}[1-x]^{b-1}}{B(a,b)} dx \\ &= \int_0^1 \frac{x^{m+a-1}[1-x]^{b-1}}{B(a,b)} dx \\ &= \frac{B(m+a,b)}{B(a,b)} \end{aligned} \quad (3.7)$$

i^{th} order statistic of Standard Uniform distribution

From (3.6),

$$g_i(x) = \frac{x^{i-1}[1-x]^{n-i}}{B(i,n-i+1)}, \quad 0<x<1 \quad (3.8)$$

The m^{th} moment of (3.8) is

$$E(X_i^m) = \frac{B(m+i,n-i+1)}{B(i,n-i+1)} \quad (3.9)$$

Putting $i=1$ in (3.8),

$$g_1(x) = n[1-x]^{n-1}, \quad 0<x<1 \quad (3.10)$$

The m^{th} moment of (3.10) is

$$E(X_1^m) = nB(m+1,n) \quad (3.11)$$

Putting $i=n$ in (3.8),

$$g_n(x) = nx^{n-1}, \quad 0<x<1 \quad (3.12)$$

The m^{th} moment of (3.12) is

$$E(X_n^m) = nB(m+n,1) \quad (3.13)$$

3.1.2 Exponentiated Standard Uniform Distribution

Putting (3.2) in (2.8) gives

$$G(x) = x^r, \quad 0 < x < 1,\ r > 0 \tag{3.14}$$

and its corresponding pdf is

$$g(x) = rx^{r-1}, \quad 0 < x < 1,\ r > 0 \tag{3.15}$$

The m^{th} moment of (3.15) is

$$\begin{aligned} E(X^m) &= r\int_0^1 x^m x^{r-1} dx \\ &= r\int_0^1 x^{m+r-1}(1-x)^{1-1} dx \\ &= rB(m+r,1) \end{aligned} \tag{3.16}$$

3.1.3 Beta Exponentiated Standard Uniform Distribution

Putting (3.14) in (2.10) gives

$$G(x) = \int_0^{x^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < 1,\ a > 0,\ b > 0,\ r > 0 \tag{3.17}$$

and its corresponding pdf is

$$g(x) = \frac{rx^{ra-1}\left[1-x^r\right]^{b-1}}{B(a,b)}, \quad 0 < x < 1,\ a > 0,\ b > 0,\ r > 0 \tag{3.18}$$

which is **GB1** with shape parameters $p = a, q = b, a = r$ and scale parameter $b = 1$.

The m^{th} moment of (3.18) is

$$E(X^m) = r\int_0^1 x^m \frac{x^{ra-1}[1-x^r]^{b-1}}{B(a,b)}dx$$
$$= r\int_0^1 \frac{x^{m+ra-1}[1-x^r]^{b-1}}{B(a,b)}dx$$

Let

$$y = x^r \Longrightarrow dy = rx^{r-1}dx$$

Thus

$$\begin{aligned}
E(X^m) &= r\int_0^1 \frac{[y^{\frac{1}{r}}]^{m+ra-1}[1-y]^{b-1}}{B(a,b)r[y^{\frac{1}{r}}]^{r-1}}dy \\
&= \int_0^1 \frac{[y^{\frac{1}{r}}]^{m+ra-1-r+1}[1-y]^{b-1}}{B(a,b)}dy \\
&= \int_0^1 \frac{y^{\frac{m}{r}+a-1}[1-y]^{b-1}}{B(a,b)}dy \\
&= \frac{B\left(\frac{m}{r}+a,b\right)}{B(a,b)}, \quad r > m
\end{aligned} \tag{3.19}$$

i^{th} order statistic of Exponentiated Standard Uniform distribution

From (3.18),

$$g_i(x) = r\frac{x^{ri-1}[1-x^r]^{n-i}}{B(i,n-i+1)}, \quad 0 < x < 1,\ r > 0 \tag{3.20}$$

The m^{th} moment of (3.20) is

$$E(X_i^m) = \frac{B(\frac{m}{r}+i,n-i+1)}{B(i,n-i+1)} \tag{3.21}$$

Putting $i = 1$ in (3.20),

$$g_1(x) = rnx^{r-1}\left[1 - x^r\right]^{n-1}, \quad 0 < x < 1,\ r > 0 \tag{3.22}$$

The m^{th} moment of (3.22) is

$$E(X_1^m) = nB\left(\frac{m}{r} + 1, n\right), \quad r > m \tag{3.23}$$

Putting $i = n$ in (3.20),

$$g_n(x) = rnx^{rn-1}, \quad 0 < x < 1,\ r > 0 \tag{3.24}$$

The m^{th} moment of (3.24) is

$$E(X_n^m) = nB\left(\frac{m}{r} + n, 1\right), \quad r > m \tag{3.25}$$

4 Distributions based on Case II of Burr Differential Equation

4.1 Burr's assumption

This is the case when $g(x,y) = g(x)$.

Therefore (2.1) becomes

$$y' = y(1-y)g(x)$$

implying that

$$\begin{aligned}
\int \frac{dy}{y(1-y)} &= \int g(x)dx \\
\int \left\{ \frac{1}{y} + \frac{1}{(1-y)} \right\} dy &= \int g(x)dx \\
\log y - \log(1-y) &= \int g(x)dx \\
\log\left(\frac{y}{1-y}\right) &= \int g(x)dx \\
\frac{y}{1-y} &= \exp\left\{ \int g(x)dx \right\}
\end{aligned}$$

Therefore

$$y = (1-y)\exp\left\{ \int g(x)dx \right\}$$

Expanding this

$$y = \exp\left\{ \int g(x)dx \right\} - y\exp\left\{ \int g(x)dx \right\}$$

Which becomes

$$y\left[1 + \exp\left\{ \int g(x)dx \right\}\right] = \exp\left\{ \int g(x)dx \right\}$$

Hence

$$y = \frac{\exp\left\{\int g(x)dx\right\}}{\left[1+\exp\left\{\int g(x)dx\right\}\right]}$$
$$= \left[\frac{1}{\left\{\frac{1}{\exp\int g(x)dx}\right\}+1}\right]$$
$$= \left[\frac{1}{\{\exp-\int g(x)dx\}+1}\right]$$

That is

$$F(x) = \left[e^{-\int g(x)dx}+1\right]^{-1} \quad \textbf{(4.1)}$$

which was obtained by Burr (1942).

4.2 $g(x) = 1$

Then (4.1) becomes

$$F(x) = \left[e^{-\int dx}+1\right]^{-1}$$

Substituting

$$\int dx = x$$

We get

$$F(x) = \left[e^{-x}+1\right]^{-1}$$
$$= \frac{1}{[e^{-x}+1]}$$

Now,

$$F(x) = 0 \Longrightarrow \frac{1}{[\infty+1]} = \frac{1}{\left[e^{-(-\infty)}+1\right]} = 0 \Longrightarrow x = -\infty$$

$$F(x) = 1 \Longrightarrow \frac{1}{[0+1]} = \frac{1}{\left[e^{-(\infty)}+1\right]} = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = \left[e^{-x}+1\right]^{-1}, \quad -\infty < x < \infty \tag{4.2}$$

and its corresponding pdf is

$$\begin{aligned} f(x) &= e^{-x}\left[e^{-x}+1\right]^{-2} \\ &= \frac{e^{-x}}{\left[e^{-x}+1\right]^{2}}, \quad -\infty < x < \infty \end{aligned} \tag{4.3}$$

which is the **Logistic distribution**.

4.2.1 Beta Logistic Distribution

Putting (4.2) in (2.3) gives

$$G(x) = \int_{0}^{\left[e^{-x}+1\right]^{-1}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad -\infty < x < \infty,\ a > 0,\ b > 0 \tag{4.4}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{\left[e^{-x}+1\right]^{-a+1}\left[1-\left(e^{-x}+1\right)^{-1}\right]^{b-1}}{B(a,b)} \frac{e^{-x}}{\left[e^{-x}+1\right]^{2}} \\ &= \frac{\left[1-\frac{1}{\left(e^{-x}+1\right)^{1}}\right]^{b-1}}{\left[e^{-x}+1\right]^{a-1}B(a,b)} \frac{e^{-x}}{\left[e^{-x}+1\right]^{2}} \\ &= \frac{\left[e^{-x}+1-1\right]^{b-1}}{B(a,b)} \frac{e^{-x}}{\left[e^{-x}+1\right]^{a-1+b-1+2}} \\ &= \frac{e^{-bx}}{B(a,b)\left[e^{-x}+1\right]^{a+b}}, \quad -\infty < x < \infty,\ a > 0,\ b > 0 \end{aligned} \tag{4.5}$$

which is the **Type IV Generalized Logistic distribution**.

i^{th} order statistic of Logistic distribution

From (4.5),

$$g_i(x) = \frac{e^{-(n-i+1)x}}{B(i, n-i+1)\,[e^{-x}+1]^{n+1}}, \quad -\infty < x < \infty \tag{4.6}$$

Putting $i = 1$ in (4.6),

$$g_1(x) = \frac{ne^{-nx}}{[e^{-x}+1]^{n+1}}, \quad -\infty < x < \infty \tag{4.7}$$

Putting $i = n$ in (4.6),

$$g_n(x) = \frac{ne^{-x}}{[e^{-x}+1]^{n+1}}, \quad -\infty < x < \infty \tag{4.8}$$

4.2.2 Exponentiated Logistic Distribution

Putting (4.2) in (2.8) gives

$$G(x) = \left[e^{-x}+1\right]^{-r}, \quad -\infty < x < \infty,\ r > 0 \tag{4.9}$$

and its corresponding pdf is

$$g(x) = \frac{re^{-x}}{[e^{-x}+1]^{r+1}}, \quad -\infty < x < \infty,\ r > 0 \tag{4.10}$$

which is the **Type I Generalized Logistic/ Burr II distribution**.

4.2.3 Beta Exponentiated Logistic Distribution

Putting (4.9) in (2.10) gives

$$G(x)=\int_{0}^{\left[e^{-x}+1\right]^{-r}}\frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt,\quad -\infty<x<\infty,\ a>0,\ b>0,\ r>0 \tag{4.11}$$

and its corresponding pdf is

$$\begin{aligned}
g(x)&=\frac{r\left[e^{-x}+1\right]^{-ra+1}\left\{1-\left[e^{-x}+1\right]^{-r}\right\}^{b-1}}{B(a,b)}\frac{e^{-x}}{\left[e^{-x}+1\right]^{2}}\\
&=\frac{r\left[1-\dfrac{1}{(e^{-x}+1)^{r}}\right]^{b-1}}{\left[e^{-x}+1\right]^{ra-1}B(a,b)}\frac{e^{-x}}{\left[e^{-x}+1\right]^{2}}\\
&=\frac{re^{-x}\left\{\left[e^{-x}+1\right]^{r}-1\right\}^{b-1}}{\left[e^{-x}+1\right]^{ra-1+rb-r+2}B(a,b)}\\
&=\frac{re^{-x}\left\{\left[e^{-x}+1\right]^{r}-1\right\}^{b-1}}{B(a,b)\left[e^{-x}+1\right]^{r(a+b-1)+1}},\quad -\infty<x<\infty,\ a>0,\ b>0,\ r>0
\end{aligned} \tag{4.12}$$

i^{th} order statistic of Exponentiated Logistic distribution

From (4.12),

$$g_i(x)=\frac{re^{-x}\left\{\left[e^{-x}+1\right]^{r}-1\right\}^{n-i}}{B(i,n-i+1)\left[e^{-x}+1\right]^{rn+1}},\quad -\infty<x<\infty,\ r>0 \tag{4.13}$$

Putting $i=1$ in (4.13),

$$g_1(x)=\frac{rne^{-x}\left\{\left[e^{-x}+1\right]^{r}-1\right\}^{n-1}}{\left[e^{-x}+1\right]^{rn+1}},\quad -\infty<x<\infty,\ r>0 \tag{4.14}$$

which is **Type I Beta Generalized Logistic distribution** with parameters $a=1, p=r, b=n$.

Putting $i=n$ in (4.13),

$$g_n(x) = \frac{nre^{-x}}{[e^{-x}+1]^{rn+1}}, \quad -\infty < x < \infty,\ r > 0 \tag{4.15}$$

4.3 $g(x) = \frac{c}{x}, \quad c > 0$

Then (4.1) becomes

$$F(x) = \left[e^{-\int \frac{c}{x} dx} + 1\right]^{-1}$$

Substituting

$$\int \frac{c}{x} dx = c \log x$$

We get

$$\begin{aligned} F(x) &= \left[e^{-c\log x} + 1\right]^{-1} \\ &= \left[e^{logx^{-c}} + 1\right]^{-1} \\ &= \left[x^{-c} + 1\right]^{-1} \\ &= \frac{1}{\frac{1}{x^c} + 1} \end{aligned}$$

Now,

$$F(x) = 0 \Longrightarrow \frac{1}{\infty + 1} = \frac{1}{\frac{1}{0^c} + 1} = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow \frac{1}{0 + 1} = \frac{1}{\frac{1}{\infty^c} + 1} = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = \left[x^{-c} + 1\right]^{-1}, \quad 0 < x < \infty \tag{4.16}$$

and its corresponding pdf is

$$\begin{aligned} f(x) &= cx^{-c-1}\left[x^{-c}+1\right]^{-2} \\ &= \frac{cx^{-c-1}}{[x^{-c}+1]^2} \\ &= \frac{cx^{-c-1+2c}}{[x^{c}+1]^2} \\ &= \frac{cx^{c-1}}{[x^{c}+1]^2}, \quad 0<x<\infty \end{aligned} \tag{4.17}$$

which is the **Log-Logistic distribution** (known as the **Fisk distribution** in economics).

The m^{th} moment of (4.17) is

$$\begin{aligned} E(X^m) &= \int_0^\infty x^m \frac{cx^{c-1}}{[x^c+1]^2}dx \\ &= c\int_0^\infty \frac{x^{m+c-1}}{[x^c+1]^2}dx \end{aligned}$$

Let

$$y = x^c \Longrightarrow dy = cx^{c-1}dx$$

Thus

$$\begin{aligned} E(X^m) &= c\int_0^\infty \frac{[y^{\frac{1}{c}}]^{m+c-1}}{[1+y]^{1+1}\,c[y^{\frac{1}{c}}]^{c-1}}dy \\ &= \int_0^\infty \frac{[y^{\frac{1}{c}}]^{m}}{[1+y]^{1+1}}dy \\ &= \int_0^\infty \frac{y^{\left(1+\frac{m}{c}\right)-1}}{[1+y]^{\left(1+\frac{m}{c}\right)+\left(1-\frac{m}{c}\right)}}dy \\ &= B\left(1+\frac{m}{c},1-\frac{m}{c}\right), \quad c>m \end{aligned} \tag{4.18}$$

4.3.1 Beta Log-Logistic Distribution

Putting (4.16) in (2.3) gives

$$G(x) = \int_{0}^{\left[x^{-c}+1\right]^{-1}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < \infty,\ a > 0,\ b > 0 \tag{4.19}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{[x^{-c}+1]^{-a+1}\left[1-(x^{-c}+1)^{-1}\right]^{b-1}}{B(a,b)} \frac{cx^{-c-1}}{[x^{-c}+1]^2} \\ &= \frac{\left[1-\frac{1}{(x^{-c}+1)^1}\right]^{b-1}}{[x^{-c}+1]^{a-1}B(a,b)} \frac{cx^{-c-1}}{[x^{-c}+1]^2} \\ &= \frac{[x^{-c}+1-1]^{b-1}}{B(a,b)} \frac{cx^{-c-1}}{[x^{-c}+1]^{a-1+b-1+2}} \\ &= \frac{cx^{-cb-1}}{B(a,b)\,[x^{-c}+1]^{a+b}} \\ &= \frac{cx^{-cb-1+ca+cb}}{B(a,b)\,[x^{c}+1]^{a+b}} \\ &= \frac{cx^{ca-1}}{B(a,b)\,[x^{c}+1]^{a+b}}, \quad 0 < x < \infty,\ a > 0,\ b > 0 \end{aligned} \tag{4.20}$$

The m^{th} moment of (4.20) is

$$\begin{aligned} E(X^m) &= \int_0^\infty x^m \frac{cx^{ca-1}}{B(a,b)\,[x^c+1]^{a+b}} dx \\ &= \frac{c}{B(a,b)} \int_0^\infty \frac{x^{m+ca-1}}{[x^c+1]^{a+b}} dx \end{aligned}$$

Let

$$y = x^c \Longrightarrow dy = cx^{c-1}dx$$

Thus

$$
\begin{aligned}
E(X^m) &= \frac{c}{B(a,b)} \int_0^\infty \frac{[y^{\frac{1}{c}}]^{m+ca-1}}{[1+y]^{a+b}\, c[y^{\frac{1}{c}}]^{c-1}} dy \\
&= \frac{1}{B(a,b)} \int_0^\infty \frac{[y^{\frac{1}{c}}]^{m+ca-c}}{[1+y]^{a+b}} dy \\
&= \frac{1}{B(a,b)} \int_0^\infty \frac{y^{\frac{m}{c}+a-1}}{[1+y]^{a+b}} dy \\
&= \frac{1}{B(a,b)} \int_0^\infty \frac{y^{\left(a+\frac{m}{c}\right)-1}}{[1+y]^{\left(a+\frac{k}{c}\right)+\left(b-\frac{m}{c}\right)}} dy \\
&= \frac{B\left(a+\dfrac{m}{c}, b-\dfrac{m}{c}\right)}{B(a,b)}, \;\; c > m
\end{aligned} \tag{4.21}
$$

i^{th} order statistic of Log-Logistic distribution

From (4.20),

$$
\begin{aligned}
g_i(x) &= \frac{cx^{-c(n-i+1)-1}}{B(i,n-i+1)\,[x^{-c}+1]^{n+1}} \\
&= \frac{cx^{ci-1}}{B(i,n-i+1)\,[x^{c}+1]^{n+1}}, \quad 0 < x < \infty
\end{aligned} \tag{4.22}
$$

The m^{th} moment of (4.22) is

$$
E(X_i^m) = \frac{B\left(i+\dfrac{m}{c}, n-i+1-\dfrac{m}{c}\right)}{B(i,n-i+1)}, \;\; c > m \tag{4.23}
$$

Putting $i = 1$ in (4.22),

$$
\begin{aligned}
g_1(x) &= \frac{ncx^{-cn-1}}{[x^{-c}+1]^{n+1}} \\
&= \frac{ncx^{c-1}}{[x^{c}+1]^{n+1}}, \quad 0 < x < \infty
\end{aligned} \tag{4.24}
$$

The m^{th} moment of (4.24) is

$$E(X_1^m) = nB\left(1+\frac{m}{c}, n-\frac{m}{c}\right), \quad c > m \tag{4.25}$$

Putting $i = n$ in (4.22),

$$\begin{aligned} g_n(x) &= \frac{ncx^{-c-1}}{[x^{-c}+1]^{n+1}} \\ &= \frac{ncx^{cn-1}}{[x^c+1]^{n+1}}, \quad 0 < x < \infty \end{aligned} \tag{4.26}$$

The m^{th} moment of (4.26) is

$$E(X_n^m) = nB\left(n+\frac{m}{c}, 1-\frac{m}{c}\right), \quad c > m \tag{4.27}$$

4.3.2 Exponentiated Log-Logistic Distribution

Putting (4.16) in (2.8) gives

$$G(x) = \left[x^{-c}+1\right]^{-r}, \quad 0 < x < \infty, \ r > 0 \tag{4.28}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= rcx^{-c-1}\left[x^{-c}+1\right]^{-r-1} \\ &= rcx^{-c-1+cr+c}\left[x^{c}+1\right]^{-r-1} \\ &= \frac{rcx^{cr-1}}{[x^c+1]^{r+1}}, \quad 0 < x < \infty, \ r > 0 \end{aligned} \tag{4.29}$$

which is the **Burr III distribution**.

The m^{th} moment of (4.29) is

$$E(X^m) = rc\int_0^\infty x^m \frac{x^{cr-1}}{[x^c+1]^{r+1}}dx$$
$$= rc\int_0^\infty \frac{x^{m+cr-1}}{[x^c+1]^{r+1}}dx$$

Let

$$y = x^c \Longrightarrow dy = cx^{c-1}dx$$

Thus

$$\begin{aligned}
E(X^m) &= rc\int_0^\infty \frac{[y^{\frac{1}{c}}]^{m+cr-1}}{[y+1]^{r+1}\, c[y^{\frac{1}{c}}]^{c-1}}dy \\
&= r\int_0^\infty \frac{\left[y^{\frac{1}{c}}\right]^{m+cr-c}}{[y+1]^{r+1}}dy \\
&= r\int_0^\infty \frac{y^{\frac{m}{c}+r-1}}{[y+1]^{r+1}}dy \\
&= r\int_0^\infty \frac{y^{\frac{m}{c}+r-1}}{[y+1]^{\left(\frac{m}{c}+r\right)+\left(1-\frac{m}{c}\right)}}dy \\
&= rB\left(r+\frac{m}{c}, 1-\frac{m}{c}\right), \quad c > m,\ r > 0
\end{aligned} \tag{4.30}$$

Extensions of Burr III distribution

Case I

Let

$$Y = \log X^c$$

where Y is Burr II distribed with cdf

$$F(y) = \left[e^{-y}+1\right]^{-r}, \quad -\infty < y < \infty$$

If $G(x)$ is the *cdf* of x, then

$$\begin{aligned} G(x) &= prob\ (X \leq x) \\ &= prob\ (e^{Y} \leq x^{c}) \\ &= prob\ (Y \leq \log x^{c}) \end{aligned}$$

Therefore

$$\begin{aligned} G(x) &= F(\log x^{c}) \\ &= \left[e^{-\log x^{c}} + 1\right]^{-r} \\ &= \left[e^{\log x^{-c}} + 1\right]^{-r} \\ &= \left[x^{-c} + 1\right]^{-r} \end{aligned}$$

Thus

$$G(x) = \left[1 + x^{-c}\right]^{-r}, \quad -\infty < x < \infty$$

and its corresponding pdf is

$$g(x) = rcx^{-c-1}\left[1 + x^{-c}\right]^{-r-1}, \quad 0 < x < \infty,\ r > 0$$

which is the same as (4.28) and (4.29), a result given by Johnson et al (1995).

Case II

Let

$$X = \alpha Y$$

where Y has Burr III distribution with cdf given by

$$G(y) = \left[1 + y^{-c}\right]^{-r}, \quad -\infty < x < \infty$$

If $F(x)$ is the *cdf* of x, then

$$\begin{aligned} F(x) &= prob\ (X \leq x) \\ &= prob\ (\alpha Y \leq x) \\ &= prob\ \left(Y \leq \frac{x}{\alpha}\right) \end{aligned}$$

Therefore

$$F(x) = G\left(\frac{x}{\alpha}\right)$$

Thus

$$F(x) = \left[1 + \left(\frac{x}{\alpha}\right)^{-c}\right]^{-r}, \quad -\infty < x < \infty \tag{4.31}$$

and its corresponding pdf is

$$f(x) = \frac{rc}{\alpha}\left(\frac{x}{\alpha}\right)^{-c-1}\left[1 + \left(\frac{x}{\alpha}\right)^{-c}\right]^{-r-1}, \quad -\infty < x < \infty \tag{4.32}$$

which is also another form of Burr III distribution known as the **Dagum distribution**, a result given by Dagum (1977).

Extension of Burr II distribution

Let

$$Y = c \log X, \quad c > 0$$

where X is Burr III distribed with cdf

$$F(x) = \left[x^{-c} + 1\right]^{-r}, \quad -\infty < x < \infty$$

If $G(y)$ is the *cdf* of y, then

$$\begin{aligned} G(y) &= prob\ (Y \leq y) \\ &= prob\ (c \log X \leq y) \\ &= prob\ (X^c \leq e^y) \\ &= prob\ (X \leq e^{\frac{y}{c}}) \end{aligned}$$

Therefore

$$G(y) = F(e^{\frac{y}{c}})$$

Thus

$$G(y) = \left[e^{-y} + 1\right]^{-r}, \quad -\infty < y < \infty$$

which is the same form as (4.9), a result given by Tadikamalla (1980).

4.3.3 Beta Exponentiated Log-Logistic Distribution

Putting (4.28) in (2.10) gives

$$G(x) = \int_{0}^{[x^{-c}+1]^{-r}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < \infty,\ a > 0,\ b > 0,\ r > 0 \tag{4.33}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{[x^{-c}+1]^{-ra+1}\left[1-(x^{-c}+1)^{-r}\right]^{b-1}}{B(a,b)} \frac{rcx^{-c-1}}{[x^{-c}+1]^{2}} \\ &= \frac{\left[1-\frac{1}{(x^{-c}+1)^{r}}\right]^{b-1}}{[x^{-c}+1]^{ra-1}B(a,b)} \frac{rcx^{-c-1}}{[x^{-c}+1]^{2}} \\ &= \frac{\left[(x^{-c}+1)^{r}-1\right]^{b-1}}{[x^{-c}+1]^{rb-r+ra-1}B(a,b)} \frac{rcx^{-c-1}}{[x^{-c}+1]^{2}} \\ &= \frac{\left[(x^{-c}+1)^{r}-1\right]^{b-1}}{B(a,b)} \frac{rcx^{-c-1}}{[x^{-c}+1]^{ra-1+rb-r+2}} \\ &= \frac{rcx^{-c-1}\left[(x^{-c}+1)^{r}-1\right]^{b-1}}{B(a,b)\,[x^{-c}+1]^{r(a+b-1)+1}}, \quad 0 < x < \infty,\ a > 0,\ b > 0,\ r > 0 \end{aligned} \tag{4.34}$$

i^{th} order statistic of Exponentiated Log-Logistic distribution

From (4.34),

$$g_i(x) = \frac{rcx^{-c-1}\left[(x^{-c}+1)^r - 1\right]^{n-i}}{B(i,n-i+1)\left[x^{-c}+1\right]^{rn+1}}, \quad 0 < x < \infty,\ r > 0 \tag{4.35}$$

Putting $i = 1$ in (4.35),

$$g_1(x) = \frac{rncx^{-c-1}\left[(x^{-c}+1)^r - 1\right]^{n-1}}{\left[x^{-c}+1\right]^{rn+1}}, \quad 0 < x < \infty,\ r > 0 \tag{4.36}$$

Putting $i = n$ in (4.35),

$$g_n(x) = \frac{rncx^{-c-1}}{\left[x^{-c}+1\right]^{rn+1}}, \quad 0 < x < \infty,\ r > 0 \tag{4.37}$$

The m^{th} moment of (4.37) is

$$\begin{aligned} E(X_n^m) &= nrc\int_0^\infty x^m \frac{x^{-c-1}}{\left[x^{-c}+1\right]^{rn+1}}dx \\ &= nrc\int_0^\infty \frac{x^{m-c-1}}{\left[x^{-c}+1\right]^{rn+1}}dx \\ &= nrc\int_0^\infty \frac{x^{m-c-1+crn+c}}{\left[x^{c}+1\right]^{rn+1}}dx \\ &= nrc\int_0^\infty \frac{x^{m-1+crn}}{\left[x^{c}+1\right]^{rn+1}}dx \end{aligned}$$

Let

$$y = x^c \Longrightarrow dy = cx^{c-1}dx$$

Thus

$$\begin{aligned}
E(X_n^m) &= rnc\int_0^\infty \frac{[y^{\frac{1}{c}}]^{m-1+crn}}{[y+1]^{rn+1}\, c[y^{\frac{1}{c}}]^{c-1}}dy \\
&= rn\int_0^\infty \frac{\left[y^{\frac{1}{c}}\right]^{m-1+crn-c+1}}{[y+1]^{rn+1}}dy \\
&= rn\int_0^\infty \frac{y^{\frac{m}{c}+rn-1}}{[y+1]^{rn+1}}dy \\
&= rn\int_0^\infty \frac{y^{\frac{m}{c}+rn-1}}{[y+1]^{\left(\frac{m}{c}+rn\right)+\left(1-\frac{m}{c}\right)}}dy \\
&= rnB\left(rn+\frac{m}{c}, 1-\frac{m}{c}\right), \quad c>m, \quad r>0
\end{aligned} \tag{4.38}$$

4.4 $g(x) = [(c-x)x]^{-1}, \quad c>0$

Rewriting

$$g(x) = \frac{1}{(c-x)x}$$

Using partial fraction method

$$\frac{A}{x} + \frac{B}{c-x} = \frac{A(c-x)+Bx}{x(c-x)}$$

Solving for A and B

$$1 \equiv Ac - Ax + Bx = Ac + (B-A)x$$
$$Ac = 1 \Longrightarrow A = \frac{1}{c}$$
$$B - A = 0 \Longrightarrow B = A = \frac{1}{c}$$

Thus

$$\begin{aligned}
\int g(x)dx &= \int \frac{dx}{x(c-x)}dx \\
&= \frac{1}{c}\int \left(\frac{1}{x} + \frac{1}{c-x}\right)dx \\
&= \frac{1}{c}\log\left(\frac{x}{c-x}\right)
\end{aligned}$$

Substituting in (4.1) we get

$$
\begin{aligned}
F(x) &= \left[e^{-\frac{1}{c}\log\left(\frac{x}{c-x}\right)}+1\right]^{-1} \\
&= \left[\exp\left\{\log\left(\frac{x}{c-x}\right)^{-\frac{1}{c}}\right\}+1\right]^{-1} \\
&= \left[\left(\frac{x}{c-x}\right)^{-\frac{1}{c}}+1\right]^{-1} \\
&= \left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-1} \\
&= \frac{1}{\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1}
\end{aligned}
$$

Now,

$$F(x)=0 \Longrightarrow \frac{1}{(\infty)^{\frac{1}{c}}+1} = \frac{1}{\left(\frac{c-0}{0}\right)^{\frac{1}{c}}+1} = 0 \Longrightarrow x=0$$

$$F(x)=1 \Longrightarrow \frac{1}{(0)^{\frac{1}{c}}+1} = \frac{1}{\left(\frac{c-c}{c}\right)^{\frac{1}{c}}+1} = 1 \Longrightarrow x=c$$

Therefore

$$F(x) = \left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-1}, \quad 0<x<c \tag{4.39}$$

and its corresponding pdf is

$$
\begin{aligned}
f(x) &= \left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-2}\left(\frac{c-x}{x}\right)^{\frac{1}{c}-1}x^{-2} \\
&= \left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-2}\frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}}, \quad 0<x<c
\end{aligned}
\tag{4.40}
$$

4.4.1 Beta Generated Distribution

Putting (4.39) in (2.3) gives

$$G(x) = \int_0^{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-1}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0<x<c,\ a>0,\ b>0 \tag{4.41}$$

and its corresponding pdf is

$$\begin{aligned}
g(x) &= \frac{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-a+1}\left[1-\left\{\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right\}^{-1}\right]^{b-1}}{B(a,b)}\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-2}\frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}} \\
&= \frac{\left[1-\frac{1}{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{1}}\right]^{b-1}}{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{a-1}B(a,b)}\frac{(c-x)^{\frac{1}{c}-1}}{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{2}x^{\frac{1}{c}+1}} \\
&= \frac{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1-1\right]^{b-1}}{B(a,b)}\frac{(c-x)^{\frac{1}{c}-1}}{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{a-1+b-1+2}x^{\frac{1}{c}+1}} \\
&= \frac{(c-x)^{\frac{1}{c}-1+\frac{b}{c}-\frac{1}{c}}}{B(a,b)\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{a+b}x^{\frac{1}{c}+1+\frac{b}{c}-\frac{1}{c}}} \\
&= \frac{(c-x)^{\frac{b}{c}-1}}{B(a,b)\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{a+b}x^{\frac{b}{c}+1}}, \quad 0<x<c,\ a>0,\ b>0
\end{aligned} \tag{4.42}$$

i^{th} order statistic of $F(x) = \left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-1}$

From (4.42),

$$g_i(x) = \frac{(c-x)^{\frac{n-i+1-c}{c}}}{B(i,n-i+1)\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{n+1} x^{\frac{n-i+1+c}{c}}}, \quad 0<x<c \tag{4.43}$$

Putting $i=1$ in (4.43),

$$g_1(x) = n\frac{(c-x)^{\frac{n}{c}-1}}{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{n+1} x^{\frac{n}{c}+1}}, \quad 0<x<c \tag{4.44}$$

Putting $i=n$ in (4.43),

$$g_n(x) = n\frac{(c-x)^{\frac{1}{c}-1}}{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{n+1} x^{\frac{1}{c}+1}}, \quad 0<x<c \tag{4.45}$$

4.4.2 Exponentiated Generated Distribution

Putting (4.39) in (2.8) gives

$$G(x) = \left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-r}, \quad 0<x<c,\ r>0 \tag{4.46}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= r\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-r-1}\left(\frac{c-x}{x}\right)^{\frac{1}{c}-1} x^{-2} \\ &= r\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-r-1}\frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}}, \quad 0<x<c,\ r>0 \end{aligned} \tag{4.47}$$

which is the **Burr IV distribution**.

4.4.3 Beta Exponentiated Generated Distribution

Putting (4.46) in (2.10) gives

$$G(x) = \int_0^{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-r}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < c,\ a > 0,\ b > 0,\ r > 0 \tag{4.48}$$

and its corresponding pdf is

$$\begin{aligned}
g(x) &= \frac{r\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-ra+1}\left[1-\left\{\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right\}^{-r}\right]^{b-1}}{B(a,b)} \left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-2} \frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}} \\
&= \frac{r\left[1-\frac{1}{\left\{\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right\}^{r}}\right]^{b-1}}{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{ra-1} B(a,b)} \frac{(c-x)^{\frac{1}{c}-1}}{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{2} x^{\frac{1}{c}+1}} \\
&= \frac{r\left[\left\{\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right\}^{r}-1\right]^{b-1}}{B(a,b)} \frac{(c-x)^{\frac{1}{c}-1}}{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{ra-1+rb-r+2} x^{\frac{1}{c}+1}} \\
&= \frac{r\left[\left\{\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right\}^{r}-1\right]^{b-1}(c-x)^{\frac{1}{c}-1}}{B(a,b)\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{r(a+b-1)+1} x^{\frac{1}{c}+1}}, \quad 0 < x < c,\ a > 0,\ b > 0,\ r > 0
\end{aligned} \tag{4.49}$$

i^{th} order statistic of Burr IV distribution

From (4.49),

$$g_i(x) = \frac{r\left[\left\{\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right\}^{r}-1\right]^{n-i}(c-x)^{\frac{1}{c}-1}}{B(i,n-i+1)\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{rn+1} x^{\frac{1}{c}+1}}, \quad 0 < x < c,\ r > 0 \tag{4.50}$$

Putting $i = 1$ in (4.50),

$$g_1(x) = \frac{rn\left[\left\{\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right\}^r-1\right]^{n-1}(c-x)^{\frac{1}{c}-1}}{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{rn+1}x^{\frac{1}{c}+1}}, \quad 0<x<c,\ r>0 \tag{4.51}$$

Putting $i = n$ in (4.50),

$$g_n(x) = \frac{rn\,(c-x)^{\frac{1}{c}-1}}{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{rn+1}x^{\frac{1}{c}+1}}, \quad 0<x<c,\ r>0 \tag{4.52}$$

4.5 $g(x) = \sec^2 x, \quad c > 0$

Let

$$t = \tan x$$
$$dt = sec^2 x\, dx$$

Thus

$$\begin{aligned}\int g(x)dx &= \int sec^2 x\, dx\\ &= \int dt\\ &= t + C\\ &= \tan x + C\end{aligned}$$

Substituting in (4.1) we get

$$F(x) = \left[e^{-\tan x - C}+1\right]^{-1}$$

Let

$$e^{-C} = k$$

Then

$$\begin{aligned}F(x) &= \left[ke^{-\tan x}+1\right]^{-1}\\ &= \frac{1}{ke^{-\tan x}+1}\end{aligned}$$

Now,

$$F(x)=0 \Longrightarrow \frac{1}{\infty+1}=\frac{1}{ke^{-(-\infty)}+1}=0 \Longrightarrow x=-\frac{\pi}{2}$$
$$F(x)=1 \Longrightarrow \frac{1}{0+1}=\frac{1}{ke^{-(\infty)}+1}=1 \Longrightarrow x=\frac{\pi}{2}$$

Therefore

$$F(x)=\left[ke^{-\tan x}+1\right]^{-1}, \quad -\frac{\pi}{2}<x<\frac{\pi}{2} \tag{4.53}$$

and its corresponding pdf is

$$f(x)=\frac{ke^{-\tan x}\sec^2 x}{[ke^{-\tan x}+1]^2}, \quad -\frac{\pi}{2}<x<\frac{\pi}{2} \tag{4.54}$$

4.5.1 Beta Generated Distribution

Putting (4.53) in (2.3) gives

$$G(x)=\int_0^{\left[ke^{-\tan x}+1\right]^{-1}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad -\frac{\pi}{2}<x<\frac{\pi}{2},\ a>0,\ b>0 \tag{4.55}$$

and its corresponding pdf is

$$\begin{aligned}
g(x)&=\frac{[ke^{-\tan x}+1]^{-a+1}\left[1-\{ke^{-\tan x}+1\}^{-1}\right]^{b-1}}{B(a,b)}ke^{-\tan x}\sec^2 x\left[ke^{-\tan x}+1\right]^{-2}\\
&=\frac{\left[1-\frac{1}{[ke^{-\tan x}+1]^1}\right]^{b-1}}{[ke^{-\tan x}+1]^{a-1}B(a,b)}\frac{ke^{-\tan x}\sec^2 x}{[ke^{-\tan x}+1]^2}\\
&=\frac{[ke^{-\tan x}+1-1]^{b-1}}{B(a,b)}\frac{ke^{-\tan x}\sec^2 x}{[ke^{-\tan x}+1]^{a-1+b-1+2}}\\
&=\frac{[ke^{-\tan x}]^{b-1}}{B(a,b)}\frac{ke^{-\tan x}\sec^2 x}{[ke^{-\tan x}+1]^{a+b}}\\
&=\frac{[ke^{-\tan x}]^{b}\sec^2 x}{B(a,b)\,[ke^{-\tan x}+1]^{a+b}}, \quad -\frac{\pi}{2}<x<\frac{\pi}{2},\ a>0,\ b>0
\end{aligned} \tag{4.56}$$

i^{th} **order statistic of** $F(x) = [ke^{-\tan x}+1]^{-1}$

From (4.56),

$$g_i(x) = \frac{[ke^{-\tan x}]^{n-i+1}\sec^2 x}{B(i,n-i+1)\,[ke^{-\tan x}+1]^{n+1}}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \tag{4.57}$$

Putting $i = 1$ in (4.57),

$$g_1(x) = \frac{n\,[ke^{-\tan x}]^{n}\sec^2 x}{[ke^{-\tan x}+1]^{n+1}}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \tag{4.58}$$

Putting $i = n$ in (4.57),

$$g_n(x) = \frac{n\,[ke^{-\tan x}]\sec^2 x}{[ke^{-\tan x}+1]^{n+1}}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \tag{4.59}$$

4.5.2 Exponentiated Generated Distribution

Putting (4.53) in (2.8) gives

$$G(x) = \left[ke^{-\tan x}+1\right]^{-r}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},\ r > 0 \tag{4.60}$$

and its corresponding pdf is

$$g(x) = \frac{rke^{-\tan x}\sec^2 x}{[ke^{-\tan x}+1]^{r+1}}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},\ r > 0 \tag{4.61}$$

which is the **Burr V distribution**.

4.5.3 Beta Exponentiated Generated Distribution

Putting (4.60) in (2.10) gives

$$G(x)=\int_{0}^{\left[ke^{-\tan x}+1\right]^{-r}}\frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt,\quad -\frac{\pi}{2}<x<\frac{\pi}{2},\ a>0,\ b>0,\ r>0 \tag{4.62}$$

and its corresponding pdf is

$$\begin{aligned}
g(x)&=\frac{r\left[ke^{-\tan x}+1\right]^{-ra+1}\left[1-\left\{ke^{-\tan x}+1\right\}^{-r}\right]^{b-1}}{B(a,b)}ke^{-\tan x}\sec^2 x\left[ke^{-\tan x}+1\right]^{-2}\\
&=\frac{r\left[1-\frac{1}{\left\{ke^{-\tan x}+1\right\}^{r}}\right]^{b-1}}{\left[ke^{-\tan x}+1\right]^{ra-1}B(a,b)}\frac{ke^{-\tan x}\sec^2 x}{\left[ke^{-\tan x}+1\right]^{2}}\\
&=\frac{r\left[\left\{ke^{-\tan x}+1\right\}^{r}-1\right]^{b-1}}{B(a,b)}\frac{ke^{-\tan x}\sec^2 x}{\left[ke^{-\tan x}+1\right]^{ra-1+rb-r+2}}\\
&=\frac{r\left[\left\{ke^{-\tan x}+1\right\}^{r}-1\right]^{b-1}ke^{-\tan x}\sec^2 x}{B(a,b)\left[ke^{-\tan x}+1\right]^{r(a+b-1)+1}},\ -\frac{\pi}{2}<x<\frac{\pi}{2},\ a>0,\ b>0,\ r>0
\end{aligned} \tag{4.63}$$

i^{th} order statistic of Burr V distribution

From (4.63),

$$g_i(x)=\frac{r\left[\left\{ke^{-\tan x}+1\right\}^{r}-1\right]^{n-i}ke^{-\tan x}\sec^2 x}{B(i,n-i+1)\left[ke^{-\tan x}+1\right]^{rn+1}},\quad -\frac{\pi}{2}<x<\frac{\pi}{2},\ r>0 \tag{4.64}$$

Putting $i=1$ in (4.64),

$$g_1(x)=\frac{rn\left[\left\{ke^{-\tan x}+1\right\}^{r}-1\right]^{n-1}ke^{-\tan x}\sec^2 x}{\left[ke^{-\tan x}+1\right]^{rn+1}},\quad -\frac{\pi}{2}<x<\frac{\pi}{2},\ r>0 \tag{4.65}$$

Putting $i=n$ in (4.64),

$$g_n(x)=\frac{rnke^{-\tan x}\sec^2 x}{\left[ke^{-\tan x}+1\right]^{rn+1}},\quad -\frac{\pi}{2}<x<\frac{\pi}{2},\ r>0 \tag{4.66}$$

4.6 $g(x) = c\cosh x, \quad c > 0$

Then

$$g(x) = c\left[\frac{e^x + e^{-x}}{2}\right]$$

Thus

$$\begin{aligned}\int g(x)dx &= c\int\left(\frac{e^x + e^{-x}}{2}\right)dx \\ &= c\left(\frac{e^x - e^{-x}}{2}\right) + D \\ &= c\sinh x + D\end{aligned}$$

Substituting in (4.1) we get

$$F(x) = \left[e^{-c\sinh x - D} + 1\right]^{-1}$$

Let

$$e^{-D} = k$$

Then

$$\begin{aligned}F(x) &= \left[ke^{-c\sinh x} + 1\right]^{-1} \\ &= \frac{1}{ke^{-c\sinh x} + 1}\end{aligned}$$

Now,

$$\begin{aligned}F(x) = 0 \Longrightarrow \frac{1}{\infty + 1} &= \frac{1}{ke^{\infty} + 1} = \frac{1}{ke^{c\left(\frac{\infty - 0}{2}\right)} + 1} = \frac{1}{ke^{-c\left(\frac{0-\infty}{2}\right)} + 1} \\ &= \frac{1}{ke^{-c\left(\frac{e^{-\infty} - e^{-(-\infty)}}{2}\right)} + 1} = 0 \Longrightarrow x = -\infty\end{aligned}$$

$$\begin{aligned}F(x) = 1 \Longrightarrow \frac{1}{0 + 1} &= \frac{1}{ke^{-\infty} + 1} = \frac{1}{ke^{c\left(\frac{-\infty + 0}{2}\right)} + 1} = \frac{1}{ke^{-c\left(\frac{\infty - 0}{2}\right)} + 1} \\ &= \frac{1}{ke^{-c\left(\frac{e^{\infty} - e^{-\infty}}{2}\right)} + 1} = 1 \Longrightarrow x = \infty\end{aligned}$$

Therefore

$$F(x)=\left[ke^{-c\sinh x}+1\right]^{-1},\quad -\infty<x<\infty \tag{4.67}$$

and its corresponding pdf is

$$f(x)=\frac{kce^{-c\sinh x}}{\cosh x\left[ke^{-c\sinh x}+1\right]^{2}},\quad -\infty<x<\infty \tag{4.68}$$

4.6.1 Beta Generated Distribution

Putting (4.67) in (2.3) gives

$$G(x)=\int_{0}^{\left[ke^{-c\sinh x}+1\right]^{-1}}\frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt,\quad -\infty<x<\infty,\ a>0,\ b>0 \tag{4.69}$$

and its corresponding pdf is

$$\begin{aligned}
g(x)&=\frac{\left[ke^{-c\sinh x}+1\right]^{-a+1}\left[1-\left\{ke^{-c\sinh x}+1\right\}^{-1}\right]^{b-1}}{B(a,b)}\frac{kce^{-c\sinh x}\cosh x}{\left[ke^{-c\sinh x}+1\right]^{2}}\\
&=\frac{\left[1-\frac{1}{\left[ke^{-c\sinh x}+1\right]^{1}}\right]^{b-1}}{\left[ke^{-c\sinh x}+1\right]^{a-1}B(a,b)}\frac{kce^{-c\sinh x}\cosh x}{\left[ke^{-c\sinh x}+1\right]^{2}}\\
&=\frac{\left[ke^{-c\sinh x}+1-1\right]^{b-1}}{B(a,b)}\frac{kce^{-c\sinh x}\cosh x}{\left[ke^{-c\sinh x}+1\right]^{a-1+b-1+2}}\\
&=\frac{\left[ke^{-c\sinh x}\right]^{b-1}}{B(a,b)}\frac{kce^{-c\sinh x}\cosh x}{\left[ke^{-c\sinh x}+1\right]^{a+b}}\\
&=\frac{c\left[ke^{-c\sinh x}\right]^{b}\cosh x}{B(a,b)\left[ke^{-c\sinh x}+1\right]^{a+b}},\quad -\infty<x<\infty,\ a>0,\ b>0
\end{aligned} \tag{4.70}$$

i^{th} **order statistic of** $F(x) = \left[ke^{-c\sinh x} + 1\right]^{-1}$

From (4.70),

$$g_i(x) = \frac{c\left[ke^{-c\sinh x}\right]^{n-i+1}\cosh x}{B(i, n-i+1)\left[ke^{-c\sinh x} + 1\right]^{n+1}}, \quad -\infty < x < \infty \tag{4.71}$$

Putting $i = 1$ in (4.71),

$$g_1(x) = \frac{nc\left[ke^{-c\sinh x}\right]^{n}\cosh x}{\left[ke^{-c\sinh x} + 1\right]^{n+1}}, \quad -\infty < x < \infty \tag{4.72}$$

Putting $i = n$ in (4.71),

$$g_n(x) = \frac{nc\left[ke^{-c\sinh x}\right]\cosh x}{\left[ke^{-c\sinh x} + 1\right]^{n+1}}, \quad -\infty < x < \infty \tag{4.73}$$

4.6.2 Exponentiated Generated Distribution

Putting (4.67) in (2.8) gives

$$G(x) = \left[ke^{-c\sinh x} + 1\right]^{-r}, \quad -\infty < x < \infty,\ r > 0 \tag{4.74}$$

and its corresponding pdf is

$$g(x) = \frac{rkce^{-c\sinh x}\cosh x}{\left[ke^{-c\sinh x} + 1\right]^{r+1}}, \quad -\infty < x < \infty,\ r > 0 \tag{4.75}$$

which is the **Burr VI distribution**.

4.6.3 Beta Exponentiated Generated Distribution

Putting (4.74) in (2.10) gives

$$G(x)=\int_{0}^{\left[ke^{-c\sinh x}+1\right]^{-r}}\frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt,\quad -\infty<x<\infty,\ a>0,\ b>0,\ r>0 \tag{4.76}$$

and its corresponding pdf is

$$\begin{aligned}
g(x)&=\frac{r\left[ke^{-c\sinh x}+1\right]^{-ra+1}\left[1-\left\{ke^{-c\sinh x}+1\right\}^{-r}\right]^{b-1}}{B(a,b)}\frac{kce^{-c\sinh x}\cosh x}{\left[ke^{-c\sinh x}+1\right]^{2}}\\
&=\frac{r\left[1-\frac{1}{\left\{ke^{-c\sinh x}+1\right\}^{r}}\right]^{b-1}}{\left[ke^{-c\sinh x}+1\right]^{ra-1}B(a,b)}\frac{kce^{-c\sinh x}\cosh x}{\left[ke^{-c\sinh x}+1\right]^{2}}\\
&=\frac{r\left[\left\{ke^{-c\sinh x}+1\right\}^{r}-1\right]^{b-1}}{B(a,b)}\frac{kce^{-c\sinh x}\cosh x}{\left[ke^{-c\sinh x}+1\right]^{ra-1+rb-r+2}}\\
&=\frac{r\left[\left\{ke^{-c\sinh x}+1\right\}^{r}-1\right]^{b-1}kce^{-c\sinh x}\cosh x}{B(a,b)\left[ke^{-c\sinh x}+1\right]^{r(a+b-1)+1}},\quad -\infty<x<\infty,\ a>0,\ b>0
\end{aligned} \tag{4.77}$$

i^{th} order statistic of Burr VI distribution

From (4.77),

$$g_i(x)=\frac{r\left[\left\{ke^{-c\sinh x}+1\right\}^{r}-1\right]^{n-i}kce^{-c\sinh x}\cosh x}{B(i,n-i+1)\left[ke^{-c\sinh x}+1\right]^{rn+1}}\quad -\infty<x<\infty,\ r>0 \tag{4.78}$$

Putting $i=1$ in (4.78),

$$g_1(x)=\frac{rn\left[\left\{ke^{-c\sinh x}+1\right\}^{r}-1\right]^{n-1}kce^{-c\sinh x}\cosh x}{\left[ke^{-c\sinh x}+1\right]^{rn+1}}\quad -\infty<x<\infty,\ r>0 \tag{4.79}$$

Putting $i=n$ in (4.78),

$$g_n(x)=\frac{rnkce^{-c\sinh x}\cosh x}{\left[ke^{-c\sinh x}+1\right]^{rn+1}}\quad -\infty<x<\infty,\ r>0 \tag{4.80}$$

4.7 $g(x) = 2$

Then (4.1) becomes

$$F(x) = \left[e^{-\int 2dx} + 1\right]^{-1}$$

Substituting

$$\int 2dx = 2x$$

We get

$$\begin{aligned} F(x) &= \left[e^{-2x} + 1\right]^{-1} \\ &= \frac{1}{[e^{-2x} + 1]} \end{aligned}$$

Now,

$$F(x) = 0 \Longrightarrow \frac{1}{[\infty + 1]} = \frac{1}{\left[e^{-(-\infty)} + 1\right]} = 0 \Longrightarrow x = -\infty$$

$$F(x) = 1 \Longrightarrow \frac{1}{[0 + 1]} = \frac{1}{\left[e^{-(\infty)} + 1\right]} = 1 \Longrightarrow x = \infty$$

Therefore

$$\begin{aligned} F(x) &= \left[e^{-2x} + 1\right]^{-1} \\ &= \left[\frac{1}{e^{-2x} + 1}\right] \\ &= \left[\frac{e^x}{e^{-x} + e^x}\right] \\ &= \left[\frac{2e^x}{2(e^{-x} + e^x)}\right] \\ &= 2^{-1}\left[\frac{2e^x}{e^{-x} + e^x}\right] \\ &= 2^{-1}\left[\frac{e^x + e^x}{e^{-x} + e^x}\right] \\ &= 2^{-1}\left[\frac{e^x + e^x + e^{-x} - e^{-x}}{e^{-x} + e^x}\right] \\ &= 2^{-1}\left[1 + \frac{e^x - e^{-x}}{e^{-x} + e^x}\right] \\ &= 2^{-1}\left[1 + \tanh x\right], \quad -\infty < x < \infty \end{aligned} \tag{4.81}$$

and its corresponding pdf is

$$f(x) = 2^{-1}\,\mathrm{sech}^2 x, \quad -\infty < x < \infty \tag{4.82}$$

4.7.1 Beta Generated Distribution

Putting (4.81) in (2.3) gives

$$G(x) = \int\limits_{0}^{2^{-1}[1+\tanh x]} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad -\infty < x < \infty,\ a > 0,\ b > 0 \tag{4.83}$$

and its corresponding pdf is

$$\begin{aligned}
g(x) &= \frac{\left[2^{-1}\left(1+\tanh x\right)\right]^{a-1}\left[1-\left\{2^{-1}\left(1+\tanh x\right)\right\}\right]^{b-1}}{B(a,b)}2^{-1}\,\mathrm{sech}^2 x \\
&= \frac{\left[\dfrac{1+\tanh x}{2}\right]^{a-1}\left[1-\left\{\dfrac{1+\tanh x}{2}\right\}\right]^{b-1}}{B(a,b)}\frac{\mathrm{sech}^2 x}{2} \\
&= \frac{[1+\tanh x]^{a-1}[2-1-\tanh x]^{b-1}}{2^{a-1}2^{b-1}B(a,b)}\frac{\mathrm{sech}^2 x}{2} \\
&= \frac{[1+\tanh x]^{a-1}[1-\tanh x]^{b-1}\,\mathrm{sech}^2 x}{2^{a-1+b-1+1}B(a,b)} \\
&= \frac{[1+\tanh x]^{a-1}[1-\tanh x]^{b-1}\,\mathrm{sech}^2 x}{2^{a+b-1}B(a,b)}, \quad -\infty < x < \infty,\ a > 0,\ b > 0
\end{aligned} \tag{4.84}$$

i^{th} order statistic of $F(x) = 2^{-1}[1+\tanh x]$

From (4.84),

$$g_i(x) = \frac{[1+\tanh x]^{i-1}[1-\tanh x]^{n-i}\,\mathrm{sech}^2 x}{2^n B(i,n-i+1)}, \quad -\infty < x < \infty \tag{4.85}$$

Putting $i = 1$ in (4.85),

$$g_1(x) = \frac{n\,[1-\tanh x]^{n-1}\,\operatorname{sech}^2 x}{2^n}, \quad -\infty < x < \infty \tag{4.86}$$

Putting $i = n$ in (4.85),

$$g_n(x) = \frac{n\,[1+\tanh x]^{n-1}\,\operatorname{sech}^2 x}{2^n}, \quad -\infty < x < \infty \tag{4.87}$$

4.7.2 Exponentiated Generated Distribution

Putting (4.81) in (2.8) gives

$$\begin{aligned} G(x) &= \left[2^{-1}\left\{1+\tanh x\right\}\right]^r \\ &= 2^{-r}\,[1+\tanh x]^r, \quad -\infty < x < \infty,\ r > 0 \end{aligned} \tag{4.88}$$

and its corresponding pdf is

$$g(x) = r2^{-r}\operatorname{sech}^2 x\,[1+\tanh x]^{r-1}, \quad -\infty < x < \infty,\ r > 0 \tag{4.89}$$

which is the **Burr VII distribution**.

4.7.3 Beta Exponentiated Generated Distribution

Putting (4.88) in (2.10) gives

$$G(x) = \int_0^{2^{-r}[1+\tanh x]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}\,dt, \quad -\infty < x < \infty,\ a > 0,\ b > 0,\ r > 0 \tag{4.90}$$

and its corresponding pdf is

$$\begin{aligned}
g(x) &= \frac{r\left[2^{-1}(1+\tanh x)\right]^{ra-1}\left[1-\{2^{-r}(1+\tanh x)^{r}\}\right]^{b-1}}{B(a,b)} 2^{-1}\operatorname{sech}^2 x \\
&= \frac{r\left[\dfrac{1+\tanh x}{2}\right]^{ra-1}\left[1-\left\{\dfrac{1+\tanh x}{2}\right\}^{r}\right]^{b-1}}{B(a,b)} \frac{\operatorname{sech}^2 x}{2} \\
&= \frac{r\left[\dfrac{1+\tanh x}{2}\right]^{ra-1}\left[\dfrac{2^{r}-\{1+\tanh x\}^{r}}{2^{r}}\right]^{b-1}}{B(a,b)} \frac{\operatorname{sech}^2 x}{2} \\
&= \frac{r[1+\tanh x]^{ra-1}\left[2^{r}-\{1+\tanh x\}^{r}\right]^{b-1}}{2^{ra-1}2^{rb-r}B(a,b)} \frac{\operatorname{sech}^2 x}{2} \\
&= \frac{r[1+\tanh x]^{ra-1}\left[2^{r}-\{1+\tanh x\}^{r}\right]^{b-1}\operatorname{sech}^2 x}{2^{ra-1+rb-r+1}B(a,b)} \\
&= \frac{r[1+\tanh x]^{ra-1}\left[2^{r}-\{1+\tanh x\}^{r}\right]^{b-1}\operatorname{sech}^2 x}{2^{r(a+b-1)}B(a,b)}, \quad -\infty < x < \infty,\ r > 0
\end{aligned} \tag{4.91}$$

i^{th} order statistic of Burr VII distribution

From (4.91),

$$g_i(x) = \frac{r[1+\tanh x]^{ri-1}\left[2^{r}-\{1+\tanh x\}^{r}\right]^{n-i}\operatorname{sech}^2 x}{2^{rn}B(i,n-i+1)}, \quad -\infty < x < \infty,\ r > 0 \tag{4.92}$$

Putting $i = 1$ in (4.92),

$$g_1(x) = \frac{rn[1+\tanh x]^{r-1}\left[2^{r}-\{1+\tanh x\}^{r}\right]^{n-1}\operatorname{sech}^2 x}{2^{rn}}, \quad -\infty < x < \infty,\ r > 0 \tag{4.93}$$

Putting $i = n$ in (4.92),

$$g_n(x) = \frac{rn[1+\tanh x]^{rn-1}\operatorname{sech}^2 x}{2^{rn}}, \quad -\infty < x < \infty,\ r > 0 \tag{4.94}$$

5 Distributions based on Case III of Burr Differential Equation

5.1 Concept of income elasticity

According to Kleiber and Kotz (2003), Stoppa (1990a) proposed a differential equation for income elasticity as

$$\eta(x,F(x)) = \frac{1-[F(x)]^{\frac{1}{\theta}}}{[F(x)]^{\frac{1}{\theta}}} g(x,F(x)), \quad x > x_0 \geqslant 0 \tag{5.1}$$

where

$$\eta(x,F(x)) = x\frac{F'(x)}{F(x)}, \textit{ is the } \textbf{income elasticity}$$

and

$$F(x) \textit{ is the cdf}$$

.

Let $g(x,F(x)) = g(x)$ and $\theta = 1$, then (5.1) becomes

$$x\frac{F'(x)}{F(x)} = \frac{1-F(x)}{F(x)} g(x)$$

which can be written as

$$x\frac{y'}{y} = \frac{1-y}{y} g(x)$$

Therefore

$$y' = y(1-y)\frac{g(x)}{xy}$$

which is (2.1) with

$$g(x,y) = \frac{g(x)}{xy}$$

Again, According to Kleiber and Kotz (2003), Dagum's differential equation is of the form

$$\frac{d\log[F(x)-\delta]}{d\log x} = \theta(x)\phi(F) \leq k, \quad 0 \leq x_0 < x < \infty \tag{5.2}$$

where

$$k > 0,\ \theta(x) > 0,\ \phi(F) > 0,\ \delta < 1$$

and

$$\frac{d\{\theta(x)\phi(F)\}}{dx} < 0.$$

When $\delta = 0,\ \theta(x) = g(x)$ and $\phi(F) = \dfrac{1-F}{F}$, (5.2) becomes

$$\frac{d\log F(x)}{d\log x} = g(x)\left[\frac{1-F}{F}\right]$$

But

$$\begin{aligned}\frac{d\log F(x)}{d\log x} &= \frac{d\log F(x)}{dx}\frac{dx}{d\log x}\\ &= \frac{F'(x)}{F(x)}\frac{1}{\frac{d\log x}{dx}}\\ &= \frac{F'(x)}{F(x)}\frac{1}{\frac{1}{x}}\\ &= x\frac{F'(x)}{F(x)}\end{aligned}$$

Therefore

$$x\frac{F'(x)}{F(x)} = g(x)\left[\frac{1-F}{F}\right]$$

which can be written as

$$x\frac{y'}{y} = \frac{1-y}{y}g(x)$$

$$y' = y(1-y)\frac{g(x)}{xy}$$

the same result as before.

Thus this Burr's equation reduces to

$$y' = (1-y)\frac{g(x)}{x}$$

which implies that

$$\begin{aligned}
\int \frac{dy}{(1-y)} &= \int \frac{g(x)}{x}dx \\
-\log(1-y) &= \int \frac{g(x)}{x}dx \\
(1-y) &= \exp\left[-\int \frac{g(x)}{x}dx\right]
\end{aligned}$$

Therefore

$$y = 1 - \exp\left[-\int \frac{g(x)}{x}dx\right]$$

That is

$$F(x) = 1 - \exp\left[-\int \frac{g(x)}{x}dx\right] \tag{5.3}$$

5.2 $g(x) = \dfrac{ckxe^{x}(1+e^{x})^{k-1}}{c\left[(1+e^{x})^{k}-1\right]+2}, \quad c>0,\ k>0$

Thus

$$\int \frac{g(x)}{x}dx = \int \frac{cke^{x}(1+e^{x})^{k-1}}{c\left[(1+e^{x})^{k}-1\right]+2}dx$$

Let

$$u = \frac{c\left[(1+e^{x})^{k}-1\right]+2}{2} \Longrightarrow 2du = ck\left(1+e^{x}\right)^{k-1}e^{x}dx$$

Then

$$\begin{aligned}\int \frac{g(x)}{x}dx &= \int \frac{cke^{x}\left(1+e^{x}\right)^{k-1}}{2u}\frac{2du}{ck\left(1+e^{x}\right)^{k-1}e^{x}} \\ &= \int \frac{1}{u}du\end{aligned}$$

Substituting in (5.3) we get

$$\begin{aligned}F(x) &= 1-\exp\left[-\int \frac{1}{u}du\right] \\ &= 1-\exp\left[-\log u\right] \\ &= 1-\exp\left[\log u^{-1}\right] \\ &= 1-u^{-1} \\ &= 1-\left\{\frac{c\left[(1+e^{x})^{k}-1\right]+2}{2}\right\}^{-1} \\ &= 1-\frac{2}{c\left[(1+e^{x})^{k}-1\right]+2}\end{aligned}$$

Now,

$$F(x)=0 \Longrightarrow 1-\frac{2}{2} = 1-\frac{2}{c\left[(1+e^{-\infty})^{k}-1\right]+2} = 0 \Longrightarrow x = -\infty$$

$$F(x)=1 \Longrightarrow 1-\frac{2}{\infty} = 1-\frac{2}{c\left[(1+e^{\infty})^{k}-1\right]+2} = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = 1-\frac{2}{c\left[(1+e^{x})^{k}-1\right]+2}, \quad -\infty < x < \infty \tag{5.4}$$

and its corresponding pdf is

$$f(x)=\frac{2kce^{x}\left(1+e^{x}\right)^{k-1}}{\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]+2\right\}^{2}},\quad -\infty<x<\infty \tag{5.5}$$

which is the **Burr IX distribution**.

5.2.1 Beta Burr IX Distribution

Putting (5.4) in (2.3) gives

$$G(x)=\int_{0}^{1-\frac{2}{c\left[\left(1+e^{x}\right)^{k}-1\right]+2}}\frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt,\quad -\infty<x<\infty,\ a>0,\ b>0 \tag{5.6}$$

and its corresponding pdf is

$$\begin{aligned}
g(x)&=\frac{\left[1-\frac{2}{c\left[\left(1+e^{x}\right)^{k}-1\right]+2}\right]^{a-1}\left[1-\left\{1-\frac{2}{c\left[\left(1+e^{x}\right)^{k}-1\right]+2}\right\}\right]^{b-1}}{B(a,b)\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]+2\right\}^{2}}2kce^{x}\left(1+e^{x}\right)^{k-1}\\
&=\frac{\left[\frac{c\left[\left(1+e^{x}\right)^{k}-1\right]+2-2}{c\left[\left(1+e^{x}\right)^{k}-1\right]+2}\right]^{a-1}\left[\frac{2}{c\left[\left(1+e^{x}\right)^{k}-1\right]+2}\right]^{b-1}}{B(a,b)}\frac{2kce^{x}\left(1+e^{x}\right)^{k-1}}{\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]+2\right\}^{2}}\\
&=\frac{\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]\right\}^{a-1}[2]^{b-1}}{\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]+2\right\}^{a-1+b-1}B(a,b)}\frac{2kce^{x}\left(1+e^{x}\right)^{k-1}}{\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]+2\right\}^{2}}\\
&=\frac{\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]\right\}^{a-1}[2]^{b-1+1}}{\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]+2\right\}^{a-1+b-1+2}B(a,b)}kce^{x}\left(1+e^{x}\right)^{k-1}\\
&=\frac{\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]\right\}^{a-1}2^{b}kce^{x}\left(1+e^{x}\right)^{k-1}}{\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]+2\right\}^{a+b}B(a,b)},\quad -\infty<x<\infty,\ a>0,\ b>0
\end{aligned} \tag{5.7}$$

i^{th} order statistic of Burr IX distribution

From (5.7),

$$g_i(x) = \frac{\left\{c\left[(1+e^x)^k-1\right]\right\}^{i-1} 2^{n-i+1} kce^x (1+e^x)^{k-1}}{\left\{c\left[(1+e^x)^k-1\right]+2\right\}^{n+1} B(i,n-i+1)}, \quad -\infty < x < \infty \tag{5.8}$$

Putting $i = 1$ in (5.8),

$$g_1(x) = \frac{n2^n kce^x (1+e^x)^{k-1}}{\left\{c\left[(1+e^x)^k-1\right]+2\right\}^{n+1}}, \quad -\infty < x < \infty \tag{5.9}$$

Putting $i = n$ in (5.8),

$$g_n(x) = \frac{n\left\{c\left[(1+e^x)^k-1\right]\right\}^{n-1} 2kce^x (1+e^x)^{k-1}}{\left\{c\left[(1+e^x)^k-1\right]+2\right\}^{n+1}}, \quad -\infty < x < \infty \tag{5.10}$$

5.2.2 Exponentiated Burr IX Distribution

Putting (5.4) in (2.8) gives

$$G(x) = \left\{1 - \frac{2}{c\left[(1+e^x)^k-1\right]+2}\right\}^r, \quad -\infty < x < \infty, \ r > 0 \tag{5.11}$$

and its corresponding pdf is

$$
\begin{aligned}
g(x) &= \frac{2rkce^{x}\left(1+e^{x}\right)^{k-1}}{\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]+2\right\}^{2}}\left\{1-\frac{2}{c\left[\left(1+e^{x}\right)^{k}-1\right]+2}\right\}^{r-1} \\
&= \frac{2rkce^{x}\left(1+e^{x}\right)^{k-1}}{\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]+2\right\}^{2}}\left\{\frac{c\left[\left(1+e^{x}\right)^{k}-1\right]+2-2}{c\left[\left(1+e^{x}\right)^{k}-1\right]+2}\right\}^{r-1} \\
&= \frac{2rkce^{x}\left(1+e^{x}\right)^{k-1}}{\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]+2\right\}^{2+r-1}}\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]\right\}^{r-1} \\
&= \frac{2rkce^{x}\left(1+e^{x}\right)^{k-1}\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]\right\}^{r-1}}{\left\{c\left[\left(1+e^{x}\right)^{k}-1\right]+2\right\}^{r+1}}, \quad -\infty<x<\infty,\ r>0
\end{aligned}
\tag{5.12}
$$

5.2.3 Beta Exponentiated Burr IX Distribution

Putting (5.11) in (2.10) gives

$$
G(x) = \int_{0}^{\left\{1-\frac{2}{c\left[\left(1+e^{x}\right)^{k}-1\right]+2}\right\}^{r}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad -\infty<x<\infty,\ a>0,\ b>0,\ r>0 \tag{5.13}
$$

and its corresponding pdf is

$$g(x)=\frac{r\left\{1-\frac{2}{c\left[(1+e^x)^k-1\right]+2}\right\}^{ra-1}\left[1-\left\{1-\frac{2}{c\left[(1+e^x)^k-1\right]+2}\right\}^r\right]^{b-1}}{B(a,b)\left\{c\left[(1+e^x)^k-1\right]+2\right\}^2}2kce^x(1+e^x)^{k-1}$$

$$=\frac{r\left\{\frac{c\left[(1+e^x)^k-1\right]+2-2}{c\left[(1+e^x)^k-1\right]+2}\right\}^{ra-1}\left[1-\left\{\frac{c\left[(1+e^x)^k-1\right]+2-2}{c\left[(1+e^x)^k-1\right]+2}\right\}^r\right]^{b-1}}{B(a,b)\left\{c\left[(1+e^x)^k-1\right]+2\right\}^2}2kce^x(1+e^x)^{k-1}$$

$$=\frac{r\left\{\frac{c\left[(1+e^x)^k-1\right]}{c\left[(1+e^x)^k-1\right]+2}\right\}^{ra-1}\left[1-\left\{\frac{c\left[(1+e^x)^k-1\right]}{c\left[(1+e^x)^k-1\right]+2}\right\}^r\right]^{b-1}}{B(a,b)\left\{c\left[(1+e^x)^k-1\right]+2\right\}^2}2kce^x(1+e^x)^{k-1}$$

$$=\frac{r\left\{c\left[(1+e^x)^k-1\right]\right\}^{ra-1}\left[1-\left\{\frac{c\left[(1+e^x)^k-1\right]}{c\left[(1+e^x)^k-1\right]+2}\right\}^r\right]^{b-1}}{B(a,b)\left\{c\left[(1+e^x)^k-1\right]+2\right\}^{ra-1+2}}2kce^x(1+e^x)^{k-1}$$

$$=\frac{r\left\{c\left[(1+e^x)^k-1\right]\right\}^{ra-1}\left[1-\left\{\frac{c\left[(1+e^x)^k-1\right]}{c\left[(1+e^x)^k-1\right]+2}\right\}^r\right]^{b-1}}{B(a,b)\left\{c\left[(1+e^x)^k-1\right]+2\right\}^{ra+1}}2kce^x(1+e^x)^{k-1}$$

(5.14)

i^{th} order statistic of Exponentiated Burr IX distribution

From (5.14),

$$g_i(x)=\frac{r\left\{c\left[(1+e^x)^k-1\right]\right\}^{ri-1}\left[1-\left\{\frac{c\left[(1+e^x)^k-1\right]}{c\left[(1+e^x)^k-1\right]+2}\right\}^r\right]^{n-i}}{B(i,n-i+1)\left\{c\left[(1+e^x)^k-1\right]+2\right\}^{ri+1}}2kce^x(1+e^x)^{k-1} \quad \textbf{(5.15)}$$

Putting $i=1$ in (5.15),

$$g_1(x)=\frac{rn\left\{c\left[(1+e^x)^k-1\right]\right\}^{r-1}\left[1-\left\{\frac{c\left[(1+e^x)^k-1\right]}{c\left[(1+e^x)^k-1\right]+2}\right\}^r\right]^{n-1}}{\left\{c\left[(1+e^x)^k-1\right]+2\right\}^{r+1}}2kce^x(1+e^x)^{k-1} \quad \textbf{(5.16)}$$

Putting $i = n$ in (5.15),

$$g_n(x) = \frac{rn\left\{c\left[(1+e^x)^k - 1\right]\right\}^{rn-1}}{\left\{c\left[(1+e^x)^k - 1\right] + 2\right\}^{rn+1}} 2kce^x (1+e^x)^{k-1}, \quad -\infty < x < \infty,\ r > 0 \tag{5.17}$$

5.3 $g(x) = 2x^2$

Then

$$\begin{aligned}\int \frac{g(x)}{x} dx &= \int 2x dx \\ &= x^2\end{aligned}$$

Substituting in (5.3) we get

$$F(x) = 1 - e^{-x^2}$$

Now,

$$F(x) = 0 \Longrightarrow [1-1] = \left[1 - e^{-0}\right] = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow [1-0] = \left[1 - e^{-(\infty)}\right] = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = 1 - e^{-x^2}, \quad 0 < x < \infty \tag{5.18}$$

and its corresponding pdf is

$$f(x) = 2xe^{-x^2}, \quad 0 < x < \infty \tag{5.19}$$

The m^{th} moment of (5.19) is

$$E(X^m) = \int_0^\infty x^m 2xe^{-x^2} dx$$

Let

$$y = x^2 \Longrightarrow dy = 2xdx$$

Thus

$$\begin{aligned} E(X^m) &= \int_0^\infty \left[y^{\frac{1}{2}}\right]^m e^{-y} 2y^{\frac{1}{2}} \frac{dy}{2y^{\frac{1}{2}}} \\ &= \int_0^\infty y^{\frac{m}{2}} e^{-y} dy \\ &= \int_0^\infty y^{\frac{m}{2}+1-1} e^{-y} dy \\ &= \Gamma\left(\frac{m}{2}+1\right) \end{aligned} \quad (5.20)$$

5.3.1 Beta Generated Distribution

Putting (5.18) in (2.3) gives

$$G(x) = \int_0^{1-e^{-x^2}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < \infty,\ a > 0,\ b > 0 \quad (5.21)$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{\left[1-e^{-x^2}\right]^{a-1}\left[1-\left\{1-e^{-x^2}\right\}\right]^{b-1}}{B(a,b)}2xe^{-x^2} \\ &= \frac{\left[1-e^{-x^2}\right]^{a-1}\left[e^{-x^2}\right]^{b-1}}{B(a,b)}2xe^{-x^2} \\ &= \frac{\left[1-e^{-x^2}\right]^{a-1}\left[e^{-x^2}\right]^{b-1+1}}{B(a,b)}2x \\ &= \frac{\left[1-e^{-x^2}\right]^{a-1}\left[e^{-x^2}\right]^{b}}{B(a,b)}2x \\ &= \frac{\left[1-e^{-x^2}\right]^{a-1}e^{-bx^2}}{B(a,b)}2x, \quad 0<x<\infty,\ a>0,\ b>0 \end{aligned} \tag{5.22}$$

i^{th} **order statistic of** $F(x) = 1-e^{-x^2}$

From (5.22),

$$g_i(x) = \frac{\left[1-e^{-x^2}\right]^{i-1}e^{-(n-i+1)x^2}}{B(i,n-i+1)}2x, \quad 0<x<\infty \tag{5.23}$$

Putting $i=1$ in (5.23),

$$g_1(x) = e^{-nx^2}2xn, \quad 0<x<\infty \tag{5.24}$$

The m^{th} moment of (5.24) is

$$E(X_1^m) = \int_0^\infty x^m e^{-nx^2}2xn dx$$

Let

$$y = nx^2 \Longrightarrow dy = 2xn dx$$

Thus

$$E(X_1^m) = \int_0^\infty \left[\left(\frac{y}{n}\right)^{\frac{1}{2}}\right]^m e^{-y} 2\left(\frac{y}{n}\right)^{\frac{1}{2}} n \frac{dy}{2\left(\frac{y}{n}\right)^{\frac{1}{2}} n}$$
$$= \frac{1}{n^{\frac{m}{2}}} \int_0^\infty y^{\frac{m}{2}} e^{-y} dy \tag{5.25}$$
$$= \frac{1}{n^{\frac{m}{2}}} \int_0^\infty y^{\frac{m}{2}+1-1} e^{-y} dy$$
$$= \frac{\Gamma(\frac{m}{2}+1)}{n^{\frac{m}{2}}}$$

Putting $i = n$ in (5.23),

$$g_n(x) = \left[1 - e^{-x^2}\right]^{n-1} e^{-x^2} 2xn, \quad 0 < x < \infty \tag{5.26}$$

5.3.2 Exponentiated Generated Distribution

Putting (5.18) in (2.8) gives

$$G(x) = \left[1 - e^{-x^2}\right]^r, \quad 0 < x < \infty, \ r > 0 \tag{5.27}$$

and its corresponding pdf is

$$g(x) = r2xe^{-x^2} \left[1 - e^{-x^2}\right]^{r-1}, \quad 0 < x < \infty, \ r > 0 \tag{5.28}$$

which is the **Burr X distribution**.

5.3.3 Beta Exponentiated Generated Distribution

Putting (5.27) in (2.10) gives

$$G(x) = \int_0^{\left[1-e^{-x^2}\right]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < \infty, \ a > 0, \ b > 0, \ r > 0 \tag{5.29}$$

and its corresponding pdf is

$$g(x) = \frac{r\left[1-e^{-x^2}\right]^{ra-1}\left[1-\left\{1-e^{-x^2}\right\}^r\right]^{b-1}}{B(a,b)} 2xe^{-x^2} \tag{5.30}$$

i^{th} **order statistic of Burr X distribution**

From (5.30),

$$g_i(x) = \frac{r\left[1-e^{-x^2}\right]^{ri-1}\left[1-\left\{1-e^{-x^2}\right\}^r\right]^{n-i}}{B(i,n-i+1)} 2xe^{-x^2} \tag{5.31}$$

Putting $i = 1$ in (5.31),

$$g_1(x) = 2xrne^{-x^2}\left[1-e^{-x^2}\right]^{r-1}\left[1-\left\{1-e^{-x^2}\right\}^r\right]^{n-1}, \quad 0 < x < \infty,\ r > 0 \tag{5.32}$$

Putting $i = n$ in (5.31),

$$g_n(x) = 2xrne^{-x^2}\left[1-e^{-x^2}\right]^{rn-1}, \quad 0 < x < \infty,\ r > 0 \tag{5.33}$$

5.4 $g(x) = \frac{ckx^c}{1+x^c}, \quad c > 0,\ k > 0$

Then

$$\int \frac{g(x)}{x} dx = \int \frac{ckx^{c-1}}{1+x^c} dx$$

Let

$$u = 1 + x^c$$
$$du = cx^{c-1} dx$$

Thus

$$\int \frac{g(x)}{x} dx = \int \frac{ckx^{c-1}}{u} \frac{du}{cx^{c-1}} = \int \frac{k}{u} du$$

Substituting in (5.3) we get

$$
\begin{aligned}
F(x) &= 1-\exp\left[-\int \frac{k}{u}du\right] \\
&= 1-\exp\left[-k\int \frac{du}{u}\right] \\
&= 1-\exp\left[-k\log u\right] \\
&= 1-\exp\left[\log u^{-k}\right] \\
&= 1-u^{-k} \\
&= 1-(1+x^c)^{-k} \\
&= 1-\frac{1}{(1+x^c)^k}
\end{aligned}
$$

Now,

$$F(x)=0 \Longrightarrow 1-\frac{1}{1}=1-\frac{1}{(1+0)^k}=0 \Longrightarrow x=0$$

$$F(x)=1 \Longrightarrow 1-0=1-\frac{1}{(1+\infty)^k}=1 \Longrightarrow x=\infty$$

Therefore

$$F(x)=1-[1+x^c]^{-k}, \quad 0<x<\infty \tag{5.34}$$

and its corresponding pdf is

$$f(x)=kcx^{c-1}[1+x^c]^{-k-1}, \quad 0<x<\infty \tag{5.35}$$

which is the **Burr XII distribution**.

The m^{th} moment of (5.35) is

$$
\begin{aligned}
E(X^m) &= \int_0^\infty x^m kcx^{(c-1)}\left[1+x^c\right]^{-(k+1)}dx \\
&= ck\int_0^\infty \frac{x^{m+c-1}}{\left[1+x^c\right]^{k+1}}dx
\end{aligned}
$$

Let

$$y = x^c \Longrightarrow dy = cx^{c-1}dx$$

Thus

$$\begin{aligned}
E(X^m) &= ck\int_0^\infty \frac{[y^{\frac{1}{c}}]^{m+c-1}}{[1+y]^{k+1}\,c[y^{\frac{1}{c}}]^{c-1}}dy \\
&= k\int_0^\infty \frac{[y^{\frac{1}{c}}]^{m}}{[1+y]^{k+1}}dy \\
&= k\int_0^\infty \frac{y^{\left(1+\frac{m}{c}\right)-1}}{[1+y]^{\left(k-\frac{m}{c}\right)+\left(1+\frac{m}{c}\right)}}dy \\
&= kB\left(k-\frac{m}{c}, 1+\frac{m}{c}\right), \quad c > m
\end{aligned} \tag{5.36}$$

5.4.1 Special Case

When c=1

$$\begin{aligned}
F(x) &= 1-[1+x]^{-k}, \quad 0 < x < \infty \\
&= 1-\left[\frac{1}{1+x}\right]^k
\end{aligned}$$

and its corresponding pdf is

$$f(x) = k[1+x]^{-k-1}, \quad 0 < x < \infty$$

which is the **Lomax/Pareto (Type II) distribution**.

When k=1

$$
\begin{aligned}
F(x) &= 1-[1+x^c]^{-1}, \quad 0<x<\infty \\
&= 1-\left[\frac{1}{1+x^c}\right] \\
&= \frac{(1+x^c)-1}{1+x^c} \\
&= \frac{x^c}{1+x^c} \\
&= \frac{1}{\frac{1}{x^c}+1} \\
&= \frac{1}{1+x^{-c}} \\
&= \left[x^{-c}+1\right]^{-1}
\end{aligned}
$$

and its corresponding pdf is

$$
f(x) = -cx^{-c-1}[x^{-c}+1]^{-2}, \quad 0<x<\infty
$$

which is the **Log-Logistic (Fisk) distribution**.

5.4.2 Extensions of Burr XII Distribution

Using Translational Transformation

Case I

Let

$$
Y = \frac{1}{X}
$$

where X has Burr XII distribution with cdf given by (5.30).

If $G(y)$ is the *cdf* of y, then

$$\begin{aligned} G(y) &= prob\ (Y \leq y) \\ &= prob\ \left(\frac{1}{X} \leq y\right) \\ &= prob\ \left(X \geq \frac{1}{y}\right) \\ &= 1 - prob\ \left(X \leq \frac{1}{y}\right) \end{aligned}$$

Therefore

$$\begin{aligned} G(y) &= 1 - F\left(\frac{1}{y}\right) \\ &= 1 - \left[1 - \left\{1 + \left(\frac{1}{y}\right)^c\right\}^{-k}\right] \\ &= \left[1 + \left(\frac{1}{y}\right)^c\right]^{-k} \end{aligned}$$

Thus

$$G(y) = \left[1 + y^{-c}\right]^{-k}, \quad 0 < y < \infty$$

and its corresponding pdf is

$$g(y) = kcy^{-c-1}\left[1 + y^{-c}\right]^{-k-1}, \quad 0 < y < \infty$$

which is the same form as (4.28) and (4.29), a result given by Tadikamalla (1980).

Case II

Let

$$X = \frac{1}{Y}$$

where Y has Burr III distribution with cdf given by

$$G(y) = (1 + y^{-c})^{-k} \quad 0 < y < \infty$$

If $F(x)$ is the *cdf* of x, then

$$\begin{aligned} F(x) &= prob\ (X \leq x) \\ &= prob\ \left(\frac{1}{Y} \leq x\right) \\ &= prob\ \left(Y \geq \frac{1}{x}\right) \\ &= 1 - prob\ \left(Y \leq \frac{1}{x}\right) \end{aligned}$$

Therefore

$$\begin{aligned} F(x) &= 1 - G\left(\frac{1}{x}\right) \\ &= 1 - \left[1 + \left(\frac{1}{x}\right)^{-c}\right]^{-k} \\ &= 1 - \left[1 + \frac{1}{x^{-c}}\right]^{-k} \end{aligned}$$

Thus

$$F(x) = 1 - [1 + x^c]^{-k}, \quad 0 < x < \infty$$

and its corresponding pdf is

$$f(x) = kcx^{c-1}[1 + x^c]^{-k-1}, \quad 0 < x < \infty$$

which is the same as (5.34) and (5.35).

Case III

Let

$$X = Y\sqrt[c]{\alpha}, \quad c > 0$$

where Y has Burr XII distribution with cdf given by

$$G(y) = 1 - [1 + y^c]^{-k}, \quad 0 < y < \infty$$

If $F(x)$ is the *cdf* of x, then

$$
\begin{aligned}
F(x) &= prob\ (X \leq x) \\
&= prob\ (\sqrt[c]{\alpha} Y \leq x) \\
&= prob\ \left(Y \leq \frac{x}{\sqrt[c]{\alpha}}\right)
\end{aligned}
$$

Therefore

$$F(x) = G\left(\frac{x}{\sqrt[c]{\alpha}}\right)$$

Thus

$$F(x) = 1 - \left[1 + \frac{x^c}{\alpha}\right]^{-k}, \quad 0 < x < \infty \tag{5.37}$$

and its corresponding pdf is

$$f(x) = \frac{kc}{\alpha} x^{c-1} \left[1 + \frac{x^c}{\alpha}\right]^{-k-1}, \quad 0 < x < \infty \tag{5.38}$$

which is another form of Burr XII distribution, a result given by Tadikamalla (1980).

Case IV

Let

$$X = \alpha Y$$

where Y has Burr XII distribution with cdf given by

$$G(y) = 1 - [1 + y^c]^{-k}, \quad 0 < y < \infty$$

If $F(x)$ is the *cdf* of x, then

$$
\begin{aligned}
F(x) &= prob\ (X \leq x) \\
&= prob\ (\alpha Y \leq x) \\
&= prob\ \left(Y \leq \frac{x}{\alpha}\right)
\end{aligned}
$$

Therefore

$$F(x) = G\left(\frac{x}{\alpha}\right)$$

Thus

$$F(x) = 1 - \left[1 + \left(\frac{x}{\alpha}\right)^{c}\right]^{-k}, \quad 0 < x < \infty \tag{5.39}$$

and its corresponding pdf is

$$f(x) = \frac{kc}{\alpha}\left(\frac{x}{\alpha}\right)^{c-1}\left[1 + \left(\frac{x}{\alpha}\right)^{c}\right]^{-k-1}, \quad 0 < x < \infty \tag{5.40}$$

which is also another form of Burr XII distribution known as the **Singh-Maddala distribution**, a result given by Tadikamalla (1980).

Using Power Transformation

Let

$$X = Y^{\frac{1}{c}}, \quad c > 0$$

where Y has Pareto II distribution with cdf given by

$$G(y) = 1 - \left[1 + \frac{y}{\beta}\right]^{-\alpha}$$

If $F(x)$ is the *cdf* of x, then

$$\begin{aligned} F(x) &= prob\ (X \leq x) \\ &= prob\ (Y^{\frac{1}{c}} \leq x) \\ &= prob\ (Y \leq x^{c}) \end{aligned}$$

Therefore

$$F(x) = G(x^{c})$$

Thus

$$F(x) = 1 - \left[1 + \frac{x^c}{\beta}\right]^{-\alpha}, \quad 0 < x < \infty$$

and its corresponding pdf is

$$f(x) = \frac{\alpha c}{\beta} x^{c-1} \left[1 + \frac{x^c}{\beta}\right]^{-\alpha-1}, \quad 0 < x < \infty$$

which is the same form as (5.37) and (5.38) with $\alpha = \beta$ and $k = \alpha$.

Using Mixtures

Case I

Let

$$f(x|\beta) = \alpha\beta x^{\alpha-1} e^{-\beta x^{\alpha}}, \quad x > 0,\ \alpha > 0,\ \beta > 0$$

where $X|\beta$ is Weibull *pdf* with shape parameter α and scale parameter $\beta^{-\frac{1}{\alpha}}$.

The *cdf* is given by

$$F(x|\beta) = \int_0^x \alpha\beta t^{\alpha-1} e^{-\beta t^{\alpha}} dt$$

Also let

$$y = e^{-\beta t^{\alpha}} \Longrightarrow dy = -\alpha\beta t^{\alpha-1} e^{-\beta t^{\alpha}} dt$$

Hence

$$\begin{aligned} F(x|\beta) &= -\int_1^{e^{-\beta x^{\alpha}}} dy \\ &= -[y]_1^{e^{-\beta x^{\alpha}}} \\ &= -[e^{-\beta x^{\alpha}} - 1] \\ &= 1 - e^{-\beta x^{\alpha}} \end{aligned}$$

If $g(\beta) = \dfrac{\delta^{\nu}}{\Gamma\nu} e^{-\delta\beta} \beta^{\nu-1}$ is a gamma *pdf*, then

$$\begin{aligned}
F(x) &= \int_0^\infty F(x|\beta)g(\beta)d\beta \\
&= \int_0^\infty [1-e^{-\beta x^\alpha}]\frac{\delta^\nu}{\Gamma\nu}e^{-\delta\beta}\beta^{\nu-1}d\beta \\
&= \frac{\delta^\nu}{\Gamma\nu}\int_0^\infty [e^{-\delta\beta}-e^{-\beta x^\alpha-\delta\beta}]\beta^{\nu-1}d\beta \\
&= \frac{\delta^\nu}{\Gamma\nu}\left\{\int_0^\infty e^{-\delta\beta}\beta^{\nu-1}d\beta-\int_0^\infty e^{-(x^\alpha+\delta)\beta}\beta^{\nu-1}d\beta\right\} \\
&= \frac{\delta^\nu}{\Gamma\nu}\left\{\frac{\Gamma\nu}{\delta^\nu}-\frac{\Gamma\nu}{(x^\alpha+\delta)^\nu}\right\} \\
&= \frac{\delta^\nu}{\delta^\nu}-\frac{\delta^\nu}{(x^\alpha+\delta)^\nu} \\
&= 1-\left[\frac{\delta}{x^\alpha+\delta}\right]^\nu \\
&= 1-\left[\frac{x^\alpha+\delta}{\delta}\right]^{-\nu} \\
&= 1-\left[1+\frac{x^\alpha}{\delta}\right]^{-\nu}, \quad 0<x<\infty
\end{aligned}$$

and its corresponding pdf is

$$f(x)=\frac{\nu\alpha}{\delta}x^{\alpha-1}\left[1+\frac{x^\alpha}{\delta}\right]^{-\nu-1}, \quad 0<x<\infty$$

which is the same form as (5.37) and (5.38) with $c=\alpha$, $\alpha=\delta$ and $k=\nu$.

Case II

Let

$$f(x|\beta)=\beta e^{-\beta x}, \quad x>0,\ \beta>0$$

where $X|\beta$ is exponential *pdf* with parameter β.

The *cdf* is given by

$$
\begin{aligned}
F(x|\beta) &= \int_0^x \beta e^{-\beta t} dt \\
&= \left[\frac{\beta e^{-\beta t}}{-\beta}\right]_0^x \\
&= -\left[e^{-\beta t}\right]_0^x \\
&= -\left[e^{-\beta x} - 1\right] \\
&= 1 - e^{-\beta x}
\end{aligned}
$$

If $g(\beta) = \dfrac{a^k}{\Gamma k} e^{-a\beta} \beta^{k-1}$ is a gamma *pdf*, then

$$
\begin{aligned}
F(x) &= \int_0^\infty F(x|\beta) g(\beta) d\beta \\
&= \int_0^\infty [1 - e^{-\beta x}] \frac{a^k}{\Gamma k} e^{-a\beta} \beta^{k-1} d\beta \\
&= \frac{a^k}{\Gamma k} \int_0^\infty [e^{-a\beta} - e^{-\beta x - a\beta}] \beta^{k-1} d\beta \\
&= \frac{a^k}{\Gamma k} \left\{ \int_0^\infty e^{-a\beta} \beta^{k-1} d\beta - \int_0^\infty e^{-(x+a)\beta} \beta^{k-1} d\beta \right\} \\
&= \frac{a^k}{\Gamma k} \left\{ \frac{\Gamma k}{a^k} - \frac{\Gamma k}{(x+a)^k} \right\} \\
&= \frac{a^k}{a^k} - \frac{a^k}{(x+a)^k} \\
&= 1 - \left[\frac{a}{x+a}\right]^k \\
&= 1 - \left[\frac{x+a}{a}\right]^{-k} \\
&= 1 - \left[1 + \frac{x}{a}\right]^{-k}, \quad 0 < x < \infty
\end{aligned}
$$

and its corresponding pdf is

$$f(x) = \frac{k}{a}\left[1+\frac{x}{a}\right]^{-k-1}, \quad 0<x<\infty$$

which is the same as (5.39) and (5.40) with $c=1$ and $\alpha=a$.

5.4.3 Beta Burr XII Distribution

Putting (5.34) in (2.3) gives

$$G(x) = \int_0^{1-[1+x^c]^{-k}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0<x<\infty,\ a>0,\ b>0 \tag{5.41}$$

and its corresponding pdf is

$$\begin{aligned}
g(x) &= \frac{\left\{1-[1+x^c]^{-k}\right\}^{a-1}\left[1-\left\{1-[1+x^c]^{-k}\right\}\right]^{b-1}}{B(a,b)}\frac{kcx^{c-1}}{[1+x^c]^{k+1}} \\
&= \frac{\left\{1-\dfrac{1}{[1+x^c]^k}\right\}^{a-1}\left\{1-1+[1+x^c]^{-k}\right\}^{b-1}}{B(a,b)}\frac{kcx^{c-1}}{[1+x^c]^{k+1}} \\
&= \frac{\left\{\dfrac{[1+x^c]^k-1}{[1+x^c]^k}\right\}^{a-1}\left\{\dfrac{1}{[1+x^c]^k}\right\}^{b-1}}{B(a,b)}\frac{kcx^{c-1}}{[1+x^c]^{k+1}} \\
&= \frac{\left\{[1+x^c]^k-1\right\}^{a-1}\{1\}^{b-1}}{[1+x^c]^{ka-k}[1+x^c]^{kb-k}B(a,b)}\frac{kcx^{c-1}}{[1+x^c]^{k+1}} \\
&= \frac{\left\{[1+x^c]^k-1\right\}^{a-1}kcx^{c-1}}{[1+x^c]^{ka-k+kb-k+k+1}B(a,b)} \\
&= \frac{\left\{[1+x^c]^k-1\right\}^{a-1}kcx^{c-1}}{[1+x^c]^{k(a+b-1)+1}B(a,b)}, \quad 0<x<\infty,\ a>0,\ b>0
\end{aligned} \tag{5.42}$$

i^{th} order statistic of Burr XII distribution

From (5.42),

$$g_i(x) = \frac{\left\{[1+x^c]^k-1\right\}^{i-1}kcx^{c-1}}{[1+x^c]^{kn+1}B(i,n-i+1)}, \quad 0<x<\infty \tag{5.43}$$

Putting $i = 1$ in (5.43),

$$g_1(x) = \frac{kncx^{c-1}}{[1+x^c]^{kn+1}}, \quad 0 < x < \infty \tag{5.44}$$

which is the **GB2** with shape parameters $p = 1, q = kn$, $a = c$ and scale parameter $b = 1$.

The m^{th} moment of (5.44) is

$$\begin{aligned} E(X_1^m) &= \int_0^\infty x^m \frac{nkcx^{(c-1)}}{[1+x^c]^{kn+1}} dx \\ &= nkc \int_0^\infty \frac{x^{m+c-1}}{[1+x^c]^{kn+1}} dx \end{aligned}$$

Let

$$y = x^c \Longrightarrow dy = cx^{c-1} dx$$

Thus

$$\begin{aligned} E(X_1^m) &= nkc \int_0^\infty \frac{[y^{\frac{1}{c}}]^{m+c-1}}{[1+y]^{kn+1} c[y^{\frac{1}{c}}]^{c-1}} dy \\ &= nk \int_0^\infty \frac{[y^{\frac{1}{c}}]^m}{[1+y]^{kn+1}} dy \\ &= nk \int_0^\infty \frac{y^{\left(1+\frac{m}{c}\right)-1}}{[1+y]^{\left(kn-\frac{m}{c}\right)+\left(1+\frac{m}{c}\right)}} dy \\ &= nkB\left(kn - \frac{m}{c}, 1 + \frac{m}{c}\right), \quad c > m \end{aligned} \tag{5.45}$$

Putting $i = n$ in (5.43),

$$g_n(x) = \frac{n\left\{[1+x^c]^k - 1\right\}^{n-1} kcx^{c-1}}{[1+x^c]^{kn+1}}, \quad 0 < x < \infty \tag{5.46}$$

5.4.4 Exponentiated Burr XII Distribution

Putting (5.34) in (2.8) gives

$$G(x)=\left\{1-[1+x^{c}]^{-k}\right\}^{r},\quad 0<x<\infty,\ r>0 \tag{5.47}$$

and its corresponding pdf is

$$g(x)=rkcx^{c-1}[1+x^{c}]^{-k-1}\left\{1-[1+x^{c}]^{-k}\right\}^{r-1},\quad 0<x<\infty,\ r>0 \tag{5.48}$$

5.4.5 Beta Exponentiated Burr XII Distribution

Putting (5.47) in (2.10) gives

$$G(x)=\int_{0}^{\left\{1-[1+x^{c}]^{-k}\right\}^{r}}\frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt,\quad 0<x<\infty,\ a>0,\ b>0,\ r>0 \tag{5.49}$$

and its corresponding pdf is

$$\begin{aligned}
g(x)&=\frac{r\left\{1-[1+x^{c}]^{-k}\right\}^{ra-1}\left[1-\left\{1-[1+x^{c}]^{-k}\right\}^{r}\right]^{b-1}}{B(a,b)}\frac{kcx^{c-1}}{[1+x^{c}]^{k+1}}\\
&=\frac{r\left\{1-\frac{1}{[1+x^{c}]^{k}}\right\}^{ra-1}\left[1-\left\{1-\frac{1}{[1+x^{c}]^{k}}\right\}^{r}\right]^{b-1}}{B(a,b)}\frac{kcx^{c-1}}{[1+x^{c}]^{k+1}}\\
&=\frac{r\left\{\frac{[1+x^{c}]^{k}-1}{[1+x^{c}]^{k}}\right\}^{ra-1}\left[1-\left\{\frac{[1+x^{c}]^{k}-1}{[1+x^{c}]^{k}}\right\}^{r}\right]^{b-1}}{B(a,b)}\frac{kcx^{c-1}}{[1+x^{c}]^{k+1}}\\
&=\frac{r\left\{[1+x^{c}]^{k}-1\right\}^{ra-1}\left[\frac{\left\{1+x^{c}\right\}^{kr}-\left\{[1+x^{c}]^{k}-1\right\}^{r}}{\left\{1+x^{c}\right\}^{kr}}\right]^{b-1}}{[1+x^{c}]^{kra-k}B(a,b)}\frac{kcx^{c-1}}{[1+x^{c}]^{k+1}}\\
&=\frac{r\left\{[1+x^{c}]^{k}-1\right\}^{ra-1}\left[\left\{1+x^{c}\right\}^{kr}-\left\{[1+x^{c}]^{k}-1\right\}^{r}\right]^{b-1}}{[1+x^{c}]^{kra-k}[1+x^{c}]^{krb-kr}B(a,b)}\frac{kcx^{c-1}}{[1+x^{c}]^{k+1}}\\
&=\frac{r\left\{[1+x^{c}]^{k}-1\right\}^{ra-1}\left[\left\{1+x^{c}\right\}^{kr}-\left\{[1+x^{c}]^{k}-1\right\}^{r}\right]^{b-1}}{[1+x^{c}]^{kra-k+krb-kr+k+1}B(a,b)}kcx^{c-1}\\
&=\frac{r\left\{[1+x^{c}]^{k}-1\right\}^{ra-1}\left[\left\{1+x^{c}\right\}^{kr}-\left\{[1+x^{c}]^{k}-1\right\}^{r}\right]^{b-1}}{[1+x^{c}]^{kr(a+b-1)+1}B(a,b)}kcx^{c-1}
\end{aligned} \tag{5.50}$$

i^{th} order statistic of Exponentiated Burr XII distribution

From (5.50),

$$g_i(x) = \frac{r\left\{[1+x^c]^k-1\right\}^{ri-1}\left[\left\{1+x^c\right\}^{kr}-\left\{[1+x^c]^k-1\right\}^r\right]^{n-i}}{[1+x^c]^{krn+1}B(i,n-i+1)}kcx^{c-1} \tag{5.51}$$

Putting $i=1$ in (5.51),

$$g_1(x) = \frac{rn\left\{[1+x^c]^k-1\right\}^{r-1}\left[\left\{1+x^c\right\}^{kr}-\left\{[1+x^c]^k-1\right\}^r\right]^{n-1}}{[1+x^c]^{krn+1}}kcx^{c-1} \tag{5.52}$$

Putting $i=n$ in (5.51),

$$g_n(x) = \frac{rn\left\{[1+x^c]^k-1\right\}^{rn-1}}{[1+x^c]^{krn+1}}kcx^{c-1}, \quad 0<x<\infty,\ r>0 \tag{5.53}$$

5.5 $g(x) = \dfrac{cx}{1-cx}, \quad c>0$

Then

$$\begin{aligned}\int \frac{g(x)}{x}dx &= c\int \frac{dx}{1-cx}\\ &= -\log(1-cx)\end{aligned}$$

Substituting in (5.3) we get

$$\begin{aligned}F(x) &= 1-\exp[\log(1-cx)]\\ &= 1-(1-cx)\\ &= cx\end{aligned}$$

Now,

$$F(x)=0 \Longrightarrow c(0)=0 \Longrightarrow x=0$$

$$F(x) = 1 \Longrightarrow c\left(\frac{1}{c}\right) = 1 \Longrightarrow x = \frac{1}{c}$$

Therefore

$$F(x) = cx, \quad 0 < x < \frac{1}{c} \tag{5.54}$$

and its corresponding pdf is

$$f(x) = c, \quad 0 < x < \frac{1}{c} \tag{5.55}$$

The m^{th} moment of (5.55) is

$$\begin{aligned} E(X^m) &= \int_0^{\frac{1}{c}} cx^m dx \\ &= c\left[\frac{x^{m+1}}{m+1}\right]_0^{\frac{1}{c}} \\ &= c\left[\frac{1}{m+1}\left(\frac{1}{c}\right)^{m+1}\right] \\ &= \frac{c}{c^{m+1}}\left[\frac{1}{m+1}\right] \\ &= \frac{1}{c^m}B(m+1,1) \end{aligned} \tag{5.56}$$

5.5.1 Beta Generated Distribution

Putting (5.54) in (2.3) gives

$$G(x) = \int_0^{cx} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0 < x < \frac{1}{c},\ a > 0,\ b > 0 \tag{5.57}$$

and its corresponding pdf is

$$
\begin{aligned}
g(x) &= \frac{[cx]^{a-1}[1-cx]^{b-1}}{B(a,b)}c \\
&= \frac{c^{a-1+1}[x]^{a-1}[1-cx]^{b-1}}{B(a,b)} \\
&= \frac{c^{a}[x]^{a-1}[1-cx]^{b-1}}{B(a,b)}, \quad 0<x<\frac{1}{c},\ a>0,\ b>0
\end{aligned}
\tag{5.58}
$$

The m^{th} moment of (5.58) is

$$
\begin{aligned}
E(X^m) &= \int_0^{\frac{1}{c}} x^m \frac{c^a[x]^{a-1}[1-cx]^{b-1}}{B(a,b)}dx \\
&= \int_0^{\frac{1}{c}} \frac{c^a x^{m+a-1}[1-cx]^{b-1}}{B(a,b)}dx
\end{aligned}
$$

Let

$$y = cx \Longrightarrow dy = cdx$$

Thus

$$
\begin{aligned}
E(X^m) &= \int_0^1 \frac{c^a\left(\frac{y}{c}\right)^{m+a-1}[1-y]^{b-1}}{cB(a,b)}dy \\
&= \int_0^1 \frac{c^a y^{m+a-1}[1-y]^{b-1}}{c^{1+m+a-1}B(a,b)}dy \\
&= \int_0^1 \frac{y^{m+a-1}[1-y]^{b-1}}{c^{1+m+a-1-a}B(a,b)}dy \\
&= \int_0^1 \frac{y^{m+a-1}[1-y]^{b-1}}{c^m B(a,b)}dy \\
&= \frac{\beta(m+a,b)}{c^m B(a,b)}
\end{aligned}
\tag{5.59}
$$

i^{th} order statistic of $F(x) = cx$

From (5.58),

$$g_i(x) = \frac{c^i\,[x]^{i-1}\,[1-cx]^{n-i}}{B(i, n-i+1)}, \quad 0 < x < \frac{1}{c} \tag{5.60}$$

The m^{th} moment of (5.60) is

$$E(X_i^m) = \frac{B(m+i, n-i+1)}{c^m B(i, n-i+1)} \tag{5.61}$$

Putting $i = 1$ in (5.60),

$$g_1(x) = nc\,[1-cx]^{n-1}, \quad 0 < x < \frac{1}{c} \tag{5.62}$$

The m^{th} moment of (5.62) is

$$E(X_1^m) = \frac{nB(m+1, n)}{c^m} \tag{5.63}$$

Putting $i = n$ in (5.60),

$$g_n(x) = nc^n\,[x]^{n-1}, \quad 0 < x < \frac{1}{c} \tag{5.64}$$

The m^{th} moment of (5.64) is

$$E(X_n^m) = \frac{nB(m+n, 1)}{c^m} \tag{5.65}$$

5.5.2 Exponentiated Generated Distribution

Putting (5.54) in (2.8) gives

$$G(x) = [cx]^r, \quad 0 < x < \frac{1}{c},\ r > 0 \tag{5.66}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= rc\,[cx]^{r-1} \\ &= rc^{1+r-1}\,[x]^{r-1} \\ &= rc^{r}\,[x]^{r-1}, \quad 0 < x < \frac{1}{c},\ r > 0 \end{aligned} \tag{5.67}$$

The m^{th} moment of (5.67) is

$$\begin{aligned} E(X^m) &= \int_0^{\frac{1}{c}} x^m rc^r\,[x]^{r-1} dx \\ &= rc^r \int_0^{\frac{1}{b}} x^{m+r-1} dx \\ &= rc^r \left[\frac{x^{m+r}}{m+r}\right]_0^{\frac{1}{b}} \\ &= rc^r \left[\frac{1}{m+r}\left(\frac{1}{b}\right)^{m+r}\right] \\ &= \frac{rc^r}{c^{m+r}}\left[\frac{1}{m+r}\right] \\ &= \frac{r}{c^m} B(m+r,1) \end{aligned} \tag{5.68}$$

5.5.3 Beta Exponentiated Generated Distribution

Putting (5.66) in (2.10) gives

$$G(x) = \int_0^{[cx]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < \frac{1}{c},\ a > 0,\ b > 0 \tag{5.69}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{r\,[cx]^{ra-1}\,[1-\{cx\}^r]^{b-1}}{B(a,b)} c \\ &= \frac{rc^{ra-1+1}\,[x]^{ra-1}\,[1-\{cx\}^r]^{b-1}}{B(a,b)} \\ &= \frac{rc^{ra}\,[x]^{ra-1}\,[1-\{cx\}^r]^{b-1}}{B(a,b)}, \quad 0 < x < \frac{1}{c},\ a > 0,\ b > 0 \end{aligned} \tag{5.70}$$

The m^{th} moment of (5.70) is

$$E(X^m) = \int_0^{\frac{1}{c}} x^m \frac{rc^{ra}[x]^{ra-1}\left[1-\{cx\}^r\right]^{b-1}}{B(a,b)}dx$$
$$= \int_0^{\frac{1}{c}} \frac{rc^{ra}x^{m+ra-1}\left[1-\{cx\}^r\right]^{b-1}}{B(a,b)}dx$$

Let

$$y = [cx]^r \Longrightarrow dy = rc^r[x]^{r-1}dx$$

Thus

$$E(X^m) = \int_0^1 \frac{rc^{ra}\left(\frac{y^{\frac{1}{r}}}{c}\right)^{m+ra-1}[1-y]^{b-1}}{rc^r\left(\frac{y^{\frac{1}{r}}}{c}\right)^{r-1}B(a,b)}dy$$
$$= \int_0^1 \frac{c^{ra-m-ra+1}y^{\frac{m+ra-1-r+1}{r}}[1-y]^{b-1}}{c^{r-r+1}B(a,b)}dy$$
$$= \int_0^1 \frac{y^{\frac{m}{r}+a-1}[1-y]^{b-1}}{c^{1+m-1}B(a,b)}dy \tag{5.71}$$
$$= \int_0^1 \frac{y^{\frac{m}{r}+a-1}[1-y]^{b-1}}{c^m B(a,b)}dy$$
$$= \frac{\beta\left(\frac{m}{r}+a,b\right)}{c^m B(a,b)}, \quad r > m$$

i^{th} order statistic of $G(x) = [cx]^r$

From (5.70),

$$g_i(x) = \frac{rc^{ri}[x]^{ri-1}\left[1-\{cx\}^r\right]^{n-i}}{B(i,n-i+1)}, \quad 0 < x < \frac{1}{c},\ r > 0 \tag{5.72}$$

The m^{th} moment of (5.72) is

$$E(X_i^m) = \frac{\beta\left(\frac{m}{r}+i, n-i+1\right)}{c^m B(i, n-i+1)} \tag{5.73}$$

Putting $i = 1$ in (5.72),

$$g_1(x) = rnc^r\,[x]^{r-1}\,[1-\{cx\}^r]^{n-1}, \quad 0 < x < \frac{1}{c},\ r > 0 \tag{5.74}$$

The m^{th} moment of (5.74) is

$$E(X_1^m) = \frac{n\beta\left(\frac{m}{r}+1, n\right)}{c^m}, \quad r > m \tag{5.75}$$

Putting $i = n$ in (5.72),

$$g_n(x) = rnc^{rn}\,[x]^{rn-1}, \quad 0 < x < \frac{1}{c},\ r > 0 \tag{5.76}$$

The m^{th} moment of (5.76) is

$$E(X_n^m) = \frac{n\beta\left(\frac{m}{r}+n, 1\right)}{c^m}, \quad r > m \tag{5.77}$$

5.6 $g(x) = cx, \quad c > 0$

Then

$$\int \frac{g(x)}{x}dx = c\int dx$$
$$= cx$$

Substituting in (5.3) we get

$$F(x) = 1 - e^{-cx}$$

Now,

$$F(x) = 0 \Longrightarrow 1-1 = 1 - \frac{1}{e^{c(0)}} = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow 1 - 0 = 1 - \frac{1}{e^{c(\infty)}} = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = 1 - e^{-cx}, \quad 0 < x < \infty \tag{5.78}$$

and its corresponding pdf is

$$f(x) = ce^{-cx}, \quad 0 < x < \infty$$

The m^{th} moment of (5.50) is

$$E(X^m) = \int_0^\infty x^m ce^{-cx} dx$$

Let

$$y = cx \Longrightarrow dy = cdx$$

Thus

$$\begin{aligned} E(X^m) &= \int_0^\infty \left(\frac{y}{c}\right)^m ce^{-y} \frac{dy}{c} \\ &= \frac{1}{c^m} \int_0^\infty y^m e^{-y} dy \\ &= \frac{1}{c^m} \int_0^\infty y^{m+1-1} e^{-y} dy \\ &= \frac{\Gamma(m+1)}{c^m} \end{aligned} \tag{5.79}$$

5.6.1 Beta Generated Distribution

Putting (5.50) in (2.3) gives

$$G(x) = \int_0^{1-e^{-cx}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < \infty, \; a > 0, \; b > 0 \tag{5.80}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{[1-e^{-cx}]^{a-1}[1-\{1-e^{-cx}\}]^{b-1}}{B(a,b)}ce^{-cx} \\ &= \frac{[1-e^{-cx}]^{a-1}[e^{-cx}]^{b-1}}{B(a,b)}ce^{-cx} \\ &= \frac{[1-e^{-cx}]^{a-1}[e^{-cx}]^{b-1+1}}{B(a,b)}c \\ &= \frac{[1-e^{-cx}]^{a-1}[e^{-cx}]^{b}}{B(a,b)}c \\ &= \frac{[1-e^{-cx}]^{a-1}ce^{-bcx}}{B(a,b)}, \quad 0<x<\infty,\ a>0,\ b>0 \end{aligned} \tag{5.81}$$

i^{th} **order statistic of** $F(x) = 1-e^{-cx}$

From (5.53),

$$g_i(x) = \frac{[1-e^{-cx}]^{i-1}ce^{-(n-i+1)cx}}{B(i,n-i+1)}, \quad 0<x<\infty \tag{5.82}$$

Putting $i=1$ in (5.54),

$$g_1(x) = nce^{-ncx}, \quad 0<x<\infty \tag{5.83}$$

The m^{th} moment of (5.55) is

$$E(X_1^m) = \int_0^\infty x^m nce^{-ncx}dx$$

Let

$$y = ncx \Longrightarrow dy = ncdx$$

Thus

$$\begin{aligned}E(X_1^m) &= \int_0^\infty \left(\frac{y}{nc}\right)^m nce^{-y}\frac{dy}{nc}\\&= \frac{1}{(nc)^m}\int_0^\infty y^m e^{-y}dy\\&= \frac{1}{(nc)^m}\int_0^\infty y^{m+1-1}e^{-y}dy\\&= \frac{\Gamma(m+1)}{(nc)^m}\end{aligned} \tag{5.84}$$

Putting $i = n$ in (5.54),

$$g_n(x) = \left[1-e^{-cx}\right]^{n-1} nce^{-cx}, \quad 0 < x < \infty \tag{5.85}$$

5.6.2 Exponentiated Generated Distribution

Putting (5.50) in (2.8) gives

$$G(x) = \left[1-e^{-cx}\right]^r, \quad 0 < x < \infty,\ r > 0 \tag{5.86}$$

and its corresponding pdf is

$$g(x) = rce^{-cx}\left[1-e^{-cx}\right]^{r-1}, \quad 0 < x < \infty,\ r > 0 \tag{5.87}$$

5.6.3 Beta Exponentiated Generated Distribution

Putting (5.58) in (2.10) gives

$$G(x) = \int_0^{\left[1-e^{-cx}\right]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0 < x < \infty,\ a > 0,\ b > 0,\ r > 0 \tag{5.88}$$

and its corresponding pdf is

$$g(x) = \frac{r\left[1-e^{-cx}\right]^{ra-1}\left[1-\{1-e^{-cx}\}^{r}\right]^{b-1}}{B(a,b)} ce^{-cx}, \;\; 0<x<\infty, \; a>0, \; b>0, \; r>0 \tag{5.89}$$

i^{th} order statistic of $G(x) = [1-e^{-cx}]^r$

From (5.61),

$$g_i(x) = \frac{r\left[1-e^{-cx}\right]^{ri-1}\left[1-\{1-e^{-cx}\}^{r}\right]^{n-i}}{B(i,n-i+1)} ce^{-cx}, \;\; 0<x<\infty \tag{5.90}$$

Putting $i=1$ in (5.62),

$$g_1(x) = rn\left[1-e^{-cx}\right]^{r-1}\left[1-\left\{1-e^{-cx}\right\}^{r}\right]^{n-1} ce^{-cx}, \;\; 0<x<\infty \tag{5.91}$$

Putting $i=n$ in (5.62),

$$g_n(x) = rn\left[1-e^{-cx}\right]^{rn-1} ce^{-cx}, \;\; 0<x<\infty \tag{5.92}$$

5.7 $g(x) = (c-x)^{-1}, \quad c>0$

Then

$$\int \frac{g(x)}{x} dx = \int \frac{dx}{x(c-x)}$$

$$\equiv \frac{A}{x} + \frac{B}{c-x} = \frac{A(c-x)+Bx}{x(c-x)}$$

Solving for A and B

$$1 \equiv Ac - Ax + Bx = Ac + (B-A)x$$

$$Ac = 1 \Longrightarrow A = \frac{1}{c}$$

$$B - A = 0 \Longrightarrow B = A = \frac{1}{c}$$

Thus

$$\begin{aligned}
\int \frac{g(x)}{x} dx &= \int \frac{dx}{x(c-x)} dx \\
&= \frac{1}{c} \int \left(\frac{1}{x} + \frac{1}{c-x} \right) dx \\
&= \frac{1}{c} [\log x - \log (c-x)] \\
&= \frac{1}{c} \log \left(\frac{x}{c-x} \right)
\end{aligned}$$

Substituting in (5.3) we get

$$\begin{aligned}
F(x) &= 1 - \exp \left[-\frac{1}{c} \log \left(\frac{x}{c-x} \right) \right] \\
&= 1 - \exp \left[\log \left(\frac{x}{c-x} \right)^{-\frac{1}{c}} \right] \\
&= 1 - \left(\frac{x}{c-x} \right)^{-\frac{1}{c}} \\
&= 1 - \left(\frac{c-x}{x} \right)^{\frac{1}{c}} \\
&= 1 - \left(\frac{c}{x} - 1 \right)^{\frac{1}{c}}
\end{aligned}$$

Now,

$$F(x) = 0 \Longrightarrow 1 - (2-1) = 1 - \left(\frac{2c}{c} - 1 \right)^{\frac{1}{c}} = 0 \Longrightarrow x = \frac{c}{2}$$

$$F(x) = 1 \Longrightarrow 1 - (1-1) = 1 - \left(\frac{c}{c} - 1 \right)^{\frac{1}{c}} = 1 \Longrightarrow x = c$$

Therefore

$$F(x) = 1 - \left(\frac{c-x}{x} \right)^{\frac{1}{c}}, \quad \frac{c}{2} < x < c \tag{5.93}$$

and its corresponding pdf is

$$f(x) = \frac{1}{x^2}\left(\frac{c-x}{x}\right)^{\frac{1}{c}-1} = \frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}}, \quad \frac{c}{2} < x < c \tag{5.94}$$

5.7.1 Beta Generated Distribution

Putting (5.93) in (2.3) gives

$$G(x) = \int_0^{1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad \frac{c}{2} < x < c,\ a > 0,\ b > 0 \tag{5.95}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{\left[1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{a-1}\left[1-\left\{1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right\}\right]^{b-1}}{B(a,b)} \frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}} \\ &= \frac{\left[1-\frac{(c-x)^{\frac{1}{c}}}{x^{\frac{1}{c}}}\right]^{a-1}\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{b-1}}{B(a,b)} \frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}} \\ &= \frac{\left[\frac{x^{\frac{1}{c}}-(c-x)^{\frac{1}{c}}}{x^{\frac{1}{c}}}\right]^{a-1}\left[\frac{(c-x)^{\frac{1}{c}}}{x^{\frac{1}{c}}}\right]^{b-1}}{B(a,b)} \frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}} \\ &= \frac{\left[x^{\frac{1}{c}}-(c-x)^{\frac{1}{c}}\right]^{a-1}(c-x)^{\frac{b}{c}-\frac{1}{c}+\frac{1}{c}-1}}{x^{\frac{a}{c}-\frac{1}{c}+\frac{b}{c}-\frac{1}{c}+\frac{1}{c}+1}B(a,b)} \\ &= \frac{\left[x^{\frac{1}{c}}-(c-x)^{\frac{1}{c}}\right]^{a-1}(c-x)^{\frac{b}{c}-1}}{x^{\frac{a+b-1}{c}+1}B(a,b)}, \quad \frac{c}{2} < x < c,\ a > 0,\ b > 0 \end{aligned} \tag{5.96}$$

i^{th} **order statistic of** $F(x) = 1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}$

From (5.96),

$$g_i(x) = \frac{\left[x^{\frac{1}{c}} - (c-x)^{\frac{1}{c}}\right]^{i-1} (c-x)^{\frac{n-i+1}{c}-1}}{x^{\frac{n}{c}+1} B(i, n-i+1)}, \quad \frac{c}{2} < x < c \tag{5.97}$$

Putting $i = 1$ in (5.97),

$$g_1(x) = \frac{n\,(c-x)^{\frac{n}{c}-1}}{x^{\frac{n}{c}+1}}, \quad \frac{c}{2} < x < c \tag{5.98}$$

Putting $i = n$ in (5.97),

$$g_n(x) = \frac{n\left[x^{\frac{1}{c}} - (c-x)^{\frac{1}{c}}\right]^{n-1} (c-x)^{\frac{1}{c}-1}}{x^{\frac{n}{c}+1}}, \quad \frac{c}{2} < x < c \tag{5.99}$$

5.7.2 Exponentiated Generated Distribution

Putting (5.93) in (2.8) gives

$$G(x) = \left[1 - \left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^r, \quad \frac{c}{2} < x < c,\ r > 0 \tag{5.100}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{r}{x^2}\left(\frac{c-x}{x}\right)^{\frac{1}{c}-1}\left[1 - \left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{r-1} \\ &= r\frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}}\left[1 - \left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{r-1}, \quad \frac{c}{2} < x < c,\ r > 0 \end{aligned} \tag{5.101}$$

5.7.3 Beta Exponentiated Generated Distribution

Putting (5.100) in (2.10) gives

$$G(x) = \int_0^{\left[1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad \frac{c}{2} < x < c,\ a > 0,\ b > 0,\ r > 0 \tag{5.102}$$

and its corresponding pdf is

$$g(x)=\frac{r\left[1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{ra-1}\left[1-\left\{1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right\}^{r}\right]^{b-1}}{B(a,b)}\frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}} \tag{5.103}$$

i^{th} **order statistic of** $G(x)=\left[1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{r}$

From (5.103),

$$g_i(x)=\frac{r\left[1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{ri-1}\left[1-\left\{1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right\}^{r}\right]^{n-i}}{B(i,n-i+1)}\frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}},\quad \frac{c}{2}<x<c,\ r>0 \tag{5.104}$$

Putting $i=1$ in (5.104),

$$g_1(x)=rn\left[1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{r-1}\left[1-\left\{1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right\}^{r}\right]^{n-1}\frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}} \tag{5.105}$$

Putting $i=n$ in (5.104),

$$g_n(x)=rn\left[1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{rn-1}\frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}},\quad \frac{c}{2}<x<c,\ r>0 \tag{5.106}$$

5.8 $g(x)=cx\sec^2 x,\quad c>0$

Then

$$\int\frac{g(x)}{x}dx=c\int\sec^2 x\,dx$$

Let

$$t=\tan x$$
$$dt=sec^2x\,dx$$

Thus

$$\begin{aligned}\int \frac{g(x)}{x} &= c\int dt\\ &= ct\\ &= c\tan x\end{aligned}$$

Substituting in (5.3) we get

$$F(x) = 1 - e^{-c\tan x}$$

Now,

$$F(x) = 0 \Longrightarrow 1 - \frac{1}{1} = 1 - \frac{1}{e^{c(0)}} = 0 \Longrightarrow x = -\frac{\pi}{2}$$

$$F(x) = 1 \Longrightarrow 1 - 0 = 1 - \frac{1}{e^{c(\infty)}} = 1 \Longrightarrow x = \frac{\pi}{2}$$

Therefore

$$F(x) = 1 - e^{-c\tan x}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \tag{5.107}$$

and its corresponding pdf is

$$f(x) = ce^{-c\tan x} sec^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \tag{5.108}$$

5.8.1 Beta Generated Distribution

Putting (5.107) in (2.3) gives

$$G(x) = \int_0^{1-e^{-c\tan x}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},\ a > 0,\ b > 0 \tag{5.109}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{\left[1-e^{-c\tan x}\right]^{a-1}\left[1-\left\{1-e^{-c\tan x}\right\}\right]^{b-1}}{B(a,b)} ce^{-c\tan x} sec^2 x \\ &= \frac{\left[1-e^{-c\tan x}\right]^{a-1}\left[e^{-c\tan x}\right]^{b-1}}{B(a,b)} ce^{-c\tan x} sec^2 x \\ &= \frac{\left[1-e^{-c\tan x}\right]^{a-1}\left[e^{-c\tan x}\right]^{b-1+1}}{B(a,b)} csec^2 x \\ &= \frac{\left[1-e^{-c\tan x}\right]^{a-1}\left[e^{-c\tan x}\right]^{b}}{B(a,b)} csec^2 x \\ &= \frac{\left[1-e^{-c\tan x}\right]^{a-1} ce^{-bc\tan x} sec^2 x}{B(a,b)}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},\ a>0,\ b>0 \end{aligned} \tag{5.110}$$

i^{th} **order statistic of** $F(x) = 1-e^{-c\tan x}$

From (5.110),

$$g_i(x) = \frac{\left[1-e^{-c\tan x}\right]^{i-1} ce^{-(n-i+1)c\tan x} sec^2 x}{B(i,n-i+1)}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \tag{5.111}$$

Putting $i = 1$ in (5.111),

$$g_1(x) = nce^{-nc\tan x} sec^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \tag{5.112}$$

Putting $i = n$ in (5.111),

$$g_n(x) = n\left[1-e^{-c\tan x}\right]^{n-1} ce^{-c\tan x} sec^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \tag{5.113}$$

5.8.2 Exponentiated Generated Distribution

Putting (5.107) in (2.8) gives

$$G(x) = \left[1-e^{-c\tan x}\right]^r, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},\ r>0 \tag{5.114}$$

and its corresponding pdf is

$$g(x) = rce^{-c\tan x}sec^2x\left[1-e^{-c\tan x}\right]^{r-1}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},\ r>0 \tag{5.115}$$

5.8.3 Beta Exponentiated Generated Distribution

Putting (5.114) in (2.10) gives

$$G(x) = \int_0^{\left[1-e^{-c\tan x}\right]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad -\frac{\pi}{2} < x < \frac{\pi}{2},\ a>0,\ b>0,\ r>0 \tag{5.116}$$

and its corresponding pdf is

$$g(x) = \frac{r\left[1-e^{-c\tan x}\right]^{ra-1}\left[1-\left\{1-e^{-c\tan x}\right\}^r\right]^{b-1}}{B(a,b)}ce^{-c\tan x}sec^2x \tag{5.117}$$

i^{th} **order statistic of** $G(x) = \left[1-e^{-c\tan x}\right]^r$

From (5.117),

$$g_i(x) = \frac{r\left[1-e^{-c\tan x}\right]^{i-1}ce^{-(n-i+1)c\tan x}sec^2x}{B(i,n-i+1)}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \tag{5.118}$$

Putting $i=1$ in (5.118),

$$g_1(x) = rnce^{nc\tan x}sec^2x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \tag{5.119}$$

Putting $i=n$ in (5.118),

$$g_n(x) = rn\left[1-e^{-c\tan x}\right]^{n-1}ce^{-c\tan x}sec^2x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \tag{5.120}$$

5.9 $g(x) = \alpha, \quad \alpha > 0$

Then

$$\begin{aligned}\int \frac{g(x)}{x} dx &= \alpha \int \frac{dx}{x} \\ &= \alpha \log x\end{aligned}$$

Substituting in (5.3) we get

$$\begin{aligned}F(x) &= 1 - \exp[-\alpha \log x] \\ &= 1 - \exp[\log x^{-\alpha}] \\ &= 1 - x^{-\alpha} \\ &= 1 - \left[x^{-1}\right]^{\alpha} \\ &= 1 - \left[\frac{1}{x}\right]^{\alpha}\end{aligned}$$

Now,

$$F(x) = 0 \Longrightarrow 1 - \frac{1}{1} = 1 - \frac{1}{1^{\alpha}} = 0 \Longrightarrow x = 1$$

$$F(x) = 1 \Longrightarrow 1 - 0 = 1 - \frac{1}{\infty^{\alpha}} = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = 1 - \left[\frac{1}{x}\right]^{\alpha}, \quad 1 < x < \infty \tag{5.121}$$

and its corresponding pdf is

$$\begin{aligned}f(x) &= \alpha x^{-\alpha-1} \\ &= \frac{\alpha}{x^{\alpha+1}}, \quad 1 < x < \infty\end{aligned} \tag{5.122}$$

which is the **Pareto (Type I) distribution**.

5.9.1 Beta Pareto (Type I) Distribution

Putting (5.121) in (2.3) gives

$$G(x) = \int_{0}^{1-\left[\frac{1}{x}\right]^{\alpha}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 1 < x < \infty,\ a > 0,\ b > 0 \tag{5.123}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{\left[1-\left[\frac{1}{x}\right]^{\alpha}\right]^{a-1}\left[1-\left\{1-\left[\frac{1}{x}\right]^{\alpha}\right\}\right]^{b-1}}{B(a,b)} \frac{\alpha}{x^{\alpha+1}} \\ &= \frac{\left[1-x^{-\alpha}\right]^{a-1}\left[1-\left\{1-x^{-\alpha}\right\}\right]^{b-1}}{B(a,b)} \alpha x^{-\alpha-1} \\ &= \frac{\left[1-x^{-\alpha}\right]^{a-1}\left[x^{-\alpha}\right]^{b-1}}{B(a,b)} \alpha x^{-\alpha-1} \\ &= \frac{\left[1-x^{-\alpha}\right]^{a-1} x^{-\alpha b+\alpha-\alpha-1}}{B(a,b)} \alpha \\ &= \frac{\left[1-x^{-\alpha}\right]^{a-1} x^{-\alpha b-1}}{B(a,b)} \alpha, \quad 1 < x < \infty,\ a > 0,\ b > 0 \end{aligned} \tag{5.124}$$

The m^{th} moment of (5.124) is

$$\begin{aligned} E(X^m) &= \int_{1}^{\infty} x^m \frac{\left[1-x^{-\alpha}\right]^{a-1}\left[x^{-\alpha}\right]^{b-1}}{B(a,b)} \alpha x^{-\alpha-1} dx \\ &= \int_{1}^{\infty} \alpha x^{m-\alpha-1} \frac{\left[1-x^{-\alpha}\right]^{a-1}\left[x^{-\alpha}\right]^{b-1}}{B(a,b)} dx \end{aligned}$$

Let

$$y = x^{-\alpha} \Longrightarrow dy = -\alpha x^{-\alpha-1} dx$$

Thus

$$
\begin{aligned}
E(X^m) &= -\int_1^0 \frac{\alpha[y^{-\frac{1}{\alpha}}]^{m-\alpha-1}\,[1-y]^{a-1}\,[y]^{b-1}}{B(a,b)\alpha[y^{-\frac{1}{\alpha}}]^{-\alpha-1}}dy \\
&= \int_0^1 \frac{[y^{-\frac{1}{\alpha}}]^{m}\,[1-y]^{a-1}\,[y]^{b-1}}{B(a,b)}dy \\
&= \int_0^1 \frac{y^{-\frac{m}{\alpha}+b-1}\,[1-y]^{a-1}}{B(a,b)}dy \\
&= \frac{B\Big(b-\dfrac{m}{\alpha},a\Big)}{B(a,b)}, \quad \alpha > m
\end{aligned} \tag{5.125}
$$

i^{th} order statistic of Pareto (Type I) distribution

From (5.124),

$$
g_i(x) = \frac{[1-x^{-\alpha}]^{i-1}\,[x^{-\alpha}]^{n-i}}{B(i,n-i+1)}\alpha x^{-\alpha-1}, \quad 1 < x < \infty \tag{5.126}
$$

The m^{th} moment of (5.126) is

$$
E(X_i^m) = \frac{B\Big(n-i+1-\dfrac{m}{\alpha},i\Big)}{B(i,n-i+1)}, \;\; \alpha > m \tag{5.127}
$$

Putting $i = 1$ in (5.126),

$$
\begin{aligned}
g_1(x) &= n\left[x^{-\alpha}\right]^{n-1}\alpha x^{-\alpha-1} \\
&= n x^{-\alpha n+\alpha-\alpha-1}\alpha \\
&= n\alpha x^{-\alpha n-1}, \quad 1 < x < \infty
\end{aligned} \tag{5.128}
$$

The m^{th} moment of (5.128) is

$$
E(X_1^m) = nB\Big(n-\frac{m}{\alpha},1\Big), \;\; \alpha > m \tag{5.129}
$$

Putting $i = n$ in (5.126),

$$g_n(x) = n\left[1 - x^{-\alpha}\right]^{n-1} \alpha x^{-\alpha-1}, \quad 1 < x < \infty \tag{5.130}$$

The m^{th} moment of (5.130) is

$$E(X_n^m) = nB\Big(1 - \frac{m}{\alpha}, n\Big), \ \ \alpha > m \tag{5.131}$$

5.9.2 Exponentiated Pareto (Type I) Distribution

Putting (5.121) in (2.8) gives

$$\begin{aligned} G(x) &= \left\{1 - \left[\frac{1}{x}\right]^{\alpha}\right\}^{r} \\ &= \left[1 - x^{-\alpha}\right]^{r}, \quad 1 < x < \infty, \ r > 0 \end{aligned} \tag{5.132}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{r\alpha}{x^{\alpha+1}}\left\{1 - \left[\frac{1}{x}\right]^{\alpha}\right\}^{r-1} \\ g(x) &= r\alpha x^{-\alpha-1}\left[1 - x^{-\alpha}\right]^{r-1}, \quad 1 < x < \infty, \ r > 0 \end{aligned} \tag{5.133}$$

The m^{th} moment of (5.133) is

$$\begin{aligned} E(X^m) &= \int_1^{\infty} x^m r\alpha x^{-\alpha-1}\left[1 - x^{-\alpha}\right]^{r-1} dx \\ &= \int_1^{\infty} r\alpha x^{m-\alpha-1}\left[1 - x^{-\alpha}\right]^{r-1} dx \end{aligned}$$

Let

$$y = x^{-\alpha} \Longrightarrow dy = -\alpha x^{-\alpha-1} dx$$

Thus

$$\begin{aligned}
E(X^m) &= -r\int_1^0 \frac{\alpha[y^{-\frac{1}{\alpha}}]^{m-\alpha-1}[1-y]^{r-1}}{\alpha[y^{-\frac{1}{\alpha}}]^{-\alpha-1}}dy \\
&= r\int_0^1 [y^{-\frac{1}{\alpha}}]^m [1-y]^{r-1}dy \\
&= r\int_0^1 y^{-\frac{m}{\alpha}+1-1}[1-y]^{r-1}dy \\
&= rB\Big(1-\frac{m}{\alpha}, r\Big), \quad \alpha > m
\end{aligned} \tag{5.134}$$

Putting $r=1$ in (5.134), we get the m^{th} moment of (5.122) as

$$E(X^m) = B\Big(1-\frac{m}{\alpha}, 1\Big), \quad \alpha > m \tag{5.135}$$

5.9.3 Beta Exponentiated Pareto (Type I) Distribution

Putting (5.132) in (2.10) gives

$$G(x) = \int_0^{\left\{1-\left[\frac{1}{x}\right]^{\alpha}\right\}^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 1<x<\infty,\ a>0,\ b>0,\ r>0 \tag{5.136}$$

and its corresponding pdf is

$$\begin{aligned}
g(x) &= \frac{r\left[1-\left(\frac{1}{x}\right)^{\alpha}\right]^{ra-1}\left[1-\left\{1-\left(\frac{1}{x}\right)^{\alpha}\right\}^{r}\right]^{b-1}}{B(a,b)}\frac{\alpha}{x^{\alpha+1}} \\
&= \frac{r\left[1-x^{-\alpha}\right]^{ra-1}\left[1-\left\{1-x^{-\alpha}\right\}^{r}\right]^{b-1}}{B(a,b)}\alpha x^{-\alpha-1}
\end{aligned} \tag{5.137}$$

i^{th} order statistic of Exponentiated Pareto (Type I) distribution

From (5.137),

$$g_i(x) = \frac{r\left[1-x^{-\alpha}\right]^{ri-1}\left[1-\{1-x^{-\alpha}\}^r\right]^{n-i}}{B(i,n-i+1)}\alpha x^{-\alpha-1}, \;\; 1<x<\infty \tag{5.138}$$

Putting $i=1$ in (5.138),

$$g_1(x) = rn\left[1-x^{-\alpha}\right]^{r-1}\left[1-\left\{1-x^{-\alpha}\right\}^r\right]^{n-1}\alpha x^{-\alpha-1}, \;\; 1<x<\infty \tag{5.139}$$

Putting $i=n$ in (5.138),

$$g_n(x) = rn\left[1-x^{-\alpha}\right]^{rn-1}\alpha x^{-\alpha-1}, \;\; 1<x<\infty \tag{5.140}$$

The m^{th} moment of (5.140) is

$$\begin{aligned} E(X_n^m) &= \int_1^\infty x^m rn\alpha x^{-\alpha-1}\left[1-x^{-\alpha}\right]^{rn-1}dx \\ &= \int_1^\infty rn\alpha x^{m-\alpha-1}\left[1-x^{-\alpha}\right]^{rn-1}dx \end{aligned}$$

Let

$$y = x^{-\alpha} \Longrightarrow dy = -\alpha x^{-\alpha-1}dx$$

Thus

$$\begin{aligned} E(X_n^m) &= -rn\int_1^0 \frac{\alpha[y^{-\frac{1}{\alpha}}]^{m-\alpha-1}[1-y]^{rn-1}}{\alpha[y^{-\frac{1}{\alpha}}]^{-\alpha-1}}dy \\ &= rn\int_0^1 [y^{-\frac{1}{\alpha}}]^m[1-y]^{rn-1}dy \\ &= rn\int_0^1 y^{-\frac{m}{\alpha}+1-1}[1-y]^{rn-1}dy \\ &= rnB\left(1-\frac{m}{\alpha}, rn\right), \quad \alpha>m \end{aligned} \tag{5.141}$$

5.10 $g(x) = \dfrac{\alpha x}{1+x}, \quad \alpha > 0$

Then

$$\begin{aligned}\int \frac{g(x)}{x} dx =& \alpha \int \frac{dx}{1+x} \\ &= \alpha \log(1+x)\end{aligned}$$

Substituting in (5.3) we get

$$\begin{aligned}F(x) &= 1 - \exp\{-\alpha \log[1+x]\} \\ &= 1 - \exp\left\{\log[1+x]^{-\alpha}\right\} \\ &= 1 - [1+x]^{-\alpha} \\ &= 1 - \left[(1+x)^{-1}\right]^{\alpha} \\ &= 1 - \left[\frac{1}{1+x}\right]^{\alpha}\end{aligned}$$

Now,

$$F(x) = 0 \Longrightarrow 1 - \frac{1}{1} = 1 - \left[\frac{1}{1+0}\right]^{\alpha} = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow 1 - 0 = 1 - \left[\frac{1}{1+\infty}\right]^{\alpha} = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = 1 - \left[\frac{1}{1+x}\right]^{\alpha}, \quad 0 < x < \infty \tag{5.142}$$

and its corresponding pdf is

$$\begin{aligned}f(x) &= \alpha[1+x]^{-\alpha-1} \\ &= \frac{\alpha}{[1+x]^{\alpha+1}}, \quad 0 < x < \infty\end{aligned} \tag{5.143}$$

which is the **Pareto (Type II) distribution**.

5.10.1 Beta Pareto (Type II) Distribution

Putting (5.142) in (2.3) gives

$$G(x) = \int_{0}^{1-\left[\frac{1}{1+x}\right]^{\alpha}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0<x<\infty,\ a>0,\ b>0 \tag{5.144}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{\left[1-\left(\frac{1}{1+x}\right)^{\alpha}\right]^{a-1}\left[1-\left\{1-\left(\frac{1}{1+x}\right)^{\alpha}\right\}\right]^{b-1}}{B(a,b)}\frac{\alpha}{[1+x]^{\alpha+1}} \\ &= \frac{\left[1-(1+x)^{-\alpha}\right]^{a-1}\left[1-\left\{1-(1+x)^{-\alpha}\right\}\right]^{b-1}}{B(a,b)}\frac{\alpha}{[1+x]^{\alpha+1}} \\ &= \frac{\left[1-(1+x)^{-\alpha}\right]^{a-1}\left[(1+x)^{-\alpha}\right]^{b-1}}{B(a,b)}\alpha\,[1+x]^{-\alpha-1} \\ &= \frac{\left[1-(1+x)^{-\alpha}\right]^{a-1}[1+x]^{-\alpha b+\alpha-\alpha-1}}{B(a,b)}\alpha \\ &= \frac{\left[1-(1+x)^{-\alpha}\right]^{a-1}[1+x]^{-\alpha b-1}}{B(a,b)}\alpha, \quad 0<x<\infty,\ a>0,\ b>0 \end{aligned} \tag{5.145}$$

i^{th} order statistic of Pareto (Type II) distribution

From (5.145),

$$g_i(x) = \frac{\left[1-(1+x)^{-\alpha}\right]^{i-1}\left[(1+x)^{-\alpha}\right]^{n-i}}{B(i,n-i+1)}\frac{\alpha}{[1+x]^{\alpha+1}}, \quad 0<x<\infty \tag{5.146}$$

Putting $i=1$ in (5.146),

$$\begin{aligned} g_1(x) &= n\left[(1+x)^{-\alpha}\right]^{n-1}\frac{\alpha}{[1+x]^{\alpha+1}} \\ &= n(1+x)^{-\alpha n+\alpha}\,\alpha\,[1+x]^{-\alpha-1} \\ &= n(1+x)^{-\alpha n+\alpha-\alpha-1}\,\alpha \\ &= n\alpha(1+x)^{-\alpha n-1}, \quad 0<x<\infty \end{aligned} \tag{5.147}$$

Putting $i=n$ in (5.146),

$$g_n(x) = n\left[1-(1+x)^{-\alpha}\right]^{n-1}\frac{\alpha}{[1+x]^{\alpha+1}}, \quad 0<x<\infty \tag{5.148}$$

5.10.2 Exponentiated Pareto (Type II) Distribution

Putting (5.142) in (2.8) gives

$$\begin{aligned} G(x) &= \left\{1-\left[\frac{1}{1+x}\right]^{\alpha}\right\}^{r} \\ &= \left[1-(1+x)^{-\alpha}\right]^{r}, \quad 0<x<\infty,\ r>0 \end{aligned} \tag{5.149}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{r\alpha}{[1+x]^{\alpha+1}}\left\{1-\left[\frac{1}{1+x}\right]^{\alpha}\right\}^{r-1} \\ g(x) &= r\alpha[1+x]^{-\alpha-1}\left[1-(1+x)^{-\alpha}\right]^{r-1}, \quad 0<x<\infty,\ r>0 \end{aligned} \tag{5.150}$$

5.10.3 Beta Exponentiated Pareto (Type II) Distribution

Putting (5.149) in (2.10) gives

$$G(x) = \int_0^{\left\{1-\left[\frac{1}{1+x}\right]^{\alpha}\right\}^{r}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0<x<\infty,\ a>0,\ b>0,\ r>0 \tag{5.151}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{r\left[1-\left(\frac{1}{1+x}\right)^{\alpha}\right]^{ra-1}\left[1-\left\{1-\left(\frac{1}{1+x}\right)^{\alpha}\right\}^{r}\right]^{b-1}}{B(a,b)}\frac{\alpha}{[1+x]^{\alpha+1}} \\ &= \frac{r\left[1-(1+x)^{-\alpha}\right]^{ra-1}\left[1-\left\{1-(1+x)^{-\alpha}\right\}^{r}\right]^{b-1}}{B(a,b)}\frac{\alpha}{[1+x]^{\alpha+1}} \end{aligned} \tag{5.152}$$

i^{th} order statistic of Exponentiated Pareto (Type II) distribution

From (5.152),

$$g_i(x) = \frac{r\left[1-(1+x)^{-\alpha}\right]^{ri-1}\left[1-\left\{1-(1+x)^{-\alpha}\right\}^{r}\right]^{n-i}}{B(i,n-i+1)}\frac{\alpha}{[1+x]^{\alpha+1}} \tag{5.153}$$

Putting $i=1$ in (5.153),

$$\begin{aligned} g_1(x) &= rn\left[1-(1+x)^{-\alpha}\right]^{r-1}\left[1-\left\{1-(1+x)^{-\alpha}\right\}^{r}\right]^{n-1}\alpha\,[1+x]^{-\alpha-1} \\ &= \frac{rn\alpha\left[1-(1+x)^{-\alpha}\right]^{r-1}\left[1-\left\{1-(1+x)^{-\alpha}\right\}^{r}\right]^{n-1}}{[1+x]^{\alpha+1}},\ \ 0<x<\infty \end{aligned} \tag{5.154}$$

Putting $i=n$ in (5.153),

$$\begin{aligned} g_n(x) &= rn\left[1-(1+x)^{-\alpha}\right]^{rn-1}\alpha\,[1+x]^{-\alpha-1} \\ &= \frac{rn\alpha\left[1-(1+x)^{-\alpha}\right]^{rn-1}}{[1+x]^{\alpha+1}},\ \ 0<x<\infty \end{aligned} \tag{5.155}$$

5.11 $g(x) = \beta x + \dfrac{\alpha x}{1+x}, \quad \alpha > 0, \ \beta > 0$

Then

$$\int \frac{g(x)}{x} dx = \beta \int dx + \alpha \int \frac{dx}{1+x}$$
$$= \beta x + \alpha \log(1+x)$$

Substituting in (5.3) we get

$$\begin{aligned} F(x) &= 1 - e^{-\beta x - \alpha \log[1+x]} \\ &= 1 - e^{-\beta x + \log[1+x]^{-\alpha}} \\ &= 1 - e^{-\beta x} e^{\log[1+x]^{-\alpha}} \\ &= 1 - e^{-\beta x}[1+x]^{-\alpha} \\ &= 1 - \frac{e^{-\beta x}}{[1+x]^{\alpha}} \end{aligned}$$

Now,

$$F(x) = 0 \Longrightarrow 1 - \frac{1}{1} = 1 - \frac{e^{0}}{[1]^{\alpha}} = 1 - \frac{e^{-\beta(0)}}{[1+0]^{\alpha}} = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow 1 - \frac{0}{\infty} = 1 - \frac{e^{-\infty}}{[\infty]^{\alpha}} = 1 - \frac{e^{-\beta(\infty)}}{[1+\infty]^{\alpha}} = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = 1 - \frac{e^{-\beta x}}{[1+x]^{\alpha}}, \quad 0 < x < \infty \tag{5.156}$$

and its corresponding pdf is

$$\begin{aligned} f(x) &= \alpha e^{-\beta x}[1+x]^{-\alpha-1} + \beta e^{-\beta x}[1+x]^{-\alpha} \\ &= \frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}} + \frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}, \quad 0 < x < \infty \end{aligned} \tag{5.157}$$

which is the **Pareto (Type III) distribution**.

5.11.1 Beta Pareto (Type III) Distribution

Putting (5.156) in (2.3) gives

$$G(x)=\int_{0}^{1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}}\frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt,\quad 0<x<\infty,\ a>0,\ b>0 \tag{5.158}$$

and its corresponding pdf is

$$\begin{aligned} g(x)&=\frac{\left[1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right]^{a-1}\left[1-\left\{1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}\right]^{b-1}}{B(a,b)}\left\{\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}}+\frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\}\\ &=\frac{\left[1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right]^{a-1}\left[\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right]^{b-1}}{B(a,b)}\left\{\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}}+\frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\} \end{aligned} \tag{5.159}$$

i^{th} order statistic of Pareto (Type III) distribution

From (5.159),

$$g_i(x)=\frac{\left[1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right]^{i-1}\left[\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right]^{n-i}}{B(i,n-i+1)}\left\{\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}}+\frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\},\quad 0<x<\infty \tag{5.160}$$

Putting $i=1$ in (5.160),

$$g_1(x)=n\left[\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right]^{n-1}\left\{\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}}+\frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\},\quad 0<x<\infty \tag{5.161}$$

Putting $i=n$ in (5.160),

$$g_n(x)=n\left[1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right]^{n-1}\left\{\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}}+\frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\},\quad 0<x<\infty \tag{5.162}$$

5.11.2 Exponentiated Pareto (Type III) Distribution

Putting (5.156) in (2.8) gives

$$G(x) = \left\{1 - \frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{r}, \quad 0 < x < \infty,\ r > 0 \tag{5.163}$$

and its corresponding pdf is

$$g(x) = r\left\{\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}} + \frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\}\left\{1 - \frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{r-1}, \quad 0 < x < \infty,\ r > 0 \tag{5.164}$$

5.11.3 Beta Exponentiated Pareto (Type III) Distribution

Putting (5.164) in (2.10) gives

$$G(x) = \int_{0}^{\left\{1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{r}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0 < x < \infty,\ a > 0,\ b > 0,\ r > 0 \tag{5.165}$$

and its corresponding pdf is

$$g(x) = \frac{r\left\{1 - \frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{ra-1}\left[1 - \left\{1 - \frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{r}\right]^{b-1}}{B(a,b)}\left\{\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}} + \frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\} \tag{5.166}$$

i^{th} order statistic of Exponentiated Pareto (Type III) distribution

From (5.166),

$$g_i(x) = \frac{r\left\{1 - \frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{ri-1}\left[1 - \left\{1 - \frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{r}\right]^{n-i}}{B(i,n-i+1)}\left\{\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}} + \frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\} \tag{5.167}$$

Putting $i = 1$ in (5.167),

$$g_1(x) = rn\left\{1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{r-1}\left[1-\left\{1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{r}\right]^{n-1}\left\{\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}}+\frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\} \tag{5.168}$$

Putting $i = n$ in (5.167),

$$g_n(x) = rn\left\{1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{rn-1}\left\{\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}}+\frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\}, \quad 0 < x < \infty \tag{5.169}$$

5.12 $g(x) = \dfrac{\alpha x(x-\mu)^{\frac{1}{\beta}-1}}{\beta[1+(x-\mu)^{\frac{1}{\beta}}]}, \quad \alpha > 0,\ \beta > 0,\ \mu > 0$

Then

$$\int \frac{g(x)}{x}dx = \int \frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta[1+(x-\mu)^{\frac{1}{\beta}}]}dx$$

Let

$$u = (x-\mu)^{\frac{1}{\beta}}$$
$$du = \frac{1}{\beta}(x-\mu)^{\frac{1}{\beta}-1}dx$$

Thus

$$\begin{aligned}\int \frac{g(x)}{x} &= \alpha\int \frac{(x-\mu)^{\frac{1}{\beta}-1}\beta}{\beta(x-\mu)^{\frac{1}{\beta}-1}[1+u]}du\\ &= \alpha\int\frac{du}{[1+u]}\\ &= \alpha\log[1+u]\\ &= \alpha\log[1+(x-\mu)^{\frac{1}{\beta}}]\end{aligned}$$

Substituting in (5.3) we get

$$
\begin{aligned}
F(x) &= 1-\exp\left\{-\alpha\log\left[1+(x-\mu)^{\frac{1}{\beta}}\right]\right\}\\
&= 1-\exp\left\{\log\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}\\
&= 1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\\
&= 1-\frac{1}{\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha}}
\end{aligned}
$$

Now,

$$
F(x)=0 \Longrightarrow 1-\frac{1}{1}=1-\frac{1}{\left[1+(\mu-\mu)^{\frac{1}{\beta}}\right]^{\alpha}}=0 \Longrightarrow x=\mu
$$

$$
F(x)=1 \Longrightarrow 1-0=1-\frac{1}{\left[1+(\infty-\mu)^{\frac{1}{\beta}}\right]^{\alpha}}=1 \Longrightarrow x=\infty
$$

Therefore

$$
F(x)=1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}, \quad \mu<x<\infty \tag{5.170}
$$

and its corresponding pdf is

$$
\begin{aligned}
f(x) &= \frac{\alpha}{\beta}(x-\mu)^{\frac{1}{\beta}-1}\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha-1}\\
&= \frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}}, \quad \mu<x<\infty
\end{aligned} \tag{5.171}
$$

which is the **Standard Pareto (Type IV) distribution**.

5.12.1 Special Case

When $\beta=1,\ \mu=1$

$$
F(x)=1-x^{-\alpha}=1-\left[\frac{1}{x}\right]^{\alpha}, \quad 1<x<\infty
$$

and its corresponding pdf is

$$\begin{aligned} f(x) &= \alpha x^{-\alpha-1} \\ &= \frac{\alpha}{x^{\alpha+1}}, \quad 1 < x < \infty \end{aligned}$$

which is the **Standard Pareto (Type I) distribution** as in (5.121) and (5.122).

When $\beta = 1,\ \mu = 0$

$$F(x) = 1 - [1+x]^{-\alpha} = 1 - \left[\frac{1}{1+x}\right]^{\alpha}, \quad 0 < x < \infty$$

and its corresponding pdf is

$$\begin{aligned} f(x) &= \alpha\,[1+x]^{-\alpha-1} \\ &= \frac{\alpha}{[1+x]^{\alpha+1}}, \quad 0 < x < \infty \end{aligned}$$

which is the **Standard Pareto (Type II) distribution** as in (5.142) and (5.143).

5.12.2 Beta Pareto (Type IV) Distribution

Putting (5.170) in (2.3) gives

$$G(x) = \int\limits_{0}^{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad \mu < x < \infty,\ a > 0,\ b > 0 \tag{5.172}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{a-1}\left[1-\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}\right]^{b-1}}{B(a,b)} \frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}} \\ &= \frac{\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{a-1}\left\{\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{b-1}}{B(a,b)} \frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}} \end{aligned} \tag{5.173}$$

i^{th} order statistic of Pareto (Type IV) distribution

From (5.173),

$$g_i(x) = \frac{\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{i-1}\left\{\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{n-i}}{B(i,n-i+1)}\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}} \tag{5.174}$$

Putting $i=1$ in (5.174),

$$\begin{aligned} g_1(x) &= n\left\{\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{n-1}\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}} \\ &= n\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha n+\alpha}\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha-1}}{\beta} \\ &= n\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha n+\alpha-\alpha-1}\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta} \\ &= n\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha n-1}\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta}, \quad \mu<x<\infty \end{aligned} \tag{5.175}$$

Putting $i=n$ in (5.174),

$$g_n(x) = n\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{n-1}\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}}, \quad \mu<x<\infty \tag{5.176}$$

5.12.3 Exponentiated Pareto (Type IV) Distribution

Putting (5.170) in (2.8) gives

$$G(x) = \left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{r}, \quad \mu<x<\infty,\ r>0 \tag{5.177}$$

and its corresponding pdf is

$$
\begin{aligned}
g(x) &= r\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{r-1}\frac{\alpha}{\beta}(x-\mu)^{\frac{1}{\beta}-1}\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha-1} \\
&= r\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{r-1}\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}}, \quad \mu<x<\infty,\ r>0
\end{aligned} \tag{5.178}
$$

5.12.4 Beta Exponentiated Pareto (Type IV) Distribution

Putting (5.177) in (2.10) gives

$$
G(x) = \int_0^{\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{r}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad \mu<x<\infty,\ a>0,\ b>0,\ r>0 \tag{5.179}
$$

and its corresponding pdf is

$$
g(x) = \frac{r\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{ra-1}\left[1-\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{r}\right]^{b-1}}{B(a,b)}\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}} \tag{5.180}
$$

i^{th} order statistic of Exponentiated Pareto (Type IV) distribution

From (5.180),

$$
g_i(x) = \frac{r\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{ri-1}\left[1-\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{r}\right]^{n-i}}{B(i,n-i+1)}\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}} \tag{5.181}
$$

Putting $i=1$ in (5.181),

$$g_1(x) = rn\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{r-1}\left[1-\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{r}\right]^{n-1}\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}} \quad \textbf{(5.182)}$$

Putting $i = n$ in (5.181),

$$g_n(x) = rn\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{rn-1}\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}}, \quad \mu < x < \infty \quad \textbf{(5.183)}$$

5.13 $g(x) = \dfrac{x}{\beta}, \quad \beta > 0$

Then

$$\int \frac{g(x)}{x}dx = \frac{1}{\beta}\int dx$$
$$= \frac{x}{\beta}$$

Substituting in (5.3) we get

$$F(x) = 1-\exp\left[-\frac{x}{\beta}\right]$$

Now,

$$F(x) = 0 \Longrightarrow 1-1 = 1-\exp\left[-\frac{0}{\beta}\right] = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow 1-0 = 1-\exp\left[-\frac{\infty}{\beta}\right] = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = 1-e^{-\frac{x}{\beta}}, \quad 0 < x < \infty \quad \textbf{(5.184)}$$

and its corresponding pdf is

$$f(x)=\frac{1}{\beta}e^{-\frac{x}{\beta}}, \quad 0<x<\infty \tag{5.185}$$

which is the **Exponential distribution** with rate parameter $\frac{1}{\beta}$.

The m^{th} moment of (5.185) is

$$E(X^m)=\int_0^\infty x^m\frac{1}{\beta}e^{-\frac{x}{\beta}}dx$$

Let

$$y=\frac{x}{\beta}\Longrightarrow dy=\frac{1}{\beta}dx$$

Thus

$$\begin{aligned}E(X^m)&=\int_0^\infty\frac{(\beta y)^m e^{-y}}{\beta}\beta dy\\&=\beta^m\int_0^\infty y^m e^{-y}dy\\&=\beta^m\int_0^\infty y^{1+m-1}e^{-y}dy\\&=\beta^m\Gamma(1+m)\end{aligned} \tag{5.186}$$

5.13.1 Beta Exponential Distribution

Putting (5.184) in (2.3) gives

$$G(x)=\int_0^{1-e^{-\frac{x}{\beta}}}\frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0<x<\infty,\ a>0,\ b>0 \tag{5.187}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{\left[1-e^{-\frac{x}{\beta}}\right]^{a-1}\left[1-\left\{1-e^{-\frac{x}{\beta}}\right\}\right]^{b-1}}{B(a,b)}\frac{1}{\beta}e^{-\frac{x}{\beta}} \\ &= \frac{\left[1-e^{-\frac{x}{\beta}}\right]^{a-1}\left[e^{-\frac{x}{\beta}}\right]^{b-1}}{B(a,b)}\frac{1}{\beta}e^{-\frac{x}{\beta}} \\ &= \frac{\left[1-e^{-\frac{x}{\beta}}\right]^{a-1}\left[e^{-\frac{x}{\beta}}\right]^{b-1+1}}{B(a,b)}\frac{1}{\beta} \\ &= \frac{\left[1-e^{-\frac{x}{\beta}}\right]^{a-1}\left[e^{-\frac{x}{\beta}}\right]^{b}}{B(a,b)}\frac{1}{\beta} \\ &= \frac{\left[1-e^{-\frac{x}{\beta}}\right]^{a-1}e^{-\frac{bx}{\beta}}}{B(a,b)}\frac{1}{\beta}, \quad 0<x<\infty,\ a>0,\ b>0 \end{aligned} \tag{5.188}$$

i^{th} order statistic of Exponential distribution

From (5.188),

$$g_i(x) = \frac{\left[1-e^{-\frac{x}{\beta}}\right]^{i-1}e^{-\frac{(n-i+1)x}{\beta}}}{B(i,n-i+1)}\frac{1}{\beta}, \quad 0<x<\infty \tag{5.189}$$

Putting $i=1$ in (5.189),

$$g_1(x) = ne^{-\frac{nx}{\beta}}\frac{1}{\beta}, \quad 0<x<\infty \tag{5.190}$$

The m^{th} moment of (5.190) is

$$E(X_1^m) = \int_0^\infty x^m \frac{1}{\beta} ne^{-\frac{nx}{\beta}} dx$$

Let

$$y = \frac{nx}{\beta} \Longrightarrow dy = \frac{n}{\beta}dx$$

Thus

$$E(X_1^m) = \int_0^\infty \frac{\left[\frac{\beta y}{n}\right]^m ne^{-y}}{\beta}\frac{\beta}{n}dy$$
$$= \frac{\beta^m}{n^m}\int_0^\infty y^m e^{-y}dy$$
$$= \frac{\beta^m}{n^m}\int_0^\infty y^{1+m-1}e^{-y}dy \qquad (5.191)$$
$$= \frac{\beta^m}{n^m}\Gamma(1+m)$$

Putting $i = n$ in (5.189),

$$g_n(x) = n\left[1-e^{-\frac{x}{\beta}}\right]^{n-1} e^{-\frac{x}{\beta}}\frac{1}{\beta}, \quad 0 < x < \infty \qquad (5.192)$$

5.13.2 Exponentiated Exponential Distribution

Putting (5.184) in (2.8) gives

$$G(x) = \left[1-e^{-\frac{x}{\beta}}\right]^r, \quad 0 < x < \infty,\ r > 0 \qquad (5.193)$$

and its corresponding pdf is

$$g(x) = \frac{r}{\beta}e^{-\frac{x}{\beta}}\left[1-e^{-\frac{x}{\beta}}\right]^{r-1}, \quad 0 < x < \infty,\ r > 0 \qquad (5.194)$$

5.13.3 Beta Exponentiated Exponential Distribution

Putting (5.193) in (2.10) gives

$$G(x) = \int_0^{\left[1-e^{-\frac{x}{\beta}}\right]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0 < x < \infty,\ a > 0,\ b > 0,\ r > 0 \qquad (5.195)$$

and its corresponding pdf is

$$g(x)=\frac{r\left[1-e^{-\frac{x}{\beta}}\right]^{ra-1}\left[1-\left\{1-e^{-\frac{x}{\beta}}\right\}^{r}\right]^{b-1}}{B(a,b)}\frac{1}{\beta}e^{-\frac{x}{\beta}} \tag{5.196}$$

i^{th} order statistic of Exponentiated Exponential distribution

From (5.196),

$$g_i(x)=\frac{r\left[1-e^{-\frac{x}{\beta}}\right]^{ri-1}\left[1-\left\{1-e^{-\frac{x}{\beta}}\right\}^{r}\right]^{n-i}}{B(i,n-i+1)}\frac{1}{\beta}e^{-\frac{x}{\beta}},\quad 0<x<\infty,\ r>0 \tag{5.197}$$

Putting $i=1$ in (5.197),

$$g_1(x)=rn\left[1-e^{-\frac{x}{\beta}}\right]^{r-1}\left[1-\left\{1-e^{-\frac{x}{\beta}}\right\}^{r}\right]^{n-1}\frac{1}{\beta}e^{-\frac{x}{\beta}},\quad 0<x<\infty,\ r>0 \tag{5.198}$$

Putting $i=n$ in (5.197),

$$g_n(x)=rn\left[1-e^{-\frac{x}{\beta}}\right]^{rn-1}\frac{1}{\beta}e^{-\frac{x}{\beta}},\quad 0<x<\infty,\ r>0 \tag{5.199}$$

5.14 $g(x)=\alpha\left(\frac{x}{\beta}\right)^{\alpha},\quad \alpha>0,\ \beta>0$

Then

$$\begin{aligned}\int\frac{g(x)}{x}dx&=\frac{\alpha}{\beta^{\alpha}}\int x^{\alpha-1}dx\\&=\frac{\alpha}{\beta^{\alpha}}\left(\frac{x^{\alpha}}{\alpha}\right)\\&=\left(\frac{x}{\beta}\right)^{\alpha}\end{aligned}$$

Substituting in (5.3) we get

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right]$$

Now,

$$F(x) = 0 \Longrightarrow 1 - 1 = 1 - \exp\left[-\left(\frac{0}{\beta}\right)^{\alpha}\right] = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow 1 - 0 = 1 - \exp\left[-\left(\frac{\infty}{\beta}\right)^{\alpha}\right] = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, \quad 0 < x < \infty \tag{5.200}$$

and its corresponding pdf is

$$\begin{aligned} f(x) &= \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \\ &= \frac{\alpha}{\beta^{\alpha}} x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \\ &= \alpha\beta^{-\alpha} x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, \quad 0 < x < \infty \end{aligned} \tag{5.201}$$

which is the **Weibull distribution** with shape parameter α and scale parameter β.

The m^{th} moment of (5.201) is

$$E(X^m) = \int_0^{\infty} x^m \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} dx$$

Let

$$y = \left(\frac{x}{\beta}\right)^{\alpha} \Longrightarrow dy = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} dx$$

Thus

$$
\begin{aligned}
E(X^m) &= \int_0^\infty \frac{\left(\beta y^{\frac{1}{\alpha}}\right)^m \frac{\alpha}{\beta}\left(y^{\frac{1}{\alpha}}\right)^{\alpha-1} e^{-y}}{\frac{\alpha}{\beta}\left(y^{\frac{1}{\alpha}}\right)^{\alpha-1}} dy \\
&= \beta^m \int_0^\infty (y)^{\frac{m}{\alpha}} e^{-y} dy \\
&= \beta^m \int_0^\infty y^{1+\frac{m}{\alpha}-1} e^{-y} dy \\
&= \beta^m \Gamma\left(1+\frac{m}{\alpha}\right)
\end{aligned}
\tag{5.202}
$$

5.14.1 Special Case

When $\alpha = 1$

$$F(x) = 1 - e^{-\frac{x}{\beta}}, \quad 0 < x < \infty$$

and its corresponding pdf is

$$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \quad 0 < x < \infty$$

which is the **Exponential distribution** with rate parameter $\frac{1}{\beta}$ as in (5.184) and (5.185).

When $\alpha = 2, \beta = 2^{\frac{1}{2}}$

$$
\begin{aligned}
F(x) &= 1 - e^{-\left(\frac{x}{2^{\frac{1}{2}}}\right)^2} \\
&= 1 - e^{-\frac{x^2}{2^{\frac{2}{2}}}} \\
&= 1 - e^{-\frac{x^2}{2}}, \quad 0 < x < \infty
\end{aligned}
\tag{5.203}
$$

and its corresponding pdf is

$$f(x) = xe^{-\frac{x^2}{2}}, \quad 0 < x < \infty \tag{5.204}$$

which is the **Rayleigh distribution** with scale parameter 1.

5.14.2 Extension of Weibull Distribution

Let

$$Y = \frac{\beta^2}{X}$$

where X has Weibull distribution with cdf given by (5.200).

If $G(y)$ is the cdf of y, then

$$\begin{aligned} G(y) &= prob\ (Y \leq y) \\ &= prob\ \left(\frac{\beta^2}{X} \leq y\right) \\ &= prob\ \left(X \geq \frac{\beta^2}{y}\right) \\ &= 1 - prob\ \left(X \leq \frac{\beta^2}{y}\right) \end{aligned}$$

Therefore

$$\begin{aligned} G(y) &= 1 - F\left(\frac{\beta^2}{y}\right) \\ &= 1 - \left[1 - e^{-\left(\frac{\beta^2}{y\beta}\right)^{\alpha}}\right] \\ &= e^{-\left(\frac{\beta}{y}\right)^{\alpha}} \\ &= e^{-\left(\frac{y}{\beta}\right)^{-\alpha}} \end{aligned}$$

Thus

$$G(y) = e^{-\left(\frac{y}{\beta}\right)^{-\alpha}}, \quad 0 < y < \infty \tag{5.205}$$

and its corresponding pdf is

$$
\begin{aligned}
g(y) &= \frac{\alpha}{\beta}\left(\frac{y}{\beta}\right)^{-\alpha-1} e^{-\left(\frac{y}{\beta}\right)^{-\alpha}} \\
&= \alpha\beta^{\alpha} y^{-\alpha-1} e^{-\left(\frac{y}{\beta}\right)^{-\alpha}}, \quad 0 < y < \infty
\end{aligned} \tag{5.206}
$$

which is the **Inverse Weibull distribution** also known as the **Fréchet distribution** by Johnson et al (1995).

The m^{th} moment of (5.206) is

$$
E(Y^m) = \int_0^{\infty} y^m \frac{\alpha}{\beta}\left(\frac{y}{\beta}\right)^{-\alpha-1} e^{-\left(\frac{y}{\beta}\right)^{-\alpha}} dy
$$

Let

$$
x = \left(\frac{y}{\beta}\right)^{-\alpha} \Longrightarrow dx = -\frac{\alpha}{\beta}\left(\frac{y}{\beta}\right)^{-\alpha-1} dy
$$

Thus

$$
\begin{aligned}
E(Y^m) &= -\int_{\infty}^{0} \frac{\left(\beta x^{-\frac{1}{\alpha}}\right)^m \frac{\alpha}{\beta}\left(x^{-\frac{1}{\alpha}}\right)^{-\alpha-1} e^{-x}}{\frac{\alpha}{\beta}\left(x^{-\frac{1}{\alpha}}\right)^{-\alpha-1}} dx \\
&= \beta^m \int_0^{\infty} (x)^{-\frac{m}{\alpha}} e^{-x} dx \\
&= \beta^m \int_0^{\infty} x^{1-\frac{m}{\alpha}-1} e^{-x} dx \\
&= \beta^m \Gamma\left(1 - \frac{m}{\alpha}\right)
\end{aligned} \tag{5.207}
$$

5.14.3 Beta Weibull Distribution

Putting (5.200) in (2.3) gives

$$
G(x) = \int_0^{1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < \infty,\ a > 0,\ b > 0 \tag{5.208}
$$

and its corresponding pdf is

$$\begin{aligned}
g(x) &= \frac{\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{a-1}\left[1-\left\{1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right\}\right]^{b-1}}{B(a,b)}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \\
&= \frac{\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{a-1}\left[e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{b-1}}{B(a,b)}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \\
&= \frac{\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{a-1}\left[e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{b-1+1}}{B(a,b)}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} \\
&= \frac{\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{a-1}\left[e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{b}}{B(a,b)}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} \\
&= \frac{\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{a-1}e^{-b\left(\frac{x}{\beta}\right)^{\alpha}}}{B(a,b)}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}, \quad 0<x<\infty,\ a>0,\ b>0
\end{aligned} \tag{5.209}$$

i^{th} order statistic of Weibull distribution

From (5.209),

$$g_i(x) = \frac{\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{i-1}e^{-(n-i+1)\left(\frac{x}{\beta}\right)^{\alpha}}}{B(i,n-i+1)}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}, \quad 0<x<\infty \tag{5.210}$$

Putting $i=1$ in (5.210),

$$g_1(x) = ne^{-n\left(\frac{x}{\beta}\right)^{\alpha}}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}, \quad 0<x<\infty \tag{5.211}$$

The m^{th} moment of (5.211) is

$$E(X_1^m) = \int_0^\infty x^m\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}ne^{-n\left(\frac{x}{\beta}\right)^{\alpha}}dx$$

Let

$$y = n\left(\frac{x}{\beta}\right)^{\alpha} \Longrightarrow dy = n\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} dx$$

Thus

$$\begin{aligned} E(X_1^m) &= \int_0^\infty \frac{\left[\beta\left(\frac{y}{n}\right)^{\frac{1}{\alpha}}\right]^m \frac{\alpha}{\beta}\left[\left(\frac{y}{n}\right)^{\frac{1}{\alpha}}\right]^{\alpha-1} ne^{-y}}{n\frac{\alpha}{\beta}\left[\left(\frac{y}{n}\right)^{\frac{1}{\alpha}}\right]^{\alpha-1}} dy \\ &= \frac{\beta^m}{n^{\frac{m}{\alpha}}}\int_0^\infty (y)^{\frac{m}{\alpha}} e^{-y} dy \\ &= \frac{\beta^m}{n^{\frac{m}{\alpha}}}\int_0^\infty y^{1+\frac{m}{\alpha}-1} e^{-y} dy \\ &= \frac{\beta^m}{n^{\frac{m}{\alpha}}}\Gamma\left(1+\frac{m}{\alpha}\right) \end{aligned} \tag{5.212}$$

Putting $i = n$ in (5.210),

$$g_n(x) = n\left[1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{n-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}, \quad 0 < x < \infty \tag{5.213}$$

5.14.4 Exponentiated Weibull Distribution

Putting (5.200) in (2.8) gives

$$G(x) = \left[1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{r}, \quad 0 < x < \infty,\ r > 0 \tag{5.214}$$

and its corresponding pdf is

$$g(x) = r\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\left[1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{r-1}, \quad 0 < x < \infty,\ r > 0 \tag{5.215}$$

5.14.5 Beta Exponentiated Weibull Distribution

Putting (5.214) in (2.10) gives

$$G(x) = \int_0^{\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < \infty,\ a > 0,\ b > 0,\ r > 0 \tag{5.216}$$

and its corresponding pdf is

$$g(x) = \frac{r\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{ra-1}\left[1-\left\{1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right\}^r\right]^{b-1}}{B(a,b)} \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \tag{5.217}$$

i^{th} order statistic of Exponentiated Weibull distribution

From (5.217),

$$g_i(x) = \frac{r\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{ri-1}\left[1-\left\{1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right\}^r\right]^{n-i}}{B(i,n-i+1)} \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \tag{5.218}$$

Putting $i = 1$ in (5.218),

$$g_1(x) = rn\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{r-1}\left[1-\left\{1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right\}^r\right]^{n-1} \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \tag{5.219}$$

Putting $i = n$ in (5.218),

$$g_n(x) = rn\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{rn-1} \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, \quad 0 < x < \infty,\ r > 0 \tag{5.220}$$

5.15 $g(x) = \alpha + 2\beta \log x, \quad \alpha > 0,\ \beta > 0$

Then

$$\int \frac{g(x)}{x} dx = \alpha \int \frac{dx}{x} + 2\beta \int \frac{\log x}{x} dx$$

Let

$$t = \log x$$
$$dt = \frac{1}{x}dx$$

Thus

$$\begin{aligned}\int \frac{g(x)}{x} &= \alpha \log x + 2\beta \int t dt \\ &= \alpha \log x + 2\beta \left(\frac{t^2}{2}\right) \\ &= \alpha \log x + \beta (\log x)^2\end{aligned}$$

Substituting in (5.3) we get

$$\begin{aligned}F(x) &= 1 - \exp\left[-\left\{\alpha \log x + \beta (\log x)^2\right\}\right] \\ &= 1 - \exp\left[-\alpha \log x - \beta (\log x)^2\right]\end{aligned}$$

Now,

$$F(x) = 0 \Longrightarrow 1 - 1 = 1 - \exp\left[-\alpha \log 1 - \beta (\log 1)^2\right] = 0 \Longrightarrow x = 1$$

$$F(x) = 1 \Longrightarrow 1 - 0 = 1 - \exp\left[-\alpha \log \infty - \beta (\log \infty)^2\right] = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = 1 - e^{-\alpha \log x - \beta (\log x)^2}, \quad 1 < x < \infty \tag{5.221}$$

and its corresponding pdf is

$$f(x) = \left[\frac{\alpha}{x} + \frac{2\beta \log x}{x}\right] e^{-\alpha \log x - \beta (\log x)^2}, \quad 1 < x < \infty \tag{5.222}$$

which is the **Benini distribution** with shape parameters $\alpha; \beta$ and scale parameter 1.

5.15.1 Special Case

When $\beta = 0$

$$
\begin{aligned}
F(x) &= 1 - e^{-\alpha \log x} \\
&= 1 - e^{\log x^{-\alpha}} \\
&= 1 - x^{-\alpha} \\
&= 1 - \left[\frac{1}{x}\right]^{\alpha}, \quad 1 < x < \infty
\end{aligned}
$$

and its corresponding pdf is

$$
\begin{aligned}
f(x) &= \alpha x^{-\alpha - 1} \\
&= \frac{\alpha}{x^{\alpha+1}}, \quad 1 < x < \infty
\end{aligned}
$$

which is the **Pareto (Type I) distribution** as in (5.121) and (5.122).

When $\alpha = 0,\ \beta = \frac{1}{2}$

$$
F(x) = 1 - e^{-\frac{(\log x)^2}{2}}, \quad 1 < x < \infty
$$

Let

$$
\begin{aligned}
u &= \log x \\
x &= e^{u}
\end{aligned}
$$

Therefore

$$
F(u) = 1 - e^{-\frac{u^2}{2}}, \quad 0 < u < \infty
$$

and its corresponding pdf is

$$f(u) = ue^{-\frac{u^2}{2}}, \quad 0 < u < \infty$$

which is the **Rayleigh distribution** with scale parameter 1, the same form as in (5.203) and (5.204).

5.15.2 Beta Benini Distribution

Putting (5.221) in (2.3) gives

$$G(x) = \int_0^{1-e^{-\alpha \log x - \beta(\log x)^2}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 1 < x < \infty,\ a > 0,\ b > 0 \tag{5.223}$$

and its corresponding pdf is

$$\begin{aligned}
g(x) &= \frac{\left[1-e^{-\alpha \log x - \beta(\log x)^2}\right]^{a-1}\left[1-\left\{1-e^{-\alpha \log x - \beta(\log x)^2}\right\}\right]^{b-1}}{B(a,b)}\left[\frac{\alpha}{x}+\frac{2\beta \log x}{x}\right] e^{-\alpha \log x - \beta(\log x)^2} \\
&= \frac{\left[1-e^{-\alpha \log x - \beta(\log x)^2}\right]^{a-1}\left[e^{-\alpha \log x - \beta(\log x)^2}\right]^{b-1}}{B(a,b)}\left[\frac{\alpha}{x}+\frac{2\beta \log x}{x}\right] e^{-\alpha \log x - \beta(\log x)^2} \\
&= \frac{\left[1-e^{-\alpha \log x - \beta(\log x)^2}\right]^{a-1}\left[e^{-\alpha \log x - \beta(\log x)^2}\right]^{b-1+1}}{B(a,b)}\left[\frac{\alpha}{x}+\frac{2\beta \log x}{x}\right] \\
&= \frac{\left[1-e^{-\alpha \log x - \beta(\log x)^2}\right]^{a-1}\left[e^{-\alpha \log x - \beta(\log x)^2}\right]^{b}}{B(a,b)}\left[\frac{\alpha}{x}+\frac{2\beta \log x}{x}\right] \\
&= \frac{\left[1-e^{-\alpha \log x - \beta(\log x)^2}\right]^{a-1} e^{-b\alpha \log x - b\beta(\log x)^2}}{B(a,b)}\left[\frac{\alpha}{x}+\frac{2\beta \log x}{x}\right], \quad 1 < x < \infty,\ a > 0,\ b > 0
\end{aligned} \tag{5.224}$$

i^{th} order statistic of Benini distribution

From (5.224),

$$g_i(x) = \frac{\left[1-e^{-\alpha \log x - \beta(\log x)^2}\right]^{i-1} e^{-(n-i+1)\alpha \log x - (n-i+1)\beta(\log x)^2}}{B(i,n-i+1)}\left[\frac{\alpha}{x}+\frac{2\beta \log x}{x}\right], \quad 1 < x < \infty \tag{5.225}$$

Putting $i = 1$ in (5.225),

$$g_1(x) = ne^{-n\alpha\log x - n\beta(\log x)^2}\left[\frac{\alpha}{x} + \frac{2\beta\log x}{x}\right], \quad 1 < x < \infty \tag{5.226}$$

Putting $i = n$ in (5.225),

$$g_n(x) = n\left[1 - e^{-\alpha\log x - \beta(\log x)^2}\right]^{n-1} e^{-\alpha\log x - \beta(\log x)^2}\left[\frac{\alpha}{x} + \frac{2\beta\log x}{x}\right], \quad 1 < x < \infty \tag{5.227}$$

5.15.3 Exponentiated Benini Distribution

Putting (5.221) in (2.8) gives

$$G(x) = \left[1 - e^{-\alpha\log x - \beta(\log x)^2}\right]^r, \quad 1 < x < \infty,\ r > 0 \tag{5.228}$$

and its corresponding pdf is

$$g(x) = r\left[\frac{\alpha}{x} + \frac{2\beta\log x}{x}\right] e^{-\alpha\log x - \beta(\log x)^2}\left[1 - e^{-\alpha\log x - \beta(\log x)^2}\right]^{r-1}, \quad 1 < x < \infty,\ r > 0 \tag{5.229}$$

5.15.4 Beta Exponentiated Benini Distribution

Putting (5.228) in (2.10) gives

$$G(x) = \int_0^{\left[1 - e^{-\alpha\log x - \beta(\log x)^2}\right]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 1 < x < \infty,\ a > 0,\ b > 0,\ r > 0 \tag{5.230}$$

and its corresponding pdf is

$$g(x) = \frac{r\left[1 - e^{-\alpha\log x - \beta(\log x)^2}\right]^{ra-1}\left[1 - \left\{1 - e^{-\alpha\log x - \beta(\log x)^2}\right\}^r\right]^{b-1}}{B(a,b)}\left[\frac{\alpha}{x} + \frac{2\beta\log x}{x}\right] e^{-\alpha\log x - \beta(\log x)^2} \tag{5.231}$$

i^{th} order statistic of Exponentiated Benini distribution

From (5.231),

$$g_i(x) = \frac{r\left[1-e^{-\alpha\log x-\beta(\log x)^2}\right]^{ri-1}\left[1-\left\{1-e^{-\alpha\log x-\beta(\log x)^2}\right\}^r\right]^{n-i}}{B(i,n-i+1)}\left[\frac{\alpha}{x}+\frac{2\beta\log x}{x}\right]e^{-\alpha\log x-\beta(\log x)^2} \tag{5.232}$$

Putting $i=1$ in (5.232),

$$g_1(x) = rn\left[1-e^{-\alpha\log x-\beta(\log x)^2}\right]^{r-1}\left[1-\left\{1-e^{-\alpha\log x-\beta(\log x)^2}\right\}^r\right]^{n-1}\left[\frac{\alpha}{x}+\frac{2\beta\log x}{x}\right]e^{-\alpha\log x-\beta(\log x)^2} \tag{5.233}$$

Putting $i=n$ in (5.232),

$$g_n(x) = rn\left[1-e^{-\alpha\log x-\beta(\log x)^2}\right]^{rn-1}\left[\frac{\alpha}{x}+\frac{2\beta\log x}{x}\right]e^{-\alpha\log x-\beta(\log x)^2} \tag{5.234}$$

6 Distributions based on Case IV of Burr Differential Equation

6.1 Reverse hazard function

Olapade (2000) states that one of the properties of Type I Generalized Logistic distribution is that it satisfies the following homogeneous differential equation:

$$(1-e^{-x})F' - be^{-x}F = 0$$

$$F' = \frac{be^{-x}}{1+e^{-x}}F$$

Therefore

$$F' = F(1-F)\left(\frac{be^{-x}}{1+e^{-x}}\right)\frac{1}{(1-F)}$$

which can be written as

$$y' = y(1-y)\left(\frac{be^{-x}}{1+e^{-x}}\right)\frac{1}{(1-y)}$$

Let

$$g(x) = \left(\frac{be^{-x}}{1+e^{-x}}\right)$$

Therefore

$$y' = y(1-y)\frac{g(x)}{(1-y)}$$

$$y' = yg(x)$$

$$g(x) = \frac{y'}{y} = \frac{f(x)}{F(x)} = r(x)$$

where $r(x)$ is the **reverse hazard function**.

Re-writing the equation

$$y' = y(1-y)\frac{r(x)}{(1-y)}$$

which is (2.1) with

$$g(x,y) = \frac{r(x)}{(1-y)}$$

This implies that

$$\begin{aligned}\int \frac{dy}{y} &= \int r(x)dx \\ \log y &= \int r(x)dx \\ y &= \exp\left[\int r(x)dx\right]\end{aligned}$$

That is

$$F(x) = \exp\left[\int r(x)dx\right] \tag{6.1}$$

6.2 $r(x) = \dfrac{1}{2\cosh x \arctan e^x}$

$$\int r(x)dx = \int \frac{1}{2\cosh x \arctan e^x} dx$$

Let

$$u = \frac{2}{\pi}\arctan e^x$$

and

$$\begin{aligned}\frac{du}{dx} &= \frac{d\frac{2}{\pi}\arctan e^x}{dx} \\ &= \frac{2}{\pi}\left[\frac{e^x}{1+e^{2x}}\right] \\ &= \frac{2}{\pi}\left[\frac{1}{e^{-x}+e^x}\right] \\ &= \frac{2}{\pi}\left[\frac{1}{\frac{2(e^{-x}+e^x)}{2}}\right] \\ &= \frac{2}{\pi}\left[\frac{2}{2(e^{-x}+e^x)}\right] \\ &= \frac{2}{\pi}\left[\frac{1}{2\cosh x}\right] \\ &= \frac{1}{\pi\cosh x}\end{aligned}$$

Thus

$$\begin{aligned}\int r(x)dx &= \int \frac{\pi\cosh x}{\pi u\cosh x}du\\ &= \int \frac{1}{u}du\end{aligned}$$

Substituting in (6.1) we get

$$\begin{aligned}F(x) &= \exp\left[\int \frac{1}{u}du\right]\\ &= \exp\left[\log u\right]\\ &= u\\ &= \frac{2}{\pi}\arctan e^{x}\end{aligned}$$

Now,

$$F(x) = 0 \Longrightarrow \frac{2}{\pi}(0) = \frac{2}{\pi}\arctan e^{-\infty} = 0 \Longrightarrow x = -\infty$$

$$F(x) = 1 \Longrightarrow \frac{2}{\pi}\left(\frac{\pi}{2}\right) = \frac{2}{\pi}\arctan e^{\infty} = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = \frac{2}{\pi}\arctan e^{x}, \quad -\infty < x < \infty \tag{6.2}$$

and its corresponding pdf is

$$f(x) = \frac{2e^{x}}{\pi}\left(\frac{1}{1+x^{2}}\right), \quad -\infty < x < \infty \tag{6.3}$$

6.2.1 Beta Generated Distribution

Putting (6.2) in (2.3) gives

$$G(x) = \int_{0}^{\frac{2}{\pi}\arctan e^{x}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad -\infty < x < \infty,\ a > 0,\ b > 0 \tag{6.4}$$

and its corresponding pdf is

$$g(x)=\frac{\left[\frac{2}{\pi}\arctan e^{x}\right]^{a-1}\left[1-\frac{2}{\pi}\arctan e^{x}\right]^{b-1}}{B(a,b)}\frac{2e^{x}}{\pi}\left(\frac{1}{1+x^{2}}\right) \tag{6.5}$$

i^{th} order statistic of $F(x)=\dfrac{2}{\pi}\arctan e^{x}$

From (6.5),

$$g_i(x)=\frac{\left[\frac{2}{\pi}\arctan e^{x}\right]^{i-1}\left[1-\frac{2}{\pi}\arctan e^{x}\right]^{n-i}}{B(i,n-i+1)}\frac{2e^{x}}{\pi}\left(\frac{1}{1+x^{2}}\right), \quad -\infty<x<\infty \tag{6.6}$$

Putting $i=1$ in (6.6),

$$g_1(x)=n\left[1-\frac{2}{\pi}\arctan e^{x}\right]^{n-1}\frac{2e^{x}}{\pi}\left(\frac{1}{1+x^{2}}\right), \quad -\infty<x<\infty \tag{6.7}$$

Putting $i=n$ in (6.6),

$$g_n(x)=n\left[\frac{2}{\pi}\arctan e^{x}\right]^{n-1}\frac{2e^{x}}{\pi}\left(\frac{1}{1+x^{2}}\right), \quad -\infty<x<\infty \tag{6.8}$$

6.2.2 Exponentiated Generated Distribution

Putting (6.2) in (2.8) gives

$$G(x)=\left[\frac{2}{\pi}\arctan e^{x}\right]^{r}, \quad -\infty<x<\infty,\ r>0 \tag{6.9}$$

and its corresponding pdf is

$$g(x)=r\frac{2e^{x}}{\pi}\left(\frac{1}{1+x^{2}}\right)\left[\frac{2}{\pi}\arctan e^{x}\right]^{r-1}, \quad -\infty<x<\infty,\ r>0 \tag{6.10}$$

which is referred to as **Burr VIII distribution**.

6.2.3 Beta Exponentiated Generated Distribution

Putting (6.9) in (2.10) gives

$$G(x) = \int_{0}^{\left[\frac{2}{\pi}\arctan e^{x}\right]^{r}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad -\infty < x < \infty,\ a > 0,\ b > 0,\ r > 0 \tag{6.11}$$

and its corresponding pdf is

$$g(x) = \frac{r\left[\frac{2}{\pi}\arctan e^{x}\right]^{ra-1}\left[1-\left(\frac{2}{\pi}\arctan e^{x}\right)^{r}\right]^{b-1}}{B(a,b)}\frac{2e^{x}}{\pi}\left(\frac{1}{1+x^{2}}\right) \tag{6.12}$$

i^{th} order statistic of Burr VIII distribution

From (6.12),

$$g_i(x) = \frac{r\left[\frac{2}{\pi}\arctan e^{x}\right]^{ri-1}\left[1-\left(\frac{2}{\pi}\arctan e^{x}\right)^{r}\right]^{n-i}}{B(i,n-i+1)}\frac{2e^{x}}{\pi}\left(\frac{1}{1+x^{2}}\right), \quad -\infty < x < \infty \tag{6.13}$$

Putting $i = 1$ in (6.13),

$$g_1(x) = rn\left[\frac{2}{\pi}\arctan e^{x}\right]^{r-1}\left[1-\left(\frac{2}{\pi}\arctan e^{x}\right)^{r}\right]^{n-1}\frac{2e^{x}}{\pi}\left(\frac{1}{1+x^{2}}\right) \tag{6.14}$$

Putting $i = n$ in (6.13),

$$g_n(x) = rn\left[\frac{2}{\pi}\arctan e^{x}\right]^{rn-1}\frac{2e^{x}}{\pi}\left(\frac{1}{1+x^{2}}\right), \quad -\infty < x < \infty,\ r > 0 \tag{6.15}$$

6.3 $r(x) = \dfrac{1-\cos 2\pi x}{x - \frac{1}{2\pi}\sin 2\pi x}$

$$\int r(x)dx = \int \frac{1-\cos 2\pi x}{x - \dfrac{1}{2\pi}\sin 2\pi x}dx$$

Let

$$\begin{aligned} u &= x - \frac{1}{2\pi}\sin 2\pi x \\ du &= 1 - \cos 2\pi x dx \end{aligned}$$

Thus

$$\begin{aligned} \int r(x)dx &= \int \frac{1-\cos 2\pi x}{(1-\cos 2\pi x)u}du \\ &= \int \frac{1}{u}du \end{aligned}$$

Substituting in (6.1) we get

$$\begin{aligned} F(x) &= \exp\left[\int \frac{1}{u}du\right] \\ &= \exp[\log u] \\ &= u \\ &= x - \frac{1}{2\pi}\sin 2\pi x \end{aligned}$$

Now,

$$F(x) = 0 \Longrightarrow 0 - 0 = 0 - \frac{1}{2\pi}\sin 2\pi(0) = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow 1 - 0 = 1 - \frac{1}{2\pi}\sin 2\pi = 1 \Longrightarrow x = 1$$

Therefore

$$F(x) = x - \frac{1}{2\pi}\sin 2\pi x, \quad 0 < x < 1 \tag{6.16}$$

and its corresponding pdf is

$$f(x) = 1 - \cos 2\pi x, \quad 0 < x < 1 \tag{6.17}$$

6.3.1 Beta Generated Distribution

Putting (6.16) in (2.3) gives

$$G(x) = \int_0^{x-\frac{1}{2\pi}\sin 2\pi x} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < 1,\ a > 0,\ b > 0 \tag{6.18}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^{a-1}\left[1-\left(x-\frac{1}{2\pi}\sin 2\pi x\right)\right]^{b-1}}{B(a,b)}\left[1-\cos 2\pi x\right] \\ &= \frac{\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^{a-1}\left[1-x+\frac{1}{2\pi}\sin 2\pi x\right]^{b-1}}{B(a,b)}\left[1-\cos 2\pi x\right] \end{aligned} \tag{6.19}$$

i^{th} order statistic of $F(x) = x - \dfrac{1}{2\pi}\sin 2\pi x$

From (6.19),

$$g_i(x) = \frac{\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^{i-1}\left[1-\left(x-\frac{1}{2\pi}\sin 2\pi x\right)\right]^{n-i}}{B(i,n-i+1)}\left[1-\cos 2\pi x\right], \quad 0 < x < 1 \tag{6.20}$$

Putting $i = 1$ in (6.20),

$$g_1(x) = n\left[1-\left(x-\frac{1}{2\pi}\sin 2\pi x\right)\right]^{n-1}\left[1-\cos 2\pi x\right], \quad 0 < x < 1 \tag{6.21}$$

Putting $i = n$ in (6.20),

$$g_n(x) = n\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^{n-1}\left[1-\cos 2\pi x\right], \quad 0 < x < 1 \tag{6.22}$$

6.3.2 Exponentiated Generated Distribution

Putting (6.16) in (2.8) gives

$$G(x) = \left[x - \frac{1}{2\pi}\sin 2\pi x\right]^r, \quad 0 < x < 1,\ r > 0 \tag{6.23}$$

and its corresponding pdf is

$$g(x) = r\,(1-\cos 2\pi x)\left[x - \frac{1}{2\pi}\sin 2\pi x\right]^{r-1}, \quad 0 < x < 1,\ r > 0 \tag{6.24}$$

which is referred to as **Burr XI distribution**.

6.3.3 Beta Exponentiated Generated Distribution

Putting (6.23) in (2.10) gives

$$G(x) = \int_0^{\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0 < x < 1,\ a > 0,\ b > 0,\ r > 0 \tag{6.25}$$

and its corresponding pdf is

$$g(x) = \frac{r\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^{ra-1}\left[1-\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^r\right]^{b-1}}{B(a,b)}\left[1-\cos 2\pi x\right] \tag{6.26}$$

i^{th} order statistic of Burr XI distribution

From (6.26),

$$g_i(x) = \frac{r\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^{ri-1}\left[1-\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^r\right]^{n-i}}{B(i,n-i+1)}\left[1-\cos 2\pi x\right] \tag{6.27}$$

Putting $i = 1$ in (6.27),

$$g_1(x) = rn\left[x - \frac{1}{2\pi}\sin 2\pi x\right]^{r-1}\left[1 - \left(x - \frac{1}{2\pi}\sin 2\pi x\right)^r\right]^{n-1}[1 - \cos 2\pi x] \tag{6.28}$$

Putting $i = n$ in (6.27),

$$g_n(x) = rn\left[x - \frac{1}{2\pi}\sin 2\pi x\right]^{rn-1}[1 - \cos 2\pi x], \quad 0 < x < 1,\ r > 0 \tag{6.29}$$

6.4 $r(x) = \dfrac{\alpha e^{-x}}{1 + e^{-x}}, \quad \alpha > 0$

$$\int r(x)dx = \alpha \int \frac{e^{-x}}{1 + e^{-x}}dx$$

Let

$$u = 1 + e^{-x}$$
$$du = -e^{-x}dx$$

Thus

$$\int r(x)dx = \alpha \int \frac{e^{-x}}{-e^{-x}u}dx$$
$$= -\alpha \int \frac{1}{u}du$$

Substituting in (6.1) we get

$$F(x) = \exp\left[-\alpha \int \frac{1}{u}du\right]$$
$$= \exp[-\alpha \log u]$$
$$= \exp\left[\log u^{-\alpha}\right]$$
$$= u^{-\alpha}$$
$$= [1 + e^{-x}]^{-\alpha}$$
$$= \frac{1}{[1 + e^{-x}]^{\alpha}}$$

Now,

$$F(x) = 0 \Longrightarrow \frac{1}{1+\infty} = \frac{1}{[1+e^{-(-\infty)}]^{\alpha}} = 0 \Longrightarrow x = -\infty$$

$$F(x) = 1 \Longrightarrow \frac{1}{1+0} = \frac{1}{[1+e^{-\infty}]^{\alpha}} = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = [1+e^{-x}]^{-\alpha}, \quad -\infty < x < \infty \tag{6.30}$$

and its corresponding pdf is

$$f(x) = \frac{\alpha e^{-x}}{1+[e^{-x}]^{\alpha+1}}, \quad -\infty < x < \infty \tag{6.31}$$

which is the **Type I Generalized Logistic distribution**.

6.4.1 Special Case

When $\alpha = 1$

$$F(x) = [1+e^{-x}]^{-1}, \quad -\infty < x < \infty$$

and its corresponding pdf is

$$f(x) = \frac{e^{-x}}{[e^{-x}+1]^{2}}, \quad -\infty < x < \infty$$

which is the **Logistic distribution** as in (4.2) and (4.3).

When $\alpha = 2$

$$F(x) = [1+e^{-x}]^{-2}, \quad -\infty < x < \infty \tag{6.32}$$

and its corresponding pdf is

$$f(x) = \frac{2e^{-x}}{[e^{-x}+1]^2\,[e^{-x}+1]}, \quad -\infty < x < \infty \tag{6.33}$$

which is the **Skew-Logistic distribution** with skewness parameter 1.

6.4.2 Extension of Type I Generalized Logistic Distribution

Introducing location and scale parameters

Let

$$Y = \sigma X + \lambda, \quad \sigma > 0,\ \lambda > 0$$

where X is Type I Generalized Logistic Distribution given as

$$F(x) = \frac{1}{[1+e^{-x}]^{\theta}}, \quad -\infty < x < \infty$$

If $G(y)$ is the *cdf* of y, then

$$\begin{aligned} G(y) &= prob\ (Y \leq y) \\ &= prob\ (\sigma X + \lambda \leq y) \\ &= prob\ \left(X \leq \frac{y-\lambda}{\sigma}\right) \end{aligned}$$

Therefore

$$\begin{aligned} G(y) &= F\left(\frac{y-\lambda}{\sigma}\right) \\ &= \frac{1}{[1+e^{-\frac{y-\lambda}{\sigma}}]^{\theta}} \\ &= [1+e^{-\frac{y-\lambda}{\sigma}}]^{-\theta} \\ &= [1+e^{-\frac{y}{\sigma}}e^{\frac{\lambda}{\sigma}}]^{-\theta} \end{aligned}$$

Let

$$\rho = e^{\frac{\lambda}{\sigma}}$$

Thus

$$G(y) = \left[1 + \rho e^{-\frac{y}{\sigma}}\right]^{-\theta}, \quad -\infty < y < \infty \tag{6.34}$$

which is a member of **Generalized Gompertz-Verhulst Family of Distributions** a result given by Ahuja and Nash (1967).

When $\theta = 1$, (6.34) becomes **Logistic distribution** with scale parameter σ a result due to Verhulst(1838).

7 Distributions based on Case V of Burr Differential Equation

7.1 Definition of hazard function

The statistical theory of survival analysis deals with survival time T which is regarded as a continous random variable. Accordingly, the survival time t is a realization of T.

As with any continous random variable, T has an associated probability density function $f(t)$. We can characterize T in terms of two other functions, namely the **survival function** $S(t)$ and the **hazard function** $h(t)$.

Since T is a continous random variable, the probability of dying at any given time is 0. A nonzero probability is obtained only when we consider the probability of dying in an interval of time.

Thus $S(t)$, $f(t)$ and $h(t)$ are defined as follows:

$S(t)$ = the probability of an individual alive at time 0 surviving until (at least) time t

$$\begin{aligned} S(t) &= prob\ (T \geq t) \\ &= 1 - prob\ (T \leq t) \\ &= 1 - F(t) \end{aligned}$$

where $F(t)$ is the *cdf*.

$f(t)$ = the instantaneous probability per unit time than an individual alive at time 0 will die at time t

$$\begin{aligned} f(t) &= \lim_{\Delta t \to 0} \frac{prob\ [t \leq T \leq t + \Delta t]}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)]}{\Delta t} \end{aligned}$$

$$
\begin{aligned}
&= \frac{dF(t)}{dt} \\
&= \frac{d[1-S(t)]}{dt} \\
&= -\frac{dS(t)}{dt}
\end{aligned}
$$

$h(t)$ = the instantaneous probability per unit time than an individual alive at time t will die in the next instant

$$
\begin{aligned}
h(t) &= \lim_{\Delta t \to 0} \frac{prob\ [t \leq T \leq t + \Delta t \mid T \geq t]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{prob\ [t \leq T \leq t, T \geq t]}{\Delta t\ \ prob\ T \geq t} \\
&= \lim_{\Delta t \to 0} \frac{prob\ [t \leq T \leq t, T \geq t]}{\Delta t\ S(t)}
\end{aligned}
$$

Since t is continous, $h(t)$ becomes

$$
\begin{aligned}
&= \frac{1}{S(t)} \lim_{\Delta t \to 0} \frac{prob\ [t \leq T \leq t, T \geq t]}{\Delta t} \\
&= \frac{f(t)}{S(t)} \\
&= \frac{f(t)}{1-F(t)}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
h(t) &= \frac{f(t)}{S(t)} \\
&= \frac{1}{S(t)} \left[-\frac{dS(t)}{dt} \right] \\
&= -\frac{dS(t)}{S(t)dt} \\
&= -\frac{d[\log S(t)]}{dt}
\end{aligned}
$$

Hence

$$
h(t)dt = -d[\log S(t)]
$$

Integrating both sides

$$-\int_0^x h(t)dt = \int_0^x d[\log S(t)]$$

$$\log S(x) - \log S(0) = -\int_0^x h(t)dt$$

$$\log\left(\frac{S(x)}{S(0)}\right) = -\int_0^x h(t)dt$$

But by definition $S(0) = prob\ (T \geq 0) = \int_0^\infty f(x)dx = 1$

Therefore

$$\log S(x) = -\int_0^x h(t)dt$$

Exponentiating both sides

$$S(x) = \exp\left[-\int_0^x h(t)dt\right]$$

Therefore

$$F(x) = 1 - \exp\left[-\int_0^x h(t)dt\right]$$

From stochastic approach, Chiang (1968) derived the hazard function as follows:

Let

$$\mu(x)\Delta x + o(\Delta x) = \textit{the probability of dying between age x and } x + \Delta x$$

and

$$\begin{aligned} prob\ (X \leq x) &= \textit{the probability of dying at or before age x} \\ &= F(x) \end{aligned}$$

Then

$$\begin{aligned} F(x+\Delta x) &= \textit{the probability of dying at or before age } x+\Delta x \\ &= \textit{the probability of dying at or before age } x \textit{ or probability} \\ &\textit{of living upto age } x \textit{ and dying between age } x \textit{ and } x+\Delta x \\ &= F(x)+[1-F(x)][\mu(x)\Delta x+o(\Delta x)] \end{aligned}$$

which becomes

$$F(x+\Delta x)-F(x)=[1-F(x)][\mu(x)\Delta x+o(\Delta x)]$$

Hence

$$\lim_{\Delta x\to 0}\frac{F(x+\Delta x)-F(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{[1-F(x)][\mu(x)\Delta x+o(\Delta x)]}{\Delta x}$$

Since

$$\lim_{\Delta x\to 0}\frac{o(\Delta x)}{\Delta x}=0$$

$$f(x)=[1-F(x)]\mu(x)$$

Therefore

$$\mu(x)=\frac{f(x)}{1-F(x)}$$

which can be written as

$$\begin{aligned} \mu(x) &= \frac{y'}{1-y} \\ y' &= (1-y)\mu(x) \\ y' &= y(1-y)\frac{\mu(x)}{y} \end{aligned}$$

which is (2.1) with

$$g(x,y)=\frac{\mu(x)}{y}$$

In demography, the hazard function $h(t)$ is called the **force of mortality** denoted by $\mu(t)$.

This implies that

$$\int \frac{dy}{(1-y)} = \int_0^x \mu(t)dt$$

$$-\log(1-y) = \int_0^x \mu(t)dt$$

$$(1-y) = \exp\left[-\int_0^x \mu(t)dt\right]$$

Therefore

$$y = 1 - \exp\left[-\int_0^x \mu(t)dt\right]$$

That is

$$F(x) = 1 - \exp\left[-\int_0^x \mu(t)dt\right] \quad \textbf{(7.1)}$$

7.2 Lomax (1954); $\mu(t) = \dfrac{1}{a+bt}, \quad a > 0,\ b > 0$

$$\begin{aligned}
\int_0^x \mu(t)dt &= \int_0^x \frac{1}{a+bt}dt \\
&= \frac{1}{b}\left[\log(a+bt)\right]_0^x \\
&= \frac{1}{b}\left[\log(a+bx) - \log a\right] \\
&= \frac{1}{b}\left[\log\left(\frac{a+bx}{a}\right)\right] \\
&= \frac{1}{b}\left[\log\left(1+\frac{b}{a}x\right)\right] \\
&= \frac{1}{b}\left[\log\left(1+\frac{x}{\frac{a}{b}}\right)\right]
\end{aligned}$$

Let

$$\frac{1}{b} = \alpha;\ \frac{a}{b} = \beta, \quad \alpha > 0,\ \beta > 0$$

Then

$$\int_0^x \mu(t)dt = \alpha\left[\log\left(1+\frac{x}{\beta}\right)\right]$$

Substituting in (7.1) we get

$$\begin{aligned}
F(x) &= 1-\exp\left\{-\alpha\left[\log\left(1+\frac{x}{\beta}\right)\right]\right\} \\
&= 1-\exp\left\{\log\left(1+\frac{x}{\beta}\right)^{-\alpha}\right\} \\
&= 1-\left(1+\frac{x}{\beta}\right)^{-\alpha} \\
&= 1-\left(\frac{\beta+x}{\beta}\right)^{-\alpha} \\
&= 1-\left(\frac{\beta}{\beta+x}\right)^{\alpha} \\
&= 1-\left(\frac{1}{1+\frac{x}{\beta}}\right)^{\alpha}
\end{aligned}$$

Now,

$$F(x)=0 \Longrightarrow 1-\frac{1}{1+0} = 1-\left(\frac{1}{1+\frac{0}{\beta}}\right)^{\alpha} = 0 \Longrightarrow x=0$$

$$F(x)=1 \Longrightarrow 1-\frac{1}{1+\infty} = 1-\left(\frac{1}{1+\frac{\infty}{\beta}}\right)^{\alpha} = 1 \Longrightarrow x=\infty$$

Therefore

$$F(x) = 1-\left(1+\frac{x}{\beta}\right)^{-\alpha}, \quad 0<x<\infty \tag{7.2}$$

and its corresponding pdf is

$$
\begin{aligned}
f(x) &= \frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha-1} \\
&= \frac{\alpha}{\beta}\left(\frac{\beta+x}{\beta}\right)^{-\alpha-1} \\
&= \frac{\alpha}{\beta}\left(\frac{\beta}{\beta+x}\right)^{\alpha+1} \\
&= \frac{\alpha\beta^{\alpha}}{(\beta+x)^{\alpha+1}}, \quad 0<x<\infty
\end{aligned} \tag{7.3}
$$

which is the **Lomax distribution**.

7.2.1 Beta Lomax Distribution

Putting (7.2) in (2.3) gives

$$
G(x) = \int_{0}^{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0<x<\infty,\ a>0,\ b>0 \tag{7.4}
$$

and its corresponding pdf is

$$
\begin{aligned}
g(x) &= \frac{\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{a-1}\left[\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{b-1}}{B(a,b)}\frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha-1} \\
&= \frac{\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{a-1}\left(1+\frac{x}{\beta}\right)^{-\alpha b-1}}{B(a,b)}\frac{\alpha}{\beta}, \quad 0<x<\infty,\ a>0,\ b>0
\end{aligned} \tag{7.5}
$$

i^{th} order statistic of Lomax distribution

From (7.5),

$$
g_i(x) = \frac{\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{i-1}\left(1+\frac{x}{\beta}\right)^{-\alpha(n-i+1)-1}}{B(i,n-i+1)}\frac{\alpha}{\beta}, \quad 0<x<\infty \tag{7.6}
$$

Putting $i = 1$ in (7.6),

$$g_1(x) = \frac{n\alpha\left(1+\frac{x}{\beta}\right)^{-\alpha n-1}}{\beta}, \quad 0 < x < \infty \tag{7.7}$$

Putting $i = n$ in (7.6),

$$g_n(x) = \frac{n\alpha\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{n-1}\left(1+\frac{x}{\beta}\right)^{-\alpha-1}}{\beta}, \quad 0 < x < \infty \tag{7.8}$$

7.2.2 Exponentiated Lomax Distribution

Putting (7.2) in (2.8) gives

$$G(x) = \left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{r}, \quad 0 < x < \infty,\ r > 0 \tag{7.9}$$

and its corresponding pdf is

$$g(x) = \frac{r\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha-1}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{r-1}, \quad 0 < x < \infty,\ r > 0 \tag{7.10}$$

7.2.3 Beta Exponentiated Lomax Distribution

Putting (7.9) in (2.10) gives

$$G(x) = \int_0^{\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{r}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0 < x < \infty,\ a > 0,\ b > 0,\ r > 0 \tag{7.11}$$

and its corresponding pdf is

$$g(x)=\frac{r\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{ra-1}\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right\}^{r}\right]^{b-1}}{B(a,b)}\frac{\alpha\beta^{\alpha}}{(\beta+x)^{\alpha+1}} \tag{7.12}$$

i^{th} order statistic of Exponentiated Lomax distribution

From (7.12),

$$g_i(x)=\frac{r\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{ri-1}\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right\}^{r}\right]^{n-i}}{B(i,n-i+1)}\frac{\alpha\beta^{\alpha}}{(\beta+x)^{\alpha+1}} \tag{7.13}$$

Putting $i=1$ in (7.13),

$$g_1(x)=rn\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{r-1}\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right\}^{r}\right]^{n-1}\frac{\alpha\beta^{\alpha}}{(\beta+x)^{\alpha+1}} \tag{7.14}$$

Putting $i=n$ in (7.13),

$$g_n(x)=rn\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{rn-1}\frac{\alpha\beta^{\alpha}}{(\beta+x)^{\alpha+1}},\quad 0<x<1,\ r>0 \tag{7.15}$$

7.3 $\mu(t)=c,\quad c>0$

$$\begin{aligned}\int_0^x \mu(t)dt &= \int_0^x cdt\\ &= [ct]_0^x\\ &= cx\end{aligned}$$

Substituting in (7.1) we get

$$F(x) = 1 - e^{-cx}$$
$$= 1 - \frac{1}{e^{cx}}$$

Now,

$$F(x) = 0 \Longrightarrow 1 - \frac{1}{\exp\{0\}} = 1 - \frac{1}{\exp\{c(0)\}} = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow 1 - \frac{1}{\exp\{\infty\}} = 1 - \frac{1}{\exp\{c(\infty)\}} = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = 1 - e^{-cx}, \quad 0 < x < \infty \tag{7.16}$$

and its corresponding pdf is

$$f(x) = ce^{-cx}, \quad 0 < x < \infty \tag{7.17}$$

which is the **Exponential distribution** with rate parameter c.

The m^{th} moment of (7.17) is

$$E(X^m) = \int_0^\infty x^m c e^{-cx} dx$$

Let

$$y = cx \Longrightarrow dy = cdx$$

Thus

$$E(X^m) = \int_0^\infty \left(\frac{y}{c}\right)^m \frac{ce^{-y}}{c} dy$$
$$= \left(\frac{1}{c}\right)^m \int_0^\infty y^m e^{-y} dy$$
$$= \left(\frac{1}{c}\right)^m \int_0^\infty y^{m+1-1} e^{-y} dy \quad \textbf{(7.18)}$$
$$= \left(\frac{1}{c}\right)^m \Gamma(m+1)$$

7.3.1 Extension of Exponential Distribution

Introducing location and scale parameters

Let

$$cX = \frac{Y-\lambda}{\sigma}, \quad c > 0,\ \sigma > 0,\ \lambda > 0$$

where X is Exponential distribution given in (7.16).

If $G(y)$ is the *cdf* of y, then

$$G(y) = prob\ (Y \le y)$$
$$= prob\ (c\sigma X + \lambda \le y)$$
$$= prob\ \left(X \le \frac{y-\lambda}{c\sigma}\right)$$

Therefore

$$G(y) = F\left(\frac{y-\lambda}{b\sigma}\right)$$
$$= 1 - e^{-b\left(\frac{y-\lambda}{b\sigma}\right)}$$
$$= 1 - e^{-\left(\frac{y-\lambda}{\sigma}\right)}$$
$$= 1 - e^{-\frac{y}{\sigma}} e^{\frac{\lambda}{\sigma}}$$

Let

$$\rho = e^{\frac{\lambda}{\sigma}}$$

Thus

$$G(y) = 1 - \rho e^{-\frac{y}{\sigma}}$$

Now,

$$G(y) = 0 \Longrightarrow 1 - \rho e^{-\frac{y}{\sigma}} = 0 \Longrightarrow \rho e^{-\frac{y}{\sigma}} = 1$$

$$\begin{aligned} e^{-\frac{y}{\sigma}} &= \frac{1}{\rho} \\ -\frac{y}{\sigma} &= \ln\frac{1}{\rho} = \ln 1 - \ln p \\ y &= \sigma \ln \rho \end{aligned}$$

$$G(y) = 1 \Longrightarrow 1 - \rho e^{-\frac{y}{\sigma}} = 1 \Longrightarrow \rho e^{-\frac{y}{\sigma}} = 0$$

$$\begin{aligned} e^{-\frac{y}{\sigma}} &= 0 = e^{-\infty} \\ \frac{y}{\sigma} &= \infty \\ y &= \infty \end{aligned}$$

Therefore

$$G(y) = 1 - \rho e^{-\frac{y}{\sigma}}, \quad \sigma \ln \rho < y < \infty \tag{7.19}$$

which is the Exponential distribution with scale parameter σ a result due to Verhulst (1847) when $\theta = 1$ on

$$[G(y)]^{\theta} = \left[1 - \rho e^{-\frac{y}{\sigma}}\right]^{\theta}, \quad \sigma \ln \rho < y < \infty \tag{7.20}$$

(7.20) is a member of the **Generalized Gompertz-Verhulst Family of Distributions** a result given by Ahuja and Nash (1967).

7.3.2 Beta Exponential Distribution

Putting (7.16) in (2.3) gives

$$G(x)=\int_{0}^{1-e^{-cx}}\frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt,\quad 0<x<\infty,\ a>0,\ b>0 \tag{7.21}$$

and its corresponding pdf is

$$\begin{aligned}
g(x)&=\frac{\left[1-e^{-cx}\right]^{a-1}\left[e^{-cx}\right]^{b-1}}{B(a,b)}ce^{-cx}\\
&=\frac{\left[1-e^{-cx}\right]^{a-1}\left[e^{-cx}\right]^{b-1+1}}{B(a,b)}c\\
&=\frac{\left[1-e^{-cx}\right]^{a-1}\left[e^{-cx}\right]^{b}}{B(a,b)}c\\
&=\frac{\left[1-e^{-cx}\right]^{a-1}e^{-bcx}}{B(a,b)}c,\quad 0<x<\infty,\ a>0,\ b>0
\end{aligned} \tag{7.22}$$

i^{th} order statistic of Exponential distribution

From (7.22),

$$g_i(x)=\frac{\left[1-e^{-cx}\right]^{i-1}e^{-(n-i+1)cx}}{B(i,n-i+1)}c,\quad 0<x<\infty \tag{7.23}$$

Putting $i=1$ in (7.23),

$$g_1(x)=nce^{-ncx},\quad 0<x<\infty \tag{7.24}$$

The m^{th} moment of (7.24) is

$$E\left(X_1^m\right)=\int_{0}^{\infty}x^m nce^{-ncx}dx$$

Let

$$y = ncx \Longrightarrow dy = ncdx$$

Thus

$$\begin{aligned} E(X_1^m) &= \int_0^\infty \frac{\left[\frac{y}{nc}\right]^m nce^{-y}}{nc} dy \\ &= \frac{1}{[nc]^m} \int_0^\infty y^m e^{-y} dy \\ &= \frac{1}{[nc]^m} \int_0^\infty y^{1+m-1} e^{-y} dy \\ &= \frac{1}{[nc]^m} \Gamma(1+m) \end{aligned} \tag{7.25}$$

Putting $i = n$ in (7.23),

$$g_n(x) = nc\left[1 - e^{-cx}\right]^{n-1} e^{-cx}, \quad 0 < x < \infty \tag{7.26}$$

7.3.3 Exponentiated Exponential Distribution

Putting (7.16) in (2.8) gives

$$G(x) = \left[1 - e^{-cx}\right]^r, \quad 0 < x < \infty, \ r > 0 \tag{7.27}$$

and its corresponding pdf is

$$g(x) = rce^{-cx}\left[1 - e^{-cx}\right]^{r-1}, \quad 0 < x < \infty, \ r > 0 \tag{7.28}$$

7.3.4 Beta Exponentiated Exponential Distribution

Putting (7.27) in (2.10) gives

$$G(x) = \int_0^{\left[1-e^{-cx}\right]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < \infty, \ a > 0, \ b > 0, \ r > 0 \tag{7.29}$$

and its corresponding pdf is

$$g(x) = \frac{r\left[1-e^{-cx}\right]^{ra-1}\left[1-\left\{1-e^{-cx}\right\}^{r}\right]^{b-1}}{B(a,b)} ce^{-cx} \tag{7.30}$$

i^{th} order statistic of Exponentiated Exponential distribution

From (7.30),

$$g_i(x) = \frac{r\left[1-e^{-cx}\right]^{ri-1}\left[1-\left\{1-e^{-cx}\right\}^{r}\right]^{n-i}}{B(i,n-i+1)} ce^{-cx}, \quad 0 < x < \infty,\ r > 0 \tag{7.31}$$

Putting $i = 1$ in (7.31),

$$g_1(x) = rn\left[1-e^{-cx}\right]^{r-1}\left[1-\left\{1-e^{-cx}\right\}^{r}\right]^{n-1} ce^{-cx}, \quad 0 < x < \infty,\ r > 0 \tag{7.32}$$

Putting $i = n$ in (7.31),

$$g_n(x) = rn\left[1-e^{-cx}\right]^{rn-1} ce^{-cx}, \quad 0 < x < \infty,\ r > 0 \tag{7.33}$$

7.4 Weibull (1951); $\mu(t) = c\alpha t^{\alpha-1}, \quad c > 0,\ \alpha > 0$

$$\begin{aligned}
\int_0^x \mu(t)dt &= \int_0^x c\alpha t^{\alpha-1}dt \\
&= c\alpha \int_0^x t^{\alpha-1}dt \\
&= \frac{c\alpha}{\alpha}\left[t^{\alpha}\right]_0^x \\
&= c\left[x^{\alpha}-0\right] \\
&= cx^{\alpha}
\end{aligned}$$

Substituting in (7.1) we get

$$\begin{aligned} F(x) &= 1 - e^{-cx^{\alpha}} \\ &= 1 - \frac{1}{e^{cx^{\alpha}}} \end{aligned}$$

Now,

$$F(x) = 0 \Longrightarrow 1 - \frac{1}{\exp\{0\}} = 1 - \frac{1}{\exp\{c(0)^{\alpha}\}} = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow 1 - \frac{1}{\exp\{\infty\}} = 1 - \frac{1}{\exp\{c(\infty)^{\alpha}\}} = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = 1 - e^{-cx^{\alpha}}, \quad 0 < x < \infty \tag{7.34}$$

and its corresponding pdf is

$$f(x) = c\alpha x^{\alpha-1} e^{-cx^{\alpha}}, \quad 0 < x < \infty \tag{7.35}$$

which is the **Weibull distribution** with shape parameter α and scale parameter $c = \beta^{-\alpha}$.

The m^{th} moment of (7.35) is

$$E(X^m) = \int_0^{\infty} x^m c\alpha x^{\alpha-1} e^{-cx^{\alpha}} dx$$

Let

$$y = cx^{\alpha} \Longrightarrow dy = c\alpha x^{\alpha-1} dx$$

Thus

$$\begin{aligned} E(X^m) &= \int_0^\infty \frac{x^m c\alpha x^{\alpha-1} e^{-y}}{c\alpha x^{\alpha-1}} dy \\ &= \int_0^\infty \left(\frac{y}{c}\right)^{\frac{m}{\alpha}} e^{-y} dy \\ &= \left(\frac{1}{c}\right)^{\frac{m}{\alpha}} \int_0^\infty y^{1+\frac{m}{\alpha}-1} e^{-y} dy \\ &= \left(\frac{1}{c}\right)^{\frac{m}{\alpha}} \Gamma\left(1+\frac{m}{\alpha}\right) \end{aligned} \tag{7.36}$$

7.4.1 Special Case

When $\alpha = 1$

$$F(x) = 1 - e^{-cx}, \quad 0 < x < \infty$$

and its corresponding pdf is

$$f(x) = ce^{-cx}, \quad 0 < x < \infty$$

which is the **Exponential distribution** with rate parameter c as in (7.16) and (7.17).

7.4.2 Beta Weibull Distribution

Putting (7.34) in (2.3) gives

$$G(x) = \int_0^{1-e^{-cx^\alpha}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < \infty,\ a > 0,\ b > 0 \tag{7.37}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{\left[1-e^{-cx^{\alpha}}\right]^{a-1}\left[1-\left\{1-e^{-cx^{\alpha}}\right\}\right]^{b-1}}{B(a,b)} c\alpha x^{\alpha-1} e^{-cx^{\alpha}} \\ &= \frac{\left[1-e^{-cx^{\alpha}}\right]^{a-1}\left[e^{-cx^{\alpha}}\right]^{b-1+1}}{B(a,b)} c\alpha x^{\alpha-1} \\ &= \frac{\left[1-e^{-cx^{\alpha}}\right]^{a-1}\left[e^{-cx^{\alpha}}\right]^{b}}{B(a,b)} c\alpha x^{\alpha-1} \\ &= \frac{\left[1-e^{-cx^{\alpha}}\right]^{a-1} e^{-bcx^{\alpha}}}{B(a,b)} c\alpha x^{\alpha-1}, \quad 0<x<\infty,\ a>0,\ b>0 \end{aligned} \tag{7.38}$$

i^{th} order statistic of Weibull distribution

From (7.38),

$$g_i(x) = \frac{\left[1-e^{-cx^{\alpha}}\right]^{i-1} e^{-(n-i+1)cx^{\alpha}}}{B(i,n-i+1)} c\alpha x^{\alpha-1}, \quad 0<x<\infty \tag{7.39}$$

Putting $i=1$ in (7.39),

$$g_1(x) = nc\alpha x^{\alpha-1} e^{-ncx^{\alpha}}, \quad 0<x<\infty \tag{7.40}$$

The m^{th} moment of (7.40) is

$$E(X_1^m) = \int_0^{\infty} x^m nc\alpha x^{\alpha-1} e^{-ncx^{\alpha}} dx$$

Let

$$y = ncx^{\alpha} \Longrightarrow dy = nc\alpha x^{\alpha-1} dx$$

Thus

$$E(X_1^m) = \int_0^\infty \frac{\left[\left(\frac{y}{nc}\right)^{\frac{1}{\alpha}}\right]^m c\alpha x^{\alpha-1} e^{-y}}{nc\alpha x^{\alpha-1}} dy$$
$$= \int_0^\infty \left(\frac{y}{nc}\right)^{\frac{m}{\alpha}} e^{-y} dy$$
$$= \left(\frac{1}{nc}\right)^{\frac{m}{\alpha}} \int_0^\infty y^{1+\frac{m}{\alpha}-1} e^{-y} dy$$
$$= \frac{\Gamma\left(1+\frac{m}{\alpha}\right)}{(nc)^{\frac{m}{\alpha}}} \tag{7.41}$$

Putting $i = n$ in (7.39),

$$g_n(x) = nc\alpha x^{\alpha-1}\left[1-e^{-cx^\alpha}\right]^{n-1} e^{-cx^\alpha}, \quad 0 < x < \infty \tag{7.42}$$

7.4.3 Exponentiated Weibull Distribution

Putting (7.34) in (2.8) gives

$$G(x) = \left[1-e^{-cx^\alpha}\right]^r, \quad 0 < x < \infty,\ r > 0 \tag{7.43}$$

and its corresponding pdf is

$$g(x) = rc\alpha x^{\alpha-1} e^{-cx^\alpha}\left[1-e^{-cx^\alpha}\right]^{r-1}, \quad 0 < x < \infty,\ r > 0 \tag{7.44}$$

7.4.4 Beta Exponentiated Weibull Distribution

Putting (7.43) in (2.10) gives

$$G(x) = \int_0^{\left[1-e^{-cx^\alpha}\right]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < \infty,\ a > 0,\ b > 0,\ r > 0 \tag{7.45}$$

and its corresponding pdf is

$$g(x)=\frac{r\left[1-e^{-cx^{\alpha}}\right]^{ra-1}\left[1-\left\{1-e^{-cx^{\alpha}}\right\}^{r}\right]^{b-1}}{B(a,b)}c\alpha x^{\alpha-1}e^{-cx^{\alpha}} \tag{7.46}$$

i^{th} order statistic of Exponentiated Weibull distribution

From (7.46),

$$g_i(x)=\frac{r\left[1-e^{-cx^{\alpha}}\right]^{ri-1}\left[1-\left\{1-e^{-cx^{\alpha}}\right\}^{r}\right]^{n-i}}{B(i,n-i+1)}c\alpha x^{\alpha-1}e^{-cx^{\alpha}},\ \ 0<x<\infty,\ r>0 \tag{7.47}$$

Putting $i=1$ in (7.47),

$$g_1(x)=rn\left[1-e^{-cx^{\alpha}}\right]^{r-1}\left[1-\left\{1-e^{-cx^{\alpha}}\right\}^{r}\right]^{n-1}c\alpha x^{\alpha-1}e^{-cx^{\alpha}},\ \ 0<x<\infty,\ r>0 \tag{7.48}$$

Putting $i=n$ in (7.47),

$$g_n(x)=rn\left[1-e^{-cx^{\alpha}}\right]^{rn-1}c\alpha x^{\alpha-1}e^{-cx^{\alpha}},\quad 0<x<\infty,\ r>0 \tag{7.49}$$

7.5 $\mu(t)=A\log t,\quad A>0$

$$\begin{aligned}\int_0^x \mu(t)dt &= \int_0^x A\log t\,dt\\ &= A\int_0^x \log t\,dt\end{aligned}$$

Let

$$u=\log t \Longrightarrow du=\frac{dt}{t}$$

and

$$dv = dt \Longrightarrow v = t$$

Then

$$\begin{aligned}
A\int_0^x \log t\,dt &= A\left([t\log t]_0^x - \int_0^x dt\right) \\
&= A\left([x\log x - 0] - [t]_0^x\right) \\
&= A\left(x\log x - [x - 0]\right) \\
&= A\left(x\log x - x\right)
\end{aligned}$$

Substituting in (7.1) we get

$$\begin{aligned}
F(x) &= 1 - \exp\{-A(x\log x - x)\} \\
&= 1 - \exp\{-Ax\log x + Ax\} \\
&= 1 - \exp\{-Ax\log x\}\exp\{Ax\} \\
&= 1 - \exp\left\{\log x^{-Ax}\right\}\exp\{Ax\} \\
&= 1 - x^{-Ax}\exp\{Ax\} \\
&= 1 - \frac{1}{x^{Ax}}\exp\{Ax\} \\
&= 1 - \left(\frac{e}{x}\right)^{Ax}
\end{aligned}$$

Therefore

$$F(x) = 1 - \left(\frac{e}{x}\right)^{Ax} \tag{7.50}$$

and its corresponding pdf is

$$f(x) = \frac{Ae^{Ax}\log x}{x^{Ax}} \tag{7.51}$$

7.6 De Moivre's Law

De Moivre (1725) postulated the existence of a maximum age ω for human beings and assumed that time T to death was uniformly distributed on $(0, \omega - x)$.

Let

$$l_{x+t} = a + bt, \quad 0 \le t \le \omega - x$$

be the number of persons living at age $x+t$.

For,

$$\begin{aligned} t = 0 &\Longrightarrow l_x = a \\ t = \omega - x &\Longrightarrow l_\omega = a + b(\omega - x) \end{aligned}$$

But

$$l_x = 0$$

Thus

$$b = -\frac{l_x}{\omega - x}$$

Therefore

$$\begin{aligned} l_{x+t} &= l_x - \frac{l_x t}{\omega - x} \\ &= l_x\left(1 - \frac{t}{\omega - x}\right), \quad 0 \le t \le \omega - x \end{aligned}$$

The force of mortality at time $x+t$ is given by:

$$\begin{aligned} \mu_{x+t} &= -\frac{1}{l_{x+t}}\frac{dl_{x+t}}{dt} \\ &= -\frac{1}{a+bt}\frac{d(a+bt)}{dt} \\ &= -\frac{b}{a+bt} \end{aligned}$$

$$
\begin{aligned}
&= -\left\{-\frac{l_x}{\omega - x}\left[\frac{1}{l_x\left(1-\frac{t}{\omega-x}\right)}\right]\right\}\\
&= \left\{\frac{l_x}{\omega - x}\left[\frac{1}{l_x - \frac{l_x t}{\omega-x}}\right]\right\}\\
&= \left\{\frac{l_x}{\omega - x}\left[\frac{1}{\frac{l_x(\omega-x)-l_x t}{\omega-x}}\right]\right\}\\
&= \left\{\frac{l_x}{\omega - x}\left[\frac{\omega-x}{l_x(\omega-x-t)}\right]\right\}\\
&= \frac{1}{\omega-x-t}, \quad 0 \le t \le \omega - x
\end{aligned}
$$

For $x = 0$

$$\mu_t = \frac{1}{\omega - t}, \quad 0 \le t \le \omega$$

Then

$$
\begin{aligned}
\int_0^x \mu(t)dt &= \int_0^x \frac{dt}{\omega - t}\\
&= [-\log(\omega - t)]_0^x\\
&= [-\log(\omega - x) - \log(\omega)]\\
&= -\log\left(\frac{\omega - x}{\omega}\right)
\end{aligned}
$$

Substituting in (7.1) we get

$$
\begin{aligned}
F(x) &= 1 - \exp\left[\log\left(\frac{\omega - x}{\omega}\right)\right]\\
&= 1 - \frac{\omega - x}{\omega}\\
&= 1 - \left(1 - \frac{x}{\omega}\right)\\
&= \frac{x}{\omega}
\end{aligned}
$$

Now,

$$F(x) = 0 \Longrightarrow \frac{0}{\omega} = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow \frac{\omega}{\omega} = 1 \Longrightarrow x = \omega$$

Therefore

$$F(x) = \frac{x}{\omega}, \quad 0 < x < \omega \tag{7.52}$$

and its corresponding pdf is

$$f(x) = \frac{1}{\omega}, \quad 0 < x < \omega \tag{7.53}$$

The m^{th} moment of (7.53) is

$$\begin{aligned} E(X^m) &= \int_0^{\omega} \frac{x^m}{\omega} dx \\ &= \frac{1}{\omega} \left[\frac{x^{m+1}}{m+1} \right]_0^{\omega} \\ &= \frac{\omega^{m+1}}{\omega} \left[\frac{1}{m+1} \right] \\ &= \omega^m B(m+1, 1) \end{aligned} \tag{7.54}$$

7.6.1 Extension of De Moivre's Law

Under De Moivre's Law, the probability of a newborn surviving at least x years is given by the survival function

$$S(x) = 1 - \frac{x}{\omega}, \quad 0 < x < \omega$$

Let (x) denote life that has survived to age x and $T(x)$, a random variable is time to death of (x). The conditional probability that (x) survives to age $x+t$ is denoted by ${}_tp_x$.

$$
\begin{aligned}
{}_t p_x &= prob\,[T(0) \geq x+t|\; T(0) \geq x] \\
&= \frac{S(x+t)}{S(x)}
\end{aligned}
$$

Under De Moivre's Law, the conditional probability that a life aged x years survives at least t more years is given by

$$
\begin{aligned}
{}_t p_x &= \frac{1-\frac{x+t}{\omega}}{1-\frac{x}{\omega}} \\
&= \frac{\frac{\omega-(x+t)}{\omega}}{\frac{\omega-x}{\omega}} \\
&= \frac{\omega-(x+t)}{\omega}\left(\frac{\omega}{\omega-x}\right) \\
&= \frac{\omega-(x+t)}{\omega-x}
\end{aligned}
$$

Therefore

$$
{}_t p_x = \frac{\omega-(x+t)}{\omega-x}, \quad 0 \leq t \leq \omega - x \tag{7.55}
$$

and the random variable $T(x)$ follows a uniform distribution on $(0, \omega - x)$.

The conditional probability that (x) dies before age $x+t$ is denoted by ${}_t q_x$.

$$
\begin{aligned}
{}_t q_x &= prob\,[0 \leq T(x) \leq t|\; T(0) \geq x] \\
&= \frac{S(x)-S(x+t)}{S(x)}
\end{aligned}
$$

Under De Moivre's Law, the conditional probability that a life aged x years fails to survive to age $x+t$ is given by

$$
\begin{aligned}
{}_tq_x &= \frac{1-\frac{x}{\omega}-\left(1-\frac{x+t}{\omega}\right)}{1-\frac{x}{\omega}} \\
&= \frac{\frac{x+t}{\omega}-\frac{x}{\omega}}{\frac{\omega-x}{\omega}} \\
&= \frac{x+t-x}{\omega}\left(\frac{\omega}{\omega-x}\right) \\
&= \frac{t}{\omega-x}
\end{aligned}
$$

Therefore

$$
{}_tq_x = \frac{t}{\omega-x}, \quad 0 \le t \le \omega - x \tag{7.56}
$$

Under De Moivre's Law, the force of mortality for a life aged x is

$$
\mu(x+t) = \frac{1}{\omega-(x+t)}, \quad 0 \le t \le \omega - x
$$

7.6.2 Beta Generated Distribution

Putting (7.52) in (2.3) gives

$$
G(x) = \int_0^{\frac{x}{\omega}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0 < x < \omega,\ a > 0,\ b > 0 \tag{7.57}
$$

and its corresponding pdf is

$$
\begin{aligned}
g(x) &= \frac{\left[\frac{x}{\omega}\right]^{a-1}\left[1-\frac{x}{\omega}\right]^{b-1}}{\omega B(a,b)} \\
&= \frac{x^{a-1}\left[1-\frac{x}{\omega}\right]^{b-1}}{\omega^{a-1}\omega B(a,b)} \\
&= \frac{x^{a-1}\left[1-\frac{x}{\omega}\right]^{b-1}}{\omega^{a}B(a,b)}, \quad 0 < x < \omega,\ a > 0,\ b > 0
\end{aligned} \tag{7.58}
$$

which is the **Three Parameter Beta distribution**.

The m^{th} moment of (7.58) is

$$E(X^m) = \int_0^{\omega} x^m \frac{\left[\frac{x}{\omega}\right]^{a-1}\left[1-\frac{x}{\omega}\right]^{b-1}}{\omega B(a,b)} dx$$

Let

$$y = \frac{x}{\omega} \Longrightarrow dy = \frac{dx}{\omega}$$

Thus

$$\begin{aligned} E(X^m) &= \int_0^1 (\omega y)^m \frac{[y]^{a-1}[1-y]^{b-1}}{\omega B(a,b)} \omega dy \\ &= \omega^m \int_0^1 \frac{[y]^{m+a-1}[1-y]^{b-1}}{B(a,b)} dy \\ &= \frac{\omega^m B(m+a,b)}{B(a,b)} \end{aligned} \tag{7.59}$$

i^{th} **order statistic of** $F(x) = \frac{x}{\omega}$

From (7.58),

$$g_i(x) = \frac{\left[\frac{x}{\omega}\right]^{i-1}\left[1-\frac{x}{\omega}\right]^{n-i}}{\omega B(i,n-i+1)}, \quad 0 < x < \omega \tag{7.60}$$

The m^{th} moment of (7.60) is

$$E(X_i^m) = \omega^m \frac{B(m+i,n-i+1)}{B(i,n-i+1)} \tag{7.61}$$

Putting $i = 1$ in (7.60),

$$g_1(x) = n\left[1-\frac{x}{\omega}\right]^{n-1}, \quad 0<x<\omega \tag{7.62}$$

The m^{th} moment of (7.62) is

$$E(X_1^m) = n\omega^m B(m+1,n) \tag{7.63}$$

Putting $i=n$ in (7.60),

$$g_n(x) = n\left[\frac{x}{\omega}\right]^{n-1}, \quad 0<x<\omega \tag{7.64}$$

The m^{th} moment of (7.64) is

$$E(X_n^m) = n\omega^m B(m+n,1) \tag{7.65}$$

7.6.3 Exponentiated Generated Distribution

Putting (7.52) in (2.8) gives

$$G(x) = \left[\frac{x}{\omega}\right]^{r}, \quad 0<x<\omega,\ r>0 \tag{7.66}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{r}{\omega}\left[\frac{x}{\omega}\right]^{r-1} \\ &= \frac{rx^{r-1}}{\omega^r}, \quad 0<x<\omega,\ r>0 \end{aligned} \tag{7.67}$$

The m^{th} moment of (7.67) is

$$\begin{aligned} E(X^m) &= \frac{r}{\omega^r}\int_0^{\omega} x^m x^{r-1} dx \\ &= \frac{r}{\omega^r}\int_0^{\omega} x^{m+r-1} dx \\ &= \frac{r}{\omega^r}\left[\frac{x^{m+r}}{m+r}\right]_0^{\omega} \\ &= \frac{r}{\omega^r}\frac{\omega^{m+r}}{m+r} \\ &= r\omega^m B(m+r,1) \end{aligned} \quad (7.68)$$

7.6.4 Beta Exponentiated Generated Distribution

Putting (7.66) in (2.10) gives

$$G(x) = \int_0^{\left[\frac{x}{\omega}\right]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < \omega,\ a > 0,\ b > 0,\ r > 0 \quad (7.69)$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{r\left[\frac{x}{\omega}\right]^{ra-1}\left\{1-\left[\frac{x}{\omega}\right]^r\right\}^{b-1}}{\omega B(a,b)} \\ &= \frac{rx^{ra-1}\left\{1-\left[\frac{x}{\omega}\right]^r\right\}^{b-1}}{\omega^{ra-1}\omega B(a,b)} \\ &= \frac{rx^{ra-1}\left\{1-\left[\frac{x}{\omega}\right]^r\right\}^{b-1}}{\omega^{ra} B(a,b)}, \quad 0 < x < \omega,\ a > 0,\ b > 0,\ r > 0 \end{aligned} \quad (7.70)$$

which is the **GB1** with shape parameters $p = a, q = b, a = r$ and scale parameter $b = \omega$.

The m^{th} moment of (7.70) is

$$E(X^m) = \int_0^{\omega} rx^m \frac{x^{ra-1}\left\{1-\left[\frac{x}{\omega}\right]^r\right\}^{b-1}}{\omega^{ra}B(a,b)}dx$$
$$= \int_0^{\omega} \frac{rx^{m+ra-1}\left\{1-\left[\frac{x}{\omega}\right]^r\right\}^{b-1}}{\omega^{ra}B(a,b)}dx$$

Let

$$y = \left[\frac{x}{\omega}\right]^r \Longrightarrow dy = \frac{rx^{r-1}dx}{\omega^r}$$

Thus

$$\begin{aligned} E(X^m) &= \int_0^1 \frac{rx^{m+ra-1}\left[1-y\right]^{b-1}}{\omega^{ra}B(a,b)rx^{r-1}}\omega^r dy \\ &= \int_0^1 \frac{x^{m+ra-1-r+1}\left[1-y\right]^{b-1}}{B(a,b)}\omega^{r-ra}dy \\ &= \int_0^1 \frac{y^{\frac{m}{r}+a-1}\left[1-y\right]^{b-1}}{B(a,b)}\omega^{r-ra+m+ra-r}dy \\ &= \frac{\omega^m B\left(\frac{m}{r}+a,b\right)}{B(a,b)}, \quad r > m \end{aligned} \tag{7.71}$$

i^{th} **order statistic of** $F(x) = \left[\frac{x}{\omega}\right]^r$

From (7.70),

$$g_i(x) = \frac{rx^{ri-1}\left\{1-\left[\frac{x}{\omega}\right]^r\right\}^{n-i}}{\omega^{ri}B(i,n-i+1)}, \quad 0 < x < \omega,\ r > 0 \tag{7.72}$$

The m^{th} moment of (7.72) is

$$E(X_i^m) = \omega^m \frac{B(\frac{m}{r}+i,n-i+1)}{B(i,n-i+1)} \tag{7.73}$$

Putting $i = 1$ in (7.72),

$$g_1(x) = \frac{rnx^{r-1}\left\{1 - \left[\frac{x}{\omega}\right]^r\right\}^{n-1}}{\omega^r}, \quad 0 < x < \omega,\ r > 0 \tag{7.74}$$

The m^{th} moment of (7.74) is

$$E(X_1^m) = n\omega^m B\left(\frac{m}{r} + 1, n\right), \quad r > m \tag{7.75}$$

Putting $i = n$ in (7.72),

$$g_n(x) = \frac{rnx^{rn-1}}{\omega^{rn}}, \quad 0 < x < \omega,\ r > 0 \tag{7.76}$$

The m^{th} moment of (7.76) is

$$E(X_n^m) = n\omega^m B\left(\frac{m}{r} + n, 1\right), \quad r > m \tag{7.77}$$

7.7 Gompertz Law

Gompertz (1825) attributed death to two causes: chance or the deterioriation of the power to withstand destruction. In deriving his law of mortality, he considered only deterioration and assumed that man's power to resist death decreases at a rate proportional to the power itself.

Since the force of mortality $\mu(t)$ is a measure of man's susceptibility to death, Gompertz used the reciprocal $\frac{1}{\mu(t)}$ as a measure of man's resistance to death and thus arrived at the formula

$$\frac{d}{dt}\frac{1}{\mu(t)} = -k\frac{1}{\mu(t)}$$

where k is a positive constant.

Hence

$$\begin{aligned}
\mu(t)\frac{d}{dt}\frac{1}{\mu(t)} &= -k \\
\frac{1}{\frac{1}{\mu(t)}}\frac{d}{dt}\frac{1}{\mu(t)} &= -k \\
\frac{d}{dt}\log\frac{1}{\mu(t)} &= -k \\
\int d\log\frac{1}{\mu(t)} &= -k\int dt
\end{aligned}$$

$$\begin{aligned}
\log\frac{1}{\mu(t)} &= -kt + d \\
\frac{1}{\mu(t)} &= e^{-kt+d} \\
\mu(t) &= e^{kt-d} \\
&= e^{kt}e^{-d}
\end{aligned}$$

Let

$$e^{-d} = B, \quad B > 0$$

and

$$e^{k} = c, \quad c > 0$$

Then

$$\mu(t) = Bc^{t}$$

is the **Gompertz Law of Mortality**.

Thus

$$
\begin{aligned}
\int_0^x \mu(t)dt &= \int_0^x Bc^t dt \\
&= \int_0^x e^{\log Bc^t} dt \\
&= \int_0^x e^{\log B + \log c^t} dt \\
&= e^{\log B} \int_0^x e^{t \log c} dt \\
&= \frac{B}{\log c} \left[e^{t \log c} \right]_0^x \\
&= \frac{B}{\log c} \left[e^{x \log c} - e^0 \right] \\
&= \frac{B}{\log c} \left[e^{\log c^x} - e^0 \right] \\
&= \frac{B}{\log c} \left[c^x - 1 \right]
\end{aligned}
$$

Substituting in (7.1) we get

$$
\begin{aligned}
F(x) &= 1 - \exp\left\{ -\frac{B}{\log c} [c^x - 1] \right\} \\
&= 1 - \frac{1}{\exp\left\{ \frac{B}{\log c} [c^x - 1] \right\}}
\end{aligned}
$$

Now,

$$
F(x) = 0 \Longrightarrow 1 - \frac{1}{\exp\{0\}} = 1 - \frac{1}{\exp\left\{ \frac{B}{\log c} [c^0 - 1] \right\}} = 0 \Longrightarrow x = 0
$$

$$
F(x) = 1 \Longrightarrow 1 - \frac{1}{\exp\{\infty\}} = 1 - \frac{1}{\exp\left\{ \frac{B}{\log c} [c^\infty - 1] \right\}} = 1 \Longrightarrow x = \infty
$$

Thus

$$F(x) = 1 - \exp\left\{-\frac{B}{\log c}\left[c^x - 1\right]\right\}, \quad 0 < x < \infty$$

Replacing $e^{-d} = B$ and $e^k = c$

$$F(x) = 1 - \exp\left\{-\frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}, \quad 0 < x < \infty \tag{7.78}$$

and its corresponding pdf is

$$f(x) = e^{-d}e^{kx}\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}, \quad 0 < x < \infty \tag{7.79}$$

which is the **Gompertz Distribution** with shape parameter $\frac{e^{-d}}{k}$ and scale parameter k.

7.7.1 Beta Gompertz Distribution

Putting (7.78) in (2.3) gives

$$G(x) = \int_0^{1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0 < x < \infty,\ a > 0,\ b > 0 \tag{7.80}$$

and its corresponding pdf is

$$\begin{aligned}
g(x) &= \frac{\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{a-1}\left[1-\left(1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)\right]^{b-1}}{B(a,b)}e^{-d}e^{kx}\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\} \\
&= \frac{\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{a-1}\left[\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{b-1}}{B(a,b)}e^{-d}e^{kx}\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\} \\
&= \frac{\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{a-1}\left[\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{b-1+1}}{B(a,b)}e^{-d}e^{kx} \\
&= \frac{\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{a-1}\left[\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{b}}{B(a,b)}e^{-d}e^{kx} \\
&= \frac{\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{a-1}\exp\left\{-\frac{be^{-d}}{k}\left[e^{kx}-1\right]\right\}}{B(a,b)}e^{-d}e^{kx}, \quad 0 < x < \infty,\ a > 0,\ b > 0
\end{aligned} \tag{7.81}$$

i^{th} order statistic of Gompertz Distribution

From (7.81),

$$g_i(x) = \frac{\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{i-1}\exp\left\{-\frac{(n-i+1)e^{-d}}{k}\left[e^{kx}-1\right]\right\}}{B(i,n-i+1)}e^{-d}e^{kx}, \quad 0<x<\infty \tag{7.82}$$

Putting $i=1$ in (7.82),

$$g_1(x) = ne^{-d}e^{kx}\exp\left\{-\frac{ne^{-d}}{k}\left[e^{kx}-1\right]\right\}, \quad 0<x<\infty \tag{7.83}$$

Putting $i=n$ in (7.82),

$$g_n(x) = ne^{-d}e^{kx}\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{n-1}\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}, \quad 0<x<\infty \tag{7.84}$$

7.7.2 Exponentiated Gompertz Distribution

Putting (7.78) in (2.8) gives

$$G(x) = \left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^r, \quad 0<x<\infty,\ r>0 \tag{7.85}$$

and its corresponding pdf is

$$g(x) = re^{-d}e^{kx}\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{r-1}, \quad 0<x<\infty,\ r>0 \tag{7.86}$$

7.7.3 Beta Exponentiated Gompertz Distribution

Putting (7.85) in (2.10) gives

$$G(x)=\int_{0}^{\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{r}}\frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt,\quad 0<x<\infty,\ a>0,\ b>0,\ r>0 \quad \textbf{(7.87)}$$

and its corresponding pdf is

$$g(x)=\frac{r\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{ra-1}\left[1-\left(1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^{r}\right]^{b-1}}{B(a,b)}e^{-d}e^{kx}\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\} \quad \textbf{(7.88)}$$

i^{th} order statistic of Exponentiated Gompertz Distribution

From (7.88),

$$g_i(x)=\frac{r\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{ri-1}\left[1-\left(1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^{r}\right]^{n-i}}{B(i,n-i+1)}e^{-d}e^{kx}\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\} \quad \textbf{(7.89)}$$

Putting $i=1$ in (7.89),

$$g_1(x)=rn\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{r-1}\left[1-\left(1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^{r}\right]^{n-1}e^{-d}e^{kx}\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\} \quad \textbf{(7.90)}$$

Putting $i=n$ in (7.89),

$$g_n(x)=rn\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{rn-1}e^{-d}e^{kx}\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\},\ 0<x<\infty,\ r>0 \quad \textbf{(7.91)}$$

7.8 Makeham's Law

Makeham (1860) suggested the modification

$$\mu(t) = A + Bc^t, \quad c > 0,\ A > 0,\ B > 0$$

which is a restoration of the missing component "chance", to the Gompertz formula.

Thus

$$\begin{aligned}
\int_0^x \mu(t)dt &= \int_0^x (A + Bc^t)dt \\
&= \int_0^x Adt + \int_0^x Bc^t dt \\
&= [At]_0^x + \int_0^x e^{\log Bc^t} dt \\
&= Ax + \int_0^x e^{\log B + \log c^t} dt \\
&= Ax + e^{\log B} \int_0^x e^{t \log c} dt \\
&= Ax + \frac{B}{\log c} \left[e^{t \log c} \right]_0^x \\
&= Ax + \frac{B}{\log c} \left[e^{x \log c} - e^0 \right] \\
&= Ax + \frac{B}{\log c} \left[e^{\log c^x} - 1 \right] \\
&= Ax + \frac{B}{\log c} [c^x - 1]
\end{aligned}$$

Substituting in (7.1) we get

$$\begin{aligned}
F(x) &= 1 - \exp\left\{ - \left(Ax + \frac{B}{\log c} [c^x - 1] \right) \right\} \\
&= 1 - \exp\left\{ -Ax - \frac{B}{\log c} [c^x - 1] \right\} \\
&= 1 - \frac{1}{\exp\left\{ Ax + \frac{B}{\log c} [c^x - 1] \right\}}
\end{aligned}$$

Now,

$$F(x)=0 \Longrightarrow 1-\frac{1}{\exp\{0\}}=1-\frac{1}{\exp\left\{A(0)+\frac{B}{\log c}\left[c^{0}-1\right]\right\}}=0 \Longrightarrow x=0$$

$$F(x)=1 \Longrightarrow 1-\frac{1}{\exp\{\infty\}}=1-\frac{1}{\exp\left\{A(\infty)+\frac{B}{\log c}\left[c^{\infty}-1\right]\right\}}=1 \Longrightarrow x=\infty$$

Thus

$$F(x)=1-\exp\left\{-Ax-\frac{B}{\log c}\left[c^{x}-1\right]\right\}, \quad 0<x<\infty$$

Replacing $e^{-d}=B$ and $e^{k}=c$

$$F(x)=1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}, \quad 0<x<\infty \qquad \textbf{(7.92)}$$

and its corresponding pdf is

$$f(x)=\left\{A+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}, \quad 0<x<\infty \qquad \textbf{(7.93)}$$

which is the **Gompertz-Makeham Distribution**.

7.8.1 Extension of Gompertz-Makeham Distribution

Elandt Johnson and Johnson (1980) gave an exercise to obtain $F(x)$ if A is exponentialy distributed. That is

$$F(x)=\int_{0}^{\infty} F(x|A)g(A)dA$$

where

$$F(x|A)=1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}, \quad 0<x<\infty$$

and

$$g(A) = \lambda e^{-\lambda A}, \quad \lambda > 0$$

Therefore

$$\begin{aligned}
F(x) &= \int_0^\infty \left(1 - \exp\left\{-Ax - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right) \lambda e^{-\lambda A} dA \\
&= \int_0^\infty \lambda e^{-\lambda A} dA - \int_0^\infty \left(\exp\left\{-Ax - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right) \lambda e^{-\lambda A} dA \\
&= -\frac{\lambda}{\lambda}\left[e^{-\lambda A}\right]_0^\infty - \lambda \int_0^\infty \left(\exp\left\{-Ax - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right) e^{-\lambda A} dA \\
&= -\left[e^{-\lambda A}\right]_0^\infty - \left(\lambda \exp\left\{-\frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\} \int_0^\infty e^{-(x+\lambda)A} dA\right) \qquad (7.94) \\
&= -[0-1] - \left(\lambda \exp\left\{-\frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\} \frac{\left[e^{-(x+\lambda)A}\right]_0^\infty}{-(x+\lambda)}\right) \\
&= 1 + \left(\frac{\lambda}{x+\lambda} \exp\left\{-\frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}[0-1]\right) \\
&= 1 - \left(\frac{\lambda}{x+\lambda} \exp\left\{-\frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right)
\end{aligned}$$

and its corresponding pdf is

$$f(x) = \lambda \exp\left\{-\frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\left[\frac{e^{-d}e^{kx}}{x+\lambda} + \frac{1}{(x+\lambda)^2}\right], \quad 0 < x < \infty \qquad (7.95)$$

7.8.2 Beta Gompertz-Makeham Distribution

Putting (7.92) in (2.3) gives

$$G(x) = \int_0^{1-\exp\left\{-Ax - \frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < \infty,\ a > 0,\ b > 0 \qquad (7.96)$$

and its corresponding pdf is

$$
\begin{aligned}
g(x) &= \frac{\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{a-1}\left[1-\left(1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)\right]^{b-1}}{B(a,b)}\left\{A+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\} \\
&= \frac{\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{a-1}\left[\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{b-1}}{B(a,b)}\left\{A+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\} \\
&= \frac{\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{a-1}\left[\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{b-1+1}}{B(a,b)}\left\{A+e^{-d}e^{kx}\right\} \\
&= \frac{\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{a-1}\left[\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{b}}{B(a,b)}\left\{A+e^{-d}e^{kx}\right\} \\
&= \frac{\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{a-1}\exp\left\{-bAx-\frac{be^{-d}}{k}\left[e^{kx}-1\right]\right\}}{B(a,b)}\left\{A+e^{-d}e^{kx}\right\}, \quad 0<x<\infty,\ a>0,\ b>0
\end{aligned}
\tag{7.97}
$$

i^{th} order statistic of Gompertz-Makeham Distribution

From (7.97),

$$
g_i(x) = \frac{\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{i-1}\exp\left\{-(n-i+1)Ax-\frac{(n-i+1)e^{-d}}{k}\left[e^{kx}-1\right]\right\}}{B(i,n-i+1)}\left\{A+e^{-d}e^{kx}\right\}, \quad 0<x<\infty \tag{7.98}
$$

Putting $i=1$ in (7.98),

$$
g_1(x) = n\left\{A+e^{-d}e^{kx}\right\}\exp\left\{-nAx-\frac{ne^{-d}}{k}\left[e^{kx}-1\right]\right\}, \quad 0<x<\infty \tag{7.99}
$$

Putting $i=n$ in (7.98),

$$
g_n(x) = n\left\{A+e^{-d}e^{kx}\right\}\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{n-1}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}, \quad 0<x<\infty \tag{7.100}
$$

7.8.3 Exponentiated Gompertz-Makeham Distribution

Putting (7.92) in (2.8) gives

$$
G(x) = \left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{r}, \quad 0<x<\infty,\ r>0 \tag{7.101}
$$

and its corresponding pdf is

$$g(x)=r\left\{A+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{r-1},\quad 0<x<\infty,\ r>0 \tag{7.102}$$

7.8.4 Beta Exponentiated Gompertz-Makeham Distribution

Putting (7.101) in (2.10) gives

$$G(x)=\int\limits_{0}^{\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{r}}\frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt,\quad 0<x<\infty,\ a>0,\ b>0,\ r>0 \tag{7.103}$$

and its corresponding pdf is

$$\begin{aligned}g(x)=&\frac{r\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{ra-1}\left[1-\left(1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^{r}\right]^{b-1}}{B(a,b)}\\&\times\left\{A+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\},\quad 0<x<\infty,\ a>0,\ b>0,\ r>0\end{aligned} \tag{7.104}$$

i^{th} order statistic of Exponentiated Gompertz-Makeham Distribution

From (7.104),

$$\begin{aligned}g_i(x)=&\frac{r\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{ri-1}\left[1-\left(1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^{r}\right]^{n-i}}{B(i,n-i+1)}\\&\times\left\{A+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\},\quad 0<x<\infty,\ r>0\end{aligned} \tag{7.105}$$

Putting $i=1$ in (7.105),

$$\begin{aligned}g_i(x)=&rn\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{r-1}\left[1-\left(1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^{r}\right]^{n-1}\\&\times\left\{A+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\},\quad 0<x<\infty,\ r>0\end{aligned} \tag{7.106}$$

Putting $i = n$ in (7.105),

$$g_n(x) = rn\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{rn-1}\left\{A+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\} \quad , \quad 0<x<\infty,\ r>0 \tag{7.107}$$

7.9 Makeham's Second Law

Makeham (1890) suggested

$$\mu(t) = A + Ht + Bc^t, \quad c>0,\ A>0,\ H>0,\ B>0$$

Thus

$$\begin{aligned}
\int_0^x \mu(t)dt &= \int_0^x \left(A+Ht+Bc^t\right)dt \\
&= \int_0^x Adt + \int_0^x Htdt + \int_0^x Bc^t dt \\
&= [At]_0^x + \left[\frac{Ht^2}{2}\right]_0^x + \int_0^x e^{\log Bc^t} dt \\
&= Ax + \frac{Hx^2}{2} + \int_0^x e^{\log B + \log c^t} dt \\
&= Ax + \frac{Hx^2}{2} + e^{\log B}\int_0^x e^{t\log c} dt \\
&= Ax + \frac{Hx^2}{2} + \frac{B}{\log c}\left[e^{t\log c}\right]_0^x \\
&= Ax + \frac{Hx^2}{2} + \frac{B}{\log c}\left[e^{x\log c} - e^0\right] \\
&= Ax + \frac{Hx^2}{2} + \frac{B}{\log c}\left[e^{\log c^x} - e^0\right] \\
&= Ax + \frac{Hx^2}{2} + \frac{B}{\log c}\left[c^x - 1\right]
\end{aligned}$$

Substituting in (7.1) we get

$$F(x) = 1 - \exp\left\{-\left(Ax + \frac{Hx^2}{2} + \frac{B}{\log c}\left[c^x - 1\right]\right)\right\}$$
$$= 1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{B}{\log c}\left[c^x - 1\right]\right\}$$
$$= 1 - \frac{1}{\exp\left\{Ax + \frac{Hx^2}{2} + \frac{B}{\log c}\left[c^x - 1\right]\right\}}$$

Now,

$$F(x) = 0 \Longrightarrow 1 - \frac{1}{\exp\{0\}} = 1 - \frac{1}{\exp\left\{A(0) + \frac{H(0)^2}{2} + \frac{B}{\log c}\left[c^0 - 1\right]\right\}} = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow 1 - \frac{1}{\exp\{\infty\}} = 1 - \frac{1}{\exp\left\{A(\infty) + \frac{H(\infty)^2}{2} + \frac{B}{\log c}\left[c^\infty - 1\right]\right\}} = 1 \Longrightarrow x = \infty$$

Thus

$$F(x) = 1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{B}{\log c}\left[c^x - 1\right]\right\}$$

Replacing $e^{-d} = B$ and $e^k = c$

$$F(x) = 1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}, \quad 0 < x < \infty \tag{7.108}$$

and its corresponding pdf is

$$f(x) = \left\{A + Hx + e^{-d}e^{kx}\right\}\exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}, \quad 0 < x < \infty \tag{7.109}$$

7.9.1 Extension of Makeham's Second Law

Van der Maén (1943) suggested a modification to the Makeham's Second Law. He defined the force of mortality as

$$\mu(t) = A + Bc^t + \frac{D}{N-t}, \quad c > 0,\ A > 0,\ B > 0,\ D > 0,\ N > 0$$

Thus

$$\begin{aligned}
\int_0^x \mu(t)dt &= \int_0^x \left(A + Bc^t + \frac{D}{N-t}\right)dt \\
&= \int_0^x Adt + \int_0^x Bc^t dt + \int_0^x \frac{D}{N-t}dt \\
&= [At]_0^x + \int_0^x e^{\log Bc^t} dt + D\int_0^x \frac{1}{N-t}dt \\
&= Ax + \int_0^x e^{\log B + \log c^t} dt - D\left[\log(N-t)\right]_0^x \\
&= Ax + e^{\log B}\int_0^x e^{t\log c} dt - D\left[\log(N-x) - \log N\right] \\
&= Ax + \frac{B}{\log c}\left[e^{t\log c}\right]_0^x + D\left[\log N - \log(N-x)\right] \\
&= Ax + \frac{B}{\log c}\left[e^{x\log c} - e^0\right] + D\log\left[\frac{N}{N-x}\right] \\
&= Ax + \frac{B}{\log c}\left[e^{\log c^x} - 1\right] + D\log\left[\frac{N}{N-x}\right] \\
&= Ax + \frac{B}{\log c}\left[c^x - 1\right] + D\log\left[\frac{N}{N-x}\right]
\end{aligned}$$

Substituting in (7.1) we get

$$\begin{aligned}
F(x) &= 1 - \exp\left\{-\left(Ax + \frac{B}{\log c}\left[c^x - 1\right] + D\log\left[\frac{N}{N-x}\right]\right)\right\} \\
&= 1 - \exp\left\{-Ax - \frac{B}{\log c}\left[c^x - 1\right] - D\log\left[\frac{N}{N-x}\right]\right\} \\
&= 1 - \left[\frac{N}{N-x}\right]^{-D}\exp\left\{-Ax - \frac{B}{\log c}\left[c^x - 1\right]\right\}
\end{aligned}$$

Replacing $e^{-d} = B$ and $e^k = c$

$$F(x) = 1 - \left[\frac{N}{N-x}\right]^{-D} \exp\left\{-Ax - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\} \tag{7.110}$$

and its corresponding pdf is

$$f(x) = \left\{\frac{D}{N}\left[\frac{N}{N-x}\right]^{-D+1} + \left[\frac{N}{N-x}\right]^{-D}\left[A + e^{-d}e^{kx}\right]\right\} \exp\left\{-Ax - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\} \tag{7.111}$$

7.9.2 Beta Generated Distribution

Putting (7.108) in (2.3) gives

$$G(x) = \int_0^{1-\exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < \infty,\ a > 0,\ b > 0 \tag{7.112}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{\left[1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right]^{a-1}\left[\exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right]^{b-1+1}}{B(a,b)}\left\{A + Hx + e^{-d}e^{kx}\right\} \\ &= \frac{\left[1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right]^{a-1}\left[\exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right]^{b}}{B(a,b)}\left\{A + Hx + e^{-d}e^{kx}\right\} \\ &= \frac{\left[1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right]^{a-1}\exp\left\{-bAx - \frac{bHx^2}{2} - \frac{be^{-d}}{k}\left[e^{kx} - 1\right]\right\}}{B(a,b)}\left\{A + Hx + e^{-d}e^{kx}\right\}, \ 0 < x < \infty,\ a > 0,\ b > 0 \end{aligned} \tag{7.113}$$

i^{th} order statistic of $F(x) = 1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}$

From (7.113),

$$g_i(x) = \frac{\left[1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right]^{i-1}\exp\left\{-(n-i+1)Ax - \frac{(n-i+1)Hx^2}{2} - \frac{(n-i+1)e^{-d}}{k}\left[e^{kx} - 1\right]\right\}}{B(i,n-i+1)}\left\{A + Hx + e^{-d}e^{kx}\right\} \tag{7.114}$$

Putting $i = 1$ in (7.114),

$$g_1(x) = n\left\{A + Hx + e^{-d}e^{kx}\right\}\exp\left\{-nAx - \frac{nHx^2}{2} - \frac{ne^{-d}}{k}\left[e^{kx} - 1\right]\right\}, \quad 0 < x < \infty \tag{7.115}$$

Putting $i = n$ in (7.114),

$$g_n(x) = n\left\{A + Hx + e^{-d}e^{kx}\right\}\left[1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right]^{n-1}\exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}, \quad 0 < x < \infty \tag{7.116}$$

7.9.3 Exponentiated Generated Distribution

Putting (7.108) in (2.8) gives

$$G(x) = \left[1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right]^r, \quad 0 < x < \infty,\ r > 0 \tag{7.117}$$

and its corresponding pdf is

$$g(x) = r\left\{A + Hx + e^{-d}e^{kx}\right\}\exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\left[1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right]^{r-1}, \quad 0 < x < \infty,\ r > 0 \tag{7.118}$$

7.9.4 Beta Exponentiated Generated Distribution

Putting (7.117) in (2.10) gives

$$G(x) = \int_0^{\left[1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0 < x < \infty,\ a > 0,\ b > 0,\ r > 0 \tag{7.119}$$

and its corresponding pdf is

$$\begin{aligned} g(x) = {} & \frac{r\left[1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right]^{ra-1}\left[1 - \left(1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}\right)^r\right]^{b-1}}{B(a,b)} \\ & \times\left\{A + Hx + e^{-d}e^{kx}\right\}\exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx} - 1\right]\right\}, \quad 0 < x < \infty,\ a > 0,\ b > 0,\ r > 0 \end{aligned} \tag{7.120}$$

i^{th} order statistic of $G(x) = \left[1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^r$

From (7.120),

$$g_i(x) = \frac{r\left[1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{ri-1}\left[1-\left(1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^r\right]^{n-i}}{B(i,n-i+1)}$$
$$\times\left\{A+Hx+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\},\quad 0<x<\infty,\ a>0,\ b>0,\ r>0 \tag{7.121}$$

Putting $i=1$ in (7.121),

$$g_i(x) = rn\left[1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{r-1}\left[1-\left(1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^r\right]^{n-1}$$
$$\times\left\{A+Hx+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\},\quad 0<x<\infty,\ r>0 \tag{7.122}$$

Putting $i=n$ in (7.121),

$$g_n(x) = rn\left[1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{rn-1}\left\{A+Hx+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$$
$$,\quad 0<x<\infty,\ r>0 \tag{7.123}$$

7.10 Double Geometric Law

Let

$$\mu(t) = A+Bc^t+Mn^t,\quad c>0,\ n>0,\ A>0,\ B>0,\ M>0$$

Thus

$$\begin{aligned}
\int_0^x \mu(t)dt &= \int_0^x \left(A + Bc^t + Mn^t\right)dt \\
&= \int_0^x Adt + \int_0^x Bc^t dt + + \int_0^x Mn^t dt \\
&= [At]_0^x + \int_0^x e^{\log Bc^t} dt + \int_0^x e^{\log Mn^t} dt \\
&= Ax + \int_0^x e^{\log B + \log c^t} dt + \int_0^x e^{\log M + \log n^t} dt \\
&= Ax + e^{\log B}\int_0^x e^{t\log c} dt + e^{\log M}\int_0^x e^{t\log n} dt \\
&= Ax + \frac{B}{\log c}\left[e^{t\log c}\right]_0^x + \frac{M}{\log n}\left[e^{t\log n}\right]_0^x \\
&= Ax + \frac{B}{\log c}\left[e^{x\log c} - e^0\right] + \frac{M}{\log n}\left[e^{x\log n} - e^0\right] \\
&= Ax + \frac{B}{\log c}\left[e^{\log c^x} - e^0\right] + \frac{M}{\log n}\left[e^{\log n^x} - e^0\right] \\
&= Ax + \frac{B}{\log c}\left[c^x - 1\right] + \frac{M}{\log n}\left[n^x - 1\right]
\end{aligned}$$

Substituting in (7.1) we get

$$\begin{aligned}
F(x) &= 1 - \exp\left\{-\left(Ax + \frac{B}{\log c}\left[c^x - 1\right] + \frac{M}{\log n}\left[n^x - 1\right]\right)\right\} \\
&= 1 - \exp\left\{-Ax - \frac{B}{\log c}\left[c^x - 1\right] - \frac{M}{\log n}\left[n^x - 1\right]\right\} \\
&= 1 - \frac{1}{\exp\left\{Ax + \frac{B}{\log c}\left[c^x - 1\right] + \frac{M}{\log n}\left[n^x - 1\right]\right\}}
\end{aligned}$$

Now,

$$F(x) = 0 \Longrightarrow 1 - \frac{1}{\exp\{0\}} = 1 - \frac{1}{\exp\left\{A(0) + \frac{B}{\log c}\left[c^0 - 1\right] + \frac{M}{\log n}\left[n^0 - 1\right]\right\}} = 0 \Longrightarrow x = 0$$

$$F(x)=1 \Longrightarrow 1-\frac{1}{\exp\{\infty\}}=1-\frac{1}{\exp\left\{A(\infty)+\frac{B}{\log c}\left[c^{\infty}-1\right]+\frac{M}{\log n}\left[n^{\infty}-1\right]\right\}}=1 \Longrightarrow x=\infty$$

Therefore

$$F(x)=1-\exp\left\{-Ax-\frac{B}{\log c}\left[c^{x}-1\right]-\frac{M}{\log n}\left[n^{x}-1\right]\right\}, \quad 0<x<\infty \qquad \textbf{(7.124)}$$

and its corresponding pdf is

$$f(x)=\{A+Bc^{x}+Mn^{x}\}\exp\left\{-Ax-\frac{B}{\log c}\left[c^{x}-1\right]-\frac{M}{\log n}\left[n^{x}-1\right]\right\}, \quad 0<x<\infty \qquad \textbf{(7.125)}$$

7.11 Opperman's Law

Gompertz laws are primarily for fitting with adult ages and not ideal for infant and child mortality.

Opperman (1870) suggested a formula for graduation of infant and child mortality by defining

$$\mu(t)=\frac{a}{\sqrt{t}}+b+ct^{\frac{1}{3}}, \quad a>0,\ b>0,\ c>0$$

Thus

$$\begin{aligned}
\int_0^x \mu(t)dt &= \int_0^x \left(\frac{a}{\sqrt{t}}+b+ct^{\frac{1}{3}}\right)dt \\
&= \int_0^x \left(at^{-\frac{1}{2}}+b+ct^{\frac{1}{3}}\right)dt \\
&= \left[\frac{at^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+bt+\frac{ct^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right]_0^x \\
&= \left[2at^{\frac{1}{2}}+bt+\frac{3}{4}ct^{\frac{4}{3}}\right]_0^x \\
&= 2ax^{\frac{1}{2}}+bx+\frac{3}{4}cx^{\frac{4}{3}} \\
&= 2a\sqrt{x}+bx+\frac{3}{4}c\left(x^{\frac{1}{3}}\right)^{4}
\end{aligned}$$

Substituting in (7.1) we get

$$
\begin{aligned}
F(x) &= 1-\exp\left\{-\left(2a\sqrt{x}+bx+\frac{3}{4}c\left(x^{\frac{1}{3}}\right)^4\right)\right\} \\
&= 1-\exp\left\{-2a\sqrt{x}-bx-\frac{3}{4}c\left(x^{\frac{1}{3}}\right)^4\right\} \\
&= 1-\frac{1}{\exp\left\{2a\sqrt{x}+bx+\frac{3}{4}c\left(x^{\frac{1}{3}}\right)^4\right\}}
\end{aligned}
$$

Now,

$$
F(x)=0 \Longrightarrow 1-\frac{1}{\exp\{0\}}=1-\frac{1}{\exp\left\{2a\sqrt{0}+b(0)+\frac{3}{4}c\left(0^{\frac{1}{3}}\right)^4\right\}}=0 \Longrightarrow x=0
$$

$$
F(x)=1 \Longrightarrow 1-\frac{1}{\exp\{\infty\}}=1-\frac{1}{\exp\left\{2a\sqrt{\infty}+b(\infty)+\frac{3}{4}c\left(\infty^{\frac{1}{3}}\right)^4\right\}}=1 \Longrightarrow x=\infty
$$

Therefore

$$
F(x)=1-\exp\left\{-2a\sqrt{x}-bx-\frac{3}{4}c\left(x^{\frac{1}{3}}\right)^4\right\}, \quad 0<x<\infty \tag{7.126}
$$

and its corresponding pdf is

$$
f(x)=\left\{\frac{a}{\sqrt{x}}+b+3cx\right\}\exp\left\{-2a\sqrt{x}-bx-\frac{3}{4}c\left(x^{\frac{1}{3}}\right)^4\right\}, \quad 0<x<\infty \tag{7.127}
$$

7.12 Thiele's Law

Thiele (1871) was of the opinion that force of mortality should take into account the differences in mortality behaviour during the major epochs of life.

He partitioned the survivorship curve into three components:

- For childhood

$$\mu_1(t) = a_1 e^{-b_1 t}$$

- For adult ages

$$\mu_2(t) = a_2 e^{-\frac{1}{2} b_2 (t-c)^2}$$

- For old ages

$$\mu_3(t) = a_3 e^{b_3 t}$$

Hence the formula meant for graduation of mortality throughout all ages was written as

$$\begin{aligned} \mu(t) &= \mu_1(t) + \mu_2(t) + \mu_3(t) \\ &= a_1 e^{-b_1 t} + a_2 e^{-\frac{1}{2} b_2 (t-c)^2} + a_3 e^{b_3 t} \end{aligned}$$

Let

$$e^{-b_1} = B_1, \quad B_1 > 0$$

$$e^{-b_2} = B_2, \quad B_2 > 0$$

and

$$e^{b_3} = B_3, \quad B_3 > 0$$

Then

$$\mu(t) = a_1 B_1^t + a_2 B_2^{\frac{1}{2}(t-c)^2} + a_3 B_3^t$$

Thus

$$\begin{aligned} \int_0^x \mu(t) dt &= \int_0^x \left(a_1 B_1^t + a_2 B_2^{\frac{1}{2}(t-c)^2} + a_3 B_3^t \right) dt \\ &= a_1 \int_0^x B_1^t dt + a_2 \int_0^x B_2^{\frac{1}{2}(t-c)^2} dt + a_3 \int_0^x B_3^t dt \\ &= a_1 \int_0^x e^{\log B^t{}_1} dt + a_2 \int_0^x e^{\log B_2^{\frac{1}{2}(t-c)^2}} dt + a_3 \int_0^x e^{\log B^t{}_3} dt \\ &= a_1 \int_0^x e^{t \log B_1} dt + a_2 \int_0^x e^{\frac{1}{2}(t-c)^2 \log B_2} dt + a_3 \int_0^x e^{t \log B_3} dt \end{aligned}$$

$$= \frac{a_1}{\log B_1}\left[e^{t\log B_1}\right]_0^x + \frac{a_2}{\log B_2}\left[\frac{e^{\frac{1}{2}(t-c)^2\log B_2}}{t-c}\right]_0^x + \frac{a_3}{\log B_3}\left[e^{t\log B_3}\right]_0^x$$

$$= \frac{a_1}{\log B_1}\left[e^{x\log B_1} - e^0\right] + \frac{a_2}{\log B_2}\left[\frac{e^{\frac{1}{2}(x-c)^2\log B_2}}{x-c} + \frac{e^{\frac{1}{2}c^2\log B_2}}{c}\right]$$

$$+ \frac{a_3}{\log B_3}\left[e^{x\log B_3} - e^0\right]$$

$$= \frac{a_1}{\log B_1}\left[e^{\log B_1{}^x} - e^0\right] + \frac{a_2}{\log B_2}\left[\frac{e^{\log B_2{}^{\frac{1}{2}(x-c)^2}}}{x-c} + \frac{e^{\log B_2{}^{\frac{1}{2}c^2}}}{c}\right] + \frac{a_3}{\log B_3}\left[e^{\log B_3{}^x} - e^0\right]$$

$$= \frac{a_1}{\log B_1}\left[B_1{}^x - 1\right] + \frac{a_2}{\log B_2}\left[\frac{B_2{}^{\frac{1}{2}(x-c)^2}}{x-c} + \frac{B_2{}^{\frac{1}{2}c^2}}{c}\right] + \frac{a_3}{\log B_3}\left[B_3{}^x - 1\right]$$

$$= \frac{a_1}{\log B_1}\left[B_1{}^x - 1\right] + \frac{a_2}{\log B_2}\left[\frac{B_2{}^{\frac{1}{2}c^2} cB_2{}^{\frac{1}{2}(x^2-2xc)} + (x-c)B_2{}^{\frac{1}{2}c^2}}{c(x-c)}\right]$$

$$+ \frac{a_3}{\log B_3}\left[B_3{}^x - 1\right]$$

$$= \frac{a_1}{\log B_1}\left[B_1{}^x - 1\right] + \frac{a_2}{\log B_2}\left[\frac{B_2{}^{\frac{1}{2}c^2}\left\{cB_2{}^{\frac{1}{2}(x^2-2xc)} + (x-c)\right\}}{c(x-c)}\right] + \frac{a_3}{\log B_3}\left[B_3{}^x - 1\right]$$

Substituting in (7.1) we get

$$F(x) = 1 - \exp\left\{-\frac{a_1}{\log B_1}\left[B_1{}^x - 1\right] - \frac{a_2}{\log B_2}\left[\frac{B_2{}^{\frac{1}{2}c^2}\left\{cB_2{}^{\frac{1}{2}(x^2-2xc)} + (x-c)\right\}}{c(x-c)}\right] - \frac{a_3}{\log B_3}\left[B_3{}^x - 1\right]\right\}$$

Now,

$$F(x) = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow x = \infty$$

Thus

$$F(x) = 1 - \exp\left\{-\frac{a_1}{\log B_1}\left[B_1{}^x - 1\right] - \frac{a_2}{\log B_2}\left[\frac{B_2{}^{\frac{1}{2}c^2}\left\{cB_2{}^{\frac{1}{2}(x^2-2xc)} + (x-c)\right\}}{c(x-c)}\right] - \frac{a_3}{\log B_3}\left[B_3{}^x - 1\right]\right\},$$
$$0 < x < \infty$$

Replacing $e^{-b_1} = B_1$, $e^{-b_2} = B_2$ and $e^{b_3} = B_3$

$$F(x) = 1 - \exp\left\{\frac{a_1}{b_1}\left[e^{-b_1x} - 1\right] + \frac{a_2}{b_2}\left[\frac{e^{-\frac{b_2c^2}{2}}\left\{ce^{\frac{-b_2(x^2-2xc)}{2}} + (x-c)\right\}}{c(x-c)}\right] - \frac{a_3}{b_3}\left[e^{b_3x} - 1\right]\right\}, \quad 0 < x < \infty \tag{7.128}$$

and its corresponding pdf is

$$f(x) = \left\{a_1e^{-b_1x} + \frac{a_2}{b_2}\left[\left\{e^{\frac{-b_2(c^2+x^2-2xc)}{2}} + e^{\frac{-b_2c^2}{2}}\right\} + \frac{\left\{ce^{\frac{-b_2c^2}{2}}\left[ce^{\frac{-b_2(x^2-2xc)}{2}} + (x-c)\right]\right\}}{[c(x-c)]^2}\right] + a_3e^{b_3x}\right\}$$
$$\times \exp\left\{\frac{a_1}{b_1}\left[e^{-b_1x} - 1\right] + \frac{a_2}{b_2}\left[\frac{e^{\frac{-b_2c^2}{2}}\left\{ce^{\frac{-b_2(x^2-2xc)}{2}} + (x-c)\right\}}{c(x-c)}\right] - \frac{a_3}{b_3}\left[e^{b_3x} - 1\right]\right\}, \quad 0 < x < \infty \tag{7.129}$$

7.12.1 Extension of Thiele's Law

Siller (1979) introduced a force of mortality for middle age of the form

$$\mu_2(t) = a_2$$

Therefore

$$F(x) = 1 - \exp\left\{\frac{a_1}{b_1}\left[e^{-b_1x} - 1\right] - a_2x - \frac{a_3}{b_3}\left[e^{b_3x} - 1\right]\right\}, \quad 0 < x < \infty \tag{7.130}$$

and its corresponding pdf is

$$f(x) = \left\{a_1e^{-b_1x} + a_2 + a_3e^{b_3x}\right\}\exp\left\{\frac{a_1}{b_1}\left[e^{-b_1x} - 1\right] - a_2x - \frac{a_3}{b_3}\left[e^{b_3x} - 1\right]\right\}, \quad 0 < x < \infty \tag{7.131}$$

7.13 Babbage's Law

Babbage (1823) assumed that $S(t)$ was quadratic (rather than linear as De Moivre had assumed). He defined

$$S(t) = 1 - ct - dt^2, \quad c > 0,\ d > 0$$

Therefore

$$\begin{aligned} \mu(t) &= \frac{-S'(t)}{S(t)} \\ &= \frac{-(-c-2dt)}{1-ct-dt^2} \\ &= \frac{c+2dt}{1-ct-dt^2} \end{aligned}$$

Thus

$$\int_0^x \mu(t)dt = \int_0^x \frac{c+2dt}{1-ct-dt^2}dt$$

Let

$$\begin{aligned} u &= 1 - ct - dt^2 \\ du &= (-c-2dt)dt \end{aligned}$$

Then

$$\begin{aligned} \int_0^x \mu(t)dt &= \int_0^x \frac{-(-c-2dt)}{u(-c-2dt)}du \\ &= -\int_0^x \frac{1}{u}du \\ &= -\left[\log u\right]_0^x \\ &= -\left[\log\left(1-ct-dt^2\right)\right]_0^x \\ &= -\left[\log\left(1-cx-dx^2\right) - \log 1\right] \\ &= -\left[\log\left(1-cx-dx^2\right) - 0\right] \\ &= -\log\left(1-cx-dx^2\right) \end{aligned}$$

Substituting in (7.1) we get

$$\begin{aligned} F(x) &= 1-\exp\left\{\log\left(1-cx-dx^2\right)\right\} \\ &= 1-(1-cx-dx^2) \\ &= cx+dx^2 \end{aligned}$$

Now,

$$F(x)=0 \Longrightarrow cx+dx^2=0$$

$$\begin{aligned} dx^2 &= -cx \\ dx &= -c \\ x &= -\frac{c}{d} \end{aligned}$$

$$F(x)=1 \Longrightarrow dx^2+cx=1$$

$$dx^2+cx-1=0$$

$$x=\frac{-c\pm\sqrt{c^2+4d}}{2d}$$

Therefore

$$F(x)=cx+dx^2, \quad \frac{-c-\sqrt{c^2+4d}}{2d}<x<\frac{-c+\sqrt{c^2+4d}}{2d} \tag{7.132}$$

and its corresponding pdf is

$$f(x)=c+2dx, \quad \frac{-c-\sqrt{c^2+4d}}{2d}<x<\frac{-c+\sqrt{c^2+4d}}{2d} \tag{7.133}$$

Adler (1867) recommended suitable values of c and d.

7.13.1 Beta Generated Distribution

Putting (7.132) in (2.3) gives

$$G(x)=\int_0^{cx+dx^2}\frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad \frac{-c-\sqrt{c^2+4d}}{2d}<x<\frac{-c+\sqrt{c^2+4d}}{2d} \tag{7.134}$$

$$, \; a>0, \; b>0$$

and its corresponding pdf is

$$
\begin{aligned}
g(x) &= \frac{\left[cx+dx^2\right]^{a-1}\left[1-(cx+dx^2)\right]^{b-1}}{B(a,b)}(c+2dx)\\
&= \frac{\left[cx+dx^2\right]^{a-1}\left[1-cx-dx^2\right]^{b-1}}{B(a,b)}(c+2dx)
\end{aligned}
\tag{7.135}
$$

i^{th} order statistic of $F(x) = cx + dx^2$

From (7.135),

$$
g_i(x) = \frac{\left[cx+dx^2\right]^{i-1}\left[1-cx-dx^2\right]^{n-i}}{B(i,n-i+1)}(c+2dx) \tag{7.136}
$$

Putting $i = 1$ in (7.136),

$$
g_1(x) = n\left[1-cx-dx^2\right]^{n-1}(c+2dx), \quad \frac{-c-\sqrt{c^2+4d}}{2d} < x < \frac{-c+\sqrt{c^2+4d}}{2d} \tag{7.137}
$$

Putting $i = n$ in (7.136),

$$
g_n(x) = n\left[cx+dx^2\right]^{n-1}(c+2dx), \quad \frac{-c-\sqrt{c^2+4d}}{2d} < x < \frac{-c+\sqrt{c^2+4d}}{2d} \tag{7.138}
$$

7.13.2 Exponentiated Generated Distribution

Putting (7.132) in (2.8) gives

$$
G(x) = \left[cx+dx^2\right]^r, \quad \frac{-c-\sqrt{c^2+4d}}{2d} < x < \frac{-c+\sqrt{c^2+4d}}{2d},\ r > 0 \tag{7.139}
$$

and its corresponding pdf is

$$g(x) = r(c+2dx)\left[cx+dx^2\right]^{r-1}, \quad \frac{-c-\sqrt{c^2+4d}}{2d} < x < \frac{-c+\sqrt{c^2+4d}}{2d}, \ r > 0 \qquad \textbf{(7.140)}$$

7.13.3 Beta Exponentiated Generated Distribution

Putting (7.139) in (2.10) gives

$$G(x) = \int_0^{\left[cx+dx^2\right]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad \frac{-c-\sqrt{c^2+4d}}{2d} < x < \frac{-c+\sqrt{c^2+4d}}{2d} \qquad \textbf{(7.141)}$$
$$, \ a > 0, \ b > 0, \ r > 0$$

and its corresponding pdf is

$$g(x) = \frac{r\left[cx+dx^2\right]^{ra-1}\left[1-\left\{cx+dx^2\right\}^r\right]^{b-1}}{B(a,b)}(c+2dx) \qquad \textbf{(7.142)}$$

i^{th} order statistic of $G(x) = \left[cx+dx^2\right]^r$

From (7.142),

$$g_i(x) = \frac{r\left[cx+dx^2\right]^{ri-1}\left[1-\left\{cx+dx^2\right\}^r\right]^{n-i}}{B(i,n-i+1)}(c+2dx) \qquad \textbf{(7.143)}$$

Putting $i = 1$ in (7.143),

$$g_1(x) = rn\left[cx+dx^2\right]^{r-1}\left[1-\left\{cx+dx^2\right\}^r\right]^{n-1}(c+2dx) \qquad \textbf{(7.144)}$$

Putting $i = n$ in (7.143),

$$g_n(x) = rn\left[cx+dx^2\right]^{rn-1}(c+2dx), \quad \frac{-c-\sqrt{c^2+4d}}{2d} < x < \frac{-c+\sqrt{c^2+4d}}{2d} \qquad \textbf{(7.145)}$$

7.14 Lambert's Law

Lambert (1776) defined

$$S(t)=\left[\frac{a-t}{t}\right]^2-b\left[e^{-\frac{t}{c}}-e^{-\frac{t}{d}}\right],\quad a>0,\ b>0,\ c>0,\ d>0$$

Therefore

$$\begin{aligned}\mu(t)&=\frac{-S'(t)}{S(t)}\\&=\frac{-\left(-\frac{2a}{t^2}\left[\frac{a-t}{t}\right]-\frac{b}{c}\left[-e^{-\frac{t}{c}}+e^{-\frac{t}{d}}\right]\right)}{\left[\frac{a-t}{t}\right]^2-b\left[e^{-\frac{t}{c}}-e^{-\frac{t}{d}}\right]}\\&=\frac{\frac{2a}{t^2}\left[\frac{a-t}{t}\right]+\frac{b}{c}\left[-e^{-\frac{t}{c}}+e^{-\frac{t}{d}}\right]}{\left[\frac{a-t}{t}\right]^2-b\left[e^{-\frac{t}{c}}-e^{-\frac{t}{d}}\right]}\end{aligned}$$

Thus

$$\int_0^x \mu(t)dt=\int_0^x\left(\frac{\frac{2a}{t^2}\left[\frac{a-t}{t}\right]+\frac{b}{c}\left[-e^{-\frac{t}{c}}+e^{-\frac{t}{d}}\right]}{\left[\frac{a-t}{t}\right]^2-b\left[e^{-\frac{t}{c}}-e^{-\frac{t}{d}}\right]}\right)dt$$

Let

$$\begin{aligned}u&=\left[\frac{a-t}{t}\right]^2-b\left[e^{-\frac{t}{c}}-e^{-\frac{t}{d}}\right]\\du&=\left(-\frac{2a}{t^2}\left[\frac{a-t}{t}\right]-\frac{b}{c}\left[-e^{-\frac{t}{c}}+e^{-\frac{t}{d}}\right]\right)dt\end{aligned}$$

Then

$$\begin{aligned}\int_0^x \mu(t)dt&=\int_0^x\left(\frac{-\left\{-\frac{2a}{t^2}\left[\frac{a-t}{t}\right]-\frac{b}{c}\left[-e^{-\frac{t}{c}}+e^{-\frac{t}{d}}\right]\right\}}{u\left\{-\frac{2a}{t^2}\left[\frac{a-t}{t}\right]-\frac{b}{c}\left[-e^{-\frac{t}{c}}+e^{-\frac{t}{d}}\right]\right\}}\right)du\\&=-\int_0^x\frac{1}{u}du\\&=-\left[\log u\right]_0^x\\&=-\left[\log\left(\left[\frac{a-t}{t}\right]^2-b\left[e^{-\frac{t}{c}}-e^{-\frac{t}{d}}\right]\right)\right]_0^x\end{aligned}$$

$$= -\left[\log\left(\left[\frac{a-x}{x}\right]^2 - b\left[e^{-\frac{x}{c}} - e^{-\frac{x}{d}}\right]\right) - \log 1\right]$$
$$= -\left[\log\left(\left[\frac{a-x}{x}\right]^2 - b\left[e^{-\frac{x}{c}} - e^{-\frac{x}{d}}\right]\right) - 0\right]$$
$$= -\log\left(\left[\frac{a-x}{x}\right]^2 - b\left[e^{-\frac{x}{c}} - e^{-\frac{x}{d}}\right]\right)$$

Substituting in (7.1) we get

$$F(x) = 1 - \exp\left\{\log\left(\left[\frac{a-x}{x}\right]^2 - b\left[e^{-\frac{x}{c}} - e^{-\frac{x}{d}}\right]\right)\right\}$$

Therefore

$$F(x) = 1 - \left(\left[\frac{a-x}{x}\right]^2 - b\left[e^{-\frac{x}{c}} - e^{-\frac{x}{d}}\right]\right) \tag{7.146}$$

and its corresponding pdf is

$$f(x) = \frac{2a}{x^2}\left[\frac{a-x}{x}\right] - \frac{b}{c}e^{-\frac{x}{c}} + \frac{b}{d}e^{-\frac{x}{d}} \tag{7.147}$$

Adler (1867) recommended suitable values of a, b, c and d.

7.15 Perk's Law

Perks (1932) proposed

$$\mu(t) = \frac{A + Bc^t}{1 + Dc^t}, \quad A > 0,\ B > 0,\ c > 0,\ D > 0$$

which is known as the **logistic curve**.

Let

$$\log c = A - \frac{B}{D}$$

Then

$$\mu(t) = \frac{Bc^t + A}{Dc^t + 1} = \frac{\frac{B}{D}c^t + \frac{A}{D}}{c^t + \frac{1}{D}} = \frac{Bc^t + A}{D\left(c^t + \frac{1}{D}\right)}$$

Using partial fraction method

$$\frac{Y}{D} + \frac{Z}{c^t + \frac{1}{D}} = \frac{Y(c^t + \frac{1}{D}) + ZD}{D\left(c^t + \frac{1}{D}\right)}$$

Solving for Y and Z

$$Bc^t + A \equiv Yc^t + \frac{Y}{D} + ZD$$

$$Yc^t = Bc^t \Longrightarrow Y = B$$

$$\frac{Y}{D} + ZD = A \Longrightarrow Z = \frac{A - \frac{Y}{D}}{D} = \frac{A}{D} - \frac{B}{D^2}$$

Hence

$$\mu(t) = \frac{B}{D} + \frac{\frac{A}{D} - \frac{B}{D^2}}{c^t + \frac{1}{D}} = \frac{B}{D} + \frac{\frac{AD-B}{D^2}}{c^t + \frac{1}{D}} = \frac{B}{D} + \frac{AD - B}{D^2\left(c^t + \frac{1}{D}\right)} = \frac{B}{D} + \frac{D\left(A - \frac{B}{D}\right)}{D^2\left(c^t + \frac{1}{D}\right)} = \frac{B}{D} + \frac{A - \frac{B}{D}}{Dc^t + 1} = \frac{B}{D} + \frac{\log c}{1 + Dc^t}$$

Thus

$$\begin{aligned}
\int_0^x \mu(t)dt &= \int_0^x \left[\frac{B}{D}+\frac{\log c}{1+Dc^t}\right]dt \\
&= \int_0^x \frac{B}{D}dt + \int_0^x \frac{\log c}{1+Dc^t}dt \\
&= \left[\frac{B}{D}t\right]_0^x + \log c\left[\frac{\log(1+c^tD)}{Dc^t\log c}\right]_0^x \\
&= \frac{B}{D}x + \frac{1}{D}\left[\frac{\log(1+c^xD)}{c^x} - \frac{\log(1+D)}{1}\right] \\
&= \frac{B}{D}x + \frac{1}{D}\left\{\log\left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]\right\} \\
&= \frac{B}{D}x + \log\left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]^{\frac{1}{D}}
\end{aligned}$$

Substituting in (7.1) we get

$$\begin{aligned}
F(x) &= 1-\exp\left\{-\frac{B}{D}x - \log\left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]^{\frac{1}{D}}\right\} \\
&= 1-\exp\left\{-\frac{B}{D}x + \log\left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]^{-\frac{1}{D}}\right\} \qquad (7.148) \\
&= 1-\left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]^{-\frac{1}{D}} \exp\left\{-\frac{B}{D}x\right\}
\end{aligned}$$

and its corresponding pdf is

$$f(x) = \left\{\frac{B}{D}+\frac{c^x\log C}{D(1+c^xD)}\right\}\left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]^{-\frac{1}{D}} \exp\left\{-\frac{B}{D}x\right\} \qquad (7.149)$$

7.15.1 Extension of Perk's Law

Martinelle (1987) suggested a modification to the Perk's law. He defined the force of mortality as

$$\mu(t) = \frac{A+Bc^t}{1+Dc^t} + Ec^t, \quad A>0,\ B>0,\ c>0,\ D>0,\ E>0$$

Thus

$$\begin{aligned}
\int_0^x Ec^t dt &= \int_0^x e^{\log Ec^t} dt \\
&= \int_0^x e^{\log E + \log c^t} dt \\
&= e^{\log E} \int_0^x e^{t \log c} dt \\
&= \frac{E}{\log c} \left[e^{t \log c} \right]_0^x \\
&= \frac{E}{\log c} \left[e^{x \log c} - e^0 \right] \\
&= \frac{E}{\log c} \left[e^{\log c^x} - 1 \right] \\
&= \frac{E}{\log c} \left[c^x - 1 \right]
\end{aligned}$$

Therefore

$$\int_0^x \mu(t) dt = \frac{B}{D} x + \log \left[\frac{(1 + c^x D)^{\frac{1}{c^x}}}{1 + D} \right]^{\frac{1}{D}} + \frac{E}{\log c} \left[c^x - 1 \right]$$

Substituting in (7.1) we get

$$\begin{aligned}
F(x) &= 1 - \exp \left\{ -\frac{B}{D} x - \log \left[\frac{(1 + c^x D)^{\frac{1}{c^x}}}{1 + D} \right]^{\frac{1}{D}} - \frac{E}{\log c} \left[c^x - 1 \right] \right\} \\
&= 1 - \exp \left\{ -\frac{B}{D} x + \log \left[\frac{(1 + c^x D)^{\frac{1}{c^x}}}{1 + D} \right]^{-\frac{1}{D}} - \frac{E}{\log c} \left[c^x - 1 \right] \right\} \qquad (7.150) \\
&= 1 - \left[\frac{(1 + c^x D)^{\frac{1}{c^x}}}{1 + D} \right]^{-\frac{1}{D}} \exp \left\{ -\frac{B}{D} x - \frac{E}{\log c} \left[c^x - 1 \right] \right\}
\end{aligned}$$

and its corresponding pdf is

$$f(x) = \left\{ \frac{B}{D} + Ec^x + \frac{c^x \log c}{D(1 + c^x D)} \right\} \left[\frac{(1 + c^x D)^{\frac{1}{c^x}}}{1 + D} \right]^{-\frac{1}{D}} \exp \left\{ -\frac{B}{D} x - \frac{E}{\log c} \left[c^x - 1 \right] \right\} \qquad (7.151)$$

7.16 Landahl's Law

According to the Landahl model

$$\mu(t) = \frac{p}{1+kt}, \quad p > 0,\ k > 0$$

where p and k are parameters representing the combined effects of all risks which may result in death.

Thus

$$\begin{aligned}
\int_0^x \mu(t)dt &= \int_0^x p(1+kt)^{-1}dt \\
&= \frac{p}{k}\left[\log(1+kt)\right]_0^x \\
&= \frac{p}{k}\left[\log(1+kx) - \ln 1\right] \\
&= \frac{p}{k}\left[\log(1+kx) - 0\right] \\
&= \frac{p}{k}\log(1+kx)
\end{aligned}$$

Substituting in (7.1) we get

$$\begin{aligned}
F(x) &= 1 - \exp\left\{-\frac{p}{k}\log(1+kx)\right\} \\
&= 1 - \exp\left\{\log(1+kx)^{-\frac{p}{k}}\right\} \\
&= 1 - (1+kx)^{-\frac{p}{k}} \\
&= 1 - \frac{1}{(1+kx)^{\frac{p}{k}}}
\end{aligned}$$

Now,

$$F(x) = 0 \Longrightarrow 1 - \frac{1}{1} = 1 - \frac{1}{(1+k\{0\})^{\frac{p}{k}}} = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow 1 - \frac{1}{\infty} = 1 - \frac{1}{(1+k\{\infty\})^{\frac{p}{k}}} = 1 \Longrightarrow x = \infty$$

Therefore

$$F(x) = 1-(1+kx)^{-\frac{p}{k}}, \quad 0<x<\infty \tag{7.152}$$

and its corresponding pdf is

$$\begin{aligned} f(x) &= p(1+kx)^{-\frac{p}{k}-1} \\ &= \frac{p}{(1+kx)^{\frac{p}{k}+1}}, \quad 0<x<\infty \end{aligned} \tag{7.153}$$

7.16.1 Beta Generated Distribution

Putting (7.152) in (2.3) gives

$$G(x) = \int_0^{1-(1+kx)^{-\frac{p}{k}}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0<x<\infty,\ a>0,\ b>0 \tag{7.154}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{\left[1-(1+kx)^{-\frac{p}{k}}\right]^{a-1}\left[1-\left\{1-(1+kx)^{-\frac{p}{k}}\right\}\right]^{b-1}}{B(a,b)}p(1+kx)^{-\frac{p}{k}-1} \\ &= \frac{\left[1-(1+kx)^{-\frac{p}{k}}\right]^{a-1}\left[(1+kx)^{-\frac{p}{k}}\right]^{b-1}}{B(a,b)}p(1+kx)^{-\frac{p}{k}-1} \\ &= \frac{\left[1-(1+kx)^{-\frac{p}{k}}\right]^{a-1}(1+kx)^{-\frac{pb}{k}+\frac{p}{k}-\frac{p}{k}-1}}{B(a,b)}p \\ &= \frac{\left[1-(1+kx)^{-\frac{p}{k}}\right]^{a-1}(1+kx)^{-\frac{pb}{k}-1}}{B(a,b)}p, \quad 0<x<\infty,\ a>0,\ b>0 \end{aligned} \tag{7.155}$$

i^{th} order statistic of $F(x) = 1-(1+kx)^{-\frac{p}{k}}$

From (7.155),

$$g_i(x) = \frac{\left[1-(1+kx)^{-\frac{p}{k}}\right]^{i-1}(1+kx)^{-\frac{p(n-i+1)}{k}-1}}{B(i,n-i+1)}p, \quad 0<x<\infty \tag{7.156}$$

Putting $i=1$ in (7.156),

$$g_1(x) = pn(1+kx)^{-\frac{pn}{k}-1}, \quad 0<x<\infty \tag{7.157}$$

Putting $i=n$ in (7.156),

$$g_n(x) = pn\left[1-(1+kx)^{-\frac{p}{k}}\right]^{n-1}(1+kx)^{-\frac{p}{k}-1}, \quad 0<x<\infty \tag{7.158}$$

7.16.2 Exponentiated Generated Distribution

Putting (7.152) in (2.8) gives

$$G(x) = \left[1-(1+kx)^{-\frac{p}{k}}\right]^r, \quad 0<x<\infty,\ r>0 \tag{7.159}$$

and its corresponding pdf is

$$g(x) = rp(1+kx)^{-\frac{p}{k}-1}\left[1-(1+kx)^{-\frac{p}{k}}\right]^{r-1}, \quad 0<x<\infty,\ r>0 \tag{7.160}$$

7.16.3 Beta Exponentiated Generated Distribution

Putting (7.159) in (2.10) gives

$$G(x) = \int_0^{\left[1-(1+kx)^{-\frac{p}{k}}\right]^r} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0<x<\infty,\ a>0,\ b>0,\ r>0 \tag{7.161}$$

and its corresponding pdf is

$$g(x) = \frac{rp\left[1-(1+kx)^{-\frac{p}{k}}\right]^{ra-1}\left[1-\left\{1-(1+kx)^{-\frac{p}{k}}\right\}^r\right]^{b-1}}{B(a,b)(1+kx)^{\frac{p}{k}+1}} \tag{7.162}$$

i^{th} **order statistic of** $G(x) = \left[1-(1+kx)^{-\frac{p}{k}}\right]^r$

From (7.162),

$$g_i(x) = \frac{rp\left[1-(1+kx)^{-\frac{p}{k}}\right]^{ri-1}\left[1-\left\{1-(1+kx)^{-\frac{p}{k}}\right\}^r\right]^{n-i}}{B(i,n-i+1)(1+kx)^{\frac{p}{k}+1}}, \quad 0<x<\infty, \ r>0 \quad \textbf{(7.163)}$$

Putting $i = 1$ in (7.163),

$$g_1(x) = \frac{rnp\left[1-(1+kx)^{-\frac{p}{k}}\right]^{r-1}\left[1-\left\{1-(1+kx)^{-\frac{p}{k}}\right\}^r\right]^{n-1}}{(1+kx)^{\frac{p}{k}+1}}, \quad 0<x<\infty, \ r>0 \quad \textbf{(7.164)}$$

Putting $i = n$ in (7.163),

$$g_n(x) = \frac{rnp\left[1-(1+kx)^{-\frac{p}{k}}\right]^{rn-1}}{(1+kx)^{\frac{p}{k}+1}}, \quad 0<x<\infty, \ r>0 \quad \textbf{(7.165)}$$

7.17 Kodlin (1967); $\mu(t) = \alpha + \beta t, \quad \alpha > 0, \ \beta > 0$

Thus

$$\begin{aligned}\int_0^x \mu(t)dt &= \int_0^x (\alpha+\beta t)dt \\ &= \int_0^x \alpha dt + \beta t dt \\ &= \left[\alpha t + \frac{\beta t^2}{2}\right]_0^x \\ &= \alpha x + \frac{\beta x^2}{2}\end{aligned}$$

Substituting in (7.1) we get

$$F(x) = 1 - \exp\left\{-\alpha x - \frac{\beta x^2}{2}\right\}$$

Now,

$$F(x) = 0 \Longrightarrow 1 - 1 = 1 - \exp\{0\} = 1 - \exp\left\{-\alpha(0) - \frac{\beta(0)^2}{2}\right\} = 0 \Longrightarrow x = 0$$

$$F(x) = 1 \Longrightarrow 1 - 0 = 1 - \exp\{-\infty\} = 1 - \exp\left\{-\alpha(\infty) - \frac{\beta(\infty)^2}{2}\right\} = 0 \Longrightarrow x = \infty$$

Therefore

$$F(x) = 1 - e^{-\alpha x - \frac{\beta x^2}{2}}, \quad 0 < x < \infty \tag{7.166}$$

and its corresponding pdf is

$$f(x) = \{\alpha + \beta x\} e^{-\alpha x - \frac{\beta x^2}{2}}, \quad 0 < x < \infty \tag{7.167}$$

which is the **Linear Exponential distribution** according to Elandt Johnson and Johnson (1980).

7.17.1 Special Case

When $\beta = 0$

$$F(x) = 1 - e^{-\alpha x}, \quad 0 < x < \infty$$

and its corresponding pdf is

$$f(x) = \alpha e^{-\alpha x}, \quad 0 < x < \infty$$

which is the **Exponential distribution** with rate parameter α, the same form as in (7.16) and (7.17).

When $\alpha = 0,\ \beta = 1$

$$F(x) = 1 - e^{-\frac{x^2}{2}}, \quad 0 < x < \infty$$

and its corresponding pdf is

$$f(x) = xe^{-\frac{x^2}{2}}, \quad 0 < x < \infty$$

which is the **Rayleigh distribution** with scale parameter 1 as in (5.203) and (5.204).

7.17.2 Extension of Linear Exponential Distribution

Flehinger and Lewis (1959) in a reliability context considered

$$F(t) = 1 - \exp\left\{-\alpha t - \frac{1}{2}\beta t^2\right\} \tag{7.168}$$

where t is the duration of the fire.

Their assumption was that the ratio of the fire loss amount x to the minimum discernible loss x_0 increases exponentially with the fire duration.

$$\frac{x}{x_0} = e^{kt}, \quad k > 0,\ x \geq x_0 > 0$$

$$\log\left(\frac{x}{x_0}\right) = \log e^{kt}$$
$$\log\left(\frac{x}{x_0}\right) = kt$$

Substituting in (7.131) we get

$$\begin{aligned} F(x) &= 1 - \exp\left\{-\alpha \log\left(\frac{x}{x_0}\right) - \beta\left[\log\left(\frac{x}{x_0}\right)\right]^2\right\} \\ &= 1 - \exp\left\{-\alpha(\log x - \log x_0) - \beta(\log x - \log x_0)^2\right\} \end{aligned}$$

Now,

$$F(x)=0\Longrightarrow 1-\exp\{0\}=1-\exp\left\{-\alpha\log\left(\frac{x_0}{x_0}\right)-\beta\left[\log\left(\frac{x_0}{x_0}\right)\right]^2\right\}=0\Longrightarrow x=x_0$$

$$F(x)=1\Longrightarrow 1-\exp\{-\infty\}=1-\exp\left\{-\alpha\log\left(\frac{\infty}{x_0}\right)-\beta\left[\log\left(\frac{\infty}{x_0}\right)\right]^2\right\}=1\Longrightarrow x=\infty$$

Therefore

$$F(x)=1-\exp\left\{-\alpha(\log x-\log x_0)-\beta(\log x-\log x_0)^2\right\},\quad x_0\leq x<\infty \tag{7.169}$$

and its corresponding pdf is

$$f(x)=\left\{\frac{\alpha}{x}+\frac{2\beta(\log x-\log x_0)}{x}\right\}\exp\left\{-\alpha(\log x-\log x_0)-\beta(\log x-\log x_0)^2\right\} \tag{7.170}$$
$$,\quad x_0\leq x<\infty$$

which Kleiber and Kotz (2003) called it the **Benini distribution** with shape parameters $\alpha;\beta$ and scale parameter x_0.

Special Case

When $\alpha=0$

$$\begin{aligned}F(x)&=1-\exp\left\{-\beta(\log x-\log x_0)^2\right\}\\&=1-\exp\{-\beta(\log x-\log x_0)(\log x-\log x_0)\}\\&=1-\exp\left\{-\beta(\log x-\log x_0)\log\left(\frac{x}{x_0}\right)\right\}\\&=1-\left(\frac{x}{x_0}\right)^{-\beta(\log x-\log x_0)},\quad x_0\leq x<\infty\end{aligned}$$

and its corresponding pdf is

$$f(x) = \left\{ \frac{2\beta(\log x - \log x_0)}{x} \right\} \exp\left\{ -\beta(\log x - \log x_0)^2 \right\}, \quad x_0 \le x < \infty$$

which Benini (1905) called it the **Benini distribution** with shape parameter β and scale parameter x_0.

7.17.3 Beta Linear Exponential Distribution

Putting (7.166) in (2.3) gives

$$G(x) = \int_0^{1-e^{-\alpha x - \frac{\beta x^2}{2}}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)} dt, \quad 0 < x < \omega,\ a > 0,\ b > 0 \tag{7.171}$$

and its corresponding pdf is

$$\begin{aligned} g(x) &= \frac{\left[1 - e^{-\alpha x - \frac{\beta x^2}{2}}\right]^{a-1} \left[1 - \left\{1 - e^{-\alpha x - \frac{\beta x^2}{2}}\right\}\right]^{b-1}}{B(a,b)} \{\alpha + \beta x\} e^{-\alpha x - \frac{\beta x^2}{2}} \\ &= \frac{\left[1 - e^{-\alpha x - \frac{\beta x^2}{2}}\right]^{a-1} \left[e^{-\alpha x - \frac{\beta x^2}{2}}\right]^{b-1}}{B(a,b)} \{\alpha + \beta x\} e^{-\alpha x - \frac{\beta x^2}{2}} \end{aligned} \tag{7.172}$$

i^{th} order statistic of Linear Exponential distribution

From (7.172),

$$g_i(x) = \frac{\left[1 - e^{-\alpha x - \frac{\beta x^2}{2}}\right]^{i-1} \left[e^{-\alpha x - \frac{\beta x^2}{2}}\right]^{n-i}}{B(i, n-i+1)} \{\alpha + \beta x\} e^{-\alpha x - \frac{\beta x^2}{2}}, \quad 0 < x < \infty \tag{7.173}$$

Putting $i = 1$ in (7.173),

$$g_1(x) = n \left[e^{-\alpha x - \frac{\beta x^2}{2}}\right]^{n-1} \{\alpha + \beta x\} e^{-\alpha x - \frac{\beta x^2}{2}}, \quad 0 < x < \infty \tag{7.174}$$

Putting $i = n$ in (7.173),

$$g_n(x) = n\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{n-1}\{\alpha+\beta x\}\,e^{-\alpha x-\frac{\beta x^2}{2}}, \quad 0<x<\infty \tag{7.175}$$

7.17.4 Exponentiated Linear Exponential Distribution

Putting (7.166) in (2.8) gives

$$G(x) = \left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{r}, \quad 0<x<\infty,\ r>0 \tag{7.176}$$

and its corresponding pdf is

$$g(x) = r\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{r-1}\{\alpha+\beta x\}\,e^{-\alpha x-\frac{\beta x^2}{2}}, \quad 0<x<\infty,\ r>0 \tag{7.177}$$

7.17.5 Beta Exponentiated Linear Exponential Distribution

Putting (7.176) in (2.10) gives

$$G(x) = \int_{0}^{\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{r}} \frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt, \quad 0<x<\infty,\ a>0,\ b>0 \tag{7.178}$$

and its corresponding pdf is

$$g(x) = \frac{r\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{ra-1}\left[1-\left\{1-e^{-\alpha x-\frac{\beta x^2}{2}}\right\}^{r}\right]^{b-1}}{B(a,b)}\{\alpha+\beta x\}\,e^{-\alpha x-\frac{\beta x^2}{2}} \tag{7.179}$$

i^{th} order statistic of Exponentiated Linear Exponential distribution

From (7.179),

$$g_i(x) = \frac{r\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{ri-1}\left[1-\left\{1-e^{-\alpha x-\frac{\beta x^2}{2}}\right\}^r\right]^{n-i}}{B(i,n-i+1)}\{\alpha+\beta x\}\,e^{-\alpha x-\frac{\beta x^2}{2}} \tag{7.180}$$

Putting $i = 1$ in (7.180),

$$g_1(x) = rn\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{r-1}\left[1-\left\{1-e^{-\alpha x-\frac{\beta x^2}{2}}\right\}^r\right]^{n-1}\{\alpha+\beta x\}\,e^{-\alpha x-\frac{\beta x^2}{2}} \tag{7.181}$$

Putting $i = n$ in (7.180),

$$g_n(x) = rn\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{rn-1}\{\alpha+\beta x\}\,e^{-\alpha x-\frac{\beta x^2}{2}}, \quad 0 < x < \infty \tag{7.182}$$

8 Conclusion

This paper has highlighted five cases of constructing probability distributions based on Burr differential equation. This study has also examined the use of generator approach to construct beta generated, exponentiated generated and beta exponentiated generated distributions. Over 75 distribution functions have been constructed out of which 69 are unique. The use of beta generated distribution was fronted by Jones in 2004 to construct order statistics of $F(x)$. Order statistics based on exponentiated distributions $[F(x)]^r$ has been obtained using a newly introduced beta exponentiated generator approach. Therefore we have extended both techniques and constructed minimum and maximum order statistics distribution for $F(x)$ and $[F(x)]^r$. A summary of results obtained is given in tables 8.1, 8.2, 8.3, 8.4, 8.5 and 8.6.

The results of this paper can further be extended by constructing more distribution functions. Extensive study on reverse hazard functions is required as this will facilitate other researchers to construct more distribution functions in future.

Table 8.1. Burr Generated Distributions Based on Differential Equation: $y' = y(1-y)g(x,y)$ **where** $y' = \frac{dy}{dx}$ **and** $y = F(x)$

$g(x), r(x), \mu(x)$	$F(x)$	$f(x)$	Support
Case I: $g(x,y) = \frac{g(x)}{y(1-y)}$			
1	x **Standard Uniform (Burr I)**	1 $E(X^m) = B(m+1,1)$	$0 < x < 1$
Case II: $g(x,y) = g(x)$			
1	$[e^{-x}+1]^{-1}$ **Logistic**	$\frac{e^{-x}}{[e^{-x}+1]^2}$	$-\infty < x < \infty$
$\frac{c}{x}$	$[x^{-c}+1]^{-1}$ **Log-Logistic (Fisk)**	$\frac{cx^{-c-1}}{[x^{-c}+1]^2}$ $E(X^m) = B\left(1+\frac{m}{c}, 1-\frac{m}{c}\right)$	$0 < x < \infty$ $c > m$
$[(c-x)x]^{-1}$	$\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-1}$	$\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-2} \frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}}$	$0 < x < c$
$\sec^2 x$	$[ke^{-\tan x}+1]^{-1}$	$\frac{ke^{-\tan x}\sec^2 x}{[ke^{-\tan x}+1]^2}$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$

$g(x), r(x), \mu(x)$	$F(x)$	$f(x)$	Support
$c\cosh x$	$\left[ke^{-c\sinh x}+1\right]^{-1}$	$\dfrac{kce^{-c\sinh x}}{\cosh x\left[ke^{-c\sinh x}+1\right]^{2}}$	$-\infty < x < \infty$
2	$2^{-1}\left[1+\tanh x\right]$	$2^{-1}\operatorname{sech}^2 x$	$-\infty < x < \infty$
Case III: $g(x,y) = \dfrac{g(x)}{xy}$			
$\frac{ckxe^{x}(1+e^{x})^{k-1}}{c\left[(1+e^{x})^{k}-1\right]+2}$	$1-\dfrac{2}{c\left[(1+e^{x})^{k}-1\right]+2}$ **Burr IX**	$\dfrac{2kce^{x}(1+e^{x})^{k-1}}{\left\{c\left[(1+e^{x})^{k}-1\right]+2\right\}^{2}}$	$-\infty < x < \infty$
$2x^2$	$1-e^{-x^2}$	$2xe^{-x^2}$ $E(X^m)=\Gamma\left(\frac{m}{2}+1\right)$	$0 < x < \infty$
$\dfrac{ckx^c}{1+x^c}$	$1-\left[1+x^c\right]^{-k}$ **Burr XII**	$kcx^{c-1}\left[1+x^c\right]^{-k-1}$ $E(X^m)=kB\left(k-\frac{m}{c},1+\frac{m}{c}\right)$	$0 < x < \infty$ $c > m$
$\dfrac{cx}{1-cx}$	cx	c $E(X^m)=\frac{1}{c^m}B(m+1,1)$	$0 < x < \dfrac{1}{c}$

$g(x), r(x), \mu(x)$	$F(x)$	$f(x)$	Support
cx	$1-e^{-cx}$	ce^{-cx} $E(X^m)=\frac{\Gamma(m+1)}{c^m}$	$0<x<\infty$
$(c-x)^{-1}$	$1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}$	$\frac{1}{x^2}\left(\frac{c-x}{x}\right)^{\frac{1}{c}-1}$	$\frac{c}{2}<x<c$
$cx\sec^2 x$	$1-e^{-c\tan x}$	$ce^{-c\tan x}sec^2 x$	$-\frac{\pi}{2}<x<\frac{\pi}{2}$
α	$1-\left[\frac{1}{x}\right]^{\alpha}$ **Pareto (Type I)**	$\frac{\alpha}{x^{\alpha+1}}$ $E(X^m)=B\left(1-\frac{m}{\alpha},1\right)$	$1<x<\infty$ $\alpha>m$
$\frac{\alpha x}{1+x}$	$1-\left[\frac{1}{1+x}\right]^{\alpha}$ **Pareto (Type II)**	$\frac{\alpha}{[1+x]^{\alpha+1}}$	$0<x<\infty$
$\beta x+\frac{\alpha x}{1+x}$	$1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}$ **Pareto (Type III)**	$\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}}+\frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}$	$0<x<\infty$
$\frac{\alpha x(x-\mu)^{\frac{1}{\beta}-1}}{\beta[1+(x-\mu)^{\frac{1}{\beta}}]}$	$1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}$ **Pareto (Type IV)**	$\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}}$	$\mu<x<\infty$

$g(x), r(x), \mu(x)$	$F(x)$	$f(x)$	Support
$\alpha\left(\frac{x}{\beta}\right)^{\alpha}$	$1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$ **Weibull**	$\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$ $E(X^m)=\beta^m\Gamma\left(1+\frac{m}{\alpha}\right)$	$0<x<\infty$
$\frac{x}{\beta}$	$1-e^{-\frac{x}{\beta}}$ **Exponential**	$\frac{1}{\beta}e^{-\frac{x}{\beta}}$ $E(X^m)=\beta^m\Gamma(1+m)$	$0<x<\infty$
$\alpha+2\beta\log x$	$1-e^{-\alpha\log x-\beta(\log x)^2}$ **Benini**	$\left[\frac{\alpha}{x}+\frac{2\beta\log x}{x}\right] e^{-\alpha\log x-\beta(\log x)^2}$	$1<x<\infty$
Case IV: $g(x,y)=\frac{r(x)}{(1-y)}$			
$\frac{1}{2\cosh x\arctan e^x}$	$\frac{2}{\pi}\arctan e^x$	$\frac{2e^x}{\pi}\left(\frac{1}{1+x^2}\right)$	$-\infty<x<\infty$
$\frac{1-\cos 2\pi x}{x-\frac{1}{2\pi}\sin 2\pi x}$	$x-\frac{1}{2\pi}\sin 2\pi x$	$1-\cos 2\pi x$	$0<x<1$
$\frac{\alpha e^{-x}}{1+e^{-x}}$	$[1+e^{-x}]^{-\alpha}$ **Type I Generalized Logistic**	$\frac{\alpha e^{-x}}{1+[e^{-x}]^{\alpha+1}}$	$-\infty<x<\infty$

$g(x), r(x), \mu(x)$	$F(x)$	$f(x)$	Support
Case V: $g(x,y) = \dfrac{\mu(x)}{y}$			
$\alpha\left[\log\left(1+\frac{x}{\beta}\right)\right]$	$1-\left(1+\frac{x}{\beta}\right)^{-\alpha}$ **Lomax**	$\dfrac{\alpha}{\beta}\left(1+\dfrac{x}{\beta}\right)^{-\alpha-1}$	$0<x<\infty$
cx	$1-e^{-cx}$ **Exponential**	ce^{-cx} $E(X^m)=\left(\frac{1}{c}\right)^m\Gamma(m+1)$	$0<x<\infty$
cx^{α}	$1-e^{-cx^{\alpha}}$ **Weibull**	$c\alpha x^{\alpha-1}e^{-cx^{\alpha}}$ $E(X^m)=\left(\frac{1}{c}\right)^{\frac{m}{\alpha}}\Gamma\left(1+\frac{m}{\alpha}\right)$	$0<x<\infty$
$\alpha x+\dfrac{\beta x^2}{2}$	$1-e^{-\alpha x-\frac{\beta x^2}{2}}$ **Linear Exponential**	$\{\alpha+\beta x\}\,e^{-\alpha x-\frac{\beta x^2}{2}}$	$0<x<\infty$
$A(x\log x-x)$	$F(x)=1-\left(\dfrac{e}{x}\right)^{Ax}$ $f(x)=\dfrac{Ae^{Ax}\log x}{x^{Ax}}$		

$g(x), r(x), \mu(x)$	$F(x)$	$f(x)$	Support
$-\log\left(\frac{\omega-x}{\omega}\right)$	$\frac{x}{\omega}$	$\frac{1}{\omega}$ $E(X^m)=\omega^m B(m+1,1)$	$0<x<\omega$
$\frac{e^{-d}}{k}\left[e^{kx}-1\right]$	$1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$ **Gompertz**	$e^{-d}e^{kx}\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$0<x<\infty$
$Ax\ +$ $\frac{e^{-d}}{k}\left[e^{kx}-1\right]$	$F(x)=1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$ $f(x)=\left\{A+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$ **Gompertz-Makeham**		$0<x<\infty$
$Ax\ +$ $\frac{e^{-d}}{k}\left[e^{kx}-1\right]+$ $D\log\left[\frac{N}{N-x}\right]$	$F(x)=1-\left[\frac{N}{N-x}\right]^{-D}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$ $f(x)=\left\{\frac{D}{N}\left[\frac{N}{N-x}\right]^{-D+1}+\left[\frac{N}{N-x}\right]^{-D}\left[A+e^{-d}e^{kx}\right]\right\}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$		
$Ax + \frac{Hx^2}{2} +$ $\frac{B}{\log c}\left[c^x-1\right]$	$F(x)=1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$ $f(x)=\left\{A+Hx+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$		$0<x<\infty$

$g(x), r(x), \mu(x)$	$F(x)$	$f(x)$	Support
$Ax + \frac{B}{\log c}[c^x - 1] + \frac{M}{\log n}[n^x - 1]$	$F(x) = 1 - \exp\left\{-Ax - \frac{B}{\log c}[c^x - 1] - \frac{M}{\log n}[n^x - 1]\right\}$	$f(x) = \{A + Bc^x + Mn^x\}\exp\left\{-Ax - \frac{B}{\log c}[c^x - 1] - \frac{M}{\log n}[n^x - 1]\right\}$	$0 < x < \infty$
$2a\sqrt{x} + bx + \frac{3}{4}c\left(x^{\frac{1}{3}}\right)^4$	$F(x) = 1 - \exp\left\{-2a\sqrt{x} - bx - \frac{3}{4}c\left(x^{\frac{1}{3}}\right)^4\right\}$	$f(x) = \left\{\frac{a}{\sqrt{x}} + b + 3cx\right\}\exp\left\{-2a\sqrt{x} - bx - \frac{3}{4}c\left(x^{\frac{1}{3}}\right)^4\right\}$	$0 < x < \infty$
$-\log(1 - cx - dx^2)$	$F(x) = cx + dx^2,$	$f(x) = c + 2dx$	$\frac{-c - \sqrt{c^2 + 4d}}{2d} < x < \frac{-c + \sqrt{c^2 + 4d}}{2d}$
$\frac{p}{k}\log(1 + kx)$	$1 - (1 + kx)^{-\frac{p}{k}}$	$p(1 + kx)^{-\frac{p}{k} - 1}$	$0 < x < \infty$

$g(x), r(x), \mu(x)$	$F(x)$	$f(x)$	Support
$\mu(x) = -\frac{a_1}{b_1}\left[e^{-b_1x}-1\right] - \frac{a_2}{b_2}\left[\frac{e^{\frac{-b_2c^2}{2}}\left\{ce^{\frac{-b_2(x^2-2xc)}{2}}+(x-c)\right\}}{c(x-c)}\right] + \frac{a_3}{b_3}\left[e^{b_3x}-1\right]$	$F(x) = 1-\exp\left\{\frac{a_1}{b_1}\left[e^{-b_1x}-1\right] + \frac{a_2}{b_2}\left[\frac{e^{\frac{-b_2c^2}{2}}\left\{ce^{\frac{-b_2(x^2-2xc)}{2}}+(x-c)\right\}}{c(x-c)}\right] - \frac{a_3}{b_3}\left[e^{b_3x}-1\right]\right\},$	$f(x) = \left\{a_1e^{-b_1x} + \frac{a_2}{b_2}\left[\left\{e^{\frac{-b_2(c^2+x^2-2xc)}{2}} + e^{\frac{-b_2c^2}{2}}\right\} + \frac{\left\{ce^{\frac{-b_2c^2}{2}}\left[ce^{\frac{-b_2(x^2-2xc)}{2}}+(x-c)\right]\right\}}{[c(x-c)]^2}\right] + a_3e^{b_3x}\right\} \times \exp\left\{\frac{a_1}{b_1}\left[e^{-b_1x}-1\right] + \frac{a_2}{b_2}\left[\frac{e^{\frac{-b_2c^2}{2}}\left\{ce^{\frac{-b_2(x^2-2xc)}{2}}+(x-c)\right\}}{c(x-c)}\right] - \frac{a_3}{b_3}\left[e^{b_3x}-1\right]\right\}$	$0 < x < \infty$
$-\frac{a_1}{b_1}\left[e^{-b_1x}-1\right] - a_2x + \frac{a_3}{b_3}\left[e^{b_3x}-1\right]$	$F(x) = 1-\exp\left\{\frac{a_1}{b_1}\left[e^{-b_1x}-1\right] - a_2x - \frac{a_3}{b_3}\left[e^{b_3x}-1\right]\right\},$	$f(x) = \left\{a_1e^{-b_1x}+a_2+a_3e^{b_3x}\right\}\exp\left\{\frac{a_1}{b_1}\left[e^{-b_1x}-1\right] - a_2x - \frac{a_3}{b_3}\left[e^{b_3x}-1\right]\right\}$	$0 < x < \infty$

$g(x), r(x), \mu(x)$	$F(x)$	$f(x)$	Support
$\mu(x) = -\log\left(\left[\frac{a-x}{x}\right]^2 - b\left[e^{-\frac{x}{c}} - e^{-\frac{x}{d}}\right]\right)$ $F(x) = 1 - \left(\left[\frac{a-x}{x}\right]^2 - b\left[e^{-\frac{x}{c}} - e^{-\frac{x}{d}}\right]\right)$ $f(x) = \frac{2a}{x^2}\left[\frac{a-x}{x}\right] - \frac{b}{c}e^{-\frac{x}{c}} + \frac{b}{d}e^{-\frac{x}{d}}$			
$\frac{B}{D}x + \log\left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]^{\frac{1}{D}}$	$F(x) = 1 - \left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]^{-\frac{1}{D}} \exp\left\{-\frac{B}{D}x\right\}$	$f(x) = \left\{\frac{B}{D} + \frac{c^x \log c}{D(1+c^xD)}\right\}\left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]^{-\frac{1}{D}} \exp\left\{-\frac{B}{D}x\right\}$	
$\frac{B}{D}x + \log\left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]^{\frac{1}{D}} + \frac{E}{\log c}\left[c^x - 1\right]$	$F(x) = 1 - \left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]^{-\frac{1}{D}} \exp\left\{-\frac{B}{D}x - \frac{E}{\log c}\left[c^x - 1\right]\right\}$	$f(x) = \left\{\frac{B}{D} + Ec^x + \frac{c^x \log c}{D(1+c^xD)}\right\}\left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]^{-\frac{1}{D}} \exp\left\{-\frac{B}{D}x - \frac{E}{\log c}\left[c^x - 1\right]\right\}$	

Table 8.2. Exponentiated Distributions Based on Burr Differential Equation

$F(x)$	$[F(x)]^r, \quad r>0$	$r[F(x)]^{r-1}, \quad r>0$	Support
x	x^r **Exponentiated Standard Uniform (Burr I)**	rx^{r-1} $E(X^m)=rB(m+r,1)$	$0<x<1$
$[e^{-x}+1]^{-1}$	$[e^{-x}+1]^{-r}$ **Type I Generalized Logistic/ Burr II**	$re^{-x}[e^{-x}+1]^{-r-1}$	$-\infty<x<\infty$
$[x^{-c}+1]^{-1}$	$[x^{-c}+1]^{-r}$ **Exponentiated Log-Logistic (Fisk)/ Burr III**	$rcx^{-c-1}[x^{-c}+1]^{-r-1}$ $E(X^m)=rB\left(r+\frac{m}{c},1-\frac{m}{c}\right)$	$0<x<\infty$ $c>m$
$\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-1}$	$\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-r}$ **Burr IV**	$r\frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}}\times \left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-r-1}$	$0<x<c$

$F(x)$	$[F(x)]^r, \quad r>0$	$r[F(x)]^{r-1}, \quad r>0$	Support
$\left[ke^{-\tan x}+1\right]^{-1}$	$\left[ke^{-\tan x}+1\right]^{-r}$ **Burr V**	$\dfrac{rke^{-\tan x}\sec^2 x}{\left[ke^{-\tan x}+1\right]^{r+1}}$	$-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$
$\left[ke^{-c\sinh x}+1\right]^{-1}$	$\left[ke^{-c\sinh x}+1\right]^{-r}$ **Burr VI**	$\dfrac{rkce^{-c\sinh x}\cosh x}{\left[ke^{-c\sinh x}+1\right]^{r+1}}$	$-\infty<x<\infty$
$2^{-1}[1+\tanh x]$	$2^{-r}[1+\tanh x]^r$ **Burr VII**	$r2^{-r}\operatorname{sech}^2 x\,[1+\tanh x]^{r-1}$	$-\infty<x<\infty$
$1-\frac{2}{c\left[(1+e^x)^k-1\right]+2}$	$\left\{1-\frac{2}{c\left[(1+e^x)^k-1\right]+2}\right\}^r$ **Exponentiated Burr IX**	$\dfrac{2rkce^x(1+e^x)^{k-1}}{\left\{c\left[(1+e^x)^k-1\right]+2\right\}^2}\times$ $\left\{1-\frac{2}{c\left[(1+e^x)^k-1\right]+2}\right\}^{r-1}$	$-\infty<x<\infty$
$1-e^{-x^2}$	$\left[1-e^{-x^2}\right]^r$ **Burr X**	$2rxe^{-x^2}\left[1-e^{-x^2}\right]^{r-1}$	$0<x<\infty$
$1-[1+x^c]^{-k}$	$\left\{1-[1+x^c]^{-k}\right\}^r$ **Exponentiated Burr XII**	$rkcx^{c-1}[1+x^c]^{-k-1}\times$ $\left\{1-[1+x^c]^{-k}\right\}^{r-1}$	$0<x<\infty$

$F(x)$	$[F(x)]^r, \quad r>0$	$r[F(x)]^{r-1}, \quad r>0$	Support
cx	$[cx]^r$	$r[cx]^{r-1}$ $E(X^m)=\frac{r}{c^m}B(m+r,1)$	$0<x<\frac{1}{c}$
$1-e^{-cx}$	$[1-e^{-cx}]^r$	$rce^{-cx}[1-e^{-cx}]^{r-1}$	$0<x<\infty$
$1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}$	$\left[1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^r$	$\frac{r}{x^2}\left(\frac{c-x}{x}\right)^{\frac{1}{c}-1}\times$ $\left[1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{r-1}$	$\frac{c}{2}<x<c$
$1-e^{-c\tan x}$	$[1-e^{-c\tan x}]^r$	$rce^{-c\tan x}sec^2x\times$ $[1-e^{-c\tan x}]^{r-1}$	$-\frac{\pi}{2}<x<\frac{\pi}{2}$
$1-\left[\frac{1}{x}\right]^{\alpha}$	$\left\{1-\left[\frac{1}{x}\right]^{\alpha}\right\}^r$ **Exponentiated Pareto (Type I)**	$\frac{r\alpha}{x^{\alpha+1}}\left\{1-\left[\frac{1}{x}\right]^{\alpha}\right\}^{r-1}$ $E(X^m)=rB\left(1-\frac{m}{\alpha},r\right)$	$1<x<\infty$ $\alpha>m$

$F(x)$	$[F(x)]^{r}, \quad r>0$	$r[F(x)]^{r-1}, \quad r>0$	Support
$1-\left[\frac{1}{1+x}\right]^{\alpha}$	$\left\{1-\left[\frac{1}{1+x}\right]^{\alpha}\right\}^{r}$ **Exponentiated Pareto (Type II)**	$\frac{r\alpha}{[1+x]^{\alpha+1}}\times$ $\left\{1-\left[\frac{1}{1+x}\right]^{\alpha}\right\}^{r-1}$	$0<x<\infty$
$1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}$	$\left\{1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{r}$ **Exponentiated Pareto (Type III)**	$r\left\{\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}}+\frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\}\times$ $\left\{1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{r-1}$	$0<x<\infty$
$1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}$	$\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{r}$ **Exponentiated Pareto (Type IV)**	$\frac{r\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}}\times$ $\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{r-1}$	$\mu<x<\infty$
$1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$	$\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{r}$ **Exponentiated Weibull**	$r\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\times$ $\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{r-1}$	$0<x<\infty$
$1-e^{-\frac{x}{\beta}}$	$\left[1-e^{-\frac{x}{\beta}}\right]^{r}$ **Exponentiated Exponential**	$\frac{r}{\beta}e^{-\frac{x}{\beta}}\left[1-e^{-\frac{x}{\beta}}\right]^{r-1}$	$0<x<\infty$

$F(x)$	$[F(x)]^r, \quad r>0$	$r[F(x)]^{r-1}, \quad r>0$	Support
$1-e^{-\alpha\log x-\beta(\log x)^2}$	$\left[1-e^{-\alpha\log x-\beta(\log x)^2}\right]^r$ **Exponentiated Benini**	$r\left[\frac{\alpha}{x}+\frac{2\beta\log x}{x}\right]e^{-\alpha\log x-\beta(\log x)^2}$ $\times\left[1-e^{-\alpha\log x-\beta(\log x)^2}\right]^{r-1}$	$1<x<\infty$
$\frac{2}{\pi}\arctan e^x$	$\left[\frac{2}{\pi}\arctan e^x\right]^r$ **Burr VIII**	$r\frac{2e^x}{\pi}\left(\frac{1}{1+x^2}\right)\times$ $\left[\frac{2}{\pi}\arctan e^x\right]^{r-1}$	$-\infty<x<\infty$
$x-\frac{1}{2\pi}\sin 2\pi x$	$\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^r$ **Burr XI**	$r(1-\cos 2\pi x)\times$ $\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^{r-1}$	$0<x<1$
$1-\left(1+\frac{x}{\beta}\right)^{-\alpha}$	$\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^r$ **Exponentiated Lomax**	$\frac{r\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha-1}\times$ $\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{r-1}$	$0<x<\infty$
$1-e^{-cx}$	$[1-e^{-cx}]^r$ **Exponentiated Exponential**	$rce^{-cx}[1-e^{-cx}]^{r-1}$	$0<x<\infty$
$1-e^{-cx^\alpha}$	$\left[1-e^{-cx^\alpha}\right]^r$ **Exponentiated Weibull**	$rc\alpha x^{\alpha-1}e^{-cx^\alpha}\left[1-e^{-cx^\alpha}\right]^{r-1}$	$0<x<\infty$

$F(x)$	$[F(x)]^r, \quad r>0$	$r[F(x)]^{r-1}, \quad r>0$	Support
$\frac{x}{\omega}$	$\left[\frac{x}{\omega}\right]^r$	$\frac{rx^{r-1}}{\omega^r}$ $E(X^m)=r\omega^m B(m+r,1)$	$0<x<\omega$
$1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^r$ **Exponentiated Gompertz**	$r\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{r-1}$ $\times\ e^{-d}e^{kx}\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$0<x<\infty$
$1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^r$ **Exponentiated Gompertz-Makeham**	$r\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{r-1}$ $\times\ \left\{A+e^{-d}e^{kx}\right\}$ $\times\ \exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$0<x<\infty$
$1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$\left[1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^r$	$r\left[1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{r-1}$ $\times\ \left\{A+Hx+e^{-d}e^{kx}\right\}$ $\times\ \exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$0<x<\infty$
$cx+dx^2$	$[F(x)]^r=\left[cx+dx^2\right]^r, \quad \frac{-c-\sqrt{c^2+4d}}{2d}<x<\frac{-c+\sqrt{c^2+4d}}{2d}$ $r[F(x)]^{r-1}=r(c+2dx)\left[cx+dx^2\right]^{r-1}$		

$F(x)$	$[F(x)]^r, \quad r>0$	$r[F(x)]^{r-1}, \quad r>0$	Support
$1-(1+kx)^{-\frac{p}{k}}$	$\left[1-(1+kx)^{-\frac{p}{k}}\right]^r$	$rp(1+kx)^{-\frac{p}{k}-1}\times$ $\left[1-(1+kx)^{-\frac{p}{k}}\right]^{r-1}$	$0<x<\infty$
$1-e^{-\alpha x-\frac{\beta x^2}{2}}$	$\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^r$ **Exponentiated Linear Exponential**	$r\{\alpha+\beta x\}e^{-\alpha x-\frac{\beta x^2}{2}}\times$ $\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{r-1}$	$0<x<\infty$

Table 8.3. Beta Generated Distributions Based on Burr Differential Equation

$F(x)$	$g(x)=\frac{[F(x)]^{a-1}[1-F(x)]^{b-1}}{B(a,b)}f(x), \quad a>0,\ b>0$	Support
x	$\frac{x^{a-1}[1-x]^{b-1}}{B(a,b)}$ **Beta Standard Uniform (Burr I)/ Beta** $E(X^m)=\frac{B(m+a,b)}{B(a,b)}$	$0<x<1$
$[e^{-x}+1]^{-1}$	$\frac{e^{-bx}}{B(a,b)[e^{-x}+1]^{a+b}}$ **Beta Logistic/ Type IV Generalized Logistic**	$-\infty<x<\infty$
$[x^{-c}+1]^{-1}$	$\frac{cx^{-cb-1}}{B(a,b)[x^{-c}+1]^{a+b}}$ **Beta Log-Logistic** $E(X^m)=\frac{B\left(a+\frac{m}{c},b-\frac{m}{c}\right)}{B(a,b)}$	$0<x<\infty$ $c>m$
$\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-1}$	$\frac{(c-x)^{\frac{b}{c}-1}}{B(a,b)\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{a+b}x^{\frac{b}{c}+1}}$	$0<x<c$
$[ke^{-\tan x}+1]^{-1}$	$\frac{[ke^{-\tan x}]^{b}\sec^2 x}{B(a,b)[ke^{-\tan x}+1]^{a+b}}$	$-\frac{\pi}{2}<x<\frac{\pi}{2}$
$[ke^{-c\sinh x}+1]^{-1}$	$\frac{c[ke^{-c\sinh x}]^{b}\cosh x}{B(a,b)[ke^{-c\sinh x}+1]^{a+b}}$	$-\infty<x<\infty$

$F(x)$	$g(x)=\frac{[F(x)]^{a-1}[1-F(x)]^{b-1}}{B(a,b)}f(x), \quad a>0,\ b>0$	Support
$2^{-1}[1+\tanh x]$	$\frac{[1+\tanh x]^{a-1}[1-\tanh x]^{b-1}\operatorname{sech}^2 x}{2^{a+b-1}B(a,b)}$	$-\infty<x<\infty$
$1-\frac{2}{c\left[(1+e^x)^k-1\right]+2}$	$\frac{\left\{c\left[(1+e^x)^k-1\right]\right\}^{a-1}2^b kce^x(1+e^x)^{k-1}}{\left\{c\left[(1+e^x)^k-1\right]+2\right\}^{a+b}B(a,b)}$ **Beta Burr IX**	$-\infty<x<\infty$
$1-e^{-x^2}$	$\frac{\left[1-e^{-x^2}\right]^{a-1}e^{-bx^2}}{B(a,b)}2x$	$0<x<\infty$
$1-[1+x^c]^{-k}$	$\frac{\left\{[1+x^c]^k-1\right\}^{a-1}kcx^{c-1}}{[1+x^c]^{k(a+b-1)+1}B(a,b)}$ **Beta Burr XII**	$0<x<\infty$
cx	$\frac{c^a[x]^{a-1}[1-cx]^{b-1}}{B(a,b)}$ $E(X^m)=\frac{B(m+a+,b)}{c^m B(a,b)}$	$0<x<\frac{1}{c}$
$1-e^{-cx}$	$\frac{[1-e^{-cx}]^{a-1}ce^{-bcx}}{B(a,b)}$	$0<x<\infty$
$1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}$	$\frac{\left[x^{\frac{1}{c}}-(c-x)^{\frac{1}{c}}\right]^{a-1}(c-x)^{\frac{b}{c}-1}}{x^{\frac{a+b-1}{c}+1}B(a,b)}$	$\frac{c}{2}<x<c$

$F(x)$	$g(x)=\dfrac{[F(x)]^{a-1}\,[1-F(x)]^{b-1}}{B(a,b)}f(x),\quad a>0,\ b>0$	Support
$1-e^{-c\tan x}$	$\dfrac{[1-e^{-c\tan x}]^{a-1}\,ce^{-bc\tan x}sec^2x}{B(a,b)}$	$-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$
$1-\left[\dfrac{1}{x}\right]^{\alpha}$	$\dfrac{[1-x^{-\alpha}]^{a-1}\,x^{-\alpha b-1}}{B(a,b)}\alpha$ **Beta Pareto (Type I)** $E(X^m)=\frac{B\left(b-\frac{m}{\alpha},a\right)}{B(a,b)}$	$1<x<\infty$ $\alpha>m$
$1-\left[\dfrac{1}{1+x}\right]^{\alpha}$	$\dfrac{\left[1-(1+x)^{-\alpha}\right]^{a-1}[1+x]^{-\alpha b-1}}{B(a,b)}\alpha$ **Beta Pareto (Type II)**	$0<x<\infty$
$1-\dfrac{e^{-\beta x}}{[1+x]^{\alpha}}$	$\dfrac{\left[1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right]^{a-1}\left[\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right]^{b-1}}{B(a,b)}\left\{\dfrac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}}+\dfrac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\}$ **Beta Pareto (Type III)**	$0<x<\infty$
$1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}$	$\dfrac{\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{a-1}\left\{\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{b-1}}{B(a,b)}\dfrac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}}$ **Beta Pareto (Type IV)**	$\mu<x<\infty$

$F(x)$	$g(x)=\frac{[F(x)]^{a-1}[1-F(x)]^{b-1}}{B(a,b)}f(x),\quad a>0,\ b>0$	Support
$1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$	$\frac{\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{a-1} e^{-b\left(\frac{x}{\beta}\right)^{\alpha}}}{B(a,b)}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}$ **Beta Weibull**	$0<x<\infty$
$1-e^{-\frac{x}{\beta}}$	$\frac{\left[1-e^{-\frac{x}{\beta}}\right]^{a-1} e^{-\frac{bx}{\beta}}}{B(a,b)}\frac{1}{\beta}$ **Beta Exponential**	$0<x<\infty$
$1-e^{-\alpha\log x-\beta(\log x)^2}$	$\frac{\left[1-e^{-\alpha\log x-\beta(\log x)^2}\right]^{a-1} e^{-b\alpha\log x-b\beta(\log x)^2}}{B(a,b)}\left[\frac{\alpha}{x}+\frac{2\beta\log x}{x}\right]$ **Beta Benini**	$1<x<\infty$
$\frac{2}{\pi}\arctan e^{x}$	$\frac{\left[\frac{2}{\pi}\arctan e^{x}\right]^{a-1}\left[1-\frac{2}{\pi}\arctan e^{x}\right]^{b-1}}{B(a,b)}\frac{2e^{x}}{\pi}\left(\frac{1}{1+x^2}\right)$	$-\infty<x<\infty$
$x-\frac{1}{2\pi}\sin 2\pi x$	$\frac{\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^{a-1}\left[1-x+\frac{1}{2\pi}\sin 2\pi x\right]^{b-1}}{B(a,b)}[1-\cos 2\pi x]$	$0<x<1$
$1-\left(1+\frac{x}{\beta}\right)^{-\alpha}$	$\frac{\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{a-1}\left(1+\frac{x}{\beta}\right)^{-\alpha b-1}}{B(a,b)}\frac{\alpha}{\beta}$ **Beta Lomax**	$0<x<\infty$

$F(x)$	$g(x) = \frac{[F(x)]^{a-1}[1-F(x)]^{b-1}}{B(a,b)} f(x), \quad a>0,\ b>0$	Support
$1-e^{-cx}$	$\frac{[1-e^{-cx}]^{a-1} e^{-bcx}}{B(a,b)} c$ **Beta Exponential**	$0<x<\infty$
$1-e^{-cx^{\alpha}}$	$\frac{\left[1-e^{-cx^{\alpha}}\right]^{a-1} e^{-bcx^{\alpha}}}{B(a,b)} c\alpha x^{\alpha-1}$ **Beta Weibull**	$0<x<\infty$
$\frac{x}{\omega}$	$\frac{x^{a-1}\left[1-\frac{x}{\omega}\right]^{b-1}}{\omega^{a} B(a,b)}$ **Three Parameter Beta** $E(X^m)=\frac{\omega^m B(m+a,b)}{B(a,b)}$	$0<x<\omega$
$1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$\frac{\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{a-1}\exp\left\{-\frac{be^{-d}}{k}\left[e^{kx}-1\right]\right\}}{B(a,b)} e^{-d}e^{kx}$ **Beta Gompertz**	$0<x<\infty$
$1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$\frac{\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{a-1}\exp\left\{-bAx-\frac{be^{-d}}{k}\left[e^{kx}-1\right]\right\}}{B(a,b)}\left\{A+e^{-d}e^{kx}\right\}$ **Beta Gompertz-Makeham**	$0<x<\infty$
$1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$\frac{\left[1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{a-1}\exp\left\{-bAx-\frac{bHx^2}{2}-\frac{be^{-d}}{k}\left[e^{kx}-1\right]\right\}}{B(a,b)}\left\{A+Hx+e^{-d}e^{kx}\right\}$	$0<x<\infty$

$F(x)$	$g(x)=\frac{[F(x)]^{a-1}[1-F(x)]^{b-1}}{B(a,b)}f(x), \quad a>0,\ b>0$	Support
$cx+dx^2$	$g(x)=\frac{\left[cx+dx^2\right]^{a-1}\left[1-cx-dx^2\right]^{b-1}}{B(a,b)}(c+2dx)$ $\frac{-c-\sqrt{c^2+4d}}{2d}<x<\frac{-c+\sqrt{c^2+4d}}{2d}$	
$1-(1+kx)^{-\frac{p}{k}}$	$\frac{\left[1-(1+kx)^{-\frac{p}{k}}\right]^{a-1}(1+kx)^{-\frac{pb}{k}-1}}{B(a,b)}p$	$0<x<\infty$
$1-e^{-\alpha x-\frac{\beta x^2}{2}}$	$\frac{\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{a-1}\left[e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{b-1}}{B(a,b)}\{\alpha+\beta x\}e^{-\alpha x-\frac{\beta x^2}{2}}$ **Beta Linear Exponential**	$0<x<\infty$

where $G(x)=\int_0^{F(x)}\frac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt$ is the Beta cdf.

Table 8.4. Beta Exponentiated Distributions Based on Burr Differential Equation

$[F(x)]^r,\ r>0$	$g(x)=\dfrac{r\,[F(x)]^{ra-1}\,[1-[F(x)]^r]^{b-1}}{B(a,b)}f(x),\ \ a>0,\ b>0$	Support
x^r	$\dfrac{rx^{ra-1}\,[1-x^r]^{b-1}}{B(a,b)}$ **Beta Exponentiated Standard Uniform/ GB1** $E(X^m)=\frac{B(\frac{m}{r}+a,b)}{B(a,b)}$	$0<x<1$ $r>m$
$[e^{-x}+1]^{-r}$	$\dfrac{re^{-x}\left\{[e^{-x}+1]^r-1\right\}^{b-1}}{B(a,b)\,[e^{-x}+1]^{r(a+b-1)+1}}$ **Beta Exponentiated Logistic**	$-\infty<x<\infty$
$[x^{-c}+1]^{-r}$	$\dfrac{rcx^{-c-1}\left[(x^{-c}+1)^r-1\right]^{b-1}}{B(a,b)\,[x^{-c}+1]^{r(a+b-1)+1}}$ **Beta Exponentiated Log-Logistic**	$0<x<\infty$
$\left[\left(\dfrac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-r}$	$\dfrac{r\left[\left\{\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right\}^r-1\right]^{b-1}(c-x)^{\frac{1}{c}-1}}{B(a,b)\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{r(a+b-1)+1}x^{\frac{1}{c}+1}}$	$0<x<c$
$[ke^{-\tan x}+1]^{-r}$	$\dfrac{r\left[\{ke^{-\tan x}+1\}^r-1\right]^{b-1}ke^{-\tan x}\sec^2 x}{B(a,b)\,[ke^{-\tan x}+1]^{r(a+b-1)+1}}$	$-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$
$\left[ke^{-c\sinh x}+1\right]^{-r}$	$\dfrac{r\left[\left\{ke^{-c\sinh x}+1\right\}^r-1\right]^{b-1}kce^{-c\sinh x}\cosh x}{B(a,b)\,[ke^{-c\sinh x}+1]^{r(a+b-1)+1}}$	$-\infty<x<\infty$

$[F(x)]^r,\ r>0$	$g(x)=\dfrac{r\,[F(x)]^{ra-1}\,[1-[F(x)]^r]^{b-1}}{B(a,b)}f(x),\ \ a>0,\ b>0$	Support
$2^{-r}\,[1+\tanh x]^{-r}$	$\dfrac{r\,[1+\tanh x]^{ra-1}\,[2^r-\{1+\tanh x\}^r]^{b-1}\,\mathrm{sech}^2\,x}{2^{r(a+b-1)}B(a,b)}$	$-\infty<x<\infty$
$\left\{1-\frac{2}{c\left[(1+e^x)^k-1\right]+2}\right\}^r$	$\dfrac{r\left\{c\left[(1+e^x)^k-1\right]\right\}^{ra-1}\left[1-\left\{\frac{c\left[(1+e^x)^k-1\right]}{c\left[(1+e^x)^k-1\right]+2}\right\}^r\right]^{b-1}}{B(a,b)\left\{c\left[(1+e^x)^k-1\right]+2\right\}^{ra+1}}2kce^x\,(1+e^x)^{k-1}$ **Beta Exponentiated Burr IX**	$-\infty<x<\infty$
$\left[1-e^{-x^2}\right]^r$	$\dfrac{r\left[1-e^{-x^2}\right]^{ra-1}\left[1-\left\{1-e^{-x^2}\right\}^r\right]^{b-1}}{B(a,b)}2xe^{-x^2}$	$0<x<\infty$
$\left\{1-[1+x^c]^{-k}\right\}^r$	$\dfrac{r\left\{[1+x^c]^k-1\right\}^{ra-1}\left[\{1+x^c\}^{kr}-\left\{[1+x^c]^k-1\right\}^r\right]^{b-1}}{[1+x^c]^{kr(a+b-1)+1}B(a,b)}kcx^{c-1}$ **Beta Exponentiated Burr XII**	$0<x<\infty$
$[cx]^r$	$\dfrac{rc^{ra}\,[x]^{ra-1}\,[1-\{cx\}^r]^{b-1}}{B(a,b)}$ $E(X^m)=\frac{B(\frac{m}{r}+a,b)}{c^mB(a,b)}$	$0<x<\dfrac{1}{c}$ $r>m$
$[1-e^{-cx}]^r$	$\dfrac{r\,[1-e^{-cx}]^{ra-1}\,\left[1-\{1-e^{-cx}\}^r\right]^{b-1}}{B(a,b)}ce^{-cx}$	$0<x<\infty$
$\left[1-\left(\dfrac{c-x}{x}\right)^{\frac{1}{c}}\right]^r$	$\dfrac{r\left[1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{ra-1}\left[1-\left\{1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right\}^r\right]^{b-1}}{B(a,b)}\,\dfrac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}}$	$\dfrac{c}{2}<x<c$

$[F(x)]^r,\ r>0$	$g(x)=\dfrac{r[F(x)]^{ra-1}[1-[F(x)]^r]^{b-1}}{B(a,b)}f(x),\ \ a>0,\ b>0$	Support
$[1-e^{-c\tan x}]^r$	$\dfrac{r[1-e^{-c\tan x}]^{ra-1}\left[1-\{1-e^{-c\tan x}\}^r\right]^{b-1}}{B(a,b)}ce^{-c\tan x}sec^2 x$	$-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$
$\left\{1-\left[\dfrac{1}{x}\right]^{\alpha}\right\}^r$	$\dfrac{r[1-x^{-\alpha}]^{ra-1}\left[1-\{1-x^{-\alpha}\}^r\right]^{b-1}}{B(a,b)}\alpha x^{-\alpha-1}$ **Beta Exponentiated Pareto (Type I)**	$1<x<\infty$
$\left\{1-\left[\dfrac{1}{1+x}\right]^{\alpha}\right\}^r$	$\dfrac{r\left[1-(1+x)^{-\alpha}\right]^{ra-1}\left[1-\left\{1-(1+x)^{-\alpha}\right\}^r\right]^{b-1}}{B(a,b)}\dfrac{\alpha}{[1+x]^{\alpha+1}}$ **Beta Exponentiated Pareto (Type II)**	$0<x<\infty$
$\left\{1-\dfrac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^r$	$\dfrac{r\left\{1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{ra-1}\left[1-\left\{1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^r\right]^{b-1}}{B(a,b)}\left\{\dfrac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}}+\dfrac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\}$ **Beta Exponentiated Pareto (Type III)**	$0<x<\infty$
$\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^r$	$\dfrac{r\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{ra-1}\left[1-\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^r\right]^{b-1}}{B(a,b)}\dfrac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}}$ **Beta Exponentiated Pareto (Type IV)**	$\mu<x<\infty$
$\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^r$	$\dfrac{r\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{ra-1}\left[1-\left\{1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right\}^r\right]^{b-1}}{B(a,b)}\dfrac{\alpha}{\beta}\left(\dfrac{x}{\beta}\right)^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$ **Beta Exponentiated Weibull**	$0<x<\infty$

$[F(x)]^r,\ r>0$	$g(x)=\dfrac{r\,[F(x)]^{ra-1}\,[1-[F(x)]^r]^{b-1}}{B(a,b)}f(x),\ \ a>0,\ b>0$	Support
$\left[1-e^{-\frac{x}{\beta}}\right]^r$	$\dfrac{r\left[1-e^{-\frac{x}{\beta}}\right]^{ra-1}\left[1-\left\{1-e^{-\frac{x}{\beta}}\right\}^r\right]^{b-1}}{B(a,b)}\dfrac{1}{\beta}e^{-\frac{x}{\beta}}$ **Beta Exponentiated Exponential**	$0<x<\infty$
$\left[1-e^{-\alpha\log x-\beta(\log x)^2}\right]^r$	$\dfrac{r\left[1-e^{-\alpha\log x-\beta(\log x)^2}\right]^{ra-1}\left[1-\left\{1-e^{-\alpha\log x-\beta(\log x)^2}\right\}^r\right]^{b-1}}{B(a,b)}\left[\frac{\alpha}{x}+\frac{2\beta\log x}{x}\right]e^{-\alpha\log x-\beta(\log x)^2}$ **Beta Exponentiated Benini**	$1<x<\infty$
$\left[\dfrac{2}{\pi}\arctan e^x\right]^r$	$\dfrac{r\left[\frac{2}{\pi}\arctan e^x\right]^{ra-1}\left[1-\left(\frac{2}{\pi}\arctan e^x\right)^r\right]^{b-1}}{B(a,b)}\dfrac{2e^x}{\pi}\left(\dfrac{1}{1+x^2}\right)$	$-\infty<x<\infty$
$\left[x-\dfrac{1}{2\pi}\sin 2\pi x\right]^r$	$\dfrac{r\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^{ra-1}\left[1-\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^r\right]^{b-1}}{B(a,b)}[1-\cos 2\pi x]$	$0<x<1$
$\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^r$	$\dfrac{r\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{ra-1}\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right\}^r\right]^{b-1}}{B(a,b)}\dfrac{\alpha\beta^\alpha}{(\beta+x)^{\alpha+1}}$ **Beta Exponentiated Lomax**	$0<x<\infty$
$[1-e^{-cx}]^r$	$\dfrac{r[1-e^{-cx}]^{ra-1}\left[1-\{1-e^{-cx}\}^r\right]^{b-1}}{B(a,b)}ce^{-cx}$ **Beta Exponentiated Exponential**	$0<x<\infty$

$[F(x)]^r,\ r>0$	$g(x)=\dfrac{r\,[F(x)]^{ra-1}\,[1-[F(x)]^r]^{b-1}}{B(a,b)}f(x),\ \ a>0,\ b>0$	Support
$\left[1-e^{-cx^\alpha}\right]^r$	$\dfrac{r\left[1-e^{-cx^\alpha}\right]^{ra-1}\left[1-\left\{1-e^{-cx^\alpha}\right\}^r\right]^{b-1}}{B(a,b)}c\alpha x^{\alpha-1}e^{-cx^\alpha}$ **Beta Exponentiated Weibull**	$0<x<\infty$
$\left[\dfrac{x}{\omega}\right]^r$	$\dfrac{rx^{ra-1}\left\{1-\left[\frac{x}{\omega}\right]^r\right\}^{b-1}}{\omega^{ra}B(a,b)}$ **GB1** $E(X^m)=\frac{\omega^m B(\frac{m}{r}+a,b)}{B(a,b)}$	$0<x<\omega$ $r>m$
$\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^r$	$g(x)=\dfrac{r\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{ra-1}\left[1-\left(1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^r\right]^{b-1}}{B(a,b)}e^{-d}e^{kx}\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$ $0<x<\infty$ **Beta Exponentiated Gompertz**	
$\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^r$	$g(x)=\dfrac{r\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{ra-1}\left[1-\left(1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^r\right]^{b-1}}{B(a,b)}\left\{A+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$ $0<x<\infty$ **Beta Exponentiated Gompertz-Makeham**	
$\left[1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^r$	$g(x)=\dfrac{r\left[1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{ra-1}\left[1-\left(1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^r\right]^{b-1}}{B(a,b)}\left\{A+Hx+e^{-d}e^{kx}\right\}$ $\times\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\},\ \ 0<x<\infty$	

$[F(x)]^r,\ r>0$	$g(x)=\dfrac{r\,[F(x)]^{ra-1}\,[1-[F(x)]^r]^{b-1}}{B(a,b)}f(x),\ \ a>0,\ b>0$	Support
$\left[cx+dx^2\right]^r$	$g(x)=\dfrac{r\left[cx+dx^2\right]^{ra-1}\left[1-\left\{cx+dx^2\right\}^r\right]^{b-1}}{B(a,b)}(c+2dx)$ $\dfrac{-c-\sqrt{c^2+4d}}{2d}<x<\dfrac{-c+\sqrt{c^2+4d}}{2d}$	
$\left[1-(1+kx)^{-\frac{p}{k}}\right]^r$	$\dfrac{rp\left[1-(1+kx)^{-\frac{p}{k}}\right]^{ra-1}\left[1-\left\{1-(1+kx)^{-\frac{p}{k}}\right\}^r\right]^{b-1}}{B(a,b)(1+kx)^{\frac{p}{k}+1}}$	$0<x<\infty$
$\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^r$	$\dfrac{r\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{ra-1}\left[1-\left\{1-e^{-\alpha x-\frac{\beta x^2}{2}}\right\}^r\right]^{b-1}}{B(a,b)}\{\alpha+\beta x\}\,e^{-\alpha x-\frac{\beta x^2}{2}}$ **Beta Exponentiated Linear Exponential**	$0<x<\infty$

where $G(x)=\int_0^{[F(x)]^r}\dfrac{t^{a-1}(1-t)^{b-1}}{B(a,b)}dt$ is the Beta Exponentiated cdf.

Table 8.5. Minimum and Maximum Order Statistics of Burr Generated Distributions

$F(x)$	$g_1(x) = n\left[1-F(x)\right]^{n-1} f(x)$	$g_n(x) = n\left[F(x)\right]^{n-1} f(x)$	Support
x **Standard Uniform (Burr I)**	$n\left[1-x\right]^{n-1}$ $E(X_1^m)=nB(m+1,n)$	nx^{n-1} $E(X_n^m)=nB(m+n,1)$	$0<x<1$
$\left[e^{-x}+1\right]^{-1}$ **Logistic**	$\dfrac{ne^{-nx}}{\left[e^{-x}+1\right]^{n+1}}$	$\dfrac{ne^{-x}}{\left[e^{-x}+1\right]^{n+1}}$	$-\infty<x<\infty$
$\left[x^{-c}+1\right]^{-1}$ **Log-Logistic**	$\dfrac{ncx^{-cn-1}}{\left[x^{-c}+1\right]^{n+1}}$ $E(X_1^m)=nB\left(1+\frac{m}{c},n-\frac{m}{c}\right)$	$\dfrac{ncx^{-c-1}}{\left[x^{-c}+1\right]^{n+1}}$ $E(X_n^m)=nB\left(n+\frac{m}{c},1-\frac{m}{c}\right)$	$0<x<\infty$ $c>m$
$\left[\left(\dfrac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-1}$	$\dfrac{n\left(c-x\right)^{\frac{n}{c}-1}}{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{n+1} x^{\frac{n}{c}+1}}$	$\dfrac{n\left(c-x\right)^{\frac{1}{c}-1}}{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{n+1} x^{\frac{1}{c}+1}}$	$0<x<c$
$\left[ke^{-\tan x}+1\right]^{-1}$	$\dfrac{n\left[ke^{-\tan x}\right]^{n}\sec^2 x}{\left[ke^{-\tan x}+1\right]^{n+1}}$	$\dfrac{n\left[ke^{-\tan x}\right]\sec^2 x}{\left[ke^{-\tan x}+1\right]^{n+1}}$	$-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$
$\left[ke^{-c\sinh x}+1\right]^{-1}$	$\dfrac{nc\left[ke^{-c\sinh x}\right]^{n}\cosh x}{\left[ke^{-c\sinh x}+1\right]^{n+1}}$	$\dfrac{nc\left[ke^{-c\sinh x}\right]\cosh x}{\left[ke^{-c\sinh x}+1\right]^{n+1}}$	$-\infty<x<\infty$

$F(x)$	$g_1(x) = n[1-F(x)]^{n-1} f(x)$	$g_n(x) = n[F(x)]^{n-1} f(x)$	Support
$2^{-1}[1+\tanh x]$	$\dfrac{n[1-\tanh x]^{n-1}\operatorname{sech}^2 x}{2^n}$	$\dfrac{n[1+\tanh x]^{n-1}\operatorname{sech}^2 x}{2^n}$	$-\infty < x < \infty$
$1-\frac{2}{c\left[(1+e^x)^k-1\right]+2}$ **Burr IX**	$\dfrac{n2^n kce^x(1+e^x)^{k-1}}{\left\{c\left[(1+e^x)^k-1\right]+2\right\}^{n+1}}$	$\dfrac{n2kce^x(1+e^x)^{k-1}}{\left\{c\left[(1+e^x)^k-1\right]+2\right\}^{n+1}} \times \left\{c\left[(1+e^x)^k-1\right]\right\}^{n-1}$	$-\infty < x < \infty$
$1-e^{-x^2}$	$e^{-nx^2}2xn$ $E(X_1^m)=\frac{\Gamma(\frac{m}{2}+1)}{n^{\frac{m}{2}}}$	$\left[1-e^{-x^2}\right]^{n-1} e^{-x^2}2xn$	$0 < x < \infty$
$1-[1+x^c]^{-k}$ **Burr XII**	$\dfrac{nkcx^{c-1}}{[1+x^c]^{kn+1}}$ **GB2** $E(X_1^m)=nkB\left(kn-\frac{m}{c},1+\frac{m}{c}\right)$	$\dfrac{n\left\{[1+x^c]^k-1\right\}^{n-1}kcx^{c-1}}{[1+x^c]^{kn+1}}$	$0 < x < \infty$ $c > m$
cx	$nc[1-cx]^{n-1}$ $E(X_1^m)=\frac{nB(m+1,n)}{c^m}$	$nc^n[x]^{n-1}$ $E(X_n^m)=\frac{nB(m+n,1)}{c^m}$	$0 < x < \dfrac{1}{c}$
$1-e^{-cx}$	nce^{-ncx} $E(X_1^m)=\frac{\Gamma(m+1)}{(nc)^m}$	$[1-e^{-cx}]^{n-1}nce^{-cx}$	$0 < x < \infty$

$F(x)$	$g_1(x) = n[1-F(x)]^{n-1} f(x)$	$g_n(x) = n[F(x)]^{n-1} f(x)$	Support
$1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}$	$\frac{n(c-x)^{\frac{n}{c}-1}}{x^{\frac{n}{c}+1}}$	$\frac{n(c-x)^{\frac{1}{c}-1}}{x^{\frac{n}{c}+1}} \times \left[x^{\frac{1}{c}} - (c-x)^{\frac{1}{c}}\right]^{n-1}$	$\frac{c}{2} < x < c$
$1-e^{-c\tan x}$	$nce^{-nc\tan x} sec^2 x$	$nce^{-c\tan x} sec^2 x \times [1-e^{-c\tan x}]^{n-1}$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$1-\left[\frac{1}{x}\right]^{\alpha}$ **Pareto (Type I)**	$n\alpha x^{-\alpha n-1}$ $E(X_1^m) = nB\left(n-\frac{m}{\alpha},1\right)$	$n\left[1-x^{-\alpha}\right]^{n-1} \alpha x^{-\alpha-1}$ $E(X_n^m) = nB\left(1-\frac{m}{\alpha},n\right)$	$1 < x < \infty$ $\alpha > m$
$1-\left[\frac{1}{1+x}\right]^{\alpha}$ **Pareto (Type II)**	$n\alpha(1+x)^{-\alpha n-1}$	$n\left[1-(1+x)^{-\alpha}\right]^{n-1} \times \frac{\alpha}{[1+x]^{\alpha+1}}$	$0 < x < \infty$
$1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}$ **Pareto (Type III)**	$n\left[\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right]^{n-1} \times \left\{\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}} + \frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\}$	$n\left[1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right]^{n-1} \times \left\{\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}} + \frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\}$	$0 < x < \infty$

$F(x)$	$g_1(x) = n\left[1-F(x)\right]^{n-1} f(x)$	$g_n(x) = n\left[F(x)\right]^{n-1} f(x)$	Support
$1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}$ **Pareto (Type IV)**	$n\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha n-1}$ $\times\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta}$	$n\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{n-1}$ $\times\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}}$	$\mu<x<\infty$
$1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$ **Weibull**	$ne^{-n\left(\frac{x}{\beta}\right)^{\alpha}}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}$ $E(X_1^m)=\frac{\beta^m}{n^{\frac{m}{\alpha}}}\Gamma\left(1+\frac{m}{\alpha}\right)$	$ne^{-\left(\frac{x}{\beta}\right)^{\alpha}}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\times$ $\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{n-1}$	$0<x<\infty$
$1-e^{-\frac{x}{\beta}}$ **Exponential**	$ne^{-\frac{nx}{\beta}}\frac{1}{\beta}$ $E(X_1^m)=\frac{\beta^m}{n^m}\Gamma(1+m)$	$n\left[1-e^{-\frac{x}{\beta}}\right]^{n-1}e^{-\frac{x}{\beta}}\frac{1}{\beta}$	$0<x<\infty$
$1-e^{-\alpha\log x-\beta(\log x)^2}$ **Benini**	$n\left[\frac{\alpha}{x}+\frac{2\beta\log x}{x}\right]\times$ $e^{-n\alpha\log x-n\beta(\log x)^2}$	$n\left[\frac{\alpha}{x}+\frac{2\beta\log x}{x}\right]e^{-\alpha\log x-\beta(\log x)^2}\times$ $\left[1-e^{-\alpha\log x-\beta(\log x)^2}\right]^{n-1}$	$1<x<\infty$
$\frac{2}{\pi}\arctan e^x$	$n\left[1-\frac{2}{\pi}\arctan e^x\right]^{n-1}\times$ $\frac{2e^x}{\pi}\left(\frac{1}{1+x^2}\right)$	$n\left[\frac{2}{\pi}\arctan e^x\right]^{n-1}\times$ $\frac{2e^x}{\pi}\left(\frac{1}{1+x^2}\right)$	$-\infty<x<\infty$

$F(x)$	$g_1(x) = n\,[1-F(x)]^{n-1}\,f(x)$	$g_n(x) = n\,[F(x)]^{n-1}\,f(x)$	Support
$x-\frac{1}{2\pi}\sin 2\pi x$	$n\left[1-\left(x-\frac{1}{2\pi}\sin 2\pi x\right)\right]^{n-1}$ $\times\ [1-\cos 2\pi x]$	$n\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^{n-1}$ $\times\ [1-\cos 2\pi x]$	$0<x<1$
$1-\left(1+\frac{x}{\beta}\right)^{-\alpha}$ **Lomax**	$\frac{n\alpha\left(1+\frac{x}{\beta}\right)^{-\alpha n-1}}{\beta}$	$\frac{n\alpha\left(1+\frac{x}{\beta}\right)^{-\alpha-1}}{\beta}\times$ $\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{n-1}$	$0<x<\infty$
$1-e^{-cx}$ **Exponential**	nce^{-ncx} $E(X_1^m)=\frac{1}{[nc]^m}\Gamma(1+m)$	$nc\,[1-e^{-cx}]^{n-1}\,e^{-cx}$	$0<x<\infty$
$1-e^{-cx^{\alpha}}$ **Weibull**	$nc\alpha x^{\alpha-1}e^{-ncx^{\alpha}}$ $E(X_1^m)=\frac{\Gamma\left(1+\frac{m}{\alpha}\right)}{(nc)^{\frac{m}{\alpha}}}$	$nc\alpha x^{\alpha-1}\left[1-e^{-cx^{\alpha}}\right]^{n-1}e^{-cx^{\alpha}}$	$0<x<\infty$
$\frac{x}{\omega}$	$n\left[1-\frac{x}{\omega}\right]^{n-1}$ $E(X_1^m)=n\omega^m B(m+1,n)$	$\frac{rx^{r-1}}{\omega^r}$ $E(X_n^m)=n\omega^m B(m+n,1)$	$0<x<\omega$

$F(x)$	$g_1(x) = n\left[1-F(x)\right]^{n-1} f(x)$	$g_n(x) = n\left[F(x)\right]^{n-1} f(x)$	Support
$1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$ne^{-d}e^{kx}\exp\left\{-\frac{ne^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$ne^{-d}e^{kx}\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\} \times$ $\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{n-1}$	$0 < x < \infty$
$1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$n\left\{A+e^{-d}e^{kx}\right\}\exp\left\{-nAx-\frac{ne^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$n\left\{A+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\} \times$ $\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{n-1}$	$0 < x < \infty$
$1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$g_1(x)=n\left\{A+Hx+e^{-d}e^{kx}\right\}\exp\left\{-nAx-\frac{nHx^2}{2}-\frac{ne^{-d}}{k}\left[e^{kx}-1\right]\right\}$ $g_n(x)=n\left\{A+Hx+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\left[1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{n-1}$		$0 < x < \infty$
$cx+dx^2$	$g_1(x) = n\left[1-cx-dx^2\right]^{n-1}(c+2dx)$, $\frac{-c-\sqrt{c^2+4d}}{2d} < x < \frac{-c+\sqrt{c^2+4d}}{2d}$ $g_n(x) = n\left[cx+dx^2\right]^{n-1}(c+2dx)$		
$1-(1+kx)^{-\frac{p}{k}}$	$pn(1+kx)^{-\frac{pn}{k}-1}$	$pn(1+kx)^{-\frac{p}{k}-1} \times$ $\left[1-(1+kx)^{-\frac{p}{k}}\right]^{n-1}$	$0 < x < \infty$
$1-e^{-\alpha x-\frac{\beta x^2}{2}}$ **Linear Exponential**	$n\left[e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{n-1} \times$ $\{\alpha+\beta x\}e^{-\alpha x-\frac{\beta x^2}{2}}$	$n\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{n-1} \times$ $\{\alpha+\beta x\}e^{-\alpha x-\frac{\beta x^2}{2}}$	$0 < x < \infty$

where $g_i(x) = \frac{[F(x)]^{i-1}[1-F(x)]^{n-i}}{B(i, n-i+1)} f(x)$ is the i^{th} order statistic of $F(x)$.

Table 8.6. Minimum and Maximum Order Statistics of Exponentiated Distributions Based on Burr Differential Equation

$[F(x)]^r, \quad r>0$	$g_1(x) = rn[F(x)]^{r-1}\,\{1-[F(x)]^r\}^{n-1} f(x)$	$g_n(x) = rn\,[F(x)]^{rn-1}\,f(x)$	Support
x^r **Exponentiated Standard Uniform (Burr I)**	$rnx^{r-1}\,[1-x^r]^{n-1}$ $E(X_1^m)=nB(\frac{m}{r}+1,n)$	rnx^{rn-1} $E(X_n^m)=nB(\frac{m}{r}+n,1)$	$0<x<1$ $r>m$
$[e^{-x}+1]^{-r}$ **Type I Generalized Logistic/ Burr II**	$\frac{rne^{-x}\left\{[e^{-x}+1]^r-1\right\}^{n-1}}{[e^{-x}+1]^{rn+1}}$ **Type I Beta Generalized Logistic**	$\frac{rne^{-x}}{[e^{-x}+1]^{rn+1}}$	$-\infty<x<\infty$
$[x^{-c}+1]^{-r}$ **Exponentiated Log-Logistic/ Burr III**	$\frac{rncx^{-c-1}\left[(x^{-c}+1)^r-1\right]^{n-1}}{[x^{-c}+1]^{rn+1}}$	$\frac{rncx^{-c-1}}{[x^{-c}+1]^{rn+1}}$ $E(X_n^m)=rnB\left(rn+\frac{m}{c},1-\frac{m}{c}\right)$	$0<x<\infty$ $c>m$

$[F(x)]^r, \quad r>0$	$g_1(x) = rn[F(x)]^{r-1}\ \{1-[F(x)]^r\}^{n-1} f(x)$	$g_n(x) = rn\,[F(x)]^{rn-1} f(x)$	Support
$\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-r}$ **Burr IV**	$\frac{rn\,(c-x)^{\frac{1}{c}-1}}{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{rn+1} x^{\frac{1}{c}+1}}\times \left[\left\{\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right\}^{r}-1\right]^{n-1}$	$\frac{rn\,(c-x)^{\frac{1}{c}-1}}{\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{rn+1} x^{\frac{1}{c}+1}}$	$0<x<c$
$[ke^{-\tan x}+1]^{-r}$ **Burr V**	$\frac{rnke^{-\tan x}\sec^2 x}{[ke^{-\tan x}+1]^{rn+1}}\times \left[\{ke^{-\tan x}+1\}^{r}-1\right]^{n-1}$	$\frac{rnke^{-\tan x}\sec^2 x}{[ke^{-\tan x}+1]^{rn+1}}$	$-\frac{\pi}{2}<x<\frac{\pi}{2}$
$\left[ke^{-c\sinh x}+1\right]^{-r}$ **Burr VI**	$\frac{rnkce^{-c\sinh x}\cosh x}{[ke^{-c\sinh x}+1]^{rn+1}}\times \left[\left\{ke^{-c\sinh x}+1\right\}^{r}-1\right]^{n-1}$	$\frac{rnkce^{-c\sinh x}\cosh x}{[ke^{-c\sinh x}+1]^{rn+1}}$	$-\infty<x<\infty$
$2^{-r}[1+\tanh x]^{r}$ **Burr VII**	$\frac{rn\,[1+\tanh x]^{r-1}\operatorname{sech}^2 x}{2^{rn}}\times [2^r-\{1+\tanh x\}^r]^{n-1}$	$\frac{rn\,[1+\tanh x]^{rn-1}\operatorname{sech}^2 x}{2^{rn}}$	$-\infty<x<\infty$
$\left\{1-\frac{2}{c\left[(1+e^x)^k-1\right]+2}\right\}^{r}$ **Exponentiated Burr IX**	$\frac{rn\left\{c\left[(1+e^x)^k-1\right]\right\}^{r-1}}{\left\{c\left[(1+e^x)^k-1\right]+2\right\}^{rn+1}}\times 2kce^x(1+e^x)^{k-1}\times \left[1-\left\{\frac{c\left[(1+e^x)^k-1\right]}{c\left[(1+e^x)^k-1\right]+2}\right\}^{r}\right]^{n-1}$	$\frac{rn\left\{c\left[(1+e^x)^k-1\right]\right\}^{rn-1}}{\left\{c\left[(1+e^x)^k-1\right]+2\right\}^{rn+1}}\times 2kce^x(1+e^x)^{k-1}$	$-\infty<x<\infty$

$[F(x)]^r, \quad r>0$	$g_1(x) = rn[F(x)]^{r-1}\{1-[F(x)]^r\}^{n-1} f(x)$	$g_n(x) = rn[F(x)]^{rn-1} f(x)$	Support
$\left[1-e^{-x^2}\right]^r$ **Burr X**	$2xrne^{-x^2}\left[1-e^{-x^2}\right]^{r-1} \times$ $\left[1-\left\{1-e^{-x^2}\right\}^r\right]^{n-1}$	$2xrne^{-x^2}\left[1-e^{-x^2}\right]^{rn-1}$	$0<x<\infty$
$\left\{1-[1+x^c]^{-k}\right\}^r$ **Exponentiated Burr XII**	$\frac{rn\left\{[1+x^c]^k-1\right\}^{r-1}}{[1+x^c]^{krn+1}} kcx^{c-1} \times$ $\left[\{1+x^c\}^{kr}-\left\{[1+x^c]^k-1\right\}^r\right]^{n-1}$	$\frac{rn\left\{[1+x^c]^k-1\right\}^{rn-1}}{[1+x^c]^{krn+1}} kcx^{c-1}$	$0<x<\infty$
$[cx]^r$	$rnc^r[x]^{r-1}[1-\{cx\}^r]^{n-1}$ $E(X_1^m)=\frac{n\beta\left(\frac{m}{r}+1,n\right)}{c^m}$	$rnc^{rn}[x]^{rn-1}$ $E(X_n^m)=\frac{n\beta\left(\frac{m}{r}+n,1\right)}{c^m}$	$0<x<\frac{1}{c}$ $r>m$
$[1-e^{-cx}]^r$	$rn[1-e^{-cx}]^{r-1}ce^{-cx} \times$ $[1-\{1-e^{-cx}\}^r]^{n-1}$	$rn[1-e^{-cx}]^{rn-1}ce^{-cx}$	$0<x<\infty$
$\left[1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^r$	$rn\left[1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{r-1} \times$ $\frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}} \times$ $\left[1-\left\{1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right\}^r\right]^{n-1}$	$rn\left[1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^{rn-1} \times$ $\frac{(c-x)^{\frac{1}{c}-1}}{x^{\frac{1}{c}+1}}$	$\frac{c}{2}<x<c$
$[1-e^{-c\tan x}]^r$	$rnce^{nc\tan x}sec^2x$	$rnce^{-c\tan x}sec^2x \times$ $[1-e^{-c\tan x}]^{n-1}$	$-\frac{\pi}{2}<x<\frac{\pi}{2}$

$[F(x)]^r, \quad r>0$	$g_1(x) = rn[F(x)]^{r-1}\ \{1-[F(x)]^r\}^{n-1} f(x)$	$g_n(x) = rn\,[F(x)]^{rn-1} f(x)$	Support
$\left\{1-\left[\frac{1}{x}\right]^{\alpha}\right\}^{r}$ **Exponentiated Pareto (Type I)**	$rn\,[1-x^{-\alpha}]^{r-1}\,\alpha x^{-\alpha-1} \times$ $\left[1-\{1-x^{-\alpha}\}^{r}\right]^{n-1}$	$rn\,[1-x^{-\alpha}]^{rn-1}\,\alpha x^{-\alpha-1}$ $E(X_n^m)=rnB\left(1-\frac{m}{\alpha},rn\right)$	$1<x<\infty$ $\alpha>m$
$\left\{1-\left[\frac{1}{1+x}\right]^{\alpha}\right\}^{r}$ **Exponentiated Pareto (Type II)**	$\frac{rn\alpha\,\left[1-(1+x)^{-\alpha}\right]^{r-1}}{[1+x]^{\alpha+1}} \times$ $\left[1-\left\{1-(1+x)^{-\alpha}\right\}^{r}\right]^{n-1}$	$\frac{rn\alpha\,\left[1-(1+x)^{-\alpha}\right]^{rn-1}}{[1+x]^{\alpha+1}}$	$0<x<\infty$
$\left\{1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{r}$ **Exponentiated Pareto (Type III)**	$rn\left\{1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{r-1}\left[1-\left\{1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{r}\right]^{n-1}$ $\times\left\{\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}}+\frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\}$	$rn\left\{1-\frac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^{rn-1}$ $\times\left\{\frac{\alpha e^{-\beta x}}{[1+x]^{\alpha+1}}+\frac{\beta e^{-\beta x}}{[1+x]^{\alpha}}\right\}$	$0<x<\infty$
$\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{r}$ **Exponentiated Pareto (Type IV)**	$rn\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{r-1}$ $\times\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}}$ $\times\left[1-\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{r}\right]^{n-1}$	$rn\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^{rn-1}$ $\times\frac{\alpha(x-\mu)^{\frac{1}{\beta}-1}}{\beta\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{\alpha+1}}$	$\mu<x<\infty$

$[F(x)]^r, \quad r>0$	$g_1(x) = rn[F(x)]^{r-1}\ \{1-[F(x)]^r\}^{n-1} f(x)$	$g_n(x) = rn\,[F(x)]^{rn-1} f(x)$	Support
$\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{r}$ **Exponentiated Weibull**	$rne^{-\left(\frac{x}{\beta}\right)^{\alpha}}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} \times$ $\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{r-1} \times$ $\left[1-\left\{1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right\}^{r}\right]^{n-1}$	$rne^{-\left(\frac{x}{\beta}\right)^{\alpha}}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} \times$ $\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^{rn-1}$	$0<x<\infty$
$\left[1-e^{-\frac{x}{\beta}}\right]^{r}$ **Exponentiated Exponential**	$rn\left[1-e^{-\frac{x}{\beta}}\right]^{r-1}\frac{1}{\beta}e^{-\frac{x}{\beta}} \times$ $\left[1-\left\{1-e^{-\frac{x}{\beta}}\right\}^{r}\right]^{n-1}$	$rn\left[1-e^{-\frac{x}{\beta}}\right]^{rn-1}\frac{1}{\beta}e^{-\frac{x}{\beta}}$	$0<x<\infty$
$\left[1-e^{-\alpha\log x-\beta(\log x)^2}\right]^{r}$ **Exponentiated Benini**	$rn\left[\frac{\alpha}{x}+\frac{2\beta\log x}{x}\right]e^{-\alpha\log x-\beta(\log x)^2}$ $\times\left[1-e^{-\alpha\log x-\beta(\log x)^2}\right]^{r-1}$ $\times\left[1-\left\{1-e^{-\alpha\log x-\beta(\log x)^2}\right\}^{r}\right]^{n-1}$	$rn\left[\frac{\alpha}{x}+\frac{2\beta\log x}{x}\right]e^{-\alpha\log x-\beta(\log x)^2}$ $\times\left[1-e^{-\alpha\log x-\beta(\log x)^2}\right]^{rn-1}$	$1<x<\infty$
$\left[\frac{2}{\pi}\arctan e^{x}\right]^{r}$ **Burr VIII**	$rn\left[1-\left(\frac{2}{\pi}\arctan e^{x}\right)^{r}\right]^{n-1} \times$ $\left[\frac{2}{\pi}\arctan e^{x}\right]^{r-1}\frac{2e^{x}}{\pi}\left(\frac{1}{1+x^2}\right)$	$rn\left[\frac{2}{\pi}\arctan e^{x}\right]^{rn-1} \times$ $\frac{2e^{x}}{\pi}\left(\frac{1}{1+x^2}\right)$	$-\infty<x<\infty$

$[F(x)]^r, \quad r>0$	$g_1(x) = rn[F(x)]^{r-1}\ \{1-[F(x)]^r\}^{n-1} f(x)$	$g_n(x) = rn\,[F(x)]^{rn-1}\, f(x)$	Support
$\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^r$ **Burr XI**	$rn\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^{r-1}$ $\times\ [1-\cos 2\pi x]$ $\times\left[1-\left(x-\frac{1}{2\pi}\sin 2\pi x\right)\right]^{n-1}$	$rn\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^{rn-1}$ $\times\ [1-\cos 2\pi x]$	$0<x<1$
$\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^r$ **Exponentiated Lomax**	$rn\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{r-1}\times$ $\frac{\alpha\beta^\alpha}{(\beta+x)^{\alpha+1}}\times$ $\left[1-\left\{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right\}^r\right]^{n-1}$	$rn\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{rn-1}\times$ $\frac{\alpha\beta^\alpha}{(\beta+x)^{\alpha+1}}$	$0<x<\infty$
$[1-e^{-cx}]^r$ **Exponentiated Exponential**	$rn\,[1-e^{-cx}]^{r-1}\,ce^{-cx}\times$ $[1-\{1-e^{-cx}\}^r]^{n-1}$	$rn\,[1-e^{-cx}]^{rn-1}\,ce^{-cx}$	$0<x<\infty$
$\left[1-e^{-cx^\alpha}\right]^r$ **Exponentiated Weibull**	$rn\left[1-e^{-cx^\alpha}\right]^{r-1}c\alpha x^{\alpha-1}e^{-cx^\alpha}$ $\left[1-\left\{1-e^{-cx^\alpha}\right\}^r\right]^{n-1}$	$rn\left[1-e^{-cx^\alpha}\right]^{rn-1}c\alpha x^{\alpha-1}e^{-cx^\alpha}$	$0<x<\infty$
$\left[\frac{x}{\omega}\right]^r$	$\frac{rnx^{r-1}\left\{1-\left[\frac{x}{\omega}\right]^r\right\}^{n-1}}{\omega^r}$ $E(X_1^m)=n\omega^m B\left(\frac{m}{r}+1,n\right)$	$\frac{rnx^{rn-1}}{\omega^{rn}}$ $E(X_n^m)=n\omega^m B\left(\frac{m}{r}+n,1\right)$	$0<x<\omega$ $r>m$

$[F(x)]^r, \quad r>0$	$g_1(x) = rn[F(x)]^{r-1}\ \{1-[F(x)]^r\}^{n-1} f(x)$	$g_n(x) = rn\,[F(x)]^{rn-1}\,f(x)$	Support
$\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^r$	$g_1(x)=rn\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{r-1}\left[1-\left(1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^r\right]^{n-1} e^{-d}e^{kx}\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$g_n(x) = rn\left[1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{rn-1} e^{-d}e^{kx}\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$,	$0<x<\infty$
$\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^r$	$g_1(x)=rn\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{r-1}\left[1-\left(1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^r\right]^{n-1}\left\{A+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$g_n(x)=rn\left[1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{rn-1}\left\{A+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$,	$0<x<\infty$
$\left[1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^r$	$g_1(x)=rn\left[1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{r-1}\left[1-\left(1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^r\right]^{n-1}$ $\times\left\{A+Hx+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$g_n(x)=rn\left[1-\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right]^{rn-1}\left\{A+Hx+e^{-d}e^{kx}\right\}\exp\left\{-Ax-\frac{Hx^2}{2}-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$,	$0<x<\infty$
$\left[cx+dx^2\right]^r$	$g_1(x) = rn\left[cx+dx^2\right]^{r-1}\left[1-\left\{cx+dx^2\right\}^r\right]^{n-1}(c+2dx)$	$g_n(x) = rn\left[cx+dx^2\right]^{rn-1}(c+2dx)$,	$\frac{-c-\sqrt{c^2+4d}}{2d}<x<\frac{-c+\sqrt{c^2+4d}}{2d}$
$\left[1-(1+kx)^{-\frac{p}{k}}\right]^r$	$\frac{rnp\left[1-(1+kx)^{-\frac{p}{k}}\right]^{r-1}}{(1+kx)^{\frac{p}{k}+1}}\times$ $\left[1-\left\{1-(1+kx)^{-\frac{p}{k}}\right\}^r\right]^{n-1}$	$\frac{rnp\left[1-(1+kx)^{-\frac{p}{k}}\right]^{rn-1}}{(1+kx)^{\frac{p}{k}+1}}$	$0<x<\infty$

$[F(x)]^r, \quad r>0$	$g_1(x) = rn[F(x)]^{r-1}\ \{1-[F(x)]^r\}^{n-1} f(x)$	$g_n(x) = rn\,[F(x)]^{rn-1} f(x)$	Support
$\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^r$ **Exponentiated Linear Exponential**	$rn\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{r-1} \times$ $\{\alpha+\beta x\}\, e^{-\alpha x-\frac{\beta x^2}{2}} \times$ $\left[1-\left\{1-e^{-\alpha x-\frac{\beta x^2}{2}}\right\}^r\right]^{n-1}$	$rn\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^{rn-1} \times$ $\{\alpha+\beta x\}\, e^{-\alpha x-\frac{\beta x^2}{2}}$	$0<x<\infty$

where $g_i(x) = \dfrac{r\,[F(x)]^{ri-1}\,\{1-[F(x)]^r\}^{n-i}}{B(i,n-i+1)} f(x)$ is the i^{th} order statistic of $[F(x)]^r$.

References

Abdel-Ghaly, A.A., Al-Dayian G.R. and Al-Kashkari F.H. (1997). The use of Burr Type XII Distribution on software reliability growth modelling. *Microelectronics Reliability*, 37(2), 305-313.

Adler, M.N. (1867). Memoir of the late Benjamin Gompertz. *Journal of the Institute of Actuaries*, 13, 1-20.

Ahsanullah, M., Nevzorov, V. B., and Shakil, M. (2013). *An introduction to order statistics*. Atlantis Press.

Ahuja, J.C. and Nash, S.W. (1967). The generalized Gompertz-Verhulst family of distributions. *Sankhyā*, A29, 141-156.

Alzaatreh A, Lee C, Famoye F. (2013). Weibull-Pareto distribution and its applications. *Communications in Statistics–Theory and Methods*, 42, 1–19.

Al-Hussaini, E.K. and Jaheen, Z. F. (1992). Bayesian estimation of the parameters, reliability and failure rate functions of the Burr type XII failure model. *Journal of Statistital Computation and Simulation*, 41, 31-40.

Al-Hussaini, E.K., Jaheen, Z.F. and Mousa, M.A.M.H. (1992). Estimamion under the Burr type XII failure model based on censored data: A comparative study, *Test*, 1, 47-60.

Al-Hussaini, E.K. and Jaheen, Z.F. (1994). Approximate Bayes estimates applied to the Burr model. *Communication in Statistics -Theory and Methods*, 23, 99-121.

Al-Hussaini, E.K. and Jaheen, Z.F. (1995). Bayesian prediction bounds for the Burr XII failure model. *Communication in Statistics -Theory and Methods*, 24, 1829-1842.

Arnold, B. C. (1983). Pareto Distributions. Fairland, MD: International Cooperative Publishing House.

Azzalini A. (1985). A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics*, 12, 171–178.

Babbage, C. (1823). On the Tables of Single and Annual Assurance Premiums. *Journal of the Institute of Actuaries*, 6, 185.

Bell, C., Klonner, S. and Moos, W. (1999). Inequality and Economic Performance in Rural India since 1955. Working paper, South-Asia Institute, Department of Development Economics, Universität Heidelberg, Germany.

Benini, R. (1905). I diagrammi a scala logaritmica (a priposito della graduazione per valore delle successioni ereditarie in Italia, Francia e Inghilterra). *Giornale degli Economisti*, Series II, 16, 222-231.

Bernoulli, James. *Acta erunditorum*, 1695, Ostwald's Klassiker, No. 46, Engelmann, Leipzig, 1894.

Bernoulli, James. *Opera omnia*, ed., G. Cramer, 4 vols, Lausanne, 1742.

Bernoulli, John. *Die Differentialrechnung aus dem Jahre*, 1691/92. Ostawald's Klassiker, No. 211. Leipzig, 1924.

Bernoulli, John. *Acta erunditorum*, 1694, Ostwald's Klassiker, No. 46. Engelmann, Leipzig, 1894.

Bernoulli, John. *Opera omnia*, ed. G. Cramer, 4 vols., Lausanne, 1742.

Bordley, R.F., McDonald, J. B. and Mantrala, A. (1996). Something new, something old: Parametric models for the size distribution of income. *Journal of Income Distribution*, 6, 91-103.

Burr, I.W. (1942). Cumulative frequency functions. *Annals of Mathematical Statistics*, 13, 215-232.

Burr, I.W. (1967 b). The effects of non-normality on constants for X and R charts. *Industrial Quality Control*, 24, 563-569.

Burr, I.W. (1968). On a general system of distributions, III. The simple range. *Journal of the American Statistical Association*, 63, 636-643.

Burr, I.W. and Cislak, P.J. (1968). On a general system of distributions: I. Its curve-shaped characteristics; II. The sample median. *Journal of the American Statistical Association*, 63, 627-635.

Chernobai, A.S., Fabozzi, F.J. and Rachev, S.T. (2007). *Operational Risk: A Guide to Basel II Capital Requirements, Models and Analysis.* John Wiley & Sons, New York, USA.

Chiang, Ch. L. (1968). Introduction to Stochastic Prcoesses in Biostatistics. Wiley & Sons, Inc., New York, London, Sydney. 301 S, 1 Abb, 25 Tab, Preis 131 s: Enderlein, G. (1970), *Biometrical Journal*, 12, 362.

Clark, R., Cox S. and Laslett G. (1999). Generalizations of power-law distributions applicable to sampled fault-trace lengths: model choice, parameter estimation and caveats. *Geophysical Journal International*, 136, 357-372.

Cook, R.D. and Johnson, M. E. (1986). Generalized Burr-Pareto-Logistic distribution with application to a uranium exploration data set. *Technometrics*, 28, 123-131.

Cooray K, Ananda MMA. (2005). Modeling actuarial data with a composite lognormal-Pareto model. *Scandinavian Actuarial Journal*, 5, 321–334.

Cowell, F.A., and Victoria-Feser, M.-P. (2006). Distributional dominance with trimmed data. *Journal of Business and Economic Statistics*, 24, 291-300.

Cummins, J.D. and Freifelder, L.R. (1978). A comparative analysis of alternanive maximum probable yearly aggregate loss estimators. *Journal of Risk and Insurance*, 35, 27-52.

Cummins, J.D., Dionne, G., McDonald, J. B. and Pritchett, B. M. (1990). Applications of the GB2 family of distributions in modeling insurance loos processes. *Insurance: Mathematics & Economics*, 9, 257-272.

Dagum, C. (1977). A new model of personal income distribution: Specification and estimation. *Economie Appliquée*, 30, 413-437.

Dagum, C. (1983). Income distribution models. In: S. Kotz, N.L. Johnson, and C. Read (eds.): *Encyclopedoa of Statistical Sciences*, Vol. 4. New York: John Wiley, pp. 27-34.

David, H. (1981). *Order statistics.* John Wiley Sons, New York.

De Moivre, A. (1725). *Annuities upon Lives: Or, The Valuation of Annuities upon Any Number of Lives: As also, of Reversions. To which is added, An Appendix Concerning the Expectations of Life, and Probabilities of Survivorship*, London: Fayram, Motte and Pearson. [10.2]

Drane, J. W., Owen, D. B. and Seibert, G. B., Jr. (1978). *Statistische Hefte*, 19, 204–210.

Dubey, S.D. (1972). Statistical contribution to reliability engineering. ARL TR 72-120, AD 751261.

Dubey, S.D. (1973). Statistical treatment of certain life testing and reliability problems. ARL TR 73-105, AD 774837.

Elandt-Johnson, R.C. and Johnson, N.L. (1980). Survival Models and Data Analysis. John Wiley & Sons, Inc., Hoboken, NJ, USA.

Eugene, N., Lee, C., and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics: Theory and Methods*, 31(4), 497-512.

Evans, R.A and Simons, G. (1975). Research on statistical procedures in reliability engineering. ARL TR, 75-154, ADAO 13687.

Famoye, F., Lee, C., and Olumolade, O. (2005). The beta-Weibull distribution. *Journal of Statistical Theory and Applications*, 4, 121-136.

Fry, T.R.L. and Onne, C.D. (1998). A generalized logistic tobit model. *Journal of Quantitative Economics*, 14, 23-32.

Gupta, R.L, Gupta, R.C. and Lvin, S.J (1996). Analysis of failure time data by Burr distribution. *Communications in Statistics-Theory and Methods*, 25, 2013-2024.

Gupta, R. D. and Kundu, D. (2001). Exponentiated Exponential Family; An Alternative to Gamma and Weibull. *Biometrical Journal*, 33, 117-130.

Gompertz, B. (1825). On the nature of the function expressive of the law of human mortality. *Philosophical Tsansactions of the Royal Society of London*, A115, 513-580.

Gove, J.H., Ducey, M.J., Leak, W.B. and Zhang, L. (2008). Rotated sigmoid structures in managed uneven-aged northern hardwork stands: a look at the Burr Type III distribution. *Forestry*, 81(2), 161-176.

Hasegawa, H. and Kozumi, H. (2003). Estimation of Lorenz curves: a Bayesian nonparametric approach. *Journal of Econometrics*, 115, 277-291.

Hastings, C., Mosteller, F., Tukey, J. W., and Winsor, C. P. (1947). Low moments for small samples: a comparative study of order statistics. *Annals of Mathematical Statistics*, 18, 413–426. (D)

Hatke, M.A. (1949). A certain cumulative probability function. *Annals of Mathematical Statistics*, 20, 461-463.

Henniger, C. and Schmitz, H.-P. (1989). Size distributions of incomes and expenditures: Testing the parametric approach. Discussion Paper A-219, SFB 303, Universität Bonn, Germany.

Hogg, R.V. and Klugman, S.A. (1983). On the estimation of long-tailed skewed distributions with acturial applications. *Journal of Econometrics*, 23, 91-102.

Jaheen, Z.F. (1996). Empirical Bayes estimation of the reliability and failure rate functions of Burr X models. *Journal of Applied Statistical Science*, 3, 281-288.

Johnson, N. L. (1949). Systems of frequency curves generated by methods of translation. *Biometrika*, 36, 149–176.

Johnson, N.L., and Kotz, S. (1970). *Continuous Univariate Distributions, Vol 1*, Houghton Mifflin, Boston.

Johnson, N.L., Kotz, S. and Balakrishnan, N. (1995). *Continuous Univariate Distributions Vol. 1, 2nd Edition*, John Wiley & Sons, New York.

Johnson, N.L., Kotz, S. and Balakrishnan, N. (1995). *Continuous Univariate Distributions Vol. 2, 2nd Edition*, John Wiley & Sons, New York.

Jones, M. (2004). Families of distributions arising from distributions of order statistics. *Test*, 13(1), 1-43.

Kleiber, C. and Kotz, S. (2003). *Statistical Size Distributions in Economics and Actuarial Sciences*, John Wiley & Sons, Hoboken, New Jersey.

Kodlin, D. (1967). A new response time distribution. *Biometrics*, 23, 227-239.

Kotz S, Vicari D. Survey of developments in the theory of continuous skewed distributions. *Metron* 2005, LXIII:225–261.

Lambert, J.H. (1776). *Dottrina degli azzardi*, Gaeta and Fontana, Milan.

Leibniz, G., *Acta Eruditorum.* A Source Book in Mathematics, 1200-1800, ed., by D.J. Struik, Princeton University Press, Princeton, N.J., 1986.

Newton,I., *Opuscula mathematica, philosophica et philologica*, 3 vols, Lausannae and Genevae, 1744.

Lindsay, S.R., Wood, G.R. and Woollons, R.C. (1996). Modeling the diameter distribution of forest stands using the Burr distribution. *Journal of Applied Statistics*, 23(6), 609-619.

Lomax, K.S. (1954). Business failure: Another example of the analysis of the failure data. *American Statistical Association Journal*, 49, 847-852.

Maén, W.J., van Der. (1943). Het berekenen van sterftekansen. *Verzekerings Archief*, 44, 4, 281-300.

Mahmoud, M.A. and Alam, F.M. (2010). The generalized linear exponential distribution. *Statistics and Probability Letters*, 80, 1005–1014.

Majumder, A. and Chakravarty, S.R. (1990). Distribution of personal income: Development of a new model and its application to U.S. income data. *Journal of Applied Eonometrics*, 5, 189-196.

Makeham, W.M. (1860). On the Law of Mortality. *Journal of the Institute of Actuaries*, 13, 325-358.

Makeham, W.M. (1890). On the further development of Gompertz's law. *Journal of the Institute of Actuaries*, 28, 152-159, 185-192, 316-332.

Martinelle, S. (1987). A generalized Perks formula for old-age mortality. Working paper of Statistika centralbyran, ISSN 0283-8680. Stockholm, Sweden.

Mielke, P.W. Jr. (1973). Another family of distribution for describing and analyzing precipitation data. *Journal of Applied Meteorology*, 12, 275-280.

Nadarajah, S., and Kotz, S. (2006) The Beta Exponential Distribution. *Reliability Engineering and System Safety*, 91, 689-697.

McDonald, J.B. (1984). Some generalized functions for the size distribution of income. *Econometrica*, 52, 647-663.

McDonald, J.B. and Mantrala, J. (1993). Apples, oranges and distribution trees. Discussion paper, Brigham Young University, Provo, Utah.

McDonald, J.B. and Mantrala, A. (1995). The distribution of income: Revisited. *Journal of Applied Econometrics*, 10, 201-204.

McDonald, J.B. and Ransom, M. (1979a). Functional forms, estimation techniques, and the distribution of income. *Econometrica*, 47, 1513-1526.

McDonald, J.B. and Xu, Y.J. (1995). A generalization of the beta distribution with applications. *Journal of Econometrics*, 66, 133-152. Erratum: *Journal of Econometrics*, 69, 427-428.

Nadarajah, S. and Kotz, S. (2006). q exponential is a Burr distribution. *Physics Letters A*, 359(6), 577-579.

Nadarajah, S. and Kotz, S. (2007). On the alternative to the Weibull function. *Engineering Fracture Mechanics*, 74(3), 451-456.

Olapade A.K. (2000). Some properties of the Type I Generalized Logistic distribution. *Intro Stats, 2.*

Olapade A.K. (2004). On extended Type I Generalized Logistic distribution. *International Journal of Mathematics and Mathematical sciences*, 57, 3069-3074.

Opperman, L. (1870). Insurance record 1870, 42-43, 46-47; *Journal of the Institute of Actuaries,* 16, (1872); H. & S. 2, 335-353.

Perks, W. (1932). On some experiments in the graduation of mortality statistics. *Journal of the Institute of Actuaries*, 63, 12-57.

Piorer, D. J. (1980). A Lagrange multiplier test for the skewness in binary logit models. *Economic Letters*, 5, 141-143.

Rodriguez, R.N. (1977). A guide to the Burr type XII distributions. *Biometrika*, 64, 129-134.

Sarhan, A.M., Ahmad, A.E.A. and Alasbahi, I.A. (2013). Exponentiated generalized linear exponential distribution. *Applied Mathematical Modeling*, 37(5), 2838–2849.

Sarhan, A. E. and Greenberg, B. G. (1962). *Contributions to order statistics.* Wiley New York.

Sarwati, A. and Abusalih, M.S. (1991). Bayesian prediction bounds for the Burr X model. *Communication in Statistics- Theory and Methods*, 20, 2307-2330.

Sherrick, B.J., Garcia, P. and Tirupattur, V. (1996). Recovering probabilistic information from option markets: Tests of distributional assumptions. *Journal of Future Markets*, 16(5), 545-560.

Siller, W. (1979). A competing risk model for animal mortality. *Ecology*, 60, 750-757.

Smith, R.J. (1989). On the use of distributional misspecification checks in limited dependent variable models. *Econometric Journal*, 99, 178-192.

Surles, J.G. and Padjett, W.J. (1998). Inference for $P[Y < X]$ in the Burr type X model. *Journal of Applied Statistical Science*, 7, 225-238.

Surles, J.G. and Padjett, W.J. (2001). Inference for reliability and stress strength for a scaled Burr X model. *Life time Data Analysis*, 7, 187-200.

Tadikamalla, P.R. (1980). A look at the Burr and related distributions. *International Statistical Review*, 48, 337-344.

Tejeda, H.A. and Goodwin, B.K. (2008). Modelling Crop Price through a Burr Distribution and analysis of correlation between crop prices and yields using copula method. *Paper presented at the Annual Meeting of the Agriculture and Applied Economics Association*, Orlando, FL, USA.

Thiele, P.N. (1871). On a mathematical formula to express the rate of mortality throughout the whole of life. *Journal of the Institute of Actuaries*, 16, 313-329.

Tukey J. W. (1977). *Exploratory Data Analysis*. Reading, MA: Addison-Wesley.

Verhulst, P.F. (1838). Notice sur la loi que la population poursuit dans son accroissement. *Correspondance mathématiqie et physique*, 10, 113-121.

Verhulst, P.F. (1847). Deuxième mémoire sus la loi d'accroissement de la population. *Mémoires de l'Académie Royale des Sciences, des Lettres et des Beaux-Arts de Belgique*, 20, 142-173.

Weibull, W. (1951). A statistical distribution function of wide applicability. *Journal of Applied Mechanics*, 18, 293-297.

Wingo, D.R. (1983). Maximum likelihood methods for fitting the Burr type-XII distribution to life test data. *Biometrika*, 25, 77-84.

Wingo, D.R. (1993). Maximum likelihood methods for fitting the Burr type-XII distribution to multiply (progressively) censored life test data. *Metrika*, 40, 203-210.

Woo J. and Ali, M.M. (1998) Right tail probability estimation in a Burr distribution. *Journal of Statistical Research*, 33, 51-55.

Zimmer, W.J., Bert Keats, J. and Wang, I.K. (1998). The Burr XII distribution in reliability analysis. *Journal of Quality Technology*, 30, 386-394.