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Research Report in Mathematics, Number 25, 2018

Henry Kimiywi MAYUNZU

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Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Statistics

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Master of Science Project

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Abstract

The dissertation applies the Markov chain theory to four types of random walks namely; simple random walk, random walk with reflecting barriers, random walks with absorbing barriers and cyclic random walks. Various methods of determining the nth power are employed where all methods yield the same results. The Computation of 2×2 transition probabilities provides results which are easily generalized. However, using the direct method of multiplication, it is difficult come up with a generalized pattern. The 3×3 transition probability matrices onwards give complex patterns which are not easy to generalize especially in the case of the cyclic random walks. The method of multiplication gives a visible pattern similar to that of the Pascal triangle, but the generalization of the nth term is difficult.

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Declaration and Approval

I the undersigned declare that this project report is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

HENRY KIMIYWI MAYUNZU
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In my capacity as a supervisor of the candidate, I certify that this report has my approval for submission.

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MAYUNZU, Henry Kimiywi

Nairobi, 2018.

1 Introduction

The aim of this dissertation is to apply the Markov chain theory to four types of random walks namely; simple random walk, random walk with reflecting barriers, random walks with absorbing barriers and cyclic random walks.

The outline of the thesis is as follows:

Chapter 1: provides background information, literature review and significance of random walks.
Chapter 2:introduces the theory of Markov chains.
Chapter 3:provides a general discussion on classification of Markov chains with a focus on two state models.
Chapter 4: applies the random walk theory to the simple random walk.
Chapter 5: applies the random walk theory to random walk with barriers.
Chapter 6: applies the random walk theory to cyclic random walks studied as doubly stochastic markov chains.

1.1 Background information

Ideally, the study of Random Walks is the study of random variables. These variables vary in complexity from simple independent distributions in one dimension to random variables of much more complicated character; with their sums subject to further conditions [Bailey60].

A random walk is a mathematical formalization of a path that consists of a succession of random steps [Kennedy60]. For example the path traced by a molecule as it travels in a liquid or gas, the search path of a foraging animal, price of a fluctuating stock and the financial status of a gambler can all be modeled as random walks, although they may not be truly random in reality.

Physical application and analogies suggest the more flexible interpretation in terms of the motion of a variable or "particle" on the x-axis [Medhi96]. The particle starts from the initial position and moves at regular time intervals unit step in the positive or negative direction depending on whether the corresponding trial resulted in success or failure.

The trials terminate when the particle for the first time reaches either o or N . We describe this by saying that the particle performs a random walk with absorbing barriers o or N . This walk is restricted to possible position $1,2,3,...,N-1$. In the absence of absorbing barriers, the random walk is called *unrestricted*. A particle starting at $i>0$ performs a random walk up to the moment when it for the first time reaches the origin [Lawler10]. In this formulation we recognize the first-passage time problem which is solved by use of generating functions in chapter III. Both the absorbing and reflecting barriers are special cases of the so called elastic barrier [Feller68].

1.2 Literature Review

A Markov chain is a randomized process where the next states is only dependent on the current one. Markov chains are intrinsic in statistical modelings and most areas of applied mathematics in the modern day [Lawler10].

The theory of Markov Chains, which is a special case of Markov processes, is named after A.A Markov, who in 1906 introduced the concept of the chains with discrete parameter and finite number of states.

A generalization to countably infinite state spaces was given by Kolmogorov in 1936. Markov chains are related Brownian motion and argotic hypothesis, two topics in physics, but Markov appears to have pursued this out of a mathematical motivation, namely the extension of the law of large numbers to dependent events. In 1913, he applied his findings for the first time to the first 20 000 letters of Pushkin's Eugene Onegin.

Kolmogorov in 1937 extended the theory for denumerable case, J.Doob in 1945 and Paul Levy in 1951 introduced continuous parameter chains. While many others have contributed to the advancement of Markov theory, Feller and K.L. Chung are among those are among those who are responsible for the present status in probability theory that the Markov chain enjoys. [Chung60] gave a comprehensive theoretical treatment of the subject and [Feller68] made a most lucid account of Markov chains for both a theoretical interest and practical point of view.

The term "random walk" was originally proposed by Karl Pearson in 1905 [Ross83]. In a letter to nature, he gave a simple model to describe a mosquito infestation in a forest. At each time step, a single mosquito moves a fixed length , at a randomly chosen angle. Pearson wanted to know the distribution of the mosquitoes after many steps had been taken. The letter was answered by Lord Rayleigh, who had already solved a more general form of this problem in 1880, in the context of sound waves in heterogeneous materials [Feller68]. Modeling a sound wave travelling through the material can be thought of as summing up a sequence of random wave-vectors of constant amplitude but random phase.

The theory of random walks was also developed by Louis Bachelier in his truly remarkable doctorial thesis, La Theorie de la Speculation, published in 1900. Bachelier proposed the random walk as the fundamental model for financial time series.

1.3 Problem statement

William Feller has described and introduced the theory of Markov Chain and has given twelve illustrative examples namely: Two state model, Random Walks with absorbing barriers, Random Walks with reflecting barriers, Cyclic Random Walks, Ehrenfest model of diffusion, Bernoulli-Laplace model of diffusion, Random placement of balls, Example from population genetics, Breeding problem, Recurrent events and residual waiting times, Another chain connected with recurrent events and Success runs. However no in-depth analysis has been done based on the theory of Markov Chains for these examples.

1.4 Objectives

The main objective of the study is to apply the Markov Chain Theory to four types of random walksnamely; simple random walks, random walks with reflecting barriers, random walks with absorbing barriers, and cyclic random walks. The specific objectives are;

(i) To classify the states of random walks

(ii) To find the asymptotic behavior of the states

(iii) To determine the n^{th}-step transition probabilities

1.5 Significance of the Study

The study of random walks and Markov chains is significant in various fields. There are many practical instances of random walks. Many processes in physics involve atomic and sub-atomic particles which migrate about the space which they inhabit, and we may often model such motion in random walk processes.

Random walks may also often be detected in non physical disguises such as in models for gambling, epidemic spread and stock market indices. We are interested in solving these problems using methods of Markov Chain technique and sums of independent identically distributed random variables technique.

A brief description and illustration of the significances of the study in various fields are given below:

In financial economics, the “random walk hypothesis” is used to model shares prices and other factors.

In population genetics, random walk describes the statistical properties of genetic drift.

In physics, random walks are used as simplified models of physical Brownian motion and diffusion such as the random movement of molecules in liquids and gases. Also in physics random walks and some of the self interacting walks play a role in quantum field theory.

In mathematical ecology, random walks are used to describe individual animal movements, to empirically support processes of biodiffusion, and occasionally to model population dynamics.

The study of Markov chains is significant in various fields. These include:

Physics in which the Markovian systems appear extensively in thermodynamics and statistical mechanics, whenever probabilities are used to represent unknown or unmodelled details of the system, if it can be assumed that the dynamics are time-invariant, and that no relevant history need be considered which is not already included in the state description.

Chemistry is often a place where Markov chains and continuous time Markov processes are especially useful because these simple physical systems tend to satisfy the Markov property quite well.

In speech recognition, hidden Markov models are the basis for most modern automatic speech recognition systems.

Markov chans are also utilized in information science, all through information processing. In his famous paper in 1948, A Mathematical Theory of Communication, Claude Shannon created the field of information theory using single steps. He opens by introducing the concept of entropy through Markov modeling of the English language. Such idealized models can capture many of the statistical regularities systems. Even without describing the full structure of the system perfectly, such signal models can make possible very effective data compression through entropy encoding techniques such as arithmetic coding. They allow effective state estimation and pattern recognition. Markov chains also play an important role in reinforcing learning.

Markov chains are also the basis for hidden Markov models, which are an important tool in such fields as telephone networks, speech recognition and bioinformatics.

In queuing theory, Markov chains are the basis for analytical treatment. (Queuing theory). Agner Krarup Erlang initiated the subject in 1917. This makes them critical for optimizing the performance of telecommunications networks, where messages must often complete for limited resources.

In internet applications, the Page Rank of a webpage as used by Google is defined by a Markov chain. It is the probability to be at page in the stationary distribution on the following Markov chain on all (known) WebPages.

Markov models have also been used to analyze web navigation behavior of users. A user's web link transition on a particular website can be modeled using first or second order Markov models and can be used to make predictions regarding future navigation and to personalize the web page for an individual user.

In statistics, Markov chain models have also become very important for generating sequences of random numbers to accurately reflect very complicated desired probability distributions, via a process called Markov chain Monte Carlo.

In Economics and finance, Markov chains are used in finance and economics to model a variety of different phenomenon, including asset prices and market crashes. The first financial model to use a Markov chain was from Prasad et al. in 1974. Another was the regime-switching model of James D. Hamilton (1989), in which Markov chain is used to model switches between periods of high volatility of asset returns.

Dynamic macroeconomics uses Markov chains intensively. An example is the use of Markov chains to exogenously model prices of equity (stock) in general equilibrium setting.

In social sciences Markov chains are generally used in describing path-dependant arguments, where current structural configurations condition future outcomes. An example is the reformulation of the idea, originally due to Karl Max's Das Kapital, tying economic development to the rise of capitalism. In current research, it is common to use a Markov chain to model how once a country reaches a specific level of economic development, the configuration of structural factors, such as size of the commercial bourgeoisie the ratio of urban to rural residence, the rate of political mobilization e.t.c., will generate a higher probability of transitioning from authoritarian to democratic regime.

In Mathematical biology, Markov chains also have many applications in biological modeling, particularly population processes, which are useful in modeling processes that are at (at least) analogous to biological populations. The Leslie matrix is one such example, though some entries are not probabilities (they may be greater than 1). Another example is the modeling of a cell shape in dividing sheets of epithelial cells. Yet another example is the state of ion channels in cell membranes.

In genetics, Markov chains have been used in population genetics in order to describe the change in gene frequencies in small populations affected by genetic drift, for example in diffusion equation method described by Motoo Kimura.

In many games of chance, Markov chains can be employed in createing models. The children's games snakes and ladder are represented exactly by Markov chains. At each urn, the player starts in a given state (on a given square) and from there has fixed odds of moving to certain other state (squares).

2 THEORY OF MARKOV CHAINS

2.1 Introduction

In this chapter various aspects of Markov chain are studied, namely: Markov property, order of Markov chain, state transition probability, and classification of states, classification of Markov chains, the Fundamental matrix and invariant (stationary) distribution.

2.2 Conditional Probability

Consider the events $E_1$and E_2 such that

$$\begin{aligned} \Pr(E_2/E_1) &= \frac{\Pr(E_1,E_2)}{\Pr(E_1)} \\ \Pr(E_1,E_2) &= \Pr(E_2/E_1)\Pr(E_1) \end{aligned}$$

In the case of three events E_1, E_2, and E_3

$$\begin{aligned} \Pr(E_3/E_2,E_1) &= \frac{\Pr(E_1,E_2,E_3)}{\Pr(E_1,E_2)} \\ \Pr(E_1,E_2,E_3) &= \Pr(E_3/E_2,E_1)\Pr(E_1,E_2) \\ &= \Pr(E_3/E_2,E_1)\Pr(E_2/E_1)\Pr(E_1) \end{aligned}$$

In the case of four events E_1, E_2, E_3, and E_4

$$\begin{aligned} \Pr(E_4/E_3,E_2,E_1) &= \frac{\Pr(E_1,E_2,E_3,E_4)}{\Pr(E_1,E_2,E_3)} \\ \Pr(E_1,E_2,E_3,E_4) &= \Pr(E_4/E_3,E_2,E_1)\Pr(E_1,E_2,E_3) \\ &= \Pr(E_4/E_3,E_2,E_1)\Pr(E_3/E_2,E_1)\Pr(E_2/E_1)\Pr(E_1) \end{aligned}$$

2.3 Markov Property

Informally, a Markov property simply means that the future depends on the immediate past and not the remote past.

Therefore

$$\begin{aligned}\Pr(E_3/E_2, E_1) &= \Pr(E_3/E_2)\\ \Pr(E_4/E_3, E_2, E_1) &= \Pr(E_4/E_3)\\ \Pr(E_1, E_2, E_3, E_4) &= \Pr(E_1)\Pr(E_2/E_1)\Pr(E_3/E_2)\Pr(E_4/E_3)\end{aligned}$$

2.4 Notations and Terminologies

Events are called states.

The set of events is called the space states.

$\Pr(E_k/E_j) =$ the transitional probability of moving from state E_k to E_j and is denoted by p_{jk}.

Therefore

$$\begin{aligned}\Pr\{E_1, E_2, E_3, E_4\} &= \Pr(E_1)p_{12}p_{23}p_{34}\\ &= p_1p_{12}p_{23}p_{34}\end{aligned}$$

Let $\Pr\{E_1\} = a_1$. In general, $\Pr\{E_j\} = a_j$ being the initial or absolute probability. Therefore,

$$\Pr\{E_1, E_2, E_3, E_4\} = a_1p_{12}p_{23}p_{34}$$

2.5 Markov Chain - Definition

A sequence of states $\{E_1, E_2, E_3, ..., E_n\}$ is called a Markov chain if the probabilities of the sample sequences are defined by

$$\Pr\{E_1, E_2, E_3, ..., E_n\} = a_{j_o}p_{j_0j_1}...p_{j_{n-1}j_n}$$

where $a_{j_o} = \Pr[E_{j_o}]$, the probability of the initial trial and $p_{j_k} = \Pr\{E_k/E_j\}$ is the probability that E_k occurs given that E_j has already occurred. In particular;

$$\Pr\{E_1, E_2, E_3, ..., E_n\} = a_1p_{12}p_{23}\ldots p_{(n-1)n}$$

2.6 Order of a Markov Chain

A Markov chain $\{X_n\}$ is said to be of order s $(s = 1, 2, 3 \ldots)$ if $\forall\ n$,

$$\begin{aligned}\Pr\{E_n &= k/E_{n-1} = j, E_{n-2} = j_1, \ldots, E_{n-s} = j_{s-1}, \ldots\} \\ &= \Pr\{E_n = k/E_{n-1} = j_{1,\ldots,}E_{n-s} = j_{s-1}, \ldots\}\end{aligned}$$

whenever the LHS is defined.

A Markov chain $\{E_n\}$ is said to be of order one (or simply a Markov chain) if

$$\begin{aligned}\Pr\{E_n &= k/E_{n-1} = j, E_{n-2} = j_1, \ldots\} \\ &= \Pr\{E_n = k/E_{n-1} = j\} \\ &= p_{jk}\end{aligned}$$

whenever

$$\Pr\{E_{n-1} = j, E_{n-2} = j_1, \ldots\} > 0$$

A chain is said to be of order zero if $\forall\ j$;

$$p_{jk} = p_k$$

This implies independence of and E_n and E_{n-1} . For example, for the Bernoulli coin tossing experiment, the transitional probability matrix is

$$\begin{pmatrix} q & p \\ q & p \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} q & p \end{pmatrix} = e \begin{pmatrix} q & p \end{pmatrix}$$

2.7 Matrix Form

In matrix form, the transitional probabilities can be represented by $\mathbf{P}$ as follows;

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} E_1 & E_2 & E_3 & E_4 & \cdots \end{array} \\ \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & \cdots \\ p_{21} & p_{22} & p_{23} & p_{24} & \cdots \\ p_{31} & p_{32} & p_{33} & p_{34} & \cdots \\ p_{41} & p_{42} & p_{43} & p_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

$\mathbf{P} = \big((p_{jk})\big)$ in short form notation

Remarks

(i) $0 \leq p_{jk} \leq 1$

(ii) Each row adds up to 1. i.e. $\sum_k p_{jk} = 1$. The rows do not change, only the columns.

(iii) (iii) The transition matrix can be finite or infinite.

A matrix whose elements are between 0 and 1 (inclusive) and each row adds up to one is called a *stochastic matrix*.

Any stochastic matrix can serve as a matrix of transition probabilities; together with our initial distribution of $\{a_k\}$, it completely defines a Markov chain with states $E_1, E_2, \ldots$.

If in addition each column adds up to one, then we have *doubly stochastic matrix*. That is, not only does the each row sum to 1, each column also sums to 1. Thus, for every column of a doubly stochastic matrix, we have $\sum_j p_{ij} = 1$.

Doubly stochastic matrices have interesting limiting state probabilities, as the following theorem:

2.8 State Transition Probability

It is customary to display the transition probabilities as the entries of a $n \times n$ matrix $\mathbf{P}$, where p_{ij} is the entry in the i^{th} row of the j^{th} column

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

$\mathbf{P}$ is called the transition probability matrix. It is a stochastic matrix because for any row i, $\sum_j p_{ij} = 1$

2.9 Higher Orders of Transition Probabilities

2.9.1 Two order transition probabilities

Let $p_{jk}^{(2)}$ be the probability of moving from state E_j to state E_k in two steps. Consider any four events E_1, E_2, E_3, and E_4,

$$\begin{aligned} p_{14}^{(2)} &= \Pr(E_1 \longrightarrow E_1 \longrightarrow E_4) + \Pr(E_1 \longrightarrow E_2 \longrightarrow E_4) \\ +\Pr(E_1 &\longrightarrow E_3 \longrightarrow E_4) + \Pr(E_1 \longrightarrow E_4 \longrightarrow E_4) \\ &= p_{11}p_{14} + p_{12}p_{24} + p_{13}p_{34} + p_{14}p_{44} \\ &= \sum_{v=1}^{4} p_{1v}p_{v4} \end{aligned}$$

In general

$$p_{jk}^{(2)} = \sum_v p_{jv}p_{vk}$$

2.9.2 Multiplication of matrices

To obtain the 2-step transition probability of moving from state E_1 to E_4 using matrices, we proceed as follows;

$$
\begin{aligned}
\mathbf{P}^2 &= \mathbf{P}\cdot\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} \\
&= \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \end{bmatrix}\begin{bmatrix} p_{14} \\ p_{24} \\ p_{34} \\ p_{44} \end{bmatrix} \\
&= \sum_{v=1}^{4} p_{1v}p_{v4}
\end{aligned}
$$

Therefore, it can be concluded that the elements of a transition matrix multiplied by itself give the probabilities of moving from one state to the other state or returning to the state in two steps. Therefore;

$$
\begin{aligned}
\mathbf{P} &= \left((p_{jk})\right) \\
\mathbf{P}^2 &= \left((p_{jk}^{(2)})\right) \\
(p_{jk}^{(2)}) &= \sum_{v} p_{jv}p_{vk}
\end{aligned}
$$

2.9.3 Three step transition probability

Write

$$
\begin{aligned}
\mathbf{P}^3 &= \mathbf{P}^2\mathbf{P} && \textbf{(1)} \\
\mathbf{P}^3 &= \mathbf{P}\mathbf{P}^2 && \textbf{(2)}
\end{aligned}
$$

where $\mathbf{P}^3 = \left((p_{jk}^{(3)})\right)$

Using (2.1) we can write $p_{jk}^{(3)} = \sum_v p_{jv}^{(2)} p_{vk}$ and using (2.2) we can write

$$
p_{jk}^{(3)} = \sum_{v} p_{vk}p_{jv}^{(2)} \qquad \textbf{(3)}
$$

Using $(2.1), (2.2)$, and (2.3) a general form can be written that

$$\mathbf{P}^n = p_{jv}^{(n-1)} p_{vk} = p_{vk} p_{jv}^{(n-1)} \tag{4}$$

where $\mathbf{P}^n = \left((p_{jk}^{(n)})\right)$ which implies that a more general form can be written as

$$\mathbf{P}^{m+n} = \mathbf{P}^m \mathbf{P}^n$$

where

$$\mathbf{P}^{m+n} = \left((p_{jk}^{(m+n)})\right)$$

and

$$\mathbf{P}_{jk}^{(m+n)} = \sum_v p_{jv}^{(m)} p_{vk}^{(n)}$$

The above formulas are referred to as the Chapman-Kolmogorov formula.

2.10 Classification of States

A state j is said to be accessible (or can be reached) from state i if, starting from state i, it is possible that the process will ever enter state j. This implies $p_{ij}^{(n)} > 0$ that for some $n > 0$. Thus, the $n-$step probability enables us to obtain reachability information between any two states of the process.

Two states that are accessible from each other are said to communicate with each other. The concept of communication divides the state space into different classes. Two states that communicate are said to be in the same class. All members of one class communicate with one another. If a class is not accessible from any state outside the class, we define the class to be a closed communicating class.

2.10.1 Return probabilities

Let be the proba $f_{jj}^{(n)}$bility of returning to state E_j in n steps for the first time and $p_{jj}^{(n)}$ the probability of returning to state in n steps but not necessarily for the first time. Therefore $p_{jj}^{(n)} \geq f_{jj}^{(n)}$.

The relationship between $p_{jj}^{(n)}$ and $f_{jj}^{(n)}$ with the assumption that $p_{jj}^{(0)} = 1$ and $f_{jj}^{(0)} = 0$

$$p_{jj}^{(n)} = \sum_{v=1}^{n} f_{jj}^{(v)} p_{jj}^{(n-v)} \qquad , n \geqq 1 \tag{5}$$

In terms of the probability generating function, let

$$\begin{aligned}\mathbf{F}(s) &= \sum_{n=0}^{\infty} f_{jj}^{(n)}\mathsf{S}^n \\ &= f_{jj}^{(0)} + \sum_{n=1}^{\infty} f_{jj}^{(n)}\mathsf{S}^n \\ &= \sum_{n=1}^{\infty} f_{jj}^{(n)}\mathsf{S}^n\end{aligned}$$

since $f_{jj}^{(0)} = 0$.

$$\begin{aligned}\mathbf{P}(s) &= \sum_{n=0}^{\infty} p_{jj}^{(n)}\mathsf{S}^n \\ &= p_{jj}^{(0)} + \sum_{n=1}^{\infty} p_{jj}^{(n)}\mathsf{S}^n \\ &= 1 + \sum_{n=1}^{\infty} p_{jj}^{(n)}\mathsf{S}^n\end{aligned}$$

$\Longrightarrow$

$$\sum_{n=1}^{\infty} p_{jj}^{(n)}\mathsf{S}^n = \mathbf{P}(s) - 1 \tag{6}$$

Therefore multiplying (2.4) by S^n and summing the result over n

$$\begin{aligned}\sum_{n=1}^{\infty} p_{jj}^{(n)}\mathsf{S}^n &= \sum_{n=1}^{\infty}\sum_{v=1}^{n} f_{jj}^{(v)} p_{jj}^{(n-v)}\mathsf{S}^n \\ &= \sum_{n=1}^{\infty}\sum_{v=1}^{\infty} f_{jj}^{(v)} p_{jj}^{(n-v)}\mathsf{S}^n \\ &= \sum_{n=1}^{\infty}\sum_{v=1}^{\infty} f_{jj}^{(v)}\mathsf{S}^v p_{jj}^{(n-v)}\mathsf{S}^{n-v}\end{aligned}$$

Therefore

$$\begin{aligned}\mathbf{P}(s) - 1 &= \sum_{v=1}^{\infty}\left[f_{jj}^{(v)}\mathsf{S}^v \sum_{n=1}^{\infty} p_{jj}^{(n-v)}\mathsf{S}^{n-v}\right] \\ &= \sum_{v=1}^{\infty}\left[f_{jj}^{(v)}\mathsf{S}^v\left(p_{jj}^{(0)}\mathsf{S}^0 + p_{jj}^{(1)}\mathsf{S}^1 + p_{jj}^{(2)}\mathsf{S}^2 + \dots\right)\right] \\ &= \sum_{v=1}^{\infty}\left[f_{jj}^{(v)}\mathsf{S}^v \sum_{n=0}^{\infty} p_{jj}^{(n)}\mathsf{S}^n\right]\end{aligned}$$

$$\begin{aligned}\mathbf{P}(s)-1 &= \mathbf{P}(s)\sum_{v=1}^{\infty} f_{jj}^{(v)}\mathsf{S}^{v}\\ &= \mathbf{P}(s)\mathsf{F}(s)\end{aligned}$$

$$\mathbf{P}(s)-\mathbf{P}(s)\mathsf{F}(s)=1$$

$$\mathbf{P}(s)=\frac{1}{1-\mathsf{F}(s)}$$

2.10.2 Persistency

A state E_j is said to be persistent if

$$f_j=\sum_{n=1}^{\infty} f_{jj}^{(n)}=1$$

Note:

$f_j=\sum f_{jj}^{n}$ is the probability of ever (or eventually) returning to state E_j. When $f_j=1$, then

$$\sum_{n=1}^{\infty} f_{jj}^{(n)}=1$$

but

$$0\leq f_{jj}^{(n)}\leq 1$$

$\Longrightarrow$

$$\{f_{jj}^{(n)}:\ n=1,2,3\ldots\}\ \text{ is a probability mass function.}$$

Since the $f_{jj}^{(n)}$ is the probability of returning to E_j in n steps for the first time we call $\{f_{jj}^{(n)}:n=1,2,3\ldots\}$ a first passage distribution. Its mean is refferred to as the mean recuurence time given by;

$$\mu_j=\sum_{n=1}^{\infty} nf_{jj}^{(n)}$$

A state E_j is persisitent if $f_j = \sum f_{jj}^{(n)} = 1$. If the mean recurrence time $\mu_j < \infty$ (finite), then E_j is a non-null persistent state. If $\mu_j = \infty$ (infinite), then E_j is a null persistent state.

2.10.3 Transiency

E_j is said to be a transient state if

$$f_j = \sum_{n=1}^{\infty} f_{jj}^{(n)} < 1$$

In this case, no further investigations are carried out. In terms of $\sum p_{jj}^{(n)}$ the following formula is used;

$$\mathbf{P}(s) = \frac{1}{1 - \mathsf{F}(s)} \quad \text{i.e.}$$

$$\sum_{n=0}^{\infty} p_{jj}^{(n)} \mathsf{S}^n = \frac{1}{1 - \sum_{n=0}^{\infty} f_{jj}^{(n)} \mathsf{S}^n}$$

Setting $S = 1$,

$$\mathbf{P}(1) = \frac{1}{1 - \mathsf{F}(1)} \quad \text{i.e.}$$

$$\begin{aligned} \sum_{n=0}^{\infty} p_{jj}^{(n)} &= \frac{1}{1 - \sum_{n=0}^{\infty} f_{jj}^{(n)}} \\ &= \frac{1}{1-1} \\ &= \frac{1}{0} = \infty \end{aligned}$$

If E_j is transient, then $f_j < 1$ hence

$$\sum_{n=0}^{\infty} p_{jj}^{(n)} = \frac{1}{1 - < 1} < \infty$$

Therefore, state E_j is persistent if

$$\sum_{n=0}^{\infty} f_{jj}^{(n)} = 1 \quad \text{or} \quad \sum_{n=0}^{\infty} p_{jj}^{(n)} = \infty$$

and E_j is transient if

$$\sum_{n=0}^{\infty} f_{jj}^{(n)} < 1 \quad \text{or} \quad \sum_{n=0}^{\infty} p_{jj}^{(n)} < \infty$$

$$(1-s)\mathbf{P}(s) = \frac{1-s}{1-\mathsf{F}(s)}$$

$$\begin{aligned}
\lim_{s\to 1}(1-s)\mathbf{P}(s) &= \lim_{s\to 1}\frac{1-s}{1-\mathsf{F}(s)} \\
&= \lim_{s\to 1}\{(1-s)\left[p_{jj}^{(0)}\mathsf{S}^0 + p_{jj}^{(1)}\mathsf{S}^1 + p_{jj}^{(2)}\mathsf{S}^2 + \ldots + p_{jj}^{(n-1)}\mathsf{S}^{n-1} + p_{jj}^{(n)}\mathsf{S}^n + \ldots\right]\} \\
&= \lim_{s\to 1}\{\left[p_{jj}^{(0)}\mathsf{S}^0 + p_{jj}^{(1)}\mathsf{S}^1 + p_{jj}^{(2)}\mathsf{S}^2 + \ldots + p_{jj}^{(n-1)}\mathsf{S}^{n-1} + p_{jj}^{(n)}\mathsf{S}^n + \ldots\right] \\
&\quad + \left[p_{jj}^{(0)}\mathsf{S}^1 - p_{jj}^{(1)}\mathsf{S}^2 - p_{jj}^{(2)}\mathsf{S}^3 - \ldots - p_{jj}^{(n-1)}\mathsf{S}^n - p_{jj}^{(n)}\mathsf{S}^{n+1} - \ldots\right]\}
\end{aligned}$$

$$LHS = \left[p_{jj}^{(0)} + p_{jj}^{(1)} + p_{jj}^{(2)}\mathsf{S}^2 + p_{jj}^{(3)}\mathsf{S}^3 + \ldots + p_{jj}^{(n-1)}\mathsf{S}^{n-1} + p_{jj}^{(n)}\mathsf{S}^n + \ldots\right]$$

$$\begin{aligned}
& p_{jj}^{(0)} - p_{jj}^{(1)} + p_{jj}^{(2)} - p_{jj}^{(3)} - \ldots - p_{jj}^{(n-1)} \\
=\ & \lim_{n\to\infty} p_{jj}^{(n)}
\end{aligned}$$

Therefore

$$\lim_{s\to 1}(1-s)\mathbf{P}(s) = \lim_{s\to 1} p_{jj}^{(n)}$$

$$\begin{aligned}
RHS &= \lim_{s\to 1}\frac{1-s}{1-\mathsf{F}(s)} \\
&= \frac{1-1}{1-\mathsf{F}(1)}
\end{aligned}$$

If E_j is persistent, then $\mathsf{F}(1) = 1$ hence

$$RHS = \frac{1-1}{1-1} = \frac{0}{0} \quad \text{undetermined}$$

Using L 'Hospital 's rule;

$$\begin{aligned}\lim_{s\to 1}(1-s)\mathbf{P}(s) &= \lim_{s\to 1}\frac{\frac{d}{ds}(1-s)}{\frac{d}{ds}[1-\mathsf{F}(s)]}\\ &= \lim_{s\to 1}\frac{-1}{1-\mathsf{F}'(s)}\\ &= \frac{1}{\mu_j}\end{aligned}$$

If E_j is transient, then $\mathsf{F}(1)<1$ therefore

$$\frac{1-1}{1-\mathsf{F}(1)}=\frac{0}{\text{a finite number}}=0$$

Thertefore,

$$\lim_{x\to\infty}p_{jj}^{(n)}=\left\{\begin{array}{c}0,\ \text{when (i) } E_j \text{ is transient and (ii) } E_j \text{ is nullpersisitent}\\ \frac{1}{\mu_j},\ \text{if } E_j \text{ is non-null persistent}\end{array}\right\}$$

called the limiting theorem.

2.10.4 Periodicity

A state $E_j$is said to be of period d if $d=GCD\left\{n:p_{jj}^{(n)}>0\right\}$

If d is 1, then E_j is said to be aperiodic.

2.10.5 Ergodicity

If a state E_j is

(i) Persistent

(ii) Non-null

(iii) Aperiodic

then it is Ergodic.

Illustrative example

Let **P** be the following matrix;

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \left[\begin{array}{ccc} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right] \end{array}$$

Consider the state E_1, to show that $f_1 = \sum_{n=1}^{\infty} f_{11}^{(n)} = 1$ or < 1, we proceed as follows;

$$\begin{aligned} f_{11}^{(1)} &= 0 \text{ i.e. } f_{11}^{(1)} = \Pr[E_1 \to E_1] = 0 \\ f_{11}^{(2)} &= \Pr[E_1 \to E_2 \to E_1] + \Pr[E_1 \to E_3 \to E_1] \\ &= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2} = \left(\frac{1}{2}\right)^1 \end{aligned}$$

$$\begin{aligned} f_{11}^{(3)} &= \Pr[E_1 \to E_2 \to E_3 \to E_1] + \Pr[E_1 \to E_3 \to E_2 \to E_1] \\ &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = \left(\frac{1}{2}\right)^2 \end{aligned}$$

$$\begin{aligned} f_{11}^{(4)} &= \Pr[E_1 \to E_2 \to E_3 \to E_2 \to E_1] + \Pr[E_1 \to E_3 \to E_2 \to E_3 \to E_1] \\ &= \frac{1}{16} + \frac{1}{16} = \frac{1}{8} = \left(\frac{1}{2}\right)^3 \end{aligned}$$

In general

$$f_{11}^{(n)} = \left\{ \begin{array}{c} 0, \quad n = 1 \\ \left(\frac{1}{2}\right)^{(n-1)}, \ n \geq 2 \end{array} \right\}$$

$$\begin{aligned} f_1 &= \sum_{n=1}^{\infty} f_{11}^{(n)} = 0 + \sum_{n=2}^{\infty} f_{11}^{(n)} \\ &= \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^{(n-1)} \\ &= \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots \end{aligned}$$

which is a geometric series

$$= \frac{1}{2}\frac{1}{1-\frac{1}{2}} = \frac{1}{2}\left[\frac{1}{\frac{1}{2}}\right] = 1$$

therefore, E_1 is persistent since $f_1 = 1$

$$\begin{aligned} \mu_1 &= \sum_{n=2}^{\infty} n f_{11}^{(n)} \\ &= 2\left(\frac{1}{2}\right)^1 + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^4 + \cdots \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{1-x}\right) &= \frac{d}{dx}(1-x)^{-1} \\ &= 1 + x + x^2 + x^3 \end{aligned}$$

$$\begin{aligned} \frac{1}{(1-x)^2} &= (1-x)^{-2} = 1 + 2x + 3x^3 \\ & \quad -1 + 1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 \\ &= 3 \end{aligned}$$

hence the mean recurrence time μ is finite and therefore E_1 is non-null.

Periodicity

$$GCD\left\{n : p_{jj}^{(n)} > 0\right\}$$

$$\begin{aligned} \text{since } p_{jj}^{(n)} &= \sum_{v=1}^{n} f_{jj}^{(v)} p_{jj}^{(n-v)} \\ &= \sum_{v=1}^{n-1} f_{jj}^{(v)} p_{jj}^{(n-v)} + f_{jj}^{(n)} p_{jj}^{(0)} \\ &= \sum_{v=1}^{n-1} f_{jj}^{(v)} p_{jj}^{(n-v)} + f_{jj}^{(n)} \end{aligned}$$

$$\text{If } f_{jj}^{(n)} > 0 \implies p_{jj}^{(n)} > 0$$

$$\begin{aligned} d &= GCD\left\{n : p_{jj}^{(n)} > 0\right\} \\ &= GCD\left\{n : f_{jj}^{(n)} > 0\right\} \\ &= \{2, 3, 4, \ldots\},\ d = 1 \end{aligned}$$

since $f_{jj}^{(n)} = \left(\frac{1}{2}\right)^{n-1} > 0$, for $n \geq 2$ and $f_{jj}^{(1)} = 0$.

Since the state E_j has been found to be persistent, non-null, and aperiodic it follows that it is ergodic.

Lemma 2.1

Let $\{f_n\}$ be a sequence such that $f_n \geq 0$, $\sum f_n = 1$, and $t(\geq 1)$ be the greatest common divisor of those n for which $f_n > 0$.

Let $\{u_n\}$ be another sequence such that $u_o = 1$ and

$$u_n = \sum_{r=1}^{n} f_r u_{n-r} \quad , n \geq 1$$

then

$$\lim_{n \to \infty} u_{n-t} = \frac{t}{\mu} \quad \text{where } \mu = \sum_{n=1}^{\infty} n f_n$$

the limit being zero as $\mu \to \infty$ and

$$\lim_{N \to \infty} u_N = 0$$

where N is not divisible by t.

Theorem 2.1

Let a_{ik} denote the probability that the chain starting with a transient state i eventually gets absorbed in an absorbing state k. If we denote the absorption probability matrix by

$$A = (a_{ik}), \quad i \in T,\ k \in S - T$$

then

$$A = (I-Q)^{-1}R = NR,\ N = (I-Q)^{-1} \tag{7}$$

Proof

$$a_{ik} = F_{ik} = \sum_n f_{ik}^n$$

Since transitions between absorbing states is impossible. Now

$$F_{ik} = \Pr[U\{X_n = k_{n\geq 1} | X_o = i\}]$$

Since state k is an absorbing state, once the chain reaches an absorbing state k after n steps, it remains there after steps $(n+1), (n+2), \ldots$

Thus

$$\{X_n = k\} \subset \{X_{n+1} = k\} \subset \{X_{n+2} = k\} \ldots$$

and using the result $A_1 \subset A_2 \subset A_3 \ldots$

$$\Pr\{U_{t\geq 1}A_t\} = \lim_{n\to\infty} \Pr[A_n]$$

to obtain

$$a_{ik} = F_{ik} = \Pr\left|_{n\geq 1} p_{ik}^{(n)}\right| = \lim_{n\to\infty} p_{ik}^{(n)} \tag{8}$$

Chapman-Kolmogorov equation can be written as;

$$p_{ik}^{(n+1)} = \sum_{j\in S} p_{ij} p_{jk}^{(n)}$$

using $p_k^{(n)} = 1$ and $p_{jk} \neq 0$ only when $j \in T$, $k \in S$, and $k \in S - T$ we obtain

$$p_{ik}^{(n+1)} = p_{ik} + \sum_{j\in T} p_{ij} p_{jk}^{(n)}$$

taking limits on both sides and as $n \to \infty$ and using (2.8) we obtain

$$a_{ik} = p_{ik} + \sum_{j\in T} p_{ij} a_{jk}$$

therefore, the matrix notation is as follows;

$$\mathbf{A} = (a_{ik}) = \mathbf{R}+\mathbf{VA}$$
$$\mathbf{A} = (\mathbf{1-V})^{-1}\mathbf{R}$$
$$= \mathbf{NR}$$

Note:

(i) The matrix $\mathbf{N} = (\mathbf{1-V})^{-1}$ is known as the fundamental matrix.

(ii) As $n \to \infty$, the limits $p_{jk}^{(n)}$ exists but are not independent of the initial state i.

Theorem 2.2

If state j is persistent and non-null, then as $n \to \infty$

(i)
$$p_{jj}^{(nt)} \to \frac{t}{\mu_{jj}}$$

when state j is periodic with period t, and

(ii)
$$p_{jj}^{(nt)} \to \frac{1}{\mu_{jj}}$$

when state j is aperiodic. In the case that j is persistent and null, (whether periodic or aperiodic), then

$$p_{jj}^{(n)} \to 0 \text{ as } n \to \infty$$

Proof

Let stat i be persistnet. Then we define

$$\mu_{jj} \leq \sum_n n f_{jj}^{(n)}$$

since

$$p_{jj}^{(n)} = \sum_{r=0}^{n} f_{jj}^{(r)} p_{jj}^{(n-r)}$$

for u_n and μ_{jj} in lemma (2.1) above.

Applying the lemma we obtain;

$$p_{jj}^{(nt)} \to \frac{1}{\mu_{jj}} \quad \text{as } n \to \infty$$

In case state j is persistent-null, $\mu_{jj} = \infty$ and $p_{jj}^{(n)} \to 0$ as $n \to \infty$.

Note:

(i) If j is persistent non-null, then

$$\lim_{n\to\infty} p_{jj}^{(n)} > 0$$

and

(ii) If j is persistent null or transient, then

$$\lim_{n\to\infty} p_{jj}^{(n)} \to 0$$

Theorem 2.3

If state k is persistent null, then for every j

$$\lim_{n\to\infty} p_{jk}^{(n)} \to 0$$

and if state k is aperiodic, persistent, and non-null then

$$\lim_{n\to\infty} p_{jk}^{(n)} \to \frac{F_{jk}}{\mu_{kk}}$$

Proof

We have

$$p_{jk}^{(n)} = \sum_{v=1}^{n} f_{jk}^{(r)} p_{kk}^{(n-r)}$$

Let $n > m$, then

$$p_{jk}^{(n)} = \sum_{v=1}^{m} f_{jk}^{(r)} p_{kk}^{(n-r)} + \sum_{v=m+1}^{n} f_{jk}^{(r)} p_{kk}^{(n-r)}$$
$$\leq \sum_{v=1}^{m} f_{jk}^{(r)} p_{kk}^{(n-r)} + \sum_{v=m+1}^{n} f_{jk}^{(r)} \tag{9}$$

Since k is persistent null,

$$p_{kk}^{(n-r)} \to 0 \text{ as } n \to \infty$$

furhther, since

$$\sum_{m=1}^{\infty} f_{jk}^{(m)} < \infty, \quad \sum_{v=m+1}^{n} f_{jk}^{(r)} \to 0 \text{ as } n, m \to \infty$$

hence as $n \to \infty$,

$$p_{jk}^{(n)} \to 0$$

from (2.9)

$$p_{jk}^{(n)} - \sum_{v=1}^{m} f_{jk}^{(r)} p_{kk}^{(n-r)} \leq \sum_{v=m+1}^{n} f_{jk}^{(r)} \tag{10}$$

since j is aperiodic, poersistent, and non-null, then by theorem (2.2)

$$p_{kk}^{(n-r)} \to \frac{1}{\mu_{kk}} \text{ as } n \to \infty$$

hence from (2.10), we get that as $n, m \to \infty$

$$p_{jk}^{(n)} \to \frac{F_{jk}}{\mu_{kk}}$$

Theorem 2.4

If a state j is persistent, then for every state k that can be reached from state j, $F_{jk} = 1$

Proof

Let a_k be the probability that starting from state j, the state k is reached without previously returning to state j. The probability of returning to state j once state k is reached is $(1 - F_{kj})$.

The probability of the compound event that starting from state j, the system reaches state k (without returning to state j) and never returns to state j is $a_k(1-F_{kj})$

If there are some other states say, $r, s, \ldots$, then we get similar terms as $a_r(1-F_{rj}), a_s(1-F_{sj}), \ldots$

Thus probability Q that starting from state j the system never returns to state j is given by

$$Q = a_k(1-F_{kj}) + a_r(1-F_{rj}) + a_s(1-F_{sj}) + \ldots$$

But since the state j is persistent, $F_{jj} = 1$ and the probability of never returning to state j is $1 - F_{jj} = 0$. Thus $Q = 0$. This implies each term is zero, so that $F_{jk} = 1$.

Theorem 2.5 (Ergodic theorem)

For a finite irreducible, aperiodic chain with transition probability matrix $\mathbf{P} = (p_{jk})$

the limits

$$v_k = \lim_{n\to\infty} p_{jk}^{(n)} \tag{11}$$

exist and are independent of the initial state j. The limits v_k are such that $v_k \geq 0$, $\sum v_k = 1$ i.e. the limits v_k define a probability distribution.

Furthermore, the limiting probability distribution $\{v_k\}$ is identical with stationary distribution for the given chain, so that

$$v_k = \sum_j v_j p_{jk}, \qquad \sum v_k = 1 \tag{12}$$

Writing

$$\mathbf{V}' = (v_1, v_2, \ldots, v_k, \ldots), \quad \sum v_k = 1 \tag{13}$$

the relation (2.) may be written as

$$\begin{aligned} \mathbf{V}' &= \mathbf{V}'\mathbf{P} \\ \mathbf{V}'(\mathbf{P} - \mathbf{1}) &= \mathbf{0} \end{aligned} \tag{14}$$

Proof

Since the states are aperiodic, persistent non-null, for each pair of $j, k, \lim_{n\to\infty} p_{jk}^{(n)}$ exists and is equal to $\frac{F_{jk}}{\mu_{kk}}$ (Theorem 2.4). Since k is persistent, $F_{jk} = 1$, so that

$$v_k = \lim_{n\to\infty} p_{jk}^{(n)} = \frac{1}{\mu_{kk}} > 0$$

and is independent of j. Since

$$\sum_k p_{jk}^{(n)} \leq 1, \qquad \forall N$$

thus

$$\sum_{k=1}^{N} \lim_{n\to\infty} p_{jk}^{(n)} \leq 1 \quad \text{ i.e. } \sum_{k=1}^{N} v_k \leq 1$$

since it holds that $\forall \sum_{k=1}^{N} v_k \leq 1$, we have

$$p_{jk}^{(n+m)} = \sum_i p_{ji}^{(n)} p_{ik}^{(m)}$$

and

$$\begin{aligned} v_k &= \lim_{n\to\infty} p_{jk}^{(n+m)} = \lim_{n\to\infty} \sum_i p_{ji}^{(n)} p_{ik}^{(m)} \\ &\geq \sum_i \left\{ \lim_{n\to\infty} p_{ji}^{(n)} \right\} p_{ik}^{(m)} \text{ by Fatou's lemma} \\ &= \sum_i v_i p_{ik}^{(m)}, \quad \forall m \end{aligned}$$

$$v_k \geq \sum_i v_i p_{ik}^{(m)}$$

i.e. suppose, if possible, that

$$v_k = \sum_i v_i p_{ik}^{(m)}$$

then summing overall k we obtain

$$\sum_k v_k > \sum_k \sum_i v_i p_{ik}^{(m)} = \sum_i v_i$$

which is impossible. Hence the equality sign holds i.e.

$$v_k = \sum_i v_i p_{ik}^{(m)}$$

when m is large we have

$$v_k = \sum v_i v_k = \left(\sum v_i\right) v_k$$

hence

$$\sum_i v_i = 1$$

This shows that $\{v_k\}$ is a probability distribution: the distribution is unique.

The distribution is known as stationary (or equilibrium) probabilities.

We state, without proof, the converse which also holds. If a chain is irreducible and aperiodic and if there exist a unique stationary distribution $\{v_k\}$ for the chain, then the chain is ergodic and

$$v_k = \frac{1}{\mu_{kk}}$$

Thus ergodicity is a necessary and sufficient condition for the existence of $\{v_k\}$ satisfying (2.) in case of an irreducible and aperiodic chain.

The distribution is known as stationary or (equilibrium) distribution of the chain and the probabilities known as stationary (or equilibrium) probabilities.

We state, without proof, the converse which also holds. If a chain is irreducible and aperiodic and if there exist a unique stationary distribution $\{v_k\}$ for the chain, then the chain is ergodic and $v_k = \frac{1}{\mu_{kk}}$.

Thus ergodicity is a necessary sufficient condition for the existence of $\{v_k\}$ satisfying (2.) in case of an irreducible and aperiodic chain

2.11 Classification of Markov Chains

2.11.1 Reachability

E_k can be reached from E_j if there exist an integer n such that $p_{jj}^{(n)} > 0$. E_j and E_k and are communicating if E_k can be reached from E_j and E_j can be reached from E_k.

In restricted random walk each state can be reached from every other state, but from an absorbing barrier no other state can be reached.

Theorem 2.6

E_k can be reached from E_j and E_j can be reached from E_i, then E_k can be reached from E_i.

Proof

If E_k can be reached from E_j, there exists $n : p_{jk}^{(n)} > 0$.

Similarly if E_j can be reached from E_i, then there exists $n : p_{ij}^{(n)} > 0$.

let

$$N = m + n, \qquad p_{ik}^{(N)} = \sum_j p_{ij}^{(m)} p_{jk}^{(n)} \geq p_{ij}^{(m)} p_{jk}^{(n)} > 0$$

$$\text{hence} \qquad p_{ik}^{(N)} > 0$$

$\Longrightarrow$

$$\exists \;\; N : \;\; p_{ik}^{(N)} > 0$$

This means that we have a Markov chain defined on a set C and this sub-chain can be studied independently of all other states.

2.11.2 Closure and closed sets

A set C of states is closed if no state outside C can be reached from any state $E_j \in C$. Thus C is closed $\iff$ $p_{jk} = 0$ whenever $E_j \in C$ and $E_k \notin C$. i.e. if $E_j \in C$ and $E_k \notin C$, then $p_{jk}^{(n)} = 0$, $n = 1, 2, 3, \ldots$

$\Longrightarrow$ $p_{jk}^{(n)} = \sum_v p_{jv} p_{vk}^{(n)} = 0$ if $E_j \in C$ and $E_k \notin C$

Illustartive example

Let the matrix $\mathbf{P}$ be given by

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \\ E_7 \\ E_8 \\ E_9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \\ 0 & * & * & 0 & * & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \end{bmatrix} \end{array}$$

Finding closed sets

$S_1 = \{E_1, E_4, E_9\}$ is a closed set

$S_2 = \{E_3, E_8\}$ is a closed set

It is also a class of a closed set whose elements communicate.

$S_3 = \{E_5\}$ is a closed set of a single element. It is an absorbing state, that is, once entered you cannot come out.

E_2, E_6, E_7 are not closed.

Note:

In the matrix $\mathbf{P}$ implies a non-zero probability.

Theorem 2.7

If in the matrix $\mathbf{P}_n$ all rows and all columns corresponding to states outside the closed set C are detected, there remain stochastic matrices for which the fundamental relations.

$$p_{jk}^{(n+1)} = \sum_{v} p_{jv} p_{vk}^{n}$$

$$\text{and } p_{jk}^{(n+m)} = \sum_v p_{jv}^n p_{vk}^m$$

again holds.

This means that we have a Markov chain defined on C, and this sub-chain can be studied independently of all other states. The state E_k is absorbing iff $p_{kk} = 1$; in this case the matrix of theorem (2.7) reduces to a single element. In general it is clear that the totality of all states E_k that can be reached from a given state $E_j$forms a closed set. (Since the closure of E_j cannot be smaller, it coincides with this set).

2.11.3 Partitioning a matrix block

Let us rearrange the transition matrix $\mathbf{P}$ by starting with absorbing state E_5 followed by the other closed sets. We will get any of the following matrices:

Case 1: Partitioning the matrix $\mathbf{P}$ into two blocks

$$\mathbf{P} = \begin{array}{c} \\ E_5 \\ E_3 \\ E_8 \\ E_1 \\ E_4 \\ E_9 \\ E_2 \\ E_6 \\ E_1 \end{array} \begin{array}{c} \begin{array}{cccccc|ccc} 5 & 3 & 8 & 1 & 4 & 9 & 2 & 6 & 1 \end{array} \\ \left[\begin{array}{cccccc|ccc} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ \hline * & * & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \end{array} \right] \end{array}$$

$\mathbf{P}$ can be written in canonical form as follows

$$\mathbf{P} = \begin{bmatrix} \mathbf{Q}_{6\times 6} & \mathbf{0}_{6\times 3} \\ \mathbf{U}_{3\times 6} & \mathbf{V}_{3\times 3} \end{bmatrix}$$

$$
\begin{aligned}
\mathbf{P}^2 &= \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{U} & \mathbf{V} \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{U} & \mathbf{V} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{Q}^2 & \mathbf{0} \\ \mathbf{UQ}+\mathbf{VU} & \mathbf{V}^2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{P}^3 &= \mathbf{PP}^2 = \mathbf{P}^2\mathbf{P} \\
&= \begin{bmatrix} \mathbf{Q}^2 & \mathbf{0} \\ \mathbf{UQ}+\mathbf{VU} & \mathbf{V}^2 \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{U} & \mathbf{V} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{Q}^3 & \mathbf{0} \\ \mathbf{UQ}^2+\mathbf{VUQ}+\mathbf{V}^2\mathbf{U} & \mathbf{V}^3 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{Q}^3 & \mathbf{0} \\ \mathbf{V}^0\mathbf{UQ}^2+\mathbf{VUQ}+\mathbf{V}^2\mathbf{UQ}^0 & \mathbf{V}^3 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{P}^4 &= \begin{bmatrix} \mathbf{Q}^3 & \mathbf{0} \\ \mathbf{V}^0\mathbf{UQ}^2+\mathbf{VUQ}+\mathbf{V}^2\mathbf{UQ}^0 & \mathbf{V}^3 \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{U} & \mathbf{V} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{Q}^4 & \mathbf{0} \\ \mathbf{UQ}^3+\mathbf{VUQ}^2+\mathbf{V}^2\mathbf{UQ}+\mathbf{V}^3\mathbf{U} & \mathbf{V}^4 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{Q}^4 & \mathbf{0} \\ \mathbf{V}^0\mathbf{UQ}^3+\mathbf{VUQ}^2+\mathbf{V}^2\mathbf{UQ}+\mathbf{V}^3\mathbf{UQ}^0 & \mathbf{V}^4 \end{bmatrix}
\end{aligned}
$$

Case 2: Partioning of the matrix P into three blocks

$\mathbf{P}$ can be partinioned into blocks of three as follows

$$\mathbf{P} = \begin{array}{c} \\ E_5 \\ E_3 \\ E_8 \\ E_1 \\ E_4 \\ E_9 \\ E_2 \\ E_6 \\ E_7 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 5 & 3 & 8 & 1 & 4 & 9 & 2 & 6 & 7 \end{array} \\ \left[\begin{array}{ccc|ccc|ccc} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \end{array} \right] \end{array}$$

Therefore, $\mathbf{P}$ can be written in canonical form as;

$$\mathbf{P} = \begin{bmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U} & \mathbf{0} \\ \mathbf{A} & \mathbf{D} & \mathbf{T} \end{bmatrix}$$

$$\begin{aligned} \mathbf{P}^2 &= \begin{bmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U} & \mathbf{0} \\ \mathbf{A} & \mathbf{D} & \mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U} & \mathbf{0} \\ \mathbf{A} & \mathbf{D} & \mathbf{T} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Q}^2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}^2 & \mathbf{0} \\ \mathbf{AQ}+\mathbf{TA} & \mathbf{DU}+\mathbf{TD} & \mathbf{T}^2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{P}^3 &= \begin{bmatrix} \mathbf{Q}^2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}^2 & \mathbf{0} \\ \mathbf{AQ}+\mathbf{TA} & \mathbf{DU}+\mathbf{TD} & \mathbf{T}^2 \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U} & \mathbf{0} \\ \mathbf{A} & \mathbf{D} & \mathbf{T} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Q}^3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}^3 & \mathbf{0} \\ \mathbf{AQ}^2+\mathbf{TAQ}+\mathbf{T}^2\mathbf{A} & \mathbf{DU}^2+\mathbf{TDU}+\mathbf{TD} & \mathbf{T}^3 \end{bmatrix} \end{aligned}$$

Which can be generalized as follows;

$$\mathbf{P}^n = \begin{bmatrix} \mathbf{Q}^n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}^n & \mathbf{0} \\ \mathbf{A}_n & \mathbf{D}_n & \mathbf{T}^n \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{A}_n &= \sum_{j=0}^{n-1} \mathbf{TAQ}^{(n-1)-j} \\ \mathbf{D}_n &= \sum_{j=0}^{n-1} \mathbf{TDU}^{(n-1)-j} \end{aligned}$$

Case 3: Partioning of the matrix P into four blocks

$$P = \begin{array}{c} \\ E_5 \\ E_3 \\ E_8 \\ E_1 \\ E_4 \\ E_9 \\ E_2 \\ E_6 \\ E_7 \end{array} \begin{array}{cccccccccc} 5 & 3 & 8 & 1 & 4 & 9 & 2 & 6 & 7 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \end{array}$$

which can be written in canonical fomr as followa;

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} & \mathbf{F} & \mathbf{G} \end{bmatrix}$$

$$
\begin{aligned}
\mathbf{P}^2 &= \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} & \mathbf{F} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} & \mathbf{F} & \mathbf{G} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{A}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}^2 & \mathbf{0} \\ \mathbf{DA}+\mathbf{GD} & \mathbf{EB}+\mathbf{GE} & \mathbf{FC}+\mathbf{GF} & \mathbf{G}^2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{P}^3 &= \begin{bmatrix} \mathbf{A}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}^2 & \mathbf{0} \\ \mathbf{DA}+\mathbf{GD} & \mathbf{EB}+\mathbf{GE} & \mathbf{FC}+\mathbf{GF} & \mathbf{G}^2 \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} & \mathbf{F} & \mathbf{G} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{A}^3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}^3 & \mathbf{0} \\ \mathbf{DA}^2+\mathbf{GDA}+\mathbf{G}^2\mathbf{D} & \mathbf{EB}^2+\mathbf{GEB}+\mathbf{G}^2\mathbf{E} & \mathbf{FC}^2+\mathbf{GFC}+\mathbf{G}^2\mathbf{F} & \mathbf{G}^3 \end{bmatrix}
\end{aligned}
$$

In general

$$
\mathbf{P}^n = \begin{bmatrix} \mathbf{A}^n & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}^n & \mathbf{0} \\ \mathbf{D}_n & \mathbf{E}_n & \mathbf{F}_n & \mathbf{G}^n \end{bmatrix}
$$

where

$$
\begin{aligned}
\mathbf{D}_n &= \sum_{j=0}^{n-1} \mathbf{G}^j \mathbf{D} \mathbf{A}^{(n-1)-j} \\
\mathbf{E}_n &= \sum_{j=0}^{n-1} \mathbf{G}^j \mathbf{E} \mathbf{B}^{(n-1)-j} \\
\mathbf{F}_n &= \sum_{j=0}^{n-1} \mathbf{G}^j \mathbf{F} \mathbf{C}^{(n-1)-j}
\end{aligned}
$$

In a more gfeneral fomr, we have

$$\mathbf{P}^n = \begin{bmatrix} \mathbf{Q}^n & \mathbf{0} \\ \mathbf{U}_n & \mathbf{V}^n \end{bmatrix}$$

$$\text{where } \mathbf{U}_n = \sum_{j=0}^{n-1} \mathbf{V}^j \mathbf{U} \mathbf{Q}^{(n-1)-j}$$

hence

$$\lim_{n\to\infty} \mathbf{P}^n = \text{ Asymptotic behavior}$$

Remarks on $\mathbf{P}^n$

1. $p_{jk}^{(n)} = 0$ for $E_j \in C$ and $E_k \notin C$ where C is the closed set.

2. The appearance of the power $\mathbf{A}^n$ indicates that when both E_j and E_k are in C the transition probabilities $p_{jk}^{(n)}$ are obtained from the recursion formula

$$p_{jk}^{(n+1)} = \sum_v p_{jv} p_{vk}^{(n)}$$

with the summation restricted to the states of the closed set C.

3. The appearance of $\mathbf{Q}^n$ indicates that the statement given in (2) remains true when C is replaced by its compliment $C\prime$, say.

4.

$$\mathbf{B}_n = \sum_{j=0}^{n-1} \mathbf{Q}^j \mathbf{B} \mathbf{A}^{n-j-1}$$

Note that, we have not assumed $\mathbf{A}$ to be irreducible. If C decomposes into several closed subsets, then $\mathbf{A}$ admits further portioning. There exist chains with infinitely many closed subsets.

2.11.4 Absorbing Markov chain

A Markov chain is absorbing if it has at least one absorbing state.

E_k is an absorbing state if

consider the 2×2 blocks partitioned matrix

$$\mathbf{P}=\begin{bmatrix}\mathbf{Q} & \mathbf{0}\\ \mathbf{U} & \mathbf{V}\end{bmatrix}$$

Let

$$\mathbf{Q}=\mathbf{I}$$

$$\mathbf{P}=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{U} & \mathbf{V}\end{bmatrix}$$

is an absorbing Markov chain, therefore

$$\mathbf{P}=\begin{bmatrix}\mathbf{Q}^n & \mathbf{0}\\ \mathbf{U}_n & \mathbf{V}^n\end{bmatrix}$$

where

$$\mathbf{U}_n=\sum_{j=0}^{n-1}\mathbf{V}^j\mathbf{U}\mathbf{Q}^{n-j-1}$$

becomes $\mathbf{Q}$ and $\mathbf{U}$ matrices where

$$\begin{aligned}\mathbf{U}_n &= \sum_{j=0}^{n-1}\mathbf{V}^j\mathbf{U}\mathbf{Q}^{n-j-1}\\ &= \sum_{j=0}^{n-1}\mathbf{V}^j\mathbf{U}\\ &= \left(\sum_{j=0}^{n-1}\mathbf{V}^j\right)\mathbf{U}\\ &= (\mathbf{V}^0+\mathbf{V}^1+\mathbf{V}^2+\mathbf{V}^3+\cdots+\mathbf{V}^{n-1})\mathbf{U}\\ &= (\mathbf{I}+\mathbf{V}^1+\mathbf{V}^2+\mathbf{V}^3+\cdots+\mathbf{V}^{n-1})\mathbf{U}\end{aligned}$$

Recall from ordinary algebra that

$$1+X+X^2+\cdots+X^{n-1}=\frac{1-X^n}{1-X}$$

in matrix form

$$(\mathbf{I}-\mathbf{V})(\mathbf{I}+\mathbf{V}^1+\mathbf{V}^2+\mathbf{V}^3+\cdots+\mathbf{V}^{n-1})=(\mathbf{I}-\mathbf{V}^n)$$

$$(\mathbf{I}+\mathbf{V}^1+\mathbf{V}^2+\mathbf{V}^3+\cdots+\mathbf{V}^{n-1}) = \frac{\mathbf{I}-\mathbf{V}^n}{\mathbf{I}-\mathbf{V}}$$

therefore

$$\mathbf{U}_n = (\mathbf{I}-\mathbf{V}^n)(\mathbf{I}-\mathbf{V})^{-1}\mathbf{U}$$

$$\begin{aligned}\lim_{n\to\infty}(\mathbf{I}+\mathbf{V}^1+\mathbf{V}^2+\mathbf{V}^3+\cdots+\mathbf{V}^{n-1}) &= \lim_{n\to\infty}(\mathbf{I}-\mathbf{V})^{-1}(\mathbf{I}-\mathbf{V}^n)\\ \lim_{n\to\infty}(\mathbf{I}+\mathbf{V}^1+\mathbf{V}^2+\mathbf{V}^3+\cdots+\mathbf{V}^{n-1}) &= (\mathbf{I}-\mathbf{V})^{-1}(\mathbf{I}-\lim_{n\to\infty}\mathbf{V}^n)\end{aligned}$$

but

$$\begin{aligned}(\mathbf{I}-\mathbf{V})(\mathbf{I}+\mathbf{V}^1+\mathbf{V}^2+\mathbf{V}^3+\cdots) &= \mathbf{I}+\mathbf{V}^1+\mathbf{V}^2+\mathbf{V}^3+\cdots\\ &= \mathbf{I}\end{aligned}$$

theerfore

$$\mathbf{I}+\mathbf{V}^1+\mathbf{V}^2+\mathbf{V}^3+\cdots = (\mathbf{I}-\mathbf{V})^{-1}$$

$$(\mathbf{I}-\mathbf{V})^{-1} = (\mathbf{I}-\mathbf{V})^{-1}(\mathbf{I}-\lim_{n\to\infty}\mathbf{V}^n)$$

$$\mathbf{I} = \mathbf{I}-\lim_{n\to\infty}\mathbf{V}^n$$

$$\lim_{n\to\infty}\mathbf{V}^n = 0$$

if

$$\text{if } \mathbf{P} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{U} & \mathbf{V}\end{bmatrix}$$

$$\mathbf{P}^n = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ (\mathbf{I}-\mathbf{V})^{-1}(\mathbf{I}-\mathbf{V}^n)\mathbf{U} & \mathbf{V}^n\end{bmatrix}$$

$$\begin{aligned}\lim_{n\to\infty}\mathbf{P}^n &= \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \lim_{n\to\infty}(\mathbf{I}-\mathbf{V})^{-1}(\mathbf{I}-\mathbf{V}^n)\mathbf{U} & \lim_{n\to\infty}\mathbf{V}^n\end{bmatrix}\\ &= \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ (\mathbf{I}-\mathbf{V})^{-1}\mathbf{U} & \mathbf{0}\end{bmatrix}\end{aligned}$$

The matrix $(\mathbf{I}-\mathbf{V})^{-1}$ is called the Fundamental Matrix of an absorbing Markov chain.

2.11.5 Illustrative example

reorganize the transition probability matrix $\mathbf{P}$ below;

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \end{array}$$

E_2 and E_4 are absorbing states

$$\mathbf{P} = \begin{array}{c} \\ E_2 \\ E_4 \\ E_1 \\ E_3 \\ E_5 \end{array} \begin{array}{c} \begin{array}{ccccc} 2 & 4 & 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

$$\mathbf{U} = \begin{bmatrix} 0 & \frac{1}{4} \\ 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \text{ and } \mathbf{V} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ (\mathbf{I}-\mathbf{V})^{-1}\mathbf{U} & \mathbf{0} \end{bmatrix}$$

$$\begin{aligned} (\mathbf{I}-\mathbf{V}) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \end{aligned}$$

Let

$$\mathbf{A} = |\mathbf{I} - \mathbf{V}|$$

$$(\mathbf{I} - \mathbf{V})^{-1} = \frac{1}{\det \mathbf{A}} \text{Adjoint } (\mathbf{A})$$

$$\det \mathbf{A} = \frac{3}{4} \begin{vmatrix} \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{3}{4} \times \frac{2}{6} = \frac{1}{4}$$

Cofactor matrix

$$= \begin{bmatrix} +\begin{vmatrix} \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} \end{vmatrix} & -\begin{vmatrix} 0 & -\frac{1}{3} \\ 0 & \frac{1}{2} \end{vmatrix} & +\begin{vmatrix} 0 & \frac{2}{3} \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} -\frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{1}{2} \end{vmatrix} & +\begin{vmatrix} \frac{3}{4} & -\frac{1}{4} \\ 0 & \frac{1}{2} \end{vmatrix} & -\begin{vmatrix} \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} -\frac{1}{4} & -\frac{1}{4} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} & -\begin{vmatrix} \frac{3}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{3} \end{vmatrix} & +\begin{vmatrix} \frac{3}{4} & -\frac{1}{4} \\ 0 & \frac{2}{3} \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{8} & \frac{3}{8} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$\text{Adjoint } \mathbf{A} = \begin{bmatrix} \frac{1}{3} & \frac{1}{8} & \frac{1}{4} \\ 0 & \frac{3}{8} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{aligned} (\mathbf{I} - \mathbf{V})^{-1} &= \frac{1}{\det \mathbf{A}} \text{Adjoint } (\mathbf{A}) \\ &= 4 \begin{bmatrix} \frac{1}{3} & \frac{1}{8} & \frac{1}{4} \\ 0 & \frac{3}{8} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{3} & \frac{1}{2} & 1 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & 2 \end{bmatrix} \end{aligned}$$

$$(\mathbf{I}-\mathbf{V})^{-1}\mathbf{U} = \begin{bmatrix} \frac{4}{3} & \frac{1}{2} & 1 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{4} \\ 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Therefore

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$$

2.11.6 Irreducible Markov chain

In the example of the 9×9 transition matrix, we partitioned the matrix into closed sets and non-closed sets.

Definition (i)

A Markov chain is irreducible if there exists no closed set other than itself.

Definition (ii)

A Markov chain is irreducible if every state can be reached from every other state.

Remark

In classifying states of a Markov chain the first thing to do is to find out whether or not the Markov chain is irreducible.

Definition of same type

Two states are of the same type if

(i) They are both persistent or both transient

(ii) If persistent, they are either both null or both non-null

(iii) They are of the same period

Theorem 2.1

In an irreducible Markov chain, all states are of the same types

Proof

Let E_j and E_k be two arbitrary states of an irreducible chain.

Every state can be reached from every other state and so there exists integers r and s such that

$$p_{jk} = \alpha > 0 \text{ and } p_{jk}^{(s)} = \beta > 0$$

it follows that

$$p_{jj}^{(n+r+s)} \geq p_{jk}^{(r)} p_{kk}^{(n)} p_{kj}^{(s)} = \alpha\beta p_{kk}^{(n)}$$

Here, $j, k,$ and s are fixed whereas n is arbitrary.

For transient E_j, the left side is the term of a convergent series, and therefore the same is true of $p_{kk}^{(n)}$. If further more, if $p_{jj}^{(n)} \to 0$, then also $p_{kk}^{(n)} \to 0$.

then also

The same statements remain true when the roles of j and k are interchanged, and also either E_i and E_k are transient or neither is; if one is a null state, so is the other.

Finally, suppose that E_j has period t. For $n = 0$, the right side in

$$p_{jj}^{(n+r+s)} \geq p_{jk}^{(r)} p_{kk}^{(n)} p_{kj}^{(s)} = \alpha\beta p_{kk}^{(n)}$$

is positive, and hence $r + s$ is a multiple of t.

But then the left side vanishes unless n is a multiple of t, and so E_k has a period which is a multiple of t. Interchanging the roles of j and k we see that these states have the same period.

Theorem 2.11

For a persistent E_j, there exists a unique irreducible closed set C containing E_i, and such that for every pair E_i, E_k of states in C

$$f_{ik} = 1 \text{ and } f_{kj} = 1$$

In other words starting from an arbitrary state E_i in C, the system is certain to pass through every other state of C, by the definition of closure no exit from C is possible.

Proof

Let E_k be a state that can be reached from E_j. It is then obviously possible to reach E_k without previously returning to E_j, and we denote the probability of the event by α.

Once E_k is reached, the probability of never returning to E_j is $(1 - f_{kj})$. The probability that, starting from E_j, the system never returns to E_j, is therefore at least $\alpha(1 - f_{kj})$. But for a persistent E_j the probability of no return is zero, and so $f_{kj} = 1$ for every E_k that can be reached from E_j.

Denote by C the aggregate of all states that can be reached from E_j. If E_i and E_k are in C, E_j can be reached from E_k, and hence also E_i can be reached from E_k. Thus every state in C can be reached from every other state in C , and so C is irreducible.

It follows that all states in C are persistent, and so every E_i can be assigned the role of E_j in the first part of the argument.

This means that $f_{ki} = 1$ for all E_k in C and so $f_{ik} = 1$ and $f_{ki} = 1$ is true. The preceding theorem implies that the closure of a persistent state is irreducible. This is not necessarily true of transient states.

2.12 Invariant (Stationary) Distribution

Definition

A distribution $\{\pi_k : k = 1,2,3\ldots\}$ is called invariant or stationary if it satisfies

$$\pi_k = \sum_{j=1} \pi_j p_{jk}; \quad j,k = 1,2,3\ldots$$

such that

$$\sum \pi_k = \sum_i \pi_{ij} p_{ij}$$

therefore

$$\begin{aligned}\pi_k &= \sum_j \pi_k p_{jk} \\ &= \sum_j \sum_i \{\pi_{ij} p_{ij}\} p_{jk}\end{aligned}$$

hence

$$\begin{aligned}\pi_k &= \sum_j \sum_i \pi_i p_{ij} p_{jk} \\ &= \sum_i \left\{ \pi_i \sum_j p_{ij} p_{jk} \right\} \\ &= \sum_i \pi_i p_{ik}^{(2)}\end{aligned}$$

By Chapman-Kolgomorov equation

$$= \sum_i \left\{ \pi_h p_{hi} p_{ik}^{(2)} \right\}$$

therefore

$$\begin{aligned}\pi_k &= \sum_i \sum_h \pi_h p_{hi} p_{ik}^{(2)} \\ &= \sum_h \left\{ \pi_h \sum_i p_{hi} p_{ik}^{(2)} \right\} \\ &= \sum_h \pi_h p_{ik}^{(3)}\end{aligned}$$

In general

$$\begin{aligned}\pi_k &= \sum_j \pi_j p_{jk} \\ &= \sum_j \pi_j p_{jk}^{(n)}\end{aligned}$$

In matrix form

$$\pi_k = \sum_{j=1}^{4} \pi_j p_{jk}, \quad k = 1,2,3\ldots$$

For $k = 1$

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21} + \pi_3 p_{31} + \pi_4 p_{41}$$

For $k=2$

$$\pi_2 = \pi_1 p_2 + \pi_2 p_{22} + \pi_3 p_{32} + \pi_4 p_{42}$$

For $k=3$

$$\pi_3 = \pi_1 p_{13} + \pi_2 p_{23} + \pi_3 p_{33} + \pi_4 p_{43}$$

For $k=3$

$$\pi_4 = \pi_1 p_{14} + \pi_2 p_{24} + \pi_3 p_{34} + \pi_4 p_{44}$$

$$\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{21} & p_{31} & p_{41} \\ p_{12} & p_{22} & p_{32} & p_{42} \\ p_{13} & p_{23} & p_{33} & p_{43} \\ p_{14} & p_{24} & p_{34} & p_{44} \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix}$$

$$\pi = \mathbf{P}'\pi \quad \text{for finite case}$$

$$\begin{aligned} \pi' &= \mathbf{P}'\pi \\ &= \pi'\mathbf{P} \quad \text{for infinite case} \end{aligned}$$

Theorem 2.12

In an irreducible ergodic Markov chain, the limits

$$\pi_k = \lim_{n\to\infty} p_{jk}^{(n)} \qquad \exists$$

and are independent of the initial state E_j. Furthermore, $\pi_k > 0$, $\sum \pi_k = 1$, and $\sum \pi_k = \sum_j \pi_j p_{jk}$.

Proof

(i) Suppose the chain is irreducible and ergodic, and define π_k by

$$\mu_k = \frac{1}{\pi_k}$$

where μ_k is the mean recurrence time of state E_k.For a persistent E_j there exist a unique irreducible closed set C containing E_j and such that for every pair E_i, E_k of states in C

$$f_{ik} = 1 \text{ and } f_{ki} = 1$$

which guarantees that $f_{ij} = 1$ for every pair of states , and so the assertion

$$\pi_k = \lim_{n\to\infty} p_{jk}^{(n)}$$

reduces to

$$p_{ij}^{(n)} \to f_{ij}\pi_j^{-1}$$

Now

$$p_{ik}^{(n+1)} = \sum_j p_{ij}^{(n)} p_{jk}$$

As $n \to \infty$, the left side approaches π_k while the general term of teh sum on the RHS tends to $\pi_k p_{jk}$. Taking only finitely many terms we infer that

$$\pi_k \geq \sum_j \pi_j p_{jk}$$

for a fixed i and n the left sides in

$$p_{ik}^{(n+1)} = \sum_j p_{ij}^{(n)} p_{jk}$$

add up to unity and hence

$$s = \pi_k \leq 1$$

summing over k in

$$\pi_k \geq \sum_j \pi_j p_{jk}$$

we get the relation $s \geq s$ in which the inequality sign is not possible. We conclude therefore that in the equation above, the inequality sign holds for all k, and so the first part of the theorem is true.

(ii) Assume $\mu_k \geq 0$ and $(\sum_{k=1}) - (\pi_k = \sum_i \pi_i p_{ij})$. By induction

$$\pi_k = \sum_i \pi_i p_{ij}^{(n)}, \quad \text{for every } n > 1$$

Since the chain is assumed to be irreducible, all states are of the same type. If they were transient or null states, the right side in $\pi_k = \sum_i \pi_i p_{ij}^{(n)}$ would tend to zero as $n \to \infty$, and this cannot be true for all k because the π_k adds to unity. Periodic chains being excluded, this means that the chain is ergodic and so the first part of the theorem applies. Thus, letting $n \to \infty$,

$$\pi_k = \sum_i \pi_i \pi_k^{-1}$$

Accordingly, the probability distribution $\{\pi_k\}$ is proportional to the probability distribution $\left\{\pi_k^{-1}\right\}$ and so $\pi_k = \pi_k^{-1}$ as asserted.

Conversely

Suppose that the chain is irreducible and aperiodic and that there exist numbers $\pi_k \geq 0$ satisfying

$$\sum \pi_k = 1 \text{ and } \pi_k = \sum_i \pi_j p_{jk}^{(n)}$$

Then the chain is ergodic and π_ks are given by

$$\lim_{n\to\infty} p_{jk}^{(n)}$$

and

$$\pi_k = \frac{1}{\mu_k}$$

Where μ_k is the mean recurrence time for E_k.

For any numbers a_k and p_{jk} satisfying these conditions, the assignment

$$P\{(E_{jo}, E_{j1}, E_{j2}, \ldots, E_{jn})\} = a_{jo}p_{joj1}p_{j1j2}p_{j2j3}\cdots p_{jn-2jn-1}p_{jn-1jn}$$

is a permissible definition of probabilities in the sample space corresponding to $n+1$ trials.

The numbers defined in $P\{(E_{jo}, E_{j1}, E_{j2}, \ldots, E_{jn})\} = a_{jo}p_{joj1}p_{j1j2}p_{j2j3}\cdots p_{jn-2jn-1}p_{jn-1jn}$ being non-negative, we need only to prove that they add to unity.

Fix first $j_o, j_1, j_2, \ldots, j_{n-1}$ and add the numbers $P\{(E_{jo}, E_{j1}, E_{j2}, \ldots, E_{jn})\} = a_{jo}p_{joj1}p_{j1j2}p_{j2j3}\cdots p_{jn-2}$ for all possible j_n. Using the fact that $p_{jo} + p_{j1} + p_{j2}\cdots = 1$ with $j = j_{n+1}$, we see immediately that the sum equals

$$a_{jo}p_{joj1}p_{j1j2}p_{j2j3}\cdots p_{jn-2jn-1}p_{jn-1jn}$$

Thus the sum over all the numbers $P\{(E_{jo}, E_{j1}, E_{j2}, \ldots, E_{jn})\} = a_{jo}p_{joj1}p_{j1j2}p_{j2j3}\cdots p_{jn-2jn-1}p_{jn-1jn}$ does not depend on n, and since $\sum a_{jo} = 1$, the sum equals unity for all n.

To obtain the probability of the event "first two trials results in (E_i, E_k), we have to fix $j_0 = k$, and add the probabilities $P\{(E_{jo}, E_{j1}, E_{j2}, \ldots, E_{jn})\} = a_{jo}p_{joj1}p_{j1j2}p_{j2j3}\cdots p_{jn-2jn-1}p_{jn-1jn}$ for all possible $j_2, j_3, \ldots, j_n$. This shows that the sum is $a_j p_{jk}$ and is independent of n.

2.13 Determining the n^{th} Power of a Transition Matrix

2.13.1 Use of generating function in determining $\mathbf{P}^n$

Recall

$$\mathbf{P} = ((p_{ij})) \implies \mathbf{P}^n = \left(\left(p_{ij}^{(n)}\right)\right)$$

Let

$$\begin{aligned} p_{ij}(s) &= \sum_{n=0}^{\infty} p_{ij}^{(n)} s^n \\ &\implies P(s) = ((p_{ij}(s)s)) \end{aligned}$$

By Chapman-Kolgomorov equation

$$p_{ij}^{(n+1)} = \sum_{k=1}^{v} p_{ik} p_{kj}^{(n)}$$

therefore

$$\sum_{n=0}^{\infty} p_{ij}^{(n+1)} s^n = \sum_{n=0}^{\infty} \sum_{k=1}^{v} p_{ik} p_{kj}^{(n)} s^n$$

$\implies$

$$\begin{aligned} \frac{1}{s} \sum_{n=0}^{\infty} p_{ij}^{(n+1)} s^{n+1} &= \sum_{n=0}^{\infty} \sum_{k=1}^{v} p_{ik} p_{kj}^{(n)} s^n \\ &= \sum_{k=1}^{v} p_{ik} \left\{ \sum_{n=0}^{\infty} p_{kj}^{(n)} s^n \right\} \\ &= \sum_{k=1}^{v} \left\{ p_{ik} p_{kj}(s) \right\} \end{aligned}$$

and in matrix form we have

$$\frac{1}{s} \left(\left(\sum_{n=0}^{\infty} p_{ij}^{(n+1)} s^{n+1} \right) \right) = \left(\left(\sum_{k=1}^{v} p_{ik} p_{kj}(s) \right) \right)$$

$\implies$

$$\frac{1}{s} \left(\left(p_{ij}(s) - p_{ij}^{(0)} \right) \right) = \left(\left(\sum_{k=1}^{v} p_{ik} p_{kj}(s) \right) \right)$$

$$\frac{1}{s} \left[((p_{ij}(s))) - \left(\left(p_{ij}^{(0)} \right) \right) \right] = \left(\left(\sum_{k=1}^{v} p_{ik} p_{kj}(s) \right) \right)$$

$$\frac{1}{s} [\mathbf{P}(s) - \mathbf{I}] = \mathbf{P}\mathbf{P}(s)$$

Therefore

$$\begin{aligned} [\mathbf{P}(s) - \mathbf{I}] &= s\mathbf{P}\mathbf{P}(s) \\ \mathbf{P}(s) - \mathbf{P}\mathbf{P}(s) &= \mathbf{I} \\ [\mathbf{I} - \mathbf{sP}] \mathbf{P}(s) &= \mathbf{I} \end{aligned}$$

Therefore

$$\mathbf{P}(s) = [\mathbf{I} - \mathbf{sP}]^{-1}$$

Suppose an $n \times n$ matrix $\mathbf{P}$ is partitioned as follows;

$$\mathbf{P} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix}$$

with $\mathbf{I}$ -$r \times r$ and $\mathbf{Q}$- $(n-r) \times (n-r)$, then

$$\begin{aligned} \mathbf{P}(s) &= [\mathbf{I} - \mathbf{sP}]^{-1} \\ &= [\mathbf{I}_n - \mathbf{sP}]^{-1} \end{aligned}$$

Using the relation

$$\begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{V} & \mathbf{W} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{T}^{-1} & \mathbf{0} \\ -\mathbf{W}^{-1}\mathbf{V}\mathbf{T}^{-1} & \mathbf{W}^{-1} \end{bmatrix}$$

then

$$\begin{aligned} \mathbf{P}(s) &= [\mathbf{I}_n - \mathbf{sP}]^{-1} \\ &= \left[\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix} - s \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{R} & \mathbf{Q} \end{pmatrix}\right]^{-1} \\ &= \begin{bmatrix} \mathbf{I}_r - s\mathbf{I}_r & \mathbf{0} \\ -s\mathbf{R} & \mathbf{I}_{n-r} - s\mathbf{Q} \end{bmatrix}^{-1} \end{aligned}$$

Suppose

$$\begin{bmatrix} \mathbf{I}_r - s\mathbf{I}_r & \mathbf{0} \\ -s\mathbf{R} & \mathbf{I}_{n-r} - s\mathbf{Q} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

then

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{I}_r - s\mathbf{I}_r & \mathbf{0} \\ -s\mathbf{R} & \mathbf{I}_{n-r} - s\mathbf{Q} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{bmatrix}$$

Therefore

$$\begin{aligned} \mathbf{A}(1-s)\mathbf{I}_r - \mathbf{B}s\mathbf{R} &= \mathbf{I}_r && \text{(i)} \\ \mathbf{B}(\mathbf{I}_{n-r} - s\mathbf{Q}) &= \mathbf{0} && \text{(ii)} \\ \mathbf{C}(1-s)\mathbf{I}_r - \mathbf{D}s\mathbf{R} &= \mathbf{0} && \text{(iii)} \\ \mathbf{D}(\mathbf{I}_{n-r} - s\mathbf{Q}) &= \mathbf{I}_{n-r} && \text{(iv)} \end{aligned}$$

From (ii)

$$\mathbf{B} = \mathbf{0}$$

since $(\mathbf{I}_{n-r} - s\mathbf{Q}) \neq \mathbf{0}$. Therefore, (i) becomes

$$(1-s)\mathbf{A} = \mathbf{I}_r$$

$\Longrightarrow$

$$\mathbf{A} = \frac{1}{(1-s)}\mathbf{I}_r$$

From (iv)

$$\mathbf{D} = (\mathbf{I}_{n-r} - s\mathbf{Q})^{-1}$$

and from (iii)

$$\begin{aligned}(1-s)\mathbf{C} &= s\mathbf{DR}\\ \mathbf{C} &= \frac{s}{1-s}(\mathbf{I}_{n-r} - s\mathbf{Q})^{-1}\mathbf{R}\end{aligned}$$

Therefore

$$\begin{bmatrix} (\mathbf{1}-s)\mathbf{I}_r & \mathbf{0} \\ -s\mathbf{R} & \mathbf{I}_{n-r} - s\mathbf{Q} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{(1-s)}\mathbf{I}_r & \mathbf{0} \\ \frac{s}{1-s}(\mathbf{I}_{n-r} - s\mathbf{Q})^{-1}\mathbf{R} & (\mathbf{I}_{n-r} - s\mathbf{Q})^{-1} \end{bmatrix}$$

2.13.2 Illustrative examples

Example 1

Reorganize the transition matrix below

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \end{array}$$

E_2 and E_4 are absorbing states hence $\mathbf{P}$ can be written as

$$\mathbf{P} = \begin{array}{c} \\ E_2 \\ E_4 \\ E_1 \\ E_3 \\ E_5 \end{array} \begin{array}{c} \begin{array}{ccccc} 2 & 4 & 1 & 3 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

$$\mathbf{R} = \begin{bmatrix} 0 & \frac{1}{4} \\ 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \text{ and } \mathbf{Q} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Using the result that

$$\begin{bmatrix} x & a & c \\ 0 & y & b \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} yz & -Qz & ab-cy \\ 0 & zx & -bx \\ 0 & 0 & xy \end{bmatrix} = xyz\mathbf{I}$$

We can find an expression for

$$\lim_{n\to\infty} \mathbf{P}^n$$

But we are given that

$$\begin{bmatrix} x & a & c \\ 0 & y & b \\ 0 & 0 & z \end{bmatrix}^{-1} = \frac{1}{xyz} \begin{bmatrix} yz & -Qz & ab-cy \\ 0 & zx & -bx \\ 0 & 0 & xy \end{bmatrix}$$

In this case;

$$\begin{aligned} x &= 1-\frac{1}{4}s,\ y = 1-\frac{1}{3}s, \text{ and } z = 1-\frac{1}{2}s \\ a &= -\frac{1}{4}s,\ b = -\frac{1}{3}s, \text{ and } c = -\frac{1}{4}s \end{aligned}$$

$$
\begin{aligned}
[\mathbf{I}-s\mathbf{Q}]^{-1} &= \begin{bmatrix} 1-\frac{1}{4}s & -\frac{1}{4}s & -\frac{1}{4}s \\ 0 & 1-\frac{1}{3}s & b=-\frac{1}{3}s \\ 0 & 0 & 1-\frac{1}{2}s \end{bmatrix}^{-1} \\
&= \frac{1}{\left(1-\frac{1}{4}s\right)\left(1-\frac{1}{3}s\right)\left(1-\frac{1}{2}s\right)} \\
&\times \begin{bmatrix} \left(1-\frac{1}{3}s\right)\left(1-\frac{1}{2}s\right) & \frac{1}{4}s\left(1-\frac{1}{2}s\right) & \frac{1}{12}s^2-\frac{1}{4}s\left(1-\frac{1}{3}s\right) \\ 0 & \left(1-\frac{1}{4}s\right)\left(1-\frac{1}{2}s\right) & \frac{1}{3}s\left(1-\frac{1}{4}s\right) \\ 0 & 0 & \left(1-\frac{1}{4}s\right)\left(1-\frac{1}{3}s\right) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{(1-\frac{1}{4}s)} & \frac{-\frac{1}{4}s}{(1-\frac{1}{4}s)(1-\frac{1}{3}s)} & \frac{\frac{1}{12}s^2+\frac{1}{4}s(1-\frac{1}{3}s)}{(1-\frac{1}{4}s)(1-\frac{1}{3}s)(1-\frac{1}{2}s)} \\ 0 & \frac{1}{(1-\frac{1}{3}s)} & \frac{\frac{1}{3}s}{(1-\frac{1}{3}s)(1-\frac{1}{2}s)} \\ 0 & 0 & \frac{1}{(1-\frac{1}{2}s)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{(1-\frac{1}{4}s)} & \frac{-\frac{1}{4}s}{(1-\frac{1}{4}s)(1-\frac{1}{3}s)} & \frac{\frac{1}{4}s}{(1-\frac{1}{4}s)(1-\frac{1}{3}s)(1-\frac{1}{2}s)} \\ 0 & \frac{1}{(1-\frac{1}{3}s)} & \frac{\frac{1}{3}s}{(1-\frac{1}{3}s)(1-\frac{1}{2}s)} \\ 0 & 0 & \frac{1}{(1-\frac{1}{2}s)} \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
[\mathbf{I}-s\mathbf{Q}]^{-1}\mathbf{R} &= \begin{bmatrix} \frac{1}{(1-\frac{1}{4}s)} & \frac{-\frac{1}{4}s}{(1-\frac{1}{4}s)(1-\frac{1}{3}s)} & \frac{\frac{1}{4}s}{(1-\frac{1}{4}s)(1-\frac{1}{3}s)(1-\frac{1}{2}s)} \\ 0 & \frac{1}{(1-\frac{1}{3}s)} & \frac{\frac{1}{3}s}{(1-\frac{1}{3}s)(1-\frac{1}{2}s)} \\ 0 & 0 & \frac{1}{(1-\frac{1}{2}s)} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{4} \\ 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\frac{1}{4}s}{(1-\frac{1}{4}s)} + \frac{\frac{1}{12}s}{(1-\frac{1}{4}s)(1-\frac{1}{3}s)} + \frac{\frac{1}{16}s}{(1-\frac{1}{4}s)(1-\frac{1}{3}s)(1-\frac{1}{2}s)} & \frac{\frac{1}{16}s}{(1-\frac{1}{4}s)(1-\frac{1}{3}s)(1-\frac{1}{2}s)} \\ \frac{\frac{1}{3}}{(1-\frac{1}{3}s)} + \frac{\frac{1}{12}s}{(1-\frac{1}{4}s)(1-\frac{1}{3}s)} & \frac{\frac{1}{12}s}{(1-\frac{1}{4}s)(1-\frac{1}{3}s)} \\ \frac{\frac{1}{4}}{(1-\frac{1}{2}s)} & \frac{\frac{1}{4}}{(1-\frac{1}{2}s)} \end{bmatrix}
\end{aligned}
$$

Illustrative example 2

$$
\mathbf{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}
$$

$$
\begin{aligned}
\mathbf{P}(s) &= [\mathbf{I} - s\mathbf{P}]^{-1} \\
&= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - s\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}\right]^{-1} \\
&= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} (1-a)\,s & as \\ bs & (1-b)\,s \end{pmatrix}\right]^{-1} \\
&= \begin{bmatrix} 1-(1-a)\,s & -as \\ -bs & 1-(1-b)\,s \end{bmatrix}^{-1} \\
&= \frac{1}{[(1-(1-a)\,s)\,(1-(1-b)\,s)-abs^2]} \begin{bmatrix} 1-(1-b)\,s & as \\ bs & 1-(1-a)\,s \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\left[(1-(1-a)\,s)\,(1-(1-b)\,s)-abs^2\right] &= \left[1-(1-a)\,s-(1-b)\,s+(1-a)(1-b)s^2-abs^2\right] \\
&= \left[1-s+as-s+bs+(1-a-b+ab)s^2-abs^2\right] \\
&= \left[1-s-(1-a-b)s+(1-a-b)s^2\right] \\
&= [(1-s)-(1-a-b)s\,(1-s)] \\
&= (1-s)\,[1-(1-a-b)s] \\
&= (1-s)\,[1-cs]
\end{aligned}
$$

where

$$
c = 1-a-b
$$

hence

$$
\mathbf{P}(s) = \frac{1}{(1-s)\,[1-cs]} \begin{bmatrix} 1-(1-b)\,s & as \\ bs & 1-(1-a)\,s \end{bmatrix}
$$

$$
\begin{aligned}
\mathbf{P}_{11}(s) &= \frac{1-(1-b)\,s}{(1-s)\,[1-cs]} \\
&= [1-(1-b)\,s]\left[1+s+s^1+s^2+\cdots s^{n-1}+s^n+\cdots\right] \\
&\quad \times \left[1+(cs)+(cs)^1+(cs)^2+\cdots(cs)^{n-1}+(cs)^n+\cdots\right] \\
&= [1-(1-b)\,s] \quad \begin{matrix} 1+(cs)+(cs)^1+(cs)^2+\cdots(cs)^{n-1} & c^n \\ s+cs^2+c^2s^3+\cdots c^{n-2} & c^{n-1} \\ \vdots & \\ + & \end{matrix}
\end{aligned}
$$

$$
\begin{aligned}
p_{11}^{(n)} &= c^n + c^{n-1} + c^{n-2} + \cdots + c + 1 \\
&= \frac{1-c^{n+1}}{1-c}
\end{aligned}
$$

$$
\begin{aligned}
p_{12}^{(n)} &= 1 - \frac{1-c^{n+1}}{1-c} \\
&= \frac{1-c}{1-c} - \frac{1-c^{n+1}}{1-c} \\
&= \frac{1-c-1+c^{n+1}}{1-c} \\
&= \frac{c^{n+1}-c}{1-c}
\end{aligned}
$$

2.13.3 Applying the relation $P(s) = \frac{1}{1-F(s)}$ in determining $\mathbf{P}^n$

$$
P = \begin{array}{c} \\ E_0 \\ E_1 \end{array}
\begin{array}{c} \begin{array}{cc} E_0 & E_1 \end{array} \\ \left[\begin{array}{cc} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{array}\right] \end{array}; \qquad \alpha_{00} + \alpha_{01} = 1 \text{ and } \alpha_{10} + \alpha_{11} = 1
$$

$$
\begin{aligned}
f_{00}^{(1)} &= \Pr[E_0 \to E_0] = \alpha_{00} \\
f_{00}^{(2)} &= \Pr[E_0 \to E_1 \to E_0] = \alpha_{01}\alpha_{10} \\
f_{00}^{(3)} &= \Pr[E_0 \to E_1 \to E_1 \to E_0] = \alpha_{01}\alpha_{11}\alpha_{10} \\
f_{00}^{(4)} &= \Pr[E_0 \to E_1 \to E_1 \to E_1 \to E_0] = \alpha_{01}\alpha_{11}^2\alpha_{10} \\
f_{00}^{(5)} &= \Pr[E_0 \to E_1 \to E_1 \to E_1 \to E_1 \to E_0] = \alpha_{01}\alpha_{11}^3\alpha_{10}
\end{aligned}
$$

In general

$$
f_{00}^{(n)} = \alpha_{01} (\alpha_{11})^{n-2} \alpha_{10}; \; n = 1, 2, 3 \ldots
$$

Multiplying both sides by S^n and summing the result over n we get

$$
\begin{aligned}
\sum_{n=2}^{\infty} f_{00}^{(n)} s^n &= \sum_{n=2}^{\infty} \alpha_{01} (\alpha_{11})^{n-2} \alpha_{10} s^n \\
&= \alpha_{01}\alpha_{10} \sum_{n=2}^{\infty} (\alpha_{11})^{n-2} s^n \\
&= \alpha_{01}\alpha_{10} s^2 \sum_{n=2}^{\infty} (\alpha_{11})^{n-2} s^{n-2} \\
&= \alpha_{01}\alpha_{10} s^2 \left[1 + \alpha_{11}s + (\alpha_{11}s)^2 + \cdots\right] \\
&= \frac{\alpha_{01}\alpha_{10} s^2}{1-\alpha_{11}s}
\end{aligned}
$$

but

$$\sum_{n=2}^{\infty} f_{00}^{(n)} s^n = F_{00}(s) - \alpha_{00}(s)$$

thus

$$\begin{aligned} F_{00}(s) - \alpha_{00}(s) &= \frac{\alpha_{01}\alpha_{10}s^2}{1-\alpha_{11}s} \\ F_{00}(s) &= \alpha_{00}(s) + \frac{\alpha_{01}\alpha_{10}s^2}{1-\alpha_{11}s} \end{aligned}$$

but we know that

$$P(s) = \frac{1}{1-F(s)}$$

$\Longrightarrow$

$$\begin{aligned} P_{00}(s) &= \frac{1}{1-\alpha_{00}s - \frac{\alpha_{01}\alpha_{10}s^2}{1-\alpha_{11}s}} \\ &= \frac{1}{\frac{1-\alpha_{11}s-\alpha_{00})(1-\alpha_{11}s)-\alpha_{01}\alpha_{10}s^2}{1-\alpha_{11}s}} \\ &= \frac{1-\alpha_{11}s}{1-\alpha_{11}s-\alpha_{00}s(1-\alpha_{11}s)-\alpha_{01}\alpha_{10}s^2} \end{aligned}$$

but

$$\alpha_{11} + \alpha_{10} = 1$$

$\Longrightarrow$

$$\alpha_{11} = 1 - \alpha_{10}$$

and

$$\begin{aligned} \alpha_{00} + \alpha_{01} &= 1 \\ \alpha_{00} &= 1 - \alpha_{01} \end{aligned}$$

thus we have

$$\begin{aligned} P_{00}(s) &= \frac{1-(1-\alpha_{10})\,s}{1-(\alpha_{11}+\alpha_{00})\,s + (\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10})\,s^2} \\ &= \frac{1-(1-\alpha_{10})\,s}{1-(1-\alpha_{10}+1-\alpha_{01})\,s + [(1-\alpha_{01})\,(1-\alpha_{10}) - \alpha_{01}\alpha_{10}]\,s^2} \\ &= \frac{1-(1-\alpha_{10})\,s}{1-(2-\alpha_{10}-\alpha_{01})\,s + [1-\alpha_{01}-\alpha_{10}+\alpha_{01}\alpha_{10}-\alpha_{01}\alpha_{10}]\,s^2} \\ &= \frac{1-(1-\alpha_{10})\,s}{1-(2-\alpha_{10}-\alpha_{01})\,s + [1-\alpha_{01}-\alpha_{10}]\,s^2} \end{aligned} \qquad \textbf{(15)}$$

Now, s is solved for by factorizing in the denominator. Let

$$\begin{aligned} a &= 1-\alpha_{01}-\alpha_{10} \\ b &= -(2-\alpha_{10}-\alpha_{01}) \\ c &= 1 \end{aligned}$$

using the quadratic formula

$$s = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\begin{aligned} -b &= --(2-\alpha_{10}-\alpha_{01}) = (2-\alpha_{10}-\alpha_{01}) \\ b^2 &= (2-\alpha_{10}-\alpha_{01})^2 = (2-(\alpha_{10}+\alpha_{01}))^2 \\ &= 4-4(\alpha_{10}+\alpha_{01})+(\alpha_{10}+\alpha_{01})^2 \\ &= 4-4\alpha_{10}-4\alpha_{01}+(\alpha_{01})^2+2\alpha_{10}\alpha_{01}+(\alpha_{10})^2 \end{aligned}$$

$$\begin{aligned} 4ac &= 4(1-\alpha_{01}-\alpha_{10})1 \\ &= 4-4\alpha_{01}-4\alpha_{10} \end{aligned}$$

therefore

$$\begin{aligned} b^2-4ac &= 4-4\alpha_{10}-4\alpha_{01}+(\alpha_{01})^2+2\alpha_{10}\alpha_{01}+(\alpha_{10})^2-(4-4\alpha_{01}-4\alpha_{10}) \\ &= (\alpha_{01})^2+2\alpha_{10}\alpha_{01}+(\alpha_{10})^2 \end{aligned}$$

$$2a = 2(1-\alpha_{01}-\alpha_{10})$$

hence

$$\begin{aligned} s &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ &= \frac{(2-\alpha_{10}-\alpha_{01}) \pm \sqrt{(\alpha_{01})^2+2\alpha_{10}\alpha_{01}+(\alpha_{10})^2}}{2(1-\alpha_{01}-\alpha_{10})} \\ &= \frac{(2-\alpha_{10}-\alpha_{01}) \pm (\alpha_{01}+\alpha_{10})}{2(1-\alpha_{01}-\alpha_{10})} \end{aligned}$$

hence

$$\begin{aligned} s &= \frac{(2-\alpha_{10}-\alpha_{01})-\alpha_{01}+\alpha_{10}}{2(1-\alpha_{01}-\alpha_{10})} \\ &= \frac{2-2\alpha_{10}-2\alpha_{01}}{2(1-\alpha_{01}-\alpha_{10})} \\ &= \frac{2(1-\alpha_{01}-\alpha_{10})}{2(1-\alpha_{01}-\alpha_{10})} \\ &= 1 \end{aligned}$$

or

$$\begin{aligned} s &= \frac{\alpha_{10}+\alpha_{01}-2+(\alpha_{01}+\alpha_{10})}{2(1-\alpha_{01}-\alpha_{10})} \\ &= \frac{2}{2(1-\alpha_{01}-\alpha_{10})} \\ &= \frac{1}{(1-\alpha_{01}-\alpha_{10})} \end{aligned}$$

In summary

$$s=1 \text{ or } s=\frac{1}{(1-\alpha_{01}-\alpha_{10})}$$
$$1-s=0 \text{ or } (1-\alpha_{01}-\alpha_{10})\,s=1$$

$\Longrightarrow$

$$1-(1-\alpha_{01}-\alpha_{10})\,s=0$$

thus

$$1-(2-\alpha_{01}-\alpha_{10})\,s+(1-\alpha_{01}-\alpha_{10})\,s^2=(1-s)\,[1-(1-\alpha_{01}-\alpha_{10})\,s]$$

hence

$$P_{00}(s)=\frac{1-(1-\alpha_{10})\,s}{(1-s)\,[1-(1-\alpha_{01}-\alpha_{10})\,s]}$$

using partial fractions, we obtain

$$\mathbf{P}_{00}(s)=\frac{A}{1-s}+\frac{B}{[1-(1-\alpha_{01}-\alpha_{10})\,s]}$$

hence

$$\frac{1-(1-\alpha_{10})\,s}{(1-s)\,[1-(1-\alpha_{01}-\alpha_{10})\,s]}=\frac{A}{1-s}+\frac{B}{[1-(1-\alpha_{01}-\alpha_{10})\,s]}$$

$$\begin{aligned} 1-(1-\alpha_{10})\,s &= A\,[1-(1-\alpha_{01}-\alpha_{10})\,s]+B(1-s) \\ 1-(1-\alpha_{10})\,s &= A-A\,(1-\alpha_{01}-\alpha_{10})\,s+B-Bs \\ 1s^0-(1-\alpha_{10})\,s^1 &= (A+B)\,s^0+[-B-A\,(1-\alpha_{01}-\alpha_{10})]\,s^1 \end{aligned}$$

from the coefficients, it follows that

$$A+B=1$$

and

$$\begin{aligned} [-B-A\,(1-\alpha_{01}-\alpha_{10})] &= -(1-\alpha_{10}) \\ [B+A\,(1-\alpha_{01}-\alpha_{10})] &= (1-\alpha_{10}) \\ B+A+A\,(-\alpha_{01}-\alpha_{10}) &= (1-\alpha_{10}) \\ (B+A)-A\,(\alpha_{01}+\alpha_{10}) &= (1-\alpha_{10}) \\ 1-A\,(\alpha_{01}+\alpha_{10}) &= (1-\alpha_{10}) \end{aligned}$$

$$\begin{aligned} A(\alpha_{01}+\alpha_{10}) &= 1-(1-\alpha_{10}) \\ A(\alpha_{01}+\alpha_{10}) &= \alpha_{10} \\ A &= \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}} \end{aligned}$$

and

$$\begin{aligned} B &= 1-A \\ &= 1-\frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}} \\ &= \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}} \end{aligned}$$

hence

$$\begin{aligned} \mathbf{p}_{00}(s) &= \frac{A}{1-s}+\frac{B}{[1-(1-\alpha_{01}-\alpha_{10})s]} \\ &= \frac{\frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}}}{1-s}+\frac{\frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}}}{[1-(1-\alpha_{01}-\alpha_{10})s]} \\ &= \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}}\left(\frac{1}{1-s}\right)+\frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}}\left(\frac{1}{[1-(1-\alpha_{01}-\alpha_{10})s]}\right) \end{aligned}$$

$$\begin{aligned} \frac{1}{1-s} &= 1+s+s^2+\cdots \\ &= \sum_{n=0}^{\infty} s^n \end{aligned}$$

$$\frac{1}{[1-(1-\alpha_{01}-\alpha_{10})s]}=\sum_{n=0}^{\infty}(1-\alpha_{01}-\alpha_{10})^n s^n$$

Therefore

$$\begin{aligned} p_{00}(s) &= \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}}\sum_{n=0}^{\infty}s^n+\frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}}\sum_{n=0}^{\infty}(1-\alpha_{01}-\alpha_{10})^n s^n \\ &= \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}}\sum_{n=0}^{\infty}s^n+\frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}}(1-\alpha_{01}-\alpha_{10})^n\sum_{n=0}^{\infty}s^n \end{aligned}$$

but $p_{00}^{(n)}$ is the coefficient of s^n in $p_{00}(s)$ hence

$$p_{00}^{(n)}=\frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}}+\frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}}(1-\alpha_{01}-\alpha_{10})^n$$

but

$$p_{00}^{(n)}+p_{01}^{(n)}=1$$

$\Longrightarrow$

$$p_{01}^{(n)} = 1 - p_{00}^{(n)}$$

hence

$$\begin{aligned} p_{01}^{(n)} &= 1 - \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}} + \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}}\left(1-\alpha_{01}-\alpha_{10}\right)^n \\ &= \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}}\left\{1-\left(1-\alpha_{01}-\alpha_{10}\right)^n\right\} \end{aligned}$$

Similarly

$$\begin{aligned} f_{11}^{(1)} &= \Pr\left[E_1 \to E_1\right] = \alpha_{11} \\ f_{11}^{(2)} &= \Pr\left[E_1 \to E_0 \to E_1\right] = \alpha_{10}\alpha_{01} \\ f_{11}^{(3)} &= \Pr\left[E_1 \to E_0 \to E_0 \to E_1\right] = \alpha_{10}\alpha_{00}\alpha_{01} \\ f_{11}^{(4)} &= \Pr\left[E_1 \to E_0 \to E_0 \to E_0 \to E_1\right] = \alpha_{10}\alpha_{00}^2\alpha_{01} \\ f_{11}^{(5)} &= \Pr\left[E_1 \to E_0 \to E_0 \to E_0 \to E_0 \to E_1\right] = \alpha_{10}\alpha_{00}^3\alpha_{01} \end{aligned}$$

In general

$$f_{11}^{(n)} = \alpha_{10}\left(\alpha_{00}\right)^{n-2}\alpha_{01}; \quad n = 2,3,\ldots$$

Multiplying both sides by S^n and summing the result over n we get

$$\begin{aligned} \sum_{n=2}^{\infty} f_{11}^{(n)} s^n &= \sum_{n=2}^{\infty} \alpha_{10}\left(\alpha_{00}\right)^{n-2}\alpha_{01}s^n \\ &= \alpha_{01}\alpha_{10}\sum_{n=2}^{\infty}\left(\alpha_{00}\right)^{n-2}s^n \\ &= \alpha_{01}\alpha_{10}s^2\sum_{n=2}^{\infty}\left(\alpha_{00}\right)^{n-2}s^{n-2} \\ &= \alpha_{01}\alpha_{10}s^2\left[1+\alpha_{00}s+\left(\alpha_{00}s\right)^2+\cdots\right] \\ &= \frac{\alpha_{01}\alpha_{10}s^2}{1-\alpha_{00}s} \end{aligned}$$

but

$$\sum_{n=2}^{\infty} f_{11}^{(n)} s^n = F_{11}(s) - \alpha_{11}(s)$$

thus

$$\begin{aligned} F_{11}(s) - \alpha_{11}(s) &= \frac{\alpha_{01}\alpha_{10}s^2}{1-\alpha_{00}s} \\ F_{11}(s) &= \alpha_{11}(s) + \frac{\alpha_{01}\alpha_{10}s^2}{1-\alpha_{00}s} \end{aligned}$$

but we know that

$$P(s) = \frac{1}{1-F(s)}$$

$\Longrightarrow$

$$
\begin{aligned}
P_{11}(s) &= \frac{1}{1-\alpha_{11}s-\frac{\alpha_{01}\alpha_{10}s^2}{1-\alpha_{00}s}} \\
&= \frac{1}{\frac{1-\alpha_{11}s-\alpha_{00})(1-\alpha_{11}s)-\alpha_{01}\alpha_{10}s^2}{1-\alpha_{00}s}} \\
&= \frac{1-\alpha_{00}s}{1-\alpha_{00}s-\alpha_{11}s(1-\alpha_{00}s)-\alpha_{01}\alpha_{10}s^2}
\end{aligned}
$$

but

$$\alpha_{00}+\alpha_{01}=1$$

$\Longrightarrow$

$$\alpha_{00}=1-\alpha_{01}$$

and

$$
\begin{aligned}
\alpha_{11}+\alpha_{10} &= 1 \\
\alpha_{11} &= 1-\alpha_{10}
\end{aligned}
$$

thus we have

$$
\begin{aligned}
P_{00}(s) &= \frac{1-(1-\alpha_{10})\,s}{1-(\alpha_{11}+\alpha_{00})\,s+(\alpha_{00}\alpha_{11}-\alpha_{01}\alpha_{10})\,s^2} \\
&= \frac{1-(1-\alpha_{10})\,s}{1-(1-\alpha_{10}+1-\alpha_{01})\,s+[(1-\alpha_{01})\,(1-\alpha_{10})-\alpha_{01}\alpha_{10}]\,s^2} \\
&= \frac{1-(1-\alpha_{10})\,s}{1-(2-\alpha_{10}-\alpha_{01})\,s+[1-\alpha_{01}-\alpha_{10}+\alpha_{01}\alpha_{10}-\alpha_{01}\alpha_{10}]\,s^2} \\
&= \frac{1-(1-\alpha_{10})\,s}{1-(2-\alpha_{10}-\alpha_{01})\,s+[1-\alpha_{01}-\alpha_{10}]\,s^2} \qquad \textbf{(16)}
\end{aligned}
$$

Now, s is solved for by factorizing in the denominator. Let

$$
\begin{aligned}
a &= 1-\alpha_{01}-\alpha_{10} \\
b &= -(2-\alpha_{10}-\alpha_{01}) \\
c &= 1
\end{aligned}
$$

using the quadratic formula

$$s=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$
\begin{aligned}
-b &= --(2-\alpha_{10}-\alpha_{01})=(2-\alpha_{10}-\alpha_{01}) \\
b^2 &= (2-\alpha_{10}-\alpha_{01})^2=(2-(\alpha_{10}+\alpha_{01}))^2 \\
&= 4-4\,(\alpha_{10}+\alpha_{01})+(\alpha_{10}+\alpha_{01})^2 \\
&= 4-4\alpha_{10}-4\alpha_{01}+(\alpha_{01})^2+2\alpha_{10}\alpha_{01}+(\alpha_{10})^2
\end{aligned}
$$

$$\begin{aligned} 4ac &= 4(1-\alpha_{01}-\alpha_{10})1 \\ &= 4-4\alpha_{01}-4\alpha_{10} \end{aligned}$$

therefore

$$\begin{aligned} b^2-4ac &= 4-4\alpha_{10}-4\alpha_{01}+(\alpha_{01})^2+2\alpha_{10}\alpha_{01}+(\alpha_{10})^2-(4-4\alpha_{01}-4\alpha_{10}) \\ &= (\alpha_{01})^2+2\alpha_{10}\alpha_{01}+(\alpha_{10})^2 \end{aligned}$$

$$2a = 2(1-\alpha_{01}-\alpha_{10})$$

hence

$$\begin{aligned} s &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{(2-\alpha_{10}-\alpha_{01})\pm\sqrt{(\alpha_{01})^2+2\alpha_{10}\alpha_{01}+(\alpha_{10})^2}}{2(1-\alpha_{01}-\alpha_{10})} \\ &= \frac{(2-\alpha_{10}-\alpha_{01})\pm(\alpha_{01}+\alpha_{10})}{2(1-\alpha_{01}-\alpha_{10})} \end{aligned}$$

hence

$$\begin{aligned} s &= \frac{(2-\alpha_{10}-\alpha_{01})-\alpha_{01}+\alpha_{10}}{2(1-\alpha_{01}-\alpha_{10})} \\ &= \frac{2-2\alpha_{10}-2\alpha_{01}}{2(1-\alpha_{01}-\alpha_{10})} \\ &= \frac{2(1-\alpha_{01}-\alpha_{10})}{2(1-\alpha_{01}-\alpha_{10})} \\ &= 1 \end{aligned}$$

or

$$\begin{aligned} s &= \frac{\alpha_{10}+\alpha_{01}-2+(\alpha_{01}+\alpha_{10})}{2(1-\alpha_{01}-\alpha_{10})} \\ &= \frac{2}{2(1-\alpha_{01}-\alpha_{10})} \\ &= \frac{1}{(1-\alpha_{01}-\alpha_{10})} \end{aligned}$$

In summary

$$s=1 \text{ or } s=\frac{1}{(1-\alpha_{01}-\alpha_{10})}$$

$$1-s=0 \text{ or } (1-\alpha_{01}-\alpha_{10})s=1$$

$\Longrightarrow$

$$1-(1-\alpha_{01}-\alpha_{10})s=0$$

thus

$$1-(2-\alpha_{01}-\alpha_{10})\,s+(1-\alpha_{01}-\alpha_{10})\,s^2=(1-s)\,[1-(1-\alpha_{01}-\alpha_{10})\,s]$$

hence

$$P_{11}(s)=\frac{1-(1-\alpha_{01})\,s}{(1-s)\,[1-(1-\alpha_{01}-\alpha_{10})\,s]}$$

using partial fractions, we obtain

$$\mathbf{P}_{11}(s)=\frac{A}{1-s}+\frac{B}{[1-(1-\alpha_{01}-\alpha_{10})\,s]}$$

hence

$$\frac{1-(1-\alpha_{01})\,s}{(1-s)\,[1-(1-\alpha_{01}-\alpha_{10})\,s]}=\frac{A}{1-s}+\frac{B}{[1-(1-\alpha_{01}-\alpha_{10})\,s]}$$

$$\begin{aligned}1-(1-\alpha_{10})\,s &= A\,[1-(1-\alpha_{01}-\alpha_{10})\,s]+B(1-s)\\ 1-(1-\alpha_{10})\,s &= A-A\,(1-\alpha_{01}-\alpha_{10})\,s+B-Bs\\ 1s^0-(1-\alpha_{10})\,s^1 &= (A+B)\,s^0+[-B-A\,(1-\alpha_{01}-\alpha_{10})]\,s^1\end{aligned}$$

from the coefficients, it follows that

$$A+B=1$$

and

$$\begin{aligned}[-B-A\,(1-\alpha_{01}-\alpha_{10})] &= -(1-\alpha_{01})\\ [B+A\,(1-\alpha_{01}-\alpha_{10})] &= (1-\alpha_{01})\\ B+A+A\,(-\alpha_{01}-\alpha_{10}) &= (1-\alpha_{01})\\ (B+A)-A\,(\alpha_{01}+\alpha_{10}) &= (1-\alpha_{01})\\ 1-A\,(\alpha_{01}+\alpha_{10}) &= (1-\alpha_{01})\end{aligned}$$

$$\begin{aligned}A\,(\alpha_{01}+\alpha_{10}) &= 1-(1-\alpha_{01})\\ A\,(\alpha_{01}+\alpha_{10}) &= \alpha_{01}\\ A &= \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}}\end{aligned}$$

and

$$\begin{aligned}B &= 1-A\\ &= 1-\frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}}\\ &= \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}}\end{aligned}$$

hence

$$
\begin{aligned}
\mathbf{p}_{00}(s) &= \frac{A}{1-s} + \frac{B}{[1-(1-\alpha_{01}-\alpha_{10})\,s]} \\
&= \frac{\frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}}}{1-s} + \frac{\frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}}}{[1-(1-\alpha_{01}-\alpha_{10})\,s]} \\
&= \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}}\left(\frac{1}{1-s}\right) + \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}}\left(\frac{1}{[1-(1-\alpha_{01}-\alpha_{10})\,s]}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{1-s} &= 1+s+s^2+\cdots \\
&= \sum_{n=0}^{\infty} s^n
\end{aligned}
$$

$$
\frac{1}{[1-(1-\alpha_{01}-\alpha_{10})\,s]} = \sum_{n=0}^{\infty} (1-\alpha_{01}-\alpha_{10})^n s^n
$$

Therefore

$$
\begin{aligned}
p_{11}(s) &= \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}} \sum_{n=0}^{\infty} s^n + \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}} \sum_{n=0}^{\infty} (1-\alpha_{01}-\alpha_{10})^n s^n \\
&= \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}} \sum_{n=0}^{\infty} s^n + \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}} (1-\alpha_{01}-\alpha_{10})^n \sum_{n=0}^{\infty} s^n
\end{aligned}
$$

but $p_{11}^{(n)}$ is the coefficient of s^n in $p_{11}(s)$ hence

$$
p_{11}^{(n)} = \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}} + \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}} (1-\alpha_{01}-\alpha_{10})^n
$$

but

$$
p_{11}^{(n)} + p_{10}^{(n)} = 1
$$

$\Longrightarrow$

$$
p_{10}^{(n)} = 1 - p_{11}^{(n)}
$$

hence

$$
\begin{aligned}
p_{10}^{(n)} &= 1 - \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}} + \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}} (1-\alpha_{01}-\alpha_{10})^n \\
&= \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}} \left\{1-(1-\alpha_{01}-\alpha_{10})^n\right\}
\end{aligned}
$$

In summary

$$
\begin{aligned}
\mathbf{P}^n &= \begin{bmatrix} p_{00}^{(n)} & p_{01}^{(n)} \\ p_{10}^{(n)} & p_{11}^{(n)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}} + \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}} (1-\alpha_{01}-\alpha_{10})^n & \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}} \left\{1-(1-\alpha_{01}-\alpha_{10})^n\right\} \\ \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}} \left\{1-(1-\alpha_{01}-\alpha_{10})^n\right\} & \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}} + \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}} (1-\alpha_{01}-\alpha_{10})^n \end{bmatrix}
\end{aligned}
$$

Consider a special case

$$\mathbf{P}^n = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

$$\begin{aligned} \alpha_{00} &= 1-\alpha, \quad \alpha_{01} = \alpha \\ \alpha_{10} &= \beta, \quad \alpha_{11} = 1-\beta \\ \alpha_{01}+\alpha_{10} &= \alpha+\beta \\ 1-\alpha_{01}-\alpha_{10} &= 1-\alpha-\beta \end{aligned}$$

$$\begin{aligned} \mathbf{P}^n &= \begin{bmatrix} \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n & \frac{\alpha}{\alpha+\beta}\left\{1-(1-\alpha-\beta)^n\right\} \\ \frac{\beta}{\alpha+\beta}\left\{1-(1-\alpha-\beta)^n\right\} & \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n \end{bmatrix} \\ &= \frac{1}{\alpha+\beta}\begin{bmatrix} \beta+\alpha(1-\alpha-\beta)^n & \alpha\left\{1-(1-\alpha-\beta)^n\right\} \\ \beta\left\{1-(1-\alpha-\beta)^n\right\} & \alpha+\beta(1-\alpha-\beta)^n \end{bmatrix} \end{aligned}$$

2.13.4 Applying the Eigen values and Eigen vectors in determining $\mathbf{P}^n$

The higher transition probabilities $p_{ij}^{(n)}$ can be computed from p_{ij} by repeated multiplications and summations, but the work involved is prohibitive even for a chain with a small number of states and for a moderate number n. For every square matrix $\mathbf{A}$, a scalar λ and a non-zero vector $\mathbf{x}$ can be found such that

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$|\mathbf{A}-\lambda\mathbf{I}| = \mathbf{0}$$

To determine λ and $\mathbf{x}$, we solve the characteristic function. For a finite Markov chain having states $1,2,3,\ldots,n$ and a stochastic matrix

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

with

$$\sum_{j=1}^{n} p_{ij} = 1, \quad i = 1,2,3,\ldots,n$$

we intorduce a characteristic matrix

$$\mathbf{A}(\lambda) = (\lambda\mathbf{I} - \mathbf{P}) = \begin{bmatrix} (\lambda - p_{11}) & -p_{12} & \dots & -p_{1n} \\ -p_{21} & (\lambda - p_{22}) & \dots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{n1} & -p_{n2} & \dots & (\lambda - p_{nn}) \end{bmatrix}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. The characteristic equation

$$|\mathbf{A}(\lambda)| = |\lambda\mathbf{I} - \mathbf{P}| = 0$$

is a polynomial in λ of degree n. The characteristic equation has n roots given by the equation $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$.They are known as the eigenvalues of the matrix $\mathbf{P}$. These eigenvalues may be real or complex numbers and some of them may be equal.

Conclusion

The three methods of determining the nth power give the same results, however the direct multiplication method is more challenging and tedious compared to the rest of the methods. Both methods give the same results as required.

3 TWO STATE MODELS

3.1 Introduction

In this chapter the random walk is studied as a Markov chain. Various cases of the two state models shall be considered; namely: simple random walk, random walk with absorbing barriers, random walks with reflecting barriers and cyclic random walks.

3.2 A Transition probability matrix for a Simple Random Walk

$$0 \leq \alpha \leq 1 \text{ and } 0 \leq \beta \leq 1$$

$$P = \begin{array}{c} \\ E_0 \\ E_1 \end{array} \begin{array}{c} \begin{array}{cc} E_0 & E_1 \end{array} \\ \left[\begin{array}{cc} 1-\alpha & \alpha \\ \beta & 1-\beta \end{array}\right] \end{array}$$

3.2.1 Classification of Markov chains

Every state can be reached from every other state hence the Markov Chain is irreducible. Thus all the states are of the same type.

3.2.2 Classification of states

Consider E_o

$$\begin{aligned} f_{00}^{(1)} &= \Pr[E_0 \to E_0] \\ &= 1-\alpha \end{aligned}$$

$$\begin{aligned} f_{00}^{(2)} &= \Pr[E_0 \to E_1 \to E_0] \\ &= \alpha\beta \end{aligned}$$

$$\begin{aligned} f_{00}^{(3)} &= \alpha\beta(1-\beta) \\ f_{00}^{(3)} &= \alpha\beta(1-\beta)^2 \end{aligned}$$

hence

$$f_{00}^{(n)} = \left\{ \begin{array}{c} \alpha\beta\,(1-\beta)^{n-2}, \quad n \geq 2 \\ (1-\alpha), \quad\quad n = 1 \\ 0, \ \text{elsewhere} \end{array} \right\}$$

For $n \geq 3$, the first return to state E_0 occurs in n transitions if

a) First transition probability is from 0 to 1, and

b) next $(n-2)$ transitions are from 1 to 1, and

c) nth transition is from 1 to 0.

$\Longrightarrow$

$$f_{00}^{(n)} = \alpha\beta\,(1-\beta)^{n-2}$$

$$\begin{aligned} f_0 &= \sum_{n=0}^{\infty} f_{00}^{(n)} \\ &= 1-\alpha+\sum_{n=2}^{\infty}\alpha\beta\,(1-\beta)^{n-2} \\ &= 1-\alpha+\alpha\beta\sum_{n=0}^{\infty}(1-\beta)^{n} \\ &= 1-\alpha+\frac{\alpha\beta}{1-(1-\beta)} \\ &= 1-\alpha+\alpha \end{aligned}$$

hence persistent

$$\begin{aligned} \mu_0 &= \sum_{n=1}^{\infty} n f_{00}^{(n)} \\ &= 1-\alpha+\alpha\beta\sum_{n=2}^{\infty} n\,(1-\beta)^{n-2} \\ &= (1-\alpha)+\alpha\beta\left[2+3\,(1-\beta)+4\,(1-\beta)^2+5\,(1-\beta)^3+\cdots\right] \end{aligned}$$

let $x = (1-\beta)$

$$\begin{aligned} \mu_0 &= (1-\alpha)+\alpha\beta\left[2+3x+4x^2+5x^3+\cdots\right] \\ &= (1-\alpha)+\frac{\alpha\beta}{x}x\left[2+3x+4x^2+5x^3+\cdots\right] \\ &= (1-\alpha)+\frac{\alpha\beta}{x}\left[2x+3x^2+4x^3+5x^4+\cdots\right] \end{aligned}$$

but

$$\begin{aligned} \frac{1}{1-x} &= 1+x+x^2+x^3+x^4+\cdots \\ \left(\frac{1}{1-x}\right)^2 &= 1+2x+3x^2+4x^3+5x^4+\cdots \end{aligned}$$

hence

$$\begin{aligned} \mu_0 &= (1-\alpha)+\frac{\alpha\beta}{x}\left[\left(\frac{1}{1-x}\right)^2-1\right] \\ &= (1-\alpha)+\frac{\alpha\beta}{x}\left[\frac{1-(1-x)^2}{(1-x)^2}\right] \\ &= (1-\alpha)+\frac{\alpha\beta}{x}\left[\frac{1-(1-2x+x^2)}{(1-x)^2}\right] \\ &= (1-\alpha)+\frac{\alpha\beta}{x}\left[\frac{2x-x^2)}{(1-x)^2}\right] \\ &= (1-\alpha)+\alpha\beta\left[\frac{2-x}{(1-x)^2}\right] \end{aligned}$$

but

$$\begin{aligned} 1-x &= 1-(1-\beta)=\beta \\ (1-x)^2 &= \beta^2 \end{aligned}$$

and

$$\begin{aligned} 2-x &= 2-(1-\beta) \\ &= 1+\beta \end{aligned}$$

hence

$$\begin{aligned} \mu_0 &= (1-\alpha)+\alpha\beta\left[\frac{1+\beta}{\beta^2}\right] \\ &= (1-\alpha)+\alpha\left[\frac{1+\beta}{\beta}\right] \\ &= 1-\alpha+\frac{\alpha}{\beta}+\alpha \\ &= 1+\frac{\alpha}{\beta} \end{aligned}$$

therefore

$$\mu_0 < \infty$$

is finite and E_0 is non-null. Now, to determine the periodicity of E_0

$$\begin{aligned} d &= GCD\left\{n : p_{jj}^{(n)} > 0\right\} \\ &= GCD\{1,2,3\ldots\} \\ &= 1 \end{aligned}$$

hence E_0 is aperiodic.

Consider E_1

$$\begin{aligned} f_{11}^{(1)} &= \Pr[E_1 \to E_1] \\ &= 1-\beta \\ f_{11}^{(2)} &= \Pr[E_1 \to E_0 \to E_1] \\ &= \alpha\beta \\ f_{11}^{(3)} &= \alpha\beta(1-\alpha) \\ f_{11}^{(3)} &= \alpha\beta(1-\alpha)^2 \end{aligned}$$

in general

$$f_{11}^{(n)} = \left\{ \begin{array}{cc} \alpha\beta(1-\alpha)^{n-2}, & n \geq 2 \\ (1-\beta), & n = 1 \\ 0, & \text{Elsewhere} \end{array} \right\}$$

$$\begin{aligned} f_1 &= \sum_{n=0}^{\infty} f_{11}^{(n)} \\ &= 1-\beta+\sum_{n=2}^{\infty}\alpha(1-\alpha)^{n-2}\beta \\ &= 1-\beta+\alpha\beta\sum_{n=0}^{\infty}(1-\alpha)^n \\ &= 1-\beta+\frac{\alpha\beta}{1-(1-\alpha)} \\ &= 1-\beta+\beta \\ &= 1 \end{aligned}$$

hence E_1 is opersistent.

$$\begin{aligned}
\mu_1 &= \sum_{n=1}^{\infty} n f_{11}^{(n)} \\
&= 1-\beta+\alpha\beta\sum_{n=2}^{\infty}(1-\alpha)^{n-2} \\
&= (1-\beta)+\alpha\beta\left[2+3(1-\alpha)+4(1-\alpha)^2+5(1-\alpha)^3+\cdots\right]
\end{aligned}$$

let $x=(1-\alpha)$

$$\begin{aligned}
\mu_1 &= (1-\beta)+\alpha\beta\left[2+3x+4x^2+5x^3+\cdots\right] \\
&= (1-\beta)+\frac{\alpha\beta}{x}x\left[2+3x+4x^2+5x^3+\cdots\right] \\
&= (1-\beta)+\frac{\alpha\beta}{x}\left[2x+3x^2+4x^3+5x^4+\cdots\right]
\end{aligned}$$

but

$$\begin{aligned}
\frac{1}{1-x} &= 1+x+x^2+x^3+x^4+\cdots \\
\left(\frac{1}{1-x}\right)^2 &= 1+2x+3x^2+4x^3+5x^4+\cdots
\end{aligned}$$

hence

$$\begin{aligned}
\mu_1 &= (1-\beta)+\frac{\alpha\beta}{x}\left[\left(\frac{1}{1-x}\right)^2-1\right] \\
&= (1-\beta)+\frac{\alpha\beta}{x}\left[\frac{1-(1-x)^2}{(1-x)^2}\right] \\
&= (1-\beta)+\frac{\alpha\beta}{x}\left[\frac{1-(1-2x+x^2)}{(1-x)^2}\right] \\
&= (1-\beta)+\frac{\alpha\beta}{x}\left[\frac{2x-x^2)}{(1-x)^2}\right] \\
&= (1-\beta)+\alpha\beta\left[\frac{2-x}{(1-x)^2}\right]
\end{aligned}$$

but

$$\begin{aligned}
1-x &= 1-(1-\alpha)=\alpha \\
(1-x)^2 &= \alpha^2
\end{aligned}$$

and

$$\begin{aligned}
2-x &= 2-(1-\alpha) \\
&= 1+\alpha
\end{aligned}$$

hence

$$\begin{aligned}\mu_1 &= (1-\beta)+\alpha\beta\left[\frac{1+\alpha}{\alpha^2}\right]\\ &= (1-\beta)+\beta\left[\frac{1+\alpha}{\alpha}\right]\\ &= 1-\beta+\frac{\beta}{\alpha}+\beta\\ &= 1+\frac{\beta}{\alpha}<\infty\end{aligned}$$

hence μ_1 is finite. Since E_0 and E_1 are communicating states, it holds that E_1 is alos ergodic. Therefore, E_0 and E_1 are persistent, non-null, and aperiodic and hence rgodic. However, the mean recurrence times are different

$$\mu_0 = 1+\frac{\alpha}{\beta} \neq \ \mu_1 = 1+\frac{\beta}{\alpha}$$

3.2.3 The stationary distribution

Consider

$$\lim_{n\to\infty}\mathbf{P}^n = \mathbf{P}'\pi$$

$$\begin{bmatrix}\pi_0\\ \pi_1\end{bmatrix} = \begin{bmatrix}1-\alpha & \alpha\\ \beta & 1-\beta\end{bmatrix}\begin{bmatrix}\pi_0\\ \pi_1\end{bmatrix}$$

$$\begin{aligned}\pi_0 &= \pi_0(1-\alpha)+\pi_1\beta\\ \pi_1 &= \pi_0\alpha+\pi_1(1-\beta)\end{aligned}$$

$$\begin{aligned}\pi_0+\pi_1 &= 1\\ \pi_1 &= 1-\pi_0\end{aligned}$$

$\Longrightarrow$

$$\begin{aligned}\pi_0 &= \pi_0(1-\alpha)+(1-\pi_0)\beta\\ &= \pi_0-\pi_0\alpha+\beta-\pi_0\beta\\ \pi_0\alpha+\pi_0\beta &= \beta\\ \pi_0 &= \frac{\beta}{\alpha+\beta}\end{aligned}$$

$$\begin{aligned}\pi_1 &= 1-\pi_0\\ &= 1-\frac{\beta}{\alpha+\beta}\\ &= \frac{\alpha}{\alpha+\beta}\end{aligned}$$

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix} = \begin{bmatrix} \frac{\beta}{\alpha+\beta} \\ \frac{\alpha}{\alpha+\beta} \end{bmatrix}$$

by theorem 2.12

$$\begin{aligned} \mu_0 &= \frac{1}{\pi_0} \\ &= \frac{\alpha+\beta}{\beta} \\ &= \frac{\alpha}{\beta}+1 \end{aligned}$$

$$\begin{aligned} \mu_1 &= \frac{1}{\pi_0} \\ &= \frac{\alpha+\beta}{\alpha} \\ &= \frac{\beta}{\alpha}+1 \end{aligned}$$

3.2.4 The n^{th} power $\mathbf{P}^n$ using the Eigen value technique

$$\mathbf{P} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

Using the Eigen value approach, we first determine the Eigen values of P. We solve the characteristic equation

$$|\mathbf{P}-\lambda\mathbf{I}| = \mathbf{0}$$

$$\begin{vmatrix} (1-\alpha)-\lambda & \alpha \\ \beta & (1-\beta)-\lambda \end{vmatrix} = \mathbf{0}$$

$$\begin{aligned} [(1-\alpha)-\lambda]\,[(1-\beta)-\lambda]-\alpha\beta &= \mathbf{0} \\ (1-\alpha)\,[(1-\beta)-\lambda]-\lambda\,[(1-\beta)-\lambda]-\alpha\beta &= \mathbf{0} \\ (1-\alpha)(1-\beta)-\lambda(1-\alpha)-\lambda(1-\beta)+\lambda^2-\alpha\beta &= \mathbf{0} \\ \lambda^2-\lambda(1-\alpha)-\lambda(1-\beta)+(1-\alpha)(1-\beta)-\alpha\beta &= \mathbf{0} \\ \lambda^2-\lambda+\lambda\alpha-\lambda+\lambda\beta+1-\alpha-\beta+\alpha\beta-\alpha\beta &= \mathbf{0} \end{aligned}$$

$$\begin{aligned} \lambda^2-\lambda+\lambda\alpha+\lambda\beta-\lambda+1-\alpha-\beta &= \mathbf{0} \\ \lambda(\lambda-1+\alpha+\beta)-(\lambda-1+\alpha+\beta) &= \mathbf{0} \\ (\lambda-1)(\lambda-1+\alpha+\beta) &= 0 \end{aligned}$$

hence

$$\lambda_1 = 1 \text{ and } \lambda_2 = 1-\alpha-\beta$$

when $\lambda_1 = 1$

$$\begin{aligned}[\mathbf{P}-\lambda\mathbf{I}]\,\mathbf{X} &= \mathbf{0} \\ \begin{bmatrix} 1-\alpha-1 & \alpha \\ \beta & 1-\beta-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}-\alpha x_1+\alpha x_2 &= 0 \\ \alpha x_1 &= \alpha x_2 \\ x_1 &= x_2\end{aligned}$$

taking, say $x_1 = x_2 = 1$

$$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

when

$$\lambda_2 = 1-\alpha-\beta$$

we have

$$\begin{aligned}[\mathbf{P}-\lambda\mathbf{I}]\,\mathbf{X} &= \mathbf{0} \\ \begin{bmatrix} (1-\alpha)-(1-\alpha-\beta) & \alpha \\ \beta & (1-\beta)-(1-\beta-\alpha) \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

$$\begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}\beta x_3 &= -\alpha x_4 \\ x_3 &= \frac{-\alpha}{\beta}x_4\end{aligned}$$

but x_4 is a free vraible. Suppose $x_4 = \beta$, then

$$x_3 = -\alpha$$

Therefore, the Eigen vector is

$$V_2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\alpha \\ \beta \end{bmatrix}$$

Recall that by spectral decomposition

$$\mathbf{P}=\mathbf{V}\mathbf{D}\mathbf{V}^{-1}$$

where $\mathbf{D}$ is teh diagonal matrix of the Eigen values of $\mathbf{P}$

$$\begin{aligned}\mathbf{V} &= \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\alpha \\ 1 & \beta \end{bmatrix}\end{aligned}$$

hence

$$\begin{aligned} P &= \begin{bmatrix} 1 & -\alpha \\ 1 & \beta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha-\beta \end{bmatrix}\begin{bmatrix} 1 & -\alpha \\ 1 & \beta \end{bmatrix}^{-1} \\ \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} &= \begin{bmatrix} 1 & -\alpha \\ 1 & \beta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha-\beta \end{bmatrix}\begin{bmatrix} 1 & -\alpha \\ 1 & \beta \end{bmatrix}^{-1}\end{aligned}$$

but

$$\mathbf{P}^n=\mathbf{V}\mathbf{D}^n\mathbf{V}^{-1}$$

$\Longrightarrow$

$$\begin{aligned}\mathbf{P}^n &= \begin{bmatrix} 1 & -\alpha \\ 1 & \beta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta)^n \end{bmatrix}\begin{bmatrix} 1 & -\alpha \\ 1 & \beta \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & -\alpha(1-\alpha-\beta)^n \\ 1 & \beta(1-\alpha-\beta)^n \end{bmatrix}\begin{bmatrix} 1 & -\alpha \\ 1 & \beta \end{bmatrix}^{-1}\end{aligned}$$

but

$$\begin{bmatrix} 1 & -\alpha \\ 1 & \beta \end{bmatrix}^{-1}=\frac{1}{\beta+\alpha}\begin{bmatrix} \beta & \alpha \\ -1 & 1 \end{bmatrix}$$

hence

$$\begin{aligned}\mathbf{P}^n &= \frac{1}{\alpha+\beta}\begin{bmatrix} 1 & -\alpha(1-\alpha-\beta)^n \\ 1 & \beta(1-\alpha-\beta)^n \end{bmatrix}\begin{bmatrix} \beta & \alpha \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{\alpha+\beta}\begin{bmatrix} \beta+\alpha(1-\alpha-\beta)^n & \alpha-\alpha(1-\alpha-\beta)^n \\ \beta-\beta(1-\alpha-\beta)^n & \alpha+\beta(1-\alpha-\beta)^n \end{bmatrix}\end{aligned}$$

which can be expressed as

$$\frac{1}{\alpha+\beta}\begin{bmatrix}\beta & \alpha\\ \beta & \alpha\end{bmatrix}+\frac{(1-\alpha-\beta)^n}{\alpha+\beta}\begin{bmatrix}\alpha & -\alpha\\ -\beta & \beta\end{bmatrix}$$

From this, it follows that the asymptotic behavior is given by

$$\begin{aligned}\lim_{n\to\infty}\mathbf{P}^n &= \lim_{n\to\infty}\frac{1}{\alpha+\beta}\begin{bmatrix}\beta & \alpha\\ \beta & \alpha\end{bmatrix}+\frac{(1-\alpha-\beta)^n}{\alpha+\beta}\begin{bmatrix}\alpha & -\alpha\\ -\beta & \beta\end{bmatrix}\\ &= \frac{1}{\alpha+\beta}\begin{bmatrix}\beta & \alpha\\ \beta & \alpha\end{bmatrix}\end{aligned}$$

3.2.5 The n^{th} power $\mathbf{P}^n$ using Chapman-Kolgomorov tehcnique

$$\mathbf{P}=\begin{bmatrix}1-\alpha & \alpha\\ \beta & 1-\beta\end{bmatrix}$$

by using Chapman-Kolgomorov equation

$$\mathbf{P}^n=\mathbf{P}^{n-1}\mathbf{P}$$

i.e.

$$\begin{bmatrix}p_{11}^{(n)} & p_{12}^{(n)}\\ p_{21}^{(n)} & p_{22}^{(n)}\end{bmatrix}=\begin{bmatrix}p_{11}^{(n-1)} & p_{12}^{(n-1)}\\ p_{21}^{(n-1)} & p_{22}^{(n-1)}\end{bmatrix}\begin{bmatrix}1-\alpha & \alpha\\ \beta & 1-\beta\end{bmatrix}$$

therefore

$$p_{11}^{(n)}=p_{11}^{(n-1)}(1-\alpha)+p_{12}^{(n-1)}\beta$$

but

$$p_{11}^{(n)}+p_{12}^{(n)}=1$$

hence

$$\begin{aligned}p_{12}^{(n)} &= 1-p_{11}^{(n)}\\ &\implies p_{12}^{(n-1)}=1-p_{11}^{(n-1)}\end{aligned}$$

and

$$\begin{aligned}p_{11}^{(n)} &= p_{11}^{(n-1)}(1-\alpha)+p_{12}^{(n-1)}\beta\\ &= p_{11}^{(n-1)}(1-\alpha)+\left(1-p_{11}^{(n-1)}\right)\beta\\ &= p_{11}^{(n-1)}(1-\alpha-\beta)+\beta, \qquad n=1,2,3\ldots \text{ and } p_{11}^{(n)}=1 \end{aligned} \tag{17}$$

Using the probability generating function technique, equation (3.1) is multiplied by s^n and the result summed over n to obtain

$$\sum_{n=1}^{\infty} p_{11}^{(n)} s^n = (1-\alpha-\beta)\sum_{n=1}^{\infty} p_{11}^{(n-1)} s^n + \beta \sum_{n=1}^{\infty} s^n$$

but

$$\begin{aligned} G(s) &= \sum_{n=0}^{\infty} p_{11}^{(n)} s^n \\ &= p_{11}^{(0)} + \sum_{n=1}^{\infty} p_{11}^{(n)} s^n \\ &= 1 + \sum_{n=1}^{\infty} p_{11}^{(n)} s^n \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} p_{11}^{(n)} s^n &= G(s) - 1 \\ &= (1-\alpha-\beta)\sum_{n=1}^{\infty} p_{11}^{(n-1)} s^n + \beta \sum_{n=1}^{\infty} s^n \\ &= (1-\alpha-\beta)\, s \sum_{n=1}^{\infty} p_{11}^{(n-1)} s^{n-1} + \beta \sum_{n=1}^{\infty} s^n \\ &= (1-\alpha-\beta)\, sG(s) + \beta s\left[1 + s + s^2 + s^3 + \cdots\right] \\ &= (1-\alpha-\beta)\, sG(s) + \frac{\beta s}{1-s} \end{aligned}$$

$$\begin{aligned} G(s) - 1 = (1-\alpha-\beta)\, sG(s) + \frac{\beta s}{1-s} \\ G(s) - (1-\alpha-\beta)\, sG(s) = 1 + \frac{\beta s}{1-s} \\ G(s)\left[1-(1-\alpha-\beta)\, s\right] = \frac{1-s+\beta s}{1-s} \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} G(s) &= \frac{1-s+\beta s}{[1-s]\left[1-(1-\alpha-\beta)\, s\right]} \\ &= \frac{1+\beta s - s}{[1-s]\left[1-(1-\alpha-\beta)\, s\right]} \\ &= \frac{1+s(\beta-1)}{[1-s]\left[1-(1-\alpha-\beta)\, s\right]} \end{aligned} \quad \textbf{(18)}$$

Using partial fractions

$$\frac{1+s(\beta-1)}{[1-s]\left[1-(1-\alpha-\beta)\, s\right]} = \frac{A}{[1-s]} + \frac{C}{\left[1-(1-\alpha-\beta)\, s\right]}$$

we now determine the values of A and C

$$\begin{aligned} 1+s(\beta-1) &= A[1-(1-\alpha-\beta)s]+C[1-s] \\ &= A-As(1-\alpha-\beta)+C-Cs \\ &= (A+C)s^0+[-A(1-\alpha-\beta)-C]s \end{aligned}$$

Equating to the corresponding powers of s

$$A+C=1$$

$$(\beta-1)=-A(1-\alpha-\beta)-C \quad \textbf{(19)}$$

$$\begin{aligned} C &= 1-A \\ -C &= A-1 \end{aligned}$$

$$\begin{aligned} (\beta-1) &= -A(1-\alpha-\beta)+(A-1) \\ &= A-A(1-\alpha-\beta)-1 \\ &= A[1-(1-\alpha-\beta)]-1 \\ &= A[\alpha+\beta]-1 \\ \beta &= A[\alpha+\beta] \\ A &= \frac{\beta}{\alpha+\beta} \end{aligned}$$

$$\begin{aligned} C &= 1-\frac{\beta}{\alpha+\beta} \\ &= \frac{\alpha}{\alpha+\beta} \end{aligned}$$

Therefore

$$\begin{aligned} G(s) &= \frac{\frac{\beta}{\alpha+\beta}}{[1-s]}+\frac{\frac{\alpha}{\alpha+\beta}}{[1-(1-\alpha-\beta)s]} \\ &= \frac{1}{\alpha+\beta}\left[\frac{\beta}{[1-s]}+\frac{\alpha}{[1-(1-\alpha-\beta)s]}\right] \end{aligned}$$

$$\begin{aligned} \frac{1}{1-s} &= 1+s+s^2+s^3+\cdots \\ &= \sum_{n=0}^{\infty} s^n \end{aligned}$$

let

$$x = (1-\alpha-\beta)$$

$$\begin{aligned} \frac{1}{1-xs} &= 1 + xs + (xs)^2 + (xs)^3 + \cdots \\ &= \sum_{n=0}^{\infty} x^n s^n \end{aligned}$$

Therefore

$$\frac{1}{1-(1-\alpha-\beta)\,s} = \sum_{n=0}^{\infty} (1-\alpha-\beta)^n\, s^n$$

and

$$\begin{aligned} G(s) &= \sum_{n=0}^{\infty} p_{11}^{(n)} s^n \\ &= \frac{1}{\alpha+\beta}\left\{\beta \sum_{n=0}^{\infty} s^n + \alpha\,(1-\alpha-\beta)^n \sum_{n=0}^{\infty} s^n\right\} \end{aligned}$$

but $p_{11}^{(n)}$ is the coefficient of s^n in $G(s)$

$$\begin{aligned} p_{11}^{(n)} &= \frac{\beta}{\alpha+\beta} + \frac{\alpha\,(1-\alpha-\beta)^n}{\alpha+\beta}, \qquad \text{for } \alpha+\beta \neq 0 \\ &= \frac{1}{\alpha+\beta}\left[\beta + \alpha\,(1-\alpha-\beta)^n\right] \end{aligned}$$

but

$$\begin{aligned} p_{11}^{(n)} + p_{12}^{(n)} &= 1 \\ p_{12}^{(n)} &= 1 - p_{11}^{(n)} \end{aligned}$$

Case II

If $\alpha+\beta = 0$, then from (3.)

$$\begin{aligned} G(s) &= \frac{1+s\,(\beta-1)}{[1-s]\,[1-(1-\alpha-\beta)\,s]} \\ &= \frac{1+s\,(\beta-1)}{[1-s]\,[1-(1-[\alpha+\beta])\,s]} \\ &= \frac{1+s\,(\beta-1)}{[1-s]\,[1-s]} \\ &= \frac{1+s\,(\beta-1)}{(1-s)^2} \end{aligned}$$

Using partial fractions

$$\frac{1+s(\beta-1)}{(1-s)^2}=\frac{A}{(1-s)}+\frac{C}{(1-s)^2}$$

multiplying across by $(1-s)^2$, we obtain

$$\begin{aligned}1+s(\beta-1) &= (1-s)A+C\\ &= A-As+C\\ &= (A+C)s^0-As\end{aligned}$$

$\Longrightarrow$

$$\begin{aligned}A+C &= 1\\ C &= 1-A\end{aligned}$$

$$A=1-\beta$$

$$\begin{aligned}C &= 1-1-\beta\\ &= \beta\end{aligned}$$

hence

$$G(s)=\frac{1-\beta}{(1-s)}+\frac{\beta}{(1-s)^2}$$

but

$$\begin{aligned}\frac{1}{1-s} &= 1+s+s^2+s^3+\cdots\\ &= \sum_{n=0}^{\infty}s^n\end{aligned}$$

$$\begin{aligned}\frac{d}{ds}\left(\frac{1}{1-s}\right) &= \frac{d}{ds}(1-s)^{-1}\\ &= \frac{1}{(1-s)^2}\\ &= \frac{d}{ds}\left(\sum_{n=0}^{\infty}s^n\right)\\ &= \sum_{n=0}^{\infty}ns^{n-1}\\ \frac{1}{(1-s)^2} &= \sum_{n=1}^{\infty}ns^{n-1}\\ &= \sum_{n=0}^{\infty}(n+1)s^n\end{aligned}$$

hence

$$G(s) = (1-\beta)\sum_{n=0}^{\infty} s^n + \beta \sum_{n=1}^{\infty} n s^{n-1}$$

but $p_{11}^{(n)}$ is the coefficient of s^n in $G(s)$, hence

$$\begin{aligned} p_{11}^{(n)} &= (1-\beta)+\beta(n+1) \\ &= 1-\beta n \end{aligned}$$

Similarly, from

$$\mathbf{P}^n = \begin{bmatrix} p_{11}^{(n)} & p_{12}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} \end{bmatrix} = \begin{bmatrix} p_{11}^{(n-1)} & p_{12}^{(n-1)} \\ p_{21}^{(n-1)} & p_{22}^{(n-1)} \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

$\Longrightarrow$

$$p_{22}^{(n)} = \alpha p_{21}^{(n-1)} + (1-\beta)\, p_{22}^{(n-1)}$$

but

$$\begin{aligned} p_{21}^{(n)} + p_{22}^{(n)} &= 1 \\ p_{21}^{(n)} &= 1 - p_{22}^{(n)} \end{aligned}$$

$\Longrightarrow$

$$p_{21}^{(n-1)} =$$

Thus

$$\begin{aligned} p_{22}^{(n)} &= \alpha\left(1 - p_{22}^{(n-1)}\right) + (1-\beta)\, p_{22}^{(n-1)} \\ &= \alpha - \alpha p_{22}^{(n-1)} + (1-\beta)\, p_{22}^{(n-1)} \\ &= \alpha + [1-\alpha-\beta]\, p_{22}^{(n-1)} \end{aligned} \quad \textbf{(20)}$$

Multiplying equation $(3.)$ by s^n and summing over n we get

$$\sum_{n=1}^{\infty} p_{22}^{(n)} s^n = \alpha \sum_{n=1}^{\infty} s^n + [1-\alpha-\beta] \sum_{n=1}^{\infty} p_{22}^{(n-1)} s^n$$

Define

$$\begin{aligned} G(s) &= \sum_{n=0}^{\infty} p_{22}^{(n)} s^n \\ &= p_{22}^{(0)} + \sum_{n=1}^{\infty} p_{22}^{(n)} s^n \\ &= 1 + \sum_{n=1}^{\infty} p_{22}^{(n)} s^n \end{aligned}$$

$\Longrightarrow$

$$\sum_{n=1}^{\infty} p_{22}^{(n)} s^n = G(s) - 1$$

thus we have

$$\begin{aligned} G(s) - 1 &= \alpha s \left[1 + s + s^2 + \cdots \right] + [1 - \alpha - \beta]\, s \sum_{n=1}^{\infty} p_{22}^{(n-1)} s^{n-1} \\ &= \frac{\alpha s}{1 - s} + [1 - \alpha - \beta]\, s G(s) \end{aligned}$$

$$G(s) - [1 - \alpha - \beta]\, s G(s) = \frac{1 - s + \alpha s}{1 - s}$$

$$\begin{aligned} G(s)\,[1 - s\,(1 - \alpha - \beta)] &= \frac{1 - s + \alpha s}{1 - s} \\ G(s) &= \frac{1 + (\alpha - 1)\, s}{(1 - s)\,[1 - (1 - \alpha - \beta)\, s]} \end{aligned}$$

Using partial fractions

$$G(s) = \frac{A}{(1 - s)} + \frac{C}{[1 - (1 - \alpha - \beta)\, s]}$$

$$\frac{1 + (\alpha - 1)\, s}{(1 - s)\,[1 - (1 - \alpha - \beta)\, s]} = \frac{A}{(1 - s)} + \frac{C}{[1 - (1 - \alpha - \beta)\, s]}$$

$$\begin{aligned} 1 + (\alpha - 1)\, s &= [1 - (1 - \alpha - \beta)\, s]\, A + (1 - s)\, C \\ &= A - (1 - \alpha - \beta)\, s A + C - C s \end{aligned}$$

$$1 s^0 + (\alpha - 1)\, s = (A + C) s^0 + [-C - A\,(1 - \alpha - \beta)]\, s$$

hence

$$\begin{aligned} A + C &= 1 \\ -C &= A - 1 \end{aligned}$$

$$\begin{aligned} (\alpha - 1) &= -C - A\,(1 - \alpha - \beta) \\ &= A - 1 - A + A\alpha + A\beta \\ &= A\,(\alpha + \beta) - 1 \end{aligned}$$

$$A = \frac{\alpha}{(\alpha + \beta)}$$

$$\begin{aligned} C &= 1-\frac{\alpha}{(\alpha+\beta)} \\ &= \frac{\beta}{(\alpha+\beta)} \end{aligned}$$

hence

$$\begin{aligned} G(s) &= \frac{\frac{\alpha}{(\alpha+\beta)}}{(1-s)}+\frac{\frac{\beta}{(\alpha+\beta)}}{[1-(1-\alpha-\beta)\,s]} \\ &= \frac{1}{(\alpha+\beta)}\left[\frac{\alpha}{(1-s)}+\frac{\beta}{[1-(1-\alpha-\beta)\,s]}\right] \\ &= \frac{1}{(\alpha+\beta)}\left[\alpha\sum_{n=0}^{\infty}s^n+(1-\alpha-\beta)^n\,\beta\sum_{n=0}^{\infty}s^n\right] \end{aligned}$$

$$\sum_{n=0}^{\infty}p_{22}^{(n)}s^n=\frac{1}{(\alpha+\beta)}\left[\alpha\sum_{n=0}^{\infty}s^n+(1-\alpha-\beta)^n\,\beta\sum_{n=0}^{\infty}s^n\right]$$

but $p_{22}^{(n)}$ is the coefficient of s^n in $G(s)$

$$\begin{aligned} p_{22}^{(n)} &= \frac{\alpha}{\alpha+\beta}+\frac{(1-\alpha-\beta)^n\,\beta}{\alpha+\beta} \\ &= \frac{\alpha+\beta\,(1-\alpha-\beta)^n}{\alpha+\beta} \end{aligned}$$

and

$$\begin{aligned} p_{21}^{(n)} &= 1-p_{22}^{(n)} \\ &= 1-\frac{\alpha+\beta\,(1-\alpha-\beta)^n}{\alpha+\beta} \\ &= \frac{(\alpha+\beta)-\left[\alpha+\beta\,(1-\alpha-\beta)^n\right]}{\alpha+\beta} \\ &= \frac{\beta-\beta\,(1-\alpha-\beta)^n}{\alpha+\beta} \end{aligned}$$

Case 2

When $\alpha+\beta=0$,

$$\begin{aligned} G(s) &= \frac{1+(\alpha-1)\,s}{(1-s)\,[1-(1-\alpha-\beta)\,s]} \\ &= \frac{1+(\alpha-1)\,s}{(1-s)\,[1-(1-[\alpha+\beta])\,s]} \\ &= \frac{1+(\alpha-1)\,s}{(1-s)\,[1-(1)\,s]} \\ &= \frac{1+(\alpha-1)\,s}{(1-s)^2} \end{aligned}$$

$$\frac{1+(\alpha-1)\,s}{(1-s)^2}=\frac{A}{(1-s)}+\frac{C}{(1-s)^2}$$

$\Longrightarrow$

$$\begin{aligned} 1+(\alpha-1)\,s &= A\,(1-s)+C \\ 1s^0+(\alpha-1)\,s &= (A+C)\,s^0-As \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} A+C &= 1 \\ C &= 1-A \end{aligned}$$

$$A=1-\alpha$$

$$\begin{aligned} C &= 1-1+\alpha \\ &= \alpha \end{aligned}$$

Hence

$$\begin{aligned} G(s) &= \frac{1-\alpha}{(1-s)}+\frac{\alpha}{(1-s)^2} \\ &= (1-\alpha)\sum_{n=0}^{\infty}s^n+\alpha\sum_{n=1}^{\infty}s^{n-1} \\ &= (1-\alpha)\sum_{n=0}^{\infty}s^n+\alpha\sum_{n=0}^{\infty}(n+1)s^n \end{aligned}$$

$$\sum_{n=0}^{\infty}p_{22}^{(n)}s^n=(1-\alpha)\sum_{n=0}^{\infty}s^n+\alpha\sum_{n=0}^{\infty}(n+1)s^n$$

but $p_{22}^{(n)}$ is the coefficient of s^n in $G(s)$

$$\begin{aligned} p_{22}^{(n)} &= (1-\alpha)+\alpha(n+1) \\ &= 1+\alpha n \end{aligned}$$

$$\begin{aligned} p_{21}^{(n)} &= 1-p_{22}^{(n)} \\ &= 1-1-\alpha n \\ &= -\alpha n \end{aligned}$$

In summary, using the Chapman-Kolgomorov equations we obtain

$$\mathbf{P}^n=\left\{\begin{array}{cc} \frac{1}{(\alpha+\beta)}\left[\begin{array}{cc} \beta+\alpha\,(1-\alpha-\beta)^n & \alpha-\alpha\,(1-\alpha-\beta)^n \\ \beta-\beta\,(1-\alpha-\beta)^n & \alpha+\beta\,(1-\alpha-\beta)^n \end{array}\right], & \text{if } \alpha+\beta\neq 0 \\ \left[\begin{array}{cc} 1- & \beta n \\ -\alpha n & 1+\alpha n \end{array}\right], & \text{if } \alpha+\beta=0 \end{array}\right\}$$

3.2.6 Applying the relation $P(s) = \frac{1}{1-F(s)}$ in determining $\mathbf{P}^n$

$$\mathbf{P} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

Here

$$\begin{aligned} \alpha_{00} &= 1-\alpha, \quad \alpha_{01} = \alpha \\ \alpha_{10} &= \beta, \quad \alpha_{11} = 1-\beta \\ \alpha_{01}+\alpha_{10} &= \alpha+\beta \\ 1-\alpha_{01}-\alpha_{10} &= 1-\alpha-\beta \end{aligned}$$

hence

$$\mathbf{P}^n = \frac{1}{\alpha+\beta} \begin{bmatrix} \beta+\alpha(1-\alpha-\beta)^n & \alpha-\alpha(1-\alpha-\beta)^n \\ \beta-\beta(1-\alpha-\beta)^n & \alpha+\beta(1-\alpha-\beta)^n \end{bmatrix}$$

3.2.7 Using the normal multiplication method

$$\mathbf{P} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

$$\begin{aligned} \mathbf{P}^2 &= \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \\ &= \begin{bmatrix} (1-\alpha)^2+\alpha\beta & \alpha(1-\alpha)+\alpha(1-\beta) \\ \beta(1-\alpha)+\beta(1-\beta) & \alpha\beta+(1-\beta)^2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{P}^3 &= \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \begin{bmatrix} (1-\alpha)^2+\alpha\beta & \alpha(1-\alpha)+\alpha(1-\beta) \\ \beta(1-\alpha)+\beta(1-\beta) & \alpha\beta+(1-\beta)^2 \end{bmatrix} \\ &= \begin{bmatrix} \begin{array}{c} (1-\alpha)^3+2\alpha\beta(1-\alpha) \\ +\alpha\beta(1-\beta) \end{array} & \begin{array}{c} \alpha(1-\alpha)^2+\alpha^2\beta+\alpha(1-\alpha)(1-\beta) \\ +\alpha(1-\beta)^2 \end{array} \\ \begin{array}{c} \beta(1-\alpha)^2+\alpha^2\beta+\beta(1-\alpha)(1-\beta) \\ +\beta(1-\beta)^2 \end{array} & \begin{array}{c} (1-\beta)^3+2\alpha\beta(1-\beta) \\ +\alpha\beta(1-\alpha) \end{array} \end{bmatrix} \end{aligned}$$

$$
\mathbf{P}^4 = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}
\times \begin{bmatrix} \left\{ \begin{array}{c} (1-\alpha)^3 + 2\alpha\beta(1-\alpha) \\ +\alpha\beta(1-\beta) \end{array} \right\} & \left\{ \begin{array}{c} \alpha(1-\alpha)^2 + \alpha^2\beta \\ +\alpha(1-\alpha)(1-\beta) + \alpha(1-\beta)^2 \end{array} \right\} \\ \left\{ \begin{array}{c} \beta(1-\alpha)^2 + \alpha^2\beta \\ +\beta(1-\alpha)(1-\beta) + \beta(1-\beta)^2 \end{array} \right\} & \left\{ \begin{array}{c} (1-\beta)^3 + 2\alpha\beta(1-\beta) \\ +\alpha\beta(1-\alpha) \end{array} \right\} \end{bmatrix}
$$

$$
= \begin{bmatrix} \left\{ \begin{array}{c} (1-\alpha)^4 + 3\alpha\beta(1-\alpha)^2 \\ +2\alpha\beta(1-\alpha)(1-\beta) \\ +\alpha^2\beta^2 + \alpha\beta(1-\beta)^2 \end{array} \right\} & \left\{ \begin{array}{c} \alpha(1-\alpha)^3 + 2\alpha^2\beta(1-\alpha) \\ +2\alpha^2\beta(1-\beta) + \alpha(1-\alpha)^2(1-\beta) \\ +\alpha(1-\alpha)(1-\beta)^2 + \alpha(1-\beta)^3 \end{array} \right\} \\ \left\{ \begin{array}{c} \beta(1-\alpha)^3 + 2\alpha\beta^2(1-\alpha) \\ +2\alpha\beta^2(1-\beta) + \alpha(1-\alpha)^2(1-\beta) \\ +\alpha(1-\alpha)(1-\beta)^2 + \beta(1-\beta)^3 \end{array} \right\} & \left\{ \begin{array}{c} (1-\beta)^4 + 3\alpha\beta(1-\beta)^2 \\ +2\alpha\beta(1-\alpha)(1-\beta) \\ +\alpha^2\beta^2 + \alpha\beta(1-\alpha)^2 \end{array} \right\} \end{bmatrix}
$$

from $\mathbf{P}^2$ above,

$$
\begin{aligned}
p_{11}^{(2)} &= (1-\alpha)^2 + \alpha\beta \\
&= \frac{\alpha+\beta}{\alpha+\beta}\left[(1-\alpha)^2 + \alpha\beta\right] \\
&= \frac{1}{\alpha+\beta}\left[(\alpha+\beta)\left(1-2\alpha+\alpha^2+\alpha\beta\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[\alpha-2\alpha^2+\alpha^3+\alpha^2\beta+\beta-2\alpha\beta+\alpha^2\beta+\alpha\beta^2\right] \\
&= \frac{1}{\alpha+\beta}\left[\beta+\alpha-2\alpha^2+\alpha^3+\alpha^2\beta-2\alpha\beta+\alpha^2\beta+\alpha\beta^2\right] \\
&= \frac{1}{\alpha+\beta}\left[\beta+\alpha\left(1-2\alpha-2\beta+\alpha^2+2\alpha\beta+\beta^2\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[\beta+\alpha\left(1-2\left(\alpha+\beta\right)+(\alpha+\beta)^2\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[\beta+\alpha\left(1-(\alpha+\beta)\right)^2\right] \\
&= \frac{1}{\alpha+\beta}\left[\beta+\alpha\left(1-\alpha-\beta\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
p_{12}^{(2)} &= \alpha(1-\alpha)+\alpha(1-\beta) \\
&= \frac{\alpha+\beta}{\alpha+\beta}\left[\alpha(1-\alpha)+\alpha(1-\beta)\right] \\
&= \frac{1}{\alpha+\beta}\left[(\alpha+\beta)\left(\alpha(1-\alpha)+\alpha(1-\beta)\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[(\alpha+\beta)\left(\alpha-\alpha^2+\alpha-\alpha\beta\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[(\alpha+\beta)\left(2\alpha-\alpha^2-\alpha\beta\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[2\alpha^2-\alpha^3-\alpha^2\beta+2\alpha\beta-\alpha^2\beta-\alpha\beta^2\right] \\
&= \frac{1}{\alpha+\beta}\left[2\alpha^2-\alpha^3-2\alpha^2\beta+2\alpha\beta-\alpha\beta^2\right] \\
&= \frac{1}{\alpha+\beta}\left[\alpha-\alpha+2\alpha^2-\alpha^3-2\alpha^2\beta+2\alpha\beta-\alpha\beta^2\right] \\
&= \frac{1}{\alpha+\beta}\left[\alpha-\alpha\left(1+2\alpha-\alpha^2+2\alpha\beta+2\beta-\beta^2\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[\alpha-\alpha\left(1+2(\alpha+\beta)+2\alpha\beta-(\alpha+\beta)^2\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[\alpha-\alpha\left(1-(\alpha+\beta)\right)^2\right] \\
&= \frac{1}{\alpha+\beta}\left[\alpha-\alpha(1-\alpha-\beta)^2\right]
\end{aligned}
$$

$$
p_{21}^{(n)} = \beta(1-\alpha)+\beta(1-\beta)
$$

$$
\begin{aligned}
p_{21}^{(2)} &= \beta(1-\alpha)+\beta(1-\beta) \\
&= \frac{\alpha+\beta}{\alpha+\beta}\left[\beta(1-\alpha)+\beta(1-\beta)\right] \\
&= \frac{1}{\alpha+\beta}\left[(\alpha+\beta)\left(\beta(1-\alpha)+\beta(1-\beta)\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[(\alpha+\beta)\left(\beta-\beta^2+\beta-\alpha\beta\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[(\alpha+\beta)\left(2\beta-\alpha\beta-\beta^2\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[2\beta^2-\alpha\beta^2-\beta^3+2\alpha\beta-\alpha^2\beta-\alpha\beta^2\right] \\
&= \frac{1}{\alpha+\beta}\left[2\beta^2-2\alpha\beta^2-\beta^3+2\alpha\beta-\alpha^2\beta\right] \\
&= \frac{1}{\alpha+\beta}\left[\beta-\beta+2\beta^2-2\alpha\beta^2-\beta^3+2\alpha\beta-\alpha^2\beta\right] \\
&= \frac{1}{\alpha+\beta}\left[\beta-\beta\left(1+2\beta-2\alpha\beta-\beta^2+2\alpha-\alpha^2\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[\beta-\beta\left(1+2\left(\alpha+\beta\right)-2\alpha\beta-\left(\alpha+\beta\right)^2\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[\beta-\beta\left(1-\left(\alpha+\beta\right)\right)^2\right] \\
&= \frac{1}{\alpha+\beta}\left[\beta-\beta\left(1-\alpha-\beta\right)^2\right]
\end{aligned}
$$

and

$$
\begin{aligned}
p_{22}^{(n)} &= (1-\beta)^2+\alpha\beta \\
&= \frac{\alpha+\beta}{\alpha+\beta}\left[(1-\beta)^2+\alpha\beta\right] \\
&= \frac{1}{\alpha+\beta}\left[(\alpha+\beta)\left(1-2\beta+\beta^2+\alpha\beta\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[\beta-2\beta^2+\beta^3+\beta^2\alpha+\alpha-2\alpha\beta+\beta^2\alpha+\beta\alpha^2\right] \\
&= \frac{1}{\alpha+\beta}\left[\alpha+\beta-2\beta^2+\beta^3+\beta^2\alpha-2\alpha\beta+\beta^2\alpha+\beta\alpha^2\right] \\
&= \frac{1}{\alpha+\beta}\left[\alpha+\beta\left(1-2\beta-2\alpha+\beta^2+2\alpha\beta+\alpha^2\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[\alpha+\beta\left(1-2\left(\alpha+\beta\right)+\left(\alpha+\beta\right)^2\right)\right] \\
&= \frac{1}{\alpha+\beta}\left[\alpha+\beta\left(1-\left(\alpha+\beta\right)\right)^2\right] \\
&= \frac{1}{\alpha+\beta}\left[\alpha+\beta\left(1-\alpha-\beta\right)^2\right]
\end{aligned}
$$

Thus

$$\mathbf{P}^n = \begin{bmatrix} \frac{1}{(\alpha+\beta)}\left[\beta+\alpha(1-\alpha-\beta)^2\right] & \frac{1}{(\alpha+\beta)}\left[\alpha-\alpha(1-\alpha-\beta)^2\right] \\ \frac{1}{(\alpha+\beta)}\left[\beta-\beta(1-\alpha-\beta)^2\right] & \frac{1}{(\alpha+\beta)}\left[\alpha+\beta(1-\alpha-\beta)^2\right] \end{bmatrix}$$
$$\frac{1}{(\alpha+\beta)}\begin{bmatrix} \beta+\alpha(1-\alpha-\beta)^2 & \alpha-\alpha(1-\alpha-\beta)^2 \\ \beta-\beta(1-\alpha-\beta)^2 & \alpha+\beta(1-\alpha-\beta)^2 \end{bmatrix}$$

Remark

It is not easy to identify the pattern by multiplication approach to enable us to have the general pattern formed.

3.3 Transition Probability Matrix for a Random Walk with Absorbing Barriers

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_2 \end{array} \begin{array}{c} \begin{array}{cc} E_1 & E_2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

$\mathbf{P}$ is the identity matrix

3.3.1 Classification of the Markov chain

Both E_1 and E_2 are absorbing Markov chains.

3.3.2 Classification of states

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_2 \end{array} \begin{array}{c} \begin{array}{cc} E_1 & E_2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

Consider E_1 and E_2 in the above Markov chain, For E_1

$$\begin{aligned} f_{11}^{(0)} &= 0 \\ f_{11}^{(1)} &= 1 \\ f_{11}^{(2)} &= 0 \\ f_{11}^{(3)} &= 0 \end{aligned}$$

In general

$$f_{11}^{(n)} = \left\{ \begin{array}{cc} 1, & n=1 \\ 0, & \text{elsewhere} \end{array} \right\}$$

$$\begin{aligned} f_1 &= \sum_{n=1}^{\infty} f_{11}^{(n)} \\ &= 1 + \sum_{n=2}^{\infty} f_{11}^{(n)} \\ &= 1 \end{aligned}$$

Hence E_1 is persistent.

$$\begin{aligned} \mu_1 &= \sum_{n=1}^{\infty} n f_{11}^{(n)} \\ &= 1 < \infty \end{aligned}$$

hence μ_1 is finite and therefore non-null.

$$\begin{aligned} d &= GCD\left\{ n : f_{jj}^{(n)} > 0 \right\} \\ &= 1 \end{aligned}$$

The same analysis applies to E_2. Therefore, E_1 and E_2 are

(i) Persistent

(ii) Non-null

(iii) Aperiodic

and are theerfore ergodic.

3.3.3 The n^{th} power $\mathbf{P}^n$

$$\mathbf{P}^n = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

3.3.4 The asymptotic behavior

$$
\begin{aligned}
\lim_{n\to\infty}\mathbf{P}^n &= \begin{bmatrix} \mathbf{I} & 0 \\ (1-\mathbf{V}^n)^{-1}\mathbf{U} & \mathbf{I} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

3.4 Transition probability matrix random walk with one reflecting barrier

$$
\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_2 \end{array}\begin{array}{c} \begin{array}{cc} E_1 & E_2 \end{array} \\ \begin{bmatrix} q & p \\ q & p \end{bmatrix} \end{array}
$$

3.4.1 Classification of the Markov chain

All states can be reached from every other state; hence the Markov Chain is irreducible. All the states are of the same type.

3.4.2 Classification of the states

Consider E_1

$$
\begin{aligned}
f_{11}^{(1)} &= \Pr[E_1 \to E_1] = p^0 q \\
f_{11}^{(2)} &= \Pr[E_1 \to E_2 \to E_1] = pq \\
f_{11}^{(3)} &= \Pr[E_1 \to E_2 \to E_2 \to E_1] = p^2 q \\
f_{11}^{(4)} &= \Pr[E_1 \to E_2 \to E_2 \to E_2 \to E_1] = p^3 q
\end{aligned}
$$

In general

$$
f_{11}^{(n)} = p^{n-1} q
$$

$$
\begin{aligned}
f_1 &= \sum_{n=1}^{\infty} f_{11}^{(n)} \\
&= q\left(1+p+p^2+\cdots\right) \\
&= \frac{q}{1-p} \\
&= 1
\end{aligned}
$$

Hence E_1 is persistent

$$\begin{aligned}\mu_1 &= \sum_{n=1}^{\infty} n f_{11}^{(n)} \\ &= \sum_{n=1}^{\infty} n p^{n-1} q \\ &= q \sum_{n=1}^{\infty} n p^{n-1}\end{aligned}$$

$$\begin{aligned}\frac{1}{1-p} &= (1+p+p^2+\cdots) \\ &= \sum_{n=0}^{\infty} p^n\end{aligned}$$

$$\begin{aligned}\frac{d}{dp}\left(\frac{1}{1-p}\right) &= 1+2p+3p^2+4p^3+\cdots \\ \frac{1}{(1-p)^2} &= \sum_{n=1}^{\infty} n p^{n-1}\end{aligned}$$

hence

$$\begin{aligned}\mu_1 &= \frac{q}{(1-p)^2} \\ &= \frac{1}{q}\end{aligned}$$

In general,

$$\mu_1 = \left\{ \begin{array}{cc} <\infty, & 0<q\leq 1 \\ \infty, & q=0 \end{array} \right\}$$

and hence E_1 is non null when $0<q\leq 1$ and null when $q=0$.

Periodicity

$$\begin{aligned}d &= GCD\left\{n : f_{jj}^{(n)} > 0\right\} \\ &= GCD\{1,2,3\ldots\} \\ &= 1\end{aligned}$$

Hence E_1 is aperiodic and hence ergodic when $0<q\leq 1$.

Consider E_2

$$\begin{aligned}
f_{22}^{(1)} &= \Pr\left[E_2 \to E_2\right] = q^0 p \\
f_{22}^{(2)} &= \Pr\left[E_2 \to E_1 \to E_2\right] = q^1 p \\
f_{22}^{(3)} &= \Pr E_2 \to E_1 \to E_1 \to E_2 = q^2 p \\
f_{22}^{(4)} &= \Pr\left[E_2 \to E_1 \to E_1 \to E_1 \to E_2\right] = q^3 p
\end{aligned}$$

generally

$$f_{22}^{(n)} = q^{n-1} p$$

$$\begin{aligned}
f_2 &= \sum_{n=1}^{\infty} f_{22}^{(n)} \\
&= p\left(1 + q + q^2 + \cdots\right) \\
&= \frac{p}{1-q} \\
&= 1
\end{aligned}$$

Hence E_2 is persistent

$$\begin{aligned}
\mu_2 &= \sum_{n=1}^{\infty} n f_{22}^{(n)} \\
&= \sum_{n=1}^{\infty} n q^{n-1} p \\
&= p \sum_{n=1}^{\infty} n q^{n-1}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{1-q} &= (1 + q + q^2 + \cdots) \\
&= \sum_{n=0}^{\infty} q^n
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dp}\left(\frac{1}{1-q}\right) &= 1 + 2q + 3q^2 + 4q^3 + \cdots \\
\frac{1}{(1-p)^2} &= \sum_{n=1}^{\infty} n p^{n-1}
\end{aligned}$$

hence

$$\begin{aligned}
\mu_2 &= \frac{p}{(1-q)^2} \\
&= \frac{1}{p}
\end{aligned}$$

In general,

$$\mu_2 = \begin{Bmatrix} <\infty, & 0<p\leq 1 \\ \infty, & p=0 \end{Bmatrix}$$

and hence E_1 is non null when $0<p\leq 1$ and null when $p=0$.

Periodicity

$$\begin{aligned} d &= GCD\left\{n : f_{jj}^{(n)} > 0\right\} \\ &= GCD\{1,2,3\ldots\} \\ &= 1 \end{aligned}$$

Hence E_2 is aperiodic and therefore ergodic when $0<p\leq 1$.

3.4.3 The asymptotic behavior of the states

$$\lim_{n\to\infty} \mathbf{P}^n = \mathbf{P}'\pi$$

$$\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} q & q \\ p & p \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}$$

$$\pi_1 = q\pi_1 + q\pi_2$$

but

$$\begin{aligned} \pi_1 + \pi_2 &= 1 \\ \pi_2 &= 1 - \pi_1 \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} \pi_1 &= q\pi_1 + q(1-\pi_1) \\ &= q \end{aligned}$$

$$\begin{aligned} \pi_2 &= p\pi_1 + p\pi_2 \\ &= p\pi_1 + p(1-\pi_1) \\ &= p \end{aligned}$$

hence

$$\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} q \\ p \end{bmatrix}$$

By theorem 2.12,

$$\begin{aligned} \mu_1 &= \frac{1}{\pi_1} \\ &= \frac{1}{q} \end{aligned}$$

and

$$\begin{aligned} \mu_2 &= \frac{1}{\pi_2} \\ &= \frac{1}{p} \end{aligned}$$

3.4.4 The n^{th} step transition probability

Method 1: Eigen Value Approach

Finding the Eigen values of the matrix $\mathbf{P}$

$$|\mathbf{P} - \lambda \mathbf{I}| = \mathbf{0}$$

$$\left| \begin{bmatrix} q & p \\ q & p \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \mathbf{0}$$

$$\left| \begin{bmatrix} q - \lambda & p \\ q & p - \lambda \end{bmatrix} \right| = \mathbf{0}$$

$$[(q-\lambda)(p-\lambda) - qp] = 0$$

$$qp - \lambda q - \lambda p + \lambda^2 - qp = 0$$

$$\begin{aligned} \lambda^2 - \lambda q - \lambda p &= 0 \\ \lambda(\lambda - q - p) &= 0 \end{aligned}$$

$$\lambda_1 = 0$$

and

$$\begin{aligned} \lambda_2 &= q + p \\ &= 1 \end{aligned}$$

since

$$q+p=1$$

When $\lambda=0$ the corresponding Eigen vectors ar

$$\mathbf{Px}=\lambda\mathbf{x}$$

$$\begin{bmatrix} q & p \\ q & p \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\mathbf{0}$$

$$qx_1+px_2=0$$

$$qx_1=-px_2$$

$$x_1=\frac{-px_2}{q}$$

Supposing $x_2=1$

$$x_1=\frac{-p}{q}$$

$\Longrightarrow$

$$v_1=\begin{bmatrix} \frac{-p}{q} \\ 1 \end{bmatrix}$$

$$V=\begin{bmatrix} 1 & \frac{-p}{q} \\ 1 & 1 \end{bmatrix}$$

When $\lambda=1$

$$\mathbf{Px}=\lambda\mathbf{x}$$

$$\begin{bmatrix} q & p \\ q & p \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} qx_1+px_2 &= x_1 \\ qx_1+px_2 &= x_2 \end{aligned}$$

$$\begin{aligned} px_2 &= x_1-qx_1 \\ &= (1-q)x_1 \\ &= px_1 \end{aligned}$$

$\Longrightarrow$

$$x_1=x_2$$

Suppose

$$x_1=x_2=1$$

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} \frac{-p}{q} & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{aligned}
\mathbf{V}^{-1} &= \frac{1}{\frac{-1}{q}} \begin{bmatrix} 1 & -1 \\ -1 & \frac{-p}{q} \end{bmatrix} \\
&= -q \begin{bmatrix} 1 & -1 \\ -1 & \frac{-p}{q} \end{bmatrix} \\
&= \begin{bmatrix} -q & q \\ q & p \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}^n &= \mathbf{VDV}^{-1} \\
&= \begin{bmatrix} \frac{-p}{q} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -q & q \\ q & p \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -q & q \\ q & p \end{bmatrix} \\
&= \begin{bmatrix} q & p \\ q & p \end{bmatrix}
\end{aligned}$$

Method 2: Chapman-Kolmogorov Approach

$$\mathbf{P}^n = \mathbf{P}^{n-1}\mathbf{P}$$

$\Longrightarrow$

$$\begin{bmatrix} p_{11}^{(n)} & p_{12}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} \end{bmatrix} = \begin{bmatrix} p_{11}^{(n-1)} & p_{12}^{(n-1)} \\ p_{21}^{(n-1)} & p_{22}^{(n-1)} \end{bmatrix} \begin{bmatrix} q & p \\ q & p \end{bmatrix}$$

Therefore

$$\begin{aligned}
p_{11}^{(n)} &= p_{11}^{(n-1)} q + p_{12}^{(n-1)} q \\
p_{11}^{(n)} &= q\left(p_{11}^{(n-1)} + p_{12}^{(n-1)}\right)
\end{aligned}$$

but

$$p_{11}^{(n-1)} + p_{12}^{(n-1)} = 1$$

$\Longrightarrow$

$$p_{11}^{(n)} = q$$

but

$$\begin{aligned} p_{11}^{(n)} + p_{12}^{(n)} &= 1 \\ p_{12}^{(n)} &= 1 - p_{11}^{(n)} \\ &= 1 - q \\ &= p \end{aligned}$$

Similarly

$$\begin{aligned} p_{22}^{(n)} &= p_{21}^{(n-1)} p + p_{22}^{(n-1)} p \\ p_{11}^{(n)} &= p\left(p_{11}^{(n-1)} + p_{12}^{(n-1)}\right) \end{aligned}$$

but

$$p_{21}^{(n-1)} + p_{22}^{(n-1)} = 1$$

$\Longrightarrow$

$$p_{22}^{(n)} = p$$

but

$$\begin{aligned} p_{22}^{(n)} + p_{21}^{(n)} &= 1 \\ p_{21}^{(n)} &= 1 - p_{22}^{(n)} \\ &= 1 - p \\ &= q \end{aligned}$$

Therefore

$$\mathbf{P}^n = \begin{bmatrix} p_{11}^{(n)} & p_{12}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} \end{bmatrix} = \begin{bmatrix} q & p \\ q & p \end{bmatrix}$$

Method 3: Direct Multiplication

$$\mathbf{P} = \begin{bmatrix} q & p \\ q & p \end{bmatrix}$$

$$\begin{aligned}
\mathbf{P}^2 &= \begin{bmatrix} q & p \\ q & p \end{bmatrix}\begin{bmatrix} q & p \\ q & p \end{bmatrix} \\
&= \begin{bmatrix} q^2+pq & qp+p^2 \\ q^2+pq & qp+p^2 \end{bmatrix} \\
&= \begin{bmatrix} q(q+p) & p(q+p) \\ q(q+p) & p(q+p) \end{bmatrix} \\
&= \begin{bmatrix} q & p \\ q & p \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}^3 &= \mathbf{P}^2\mathbf{P} \\
&= \begin{bmatrix} q & p \\ q & p \end{bmatrix}\begin{bmatrix} q & p \\ q & p \end{bmatrix} \\
&= \begin{bmatrix} q & p \\ q & p \end{bmatrix}
\end{aligned}$$

which $\Longrightarrow$

$$\mathbf{P}^n = \begin{bmatrix} p_{11}^{(n)} & p_{12}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} \end{bmatrix} = \begin{bmatrix} q & p \\ q & p \end{bmatrix}$$

Method 4: Applying the relation $P(s) = \frac{1}{1-F(s)}$

Using the relation arrived at in section 2.1.3 that

$$\begin{aligned}
\mathbf{P}^n &= \begin{bmatrix} p_{00}^{(n)} & p_{01}^{(n)} \\ p_{10}^{(n)} & p_{11}^{(n)} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}} + \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}}(1-\alpha_{01}-\alpha_{10})^n & \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}}\left\{1-(1-\alpha_{01}-\alpha_{10})^n\right\} \\ \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}}\left\{1-(1-\alpha_{01}-\alpha_{10})^n\right\} & \frac{\alpha_{01}}{\alpha_{01}+\alpha_{10}} + \frac{\alpha_{10}}{\alpha_{01}+\alpha_{10}}(1-\alpha_{01}-\alpha_{10})^n \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\alpha_{00} &= \alpha_{10} = q, \qquad \alpha_{01} = \alpha_{11} = p \\
\alpha_{01}+\alpha_{10} &= p+q = 1 \\
1-\alpha_{01}-\alpha_{10} &= 1-p-q \\
&= 1-(p+q) \\
&= 0
\end{aligned}$$

Hence subsituting to equation $(3.)$ we have

$$\mathbf{P}^n = \begin{bmatrix} q & p \\ q & p \end{bmatrix}$$

3.5 Transition Probability Matrix for Random Walks with Two Reflecting Barriers

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} a & 1-a \\ 1-b & b \end{bmatrix} \end{array}$$

3.5.1 Classification of the Markov chain

All states can be reached from every other state; hence the Markov Chain is irreducible. Thus all the states are of the same type.

3.5.2 Classification of the states

Consider E_0

$$\begin{aligned}
f_{00}^{(1)} &= \Pr[E_0 \to E_0] = a \\
f_{00}^{(2)} &= \Pr[E_0 \to E_1 \to E_0] = (1-a)(1-b) \\
f_{00}^{(3)} &= \Pr[E_0 \to E_1 \to E_1 \to E_0] = (1-a)\,b\,(1-b) \\
f_{00}^{(4)} &= \Pr[E_0 \to E_1 \to E_1 \to E_1 \to E_0] = (1-a)\,b^2\,(1-b) \\
f_{00}^{(5)} &= \Pr[E_0 \to E_1 \to E_1 \to E_1 \to E_1 \to E_0] = (1-a)\,b^3\,(1-b)
\end{aligned}$$

On general

$$f_{00}^{(n)} = \left\{ \begin{array}{c} a, \quad n = 1 \\ (1-a)\,b^{n-2}\,(1-b), \quad n \geq 2 \end{array} \right\}$$

$$
\begin{aligned}
f_0 &= \sum_{n=1}^{\infty} f_{00}^{(n)} \\
&= a+(1-a)(1-b)\sum_{n=2}^{\infty} b^{n-2} \\
&= a+(1-a)(1-b)\left[1+b+b^2+\cdots\right] \\
&= a+\frac{(1-a)(1-b)}{1-b} \\
&= 1
\end{aligned}
$$

Hence E_0 is persistent.

$$
\begin{aligned}
\mu_0 &= \sum_{n=0}^{\infty} n f_{00}^{(n)} \\
&= 0+a+\sum_{n=2}^{\infty} n(1-a)(1-b)b^{n-2} \\
&= a+2(1-a)(1-b)+3(1-a)(1-b)b+4(1-a)(1-b)b^2+\cdots \\
&= a+(1-a)(1-b)\left[2+3b+4b^2+\cdots\right] \\
&= a+\frac{(1-a)(1-b)}{b}\left[2b+3b^2+4b^3+\cdots\right] \\
&= a+\frac{(1-a)(1-b)}{b}\left[\frac{1}{(1-b)^2}-1\right] \\
&= a+\frac{(1-a)(1-b)}{b}\left[\frac{1-(1-b)^2}{(1-b)^2}\right] \\
&= a+\frac{(1-a)}{b}\left[\frac{1-\left(1-2b+b^2\right)}{(1-b)}\right] \\
&= a+\frac{(1-a)}{b}\left[\frac{b(2-b)}{(1-b)}\right] \\
&= a+\frac{(1-a)(2-b)}{(1-b)} \\
&= \frac{2-a-b}{1-b}
\end{aligned}
$$

State E_0 is non-null iff $b \neq 1$

Periodicity

$$
\begin{aligned}
d &= GCD\left\{n : f_{jj}^{(n)} > 0\right\} \\
&= GCD\{1,2,3\ldots\} \\
&= 1
\end{aligned}
$$

Thus E_0 is aperiodic and persistent, hence ergodic.

$$
\begin{aligned}
\mu_1 &= \sum_{n=0}^{\infty} n f_{11}^{(n)} \\
&= 0 + b + \sum_{n=2}^{\infty} n(1-a)(1-b)a^{n-2} \\
&= b + 2(1-a)(1-b) + 3(1-a)(1-b)a + 4(1-a)(1-b)a^2 + \cdots \\
&= b + (1-a)(1-b)\left[2 + 3a + 4a^2 + \cdots\right] \\
&= b + \frac{(1-a)(1-b)}{a}\left[2a + 3a^2 + 4a^3 + \cdots\right] \\
&= b + \frac{(1-a)(1-b)}{a}\left[\frac{1}{(1-a)^2} - 1\right] \\
&= b + \frac{(1-a)(1-b)}{a}\left[\frac{1-(1-a)^2}{(1-a)^2}\right] \\
&= b + \frac{(1-a)}{a}\left[\frac{1-\left(1-2a+a^2\right)}{(1-a)}\right] \\
&= b + \frac{(1-a)}{a}\left[\frac{a(2-a)}{(1-a)}\right] \\
&= b + \frac{(1-a)(2-b)}{(1-a)} \\
&= \frac{2-a-b}{1-a}
\end{aligned}
$$

State E_1 is non-null iff $a \neq 1$.

Periodicity

$$
\begin{aligned}
d &= GCD\left\{n : f_{jj}^{(n)} > 0\right\} \\
&= GCD\{1,2,3\ldots\} \\
&= 1
\end{aligned}
$$

Thus E_1 is aperiodic, and persistent, hence ergodic when $a \neq 1$.

3.5.3 The asymptotic behavior

$$\lim_{n\to\infty} \mathbf{P}^n = \mathbf{P}'\pi$$

$$\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} a & 1-a \\ 1-b & b \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}$$

$$\begin{aligned} \pi_1 &= a\pi_1 + (1-a)\pi_2 \\ \pi_2 &= (1-b)\pi_1 + b\pi_2 \end{aligned}$$

but

$$\begin{aligned} \pi_1 + \pi_2 &= 1 \\ \pi_2 &= 1 - \pi_1 \end{aligned}$$

$$\begin{aligned} \pi_1 &= a\pi_1 + (1-a)(1-\pi_1) \\ &= a\pi_1 + (1-\pi_1) - a(1-\pi_1) \\ &= a\pi_1 + 1 - \pi_1 - a + a\pi_1 \end{aligned}$$

$$2\pi_1 - 2a\pi_1 = 1 - a$$

$$2\pi_1(1-a) = (1-a)$$

$$\begin{aligned} 2\pi_1 &= 1 \\ \pi_1 &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \pi_2 &= 1 - \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

Therefore

$$\begin{aligned} \mu_k &= \frac{1}{\pi_k} \\ &= \frac{1}{0.5} \\ &= 2 \end{aligned}$$

Since μ is finite and thus E_1 is non-null.

Periodicity

$$\begin{aligned} d &= GCD\left\{n : f_{jj}^{(n)} > 0\right\} \\ &= GCD\{1,2,3\ldots\} \\ &= 1 \end{aligned}$$

Thus E_1 and E_2 are persistent, aperiodic, and non-null hence ergodic.

3.5.4 The n^{th} power $\mathbf{P}^n$

Method 1: Eigen values approach

$$|\mathbf{P}-\lambda\mathbf{I}| = \mathbf{0}$$

$$\left|\begin{bmatrix} a & 1-a \\ 1-b & b \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = \mathbf{0}$$

$$\left|\begin{bmatrix} a-\lambda & 1-a \\ 1-b & b-\lambda \end{bmatrix}\right| = \mathbf{0}$$

$$[(a-\lambda)(b-\lambda)-(1-b)(1-a)] = 0$$

$$\begin{aligned} ab-\lambda a-\lambda b+\lambda^2-[1-a-b+ab] &= 0 \\ ab-\lambda a-\lambda b+\lambda^2-1+a+b-ab &= 0 \\ -\lambda a-\lambda b+\lambda^2-1+a+b &= 0 \\ \lambda^2-(a+b)\lambda+(a+b-1) &= 0 \\ \lambda^2-\lambda-(a+b-1)\lambda+(a+b-1) &= 0 \\ \lambda[\lambda-1]-(a+b-1)[\lambda-1] &= 0 \\ [\lambda-1][\lambda-(a+b-1)] &= 0 \end{aligned}$$

$\Longrightarrow$

$$\lambda = 1 \text{ or } \lambda = a+b-1$$

when $\lambda = 1$

$$\mathbf{Px} = \lambda\mathbf{x}$$

$$\begin{bmatrix} a & 1-a \\ 1-b & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} ax_1 + (1-a)x_2 &= x_1 \\ (1-b)x_1 + bx_2 &= x_2 \end{aligned}$$

$$\begin{aligned} ax_1 + x_2 - ax_2 &= x_1 \\ ax_1 + (1-a)\,x_2 &= x_1 \\ (1-a)\,x_2 &= x_1 - ax_1 \\ (1-a)\,x_2 &= (1-a)\,x_1 \\ x_2 &= x_1 \end{aligned}$$

Taking a case, say $x_2 = x_1 = 1$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

When $\lambda = a + b - 1$

$$\mathbf{Px} = \lambda \mathbf{x}$$

$$\begin{bmatrix} a & 1-a \\ 1-b & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [a+b-1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} ax_1 + (1-a)x_2 &= [a+b-1]\,x_1 \\ (1-b)x_1 + bx_2 &= [a+b-1]\,x_2 \end{aligned}$$

$$\begin{aligned} (1-a)x_2 &= [a+b-1-a]\,x_1 \\ (1-a)x_2 &= (b-1)\,x_1 \end{aligned}$$

$$x_1 = \frac{(1-a)}{(b-1)} x_2$$

$$(1-b)\frac{(1-a)}{(b-1)} x_2 + bx_2 = [a+b-1]\,x_2$$

$$\begin{aligned} \left[\frac{(1-b)(1-a)}{(b-1)} + b\right] x_2 &= (a+b-1)\,x_2 \\ \left[\frac{(b-1)(a-1)}{(b-1)} + b\right] x_2 &= (a+b-1)\,x_2 \\ [a-1+b]\,x_2 &= (a+b-1)\,x_2 \end{aligned}$$

therefore x_2 is a free variable and therefore can take any value. Suppose

$$x_2 = 1$$

$$x_1 = \frac{(1-a)}{(b-1)}$$

$$v_2 = \begin{bmatrix} \frac{(1-a)}{(b-1)} \\ 1 \end{bmatrix}$$

hence

$$\begin{aligned} \mathbf{V} &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{(1-a)}{(b-1)} \\ 1 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{V}^{-1} &= \frac{1}{1-\frac{(1-a)}{(b-1)}} \begin{bmatrix} 1 & \frac{(a-1)}{(b-1)} \\ -1 & 1 \end{bmatrix} \\ &= \frac{b-1}{(a-1)+(b+1)} \begin{bmatrix} 1 & \frac{(a-1)}{(b-1)} \\ -1 & 1 \end{bmatrix} \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{P}^n &= \mathbf{VDV}^{-1} \\ &= \frac{b-1}{(a-1)+(b+1)} \begin{bmatrix} 1 & \frac{(1-a)}{(b-1)} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (a+b-1)^n \end{bmatrix} \begin{bmatrix} 1 & \frac{(a-1)}{(b-1)} \\ -1 & 1 \end{bmatrix} \\ &= \frac{b-1}{(a-1)+(b+1)} \begin{bmatrix} 1 & \frac{(1-a)}{(b-1)}(a+b-1)^n \\ 1 & (a+b-1)^n \end{bmatrix} \begin{bmatrix} 1 & \frac{(a-1)}{(b-1)} \\ -1 & 1 \end{bmatrix} \\ &= \frac{b-1}{(a-1)+(b+1)} \begin{bmatrix} 1-\frac{(1-a)}{(b-1)}(a+b-1)^n & \frac{(a-1)}{(b-1)}+\frac{(1-a)}{(b-1)}(a+b-1)^n \\ 1-(a+b-1)^n & \frac{(a-1)}{(b-1)}+(a+b-1)^n \end{bmatrix} \\ &= \frac{1}{(a-1)+(b+1)} \begin{bmatrix} (b-1)+(a-1)(a+b-1)^n & (a-1)-(a-1)(a+b-1)^n \\ (b-1)-(b-1)(a+b-1)^n & (a-1)+(b-1)(a+b-1)^n \end{bmatrix} \end{aligned}$$

Method 2: Applying the relation $P(s) = \frac{1}{1-F(s)}$ in determining $\mathbf{P}^n$

$$\mathbf{P}^n = \frac{1}{\alpha_{01}+\alpha_{10}} \begin{bmatrix} \alpha_{10}+\alpha_{01}(1-\alpha_{01}-\alpha_{10})^n & \alpha_{01}-\alpha_{01}(1-\alpha_{01}-\alpha_{10})^n \\ \alpha_{10}-\alpha_{10}(1-\alpha_{01}-\alpha_{10})^n & \alpha_{01}+\alpha_{10}(1-\alpha_{01}-\alpha_{10})^n \end{bmatrix}$$

here

$$\begin{aligned}\alpha_{00} &= a, \quad \alpha_{10} = 1-b, \quad \alpha_{01} = 1-a, \quad \alpha_{11} = b \\ 1-\alpha_{01}-\alpha_{10} &= 1-(1-a)-(1-b) \\ &= 1-1+a-1+b \\ &= a+b-1\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{P}^n &= \frac{1}{(1-a)+(1-b)}\left[\begin{array}{cc}(1-b)+(1-a)\,(a+b-1)^n & (1-a)-(1-a)\,(a+b-1)^n \\ (1-b)-(1-b)\,(a+b-1)^n & (1-a)+(1-b)\,(a+b-1)^n\end{array}\right] \\ &= \frac{1}{(a-1)+(b-1)}\left[\begin{array}{cc}(b-1)+(a-1)(a+b-1)^n & (a-1)-(a-1)(a+b-1)^n \\ (b-1)-(b-1)\,(a+b-1)^n & (a-1)+(b-1)\,(a+b-1)^n\end{array}\right]\end{aligned}$$

Method 3: Chapman-Kolmogorov approach

$$\mathbf{P}^n = \mathbf{P}^{n-1}\mathbf{P}$$

$\Longrightarrow$

$$\left[\begin{array}{cc}p_{11}^{(n)} & p_{12}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)}\end{array}\right] = \left[\begin{array}{cc}p_{11}^{(n-1)} & p_{12}^{(n-1)} \\ p_{21}^{(n-1)} & p_{22}^{(n-1)}\end{array}\right]\left[\begin{array}{cc}a & 1-a \\ 1-b & b\end{array}\right]$$

Therefore

$$P_{11}^{(n)} = aP_{11}^{(n-1)} + (1-b)P_{12}^{(n-1)}$$

but

$$\begin{aligned}P_{11}^{(n-1)} + P_{12}^{(n-1)} &= 1 \\ P_{12}^{(n-1)} &= 1-P_{11}^{(n-1)}\end{aligned}$$

$\Longrightarrow$

$$\begin{aligned}P_{11}^{(n)} &= aP_{11}^{(n-1)} + (1-b)\left(1-P_{11}^{(n-1)}\right) \\ &= aP_{11}^{(n-1)} + (1-b) - (1-b)P_{11}^{(n-1)} \\ &= (a+b-1)P_{11}^{(n-1)} + (1-b) \qquad \textbf{(21)}\end{aligned}$$

Using the probability generating function technique, equation $(3.)$ is multiplied by s^n and the result summed over n

$$\sum_{n=1}^{\infty} P_{11}^{(n)} s^n = (a+b-1)\sum_{n=1}^{\infty} P_{11}^{(n-1)} s^n + (1-b)\sum_{n=1}^{\infty} s^n$$

but

$$
\begin{aligned}
G(s) &= \sum_{n=0}^{\infty} P_{11}^{(n)} s^n \\
&= P_{11}^{(0)} + \sum_{n=1}^{\infty} P_{11}^{(n)} s^n \\
&= 1 + \sum_{n=1}^{\infty} P_{11}^{(n)} s^n
\end{aligned}
$$

$$
\sum_{n=1}^{\infty} P_{11}^{(n)} s^n = G(s) - 1
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} P_{11}^{(n)} s^n &= G(s) - 1 \\
&= (a+b-1)s \sum_{n=1}^{\infty} P_{11}^{(n-1)} s^{n-1} + (1-b) \sum_{n=1}^{\infty} s^n \\
&= (a+b-1)sG(s) + (1-b)s\left[1 + s + s^2 + \cdots\right] \\
&= (a+b-1)sG(s) + \frac{(1-b)s}{1-s}
\end{aligned}
$$

$$
\begin{aligned}
\left[1-(a+b-1)s\right] G(s) &= 1 + \frac{(1-b)s}{1-s} \\
&= \frac{1-s+(1-b)s}{1-s}
\end{aligned}
$$

$$
\begin{aligned}
G(s) &= \frac{1}{\left[1-(a+b-1)s\right]} \frac{1-s+s-bs}{1-s} \\
&= \frac{1-bs}{\left[1-(a+b-1)s\right](1-s)}
\end{aligned}
\tag{22}
$$

Using partial fractions

$$
\frac{1-bs}{\left[1-(a+b-1)s\right](1-s)} = \frac{A}{(1-s)} + \frac{B}{\left[1-(a+b-1)s\right]}
$$

$\Longrightarrow$

$$
\begin{aligned}
1-bs &= \left[1-(a+b-1)s\right] A + (1-s) B \\
&= A - (a+b-1)sA + B - Bs \\
&= (A+B)s^0 + \left[-B-(a+b-1)A\right] s
\end{aligned}
$$

$\Longrightarrow$

$$A+B=1$$

and

$$-b=-B-(a+b-1)A$$

$$-B=A-1$$

$$\begin{aligned}-b &= A-1-(a+b-1)A \\ &= A[1-(a+b-1)]-1 \\ &= A[2-a-b)]-1\end{aligned}$$

$$A=\frac{1-b}{(2-a-b)}$$

$$\begin{aligned}B &= 1-A \\ &= 1-\frac{1-b}{(2-a-b)} \\ &= \frac{1-a}{(2-a-b)}\end{aligned}$$

Therefore

$$\begin{aligned}G(s) &= \frac{\frac{1-b}{(2-a-b)}}{(1-s)}+\frac{\frac{1-a}{(2-a-b)}}{[1-(a+b-1)s]} \\ &= \frac{1}{(2-a-b)}\left[\frac{1-b}{(1-s)}+\frac{1-a}{[1-(a+b-1)s]}\right]\end{aligned}$$

$$\begin{aligned}\frac{1}{1-s} &= 1+s+s^2+s^3+\cdots \\ &= \sum_{n=0}^{\infty} s^n\end{aligned}$$

Let

$$x=(a+b-1)$$

$$\begin{aligned}\frac{1}{1-xs} &= 1+xs+xs^2+xs^3+\cdots \\ &= \sum_{n=0}^{\infty} x^n s^n\end{aligned}$$

hence

$$\frac{1}{[1-(a+b-1)s]}=\sum_{n=0}^{\infty}(a+b-1)^n s^n$$

and

$$G(s)=\frac{1}{(2-a-b)}\left\{(1-b)\sum_{n=0}^{\infty}s^n+(1-a)\sum_{n=0}^{\infty}(a+b-1)^n s^n\right\}$$

$$\sum_{n=0}^{\infty}P_{11}^{(n)}s^n=\frac{1}{(2-a-b)}\left\{(1-b)\sum_{n=0}^{\infty}s^n+(1-a)\sum_{n=0}^{\infty}(a+b-1)^n s^n\right\}$$

$P_{11}^{(n)}$ is the coefficient of s^n

$$\begin{aligned}P_{11}^{(n)} &= \frac{1}{(2-a-b)}\{(1-b)+(1-a)(a+b-1)^n\}\\ &= \frac{1}{(1-a)+(1-b)}[(1-b)+(1-a)(a+b-1)^n]\\ &= \frac{1}{(a-1)+(b-1)}[(b-1)+(a-1)(a+b-1)^n]\end{aligned}$$

but

$$P_{11}^{(n)}+P_{12}^{(n)}=1$$

$$\begin{aligned}P_{12}^{(n)} &= 1-P_{11}^{(n)}\\ &= 1-\frac{1}{(a-1)+(b-1)}[(b-1)+(a-1)(a+b-1)^n]\\ &= \frac{(a-1)+(b-1)-[(b-1)+(a-1)(a+b-1)^n]}{(a-1)+(b-1)}\\ &= \frac{(a-1)+(b-1)-(b-1)-(a-1)(a+b-1)^n}{(a-1)+(b-1)}\\ &= \frac{(a-1)-(a-1)(a+b-1)^n}{(a-1)+(b-1)}\end{aligned}$$

Similarly, from

$$\mathbf{P}^n=\mathbf{P}^{n-1}\mathbf{P}$$

$$\begin{bmatrix}p_{11}^{(n)} & p_{12}^{(n)}\\ p_{21}^{(n)} & p_{22}^{(n)}\end{bmatrix}=\begin{bmatrix}p_{11}^{(n-1)} & p_{12}^{(n-1)}\\ p_{21}^{(n-1)} & p_{22}^{(n-1)}\end{bmatrix}\begin{bmatrix}a & 1-a\\ 1-b & b\end{bmatrix}$$

Therefore

$$P_{22}^{(n)}=(1-a)P_{21}^{(n-1)}+bP_{22}^{(n-1)}$$

but

$$\begin{aligned}P_{21}^{(n-1)}+P_{22}^{(n-1)} &= 1\\ P_{21}^{(n-1)} &= 1-P_{22}^{(n-1)}\end{aligned}$$

$\Longrightarrow$

$$
\begin{aligned}
P_{22}^{(n)} &= (1-a)\left(1-P_{22}^{(n-1)}\right)+bP_{22}^{(n-1)} \\
&= bP_{22}^{(n-1)}+(1-a)-(1-a)P_{22}^{(n-1)} \\
&= (a+b-1)P_{22}^{(n-1)}+(1-a) \qquad \textbf{(23)}
\end{aligned}
$$

Using the probability generating function technique, equation $(3.)$ is multiplied by s^n and the result summed over n

$$\sum_{n=1}^{\infty} P_{22}^{(n)} s^n = (a+b-1)\sum_{n=1}^{\infty} P_{22}^{(n-1)} s^n + (1-a)\sum_{n=1}^{\infty} s^n$$

but

$$
\begin{aligned}
G(s) &= \sum_{n=0}^{\infty} P_{22}^{(n)} s^n \\
&= P_{22}^{(0)} + \sum_{n=1}^{\infty} P_{22}^{(n)} s^n \\
&= 1+\sum_{n=1}^{\infty} P_{22}^{(n)} s^n
\end{aligned}
$$

$$\sum_{n=1}^{\infty} P_{22}^{(n)} s^n = G(s)-1$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} P_{22}^{(n)} s^n &= G(s)-1 \\
&= (a+b-1)s\sum_{n=1}^{\infty} P_{22}^{(n-1)} s^{n-1} + (1-a)\sum_{n=1}^{\infty} s^n \\
&= (a+b-1)sG(s)+(1-a)s\left[1+s+s^2+\cdots\right] \\
&= (a+b-1)sG(s)+\frac{(1-a)s}{1-s}
\end{aligned}
$$

$$
\begin{aligned}
\left[1-(a+b-1)s\right]G(s) &= 1+\frac{(1-a)s}{1-s} \\
&= \frac{1-s+(1-a)s}{1-s}
\end{aligned}
$$

$$
\begin{aligned}
G(s) &= \frac{1}{\left[1-(a+b-1)s\right]}\frac{1-s+s-as}{1-s} \\
&= \frac{1-as}{\left[1-(a+b-1)s\right](1-s)} \qquad \textbf{(24)}
\end{aligned}
$$

Using partial fractions

$$\frac{1-as}{[1-(a+b-1)s](1-s)} = \frac{A}{(1-s)} + \frac{B}{[1-(a+b-1)s]}$$

$\Longrightarrow$

$$\begin{aligned} 1-as &= [1-(a+b-1)s]A + (1-s)B \\ &= A-(a+b-1)sA + B - Bs \\ &= (A+B)s^0 + [-B-(a+b-1)A]s \end{aligned}$$

$\Longrightarrow$

$$A+B=1$$

and

$$-a = -B-(a+b-1)A$$

$$-B = A-1$$

$$\begin{aligned} -a &= A-1-(a+b-1)A \\ &= A[1-(a+b-1)]-1 \\ &= A[2-a-b)]-1 \end{aligned}$$

$$A = \frac{1-a}{(2-a-b)}$$

$$\begin{aligned} B &= 1-A \\ &= 1-\frac{1-a}{(2-a-b)} \\ &= \frac{1-b}{(2-a-b)} \end{aligned}$$

Therefore

$$\begin{aligned} G(s) &= \frac{\frac{1-a}{(2-a-b)}}{(1-s)} + \frac{\frac{1-b}{(2-a-b)}}{[1-(a+b-1)s]} \\ &= \frac{1}{(2-a-b)}\left[\frac{1-b}{(1-s)} + \frac{1-a}{[1-(a+b-1)s]}\right] \end{aligned}$$

$$\begin{aligned} \frac{1}{1-s} &= 1+s+s^2+s^3+\cdots \\ &= \sum_{n=0}^{\infty} s^n \end{aligned}$$

Let

$$x = (a+b-1)$$

$$\begin{aligned} \frac{1}{1-xs} &= 1+xs+xs^2+xs^3+\cdots \\ &= \sum_{n=0}^{\infty} x^n s^n \end{aligned}$$

hence

$$\frac{1}{[1-(a+b-1)s]} = \sum_{n=0}^{\infty}(a+b-1)^n s^n$$

and

$$G(s) = \frac{1}{(2-a-b)}\left\{(1-a)\sum_{n=0}^{\infty} s^n + (1-b)\sum_{n=0}^{\infty}(a+b-1)^n s^n\right\}$$

$$\sum_{n=0}^{\infty} P_{22}^{(n)} s^n = \frac{1}{(2-a-b)}\left\{(1-a)\sum_{n=0}^{\infty} s^n + (1-b)\sum_{n=0}^{\infty}(a+b-1)^n s^n\right\}$$

$P_{22}^{(n)}$ is the coefficient of s^n

$$\begin{aligned} P_{22}^{(n)} &= \frac{1}{(2-a-b)}\{(1-a)+(1-b)(a+b-1)^n\} \\ &= \frac{1}{(1-a)+(1-b)}[(1-a)+(1-b)(a+b-1)^n] \\ &= \frac{1}{(a-1)+(b-1)}[(a-1)+(b-1)(a+b-1)^n] \end{aligned}$$

but

$$P_{22}^{(n)} + P_{21}^{(n)} = 1$$

$$\begin{aligned} P_{21}^{(n)} &= 1 - P_{22}^{(n)} \\ &= 1 - \frac{1}{(a-1)+(b-1)}[(a-1)+(b-1)(a+b-1)^n] \\ &= \frac{(a-1)+(b-1)-[(a-1)+(b-1)(a+b-1)^n]}{(a-1)+(b-1)} \\ &= \frac{(a-1)+(b-1)-(a-1)-(a-1)(a+b-1)^n}{(a-1)+(b-1)} \\ &= \frac{(b-1)-(b-1)(a+b-1)^n}{(a-1)+(b-1)} \end{aligned}$$

Therefore

$$\mathbf{P}^n = \frac{1}{(a-1)+(b-1)}\begin{bmatrix} (b-1)-(a-1)(a+b-1)^n & (a-1)-(a-1)(a+b-1)^n \\ (b-1)-(b-1)(a+b-1)^n & (a-1)-(b-1)(a+b-1)^n \end{bmatrix}$$

4 SIMPLE RANDOM WALKS

4.1 Introduction

In this chapter the Simple random walk is studied as a Markov chain. We will look at the classification of states, asymptotic behaviour and the nth power of the transition probability matrix.

Let $(X_i : i \geq 1)$ be a collection of independent and identically distributed random variables taking the values $+1$ and -1 with probabilities p and $q = 1 - p$ respectively. Then the collection $(Sn \geq 0)$ where

$$S_n = S_o + \sum_{i=1}^{\infty} X_i$$

is called a simple random walk when $S_o = 0$. The random walk is said to be simpe if

$$X_i = \pm 1, \text{ with } \Pr[X_i = 1] = p \quad \text{and } \Pr[X_i = -1] = 1 - p = q$$

Consider a coin tossing experiment consisting of an infinite series of coin tosses. Each coin toss lands heads up with probability p and causes a player to win one dollar. With probability $1 - p$ the player loses a dollar with the provision that the net worth of the player cannot drop below zero. We can think of the game as being played by a "benign adversary" who does not debit the player one dollar when tails lands up and the player has a state of zero.

If we let

$$X_n, n = 0, 1, 2 \ldots$$

be the total amount of money that the player has at the n^{th} game, where we set

$$X_o = 0$$

then X_n is a Markov chain. The state space of the chain is given by

$$S = \{0, 1, 2 \ldots\}$$

and the transition matrix is given by

$$\mathbf{P} = \begin{bmatrix} 1-p & p & 0 & 0 & 0 & \cdots \\ 1-p & 0 & p & 0 & 0 & \cdots \\ 0 & 1-p & 0 & p & 0 & \cdots \\ 0 & 0 & 1-p & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

This matrix is tridiagonal and although it is infinite, it can be solved with ease.

$$\left(\pi_k = \sum_{k=0}^{\infty} \pi_j p_{jk}, \quad j = 1,2,3,\ldots\right)$$

$$\begin{aligned} \pi_o &= (1-p)\pi_o + (1-p)\pi_1 \\ \pi_o[1-(1-p)] &= (1-p)\pi_1 \\ \pi_o p &= (1-p)\pi_1 \\ \pi_o &= \frac{(1-p)}{p}\pi_1 \end{aligned} \tag{25}$$

$$\pi_i = p\pi_{i-1} + (1-p)\pi_{i+1}, \quad i = 1,2,3,\ldots \tag{26}$$

A simpler method for solving these equations involves expressing the state probabilities in terms of π_o and determining its value by using the fact that the sum of the probabilities adds up to one, Hence from (4.1) above

$$\pi_1 = \frac{p}{(1-p)}\pi_o \tag{27}$$

and (4.2) can be expressed as follows

$$\pi_{i+1} = \frac{\pi_i}{1-p} - \frac{p}{(1-p)}\pi_{i-1} \tag{28}$$

setting $i = 1$ on equation (4.4) and using (4.3)

$$\begin{aligned} \pi_2 &= \frac{\pi_1}{1-p} - \frac{p}{(1-p)}\pi_o \\ &= \frac{p}{(1-p)^2}\pi_o - \frac{p}{(1-p)}\pi_o \\ &= \frac{p^2}{(1-p)^2}\pi_o \end{aligned} \tag{29}$$

Comparing (4.3) and (4.5) suggests that

$$\pi_i = \alpha^i \pi_o, \quad i = 1,2,3\ldots \tag{30}$$

where

$$\alpha = \frac{p}{(1-p)}$$

To check that this is correct, we substitute (4.6) into (4.4)

$$\alpha^{i+1}\pi_{i+1} = \frac{\alpha^i \pi_i}{1-p} - \frac{p}{(1-p)}\alpha^{i-1}\pi_{i-1}$$

Cancelling common terms shows that this is satisfied if

$$\alpha^2(1-p) - \alpha + p = 0$$

$\Longrightarrow$

$$\begin{aligned} \alpha &= \frac{1 \pm \sqrt{1-4p(1-p)}}{2(1-p)} \\ &= \frac{1 \pm (1-2p)}{2(1-p)} \end{aligned}$$

the positiove radical yielding

$$\alpha = 1$$

which is eliminated since there is no way the probabilities can sum to 1 with this solution. Taking the negative radical implies that

$$\alpha = \frac{p}{1-p}$$

as hypothesized, summing (4.6) yields

$$\begin{aligned} 1 &= \pi_o \sum_{i=0}^{\infty} \alpha^i \\ &= \frac{\pi_o}{1-\alpha} \end{aligned}$$

and hence the solution is given by

$$\pi_i = (1-\alpha)\alpha^i, \qquad i = 1, 2, 3 \ldots$$

The stationary probabilities are thus geometric with parameter $(1-\alpha)$.

4.2 A 3×3 Transition Probability Matrix

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \end{array} \begin{array}{c} \begin{array}{ccc} E_0 & E_1 & E_2 \end{array} \\ \left[\begin{array}{ccc} 1-\alpha & \alpha & 0 \\ 1-\alpha & 0 & \alpha \\ 0 & 1-\alpha & \alpha \end{array} \right] \end{array}$$

4.2.1 Clasiification of the states

Every state can be reached from every other state hence the Markov Chain is irreducible. Thus all the states are of the same type.

4.2.2 Classification of the states

Consider E_o

$$\begin{aligned}
f_{00}^{(1)} &= \Pr[E_0 \to E_0] = 1-\alpha \\
f_{00}^{(2)} &= \Pr[E_0 \to E_1 \to E_0] = \alpha(1-\alpha) \\
f_{00}^{(3)} &= 0 \\
f_{00}^{(4)} &= \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_0] = \alpha^2(1-\alpha)^2 \\
f_{00}^{(5)} &= \Pr[E_0 \to E_1 \to E2 \to E_2 \to E_1 \to E_0] = \alpha^3(1-\alpha)^2
\end{aligned}$$

$$\begin{aligned}
f_{00}^{(6)} &= \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_2 \to E_1 \to E_0] \\
&= \alpha^4(1-\alpha)^2 + \alpha^3(1-\alpha)^3
\end{aligned}$$

$$\begin{aligned}
f_{00}^{(7)} &= \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_2 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_2 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_1 \to E_2 \to E_1 \to E_0] \\
&= \alpha^5(1-\alpha)^2 + 2\alpha^4(1-\alpha)^3
\end{aligned}$$

$$
\begin{aligned}
f_o &= \sum_{n=1}^{\infty} f_{00}^{(n)} \\
&= (1-\alpha)+\alpha(1-\alpha)+\alpha^2(1-\alpha)^2+\alpha^3(1-\alpha)^2+\alpha^4(1-\alpha)^2+\alpha^3(1-\alpha)^3 \\
&\quad +\alpha^5(1-\alpha)^2+2\alpha^4(1-\alpha)^3+\alpha^6(1-\alpha)^2+3\alpha^5(1-\alpha)^3+\alpha^7(1-\alpha)^2 \\
&\quad +4\alpha^6(1-\alpha)^3\cdots \\
&= \{(1-\alpha)+\alpha(1-\alpha)\}+\left\{\begin{array}{c}\alpha^2(1-\alpha)^2+\alpha^3(1-\alpha)^2+\alpha^4(1-\alpha)^2+\alpha^5(1-\alpha)^2 \\ +\alpha^6(1-\alpha)^2+\alpha^7(1-\alpha)^2\cdots\end{array}\right\} \\
&\quad +\left\{\alpha^3(1-\alpha)^3+2\alpha^4(1-\alpha)^3+3\alpha^5(1-\alpha)^3+4\alpha^6(1-\alpha)^3\cdots\right\} \\
&= \{(1-\alpha)\}+\alpha^2(1-\alpha)^2\left\{1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\cdots\right\} \\
&\quad +\alpha^3(1-\alpha)^3\left\{1+2\alpha+3\alpha^2+4\alpha^3+\cdots\right\} \\
&= (1-\alpha)+\alpha^2(1-\alpha)^2\frac{1}{1-\alpha}+\alpha^3(1-\alpha)^3\frac{1}{(1-\alpha)^2} \\
&= (1-\alpha)\left[1+\alpha^2(1-\alpha)\frac{1}{1-\alpha}+\alpha^3(1-\alpha)^2\frac{1}{(1-\alpha)^2}\right] \\
&= (1-\alpha)\left[1+\alpha^2+\alpha^3\right] \\
&= (1-\alpha)\frac{(1-\alpha)^4}{(1-\alpha)} \\
&= (1-\alpha)^4
\end{aligned}
$$

hence

$$f_o < 1$$

hence E_o is transient

4.2.3 Asymptotic behavior

$$\lim_{n\to\infty}\mathbf{P}^n = \mathbf{P}'\pi$$

$$
\begin{bmatrix} \pi_o \\ \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} 1-\alpha & 1-\alpha & 0 \\ \alpha & 0 & 1-\alpha \\ 0 & \alpha & \alpha \end{bmatrix} \begin{bmatrix} \pi_o \\ \pi_1 \\ \pi_2 \end{bmatrix}
$$

$$
\begin{aligned}
\pi_o &= (1-\alpha)\pi_o+(1-\alpha)\pi_1 && \text{(i)} \\
\pi_1 &= \alpha\pi_o+(1-\alpha)\pi_2 && \text{(ii)} \\
\pi_2 &= \alpha\pi_1+\alpha\pi_2 && \text{(iii)}
\end{aligned}
$$

From (i)

$$\begin{aligned}\alpha\pi_o &= (1-\alpha)\pi_1 \\ \pi_1 &= \frac{\alpha}{(1-\alpha)}\pi_o\end{aligned}$$

from (ii)

$$\begin{aligned}\frac{\alpha}{(1-\alpha)}\pi_o &= \alpha\pi_o + (1-\alpha)\pi_2 \\ \frac{\alpha-\alpha+\alpha^2}{(1-\alpha)}\pi_o &= (1-\alpha)\pi_2 \\ \pi_2 &= \left(\frac{\alpha}{1-\alpha}\right)^2\pi_o\end{aligned}$$

Deriving an expression in terms of α only for the value π_o

$$\sum_{k=0}^{\infty}\pi_k = 1$$

$$\left[1+\frac{\alpha}{(1-\alpha)}+\left(\frac{\alpha}{1-\alpha}\right)^2\right]\pi_o = 1$$

let

$$x = \frac{\alpha}{1-\alpha}$$

from

$$1+x+x^2+\cdots+x^{n-1} = \frac{1-x^n}{1-x}$$

hence

$$\left[\frac{1-\left(\frac{\alpha}{1-\alpha}\right)^3}{1-\frac{\alpha}{1-\alpha}}\right]\pi_o = 1$$

$\Longrightarrow$

$$\pi_o = \frac{(1-\alpha)(1-2\alpha)}{(1-\alpha)^3-\alpha}$$

$\Longrightarrow$

$$\begin{aligned}\pi_1 &= \frac{\alpha}{(1-\alpha)}\pi_o \\ &= \frac{\alpha}{(1-\alpha)}\times\frac{(1-\alpha)(1-2\alpha)}{(1-\alpha)^3-\alpha} \\ &= \frac{\alpha(1-2\alpha)}{(1-\alpha)^3-\alpha}\end{aligned}$$

and

$$\begin{aligned}
\pi_2 &= \frac{\alpha^2}{(1-\alpha)^2}\pi_o \\
&= \frac{\alpha^2}{(1-\alpha)^2} \times \frac{(1-\alpha)(1-2\alpha)}{(1-\alpha)^3-\alpha} \\
&= \frac{\alpha^2(1-2\alpha)}{(1-\alpha)\left[(1-\alpha)^3-\alpha\right]} \\
&= \frac{\alpha^2(1-2\alpha)}{(1-\alpha)^4-\alpha(1-\alpha)}
\end{aligned}$$

4.2.4 The n^{th} power $\mathbf{P}^n$

$$\mathbf{P} = \begin{bmatrix} 1-\alpha & \alpha & 0 \\ 1-\alpha & 0 & \alpha \\ 0 & 1-\alpha & \alpha \end{bmatrix}$$

Eigen values are obtained by solving

$$|\mathbf{P}-\lambda\mathbf{I}| = \mathbf{0}$$

$$\begin{aligned}
|\mathbf{P}-\lambda\mathbf{I}| &= \begin{vmatrix} 1-\alpha-\lambda & \alpha & 0 \\ 1-\alpha & -\lambda & \alpha \\ 0 & 1-\alpha & \alpha-\lambda \end{vmatrix} \\
&= (1-\alpha-\lambda)\begin{vmatrix} -\lambda & \alpha \\ 1-\alpha & \alpha-\lambda \end{vmatrix} - \alpha\begin{vmatrix} 1-\alpha & \alpha \\ 0 & \alpha-\lambda \end{vmatrix} \\
&= (1-\alpha-\lambda)\left[-\lambda(\alpha-\lambda)-\alpha(1-\alpha)\right]-\alpha\left[(1-\alpha)(\alpha-\lambda)\right] \\
&= (1-\alpha-\lambda)\left[-\lambda\alpha+\lambda^2-\alpha+\alpha^2\right]-\alpha\left[\alpha-\lambda-\alpha^2+\alpha\lambda\right] \\
&= -\lambda\alpha+\lambda^2-\alpha+\alpha^2+\lambda\alpha^2-\lambda^2\alpha+\alpha^2-\alpha^3+\lambda^2\alpha-\lambda^3 \\
&\quad +\lambda\alpha-\lambda-\alpha^2+\alpha\lambda+\alpha^3-\lambda\alpha^2 \\
&= \lambda^3-\lambda\alpha+\lambda\alpha^2-\lambda^2+\alpha-\alpha^2
\end{aligned}$$

solving

$$\lambda^3-\lambda\alpha+\lambda\alpha^2-\lambda^2+\alpha-\alpha^2 = 0$$

$$\lambda\left(\lambda^2-\alpha+\alpha^2\right)-\left(\lambda^2-\alpha+\alpha^2\right) = 0$$

$$(\lambda-1)\left(\lambda^2-\alpha+\alpha^2\right) = 0$$

hence

$$\lambda = 1 \text{ or } \lambda = \pm\sqrt{\alpha - \alpha^2}$$

and the Eigen values are $\lambda_1 = 1,\ \ \lambda_2 = +\sqrt{\alpha-\alpha^2},$ and $\lambda_3 = -\sqrt{\alpha^2-\alpha}$. When $\lambda = 1$

$$\mathbf{Px} = \lambda\mathbf{x}$$

$$\begin{bmatrix} 1-\alpha & \alpha & 0 \\ 1-\alpha & 0 & \alpha \\ 0 & 1-\alpha & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned} (1-\alpha)x_1 + \alpha x_2 &= x_1 && \text{(i)} \\ (1-\alpha)x_1 + \alpha x_3 &= x_2 && \text{(ii)} \\ (1-\alpha)x_2 + \alpha x_3 &= x_3 && \text{(iii)} \end{aligned}$$

from (i),

$$\begin{aligned} x_1 - \alpha x_1 + \alpha x_2 &= x_1 \\ \alpha x_1 &= \alpha x_2 \\ x_1 &= x_2 \end{aligned}$$

from (ii)

$$\begin{aligned} (1-\alpha)x_1 + \alpha x_3 &= x_2 \\ (1-\alpha)x_2 + \alpha x_3 &= x_2 \\ x_2 - \alpha x_2 + \alpha x_3 &= x_2 \\ \alpha x_2 &= \alpha x_3 \\ x_2 &= x_3 \end{aligned}$$

hence $x_1 = x_2 = x_3$. Suppose $x_1 = x_2 = x_3 = 1$ then

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

When $\lambda = +\sqrt{\alpha^2 - \alpha}$

$$\begin{bmatrix} 1-\alpha & \alpha & 0 \\ 1-\alpha & 0 & \alpha \\ 0 & 1-\alpha & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = +\sqrt{\alpha-\alpha^2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned}(1-\alpha)x_1+\alpha x_2 &= +\sqrt{\alpha-\alpha^2}x_1 \qquad \text{(i)}\\(1-\alpha)x_1+\alpha x_3 &= +\sqrt{\alpha-\alpha^2}x_2 \qquad \text{(ii)}\\(1-\alpha)x_2+\alpha x_3 &= +\sqrt{\alpha-\alpha^2}x_3 \qquad \text{(iii)}\end{aligned}$$

from (iii)

$$\begin{aligned}x_2 &= \left(\frac{-\alpha+\sqrt{\alpha-\alpha^2}}{(1-\alpha)}\right)x_3\\&= \left(\frac{-\alpha+\sqrt{\alpha-\alpha^2}}{-(\alpha-1)}\right)x_3\\&= -\left(\frac{-\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\right)x_3\\&= \left(\frac{\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\right)x_3\end{aligned}$$

from (ii)

$$\begin{aligned}(1-\alpha)x_1+\alpha x_3 &= +\sqrt{\alpha^2-\alpha}x_2\\(1-\alpha)x_1 &= -\alpha x_3+\sqrt{\alpha^2-\alpha}x_2\end{aligned}$$

$$\begin{aligned}(1-\alpha)x_1 &= -\alpha x_3+\sqrt{\alpha^2-\alpha}\left(\frac{\alpha-\sqrt{\alpha(\alpha-1)}}{(\alpha-1)}\right)x_3\\&= \left[-\alpha+\sqrt{\alpha^2-\alpha}\left(\frac{\alpha-\sqrt{\alpha(\alpha-1)}}{(\alpha-1)}\right)\right]x_3\end{aligned}$$

$$\begin{aligned}x_1 &= \frac{1}{(1-\alpha)}\left[-\alpha+\left(\frac{\alpha\sqrt{\alpha^2-\alpha}-\left(\alpha^2-\alpha\right)}{(\alpha-1)}\right)\right]x_3\\&= \frac{1}{(\alpha-1)}\left[\left(\frac{-\alpha(\alpha-1)+\alpha\sqrt{\alpha^2-\alpha}-\left(\alpha^2-\alpha\right)}{(\alpha-1)}\right)\right]x_3\\&= \left(\frac{\alpha\sqrt{\alpha^2-\alpha}}{(\alpha-1)^2}\right)x_3\end{aligned}$$

but x_3 is a free variable. Suppose that $x_3=t$, where t is a real number, then

$$v_2=\begin{bmatrix}\frac{-\alpha\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)^2}\\\frac{-\alpha+\alpha\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)}\\1\end{bmatrix}t$$

assuming that $t=1$,

$$v_2 = \begin{bmatrix} \frac{-\alpha\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)^2} \\ \frac{-\alpha+\alpha\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)} \\ 1 \end{bmatrix}$$

For $\lambda = \sqrt{\alpha-\alpha^2}$

$$\begin{bmatrix} 1-\alpha & \alpha & 0 \\ 1-\alpha & 0 & \alpha \\ 0 & 1-\alpha & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\sqrt{\alpha-\alpha^2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned} (1-\alpha)x_1+\alpha x_2 &= -\sqrt{\alpha-\alpha^2}x_1 && \text{(i)} \\ (1-\alpha)x_1+\alpha x_3 &= -\sqrt{\alpha-\alpha^2}x_2 && \text{(ii)} \\ (1-\alpha)x_2+\alpha x_3 &= -\sqrt{\alpha-\alpha^2}x_3 && \text{(iii)} \end{aligned}$$

from (iii)

$$\begin{aligned} (1-\alpha)x_2 &= -\alpha x_3 - \sqrt{\alpha-\alpha^2}x_3 \\ &= \left(-\alpha-\sqrt{\alpha-\alpha^2}\right)x_3 \end{aligned}$$

$$\begin{aligned} x_2 &= \left(\frac{-\alpha-\sqrt{\alpha-\alpha^2}}{(1-\alpha)}\right)x_3 \\ &= \left(-\frac{-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\right)x_3 \end{aligned}$$

from (ii)

$$\begin{aligned} (1-\alpha)x_1 &= -\alpha x_3 - \sqrt{\alpha-\alpha^2}x_2 \\ &= -\alpha x_3 - \sqrt{\alpha-\alpha^2}\left(-\frac{-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\right)x_3 \\ &= \left[-\alpha-\sqrt{\alpha-\alpha^2}\left(-\frac{-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\right)\right]x_3 \\ &= \left(-\alpha+\frac{\alpha\sqrt{\alpha-\alpha^2}+\alpha-\alpha^2}{1-\alpha}\right)x_3 \\ &= \left(\frac{-\alpha(1-\alpha)+\alpha\sqrt{\alpha-\alpha^2}+\alpha-\alpha^2}{1-\alpha}\right)x_3 \\ &= \left(\frac{\alpha\sqrt{\alpha-\alpha^2}}{-(-1+\alpha)}\right)x_3 \end{aligned}$$

$$x_1 = -\frac{\alpha\sqrt{\alpha-\alpha^2}}{(-1+\alpha)^2}$$

but x_3 is a free variable. Suppose that $x_3 = 1$, then

$$v_3 = \begin{bmatrix} \frac{\alpha\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)^2} \\ -\frac{-\alpha-\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)} \\ 1 \end{bmatrix}$$

hence

$$\begin{aligned} \mathbf{V} &= \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\frac{\alpha\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)^2} & \frac{\alpha\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)^2} \\ 1 & \frac{-\alpha-\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)} & -\frac{-\alpha-\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)} \\ 1 & 1 & 1 \end{bmatrix} \end{aligned}$$

using the spectral decomposition matrix

$$\mathbf{P}^n = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$$

$$\begin{aligned} \mathbf{P}^n &= \begin{bmatrix} 1 & -\frac{\alpha\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)^2} & \frac{\alpha\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)^2} \\ 1 & \frac{-\alpha-\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)} & -\frac{-\alpha-\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)} \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\sqrt{\alpha-\alpha^2}\right)^n & 0 \\ 0 & 0 & \left(-\sqrt{\alpha-\alpha^2}\right)^n \end{bmatrix} \\ &\times \begin{bmatrix} 1 & -\frac{\alpha\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)^2} & \frac{\alpha\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)^2} \\ 1 & \frac{-\alpha-\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)} & -\frac{-\alpha-\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)} \\ 1 & 1 & 1 \end{bmatrix}^{-1} \end{aligned}$$

$$
\begin{aligned}
|\mathbf{V}| &= 1\begin{vmatrix} \frac{-\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)} & \frac{\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)} \\ 1 & 1 \end{vmatrix} + \frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\begin{vmatrix} 1 & \frac{\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)} \\ 1 & 1 \end{vmatrix} \\
&\quad + \frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\begin{vmatrix} 1 & \frac{\alpha-\alpha\sqrt{\alpha-\alpha^2}}{(-1+\alpha)} \\ 1 & 1 \end{vmatrix} \\
&= \left[\frac{\alpha-\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)} - \frac{\alpha+\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\right] + \left[\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\left(\frac{\alpha-1-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\right)\right] \\
&\quad + \left[\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\left(\frac{\alpha-1-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\right)\right] \\
&= \frac{-2\sqrt{\alpha-\alpha^2}}{\alpha-1} + \frac{-\alpha\sqrt{\alpha-\alpha^2}-\alpha(\alpha-\alpha^2)}{(\alpha-1)^3} + \frac{\alpha\sqrt{\alpha-\alpha^2}-\alpha(\alpha-\alpha^2)}{(\alpha-1)^3} \\
&= \frac{-2\sqrt{\alpha-\alpha^2}}{\alpha-1} + \frac{-2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3}
\end{aligned}
$$

cofactor matrix.

Let

$$
\begin{aligned}
a_{11} &= \begin{vmatrix} \frac{\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)} & \frac{\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)} \\ 1 & 1 \end{vmatrix} \\
a_{12} &= \begin{vmatrix} 1 & \frac{\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)} \\ 1 & 1 \end{vmatrix} \\
a_{13} &= \begin{vmatrix} 1 & \frac{\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)} \\ 1 & 1 \end{vmatrix} \\
a_{21} &= \begin{vmatrix} \frac{-\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & \frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\ 1 & 1 \end{vmatrix} \\
a_{22} &= \begin{vmatrix} 1 & \frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\ 1 & 1 \end{vmatrix} \\
a_{23} &= \begin{vmatrix} 1 & \frac{-\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\ 1 & 1 \end{vmatrix} \\
a_{31} &= \begin{vmatrix} \frac{-\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & \frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\ \frac{\alpha-\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)} & \frac{\alpha+\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)} \end{vmatrix} \\
a_{32} &= \begin{vmatrix} 1 & \frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\ 1 & \frac{\alpha+\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \end{vmatrix} \\
a_{33} &= \begin{vmatrix} 1 & \frac{-\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\ 1 & \frac{\alpha-\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
a_{11} &= \frac{\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)}-\frac{\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)} \\
&= \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} \\
a_{12} &= 1-\frac{\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)} \\
&= \frac{-\sqrt{\alpha-\alpha^2}-1}{(\alpha-1)} \\
a_{13} &= 1-\frac{\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)} \\
&= \frac{\sqrt{\alpha-\alpha^2}-1}{(\alpha-1)} \\
a_{21} &= \frac{-\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\
&= \frac{-2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\
a_{22} &= 1-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\
a_{23} &= 1+\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\
a_{31} &= \left(\frac{-\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\frac{\alpha+\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\right)-\left(\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\frac{\alpha-\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\right) \\
&= \frac{1}{(\alpha-1)^3}\left[\left(-\alpha\sqrt{\alpha-\alpha^2}\right)\left(\alpha+\alpha\sqrt{\alpha-\alpha^2}\right)\right]-\left[\left(\alpha\sqrt{\alpha-\alpha^2}\right)\left(\alpha-\alpha\sqrt{\alpha-\alpha^2}\right)\right] \\
&= \frac{1}{(\alpha-1)^3}\left[\alpha^2\sqrt{\alpha-\alpha^2}+\alpha\left(\alpha-\alpha^2\right)+\alpha^2\sqrt{\alpha-\alpha^2}-\alpha\left(\alpha-\alpha^2\right)\right] \\
&= \frac{1}{(\alpha-1)^3}\left[2\alpha^2\sqrt{\alpha-\alpha^2}\right] \\
&= \frac{2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3} \\
a_{32} &= \frac{\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)}-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\
&= \frac{(\alpha-1)\left(\alpha+\sqrt{\alpha-\alpha^2}\right)-\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\
&= \frac{\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\
a_{33} &= \frac{\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)}+\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\
&= \frac{(\alpha-1)\left(\alpha-\sqrt{\alpha-\alpha^2}\right)+\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\
&= \frac{\alpha^2-\alpha+\sqrt{\alpha-\alpha^2}-\alpha\sqrt{\alpha-\alpha^2}+\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}
\end{aligned}
$$

Cofactor matrix of $\mathbf{V}$

$$\begin{bmatrix} \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} & \frac{-\sqrt{\alpha-\alpha^2}-1}{(\alpha-1)} & \frac{\sqrt{\alpha-\alpha^2}-1}{(\alpha-1)} \\ \frac{-2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & 1-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & 1+\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\ \frac{2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3} & \frac{\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & \frac{\alpha^2-\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \end{bmatrix}$$

adjoint of $\mathbf{V}$ is

$$\begin{bmatrix} \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} & \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & \frac{2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3} \\ \frac{\sqrt{\alpha-\alpha^2}+1}{(\alpha-1)} & 1-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & -\frac{\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\ \frac{\sqrt{\alpha-\alpha^2}-1}{(\alpha-1)} & -1-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & \frac{\alpha^2-\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \end{bmatrix}$$

hence from

$$\mathbf{V}^{-1} = \frac{1}{\det \mathbf{V}} \text{adjoint } \mathbf{V}$$

$$\mathbf{V}^{-1} = \frac{1}{\frac{-2\sqrt{\alpha-\alpha^2}}{\alpha-1}+\frac{-2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3}} \begin{bmatrix} \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} & \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & \frac{2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3} \\ \frac{\sqrt{\alpha-\alpha^2}+1}{(\alpha-1)} & 1-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & -\frac{\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\ \frac{\sqrt{\alpha-\alpha^2}-1}{(\alpha-1)} & -1-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & \frac{\alpha^2-\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \end{bmatrix}$$

$$\begin{aligned} \mathbf{P}^n \quad = \quad & \frac{1}{\frac{-2\sqrt{\alpha-\alpha^2}}{\alpha-1}+\frac{-2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3}} \begin{bmatrix} 1 & -\frac{\alpha\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)^2} & \frac{\alpha\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)^2} \\ 1 & \frac{-\alpha-\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)} & -\frac{-\alpha-\sqrt{-(-1+\alpha)\alpha}}{(-1+\alpha)} \\ 1 & 1 & 1 \end{bmatrix} \\ & \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\sqrt{\alpha-\alpha^2}\right)^n & 0 \\ 0 & 0 & \left(-\sqrt{\alpha-\alpha^2}\right)^n \end{bmatrix} \\ & \times \begin{bmatrix} \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} & \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & \frac{2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3} \\ \frac{\sqrt{\alpha-\alpha^2}+1}{(\alpha-1)} & 1-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & -\frac{\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\ \frac{\sqrt{\alpha-\alpha^2}-1}{(\alpha-1)} & -1-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & \frac{\alpha^2-\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \end{bmatrix} \end{aligned}$$

let

$$\begin{aligned} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \quad = \quad & \begin{bmatrix} 1 & -\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & \frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\ 1 & \frac{-\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)} & -\frac{-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)} \\ 1 & 1 & 1 \end{bmatrix} \\ & \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\sqrt{\alpha-\alpha^2}\right)^n & 0 \\ 0 & 0 & \left(-\sqrt{\alpha-\alpha^2}\right)^n \end{bmatrix} \end{aligned}$$

$$b_{11} = 1$$

$$\begin{aligned}
b_{12} &= -\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\left(\sqrt{\alpha-\alpha^2}\right)^n \\
&= \frac{-\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2} \\
b_{13} &= \frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\left(-\sqrt{\alpha-\alpha^2}\right)^n \\
&= \frac{-\alpha i\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\left(\alpha-\alpha^2\right)^{\frac{n}{2}} \\
&= \frac{-\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2} \\
b_{21} &= 1 \\
b_{22} &= \frac{-\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\left(\alpha-\alpha^2\right)^{\frac{n}{2}} \\
&= \frac{-\alpha\left(\alpha-\alpha^2\right)^{\frac{n}{2}}+\left(\alpha-\alpha^2\right)^{\frac{n}{2}}\sqrt{\alpha-\alpha^2}}{(\alpha-1)} \\
&= \frac{-\alpha\left(\alpha-\alpha^2\right)^{\frac{n}{2}}-i\left(\alpha-\alpha^2\right)^{\frac{n}{2}}\sqrt{\alpha-\alpha^2}}{(\alpha-1)} \\
&= \frac{\alpha\left(\alpha-\alpha^2\right)^{\frac{n}{2}}-i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)} \\
b_{23} &= -\left(\alpha-\alpha^2\right)^{\frac{n}{2}}\frac{\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)} \\
&= \frac{-\alpha\left(\alpha-\alpha^2\right)^{\frac{n}{2}}-i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)} \\
b_{31} &= 1 \\
b_{32} &= \left(\alpha-\alpha^2\right)^{\frac{n}{2}} \\
&= i\sqrt{\alpha^2-\alpha} \\
b_{33} &= -\left(\alpha^2-\alpha\right)^{\frac{n}{2}}
\end{aligned}$$

$$\mathbf{P}^n = \frac{1}{\frac{-2\sqrt{\alpha-\alpha^2}}{\alpha-1}+\frac{-2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3}}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33}\end{bmatrix}$$
$$\times\begin{bmatrix} \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} & \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & \frac{2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3} \\ \frac{\sqrt{\alpha-\alpha^2}+1}{(\alpha-1)} & 1-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & -\frac{\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\ \frac{\sqrt{\alpha-\alpha^2}-1}{(\alpha-1)} & -1-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & \frac{\alpha^2-\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\end{bmatrix}$$

Let

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33}\end{bmatrix} = \begin{bmatrix} 1 & \frac{-\alpha i(\alpha^2-\alpha)^{\frac{n+1}{2}}}{(\alpha-1)^2} & \frac{-\alpha i(\alpha^2-\alpha)^{\frac{n+1}{2}}}{(\alpha-1)^2} \\ 1 & \frac{\alpha(\alpha^2-\alpha)^{\frac{n}{2}}-i(\alpha^2-\alpha)^{\frac{n+1}{2}}}{(\alpha-1)} & \frac{-\alpha(\alpha^2-\alpha)^{\frac{n}{2}}-i(\alpha^2-\alpha)^{\frac{n+1}{2}}}{(\alpha-1)} \\ 1 & \left(\alpha^2-\alpha\right)^{\frac{n}{2}} & -\left(\alpha^2-\alpha\right)^{\frac{n}{2}}\end{bmatrix}$$
$$\times\begin{bmatrix} \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} & \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & \frac{2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3} \\ \frac{\sqrt{\alpha-\alpha^2}+1}{(\alpha-1)} & 1-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & -\frac{\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} \\ \frac{\sqrt{\alpha-\alpha^2}-1}{(\alpha-1)} & -1-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} & \frac{\alpha^2-\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\end{bmatrix}$$

$$\begin{aligned} c_{11} &= \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} - \left[\frac{\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2}\frac{\sqrt{\alpha-\alpha^2}+1}{(\alpha-1)}\right] - \left[\frac{-\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2}\frac{\sqrt{\alpha-\alpha^2}-1}{(\alpha-1)}\right] \\ &= \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} - \frac{\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2}\left[i\left(\alpha^2-\alpha\right)+1-i\left(\alpha^2-\alpha\right)+1\right] \\ &= \frac{-2i\sqrt{\alpha-\alpha^2}}{(\alpha-1)} - \frac{2\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2} \\ &= \frac{-2i\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\left[1+\frac{\alpha\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^2}\right] \end{aligned}$$

$$\begin{aligned}
c_{12} &= \frac{-2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)} - \frac{\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2}\left(1-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\right) - \frac{\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2}\left(-1-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\right) \\
&= \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)} - \frac{\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2}\left[\frac{(\alpha-1)^2-\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\right] + \frac{\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2}\left(1+\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\right) \\
&= \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)} - \frac{\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2}\left[\alpha^2-2\alpha+1-\alpha\sqrt{\alpha-\alpha^2}\right] \\
&\quad + \frac{\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2}\left(\frac{\alpha^2-2\alpha+1+\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\right) \\
&= \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)} - \frac{\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^4}\left[\alpha^2-2\alpha+1-\alpha\sqrt{\alpha-\alpha^2}-\alpha^2+2\alpha-1-\alpha\sqrt{\alpha-\alpha^2}\right] \\
&= \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)} - \frac{\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^4}\left[-2\alpha\sqrt{\alpha-\alpha^2}\right] \\
&= \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)} + \frac{\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^4}\left[2i\alpha\sqrt{\alpha^2-\alpha}\right] \\
&= \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)} + \frac{2\alpha^2 i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}\left(\alpha^2-\alpha\right)}{(\alpha-1)^4} \\
&= \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)} + \left[i-\frac{\alpha\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2}\right]
\end{aligned}$$

$$
\begin{aligned}
c_{13} &= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3} - \frac{\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2}\left(\frac{\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\right) \\
&\quad - \frac{\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2}\left(\frac{\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\right) \\
&= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3} - \frac{\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^4}\left(\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}-\alpha^2+\alpha-\sqrt{\alpha-\alpha^2}\right) \\
&= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3} - \frac{\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^4}\left(-2\sqrt{\alpha-\alpha^2}\right) \\
&= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3} + \frac{2\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}\sqrt{\alpha-\alpha^2}}{(\alpha-1)^4} \\
&= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3}\left[\alpha i + \frac{\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)}\right]
\end{aligned}
$$

$$
\begin{aligned}
c_{21} &= \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} + \left[\frac{\alpha\left(\alpha^2-\alpha\right)^{\frac{n}{2}} - i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)}\right]\left[\frac{\sqrt{\alpha-\alpha^2}+1}{(\alpha-1)}\right] \\
&\quad - \left[\frac{\alpha\left(\alpha^2-\alpha\right)^{\frac{n}{2}} - i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)}\right]\left[\frac{\sqrt{\alpha-\alpha^2}-1}{(\alpha-1)}\right] \\
&= \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} + \frac{1}{(\alpha-1)^2}\left\{\begin{array}{c} \left[\alpha\left(\alpha^2-\alpha\right)^{\frac{n}{2}} - i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}\right]\left[\sqrt{\alpha-\alpha^2}+1\right] \\ -\left[\alpha\left(\alpha^2-\alpha\right)^{\frac{n}{2}} - i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}\right]\left[\sqrt{\alpha-\alpha^2}-1\right] \end{array}\right\} \\
&= \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} + \frac{\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^2}\left\{\begin{array}{c} \left[\alpha - i\left(\alpha^2-\alpha\right)^{\frac{1}{2}}\right]\left[\sqrt{\alpha-\alpha^2}+1\right] \\ -\left[\alpha + i\left(\alpha^2-\alpha\right)^{\frac{1}{2}}\right]\left[\sqrt{\alpha-\alpha^2}-1\right] \end{array}\right\} \\
&= \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} + \frac{\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^2}\left\{\begin{array}{c} \alpha\sqrt{\alpha-\alpha^2}+\alpha-i\left(\alpha^2-\alpha\right)+\alpha-i\left(\alpha^2-\alpha\right)^{\frac{1}{2}} \\ -\alpha\sqrt{\alpha-\alpha^2}+\alpha-i\left(\alpha^2-\alpha\right)+i\left(\alpha^2-\alpha\right)^{\frac{1}{2}} \end{array}\right\} \\
&= \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} + \frac{\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^2}\left\{2\alpha - 2i\sqrt{\alpha^2-\alpha}\, i\left(\alpha^2-\alpha\right)^{\frac{1}{2}}\right\} \\
&= \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} + \frac{\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^2}\left[2\alpha - 2i^2\left(\alpha^2-\alpha\right)\right] \\
&= \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} + \frac{\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^2}\left[2\alpha + 2\alpha^2 - 2\alpha\right] \\
&= \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} + \frac{2\alpha^2\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^2} \\
&= \frac{1}{(\alpha-1)}\left[\frac{2\alpha^2\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)} - 2\sqrt{\alpha-\alpha^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
c_{22} &= \frac{-2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} + \left[\frac{\alpha\left(\alpha^2-\alpha\right)-i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)}\right] \\
&\quad \times \left[1-\frac{\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}-\frac{\alpha\left(\alpha^2-\alpha\right)^{\frac{n}{2}}-i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)}\right]\left[1+\frac{\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\right] \\
&= \frac{-2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} + \left[\frac{\alpha\left(\alpha-\alpha^2\right)^{\frac{n}{2}}-i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)}\right]\left[\frac{\alpha^2-2\alpha+1-\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\right] \\
&\quad + \left[\frac{\alpha\left(\alpha^2-\alpha\right)^{\frac{n}{2}}+i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)}\right]\left[\frac{\alpha^2-2\alpha+1+\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\right] \\
&= \frac{-2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} + \frac{\left(\alpha-\alpha^2\right)^{\frac{n}{2}}}{(\alpha-1)^3}\left[\left(\alpha-i\left(\alpha^2-\alpha\right)^{\frac{1}{2}}\right)\left(\alpha^2-2\alpha+1-\alpha\sqrt{\alpha-\alpha^2}\right)\right] \\
&\quad + \left[\left(\alpha+i\left(\alpha^2-\alpha\right)^{\frac{1}{2}}\right)\left(\alpha^2-2\alpha+1+\alpha\sqrt{\alpha-\alpha^2}\right)\right] \\
&= \frac{-2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} + \frac{\left(\alpha-\alpha^2\right)^{\frac{n}{2}}}{(\alpha-1)^3}\left[2\alpha^3-4\alpha^2+2\alpha+2\alpha i\sqrt{\alpha-\alpha^2}\sqrt{\alpha^2-\alpha}\right] \\
&= \frac{-2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} + \frac{\left(\alpha-\alpha^2\right)^{\frac{n}{2}}}{(\alpha-1)^3}\left[2\alpha\left(\alpha^2-2\alpha+1\right)+2\alpha i\left(\alpha^2-\alpha\right)\right] \\
&= \frac{-2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} + \frac{\left(\alpha-\alpha^2\right)^{\frac{n}{2}}}{(\alpha-1)^3}\left[2\alpha\left(\alpha^2-2\alpha+1\right)-2\alpha^3-2\alpha^2\right] \\
&= \frac{-2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} + \frac{\left(\alpha-\alpha^2\right)^{\frac{n}{2}}}{(\alpha-1)^3}\left[2\alpha-2\alpha^2\right] \\
&= \frac{2\alpha}{(\alpha-1)^2} + \left[\sqrt{\alpha-\alpha^2}-\left(\alpha^2-\alpha\right)^{\frac{n}{2}}\right]
\end{aligned}
$$

$$
\begin{aligned}
c_{23} &= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3}+\frac{\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\left[\frac{\alpha\left(\alpha^2-\alpha\right)^{\frac{n}{2}}-i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)}\right] \\
&\quad+\frac{\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\left[\frac{-\alpha\left(\alpha^2-\alpha\right)^{\frac{n}{2}}-i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)}\right] \\
&= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3}+\frac{1}{(\alpha-1)^3}\left\{\begin{array}{c}\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}\left[\alpha\left(\alpha^2-\alpha\right)^{\frac{n}{2}}-i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}\right] \\ -\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}\left[-\alpha\left(\alpha^2-\alpha\right)^{\frac{n}{2}}+i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}\right]\end{array}\right\} \\
&= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3}+\frac{1}{(\alpha-1)^3}\left\{\begin{array}{c}\left(\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}\right)\left(\alpha^2-\alpha\right)^{\frac{n}{2}}\left[\alpha-i\left(\alpha^2-\alpha\right)^{\frac{1}{2}}\right] \\ -\left(-\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}\right)\left(\alpha^2-\alpha\right)^{\frac{n}{2}}\left[-\alpha+i\left(\alpha^2-\alpha\right)^{\frac{1}{2}}\right]\end{array}\right\} \\
&= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3}+\frac{\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^3}\left\{\begin{array}{c}\left[\begin{array}{c}\alpha^3-\alpha^2\sqrt{\alpha-\alpha^2}-\alpha^2+\alpha\sqrt{\alpha-\alpha^2} \\ -\alpha\sqrt{\alpha-\alpha^2}+\left(\alpha^2-\alpha\right)\end{array}\right] \\ -\left[\begin{array}{c}\alpha^3+\alpha^2\sqrt{\alpha-\alpha^2}-\alpha^2-\alpha\sqrt{\alpha-\alpha^2} \\ +\alpha\sqrt{\alpha-\alpha^2}+\left(\alpha^2-\alpha\right)\end{array}\right]\end{array}\right\} \\
&= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3}+\frac{\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^3}\left[-2\alpha\sqrt{\alpha-\alpha^2}\right] \\
&= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3}-\frac{\left(\alpha^2-\alpha\right)^{\frac{n}{2}}\left(2\alpha\sqrt{\alpha-\alpha^2}\right)}{(\alpha-1)^3} \\
&= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3}-\frac{\left(\alpha^2-\alpha\right)^{\frac{n}{2}}2\alpha i\sqrt{\alpha^2-\alpha}}{(\alpha-1)^3} \\
&= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3}-\frac{2\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^3} \\
&= \frac{-2\alpha^2 i\sqrt{\alpha^2-\alpha}}{(\alpha-1)^3}-\frac{2\alpha i\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^3} \\
&= \frac{-2\alpha^2 i\sqrt{\alpha^2-\alpha}}{(\alpha-1)^3}\left[1+\left(\alpha^2-\alpha\right)^{\frac{n}{2}}\right]
\end{aligned}
$$

$$
\begin{aligned}
c_{31} &= \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} + \left(\alpha^2-\alpha\right)^{\frac{n}{2}}\left(\frac{\sqrt{\alpha-\alpha^2}+1}{(\alpha-1)}\right) - \left(\alpha^2-\alpha\right)^{\frac{n}{2}}\left(\frac{\sqrt{\alpha-\alpha^2}-1}{(\alpha-1)}\right) \\
&= \frac{-2\sqrt{\alpha-\alpha^2}}{(\alpha-1)} + \frac{\left(\alpha^2-\alpha\right)^{\frac{n}{2}}\sqrt{\alpha-\alpha^2}+\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)} - \frac{\left(\alpha^2-\alpha\right)^{\frac{n}{2}}\sqrt{\alpha-\alpha^2}+\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(1-\alpha)} \\
&= \frac{1}{\alpha-1}\left[-2\sqrt{\alpha-\alpha^2}+2\left(\alpha^2-\alpha\right)^{\frac{n}{2}}\right] \\
&= \frac{2}{\alpha-1}\left[\left(\alpha^2-\alpha\right)^{\frac{n}{2}}-\sqrt{\alpha-\alpha^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
c_{32} &= \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} + \left(\alpha^2-\alpha\right)^{\frac{n}{2}}\left(1-\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\right) + \left(\alpha^2-\alpha\right)^{\frac{n}{2}}\left(1+\frac{\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\right) \\
&= \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} + \left(\alpha^2-\alpha\right)^{\frac{n}{2}}\left(\frac{\alpha^2-2\alpha+1-\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\right) \\
&\quad + \left(\alpha^2-\alpha\right)^{\frac{n}{2}}\left(\frac{\alpha^2-2\alpha+1+\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\right) \\
&= \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} + \frac{\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^2}\left(\alpha^2-2\alpha+1-\alpha\sqrt{\alpha-\alpha^2}+\alpha^2-2\alpha+1+\alpha\sqrt{\alpha-\alpha^2}\right) \\
&= \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} + \frac{\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^2}\left(2\alpha^2-4\alpha+2\right) \\
&= \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} + \frac{2\left(\alpha^2-\alpha\right)^{\frac{n}{2}}\left(\alpha^2-2\alpha+1\right)}{(\alpha-1)^2} \\
&= \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} + \frac{2\left(\alpha^2-\alpha\right)^{\frac{n}{2}}(\alpha-1)^2}{(\alpha-1)^2} \\
&= \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2} + 2\left(\alpha^2-\alpha\right)^{\frac{n}{2}} \\
&= \frac{2}{(\alpha-1)^2}\left[\alpha\sqrt{\alpha-\alpha^2}+2(\alpha-1)^2\left(\alpha^2-\alpha\right)^{\frac{n}{2}}\right]
\end{aligned}
$$

$$
\begin{aligned}
c_{33} &= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3} + \left(\alpha^2-\alpha\right)^{\frac{n}{2}} \left(\frac{\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\right) - \left(\alpha^2-\alpha\right)^{\frac{n}{2}} \left(\frac{\alpha^2-\alpha+\sqrt{\alpha-\alpha^2}}{(\alpha-1)^2}\right) \\
&= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3} + \frac{\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^2} \left(\alpha^2-\alpha-\sqrt{\alpha-\alpha^2}-\alpha^2+\alpha-\sqrt{\alpha-\alpha^2}\right) \\
&= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3} + \frac{\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^2} \left(-2\sqrt{\alpha-\alpha^2}\right) \\
&= \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3} - \frac{2i\left(\alpha^2-\alpha\right)\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^2} \\
&= \frac{-2\alpha^2 i\left(\alpha^2-\alpha\right)}{(\alpha-1)^3} - \frac{2i\left(\alpha^2-\alpha\right)\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^2} \\
&= \frac{-2\alpha^3 i(\alpha-1)}{(\alpha-1)^3} - \frac{2\alpha i(\alpha-1)\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^2} \\
&= \frac{-2\alpha i}{(\alpha-1)}\left[\left(\alpha^2-\alpha\right)^{\frac{n}{2}} + \frac{\alpha^2}{\alpha-1}\right]
\end{aligned}
$$

and

$$
\mathbf{P}^n = \frac{1}{\frac{-2\sqrt{\alpha-\alpha^2}}{\alpha-1} + \frac{-2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3}} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \text{ where}
$$

$$
\begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} = \begin{bmatrix} \frac{-2i\sqrt{\alpha-\alpha^2}}{(\alpha-1)}\left[1+\frac{\alpha\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)^2}\right] \\ \frac{1}{(\alpha-1)}\left[\frac{2\alpha^2\left(\alpha^2-\alpha\right)^{\frac{n}{2}}}{(\alpha-1)} - 2\sqrt{\alpha-\alpha^2}\right] \\ \frac{2}{\alpha-1}\left[\left(\alpha^2-\alpha\right)^{\frac{n}{2}} - \sqrt{\alpha-\alpha^2}\right] \end{bmatrix}
$$

$$
\begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} = \begin{bmatrix} \frac{2\alpha\sqrt{\alpha-\alpha^2}}{(\alpha-1)} + \left[i - \frac{\alpha\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)^2}\right] \\ \frac{2\alpha}{(\alpha-1)^2} + \left[\sqrt{\alpha-\alpha^2} - \left(\alpha^2-\alpha\right)^{\frac{n}{2}}\right] \\ \frac{2}{(\alpha-1)^2}\left[\alpha\sqrt{\alpha-\alpha^2} + 2(\alpha-1)^2\left(\alpha^2-\alpha\right)^{\frac{n}{2}}\right] \end{bmatrix}
$$

$$
\begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix} = \begin{bmatrix} \frac{-2\alpha^2\sqrt{\alpha-\alpha^2}}{(\alpha-1)^3}\left[\alpha i + \frac{\left(\alpha^2-\alpha\right)^{\frac{n+1}{2}}}{(\alpha-1)}\right] \\ \frac{-2\alpha^2 i\sqrt{\alpha^2-\alpha}}{(\alpha-1)^3}\left[1+\left(\alpha^2-\alpha\right)^{\frac{n}{2}}\right] \\ \left[\left(\alpha^2-\alpha\right)^{\frac{n}{2}} + \frac{\alpha^2}{\alpha-1}\right] \end{bmatrix}
$$

4.3 A 4×4 Transition Probability Matrix

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{cccc} E_0 & E_1 & E_2 & E_3 \end{array} \\ \left[\begin{array}{cccc} 1-\alpha & \alpha & 0 & 0 \\ 1-\alpha & 0 & \alpha & 0 \\ 0 & 1-\alpha & 0 & \alpha \\ 0 & 0 & 1-\alpha & \alpha \end{array} \right] \end{array}$$

4.3.1 Classification of the Markov Chain

Every state can be reached from every other state hence the Markov Chain is irreducible. Thus all the states are of the same type.

4.3.2 Classification of the states

Consider E_o

$$\begin{aligned}
f_{oo}^{(1)} &= \Pr[E_o \to E_o] = 1-\alpha \\
f_{oo}^{(2)} &= \Pr[E_o \to E_1 \to E_o] = \alpha(1-\alpha) \\
f_{oo}^{(3)} &= \Pr[E_o \to E_1 \to E_2 \to E_o] = 0 \\
f_{oo}^{(4)} &= \Pr[E_o \to E_1 \to E_2 \to E_1 \to E_o] = \alpha^2(1-\alpha)^2 \\
f_{oo}^{(5)} &= 0 \\
f_{oo}^{(6)} &= \Pr[E_o \to E_1 \to E_2 \to E_1 \to E_2 \to E_1 \to E_o] = \alpha^3(1-\alpha)^3
\end{aligned}$$

In general

$$f_{00}^{(n)} = \left\{ \begin{array}{c} 1-\alpha, \quad n=1 \\ \alpha^{\frac{n}{2}}(1-\alpha)^{\frac{n}{2}}, \quad n \text{ is even} \\ 0, \quad \text{elsewhere} \end{array} \right\}$$

$$\begin{aligned}
f_0 &= \sum_{n=1}^{\infty} f_{00}^{(n)} \\
&= 1-\alpha+\alpha(1-\alpha)+\alpha^2(1-\alpha)^2+\alpha^2(1-\alpha)^2+\cdots < 1
\end{aligned}$$

hence E_o is transient.

4.3.3 Asymptotic behavior

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 1-\alpha & \alpha & 0 & 0 \\ 1-\alpha & 0 & \alpha & 0 \\ 0 & 1-\alpha & 0 & \alpha \\ 0 & 0 & 1-\alpha & \alpha \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}$$

solving for π_i in terms of π_0

$$\begin{aligned} \pi_0 &= (1-\alpha)\pi_0 + (1-\alpha)\pi_1 \qquad \text{(i)} \\ \pi_1 &= \alpha\pi_0 + (1-\alpha)\pi_2 \qquad \text{(ii)} \\ \pi_2 &= \alpha\pi_1 + (1-\alpha)\pi_3 \qquad \text{(iii)} \\ \pi_3 &= \alpha\pi_2 + \alpha\pi_3 \qquad \text{(iv)} \end{aligned}$$

From (i)

$$\begin{aligned} \alpha\pi_0 &= (1-\alpha)\pi_1 \\ \pi_1 &= \frac{\alpha}{(1-\alpha)}\pi_0 \end{aligned}$$

from (ii)

$$\begin{aligned} \frac{\alpha}{(1-\alpha)}\pi_0 &= \alpha\pi_0 + (1-\alpha)\pi_2 \\ \left[\frac{\alpha}{(1-\alpha)} - \alpha\right]\pi_0 &= (1-\alpha)\pi_2 \\ \left[\frac{\alpha-\alpha-\alpha^2}{(1-\alpha)}\right]\pi_0 &= (1-\alpha)\pi_2 \\ \pi_2 &= \left(\frac{\alpha}{1-\alpha}\right)^2\pi_0 \end{aligned}$$

from (iii)

$$\begin{aligned} \left(\frac{\alpha}{1-\alpha}\right)^2\pi_0 &= \alpha\frac{\alpha}{(1-\alpha)}\pi_0 + (1-\alpha)\pi_3 \\ \left[\left(\frac{\alpha}{1-\alpha}\right)^2 - \frac{\alpha^2}{(1-\alpha)}\right]\pi_0 &= (1-\alpha)\pi_3 \\ \pi_3 &= \frac{1}{(1-\alpha)}\left[\left(\frac{\alpha}{1-\alpha}\right)^2 - \frac{\alpha^2}{(1-\alpha)}\right]\pi_0 \\ &= \frac{1}{(1-\alpha)^3}\left[\alpha^2 - \alpha^2 + \alpha^3\right]\pi_0 \\ \pi_3 &= \left(\frac{\alpha}{1-\alpha}\right)^3\pi_0 \end{aligned}$$

Deriving π_o in terms of α only

$$\begin{aligned}\sum_{k=0}^{\infty}\pi_k &= 1\\ \Longrightarrow & \left[1+\left(\frac{\alpha}{1-\alpha}\right)+\left(\frac{\alpha}{1-\alpha}\right)^2+\left(\frac{\alpha}{1-\alpha}\right)^3\right]\pi_0=1\end{aligned}$$

let

$$x=\frac{\alpha}{1-\alpha}$$

and recall

$$1+x+x^2+x^3+\cdots+x^{n-1}=\frac{1-x^n}{1-x}$$

hence

$$\frac{1-\left(\frac{\alpha}{1-\alpha}\right)^4}{1-\left(\frac{\alpha}{1-\alpha}\right)}\pi_0=1$$

$$\begin{aligned}\pi_0 &= \frac{1-\left(\frac{\alpha}{1-\alpha}\right)}{1-\left(\frac{\alpha}{1-\alpha}\right)^4}\\ &= \frac{\frac{1-\alpha-\alpha}{1-\alpha}}{\frac{(1-\alpha)^4-\alpha^4}{(1-\alpha)^4}}\\ &= \frac{(1-2\alpha)(1-\alpha)^4}{(1-\alpha)\left[(1-\alpha)^4-\alpha^4\right]}\\ \pi_0 &= \frac{(1-2\alpha)(1-\alpha)^3}{(1-\alpha)^4-\alpha^4}\end{aligned}$$

hence

$$\begin{aligned}\pi_1 &= \frac{\alpha}{(1-\alpha)}\frac{(1-2\alpha)(1-\alpha)^3}{(1-\alpha)^4-\alpha^4}\\ &= \frac{\alpha(1-2\alpha)(1-\alpha)^2}{(1-\alpha)^4-\alpha^4}\end{aligned}$$

$$\begin{aligned}\pi_2 &= \left(\frac{\alpha}{1-\alpha}\right)^2\frac{(1-2\alpha)(1-\alpha)^3}{(1-\alpha)^4-\alpha^4}\\ &= \frac{\alpha^2(1-2\alpha)(1-\alpha)}{(1-\alpha)^4-\alpha^4}\end{aligned}$$

$$\begin{aligned}\pi_3 &= \left(\frac{\alpha}{1-\alpha}\right)^3\frac{(1-2\alpha)(1-\alpha)^3}{(1-\alpha)^4-\alpha^4}\\ &= \frac{\alpha^3(1-2\alpha)}{(1-\alpha)^4-\alpha^4}\end{aligned}$$

4.4 A 6×6 Transition Probability Matrix

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{array} \begin{array}{c} \begin{array}{cccccc} E_0 & E_1 & E_2 & E_3 & E_4 & E_5 \end{array} \\ \begin{bmatrix} 1-\alpha & \alpha & 0 & 0 & 0 & 0 \\ 1-\alpha & 0 & \alpha & 0 & 0 & 0 \\ 0 & 1-\alpha & 0 & \alpha & 0 & 0 \\ 0 & 0 & 1-\alpha & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1-\alpha & 0 & \alpha \\ 0 & 0 & 0 & 0 & 1-\alpha & \alpha \end{bmatrix} \end{array}$$

4.4.1 Classification of the Markov chain

Every state can be reached from every other state hence the Markov Chain is irreducible. Thus all the states are of the same type.

4.4.2 Asymptotic behavior

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \end{bmatrix} = \begin{bmatrix} 1-\alpha & \alpha & 0 & 0 & 0 & 0 \\ 1-\alpha & 0 & \alpha & 0 & 0 & 0 \\ 0 & 1-\alpha & 0 & \alpha & 0 & 0 \\ 0 & 0 & 1-\alpha & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1-\alpha & 0 & \alpha \\ 0 & 0 & 0 & 0 & 1-\alpha & \alpha \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \end{bmatrix}$$

solving for π_i in terms of π_0

$$\begin{aligned} \pi_0 &= (1-\alpha)\pi_0 + (1-\alpha)\pi_1 && \text{(i)} \\ \pi_1 &= \alpha\pi_0 + (1-\alpha)\pi_2 && \text{(ii)} \\ \pi_2 &= \alpha\pi_1 + (1-\alpha)\pi_3 && \text{(iii)} \\ \pi_3 &= \alpha\pi_2 + (1-\alpha)\pi_4 && \text{(iv)} \\ \pi_4 &= \alpha\pi_3 + (1-\alpha)\pi_5 && \text{(v)} \\ \pi_5 &= \alpha\pi_4 + \alpha\pi_5 && \text{(vi)} \end{aligned}$$

From (i)

$$\pi_0 = (1-\alpha)\pi_0 + (1-\alpha)\pi_1$$

$$\begin{aligned}\alpha\pi_0 &= (1-\alpha)\,\pi_1\\ \pi_1 &= \frac{\alpha}{(1-\alpha)}\pi_0\end{aligned}$$

from (ii)

$$\begin{aligned}\frac{\alpha}{(1-\alpha)}\pi_0 &= \alpha\pi_0+(1-\alpha)\,\pi_2\\ \left[\frac{\alpha}{(1-\alpha)}-\alpha\right]\pi_0 &= (1-\alpha)\,\pi_2\\ \left[\frac{\alpha-\alpha-\alpha^2}{(1-\alpha)}\right]\pi_0 &= (1-\alpha)\,\pi_2\\ \pi_2 &= \left(\frac{\alpha}{1-\alpha}\right)^2\pi_0\end{aligned}$$

from (iii)

$$\begin{aligned}\left(\frac{\alpha}{1-\alpha}\right)^2\pi_0 &= \alpha\frac{\alpha}{(1-\alpha)}\pi_0+(1-\alpha)\,\pi_3\\ \left[\left(\frac{\alpha}{1-\alpha}\right)^2-\frac{\alpha^2}{(1-\alpha)}\right]\pi_0 &= (1-\alpha)\,\pi_3\\ \pi_3 &= \frac{1}{(1-\alpha)}\left[\left(\frac{\alpha}{1-\alpha}\right)^2-\frac{\alpha^2}{(1-\alpha)}\right]\pi_0\\ &= \frac{1}{(1-\alpha)^3}\left[\alpha^2-\alpha^2+\alpha^3\right]\pi_0\\ \pi_3 &= \left(\frac{\alpha}{1-\alpha}\right)^3\pi_0\end{aligned}$$

From (iv)

$$\begin{aligned}\left(\frac{\alpha}{1-\alpha}\right)^3\pi_0 &= \frac{\alpha^3}{(1-\alpha)^2}\pi_0+(1-\alpha)\,\pi_4\\ \frac{\alpha^3-\alpha^3+\alpha^4}{(1-\alpha)^3}\pi_0 &= (1-\alpha)\,\pi_4\\ \pi_4 &= \left(\frac{\alpha}{1-\alpha}\right)^4\pi_0\end{aligned}$$

from (v)

$$\begin{aligned}\left(\frac{\alpha}{1-\alpha}\right)^4\pi_0 &= \frac{\alpha^4}{(1-\alpha)^3}\pi_0+(1-\alpha)\,\pi_5\\ \frac{\alpha^4-\alpha^4+\alpha^5}{(1-\alpha)^4}\pi_0 &= (1-\alpha)\,\pi_5\\ \pi_5 &= \left(\frac{\alpha}{1-\alpha}\right)^5\pi_0\end{aligned}$$

Deriving π_o in terms of α only

$$\sum_{k=0}^{\infty}\pi_k = 1$$

hence

$$\left[1+\left(\frac{\alpha}{1-\alpha}\right)+\left(\frac{\alpha}{1-\alpha}\right)^2+\left(\frac{\alpha}{1-\alpha}\right)^3+\left(\frac{\alpha}{1-\alpha}\right)^4+\left(\frac{\alpha}{1-\alpha}\right)^5\right]\pi_0 = 1$$

let

$$x = \frac{\alpha}{1-\alpha}$$

and recall

$$1+x+x^2+x^3+\cdots+x^{n-1} = \frac{1-x^n}{1-x}$$

hence

$$\frac{1-\left(\frac{\alpha}{1-\alpha}\right)^6}{1-\left(\frac{\alpha}{1-\alpha}\right)}\pi_0 = 1$$

$$\begin{aligned}
\pi_0 &= \frac{1-\left(\frac{\alpha}{1-\alpha}\right)}{1-\left(\frac{\alpha}{1-\alpha}\right)^6} \\
&= \frac{(1-2\alpha)(1-\alpha)^6}{(1-\alpha)\left[(1-\alpha)^6-\alpha^6\right]} \\
\pi_0 &= \frac{(1-2\alpha)(1-\alpha)^5}{(1-\alpha)^6-\alpha^6}
\end{aligned}$$

hence

$$\begin{aligned}
\pi_1 &= \frac{\alpha}{(1-\alpha)}\frac{(1-2\alpha)(1-\alpha)^5}{(1-\alpha)^6-\alpha^6} \\
&= \frac{\alpha(1-2\alpha)(1-\alpha)^4}{(1-\alpha)^6-\alpha^6}
\end{aligned}$$

$$\begin{aligned}
\pi_2 &= \left(\frac{\alpha}{1-\alpha}\right)^2\frac{(1-2\alpha)(1-\alpha)^5}{(1-\alpha)^6-\alpha^6} \\
&= \frac{\alpha^2(1-2\alpha)(1-\alpha)^3}{(1-\alpha)^6-\alpha^6}
\end{aligned}$$

$$\begin{aligned}
\pi_3 &= \left(\frac{\alpha}{1-\alpha}\right)^3\frac{(1-2\alpha)(1-\alpha)^5}{(1-\alpha)^6-\alpha^6} \\
&= \frac{\alpha^3(1-2\alpha)^2}{(1-\alpha)^6-\alpha^6}
\end{aligned}$$

$$\begin{aligned}
\pi_4 &= \left(\frac{\alpha}{1-\alpha}\right)^4 \frac{(1-2\alpha)\,(1-\alpha)^5}{(1-\alpha)^6-\alpha^6} \\
&= \frac{\alpha^4\,(1-2\alpha)}{(1-\alpha)^6-\alpha^6}
\end{aligned}$$

$$\begin{aligned}
\pi_5 &= \left(\frac{\alpha}{1-\alpha}\right)^5 \frac{(1-2\alpha)\,(1-\alpha)^5}{(1-\alpha)^6-\alpha^6} \\
&= \frac{\alpha^5\,(1-2\alpha)}{(1-\alpha)^6-\alpha^6}
\end{aligned}$$

In general

$$\pi_n = \frac{\alpha^{n-1}(1-2\alpha)}{(1-\alpha)^n-\alpha^n}$$

This analysis can help the insurance company to set higher premiums for clients who have more claims and lower premiums for those who have fewer or no claims at all. A simple random walk is systematic if the particle has the same probability for each of the neighbours. That is when $p=q=\frac{1}{2}$ the random walk is systematic. We assume $0<p<1$, that otherwise the simple random walk becomes trivial.

5 RANDOM WALKS WITH BARRIERS

5.1 Introduction

Sometimes the walker cannot go outside some defined boundaries, in which case the walk is said to be a restricted random walk and the boundaries are called barriers. These barriers can impose different characteristics on the walk process. For example, they can be reflecting barriers, which mean that on hitting them the walk turns around and continues. They can also be absorbing barriers, which means that the walk ends when a barrier is hit.

5.2 Random Walks with Absorbing Barriers

Let the possible states be $E_1, E_2, \ldots, E_p$ and consider the matrix of transition probabilities.

5.2.1 A 3×3 transition probability matrix

For a 3×3 transition probability matrix, we have

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ q & 0 & p \\ 0 & 0 & 1 \end{array}\right] \end{array}$$

Rearranging and partioning the matrix, we obtain

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_3 \\ E_2 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_3 & E_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline q & p & 0 \end{array}\right] \end{array}$$

where

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} q & p \end{bmatrix}, \quad \text{and } \mathbf{V} = 0$$

Classification of the Markov chain

E_1 and E_2 are absorbing Markov chains.

Classification of the states

$E_1$and E_3 are absorbing states and are trivially persistent, non-null and aperiodic and hence Ergodic.

Consider E_2

$$\begin{aligned} f_{22}^{(1)} &= \Pr[E_2 \to E_2] \\ &= r \\ f_{22}^{(2)} &= 0 \\ f_{22}^{(3)} &= 0 \end{aligned}$$

In general

$$f_{22}^{(n)} = \left\{ \begin{array}{cc} r, & n=1 \\ 0, & \text{elsewhere} \end{array} \right\}$$

$$\begin{aligned} f_2 &= \sum_{n=1}^{\infty} f_{22}^{(n)} \\ &= r < 1 \end{aligned}$$

Hence E_2 is transient. In this case, there is no further investigation on E_2.

The n^{th} power $\mathbf{P}^n$

Since we are dealing with a reducible Markov chain, $\mathbf{P}$ can be expressed as

$$\left[\begin{array}{cc} \mathbf{I} & \mathbf{0} \\ \mathbf{U} & \mathbf{V} \end{array} \right]$$

where

$$\mathbf{I} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \quad \mathbf{U} = \left[\begin{array}{cc} q & p \end{array} \right], \quad \text{and } \mathbf{V} = 0$$

Therefore

$$\mathbf{P}^n = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{U}_n & \mathbf{V}^n \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ (\mathbf{I}-\mathbf{V})^{-1}(\mathbf{I}-\mathbf{V}^n)\mathbf{U} & \mathbf{V}^n \end{bmatrix}$$

But

$$\mathbf{V} = \mathbf{0} \implies \mathbf{V}^n = \mathbf{0}$$

$$\mathbf{U} = \begin{bmatrix} q & p \end{bmatrix}$$

$$(\mathbf{I}-\mathbf{V}) = (\mathbf{I}-\mathbf{0}) = \mathbf{I}$$

$$(\mathbf{I}-\mathbf{V})^{-1} = \mathbf{0}$$

$$(\mathbf{I}-\mathbf{V}^n) = \mathbf{0}$$

Therefore

$$\mathbf{P}^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q & p & 0 \end{bmatrix}$$

Asymptotic behavior

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ (\mathbf{I}-\mathbf{V})^{-1}(\mathbf{I}-\lim_{\mathbf{n}\to\infty}\mathbf{V}^n)\mathbf{U} & \mathbf{V}^n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q & p & 0 \end{bmatrix}$$

5.2.2 A 4×4 transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_3 \\ E_2 \\ E_4 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_3 & E_2 & E_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & 0 & 1 \end{array} \right] \end{array}$$

The matrix can be rearranged and partioned as follows

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_4 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_4 & E_2 & E_3 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ q & 0 & 0 & p \\ 0 & p & q & 0 \end{array} \right] \end{array}$$

where

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix}, \quad \text{and } \mathbf{V} = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}$$

Classification of the states

E_1 and E_4 are absorbing states. They are trivially persistent, non-null and aperiodic and hence they are ergodic. We now consider the remaining states E_2 and E_3.

$$\begin{aligned} f_{22}^{(1)} &= 0 \\ f_{22}^{(2)} &= \Pr[E_2 \to E_3 \to E_2] = pq \\ f_{22}^{(3)} &= 0 \\ f_{22}^{(4)} &= 0 \end{aligned}$$

In general

$$f_{22}^{(n)} = \left\{ \begin{array}{cc} pq, & n = 2 \\ 0, & \text{elsewhere} \end{array} \right\}$$

$$f_2 = \sum_{n=1}^{\infty} f_{22}^{(n)}$$

but $0 < p < 1$ and $0 < q < 1 \implies f_2 = pq < 1$ hence E_2 is transient, thus no further investigation is required.

Let us consider E_3

$$\begin{aligned} f_{33}^{(1)} &= 0 \\ f_{33}^{(2)} &= \Pr[E_3 \to E_2 \to E_3] = pq \\ f_{33}^{(3)} &= 0 \\ &\vdots \end{aligned}$$

In general

$$f_{33}^{(n)} = \left\{ \begin{array}{cc} pq, & n = 2 \\ 0, & \text{elsewhere} \end{array} \right\}$$

$$\begin{aligned} f_3 &= \sum_{n=1}^{\infty} f_{33}^{(n)} \\ &= pq < 1 \end{aligned}$$

Hence E_3 is transient thus no further investigation is required. We now wish to determine $\mathbf{P}^n$ and the asymptotic behaviour $\lim_{n\to\infty} \mathbf{P}^n$. Since we are dealing with a reducible Markov chain, $\mathbf{P}$ can be expressed as

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{U} & \mathbf{V} \end{bmatrix}$$

where

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix}, \quad \text{and } \mathbf{V} = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}$$

$$\begin{aligned} P^n &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{U}_n & \mathbf{V}^n \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ (\mathbf{I}-\mathbf{V})^{-1}(\mathbf{I}-\mathbf{V}^n)\mathbf{U} & \mathbf{V}^n \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
\mathbf{V} &= \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} \\
\mathbf{V}^2 &= \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}\begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} \\
&= \begin{bmatrix} pq & 0 \\ 0 & pq \end{bmatrix} \\
\mathbf{V}^3 &= \begin{bmatrix} pq & 0 \\ 0 & pq \end{bmatrix}\begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & p^2q \\ pq^2 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}^4 &= \begin{bmatrix} 0 & p^2q \\ pq^2 & 0 \end{bmatrix}\begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix} \\
&= \begin{bmatrix} p^2q^2 & 0 \\ 0 & p^2q^2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}^5 &= \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}\begin{bmatrix} p^2q^2 & 0 \\ 0 & p^2q^2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & p^3q \\ pq^3 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}^6 &= \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}\begin{bmatrix} 0 & p^3q \\ pq^3 & 0 \end{bmatrix} \\
&= \begin{bmatrix} p^3q^3 & 0 \\ 0 & p^3q^3 \end{bmatrix}
\end{aligned}$$

In general

$$\mathbf{V}^n = \left\{ \begin{array}{c} \begin{bmatrix} (pq)^{\frac{n}{2}} & 0 \\ 0 & (pq)^{\frac{n}{2}} \end{bmatrix}, \quad \text{when } n \text{ is even} \\ \begin{bmatrix} 0 & p\,(pq)^{\frac{n-1}{2}} \\ q\,(pq)^{\frac{n-1}{2}} & 0 \end{bmatrix}, \quad \text{when } n \text{ is odd} \end{array} \right\}$$

The asymptotic behavior

$$\lim_{n\to\infty}\mathbf{P}^n = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ (\mathbf{I}-\mathbf{V})^{-1}\mathbf{U} & \mathbf{V}^n \end{bmatrix}$$

$$(\mathbf{I}-\mathbf{V})^{-1} = \frac{1}{1-pq}\begin{bmatrix} 1 & p \\ q & 1 \end{bmatrix}$$

$$\begin{aligned}(\mathbf{I}-\mathbf{V})^{-1}\mathbf{U} &= \frac{1}{1-pq}\begin{bmatrix} 1 & p \\ q & 1 \end{bmatrix}\begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix} \\ &= \frac{1}{1-pq}\begin{bmatrix} q & p^2 \\ q^2 & p \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}\mathbf{P}^n &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{q}{1-pq} & \frac{p^2}{1-pq} & 0 & 0 \\ \frac{q^2}{1-pq} & \frac{p}{1-pq} & 0 & 0 \end{bmatrix} \\ &= \frac{1}{1-pq}\begin{bmatrix} 1-pq & 0 & 0 & 0 \\ 0 & 1-pq & 0 & 0 \\ q & p^2 & 0 & 0 \\ q^2 & p & 0 & 0 \end{bmatrix}\end{aligned}$$

but

$$\begin{aligned}1-pq &= 1-p(1-p) \\ &= 1-p+p^2 \\ &= q+p^2\end{aligned}$$

hence

$$\lim_{n\to\infty}\mathbf{P}^n = \frac{1}{q+p^2}\begin{bmatrix} q+p^2 & 0 & 0 & 0 \\ 0 & q+p^2 & 0 & 0 \\ q & p^2 & 0 & 0 \\ q^2 & p & 0 & 0 \end{bmatrix}$$

5.2.3 A 5×5 Transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{array} \begin{array}{c} \begin{array}{ccccc} E_1 & E_2 & E_3 & E_4 & E_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The matrix can be rearranged and partioned as follows

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_5 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{array}{c} \begin{array}{ccccc} E_1 & E_5 & E_2 & E_3 & E_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ q & 0 & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & p & 0 & q & 0 \end{bmatrix} \end{array}$$

where

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} q & 0 \\ 0 & 0 \\ 0 & p \end{bmatrix}, \quad \text{and } \mathbf{V} = \begin{bmatrix} 0 & p & 0 \\ q & 0 & p \\ 0 & q & 0 \end{bmatrix}$$

Classification of the states

E_1 and E_5 are absorbing states. They are trivially persistent, non-null and aperiodic and hence they are ergodic. We now consider the remaining states E_2, E_3 and E_4. Consider E_2

$$\begin{aligned} f_{22}^{(1)} &= 0 \\ f_{22}^{(2)} &= \Pr[E_2 \to E_3 \to E_2] = pq \\ f_{22}^{(3)} &= 0 \end{aligned}$$

$$f_{22}^{(n)} = \begin{Bmatrix} pq, & n = 2 \\ 0, & \text{elsewhere} \end{Bmatrix}$$

$$\begin{aligned} f_2 &= \sum_{n=1}^{\infty} f_{22}^{(n)} \\ &= pq < 1 \end{aligned}$$

Hence E_2 is transient. Now Consider E_3

$$\begin{aligned} f_{33}^{(1)} &= 0 \\ f_{33}^{(2)} &= \Pr[E_3 \to E_2 \to E_3] + \Pr[E_3 \to E_4 \to E_3] = 2pq \\ f_{33}^{(3)} &= 0 \end{aligned}$$

$$f_{33}^{(n)} = \left\{ \begin{array}{cc} 2pq, & n = 2 \\ 0, & \text{elsewhere} \end{array} \right\}$$

$$\begin{aligned} f_3 &= \sum_{n=1}^{\infty} f_{33}^{(n)} \\ &= 2pq < 1 \end{aligned}$$

Hence E_3 is transient. Now Consider E_4

$$\begin{aligned} f_{44}^{(1)} &= 0 \\ f_{44}^{(2)} &= \Pr[E_4 \to E_3 \to E_4] = pq \\ f_{44}^{(3)} &= 0 \end{aligned}$$

$$f_{44}^{(n)} = \left\{ \begin{array}{cc} pq, & n = 2 \\ 0, & \text{elsewhere} \end{array} \right\}$$

$$\begin{aligned} f_4 &= \sum_{n=1}^{\infty} f_{44}^{(n)} \\ &= pq < 1 \end{aligned}$$

The asymptotic behavior

$$\begin{aligned} \mathbf{I} - \mathbf{V} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & p & 0 \\ q & 0 & p \\ 0 & q & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -p & 0 \\ -q & 1 & -p \\ 0 & -q & 1 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
|\mathbf{I}-\mathbf{V}| &= \begin{vmatrix} 1 & -p \\ -q & 1 \end{vmatrix} + \begin{vmatrix} -q & -p \\ 0 & 1 \end{vmatrix} \\
&= 1-qp-qp \\
&= 1-2qp
\end{aligned}
$$

but

$$(p+q)^2 = p^2+q^2+2pq$$

$\Longrightarrow$

$$1-2pq = p^2+q^2$$

since

$$p+q=1$$

hence

$$|\mathbf{I}-\mathbf{V}| = p^2+q^2$$

$$
\begin{aligned}
\mathbf{N} &= (\mathbf{I}-\mathbf{V})^{-1} \\
&= \frac{1}{p^2+q^2}\begin{bmatrix} p+q^2 & p & p^2 \\ q & 1 & p \\ q^2 & q & p^2+q \end{bmatrix}
\end{aligned}
$$

$$
N = \begin{bmatrix} \frac{p+q^2}{p^2+q^2} & \frac{p}{p^2+q^2} & \frac{p^2}{p^2+q^2} \\ \frac{q}{p^2+q^2} & \frac{1}{p^2+q^2} & \frac{p}{p^2+q^2} \\ \frac{q^2}{p^2+q^2} & \frac{q}{p^2+q^2} & \frac{p^2+q}{p^2+q^2} \end{bmatrix}
$$

$$
\begin{aligned}
\mathbf{NU} &= \begin{bmatrix} \frac{p+q^2}{p^2+q^2} & \frac{p}{p^2+q^2} & \frac{p^2}{p^2+q^2} \\ \frac{q}{p^2+q^2} & \frac{1}{p^2+q^2} & \frac{p}{p^2+q^2} \\ \frac{q^2}{p^2+q^2} & \frac{q}{p^2+q^2} & \frac{p^2+q}{p^2+q^2} \end{bmatrix}\begin{bmatrix} q & 0 \\ 0 & 0 \\ 0 & p \end{bmatrix} \\
&= \begin{bmatrix} \frac{(p+q^2)q}{p^2+q^2} & \frac{p3}{p^2+q^2} \\ \frac{q^2}{p^2+q^2} & \frac{p^2}{p^2+q^2} \\ \frac{q^3}{p^2+q^2} & \frac{(p^2+q)p}{p^2+q^2} \end{bmatrix}
\end{aligned}
$$

$$\lim_{n\to\infty}\mathbf{P}^n = \begin{bmatrix} \mathbf{I} & 0 \\ (\mathbf{I}-\mathbf{V})^{-1}\mathbf{U} & 0 \end{bmatrix}$$

$$= \frac{1}{p^2+q^2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \left(p+q^2\right)q & p3 & 0 & 0 & 0 \\ q^2 & p^2 & 0 & 0 & 0 \\ q^3 & \left(p^2+q\right)p & 0 & 0 & 0 \end{bmatrix}$$

5.2.4 A 6×6 transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_3 \\ E_2 \\ E_4 \\ E_5 \\ E_6 \end{array} \begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & 0 \\ 0 & 0 & q & 0 & p & 0 \\ 0 & 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The matrix can be rearranged and partioned as follows

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_6 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{array} \begin{array}{c} \begin{array}{cccccc} E_1 & E_6 & E_2 & E_3 & E_4 & E_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ q & 0 & 0 & p & 0 & 0 \\ 0 & 0 & q & 0 & p & 0 \\ 0 & 0 & 0 & q & 0 & p \\ 0 & p & 0 & 0 & q & 0 \end{bmatrix} \end{array}$$

where

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} q & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & p \end{bmatrix}, \quad \text{and } \mathbf{V} = \begin{bmatrix} 0 & p & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & q & 0 \end{bmatrix}$$

$$\mathbf{I}-\mathbf{V}=\begin{bmatrix} 1 & -p & 0 & 0 \\ -q & 1 & -p & 0 \\ 0 & -q & 1 & -p \\ 0 & 0 & -q & 1 \end{bmatrix}$$

Asymptotic behavior

$$\begin{aligned}
|\mathbf{I}-\mathbf{V}| &= 1\left[\begin{vmatrix} 1 & -p & 0 \\ -q & 1 & -p \\ 0 & -q & 1 \end{vmatrix} + p\begin{vmatrix} -q & -p & 0 \\ 0 & 1 & -p \\ 0 & -q & 1 \end{vmatrix}\right] \\
&= 1\left[1\begin{vmatrix} 1 & -p \\ -q & 1 \end{vmatrix} + p\begin{vmatrix} -q & -p \\ 0 & 1 \end{vmatrix}\right] + p\left[-q\begin{vmatrix} 1 & -p \\ -q & 1 \end{vmatrix} + p\begin{vmatrix} 0 & -p \\ 0 & 1 \end{vmatrix}\right] \\
&= 1-pq-pq+-pq(1-pq) \\
&= 1-2pq-pq-p^2q^2 \\
&= p^2+q^2-pq-p^2q^2
\end{aligned}$$

Minor matrices

$$\begin{vmatrix} 1 & -p & 0 \\ -q & 1 & -p \\ 0 & -q & 1 \end{vmatrix}\begin{vmatrix} -q & -p & 0 \\ 0 & 1 & -p \\ 0 & -q & 1 \end{vmatrix}\begin{vmatrix} -q & 1 & 0 \\ 0 & -q & -p \\ 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} -q & 1 & -p \\ 0 & -q & 1 \\ 0 & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & 0 \\ -q & 1 & -p \\ 0 & -q & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -p \\ 0 & -q & 1 \end{vmatrix}\begin{vmatrix} 1 & -p & 0 \\ 0 & -q & -p \\ 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & -p & 0 \\ 0 & -q & 1 \\ 0 & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & 0 \\ 1 & -p & 0 \\ 0 & -q & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ -q & -p & 0 \\ 0 & -q & 1 \end{vmatrix}\begin{vmatrix} 1 & -p & 0 \\ -q & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & -p & 0 \\ -q & 1 & -p \\ 0 & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & 0 \\ 1 & -p & 0 \\ -q & 1 & -p \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ -q & -p & 0 \\ 0 & 1 & -p \end{vmatrix}\begin{vmatrix} 1 & -p & 0 \\ -q & 1 & 0 \\ 0 & -q & -p \end{vmatrix}\begin{vmatrix} 1 & -p & 0 \\ -q & 1 & -p \\ 0 & -q & 1 \end{vmatrix}$$

to obtain

$$\begin{bmatrix} p^2+q^2 & pq^2-q & q^2 & q^3 \\ pq^2-q & 1-pq & -q & q^2 \\ p^2 & -p & 1 & pq^2-q \\ p^3 & p^2 & -p & p^2+q^2 \end{bmatrix}$$

Co-factor matrix

$$\begin{bmatrix} p^2+q^2 & pq^2-q & p^2 & p^3 \\ pq^2-q & 1-pq & -p & p^2 \\ q^2 & -q & 1 & -p \\ q^3 & q^2 & pq^2-q & p^2+q^2 \end{bmatrix}$$

$$(\mathbf{1}-\mathbf{V})^{-1} = \frac{1}{p^2+q^2-pq-p^2q^2}\begin{bmatrix} p^2+q^2 & pq^2-q & p^2 & p^3 \\ pq^2-q & 1-pq & -p & p^2 \\ q^2 & -q & 1 & -p \\ q^3 & q^2 & pq^2-q & p^2+q^2 \end{bmatrix}$$

$$\begin{aligned} (\mathbf{1}-\mathbf{V})^{-1} &= \frac{1}{p^2+q^2-pq-p^2q^2}\begin{bmatrix} p^2+q^2 & pq^2-q & p^2 & p^3 \\ pq^2-q & 1-pq & -p & p^2 \\ q^2 & -q & 1 & -p \\ q^3 & q^2 & pq^2-q & p^2+q^2 \end{bmatrix}\begin{bmatrix} q & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & p \end{bmatrix} \\ &= \frac{1}{p^2+q^2-pq-p^2q^2}\begin{bmatrix} \left(p^2+q^2\right)q & p^4 \\ \left(pq^2-q\right)q & p^3 \\ q^3 & -p \\ q^4 & \left(p^2+q^2\right)p \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty}\mathbf{P}^n &= \begin{bmatrix} \mathbf{I} & 0 \\ (\mathbf{I}-\mathbf{V})^{-1}\mathbf{U} & 0 \end{bmatrix} \\ &= \frac{1}{p^2+q^2-pq-p^2q^2}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \left(p^2+q^2\right)q & p^4 & 0 & 0 & 0 & 0 \\ \left(pq^2-q\right)q & p^3 & 0 & 0 & 0 & 0 \\ q^3 & -p & 0 & 0 & 0 & 0 \\ q^4 & \left(p^2+q^2\right)p & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

5.2.5 General form, $\rho \times \rho$ transition probability matrix

In general, the transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_{\rho-1} \end{array} \begin{array}{c} \begin{array}{cccccc} E_0 & E_1 & E_2 & E_3 & \dots & E_{\rho-1} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ q & 0 & p & 0 & \dots & 0 \\ 0 & q & 0 & p & \dots & 0 \\ 0 & 0 & q & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \end{array}$$

From each of "interior" states $E_0, E_1, \dots, E_{\rho-1}$ transitions are possible to the right and to the left neighbours (with $p_{j,j+1} = p$, and $p_{j,j-1} = q$, and $p+q = 1$). However no transition is possible from either E_0 or $E_{\rho-1}$ to any other state. The system may move from one state to another but once in E_0 or $E_{\rho-1}$ is reached; the system stays there fixed forever. This Markov chain differs from terminologically from the model of random walk with the absorbing barriers at 0 and ρ.

The fundamental matrix

The fundamental matrix gives the expected number of visits to each state before absorption occurs. For an absorbing Markov chain we define the fundamental matrix to be

$$\mathbf{N} = (\mathbf{I} - \mathbf{V})^{-1}$$

we now derive the probabilistic interpretation of $\mathbf{N}$.We define n_{ij} to be the function giving the total number of times that a process starting from E_i is in E_j (this is defined only for transient states E_i and E_j). $X_{ij}^{(k)}$ is defined as the function that is 1 if the process is in state E_j after k steps and zero otherwise. i.e.

n_{ij} =The number of times that a process starting from E_i is in E_j and

$$X_{ij}^{(k)} = \left\{ \begin{array}{c} 1 \text{ , if the process is in state } E_j \text{ after } k \text{ steps starting form } E_i \\ 0, \text{ otherwise} \end{array} \right\}$$

$\Longrightarrow$

$$n_{ij} = \sum_{k=0}^{\infty} X_{ij}^{(k)}$$

thus

$$E\left[n_{ij}\right]=\sum_{k=0}^{\infty}E\left[X_{ij}^{(k)}\right]$$

by definition

$$\begin{aligned}E[X_{ij}] &= \sum_{k=0}^{\infty}\left[1\cdot\Pr\left(X_{ij}^{(k)}=1\right)+0\cdot\Pr\left(X_{ij}^{(k)}=0\right)\right]\\ &= \sum_{k=0}^{\infty}\left[1\cdot\Pr\left(X_{ij}^{(k)}=1\right)\right]\\ &= \sum_{k=0}^{\infty}p_{ij}^{(k)}\end{aligned}$$

In matrix form, we have

$$\begin{aligned}\left(\left(E\left(n_{ij}\right)\right)\right) &= \sum_{k=0}^{\infty}\left(\left(p_{ij}^{(k)}\right)\right)\\ &= \sum_{k=0}^{\infty}Q^{k}\end{aligned}$$

since E_i and E_j are transient

$$\begin{aligned}&= \mathbf{I}+\mathbf{V}+\mathbf{V}^2+\cdots\\ &= (\mathbf{I}-\mathbf{V})^{-1}\\ &= \mathbf{N}\end{aligned}$$

$\Longrightarrow$

$$\left(\left(E\left(n_{ij}\right)\right)\right)=\mathbf{N}$$

Thus $E\left[n_{ij}\right]$ is the average number of times that the system takes in state E_j before the absorption given that it starts in state E_i and this means are the entry of fundamental matrix $\mathbf{N}=(\mathbf{I}-\mathbf{V})^{-1}$.

Remark

The mean of the total number of times the process is in a given transient state is always finite and that these means are given by $\mathbf{N}$. Alternatively, to compute $E\left[n_{ij}\right]$ we may add up the original position contribution plus each state's contribution.

Let

$$\delta_{ij}=\left\{\begin{array}{l}1,\ \text{if } i=j\\ 0,\ \text{if } i\neq j\end{array}\right\}$$

Thus δ_{ij} is the contribution to the original position. After one step, we move to E_k say with probability p_{ik}. Let $X_{ij}^{(n+1)}$ be a function that is 1 if the process is in state E_j after steps $(n+1)$ and 0 otherwise.

Then by Chapman-Kolmogorov equations

$$X_{ij}^{(n+1)} = \sum_{k=1}^{n} X_{ik} X_{kj}^{(n)}$$

$$\sum_{n=0}^{\infty} X_{ij}^{(n+1)} = \sum_{n=0}^{\infty} \left[\sum_{k=1}^{n} X_{ik} X_{kj}^{(n)} \right]$$

$\Longrightarrow$

$$\begin{aligned} \sum_{n=0}^{\infty} X_{ij}^{(n+1)} &= \sum_{k=1}^{\infty} X_{ik} \sum_{n=0}^{\infty} X_{kj}^{(n)} \\ &= \sum_{k=1}^{\infty} X_{ik} E\left[n_{kj}\right] \end{aligned}$$

then

$$\sum_{n=0}^{\infty} X_{ij}^{(n+1)} - X_{kj}^{(0)} = \sum_{k=1}^{\infty} X_{ik} E\left[n_{kj}\right] \tag{31}$$

with the conventions that

$$\begin{aligned} X_{ij}^{(0)} &= \\ \delta_{ij} &= \left\{ \begin{array}{l} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{array} \right\} \end{aligned}$$

hence equation (3.1) becomes

$$\sum_{n=0}^{\infty} X_{ij}^{(n+1)} - \delta_{ij} = \sum_{k=1}^{\infty} X_{ik} E\left[n_{kj}\right]$$

and in matrix form

$$((E[n_{ij}])) - ((\delta_{ij})) = \left(\left(\sum_{k=1}^{\infty} X_{ik} E\left[n_{kj}\right] \right) \right)$$

$\Longrightarrow$

$$((E[n_{ij}])) - \mathbf{I} = \mathbf{V} \left(\left(E\left[n_{kj}\right] \right) \right)$$

$\Longrightarrow$

$$\left(\left(E\left[n_{kj}\right] \right) \right) [\mathbf{I} - \mathbf{V}] = \mathbf{I}$$

Thus

$$\left(\left(E\left[n_{kj}\right]\right)\right) = [\mathbf{I}-\mathbf{V}]^{-1}$$
$$= \mathbf{N}$$

Theorem 3.1

If b_{ij} is the probability that the process starting in transient state ends up in absorbing state E_k, then

$$b_{ij} = \mathbf{B}$$
$$= \mathbf{NU}$$
$$= [\mathbf{I}-\mathbf{V}]^{-1}$$

Proof

Define

(i) E_{ij} occurs iff a one-step transition $E_i \to E_j$ occurs

(ii) A_{ik} occurs iff absorption into k from j occurs

Here E_i and E_j and are transient while E_k is an absorbing state. The event A_{ik} can occur in two mutually exclusive ways.

(i) A direct transition occurs $E_i \to E_k$ or

(ii) Some direct absorption occurs via first transition event E_{ij}; transient and then A_{jk} occurs for some j.

Hence

$$A_{ik} = E_{ik} \cup \left[\cup_j E_{ij} \cap A_{jk}\right] \quad (j \text{ transient})$$

and

$$\Pr[A_{ik}] = \Pr[E_{ik}] + \Pr\left[\cup_j E_{ij} \cap A_{jk}\right]$$
$$= \Pr[E_{ik}] + \sum_j \Pr\left[E_{ij} \cap A_{jk}\right]$$
$$= \Pr[E_{ik}] + \sum_j \Pr[E_{ij}] \Pr\left[A_{jk}\right]$$

Now let

$$b_{ik} = \Pr\left[A_{jk}\right]$$

then we have

$$b_{ik} = p_{ik} + \sum_j p_{ij} b_{jk} \quad (j \text{ transient})$$

In matrix form

Since both i and j are transient and k is absorbing

$$((b_{ik})) = ((p_{ik})) + \left(\left(\sum_j p_{ij} b_{jk}\right)\right)$$

$\Longrightarrow$

$$\mathbf{B} = \mathbf{U} + \mathbf{VB}$$

hence

$$\begin{aligned} \mathbf{B} - \mathbf{VB} &= \mathbf{U} \\ &= \mathbf{I} - \mathbf{V} \end{aligned}$$

$\Longrightarrow$

$$\mathbf{B} = (\mathbf{I} - \mathbf{V})^{-1} \mathbf{U}$$

which is interpretable as the matrix of absorption probabilities.

Application of the fundamental matrix

A number of quantities can be expressed in terms of fundamental matrix. This result will be illustrated in terms of random walk.

Theorem 3.2

$$((var(n_{ij}))) = \mathbf{N}\left[2\mathbf{N}_{dg} - \mathbf{I}\right] - \mathbf{N}_{sq} (s \times s) \text{ matrix}$$

where $\mathbf{N}_{dg}$ is a matrix that results from $\mathbf{N}$ by offsetting the diagonal entries equal to zero.$\mathbf{N}_{sq}$ is the matrix that results from $\mathbf{N}$ by squaring each entry.$\mathbf{N} = (\mathbf{I} - \mathbf{V})^{-1}$ is a fundamental matrix.

Proof

By definition, $((var(n_{ij}))) = E\left[n_{ij}^2\right] - [E(n_{ij})]^2$. Assume that these means are finite. Since $((E(n_{ij}))) = \mathbf{N}$, then we note that $\left(\left([E(n_{ij})]^2\right)\right) = \mathbf{N}_{sq}$. To compute $E\left[n_{ij}^2\right]$ we again ask where the process can go in one step from its starting position E_i. We note that it can go to E_k with probability p_{ik}. If the new state is absorbing, then we can never reach E_j again. The only possible contribution is from the initial state which is δ_{ij} defined as

$$\delta_{ij} = \left\{ \begin{array}{l} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{array} \right\}$$

If the new state is transient, we will be in E_j δ_{ij} times from the original position and n_{ij} times from the later steps. Hence

$$\begin{aligned} \left(\left([E(n_{ij})]^2\right)\right) &= \left(\left(\sum_{E_k} p_{ik}\delta_{ij}^2 + \sum_{E_k} p_{ik} E\left(n_{kj} + \delta_{ij}\right)^2\right)\right) \\ &= \left(\left(\sum_{E_k} p_{ik}\left[n_{ij}^2 + 2\delta_{ij}E\left(n_{kj}\right)\right] + \delta_{ij}\right)\right) \quad \text{where } \delta_{ij}^2 = \delta_{ij} \\ &= \left(\left(\sum_{E_k} p_{ik}\right)\right) \left(\left(n_{ij}^2\right)\right) + 2\left(\left(E\left(n_{ij}\right)\right)\right)\left(\left(\delta_{ij}\right)\right) + \left(\left(\delta_{ij}\right)\right) \\ &= \mathbf{V}\left(\left(E\left(n_{ij}^2\right)\right)\right) + 2\left(\mathbf{VN}\right)_{dg} + \mathbf{I} \end{aligned}$$

$\Longrightarrow$

$$\left(\left(E\left(n_{ij}^2\right)\right)\right)[\mathbf{I} - \mathbf{V}] = 2\left(\mathbf{VN}\right)_{dg} + \mathbf{I}$$

$\Longrightarrow$

$$\begin{aligned} \left(\left(E\left(n_{ij}^2\right)\right)\right) &= \left[2\left(\mathbf{VN}\right)_{dg} + \mathbf{I}\right][\mathbf{I} - \mathbf{V}]^{-1} \\ &= \left[2\left(\mathbf{VN}\right)_{dg} + \mathbf{I}\right]\mathbf{N} \end{aligned} \qquad \textbf{(32)}$$

but

$$\begin{aligned} \mathbf{VN} &= \mathbf{NV} \\ &= \mathbf{V}\left[\mathbf{I} + \mathbf{V} + \mathbf{V}^2 + \cdots\right] \\ &= \mathbf{V} + \mathbf{V}^2 + \mathbf{V}^3 + \cdots \\ &= (\mathbf{I} - \mathbf{V})^{-1} - \mathbf{I} \\ &= \mathbf{N} - \mathbf{I} \end{aligned}$$

hence (3.2) becomes

$$\left[2\left(\mathbf{VN}\right)_{dg} + \mathbf{I}\right]\mathbf{N} = \left[2\mathbf{N}_{dg} - \mathbf{I}\right]$$

and therefore

$$\begin{aligned} \left(\left(var\left(n_{ij}\right)\right)\right) &= E\left[n_{ij}^2\right] - \left[E\left(n_{ij}\right)\right]^2 \\ &= \mathbf{N}\left[2\mathbf{N}_{dg} - \mathbf{I}\right] - \mathbf{N}_{sq} \end{aligned}$$

5.2.6 The fundamental matrix for Markov chain

Consider a Markov chain with transitional probability matrix

Case 1

Consider a 2×2 transitional probability matrix

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_1 \end{array} \begin{array}{c} \begin{array}{cc} E_0 & E_1 \end{array} \\ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \end{array}$$

This is an identity matrix. We observe that $\{E_0\}$ forms an absorbing state and $\{E_1\}$ forms a closed set. E_0 and E_1 are absorbing states. They are trivially persistent, non-null and aperiodic and hence they are ergodic. Next we determine n^{th} power $\mathbf{P}^n$.

$$\mathbf{P}^n = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

The asymptotic behavior

$$\lim_{n\to\infty} \mathbf{P}^n = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

Case 2

Consider a 3×3 transitional probability matrix

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \end{array} \begin{array}{c} \begin{array}{ccc} E_0 & E_1 & E_2 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ q & r & p \\ 0 & 0 & 1 \end{array} \right] \end{array}$$

where

$$\begin{aligned} p+q+r &= 1 \\ 0 &< p<1, \;\; 0<q<1, \;\; 0<r<1, \end{aligned}$$

We observe that $\{E_0, E_2\}$ forms an absorbing state $\{E_1\}$ and forms a closed set. We can rearrange and partition the matrix in the following order

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_2 \\ E_1 \end{array} \begin{array}{c} \begin{array}{ccc} E_0 & E_2 & E_1 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline q & p & r \end{array}\right] \end{array}$$

where

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} q & p \end{bmatrix}, \quad \text{and } \mathbf{V} = r$$

Classification of the states

E_0 and E_2 are trivially

(i) Persistent

(ii) Non null

(iii) Aperiodic

Hence they are Ergodic

Consider E_1

$$\begin{aligned} f_{11}^{(1)} &= \Pr\left[E_1 \to E_1\right] = r \\ f_{11}^{(2)} &= 0 \\ f_{11}^{(3)} &= 0 \end{aligned}$$

In general

$$f_{11}^{(n)} = \left\{ \begin{array}{ll} r, & \text{when } n = 1 \\ 0, & \text{elsewhere} \end{array} \right\}$$

$$\begin{aligned} f_1 &= \sum_{n=1}^{\infty} f_{11}^{(n)} \\ &= r < 1 \end{aligned}$$

hence E_1 is transient.

Case 3

Consider a 4×4 transitional probability matrix

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{cccc} E_0 & E_1 & E_2 & E_3 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ q & r & p & 0 \\ 0 & q & r & p \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

where

$$\begin{aligned} p+q+r &= 1 \\ 0 &< p<1, \;\; 0<q<1, \;\; 0<r<1, \end{aligned}$$

We observe that $\{E_0, E_3\}$ forms an absorbing state $\{E_1, E_2\}$ and forms a closed set. We can rearrange and partition the matrix in the following order

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_3 \\ E_1 \\ E_2 \end{array} \begin{array}{c} \begin{array}{cccc} E_0 & E_3 & E_1 & E_2 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ q & 0 & r & p \\ 0 & p & q & r \end{array}\right] \end{array}$$

where

$$\mathbf{I} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right], \quad \mathbf{U} = \left[\begin{array}{cc} q & 0 \\ 0 & p \end{array}\right], \text{ and } \mathbf{V} = \left[\begin{array}{cc} r & p \\ q & r \end{array}\right]$$

Classification of the states

E_0 and E_3 are trivially

(i) Persistent

(ii) Non null

(iii) Aperiodic

Hence they are Ergodic

Consider E_1

$$\begin{aligned} f_{11}^{(1)} &= \Pr[E_1 \to E_1] = r \\ f_{11}^{(2)} &= \Pr[E_1 \to E_2 \to E_1] = pq \\ f_{11}^{(3)} &= \Pr[E_1 \to E_2 \to E_2 \to E_1] = pqr \\ f_{11}^{(4)} &= \Pr[E_1 \to E_2 \to E_2 \to E_2 \to E_1] = pqr^2 \end{aligned}$$

$$f_{11}^{(n)} = \left\{ \begin{array}{c} r, \quad \text{when } n = 1 \\ pqr^{n-2}, \quad n \geq 2 \end{array} \right\}$$

$$\begin{aligned} f_1 &= \sum_{n=1}^{\infty} f_{11}^{(n)} \\ &= r + pq + pqr + pqr^2 + \cdots \\ &= r + pq + pqr\left[1 + r + r^2 + \cdots\right] \\ &= r + pq + \frac{pqr}{1-r} < 1 \end{aligned}$$

hence E_1 is transient. Now, consider E_2

$$\begin{aligned} f_{22}^{(1)} &= \Pr[E_2 \to E_2] = r \\ f_{22}^{(2)} &= \Pr[E_2 \to E_1 \to E_2] = pq \\ f_{22}^{(3)} &= \Pr[E_2 \to E_1 \to E_1 \to E_2] = pqr \\ f_{22}^{(4)} &= \Pr[E_2 \to E_1 \to E_1 \to E_1 \to E_2] = pqr^2 \end{aligned}$$

In general

$$f_{22}^{(n)} = \left\{ \begin{array}{c} r, \quad \text{when } n = 1 \\ pqr^{n-2}, \quad n \geq 2 \end{array} \right\}$$

$$\begin{aligned} f_2 &= \sum_{n=1}^{\infty} f_{22}^{(n)} \\ &= r + pq + pqr + pqr^2 + \cdots \\ &= r + pq + pqr\left[1 + r + r^2 + \cdots\right] \\ &= r + pq + \frac{pqr}{1-r} < 1 \end{aligned}$$

hence E_2 is transient. In general, the transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ \vdots \\ E_{\rho-1} \\ E_\rho \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & \ldots & \rho-2 & \rho-1 & \rho \end{array} \\ \left[\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ q & r & p & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & q & r & p & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & r & p \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array}\right] \end{array}$$

We observe that $\{E_0, E_\rho\}$ forms an absorbing state $\{E_1, E_2, \ldots, E_{\rho-1}\}$ and forms a closed set. We can rearrange and partition the matrix in the following order

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_\rho \\ E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_{\rho-3} \\ E_{\rho-2} \\ E_{\rho-1} \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 0 & \rho & 1 & 2 & 3 & 4 & \ldots & \rho-4 & \rho-3 & \rho-2 & \rho-1 \end{array} \\ \left[\begin{array}{ccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ q & 0 & r & p & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & q & r & p & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & r & p & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & q & r & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & q & r & p \\ 0 & p & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & q & r \end{array}\right] \end{array}$$

Hence the transitional matrix is in the form

$$P = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{U} & \mathbf{V} \end{bmatrix}$$

where

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{U} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_{\rho-3} \\ E_{\rho-2} \\ E_{\rho-1} \end{array} \begin{array}{c} \rho \quad 0 \\ \left[\begin{array}{cc} q & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \\ 0 & p \end{array}\right] \end{array}$$

and

$$\mathbf{V} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_{\rho-3} \\ E_{\rho-2} \\ E_{\rho-1} \end{array} \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \quad \dots \quad \rho-4 \quad \rho-3 \quad \rho-2 \quad \rho-1 \\ \left[\begin{array}{ccccccccc} r & p & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ q & r & p & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & q & r & p & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & q & r & p & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & q & r & p \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & q & r \end{array}\right] \end{array}$$

and $\mathbf{0}$ is a $2 \times (\rho - 2)$ matrix. Then the fundamental matrix $\mathbf{N}$ is given by

$$\mathbf{N} = (\mathbf{I} - \mathbf{V})^{-1}$$

thus

$$\mathbf{N} = \frac{adj(\mathbf{I} - \mathbf{V})}{|\mathbf{I} - \mathbf{V}|}$$

To solve for $(\mathbf{I} - \mathbf{V})^{-1}$ from $(\mathbf{I} - \mathbf{V})$, consider a 2×2 matrix i.e.

$$\begin{bmatrix} p+q & -p \\ -q & p+q \end{bmatrix}$$

then

$$\begin{aligned} \det(\mathbf{I} - \mathbf{V})_{2\times 2} &= (p+q)^2 - pq \\ &= p^2 + pq + q^2 \\ &= p^2 q^0 + pq + p^0 q^2 \\ &= \sum_{i=0}^{2} p^{2-i} q^i \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned}(\mathbf{I}-\mathbf{V})^{-1} &= \frac{1}{p^2+pq+q^2}\begin{bmatrix} p+q & -p \\ -q & p+q \end{bmatrix} \\ &= \frac{1}{\sum_{i=0}^{2} p^{2-i}q^i}\begin{bmatrix} p+q & -p \\ -q & p+q \end{bmatrix}\end{aligned}$$

For a 3×3 transition probability matrix

$$(\mathbf{I}-\mathbf{V})^{-1} = \begin{bmatrix} p+q & -p & 0 \\ -q & p+q & -p \\ 0 & -q & p+q \end{bmatrix}$$

$\Longrightarrow$

$$\begin{aligned}\det(\mathbf{I}-\mathbf{V})_{3\times3} &= \begin{vmatrix} p+q & -p & 0 \\ -q & p+q & -p \\ 0 & -q & p+q \end{vmatrix} \\ &= (p+q)\begin{vmatrix} p+q & -p \\ -q & p+q \end{vmatrix} + p\begin{vmatrix} -q & -p \\ 0 & p+q \end{vmatrix} \\ &= (p+q)\left[(p+q)^2-pq\right]-pq\,(p+q) \\ &= (p+q)\left[(p+q)^2-pq-pq\right] \\ &= (p+q)\left[p^2+q^2+2pq-2pq\right] \\ &= (p+q)\left[p^2+q^2\right] \\ &= p^3+p^2q+pq^2+q^3 \\ &= \sum_{i=0}^{3} p^{3-i}q^i\end{aligned}$$

$\Longrightarrow$

$$(\mathbf{I}-\mathbf{V})^{-1} = \frac{1}{\sum_{i=0}^{3} p^{3-i}q^i}\begin{bmatrix} (p+q)^2-pq & p\,(p+q) & p^2 \\ q\,(p+q) & (p+q)^2 & p\,(p+q) \\ q^2 & q\,(p+q) & (p+q)^2-pq \end{bmatrix}$$

Considering a 4×4 matrix

$$(\mathbf{I}-\mathbf{V})_{4\times4} = \begin{bmatrix} p+q & -p & 0 & 0 \\ -q & p+q & -p & 0 \\ 0 & -q & p+q & -p \\ 0 & 0 & -q & p+q \end{bmatrix}$$

then

$$\det(\mathbf{I}-\mathbf{V})_{4\times4} = \begin{vmatrix} p+q & -p & 0 & 0 \\ -q & p+q & -p & 0 \\ 0 & -q & p+q & -p \\ 0 & 0 & -q & p+q \end{vmatrix}$$
$$= (p+q)(p+q)\left(p^2+q^2\right)+p\left\{-q\left[(p+q)^2-pq\right]\right\}$$
$$= (p+q)^2\left(p^2+q^2\right)-pq\left[(p+q)^2-pq\right]$$
$$= (p+q)^2\left(p^2+q^2\right)-pq\left[p^2+q^2+pq\right]$$
$$= \left(p^2+pq+q^2\right)\left(p^2+q^2\right)-qp^3-p^2q^2-pq^3$$
$$= p^4+p^3q+p^2q^2+pq^3+q^4$$
$$= \sum_{i=0}^{4} p^{4-i}q^i$$

Therefore

$$(\mathbf{I}-\mathbf{V})^{-1}_{4\times4} = \frac{1}{\sum_{i=0}^{4} p^{4-i}q^i} \times \begin{bmatrix} (p+q)\left(p^2+q^2\right) & p\left[(p+q)^2-pq\right] & p^2(p+q) & p^3 \\ q\left[(p+q)^2-pq\right] & (p+q)\left[(p+q)-pq\right] & p(p+q)^2 & p^2(p+q) \\ q^2(p+q) & q(p+q)^2 & (p+q)^3 & p(p+q)^2 \\ q^3 & q^2(p+q) & q(p+q)^3 & (p+q)\left(p^2+q^2\right) \end{bmatrix}$$

Now

$$(\mathbf{I}-\mathbf{V}) = \begin{bmatrix} 1-r & -p & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -q & 1-r & -p & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -q & 1-r & -p & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -q & 1-r & -p & \ddots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -q & 1-r & -p & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -q & 1-r & -p \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -q & 1-r \end{bmatrix}$$

but

$$p+q+r=1$$

$\Longrightarrow$

$$1-r=p+q$$

hence

$$(\mathbf{I}-\mathbf{V})=\begin{bmatrix} p+q & -p & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -q & p+q & -p & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -q & p+q & -p & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -q & p+q & -p & \ddots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -q & p+q & -p & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -q & p+q & -p \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -q & p+q \end{bmatrix}$$

For general case, the fundamental matrix is goven by $\mathbf{N}=(\mathbf{I}-\mathbf{V})^{-1}=((n_{ij}))$ where

$$n_{ij}=\frac{1}{(p-q)(r^n-1)}\left\{\begin{array}{l} \left(r^j-1\right)\left(r^{n-i}-1\right), \text{ if } j\le i \\ \left(r^i-1\right)\left(r^{n-i}-r^{j-i}\right), \text{if } j\ge i \end{array}\right\}$$

where

$$r=\frac{p}{q}$$

$$\lim_{n\to\infty}\mathbf{P}^n=\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ (\mathbf{I}-\mathbf{V})^{-1}\mathbf{U} & \mathbf{0} \end{bmatrix}$$

$$\begin{aligned} (\mathbf{I}-\mathbf{V})^{-1}\mathbf{U}_{2\times 2} &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r & p \\ q & r \end{bmatrix}\right)^{-1}\begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix} \\ &= \frac{1}{p^2+pq+q^2}\begin{bmatrix} p+q & p \\ q & p+q \end{bmatrix}\begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix} \\ &= \frac{1}{p^2+pq+q^2}\begin{bmatrix} (p+q)\,q & p^2 \\ q^2 & (p+q)\,p \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
(\mathbf{I}-\mathbf{V})^{-1}\mathbf{U}_{3\times 3} &= \frac{1}{\sum_{i=0}^{3} p^{3-i}q^{i}} \\
&\times \begin{bmatrix} (p+q)^2-pq & p(p+q) & p^2 \\ q(p+q) & (p+q)^2 & p(p+q) \\ q^2 & q(p+q) & (p+q)^2-pq \end{bmatrix} \begin{bmatrix} q & 0 \\ 0 & 0 \\ 0 & p \end{bmatrix} \\
&= \frac{1}{\sum_{i=0}^{3} p^{3-i}q^{i}} \begin{bmatrix} \left[(p+q)^2-pq\right]q & p^3 \\ q^2(p+q) & p^2(p+q) \\ q^3 & \left[(p+q)^2-pq\right]p \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{I}-\mathbf{V})^{-1}\mathbf{U} &= \frac{1}{\sum_{i=0}^{4} p^{4-i}q^{i}} \\
&\times \begin{bmatrix} (p+q)\left(p^2+q^2\right) & p\left[(p+q)^2-pq\right] & p^2(p+q) & p^3 \\ q\left[(p+q)^2-pq\right] & (p+q)\left[(p+q)-pq\right] & p(p+q)^2 & p^2(p+q) \\ q^2(p+q) & q(p+q)^2 & (p+q)^3 & p(p+q)^2 \\ q^3 & q^2(p+q) & q(p+q)^3 & (p+q)\left(p^2+q^2\right) \end{bmatrix} \\
&\times \begin{bmatrix} q & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & p \end{bmatrix} \\
&= \frac{1}{\sum_{i=0}^{4} p^{4-i}q^{i}} \begin{bmatrix} q(p+q)\left(p^2+q^2\right) & p^4 \\ q^2\left[(p+q)^2-pq\right] & p^3(p+q) \\ q^3(p+q) & p^2(p+q)^2 \\ q^4 & p(p+q)\left(p^2+q^2\right) \end{bmatrix}
\end{aligned}
$$

5.3 Application of the Fundamental Matrix to Random Walks

Illustrative Eaxample (i)

Consider a typical example from a random walk. Let's take the case of five states E_0 and E_4 being the absorbing states and $\{E_1, E_2, E_3\}$ forms a closed set ("interior state"). Consider $p_{i,i+1} = p$ and $p_{i,i-1} = q$, $i = 0, 1, 2, 3, 4$. Then the transitional probability matrix is given as

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{array}{c} \begin{array}{ccccc} E_0 & E_1 & E_2 & E_3 & E_4 \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

hence the matrix can be rearranged in the following manner

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{array} \begin{array}{c} \begin{array}{ccccc} E_0 & E_1 & E_2 & E_3 & E_4 \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ q & 0 & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & p & 0 & q & 0 \end{array}\right] \end{array}$$

In canonical form

$$\mathbf{P} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{U} & \mathbf{V} \end{bmatrix}$$

where 0 is the zero matrix and

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} q & 0 \\ 0 & 0 \\ 0 & p \end{bmatrix} \text{ and } \mathbf{V} = \begin{bmatrix} 0 & p & 0 \\ q & 0 & p \\ 0 & q & 0 \end{bmatrix}$$

then

$$\begin{aligned} \mathbf{I} - \mathbf{V} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & p & 0 \\ q & 0 & p \\ 0 & q & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -p & 0 \\ -q & 1 & -p \\ 0 & -q & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}|\mathbf{I}-\mathbf{V}| &= \begin{vmatrix} 1 & -p \\ -q & 1 \end{vmatrix} + \begin{vmatrix} -q & -p \\ 0 & 1 \end{vmatrix} \\ &= 1-qp-qp \\ &= 1-2qp\end{aligned}$$

but

$$(p+q)^2 = p^2+q^2+2pq$$

$\Longrightarrow$

$$1-2pq = p^2+q^2$$

since

$$p+q=1$$

hence

$$|\mathbf{I}-\mathbf{V}| = p^2+q^2$$

$$\begin{aligned}\mathbf{N} &= (\mathbf{I}-\mathbf{V})^{-1} \\ &= \frac{1}{p^2+q^2}\begin{bmatrix} p+q^2 & p & p^2 \\ q & 1 & p \\ q^2 & q & p^2+q \end{bmatrix}\end{aligned}$$

$$N = \begin{bmatrix} \frac{p+q^2}{p^2+q^2} & \frac{p}{p^2+q^2} & \frac{p^2}{p^2+q^2} \\ \frac{q}{p^2+q^2} & \frac{1}{p^2+q^2} & \frac{p}{p^2+q^2} \\ \frac{q^2}{p^2+q^2} & \frac{q}{p^2+q^2} & \frac{p^2+q}{p^2+q^2} \end{bmatrix}$$

We see that the process starts from E_2 (the middle state), then it will be in the middle state an average of $\frac{1}{p^2+q^2}$ times. This quantity is always between 1 and 2. The minimum of 1 is achieved if $p=0$ or 1. The maximum of 2 is achieved if $p=\frac{1}{2}$. In the former case the process starts at E_2 and goes directly to one of the boundaries, hence it will be in state E_2 only at the beginning. But even in the case $p=\frac{1}{2}$ we will expect the process to return only once on average.

Illustrative example (ii)

We will give the fundamental matrix for the case $p=\frac{2}{3}$ i.e. when it is twice as likely to move to the right as to the left. Then

$$\mathbf{N} = \begin{bmatrix} \frac{7}{5} & \frac{6}{5} & \frac{4}{5} \\ \frac{3}{5} & \frac{9}{5} & \frac{6}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{7}{5} \end{bmatrix}$$

from theorem 2.4, we have

$$((var(n_{ij}))) = \mathbf{N}\left[2\mathbf{N}_{dg} - \mathbf{I}\right] - \mathbf{N}_{sq}$$

where

$\mathbf{N}_{dg}$ is the matrix that results from $\mathbf{N}$ by setting off diagonal entries equal to zero

$\mathbf{N}_{sq}$ is the matrix that results from $\mathbf{N}$ by squaring each entry and

$\mathbf{N} = (\mathbf{I} - \mathbf{V})^{-1}$ is the fundamental matrix.

Therefore

$$\mathbf{N}_{sq} = \begin{bmatrix} \frac{49}{25} & \frac{36}{25} & \frac{16}{25} \\ \frac{9}{25} & \frac{81}{25} & \frac{36}{25} \\ \frac{1}{25} & \frac{9}{25} & \frac{49}{25} \end{bmatrix}, \quad \mathbf{N}_{dg} = \begin{bmatrix} \frac{7}{5} & 0 & 0 \\ 0 & \frac{9}{5} & 0 \\ 0 & 0 & \frac{7}{5} \end{bmatrix}$$

$$\begin{aligned} 2\mathbf{N}_{dg} - \mathbf{I} &= 2\begin{bmatrix} \frac{7}{5} & 0 & 0 \\ 0 & \frac{9}{5} & 0 \\ 0 & 0 & \frac{7}{5} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{9}{5} & 0 & 0 \\ 0 & \frac{13}{5} & 0 \\ 0 & 0 & \frac{9}{5} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{N}\left[2\mathbf{N}_{dg} - \mathbf{I}\right] &= \begin{bmatrix} \frac{49}{25} & \frac{36}{25} & \frac{16}{25} \\ \frac{9}{25} & \frac{81}{25} & \frac{36}{25} \\ \frac{1}{25} & \frac{9}{25} & \frac{49}{25} \end{bmatrix} \begin{bmatrix} \frac{9}{5} & 0 & 0 \\ 0 & \frac{13}{5} & 0 \\ 0 & 0 & \frac{9}{5} \end{bmatrix} \\ &= \begin{bmatrix} \frac{63}{25} & \frac{78}{25} & \frac{36}{25} \\ \frac{27}{25} & \frac{117}{25} & \frac{54}{25} \\ \frac{9}{25} & \frac{39}{25} & \frac{63}{25} \end{bmatrix} \end{aligned}$$

Therefore

$$\begin{aligned}((var(n_{ij}))) &= \mathbf{N}\left[2\mathbf{N}_{dg}-\mathbf{I}\right]-\mathbf{N}_{sq}\\ &= \begin{bmatrix}\frac{63}{25} & \frac{78}{25} & \frac{36}{25}\\ \frac{27}{25} & \frac{117}{25} & \frac{54}{25}\\ \frac{9}{25} & \frac{39}{25} & \frac{63}{25}\end{bmatrix} - \begin{bmatrix}\frac{49}{25} & \frac{36}{25} & \frac{16}{25}\\ \frac{9}{25} & \frac{81}{25} & \frac{36}{25}\\ \frac{1}{25} & \frac{9}{25} & \frac{49}{25}\end{bmatrix}\\ &= \begin{bmatrix}\frac{14}{25} & \frac{42}{25} & \frac{20}{25}\\ \frac{18}{25} & \frac{36}{25} & \frac{18}{25}\\ \frac{18}{25} & \frac{30}{25} & \frac{14}{25}\end{bmatrix}\end{aligned}$$

i.e

$$((var(n_{ij}))) = \begin{matrix} & E_1 & E_2 & E_3\\ E_1 & \frac{14}{25} & \frac{42}{25} & \frac{20}{25}\\ E_2 & \frac{18}{25} & \frac{36}{25} & \frac{18}{25}\\ E_3 & \frac{18}{25} & \frac{30}{25} & \frac{14}{25}\end{matrix}$$

Thus we see that for any state as an initial state, the variance is largest for the middle state. We also note that $((var(n_{ij})))$ is quite large compared to $\mathbf{N}_{sq}$. Hence the means are fairly unreliable estimates for this Markov chain.

5.4 Random Walks with Reflecting Barriers

5.4.1 Random walks with one reflecting barrier

Consider a random walk matrix with

$$\begin{aligned}p_{j,j+1} &= p; p_{j,j-1} = q \quad \text{for } j = 2,3,4\ldots \quad \text{and}\\ p_{12} &= p; p_{11} = q\end{aligned}$$

Considering a 3×3 transition probability matrix

$$\mathbf{P} = \begin{matrix} & E_0 & E_1 & E_2\\ E_0 & q & p & 0\\ E_1 & q & 0 & p\\ E_2 & 0 & q & p\end{matrix}$$

Classification of the Markov chain

Every state can be reached from every other state hence the Markov Chain is irreducible. Thus all the states are of the same type.

Classification of the states

Every state can be reached from every other state hence the Markov Chain is irreducible. Thus all the states are of the same type.

Consider E_0

$$
\begin{aligned}
f_{00}^{(1)} &= \Pr[E_0 \to E_0] = q \\
f_{00}^{(2)} &= \Pr[E_0 \to E_1 \to E_0] = pq \\
f_{00}^{(3)} &= 0 \\
f_{00}^{(4)} &= \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_0] = p^2q^2 \\
f_{00}^{(5)} &= \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_1 \to E_0] = p^3q^2 \\
f_{00}^{(6)} &= \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_2 \to E_1 \to E_0] \\
&= p^4q^2 + p^3q^3 \\
f_{00}^{(7)} &= \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_2 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_2 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_1 \to E_2 \to E_1 \to E_0] \\
&= p^5q^2 + 2p^4q^3 \\
f_{00}^{(8)} &= \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_2 \to E_2 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_1 \to E_2 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_2 \to E_2 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_2 \to E_1 \to E_2 \to E_1 \to E_0]
\end{aligned}
$$

$$
\begin{aligned}
f_0 &= \sum_{n=0}^{\infty} q + pq + p^2q^2 + p^3p^2 + p^4q^3 + p^5q^2 + 2p^4q^3 \\
&\quad + 4p^5q^3 + p^6q^2 + 5p^6q^3 + p^4q^4 + p^5q^4 + p^7q^2
\end{aligned}
$$

Asymptotic behavior

$$\mathbf{P}^n = \mathbf{P}'\pi$$

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} q & p & 0 \\ q & 0 & p \\ 0 & q & p \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{bmatrix}$$

$$\begin{aligned} \pi_0 &= q\pi_0 + q\pi_1 && \text{(i)} \\ \pi_1 &= p\pi_0 + q\pi_2 && \text{(ii)} \\ \pi_2 &= p\pi_1 + p\pi_2 && \text{(iii)} \\ \pi_0 + \pi_1 + \pi_2 &= 1 && \text{(iv)} \end{aligned}$$

$$\begin{aligned} \pi_0 &= (1-p)\,\pi_0 + q\pi_1 \\ \pi_1 &= \left(\frac{p}{1-p}\right)\pi_0 \end{aligned}$$

$$\begin{aligned} \left(\frac{p}{1-p}\right)\pi_0 &= p\pi_0 + (1-p)\,\pi_2 \\ \pi_2 &= \left(\frac{p}{1-p}\right)^2 \pi_0 \end{aligned}$$

Deriving π_0 in terms of p only

$$\sum_{k=0}^{\infty} \pi_k = 1$$

$$\left[1 + \left(\frac{p}{1-p}\right) + \left(\frac{p}{1-p}\right)^2\right]\pi_0 = 1$$

let

$$x = \frac{p}{1-p}$$

$$\begin{aligned} \left[1 + x + x^2\right]\pi_0 &= 1 \\ \left(\frac{1-x^3}{1-x}\right)\pi_0 &= 1 \\ \pi_0 &= \left(\frac{1-x}{1-x^3}\right) \end{aligned}$$

$$\begin{aligned}
\pi_0 &= \frac{(1-p)^2(1-2p)}{(1-p)^3-p^3} \\
\pi_1 &= \left(\frac{p}{1-p}\right)\frac{(1-p)^2(1-2p)}{(1-p)^3-p^3} \\
&= \frac{p(1-p)(1-2p)}{(1-p)^3-p^3} \\
\pi_2 &= \left(\frac{p}{1-p}\right)^2\frac{(1-p)^2(1-2p)}{(1-p)^3-p^3} \\
&= \frac{p^2(1-2p)}{(1-p)^3-p^3}
\end{aligned}$$

The n^{th} power $\mathbf{P}^n$

Determining the eigen values

$$\mathbf{P} = \begin{bmatrix} q & p & 0 \\ q & 0 & p \\ 0 & q & p \end{bmatrix}$$

$$|\mathbf{P}-\lambda\mathbf{I}| = \mathbf{0}$$

$$\begin{bmatrix} q & p & 0 \\ q & 0 & p \\ 0 & q & p \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \mathbf{0}$$

$$\begin{vmatrix} q-\lambda & p & 0 \\ q & -\lambda & p \\ 0 & q & p-\lambda \end{vmatrix} = 0$$

$$\begin{aligned}
(q-\lambda)\left[-\lambda(p-\lambda)-pq\right]-p\left[q(p-\lambda)\right] &= 0 \\
(q-\lambda)\left[-\lambda p+\lambda^2-pq\right]-p\left[qp-q\lambda\right] &= 0 \\
(q-\lambda)\left[-\lambda p+\lambda^2-pq\right]-qp^2+pq\lambda &= 0
\end{aligned}$$

$$
\begin{aligned}
q\left[-\lambda p+\lambda^2-pq\right]-\lambda\left[-\lambda p+\lambda^2-pq\right]-qp^2+pq\lambda &= 0\\
-\lambda pq+q\lambda^2-pq^2+\lambda^2 p-\lambda^3+\lambda pq-qp^2+pq\lambda &= 0\\
-\lambda^3+q\lambda^2+\lambda^2 p+\lambda pq-\lambda pq+pq\lambda-pq^2-qp^2 &= 0\\
\lambda^3-q\lambda^2-\lambda^2 p+pq^2+qp^2-pq\lambda &= 0\\
\lambda^2(\lambda-q-p)+pq(q+p-\lambda) &= 0\\
\lambda^2(\lambda-q-p)-pq(\lambda-q-p) &= 0\\
\left(\lambda^2-pq\right)(\lambda-q-p) &= 0
\end{aligned}
$$

$\Longrightarrow$

$$\lambda^2=pq$$

$$\lambda=\pm\sqrt{pq}$$

or

$$
\begin{aligned}
\lambda &= q+p\\
&= 1
\end{aligned}
$$

Determining the corresponding eigen vectors. For $\lambda=1$

$$\mathbf{Px}=\lambda\mathbf{x}$$

$$
\begin{bmatrix} q & p & 0\\ q & 0 & p\\ 0 & q & p \end{bmatrix}
\begin{bmatrix} x_1\\ x_2\\ x_3 \end{bmatrix}
=1\begin{bmatrix} x_1\\ x_2\\ x_3 \end{bmatrix}
$$

$$
\begin{aligned}
qx_1+px_2 &= x_1 && \text{(i)}\\
qx_1+px_3 &= x_2 && \text{(ii)}\\
qx_2+px_3 &= x_3 && \text{(iii)}
\end{aligned}
$$

from (i)

$$
\begin{aligned}
qx_2 &= x_1-px_1\\
&= (1-p)x_1\\
&= qx_1\\
x_1 &= x_2
\end{aligned}
$$

from (iii)

$$
\begin{aligned}
qx_2 &= x_3-px_3\\
&= (1-p)x_3\\
&= qx_3\\
x_1 &= x_3
\end{aligned}
$$

$\Longrightarrow$

$$x_1 = x_2 = x_3$$

Let $x_1 = x_2 = x_3 = 1$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

when $\lambda = +\sqrt{pq}$

$$\begin{bmatrix} q & p & 0 \\ q & 0 & p \\ 0 & q & p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = +\sqrt{pq} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned} qx_1 + px_2 &= +\sqrt{pq}x_1 && \text{(i)} \\ qx_1 + px_3 &= +\sqrt{pq}x_2 && \text{(ii)} \\ qx_2 + px_3 &= +\sqrt{pq}x_3 && \text{(iii)} \end{aligned}$$

from (i)

$$\begin{aligned} px_2 &= \sqrt{pq}x_1 - qx_1 \\ &= (\sqrt{pq} - q)\,x_1 \\ x_2 &= \frac{\sqrt{pq} - q}{p}x_1 \end{aligned}$$

using (iii)

$$\begin{aligned} qx_1 + px_3 &= \frac{\sqrt{pq} - q}{p}x_1 \\ px_3 &= \frac{\sqrt{pq} - q}{p}x_1 - qx_1 \\ &= \left[(\sqrt{pq})\,\frac{\sqrt{pq} - q}{p} - q\right]x_1 \\ x_3 &= \left(\frac{\sqrt{pq} - q - pq}{p^2}\right)x_1 \end{aligned}$$

$\Longrightarrow$ let $x_3 = 1$, say then

$$v_2 = \begin{bmatrix} 1 \\ \frac{\sqrt{pq}-q}{p} \\ \frac{\sqrt{pq}-q-pq}{p^2} \end{bmatrix}$$

when $\lambda = -\sqrt{pq}$

$$\begin{bmatrix} q & p & 0 \\ q & 0 & p \\ 0 & q & p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\sqrt{pq} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned} qx_1 + px_2 &= -\sqrt{pq}x_1 && \text{(i)} \\ qx_1 + px_3 &= -\sqrt{pq}x_2 && \text{(ii)} \\ qx_2 + px_3 &= -\sqrt{pq}x_3 && \text{(iii)} \end{aligned}$$

from (i)

$$\begin{aligned} px_2 &= -\sqrt{pq}x_1 - qx_1 \\ &= (-\sqrt{pq} - q)\,x_1 \\ x_2 &= \frac{-\sqrt{pq} - q}{p}x_1 \end{aligned}$$

$$\begin{aligned} \frac{-\sqrt{pq} - q}{p}x_1 &= qx_1 + px_3 \\ x_3 &= \left(\frac{-\sqrt{pq} - q}{p^2} - p \right) x_1 \\ &= \left(\frac{-\sqrt{pq} - q - pq}{p^2} \right) x_1 \end{aligned}$$

let $x_1 = 1$, say then

$$v_3 = \begin{bmatrix} 1 \\ \frac{-\sqrt{pq}-q}{p} \\ \frac{-\sqrt{pq}-q-pq}{p^2} \end{bmatrix}$$

hence

$$V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{\sqrt{pq}-q}{p} & \frac{-\sqrt{pq}-q}{p} \\ 1 & \frac{\sqrt{pq}-q-pq}{p^2} & \frac{-\sqrt{pq}-q-pq}{p^2} \end{bmatrix}$$

by spectral decomposition

$$\mathbf{P}^n = \mathbf{V}\mathbf{D}^n\mathbf{V}^{-1}$$

$$\mathbf{P}^n = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{\sqrt{pq}-q}{p} & \frac{-\sqrt{pq}-q}{p} \\ 1 & \frac{\sqrt{pq}-q-pq}{p^2} & \frac{-\sqrt{pq}-q-pq}{p^2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (pq)^{\frac{n}{2}} & 0 \\ 0 & 0 & (-pq)^{\frac{n}{2}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{\sqrt{pq}-q}{p} & \frac{-\sqrt{pq}-q}{p} \\ 1 & \frac{\sqrt{pq}-q-pq}{p^2} & \frac{-\sqrt{pq}-q-pq}{p^2} \end{bmatrix}^{-1}$$

$$
\begin{aligned}
|\mathbf{V}| &= 1\begin{vmatrix} \frac{\sqrt{pq}-q}{p} & \frac{-\sqrt{pq}-q}{p} \\ \frac{\sqrt{pq}-q-pq}{p^2} & \frac{-\sqrt{pq}-q-pq}{p^2} \end{vmatrix} - 1\begin{vmatrix} 1 & \frac{-\sqrt{pq}-q}{p} \\ 1 & \frac{-\sqrt{pq}-q-pq}{p^2} \end{vmatrix} + 1\begin{vmatrix} 1 & \frac{\sqrt{pq}-q}{p} \\ 1 & \frac{\sqrt{pq}-q-pq}{p^2} \end{vmatrix} \\
&= \left(\frac{-2pq\sqrt{pq}}{p^3}\right) - \left(\frac{-q\sqrt{pq}-q}{p^2}\right) + \left(\frac{q\sqrt{pq}-q}{p^2}\right) \\
&= \frac{2pq}{p^3} \\
&= \frac{2q}{p^2}
\end{aligned}
$$

Minor matrices

$$
\begin{array}{ccc}
\begin{vmatrix} \frac{\sqrt{pq}-q}{p} & \frac{-\sqrt{pq}-q}{p} \\ \frac{\sqrt{pq}-q-pq}{p^2} & \frac{-\sqrt{pq}-q-pq}{p^2} \end{vmatrix} & \begin{vmatrix} 1 & \frac{-\sqrt{pq}-q}{p} \\ 1 & \frac{-\sqrt{pq}-q-pq}{p^2} \end{vmatrix} & \begin{vmatrix} 1 & \frac{\sqrt{pq}-q}{p} \\ 1 & \frac{\sqrt{pq}-q-pq}{p^2} \end{vmatrix} \\
\begin{vmatrix} 1 & 1 \\ \frac{\sqrt{pq}-q-pq}{p^2} & \frac{-\sqrt{pq}-q-pq}{p^2} \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & \frac{-\sqrt{pq}-q-pq}{p^2} \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & \frac{\sqrt{pq}-q-pq}{p^2} \end{vmatrix} \\
\begin{vmatrix} 1 & 1 \\ \frac{\sqrt{pq}-q}{p} & \frac{-\sqrt{pq}-q}{p} \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & \frac{-\sqrt{pq}-q}{p} \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & \frac{\sqrt{pq}-q}{p} \end{vmatrix}
\end{array}
$$

$$
= \begin{bmatrix} \frac{-2pq\sqrt{pq}}{p^3} & \frac{-2\sqrt{pq}}{p^2} & \frac{-2\sqrt{pq}}{p} \\ \frac{-q\sqrt{pq}-q}{p^2} & \frac{-\sqrt{pq}-1}{p^2} & \frac{-\sqrt{pq}-1}{p} \\ \frac{q\sqrt{pq}-q}{p^2} & \frac{\sqrt{pq}-1}{p^2} & \frac{\sqrt{pq}-1}{p} \end{bmatrix}
$$

$$
\begin{aligned}
\mathbf{P}^n &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{\sqrt{pq}-q}{p} & \frac{-\sqrt{pq}-q}{p} \\ 1 & \frac{\sqrt{pq}-q-pq}{p^2} & \frac{-\sqrt{pq}-q-pq}{p^2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (pq)^{\frac{n}{2}} & 0 \\ 0 & 0 & (-pq)^{\frac{n}{2}} \end{bmatrix} \begin{bmatrix} \frac{-2pq\sqrt{pq}}{p^3} & \frac{-2\sqrt{pq}}{p^2} & \frac{-2\sqrt{pq}}{p} \\ \frac{-q\sqrt{pq}-q}{p^2} & \frac{-\sqrt{pq}-1}{p^2} & \frac{-\sqrt{pq}-1}{p} \\ \frac{q\sqrt{pq}-q}{p^2} & \frac{\sqrt{pq}-1}{p^2} & \frac{\sqrt{pq}-1}{p} \end{bmatrix} \\
&= \begin{bmatrix} 1 & (pq)^{\frac{n}{2}} & (-pq)^{\frac{n}{2}} \\ 1 & \frac{\sqrt{pq}-q}{p}(pq)^{\frac{n}{2}} & \frac{-\sqrt{pq}-q}{p}(-pq)^{\frac{n}{2}} \\ 1 & \frac{\sqrt{pq}-q-pq}{p^2}(pq)^{\frac{n}{2}} & \frac{-\sqrt{pq}-q-pq}{p^2}(-pq)^{\frac{n}{2}} \end{bmatrix} \begin{bmatrix} \frac{-2pq\sqrt{pq}}{p^3} & \frac{-2\sqrt{pq}}{p^2} & \frac{-2\sqrt{pq}}{p} \\ \frac{-q\sqrt{pq}-q}{p^2} & \frac{-\sqrt{pq}-1}{p^2} & \frac{-\sqrt{pq}-1}{p} \\ \frac{q\sqrt{pq}-q}{p^2} & \frac{\sqrt{pq}-1}{p^2} & \frac{\sqrt{pq}-1}{p} \end{bmatrix}
\end{aligned}
$$

Consider a 4×4 transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} q & p & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & q & p \end{array}\right] \end{array}$$

Classification of the Markov chain

Every state can be reached from every other state hence the Markov Chain is irreducible. Thus all the states are of the same type.

Classification of the states

$$\begin{aligned}
f_{00}^{(1)} &= \Pr[E_0 \to E_0] = q \\
f_{00}^{(2)} &= \Pr[E_0 \to E_1 \to E_0] = pq \\
f_{00}^{(3)} &= 0 \\
f_{00}^{(4)} &= \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_0] = p^2q^2 \\
f_{00}^{(5)} &= \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_1 \to E_0] = p^3q^2 \\
f_{00}^{(6)} &= \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_2 \to E_1 \to E_0] \\
&= p^4q^2 + p^3q^3 \\
f_{00}^{(7)} &= \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_2 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_2 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_1 \to E_2 \to E_1 \to E_0] \\
&= p^5q^2 + 2p^4q^3 \\
f_{00}^{(8)} &= \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_2 \to E_2 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_1 \to E_2 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_2 \to E_2 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_2 \to E_2 \to E_1 \to E_2 \to E_1 \to E_0]
\end{aligned}$$

Asymptotic behavior

$$\lim_{n\to\infty}\mathbf{P}^n = \mathbf{P}'\pi$$

$$\begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \pi_3\end{bmatrix} = \begin{bmatrix} q & q & 0 & 0\\ p & 0 & q & 0\\ 0 & p & 0 & q\\ 0 & 0 & p & p\end{bmatrix}\begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \pi_3\end{bmatrix}$$

$$\begin{aligned}
\pi_0 &= q\pi_0 + q\pi_1 && \text{(i)}\\
\pi_1 &= p\pi_0 + q\pi_2 && \text{(ii)}\\
\pi_2 &= p\pi_1 + q\pi_3 && \text{(iii)}\\
\pi_3 &= p\pi_2 + p\pi_3 && \text{(iv)}\\
\pi_0+\pi_1+\pi_2+\pi_3 &= 1 && \text{(v)}
\end{aligned}$$

$$\begin{aligned}
\pi_0 &= (1-p)\,\pi_0 + (1-p)\,\pi_1\\
\pi_1 &= \left(\frac{p}{1-p}\right)\pi_0
\end{aligned}$$

$$\begin{aligned}
\pi_1 &= p\pi_0 + q\pi_2\\
\left(\frac{p}{1-p}\right)\pi_0 &= p\pi_0 + q\pi_2\\
\pi_2 &= \left(\frac{p}{1-p}\right)^2\pi_0
\end{aligned}$$

$$\begin{aligned}
\pi_3 &= \left(\frac{p}{1-p}\right)\left(\frac{p}{1-p}\right)^2\pi_0\\
&= \left(\frac{p}{1-p}\right)^3\pi_0
\end{aligned}$$

Deriving π_0 in terms of p only

$$\sum_{k=0}^{\infty}\pi_k = 1$$

$$\left[1+\left(\frac{p}{1-p}\right)+\left(\frac{p}{1-p}\right)^2+\left(\frac{p}{1-p}\right)^3\right]\pi_0 = 1$$

let

$$x = \frac{p}{1-p}$$

$$\begin{aligned}\left[1+x+x^2+x^3\right]\pi_0 &= 1\\ \left(\frac{1-x^4}{1-x}\right)\pi_0 &= 1\\ \pi_0 &= \left(\frac{1-x}{1-x^4}\right)\end{aligned}$$

$$\begin{aligned}\pi_0 &= \frac{(1-p)^3(1-2p)}{(1-p)^4-p^4}\\ \pi_1 &= \left(\frac{p}{1-p}\right)\frac{(1-p)^3(1-2p)}{(1-p)^4-p^4}\\ &= \frac{p(1-p)^2(1-2p)}{(1-p)^4-p^4}\\ \pi_2 &= \left(\frac{p}{1-p}\right)^2\frac{(1-p)^3(1-2p)}{(1-p)^4-p^4}\\ &= \frac{p^2(1-p)(1-2p)}{(1-p)^4-p^4}\\ \pi_3 &= \left(\frac{p}{1-p}\right)^2\frac{(1-p)^3(1-2p)}{(1-p)^4-p^4}\\ &= \frac{p^3(1-2p)}{(1-p)^4-p^4}\end{aligned}$$

In general

$$\pi_n = \frac{p^{n-1}(1-2p)}{(1-p)^n-p^n}$$

The n^{th} power

$$\mathbf{A}(\lambda) = \mathbf{P}-\lambda\mathbf{I}$$

$$\begin{bmatrix} q & p & 0 & 0\\ q & 0 & p & 0\\ 0 & q & 0 & p\\ 0 & 0 & q & p\end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0\\ q & \lambda & 0 & 0\\ 0 & q & \lambda & 0\\ 0 & 0 & q & \lambda\end{bmatrix} = \begin{bmatrix} -\lambda+q & p & 0 & 0\\ q & -\lambda & p & 0\\ 0 & q & -\lambda & p\\ 0 & 0 & q & -\lambda+p\end{bmatrix}$$

We determine the corresponding eigen values by solving the charactersitic function

$$|\mathbf{P}-\lambda\mathbf{I}| = \mathbf{0}$$

$$\begin{vmatrix} -\lambda+q & p & 0 & 0\\ q & -\lambda & p & 0\\ 0 & q & -\lambda & p\\ 0 & 0 & q & -\lambda+p\end{vmatrix} = \mathbf{0}$$

$$(-\lambda+p+q)\begin{vmatrix} 1-p & -p & 0 & 0 \\ 1 & \lambda & -p & 0 \\ 1 & q & \lambda & -p \\ 1 & 0 & q & \lambda-p \end{vmatrix}=0$$

$$(-\lambda+p+q)=0$$

$$\lambda=1$$

$$\begin{vmatrix} 1-p & -p & 0 & 0 \\ q & \lambda & -p & 0 \\ 0 & q & \lambda & -p \\ 0 & 0 & q & \lambda-p \end{vmatrix}=0$$

$$\left[1\begin{vmatrix} -\lambda & p & 0 \\ q & -\lambda & p \\ 0 & q & p-\lambda \end{vmatrix} - p\begin{vmatrix} 1 & p & 0 \\ 1 & -\lambda & p \\ 1 & q & p-\lambda \end{vmatrix}\right]=0$$

$$-\lambda\left[\begin{vmatrix} -\lambda & p \\ q & p-\lambda \end{vmatrix} - p\begin{vmatrix} q & p \\ 0 & p-\lambda \end{vmatrix}\right] - p\left[1\begin{vmatrix} -\lambda & p \\ q & p-\lambda \end{vmatrix} - p\begin{vmatrix} 1 & p \\ 1 & p-\lambda \end{vmatrix}\right]=0$$

$$\begin{aligned} -\lambda\left[-\lambda p+\lambda^2-pq\right]-p\left[-\lambda p+\lambda^2-pq\right]-p\left[p-\lambda-p\right] &= 0 \\ -\lambda^3+2\lambda pq &= 0 \\ \lambda(-\lambda^2+2\lambda pq) &= 0 \end{aligned}$$

hence

$$\lambda=0 \text{ or } \lambda=\sqrt{2pq} \text{ or } \lambda=-\sqrt{2pq}$$

Therefore

$$\lambda_1=1,\ \lambda_2=0 \text{ or } \lambda_3=\sqrt{2pq} \text{ or } \lambda_4=-\sqrt{2pq}$$

when $\lambda = 1$

$$\begin{aligned}
|\mathbf{A}(\lambda)| &= |\mathbf{P} - \lambda \mathbf{I}| \\
&= \begin{vmatrix} \lambda - q & -p & 0 & 0 \\ -q & \lambda & -p & 0 \\ 0 & -q & \lambda & -p \\ 0 & 0 & -q & \lambda - p \end{vmatrix} \\
&= \begin{vmatrix} 1 - q & -p & 0 & 0 \\ -q & 1 & -p & 0 \\ 0 & -q & 1 & -p \\ 0 & 0 & -q & 1 - p \end{vmatrix} \\
&= \begin{vmatrix} p & -p & 0 & 0 \\ -q & 1 & -p & 0 \\ 0 & -q & 1 & -p \\ 0 & 0 & -q & q \end{vmatrix}
\end{aligned}$$

$$\begin{vmatrix} 1 & -p & 0 \\ -q & 1 & -p \\ 0 & -q & q \end{vmatrix} \begin{vmatrix} -q & -p & 0 \\ 0 & 1 & -p \\ 0 & -q & q \end{vmatrix} \begin{vmatrix} -q & 1 & 0 \\ 0 & -q & -p \\ 0 & 0 & q \end{vmatrix} \begin{vmatrix} -q & 1 & -p \\ 0 & -q & 1 \\ 0 & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & 0 \\ -q & 1 & -p \\ 0 & -q & q \end{vmatrix} \begin{vmatrix} p & 0 & 0 \\ 0 & 1 & -p \\ 0 & -q & q \end{vmatrix} \begin{vmatrix} p & -p & 0 \\ 0 & -q & -p \\ 0 & 0 & q \end{vmatrix} \begin{vmatrix} p & -p & 0 \\ 0 & -q & 1 \\ 0 & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & 0 \\ 1 & -p & 0 \\ 0 & -q & q \end{vmatrix} \begin{vmatrix} p & 0 & 0 \\ -q & -p & 0 \\ 0 & -q & q \end{vmatrix} \begin{vmatrix} p & -p & 0 \\ -q & 1 & 0 \\ 0 & 0 & q \end{vmatrix} \begin{vmatrix} p & -p & 0 \\ -q & 1 & -p \\ 0 & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & 0 \\ 1 & -p & 0 \\ -q & 1 & -p \end{vmatrix} \begin{vmatrix} p & 0 & 0 \\ -q & -p & 0 \\ 0 & 1 & -p \end{vmatrix} \begin{vmatrix} p & -p & 0 \\ -q & 1 & 0 \\ 0 & -q & -p \end{vmatrix} \begin{vmatrix} p & -p & 0 \\ -q & 1 & -p \\ 0 & -q & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & -p & 0 \\ -q & 1 & -p \\ 0 & -q & q \end{vmatrix} = 1 \begin{vmatrix} 1 & -p \\ -q & q \end{vmatrix} + p \begin{vmatrix} -q & -p \\ 0 & q \end{vmatrix} = q(1-p) + p(-q^2) = q^3$$

$$\begin{vmatrix} -q & -p & 0 \\ 0 & 1 & -p \\ 0 & -q & q \end{vmatrix} = -q\begin{vmatrix} 1 & -p \\ -q & q \end{vmatrix} + p\begin{vmatrix} 0 & -p \\ 0 & q \end{vmatrix} = -q(q-pq) = -q^3$$

$$\begin{vmatrix} -q & 1 & 0 \\ 0 & -q & -p \\ 0 & 0 & q \end{vmatrix} = -q\begin{vmatrix} -q & -p \\ 0 & q \end{vmatrix} - \begin{vmatrix} -q & -p \\ 0 & q \end{vmatrix} = -q(-q^2) = q^3$$

$$\begin{vmatrix} -q & 1 & -p \\ 0 & -q & 1 \\ 0 & 0 & -q \end{vmatrix} = -q\begin{vmatrix} -q & -p \\ 0 & q \end{vmatrix} - \begin{vmatrix} 0 & -p \\ 0 & q \end{vmatrix} = -q(q^2) = -q^3$$

$$\begin{vmatrix} -p & 0 & 0 \\ -q & 1 & -p \\ 0 & -q & q \end{vmatrix} = -p\begin{vmatrix} 1 & -p \\ -q & q \end{vmatrix} - p(q-pq) = -pq^2$$

$$\begin{vmatrix} p & 0 & 0 \\ 0 & 1 & -p \\ 0 & -q & q \end{vmatrix} = p\begin{vmatrix} 1 & -p \\ -q & q \end{vmatrix} = p(q-pq) = -pq^2$$

$$\begin{vmatrix} p & -p & 0 \\ 0 & -q & -p \\ 0 & 0 & q \end{vmatrix} = p\begin{vmatrix} -q & -p \\ 0 & q \end{vmatrix} + p\begin{vmatrix} 0 & -p \\ 0 & q \end{vmatrix} = -pq^2$$

$$\begin{vmatrix} p & -p & 0 \\ 0 & -q & 1 \\ 0 & 0 & -q \end{vmatrix} = p\begin{vmatrix} -q & 1 \\ 0 & -q \end{vmatrix} + p\begin{vmatrix} 0 & 1 \\ 0 & -q \end{vmatrix} = pq^2$$

$$\begin{vmatrix} -p & 0 & 0 \\ 1 & -p & 0 \\ 0 & -q & q \end{vmatrix} = -p\begin{vmatrix} -p & 0 \\ -q & q \end{vmatrix} = p^2q$$

$$\begin{vmatrix} p & 0 & 0 \\ -q & -p & 0 \\ 0 & -q & q \end{vmatrix} = p\begin{vmatrix} -p & 0 \\ -q & q \end{vmatrix} = -p^2q$$

$$\begin{vmatrix} p & -p & 0 \\ -q & 1 & 0 \\ 0 & 0 & q \end{vmatrix} = p\begin{vmatrix} 1 & 0 \\ 0 & q \end{vmatrix} + p\begin{vmatrix} -q & 0 \\ 0 & q \end{vmatrix} = pq - pq^2 = p^2q$$

$$\begin{vmatrix} p & -p & 0 \\ -q & 1 & -p \\ 0 & 0 & -q \end{vmatrix} = p\begin{vmatrix} 1 & -p \\ 0 & -q \end{vmatrix} + p\begin{vmatrix} -q & -p \\ 0 & -q \end{vmatrix} = -pq + pq^2 = -p^2q$$

$$\begin{vmatrix} -p & 0 & 0 \\ 1 & -p & 0 \\ -q & 1 & -p \end{vmatrix} = -p\begin{vmatrix} -p & 0 \\ 1 & -p \end{vmatrix} = -p^3$$

$$\begin{vmatrix} p & 0 & 0 \\ -q & -p & 0 \\ 0 & 1 & -p \end{vmatrix} = p\begin{vmatrix} -p & 0 \\ 1 & -p \end{vmatrix} = p^3$$

$$\begin{vmatrix} p & -p & 0 \\ -q & 1 & 0 \\ 0 & -q & -p \end{vmatrix} = p\begin{vmatrix} 1 & 0 \\ -q & -p \end{vmatrix} + p\begin{vmatrix} -q & 0 \\ 0 & -p \end{vmatrix} = -p^2 + p^2q = -p^3$$

$$\begin{aligned} \begin{vmatrix} p & -p & 0 \\ -q & 1 & -p \\ 0 & -q & 1 \end{vmatrix} &= p\begin{vmatrix} 1 & -p \\ -q & 1 \end{vmatrix} + p\begin{vmatrix} -q & -p \\ 0 & 1 \end{vmatrix} = p(1-qp) - pq = p - p^2q - pq \\ &= p - pq - p^2q = p(1-q) - p^2q = p^2 - p^2q \\ &= p^2(1-q) = p^3 \end{aligned}$$

$$|\mathbf{A}(\lambda)| = |\mathbf{P} - \lambda\mathbf{I}| = \begin{bmatrix} q^3 & -q^3 & q^3 & -q^3 \\ -pq^2 & pq^2 & -pq^2 & pq^2 \\ p^2q & -p^2q & p^2q & -p^2q \\ -p^3 & p^3 & -p^3 & p^3 \end{bmatrix}$$

adjoint

$$= \begin{bmatrix} q^3 & -pq^2 & p^2q & -p^3 \\ -q^3 & pq^2 & -p^2q & p^3 \\ q^3 & -pq^2 & p^2q & -p^3 \\ -q^3 & pq^2 & -p^2q & p^3 \end{bmatrix}$$

when $\lambda = \sqrt{2pq}$

$$\begin{aligned}|\mathbf{A}(\lambda)| &= |\mathbf{P}-\lambda\mathbf{I}| \\ &= \begin{vmatrix} \sqrt{2pq}-q & -p & 0 & 0 \\ -q & \sqrt{2pq} & -p & 0 \\ 0 & -q & \sqrt{2pq} & -p \\ 0 & 0 & -q & \sqrt{2pq}-p \end{vmatrix}\end{aligned}$$

$$\begin{vmatrix} \sqrt{2pq} & -p & 0 \\ -q & \sqrt{2pq} & -p \\ 0 & -q & \sqrt{2pq}-p \end{vmatrix} \begin{vmatrix} -q & -p & 0 \\ 0 & \sqrt{2pq} & -p \\ 0 & -q & \sqrt{2pq}-p \end{vmatrix} \begin{vmatrix} -q & \sqrt{2pq} & 0 \\ 0 & -q & -p \\ 0 & 0 & \sqrt{2pq}-p \end{vmatrix}$$

$$\begin{vmatrix} -q & \sqrt{2pq} & -p \\ 0 & -q & \sqrt{2pq} \\ 0 & 0 & -q \end{vmatrix} \begin{vmatrix} -p & 0 & 0 \\ -q & \sqrt{2pq} & -p \\ 0 & -q & \sqrt{2pq}-p \end{vmatrix}$$

$$\begin{vmatrix} \sqrt{2pq}-q & 0 & 0 \\ 0 & \sqrt{2pq} & -p \\ 0 & -q & \sqrt{2pq}-p \end{vmatrix} \begin{vmatrix} \sqrt{2pq}-q & -p & 0 \\ 0 & -q & -p \\ 0 & 0 & \sqrt{2pq}-p \end{vmatrix}$$

$$\begin{vmatrix} \sqrt{2pq}-q & -p & 0 \\ 0 & -q & \sqrt{2pq} \\ 0 & 0 & -q \end{vmatrix} \begin{vmatrix} -p & 0 & 0 \\ \sqrt{2pq} & -p & 0 \\ 0 & -q & \sqrt{2pq}-p \end{vmatrix}$$

$$\begin{vmatrix} \sqrt{2pq}-q & 0 & 0 \\ -q & -p & 0 \\ 0 & -q & \sqrt{2pq}-p \end{vmatrix} \begin{vmatrix} \sqrt{2pq}-q & -p & 0 \\ -q & \sqrt{2pq} & 0 \\ 1 & 0 & \sqrt{2pq}-p \end{vmatrix} \begin{vmatrix} \sqrt{2pq}-q & -p & 0 \\ -q & \sqrt{2pq} & -p \\ 1 & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & 0 \\ \sqrt{2pq} & -p & 0 \\ -q & \sqrt{2pq} & -p \end{vmatrix} \begin{vmatrix} \sqrt{2pq}-q & 0 & 0 \\ -q & -p & 0 \\ 0 & \sqrt{2pq} & -p \end{vmatrix} \begin{vmatrix} \sqrt{2pq}-q & -p & 0 \\ -q & \sqrt{2pq} & 0 \\ 0 & -q & -p \end{vmatrix}$$

$$\begin{vmatrix} \sqrt{2pq}-q & -p & 0 \\ -q & \sqrt{2pq} & -p \\ 0 & -q & \sqrt{2pq} \end{vmatrix}$$

$$\begin{vmatrix} \sqrt{2pq} & -p & 0 \\ -q & \sqrt{2pq} & -p \\ 0 & -q & \sqrt{2pq}-p \end{vmatrix} = \sqrt{2pq}\begin{vmatrix} \sqrt{2pq} & -p \\ -q & \sqrt{2pq}-p \end{vmatrix} + p\begin{vmatrix} -q & -p \\ 0 & \sqrt{2pq}-p \end{vmatrix}$$

$$= \sqrt{2pq}\left(\sqrt{2pq}\left(\sqrt{2pq}-p\right)-pq\right)-pq\left(\sqrt{2pq}-p\right)$$

$$= -p^2q$$

$$\begin{vmatrix} -q & -p & 0 \\ 0 & \sqrt{2pq} & -p \\ 0 & -q & \sqrt{2pq}-p \end{vmatrix} = -q\begin{vmatrix} \sqrt{2pq} & -p \\ -q & \sqrt{2pq}-p \end{vmatrix} + p\begin{vmatrix} 0 & -p \\ 0 & \sqrt{2pq}-p \end{vmatrix}$$

$$= -q\left(\sqrt{2pq}\left(\sqrt{2pq}-p\right)-pq\right)$$

$$= -q\left(pq-p\sqrt{2pq}\right)$$

$$\begin{vmatrix} -q & \sqrt{2pq} & 0 \\ 0 & -q & -p \\ 0 & 0 & \sqrt{2pq}-p \end{vmatrix} = -q\begin{vmatrix} -q & -p \\ 0 & \sqrt{2pq}-p \end{vmatrix} - \sqrt{2pq}\begin{vmatrix} 0 & -p \\ 0 & \sqrt{2pq}-p \end{vmatrix}$$

$$= -q(-q(\sqrt{2pq}-p) = -q^2\left(p-\sqrt{2pq}\right)$$

$$\begin{vmatrix} -q & \sqrt{2pq} & -p \\ 0 & -q & \sqrt{2pq} \\ 0 & 0 & -q \end{vmatrix} = -q\begin{vmatrix} -q & \sqrt{2pq} \\ 0 & -q \end{vmatrix} = -q^3$$

$$\begin{vmatrix} -p & 0 & 0 \\ -q & \sqrt{2pq} & -p \\ 0 & -q & \sqrt{2pq}-p \end{vmatrix} = -p\begin{vmatrix} \sqrt{2pq} & -p \\ -q & \sqrt{2pq}-p \end{vmatrix}$$

$$= -p\left[\sqrt{2pq}(\sqrt{2pq}-p)-pq\right]$$

$$= -p^2\left(q-\sqrt{2pq}\right)$$

$$
\begin{aligned}
\begin{vmatrix} \sqrt{2pq}-q & 0 & 0 \\ 0 & \sqrt{2pq} & -p \\ 0 & -q & \sqrt{2pq}-p \end{vmatrix} &= \left(\sqrt{2pq}-q\right)\begin{vmatrix} \sqrt{2pq} & -p \\ -q & \sqrt{2pq}-p \end{vmatrix} \\
&= \left(\sqrt{2pq}-q\right)\left[\sqrt{2pq}\left(\sqrt{2pq}-p\right)-pq\right] \\
&= \left(\sqrt{2pq}-q\right)\left[\left(2pq-p\sqrt{2pq}\right)-pq\right] \\
&= p\left(\sqrt{2pq}-q\right)\left[q-\sqrt{2pq}\right] \\
&= -p\left(\sqrt{2pq}-q\right)^2 \\
&= -p\left(2pq-2q\sqrt{2pq}+q^2\right) \\
&= -pq\left(2p-2\sqrt{2pq}+q\right) \\
&= -pq\left(p-2\sqrt{2pq}+p+q\right) \\
&= -pq\left(-2\sqrt{2pq}+p+1\right)
\end{aligned}
$$

$$
\begin{aligned}
\begin{vmatrix} \sqrt{2pq}-q & -p & 0 \\ 0 & -q & -p \\ 0 & 0 & \sqrt{2pq}-p \end{vmatrix} &= \left(\sqrt{2pq}-q\right)\begin{vmatrix} -q & -p \\ 0 & \sqrt{2pq}-p \end{vmatrix} + p\begin{vmatrix} 0 & -p \\ 0 & \sqrt{2pq}-p \end{vmatrix} \\
&= -q\left(\sqrt{2pq}-q\right)\left(\sqrt{2pq}-p\right) \\
&= -q\left(2pq-q\sqrt{2pq}-p\sqrt{2pq}+pq\right) \\
&= -q\left(3pq-(q+p)\sqrt{2pq}\right) \\
&= -q\left(3pq-\sqrt{2pq}\right) = q(\sqrt{2pq}-3pq)
\end{aligned}
$$

$$
\begin{aligned}
\begin{vmatrix} \sqrt{2pq}-q & -p & 0 \\ 0 & -q & \sqrt{2pq} \\ 0 & 0 & -q \end{vmatrix} &= \left(\sqrt{2pq}-q\right)\begin{vmatrix} -q & \sqrt{2pq} \\ 0 & -q \end{vmatrix} + p\begin{vmatrix} 0 & \sqrt{2pq} \\ 0 & -q \end{vmatrix} \\
&= q^2\left(\sqrt{2pq}-q\right)
\end{aligned}
$$

$$
\begin{vmatrix} -p & 0 & 0 \\ \sqrt{2pq} & -p & 0 \\ 0 & -q & \sqrt{2pq}-p \end{vmatrix} = -p\begin{vmatrix} -p & 0 \\ -q & \sqrt{2pq}-p \end{vmatrix} = p^2\left(\sqrt{2pq}-p\right)
$$

$$\begin{vmatrix} \sqrt{2pq}-q & 0 & 0 \\ -q & -p & 0 \\ 0 & -q & \sqrt{2pq}-p \end{vmatrix} = \left(\sqrt{2pq}-q\right)\begin{vmatrix} -p & 0 \\ -q & \sqrt{2pq}-p \end{vmatrix}$$

$$\begin{aligned} &= -p\left(\sqrt{2pq}-q\right)\left(\sqrt{2pq}-p\right) \\ &= -p\left(2pq-p\sqrt{2pq}-q\sqrt{2pq}+pq\right) \\ &= -p\left(3pq-(p+q)\sqrt{2pq}\right) \\ &= p\left(\sqrt{2pq}-3pq\right) \end{aligned}$$

$$\begin{vmatrix} \sqrt{2pq}-q & -p & 0 \\ -q & \sqrt{2pq} & 0 \\ 0 & 0 & \sqrt{2pq}-p \end{vmatrix} = \left(\sqrt{2pq}-q\right)\begin{vmatrix} \sqrt{2pq} & 0 \\ 0 & \sqrt{2pq}-p \end{vmatrix} + p\begin{vmatrix} -q & 0 \\ 0 & \sqrt{2pq}-p \end{vmatrix}$$

$$\begin{aligned} &= \sqrt{2pq}\left(\sqrt{2pq}-q\right)\left(\sqrt{2pq}-p\right)-pq\left(\sqrt{2pq}-p\right) \\ &= \sqrt{2pq}\left(\sqrt{2pq}-p\right)\left[\sqrt{2pq}-q-pq\right] \\ &= \left(\sqrt{2pq}-p\right)\left[2pq-q\sqrt{2pq}-pq\sqrt{2pq}\right] \\ &= q\left(\sqrt{2pq}-p\right)\left[2p-\sqrt{2pq}-p\sqrt{2pq}\right] \\ &= pq(2\sqrt{2pq}-q-1 \end{aligned}$$

$$\begin{vmatrix} \sqrt{2pq}-q & -p & 0 \\ -q & \sqrt{2pq} & -p \\ 0 & 0 & -q \end{vmatrix} = \left(\sqrt{2pq}-q\right)\begin{vmatrix} \sqrt{2pq} & -p \\ 0 & -q \end{vmatrix} + p\begin{vmatrix} -q & -p \\ 0 & -q \end{vmatrix}$$

$$\begin{aligned} &= -q\sqrt{2pq}\left(\sqrt{2pq}-q\right)+pq^2 \\ &= -2pq^2+q^2\sqrt{2pq}+pq^2 \\ &= -pq^2+q^2\sqrt{2pq} \\ &= q^2\left(\sqrt{2pq}-p\right) \end{aligned}$$

$$\begin{vmatrix} -p & 0 & 0 \\ \sqrt{2pq} & -p & 0 \\ -q & \sqrt{2pq} & -p \end{vmatrix} = p^3$$

$$\begin{vmatrix} \sqrt{2pq}-q & 0 & 0 \\ -q & -p & 0 \\ 0 & \sqrt{2pq} & -p \end{vmatrix} = p^2\left(\sqrt{2pq}-q\right)$$

$$\begin{vmatrix} \sqrt{2pq}-q & -p & 0 \\ -q & \sqrt{2pq} & 0 \\ 0 & -q & -p \end{vmatrix} \begin{aligned}[t] &= \left(\sqrt{2pq}-q\right)\begin{vmatrix} \sqrt{2pq} & 0 \\ -q & -p \end{vmatrix} + p\begin{vmatrix} -q & 0 \\ 0 & -p \end{vmatrix} \\ &= -p\sqrt{2pq}\left(\sqrt{2pq}-q\right)+p^2q \\ &= -2p^2q+pq\sqrt{2pq}+p^2q \\ &= pq(\sqrt{2pq}-p) \end{aligned}$$

$$\begin{vmatrix} \sqrt{2pq}-q & -p & 0 \\ -q & \sqrt{2pq} & -p \\ 0 & -q & \sqrt{2pq} \end{vmatrix} \begin{aligned}[t] &= \left(\sqrt{2pq}-q\right)\begin{vmatrix} \sqrt{2pq} & -p \\ -q & \sqrt{2pq} \end{vmatrix} + p\begin{vmatrix} -q & -p \\ 0 & \sqrt{2pq} \end{vmatrix} \\ &= \left(\sqrt{2pq}-q\right)[2pq-pq]-pq\sqrt{2pq} \\ &= pq\left(\sqrt{2pq}-q\right)-pq\sqrt{2pq} \\ &= -pq^2 \end{aligned}$$

$$= \begin{bmatrix} -p^2q & -pq(\sqrt{2pq}-q) & q^2(\sqrt{2pq}-p) & p^3 \\ p^2(\sqrt{2pq}-q) & -pq(2\sqrt{2pq}-1-p) & q(\sqrt{2pq}-3pq) & -p^2(\sqrt{2pq}-q) \\ p^2(\sqrt{2pq}-p) & -p(\sqrt{2pq}-3pq) & pq(2\sqrt{2pq}-1-q) & pq(\sqrt{2pq}-p) \\ q^3 & -q^2(\sqrt{2pq}-p) & -pq(\sqrt{2pq}-p) & -pq^2 \end{bmatrix}$$

When $\lambda = -\sqrt{2pq}$

$$\begin{aligned} |\mathbf{A}(\lambda)| &= |\mathbf{P}-\lambda\mathbf{I}| \\ &= \begin{vmatrix} -\sqrt{2pq}-q & -p & 0 & 0 \\ -q & -\sqrt{2pq} & -p & 0 \\ 0 & -q & -\sqrt{2pq} & -p \\ 0 & 0 & -q & -\sqrt{2pq}-p \end{vmatrix} \end{aligned}$$

$$\begin{vmatrix} -\sqrt{2pq} & -p & 0 \\ -q & -\sqrt{2pq} & -p \\ 0 & -q & -\sqrt{2pq}-p \end{vmatrix} \begin{vmatrix} -q & -p & 0 \\ 0 & -\sqrt{2pq} & -p \\ 0 & -q & -\sqrt{2pq}-p \end{vmatrix}$$

$$\begin{vmatrix} -q & -\sqrt{2pq} & 0 \\ 0 & -q & -p \\ 0 & 0 & -\sqrt{2pq}-p \end{vmatrix} \begin{vmatrix} -q & -\sqrt{2pq} & -p \\ 0 & -q & -\sqrt{2pq} \\ 0 & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & 0 \\ -q & -\sqrt{2pq} & -p \\ 0 & -q & -\sqrt{2pq}-p \end{vmatrix} \begin{vmatrix} -\sqrt{2pq}-q & 0 & 0 \\ 0 & -\sqrt{2pq} & -p \\ 0 & -q & -\sqrt{2pq}-p \end{vmatrix}$$

$$\begin{vmatrix} -\sqrt{2pq}-q & -p & 0 \\ 0 & -q & -p \\ 0 & 0 & -\sqrt{2pq}-p \end{vmatrix} \begin{vmatrix} -\sqrt{2pq}-q & -p & 0 \\ 0 & -q & -\sqrt{2pq} \\ 0 & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & 0 \\ -\sqrt{2pq} & -p & 0 \\ 0 & -q & -\sqrt{2pq}-p \end{vmatrix} \begin{vmatrix} -\sqrt{2pq}-q & 0 & 0 \\ -q & -p & 0 \\ 0 & -q & -\sqrt{2pq}-p \end{vmatrix}$$

$$\begin{vmatrix} -\sqrt{2pq}-q & -p & 0 \\ -q & -\sqrt{2pq} & 0 \\ 0 & 0 & -\sqrt{2pq}-p \end{vmatrix} \begin{vmatrix} -\sqrt{2pq}-q & -p & 0 \\ -q & -\sqrt{2pq} & -p \\ 0 & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & 0 \\ -\sqrt{2pq} & -p & 0 \\ -q & -\sqrt{2pq} & -p \end{vmatrix} \begin{vmatrix} -\sqrt{2pq}-q & 0 & 0 \\ -q & -p & 0 \\ 0 & -\sqrt{2pq} & -p \end{vmatrix}$$

$$\begin{vmatrix} -\sqrt{2pq}-q & -p & 0 \\ -q & -\sqrt{2pq} & 0 \\ 0 & -q & -p \end{vmatrix} \begin{vmatrix} -\sqrt{2pq}-q & -p & 0 \\ -q & -\sqrt{2pq} & -p \\ 0 & -q & -\sqrt{2pq} \end{vmatrix}$$

$$\begin{aligned} \begin{vmatrix} -\sqrt{2pq} & -p & 0 \\ -q & -\sqrt{2pq} & -p \\ 0 & -q & -\sqrt{2pq}-p \end{vmatrix} &= -\sqrt{2pq}\begin{vmatrix} -\sqrt{2pq} & -p \\ -q & -\sqrt{2pq}-p \end{vmatrix} + p\begin{vmatrix} -q & -p \\ 0 & -\sqrt{2pq}-p \end{vmatrix} \\ &= 2pq\left(-\sqrt{2pq}-p\right) - pq - pq(-\sqrt{2pq}-p) \\ &= -p^2q \end{aligned}$$

$$\begin{aligned}\begin{vmatrix} -q & -p & 0 \\ 0 & -\sqrt{2pq} & -p \\ 0 & -q & -\sqrt{2pq}-p \end{vmatrix} &= q\begin{vmatrix} -\sqrt{2pq} & -p \\ -q & -\sqrt{2pq}-p \end{vmatrix} \\ &= -q\left[-\sqrt{2pq}\left(-\sqrt{2pq}-p\right)-qp\right] \\ &= -q(pq+\sqrt{2pq})\end{aligned}$$

$$\begin{aligned}\begin{vmatrix} -q & -\sqrt{2pq} & 0 \\ 0 & -q & -p \\ 0 & 0 & -\sqrt{2pq}-p \end{vmatrix} &= -q\begin{vmatrix} -q & -p \\ 0 & -\sqrt{2pq}-p \end{vmatrix} + \sqrt{2pq}\begin{vmatrix} 0 & -p \\ 0 & -\sqrt{2pq}-p \end{vmatrix} \\ &= -q^2\left(-\sqrt{2pq}-p\right) \\ &= q^2\left(\sqrt{2pq}+p\right)\end{aligned}$$

$$\begin{aligned}\begin{vmatrix} -q & -\sqrt{2pq} & -p \\ 0 & -q & -\sqrt{2pq} \\ 0 & 0 & -q \end{vmatrix} &= -q\begin{vmatrix} -q & -\sqrt{2pq} \\ 0 & -q \end{vmatrix} \\ &= -q^3\end{aligned}$$

$$\begin{aligned}\begin{vmatrix} -p & 0 & 0 \\ -q & -\sqrt{2pq} & -p \\ 0 & -q & -\sqrt{2pq}-p \end{vmatrix} &= -p\begin{vmatrix} -\sqrt{2pq} & -p \\ -q & -\sqrt{2pq}-p \end{vmatrix} \\ &= -p\left[-\sqrt{2pq}\left(-\sqrt{2pq}-p\right)-pq\right] \\ &= -p\left[2pq+p\sqrt{2pq}-pq\right] \\ &= -p^2\left[q+\sqrt{2pq}\right]\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} -\sqrt{2pq}-q & 0 & 0 \\ 0 & -\sqrt{2pq} & -p \\ 0 & -q & -\sqrt{2pq}-p \end{vmatrix} &= \left(-\sqrt{2pq}-q\right)\begin{vmatrix} -\sqrt{2pq} & -p \\ -q & -\sqrt{2pq}-p \end{vmatrix} \\
&= \left(-\sqrt{2pq}-q\right)\left[-\sqrt{2pq}\left(-\sqrt{2pq}-p\right)-pq\right] \\
&= \left(-\sqrt{2pq}-q\right)\left[\left(2pq+p\sqrt{2pq}\right)-pq\right] \\
&= -p\left(q+\sqrt{2pq}\right)^2 \\
&= -p\left(q^2+2\sqrt{2pq}+2pq\right) \\
&= -pq\left(q+2\sqrt{2pq}+2p\right) \\
&= -pq\left(2\sqrt{2pq}+p+1\right)
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} -\sqrt{2pq}-q & -p & 0 \\ 0 & -q & -p \\ 0 & 0 & -\sqrt{2pq}-p \end{vmatrix} &= \left(-\sqrt{2pq}-q\right)\begin{vmatrix} -q & -p \\ 0 & -\sqrt{2pq}-p \end{vmatrix} \\
&= -q\left(-\sqrt{2pq}-q\right)\left(-\sqrt{2pq}-p\right) \\
&= -q(2pq+p\sqrt{2pq}+q\sqrt{2pq}+pq) \\
&= -q(2pq+(p+q)\sqrt{2pq}+pq) \\
&= -q(\sqrt{2pq}+3pq)
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} -\sqrt{2pq}-q & -p & 0 \\ 0 & -q & -\sqrt{2pq} \\ 0 & 0 & -q \end{vmatrix} &= \left(-\sqrt{2pq}-q\right)\begin{vmatrix} -q & -\sqrt{2pq} \\ 0 & -q \end{vmatrix} \\
&= -q^2\left(-\sqrt{2pq}-q\right)
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} -p & 0 & 0 \\ -\sqrt{2pq} & -p & 0 \\ 0 & -q & -\sqrt{2pq}-p \end{vmatrix} &= -p\begin{vmatrix} -p & 0 \\ -q & -\sqrt{2pq}-p \end{vmatrix} \\
&= p^2\left(-\sqrt{2pq}-p\right)
\end{aligned}$$

$$\begin{vmatrix} -\sqrt{2pq}-q & 0 & 0 \\ -q & -p & 0 \\ 0 & -q & -\sqrt{2pq}-p \end{vmatrix} = \left(-\sqrt{2pq}-q\right)\begin{vmatrix} -p & 0 \\ -q & -\sqrt{2pq}-p \end{vmatrix}$$

$$= -p\left(-\sqrt{2pq}-q\right)\left(-\sqrt{2pq}-p\right)$$

$$= -p\left(-\sqrt{2pq}-3pq\right)$$

$$-1\begin{vmatrix} \sqrt{2pq}-q & p & 0 \\ q & \sqrt{2pq} & 0 \\ 0 & 0 & \sqrt{2pq}-p \end{vmatrix} = \left(-\sqrt{2pq}-q\right)\begin{vmatrix} -\sqrt{2pq} & 0 \\ 0 & -\sqrt{2pq}-p \end{vmatrix}$$

$$+p\begin{vmatrix} -q & 0 \\ 0 & -\sqrt{2pq}-p \end{vmatrix}$$

$$= -\sqrt{2pq}\left(-\sqrt{2pq}-q\right)\left(-\sqrt{2pq}-p\right)-pq\left(-\sqrt{2pq}-p\right)$$

$$= \left(-\sqrt{2pq}-p\right)\left[2pq+q\sqrt{2pq}-pq\right]$$

$$= q\left(-\sqrt{2pq}-p\right)\left[p+\sqrt{2pq}\right]$$

$$= q\left[p+\sqrt{2pq}\right]^2$$

$$= q\left(p^2+2p\sqrt{2pq}+2pq\right)$$

$$= qp\left(p+2\sqrt{2pq}+2q\right)$$

$$= pq\left(2\sqrt{2pq}+q+1\right)$$

$$\begin{vmatrix} -\sqrt{2pq}-q & -p & 0 \\ -q & -\sqrt{2pq} & -p \\ 0 & 0 & -q \end{vmatrix} = -q\begin{vmatrix} -\sqrt{2pq}-q & -p \\ -q & -\sqrt{2pq} \end{vmatrix}$$

$$= -q\left[-\sqrt{2pq}\left(-\sqrt{2pq}-q\right)-pq\right]$$

$$= -q\left[2pq+q\sqrt{2pq}-pq\right]$$

$$= -q^2\left[p+\sqrt{2pq}\right]$$

$$\begin{vmatrix} -p & 0 & 0 \\ -\sqrt{2pq} & -p & 0 \\ -q & -\sqrt{2pq} & -p \end{vmatrix} = -p\begin{vmatrix} -p & 0 \\ -\sqrt{2pq} & -p \end{vmatrix} = -p^3$$

$$\begin{vmatrix} -\sqrt{2pq}-q & 0 & 0 \\ -q & -p & 0 \\ 0 & -\sqrt{2pq} & -p \end{vmatrix} = -p\begin{vmatrix} -\sqrt{2pq}-q & 0 \\ -q & -p \end{vmatrix}$$
$$= p^2\left(-\sqrt{2pq}-q\right)$$

$$\begin{vmatrix} -\sqrt{2pq}-q & -p & 0 \\ -q & -\sqrt{2pq} & 0 \\ 0 & -q & -p \end{vmatrix} = -p\begin{vmatrix} -\sqrt{2pq}-q & -p \\ -q & -\sqrt{2pq} \end{vmatrix}$$
$$= -p\left[\left(-\sqrt{2pq}\right)\left(-\sqrt{2pq}-q\right)-pq\right]$$
$$= -p\left(2pq+q\sqrt{2pq}-pq\right)$$
$$= -pq\left(p+\sqrt{2pq}\right)$$

$$\begin{vmatrix} -\sqrt{2pq}-q & -p & 0 \\ -q & -\sqrt{2pq} & -p \\ 0 & -q & -\sqrt{2pq} \end{vmatrix} = q\begin{vmatrix} -\sqrt{2pq}-q & 0 \\ -q & -p \end{vmatrix} - \sqrt{2pq}\begin{vmatrix} -\sqrt{2pq}-q & -p \\ -q & -\sqrt{2pq} \end{vmatrix}$$
$$= pq\left(\sqrt{2pq}+q\right)-\sqrt{2pq}\,[2pq-pq]$$
$$= pq\left[\sqrt{2pq}-\sqrt{2pq}+q\right]$$
$$= pq^2$$

$$= \begin{bmatrix} -p^2q & -pq(\sqrt{2pq}-q) & q^2(\sqrt{2pq}-p) & p^3 \\ p^2(\sqrt{2pq}-q) & -pq(2\sqrt{2pq}-1-p) & q(\sqrt{2pq}-3pq) & -p^2(\sqrt{2pq}-q) \\ p^2(\sqrt{2pq}-p) & -p(\sqrt{2pq}-3pq) & pq(2\sqrt{2pq}-1-q) & pq(\sqrt{2pq}-p) \\ q^3 & -q^2(\sqrt{2pq}-p) & -pq(\sqrt{2pq}-p) & -pq^2 \end{bmatrix}$$

Therefore.eigen values are

$$\lambda_1 = 0,\ \lambda_2 = 1 \text{ or } \lambda_3 = \sqrt{2pq} \text{ or } \lambda_4 = -\sqrt{2pq}$$

Therefore

$$\begin{aligned}
\prod_{m=2}^{4}(\lambda_1-\lambda_m) &= (0-1)(0-\sqrt{2pq})(0+\sqrt{2pq}) = \left(\sqrt{2pq}\right)^2 = 2pq \\
\prod_{m=1,m\neq 2}^{4}(\lambda_2-\lambda_m) &= (1-0)(1-\sqrt{2pq})(1+\sqrt{2pq}) = (1-2pq) = p^2+q^2 \\
\prod_{m=1,m\neq 3}^{4}(\lambda_3-\lambda_m) &= (\sqrt{2pq}-0)(\sqrt{2pq}-1)(\sqrt{2pq}+\sqrt{2pq}) \\
&= 2\left(\sqrt{2pq}\right)^2\left(\sqrt{2pq}-1\right) \\
&= -4pq\left(1-\sqrt{2pq}\right) \\
\prod_{m=1,}^{3}(\lambda_4-\lambda_m) &= (-\sqrt{2pq}-0)(-\sqrt{2pq}-1)(-\sqrt{2pq}-\sqrt{2pq}) \\
&= -2\left(-\sqrt{2pq}\right)^2\left(-\sqrt{2pq}-1\right) \\
&= -4pq\left(1+\sqrt{2pq}\right)
\end{aligned}$$

The adjoint of $\mathbf{A}(\lambda_2)=\mathbf{A}(\lambda_1)$ has the identical row

$$\begin{bmatrix} q^3 & pq^2 & p^3 & qp^2 \end{bmatrix}$$

This means that

$$\begin{bmatrix} A_{11}(1) & A_{22}(1) & A_{33}(1) & A_{44}(1) \end{bmatrix} = \begin{bmatrix} q^3 & pq^2 & p^3 & qp^2 \end{bmatrix}$$

Using

$$\begin{pmatrix} \pi_1 & \pi_1 & \pi_1 & \pi_1 \end{pmatrix} = \begin{pmatrix} \frac{q^3}{p^2+q^2} & \frac{pq^2}{p^2+q^2} & \frac{p^3}{p^2+q^2} & \frac{qp^2}{p^2+q^2} \end{pmatrix}$$

Since

$$\begin{aligned}
\sum_{k=1}^{4} A_{kk}(l) &= \begin{bmatrix} q^3 & +pq^2 & +p^3 & +qp^2 \end{bmatrix} \\
&= q^3+q^2(1-q)+p^3+p^2(1-p) \\
&= p^2+q^2
\end{aligned}$$

The mean recurrence times is

$$(\mu_1,\mu_2,\mu_3,\mu_4) = \begin{pmatrix} \frac{p^2+q^2}{q^3} & \frac{p^2+q^2}{pq^2} & \frac{p^2+q^2}{p^3} & \frac{p^2+q^2}{qp^2} \end{pmatrix}$$

If

$$p = q = \frac{1}{2}$$

then

$$\left(\begin{array}{cccc} \pi_1 & \pi_1 & \pi_1 & \pi_1 \end{array} \right) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right)$$

and

$$(\mu_1, \mu_2, \mu_3, \mu_4) = (4, 4, 4, 4)$$

The adjoint matrices of the roots $\lambda_3 = \sqrt{2pq}$ and $\lambda_4 = -\sqrt{2pq}$ are alike

$$\left[\begin{array}{cccc} -p^2 q & -pq(\lambda - q) & q^2(\lambda - p) & p^3 \\ p^2(\lambda - q) & -pq(2\lambda - 1 - p) & q(\lambda - 3pq) & -p^2(\lambda - q) \\ p^2(\lambda - p) & -p(\lambda - 3pq) & pq(2\lambda - 1 - q) & pq(\lambda - p) \\ q^3 & -q^2(\lambda - p) & -pq(\lambda - p) & -pq^2 \end{array} \right]$$

Using the foprmula

$$p_{ij}^{(n)} = \sum_{t=1}^{4} A_{ji}(\lambda_t) \lambda_t^{(n)} \frac{1}{\prod_{m=1, m \neq t} (\lambda_t - \lambda_m)}$$

we obtain the transition probability matrices $p_{ij}^{(n)}$ for $n = 2k$ and for $n = 2k+1$

$$\begin{aligned} \mathbf{P}^n &= \left[\begin{array}{cccc} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \end{array} \right] \\ \mathbf{p}_1 &= \left[\begin{array}{c} q^3 + p^2 q \Delta^{k-1} \\ q^3 + pq^2 (p - q) \Delta^{k-1} \\ q^3 + pq^2 (p - q) \Delta^{k-1} \\ q^3 \left(1 - \Delta^{k-1} \right) \end{array} \right], \mathbf{p}_2 = \left[\begin{array}{c} pq^2 + p^2 q (p - q) \Delta^{k-1} \\ pq^2 + pq \left[(p - q)^2 + p \right] \Delta^{k-1} \\ pq^2 \left(1 - \Delta^{k-1} \right) \\ pq^2 - q^3 (p - q) \Delta^{k-1} \end{array} \right] \\ \mathbf{p}_3 &= \left[\begin{array}{c} pq^2 + p^3 (p - q) \Delta^{k-1} \\ p^2 q \left(1 - \Delta^{k-1} \right) \\ p^2 q + pq \left[(p - q)^2 + q \right] \Delta^{k-1} \\ p^2 q - pq^2 (p - q) \Delta^{k-1} \end{array} \right], \mathbf{p}_4 = \left[\begin{array}{c} p^3 \left(1 - \Delta^{k-1} \right) \\ p^3 - p^2 q (p - q) \Delta^{k-1} \\ p^3 - p^2 q (p - q) \Delta^{k-1} \\ p^3 + pq^2 \Delta^{k-1} \end{array} \right] \end{aligned}$$

For $k = 1,2,3\ldots$ and $\Delta = 2pq$ and

$$P^{2k+1} = \frac{1}{1-2pq}\left[\begin{array}{cccc} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 \end{array}\right]$$

$$\mathbf{p}_1 = \left[\begin{array}{c} q^3 + p^2 q\Delta^k \\ q^3 + pq^2\,(p-q)\,\Delta^k \\ q^3 + pq^2\,(p-q)\,\Delta^k \\ q^3\left(1-\Delta^k\right) \end{array}\right], \mathbf{p}_2 = \left[\begin{array}{c} pq^2 + p^2q(p-q)\Delta^k \\ pq^2 + pq\left[(p-q)^2 + p\right]\Delta^k \\ pq^2\left(1-\Delta^k\right) \\ pq^2 - q^3(p-q)\Delta^k \end{array}\right]$$

$$\mathbf{p}_3 = \left[\begin{array}{c} pq^2 + p^3(p-q)\Delta^k \\ p^2q\left(1-\Delta^k\right) \\ p^2q + pq\left[(p-q)^2 + q\right]\Delta^k \\ p^2q - pq^2(p-q)\Delta^k \end{array}\right], \mathbf{p}_4 = \left[\begin{array}{c} p^3\left(1-\Delta^k\right) \\ p^3 - p^2q\,(p-q)\,\Delta^k \\ p^3 - p^2q\,(p-q)\,\Delta^k \\ p^3 + pq^2\Delta^k \end{array}\right]$$

for $k = 0,1,2,3\ldots$

Consider a chain with possible states $E_0, E_1, E_2, \ldots, E_{\rho-1}$ and transition probabilities

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ \vdots \\ E_{\rho-2} \\ E_{\rho-1} \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & \ldots & \rho-4 & \rho-3 & \rho-2 & \rho-1 \end{array} \\ \left[\begin{array}{ccccccccccc} q & p & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & q & 0 & p & p & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & p & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & p & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & q & p \end{array}\right] \end{array}$$

This can be interpreted in gambling language by considering two players for unit stakes with agreement that every time a player loses his last dollar his adversary returns it so that the game can continue forever. We suppose that the players own between them $\rho+1$ dollars and say that the system is in state E_k if the two capitals are k and $\rho-k+1$, respectively. The transition probabilities are then given by our matrix .Our chain represents a random walk with reflecting barriers at the points $\frac{1}{2}$ and $\rho+\frac{1}{2}$.

For $2 \leq k \leq \rho-1$ we have $p_{k,k+1} = p$ and $p_{k,k-1} = q$; the first and the last rows are defined by $(q,p,0,0,\ldots,0)$ and $(0,\ldots,q,p)$.

5.5 Random Walks with Two Reflecting Barriers: General Case

Consider that a particle may be at any position r,$r = 0,1,2\ldots,k(\geq 1)$ of the x-axis. From state r it moves to state $r+1$, $1 \leq r \leq k-1$ with probability a and is reflected to state 1 with probability $(1-a),(0 < a < 1)$; if it reaches state k it remains there with probability b and is reflected to $k-1$ with probability $(1-b)$ with state space $S = \{0,1,2\ldots,k\}$.

5.5.1 A 3×3 transitional probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} a & 1-a & 0 \\ q & 0 & p \\ 0 & 1-b & b \end{bmatrix} \end{array}$$

Classification of the states Consider E_0

$$\begin{aligned} f_{00}^{(1)} &= \Pr[E_0 \longrightarrow E_0] = a \\ f_{00}^{(2)} &= \Pr[E_0 \longrightarrow E_1 \longrightarrow E_0] = q(1-a) \\ f_{00}^{(3)} &= 0 \end{aligned}$$

In general

$$f_{00}^{(n)} = \left\{ \begin{array}{c} a, \quad n=1 \\ q(1-a), \quad n=2 \\ 0, \quad \text{elsewhere} \end{array} \right\}$$

$$\begin{aligned} f_0 &= \sum_{n=1}^{\infty} f_{00}^{(n)} \\ &= a + q(1-a) < 1 \end{aligned}$$

Hence E_0 is transient.

Consider E_1

$$
\begin{aligned}
f_{11}^{(1)} &= \Pr[E_1 \longrightarrow E_1] = 0 \\
f_{11}^{(2)} &= \Pr[E_1 \longrightarrow E_0 \longrightarrow E_1] + \Pr[E_1 \longrightarrow E_2 \longrightarrow E_1] \\
&= q(1-a) + p(1-b) \\
f_{11}^{(3)} &= \Pr[E_1 \longrightarrow E_0 \longrightarrow E_0 \longrightarrow E_1] + \Pr[E_1 \longrightarrow E_2 \longrightarrow E_2 \longrightarrow E_1] \\
&= qa(1-a) + pb(1-b) \\
f_{11}^{(4)} &= \Pr[E_1 \longrightarrow E_0 \longrightarrow E_0 \longrightarrow E_0 \longrightarrow E_1] + \Pr[E_1 \longrightarrow E_2 \longrightarrow E_2 \longrightarrow E_2 \longrightarrow E_1] \\
&= qa^2(1-a) + pb^2(1-b)
\end{aligned}
$$

In general,

$$
f_{11}^{(n)} = \left\{ \begin{array}{c} 0, \quad n = 1 \\ qa^{n-2}(1-a) + pb^{n-2}(1-b), \quad n = 2 \end{array} \right\}
$$

$$
\begin{aligned}
f_1 &= qa^0(1-a) + pb^0(1-b) + qa^1(1-a) + pb^1(1-b) + qa^1(1-a) + pb^1(1-b) \\
&\quad + qa^3(1-a) + pb^3(1-b) + \cdots \\
&= q(1-a) + p(1-b) + qa(1-a)\left[1 + a + a^2 + \cdots\right] + \\
&\quad pb(1-b)\left[1 + b + b^2 + \cdots\right] \\
&= q(1-a) + p(1-b) + \frac{qa(1-a)}{1-a} + \frac{pb(1-b)}{1-b} \\
&= q(1-a) + p(1-b) + qa + pb \\
&= q - qa + p - pb + qa + pb \\
&= q + p \\
&= 1
\end{aligned}
$$

Consider

$$
\begin{aligned}
f_{22}^{(1)} &= \Pr[E_2 \longrightarrow E_2] = b \\
f_{22}^{(2)} &= \Pr[E_2 \longrightarrow E_1 \longrightarrow E_2] = p(1-b) \\
f_{22}^{(3)} &= 0
\end{aligned}
$$

In general

$$
f_{22}^{(n)} = \left\{ \begin{array}{c} b, \quad n = 1 \\ p(1-b), \quad n = 2 \\ 0, \quad \text{elsewhere} \end{array} \right\}
$$

$$
\begin{aligned}
f_2 &= \sum_{n=1}^{\infty} f_{22}^{(n)} \\
&= b + p(1-b) < 1
\end{aligned}
$$

Hence E_2 is transient.

Asymptotic behavior

$$\lim_{n\longrightarrow\infty}\mathbf{P}^n=\mathbf{P}'\pi$$

$$\begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\end{bmatrix}=\begin{bmatrix}a & q & \\ 1-a & & 1-b\\ 0 & p & b\end{bmatrix}\begin{bmatrix}\pi_0\\ \pi_1\\ \pi_2\end{bmatrix}$$

$$\begin{aligned}\pi_0 &= a\pi_0+q\pi_1\\ \pi_1 &= (1-a)\,\pi_0+(1-b)\pi_2\\ \pi_2 &= p\pi_1+b\pi_2\end{aligned}$$

5.5.2 A 4×4 transition probability matrix

$$\mathbf{P}=\begin{array}{c}\\ 0\\ 1\\ 2\\ 3\end{array}\begin{array}{c}\begin{array}{cccc}0 & 1 & 2 & 3\end{array}\\ \begin{bmatrix}a & 1-a & 0 & 0\\ q & 0 & p & 0\\ 0 & q & 0 & p\\ 0 & 0 & 1-b & b\end{bmatrix}\end{array}$$

Classification of the states

$$\begin{aligned}f_{00}^{(1)} &= \Pr[E_0\longrightarrow E_0]=a\\ f_{00}^{(2)} &= \Pr[E_0\longrightarrow E_1\longrightarrow E_0]=q(1-a)\\ f_{00}^{(3)} &= 0\end{aligned}$$

In general

$$f_{00}^{(n)}=\left\{\begin{array}{c}a,\quad n=1\\ q(1-a),\quad n=2\\ 0,\quad \text{elsewhere}\end{array}\right\}$$

$$\begin{aligned} f_0 &= \sum_{n=1}^{\infty} f_{00}^{(n)} \\ &= a + q(1-a) < 1 \end{aligned}$$

Hence E_0 is transient. Now consider E_1

$$\begin{aligned} f_{11}^{(1)} &= \Pr[E_1 \longrightarrow E_1] = 0 \\ f_{11}^{(2)} &= \Pr[E_1 \longrightarrow E_0 \longrightarrow E_1] + \Pr[E_1 \longrightarrow E_2 \longrightarrow E_1] \\ &= q(1-a) + pq \\ f_{11}^{(3)} &= \Pr[E_1 \longrightarrow E_0 \longrightarrow E_0 \longrightarrow E_1] \\ &= qa(1-a) \\ f_{11}^{(4)} &= \Pr[E_1 \longrightarrow E_0 \longrightarrow E_0 \longrightarrow E_0 \longrightarrow E_1] \\ &= qa^2(1-a) \end{aligned}$$

In general

$$f_{11}^{(n)} = \left\{ \begin{array}{cc} 0, & n = 1 \\ q(1-a) + pq, & n = 2 \\ qa^{n-1}(1-a), & n \geq 3 \end{array} \right\}$$

$$\begin{aligned} f_1 &= \sum_{n=1}^{\infty} f_{11}^{(n)} \\ &= pq + q(1-a) + qa(1-a) + qa^2(1-a) + qa^3(1-a) + \cdots \\ &= pq + q(1-a)\left[1 + a + a^2 + \cdots\right] \\ &= pq + \frac{q(1-a)}{1-a} \\ &= pq + q < 1 \end{aligned}$$

hence E_1 is transient. Consider E_2

$$\begin{aligned} f_{22}^{(1)} &= \Pr[E_2 \longrightarrow E_2] = 0 \\ f_{22}^{(2)} &= \Pr[E_2 \longrightarrow E_1 \longrightarrow E_2] + \Pr[E_2 \longrightarrow E_3 \longrightarrow E_2] = pq + p(1-b) \\ f_{22}^{(3)} &= \Pr[E_2 \longrightarrow E_1 \longrightarrow E_1 \longrightarrow E_2] = pb(1-b) \\ f_{22}^{(4)} &= \Pr[E_2 \longrightarrow E_1 \longrightarrow E_1 \longrightarrow E_1 \longrightarrow E_2] = pb^2(1-b) \end{aligned}$$

Consider E_3

$$\begin{aligned} f_{33}^{(1)} &= \Pr[E_3 \longrightarrow E_3] = b \\ f_{33}^{(2)} &= \Pr[E_3 \longrightarrow E_2 \longrightarrow E_3] = p(1-b) \\ f_{33}^{(3)} &= 0 \end{aligned}$$

In general

$$f_{33}^{(n)} = \left\{ \begin{array}{c} b,\ n=1 \\ (1-b)p,\ n=2 \\ 0,\ \text{elsewhere} \end{array} \right\}$$

Hence E_3 is transient. In general, the transition probability matrix

$$\mathbf{P} = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ k-1 \\ k \end{array} \left[\begin{array}{cccccccccccc} a & 1-a & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & p & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & p & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1-b & b \end{array} \right]$$

If $a=1$, then 0 is an absorbing barrier and if $a=0$, 0 is a reflecting barrier, if $0<a<1$, 0 is an elastic barrier. Similar is the case with state k . The case when both 0 and k are absorbing barriers corresponds to the familiar Gambler's ruin problem (with the total capital between the two gamblers amounting to k).

5.6 Invariant Distributions for Random Walks with Reflecting Barriers

5.6.1 Two reflecting Barriers

The application of Markov chains will now be illustrated by discussion of a random walk with states $1,2,...,\rho$ with two reflecting barriers.

Consider a chain with states $0,1,2,...,\rho-1$ whose first and last rows are $(q,p,0,0,\dots,0)$ and $(0,0,\dots,0,q,p)$ respectively. In all other rows $p_{k,k+1}=p; p_{k,k-1}=q$ for $2\leq k\leq \rho$

and $p_{kk}=0$. Then we have the transitional probability matrix as

$$\mathbf{P}=\begin{array}{c} \\ E_0\\E_1\\E_2\\E_3\\E_4\\\vdots\\E_{\rho-2}\\E_{\rho-1}\end{array}\begin{array}{c}\begin{array}{cccccccccccc}0&1&2&3&4&5&\dots&\rho-4&\rho-3&\rho-2&\rho-1\end{array}\\ \left[\begin{array}{ccccccccccc}
q & p & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\
q & 0 & p & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\
0 & q & 0 & p & p & 0 & \dots & 0 & 0 & 0 & 0\\
0 & 0 & q & 0 & p & 0 & \dots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & q & 0 & p & \dots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & q & 0 & p\\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & q & p
\end{array}\right]\end{array}$$

By definition, invarinat distribution is given by

$$\pi_j=\sum\pi_i p_{ij} \tag{33}$$

Writing (2.3) in matrix form we have

$$\pi=\mathbf{P}'\pi$$

$\Longrightarrow$

$$\pi'=\pi'\mathbf{P}$$

Then

$$(\pi_0,\pi_1,\dots,\pi_{\rho-1})=(\pi_0,\pi_1,\dots,\pi_{\rho-1})\left[\begin{array}{ccccccc}
q & p & 0 & 0 & \dots & 0 & 0\\
q & 0 & p & 0 & \dots & 0 & 0\\
0 & q & 0 & p & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \dots & 0 & p\\
0 & 0 & 0 & 0 & \dots & q & p
\end{array}\right]$$

then we have

$$\pi_0 = q\pi_0+q\pi_1 \tag{34}$$
$$\pi_{\rho-1} = p\pi_{\rho-2}+p\pi_{\rho-1} \tag{35}$$
$$\pi_k = p\pi_{k-1}+q\pi_{k+1},\ \text{for } 2\le k\le\rho-2 \tag{36}$$

Solving for $\pi_1$in terms of π_0

$$\pi_1=\left(\frac{p}{q}\right)\pi_0$$

From

$$\pi_k = p\pi_{k-1} + q\pi_{k+1}$$

$$\pi_{k+1} = \frac{1}{q}\left[\pi_k - p\pi_{k-1}\right]$$

when $k = 1$

$$\pi_2 = \frac{1}{q}\left[\pi_1 - p\pi_0\right]$$

but

$$\pi_1 = \left(\frac{p}{q}\right)\pi_0$$

hence

$$\begin{aligned}
\pi_2 &= \frac{1}{q}\left[\left(\frac{p}{q}\right)\pi_0 - p\pi_0\right] \\
&= \frac{1}{q}\left[\left(\frac{p}{q}\right) - p\right]\pi_0 \\
&= \frac{p}{q}\left[\frac{1}{q} - 1\right]\pi_0 \\
&= \left(\frac{p}{q}\right)^2 \pi_0
\end{aligned}$$

for $k = 2$

$$\begin{aligned}
\pi_3 &= \frac{1}{q}\left[\pi_2 - p\pi_1\right] \\
&= \frac{1}{q}\left[\left(\frac{p}{q}\right)^2 \pi_0 - p\left(\frac{p}{q}\right)\pi_0\right] \\
&= \left(\frac{p}{q}\right)^3 \pi_0
\end{aligned}$$

In general

$$\pi_k = \left(\frac{p}{q}\right)^k \pi_0, \; k = 1, 2, 3, \ldots, \rho - 1$$

$$\pi_{\rho-1} = p\pi_{\rho-2} + p\pi_{\rho-1}$$

$\Longrightarrow$

$$\begin{aligned}
(1-p)\pi_{\rho-1} &= p\pi_{\rho-2} \\
&= p\left(\frac{p}{q}\right)^{\rho-2}
\end{aligned}$$

hence

$$\begin{aligned}\pi_{\rho-1} &= \frac{p}{q}\left(\frac{p}{q}\right)^{\rho-2}\\ &= \left(\frac{p}{q}\right)^{\rho-1}\end{aligned}$$

We require that $\sum_k \pi_k = 1$ for it to be a probability distribution. Then

$$\begin{aligned}\sum_{k=0}^{\rho-1}\pi_k &= \sum_{k=0}^{\rho-1}\left(\frac{p}{q}\right)^k \pi_0\\ &= \pi_0\sum_{k=0}^{\rho-1}\left(\frac{p}{q}\right)^k\\ &= 1\\ &= \pi_0\left[1+\left(\frac{p}{q}\right)^1+\left(\frac{p}{q}\right)^2+\left(\frac{p}{q}\right)^3+\cdots\left(\frac{p}{q}\right)^{\rho-1}\right]\\ &= \pi_0\left[\frac{1-\left(\frac{p}{q}\right)^\rho}{1-\left(\frac{p}{q}\right)}\right]\end{aligned}$$

Hence

$$\pi_0\left[\frac{1-\left(\frac{p}{q}\right)^\rho}{1-\left(\frac{p}{q}\right)}\right]=1$$

$$\pi_0=\frac{1-\left(\frac{p}{q}\right)}{1-\left(\frac{p}{q}\right)^\rho}$$

Hence

$$\pi_k=\frac{1-\left(\frac{p}{q}\right)}{1-\left(\frac{p}{q}\right)^\rho}\left(\frac{p}{q}\right)^k,\ k=1,2,3,\ldots,\rho-1$$

which is the stationary distribution.

We observe that

If $p>q$, then $\{\pi_k\}$ decreases geometrically away from the upper barrier, then the states are transient.

If $p<q$, then $\{\pi_k\}$ decreases geometrically away from the lower barrier, hence the states are ergodic.

If $p=q$, then

$$\pi_k=\pi_0=\frac{1}{\rho}\quad\forall k$$

Hence the states are persistent null.

5.6.2 One reflecting barrier

Consider a random walk matrix with $p_{j,j+1} = p; p_{j,j-1} = q$ for $j = 2,3,4,\ldots$ where $p+q = 1$. We have

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ \vdots \end{array} \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & \ldots \end{matrix} \\ \begin{bmatrix} q & p & 0 & 0 & 0 & \ldots \\ q & 0 & p & 0 & 0 & \ldots \\ 0 & q & 0 & p & p & \ldots \\ 0 & 0 & q & 0 & p & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

By definition, the invariant distribution is given by

$$\pi_j = \sum \pi_i p_{ij} \qquad \textbf{(37)}$$

Writing (2.3) in matrix form we have

$$\pi = \mathbf{P}'\pi$$

$\Longrightarrow$

$$\pi' = \pi'\mathbf{P}$$

Then

$$(\pi_1, \pi_2, \ldots) = (\pi_1, \pi_2, \ldots) \begin{bmatrix} q & p & 0 & 0 & \ldots \\ q & 0 & p & 0 & \ldots \\ 0 & q & 0 & p & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

then we have

$$\begin{aligned} \pi_0 &= q\pi_0 + q\pi_1 \\ \pi_2 &= p\pi_1 + q\pi_3 \\ \pi_3 &= p\pi_2 + q\pi_4 \\ \pi_k &= p\pi_{k-1} + q\pi_{k+1}, \text{ for } k = 2,3,4\ldots \end{aligned} \qquad \textbf{(38)}$$

Solving for $\pi_2$in terms of π_1

$$\pi_2 = \left(\frac{p}{q}\right)\pi_1$$

From

$$\pi_k = p\pi_{k-1} + q\pi_{k+1}$$

$$\pi_{k+1} = \frac{1}{q}\left[\pi_k - p\pi_{k-1}\right]$$

when $k = 2$

$$\pi_3 = \frac{1}{q}\left[\pi_1 - p\pi_1\right]$$

but

$$\pi_2 = \left(\frac{p}{q}\right)\pi_1$$

hence

$$\begin{aligned}
\pi_3 &= \frac{1}{q}\left[\left(\frac{p}{q}\right)\pi_1 - p\pi_1\right] \\
&= \frac{1}{q}\left[\left(\frac{p}{q}\right) - p\right]\pi_1 \\
&= \frac{p}{q}\left[\frac{1}{q} - 1\right]\pi_1 \\
&= \left(\frac{p}{q}\right)^2\pi_1
\end{aligned}$$

for $k = 3$

$$\begin{aligned}
\pi_4 &= \frac{1}{q}\left[\pi_3 - p\pi_2\right] \\
&= \frac{1}{q}\left[\left(\frac{p}{q}\right)^2\pi_1 - p\left(\frac{p}{q}\right)\pi_1\right] \\
&= \left(\frac{p}{q}\right)^3\pi_1
\end{aligned}$$

In general

$$\pi_k = \left(\frac{p}{q}\right)^{k-1}\pi_0,\ k = 2, 3, \ldots, \rho - 1$$

$$\pi_{\rho-1} = p\pi_{\rho-2} + p\pi_{\rho-1}$$

We require that $\sum_k \pi_k = 1$ for it to be a probability distribution. Then

$$\begin{aligned}
\sum_{k=1}^{\infty}\pi_k &= \sum_{k=1}^{\infty}\left(\frac{p}{q}\right)^{k-1}\pi_1 \\
&= \pi_1\sum_{k=1}^{\infty}\left(\frac{p}{q}\right)^{k-1} \\
&= 1 \\
&= \pi_1\left[1 + \left(\frac{p}{q}\right)^1 + \left(\frac{p}{q}\right)^2 + \left(\frac{p}{q}\right)^3 + \cdots\right]
\end{aligned}$$

but

$$
\begin{aligned}
\left[1+\left(\frac{p}{q}\right)^1+\left(\frac{p}{q}\right)^2+\left(\frac{p}{q}\right)^3+\cdots\right] &= \frac{1}{1-\frac{p}{q}} \\
&= \frac{q}{q-p}
\end{aligned}
$$

Hence

$$\pi_1 = \frac{q-p}{q}$$

and

$$\pi_k = \frac{q-p}{q}\left(\frac{p}{q}\right)^k,\ k = 1,2,3,\ldots$$

which is the stationary distribution.

6 CYCLIC RANDOM WALKS AS DOUBLY STOCHASTIC MARKOV CHAINS

6.1 Introduction

In this chapter we are going to look at the definition of a double stochastic matrix, classification of the states of a cyclic random walk, the asymptotic behaviour of a cyclic random walk and its n^{th} power. A matrix whose elements are between 0 and 1(inclusive) and each row adds up to one is called a *stochastic matrix*. Any stochastic matrix can serve as a matrix of transition probabilities; together with our initial distribution $\{a_k\}$ it completely defines a Markov chain with states $E_1, E_2, \ldots$.

If in addition to the above each column adds up to one, then we have a *doubly stochastic matrix*. This implies that, not only does the each row add to 1, each column also adds to 1. Thus, for every column j of a doubly stochastic matrix, we have

$$\sum_j p_{ij} = 1$$

Let the possible states be $E_1, E_2, \ldots E_\rho$ but order them cyclically so that E_ρ has the neighbours $E_{\rho-1}$ and E_1 . If as before the system always passes either to the right or to the left neighbours. The rows of the matrix $\mathbf{P}$ are as in 3.1 except that the first and the last rows are $(0, p, 0, 0, \ldots, q)$ and $(p, 0, 0, \ldots, q, 0)$ respectively. More generally, we may permit transitions between any two states. Let $q_0, q_1, q_2, \ldots, q_{\rho-1}$ be, respectively, the probability of staying fixed or moving $1, 2, \ldots, \rho-1$ units to the right (where k units to the right is the same as $\rho-k$ units to the left), then $\mathbf{P}$ is the cyclical matrix.

6.2 Transition Probability Matrix

The matrix below is a 3×3 transitional probability matrix of a cyclic random walk. The sum of the rows and columns is one and is therefore a doubly stochastic matrix.

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \end{array} \begin{array}{c} \begin{array}{ccc} E_0 & E_1 & E_2 \end{array} \\ \left[\begin{array}{ccc} 0 & p & q \\ q & 0 & p \\ p & q & 0 \end{array}\right] \end{array}$$

The figure below shows the diagrammatic representation of the states in the 3×3 transition probability matrix $\mathbf{P}$ above.

6.2.1 Classification of the Markov chain

All the states can be reached from every other state. The Markov chain is therefore irreducible. Hence all the states are of the same type.

6.2.2 Classification of the states

Consider E_0

$$
\begin{aligned}
f_{00}^{(1)} &= \Pr[E_0 \longrightarrow E_0] = 0 \\
f_{00}^{(2)} &= \Pr[E_0 \longrightarrow E_1 \longrightarrow E_0] + \Pr[E_0 \longrightarrow E_2 \longrightarrow E_0] = pq + qp = 2pq \\
f_{00}^{(3)} &= \Pr[E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_0] + \Pr[E_0 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0] = p^3 + q^3 \\
f_{00}^{(4)} &= \Pr[E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0] + \Pr[E_0 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_0] \\
&= 2p^2q^2 \\
f_{00}^{(5)} &= \Pr[E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_0] \\
&\quad + \Pr[E_0 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0] \\
&= p^4q + q^4p \\
f_{00}^{(6)} &= \Pr[E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0] \\
&\quad + \Pr[E_0 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_0] \\
&= 2p^3q^3 \\
f_{00}^{(7)} &= \Pr[E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_0] \\
&\quad + \Pr[E_0 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0] \\
&= p^5q^2 + q^5p^2 \\
f_{00}^{(8)} &= \Pr[E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0] \\
&\quad + \Pr[E_0 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_0] \\
&= 2p^4q^4
\end{aligned}
$$

In general

$$
f_{00}^{(n)} = \left\{ \begin{array}{c} 0,\ n = 0, 1 \\ 2\,(pq)^{\frac{n}{2}}\,,\quad n \text{ is even} \\ \left(p^3 + q^3\right)(pq)^{\frac{n-3}{2}}\,,\quad n \geq 3 \text{ and odd} \end{array} \right\}
$$

$$
\begin{aligned}
f_0 &= \sum_{n=1}^{\infty} f_{00}^{(n)} \\
&= 2pq + 2p^2q^2 + 2p^3q^3 + 2p^4q^4 + \cdots + p^3 + p^4q + p^5q^2 + \cdots + q^3 + q^4p + q^5q^2 + \cdots \\
&= 2pq\left[1 + pq + p^2q^2 + p^3q^3 + \cdots\right] + p^3\left[1 + pq + p^2q^2 + \cdots\right] + q^3\left[1 + qp + q^2q^2 + \cdots\right] \\
&= \frac{2pq}{1-pq} + \frac{p^3}{1-pq} + \frac{q^3}{1-pq} \\
&= \frac{2p(1-p) + (1-p)^3 + p^3}{1-p(1-p)}
\end{aligned}
$$

Hence E_0 is transient.

6.2.3 The asymptotic behavior

$$\pi = \mathbf{P}'\pi$$

$$
\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} 0 & q & p \\ p & 0 & q \\ q & p & 0 \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{bmatrix}
$$

$$
\begin{aligned}
\pi_0 &= q\pi_1 + p\pi_2 && \text{(i)} \\
\pi_1 &= p\pi_0 + q\pi_2 && \text{(ii)} \\
\pi_2 &= q\pi_0 + p\pi_1 && \text{(iii)} \\
1 &= \pi_0 + \pi_1 + \pi_2 && \text{(iv)}
\end{aligned}
$$

Let

$$\pi_2 = 1 - \pi_0 - \pi_1$$

using equations (i) and (ii)

$$
\begin{aligned}
\pi_1 &= p\left[q\pi_1 + p\pi_2\right] + q\pi_2 \\
&= pq\pi_1 + p^2\pi_2 + q\pi_2 \\
(1-pq)\pi_1 &= \left(p^2 + q\right)\pi_2 \\
\pi_1 &= \frac{p^2+q}{1-pq}\pi_2
\end{aligned}
$$

hence

$$
\begin{aligned}
\pi_0 &= \frac{q\left(p^2+q\right)}{1-pq}\pi_2 + p\pi_2 \\
&= \left[\frac{q\left(p^2+q\right)}{(1-pq)} + p\right]\pi_2
\end{aligned}
$$

but

$$\begin{aligned}\pi_2 &= 1-\pi_0-\pi_1 \\ &= 1-\left[\frac{q\left(p^2+q\right)}{(1-pq)}+p\right]\pi_2-\frac{p^2+q}{1-pq}\pi_2\end{aligned}$$

$$\begin{aligned}\pi_2\left(1+\left[\frac{q\left(p^2+q\right)}{(1-pq)}+p\right]+\frac{p^2+q}{1-pq}\right) &= 1 \\ \pi_2\left[\frac{1-pq+p-p^2q+p^2q+q^2+p^2+q}{(1-pq)}\right] &= 1\end{aligned}$$

but

$$p^2+q^2=1-2pq$$

$$\begin{aligned}\pi_2\left[\frac{1-pq+p+q+1-2pq}{(1-pq)}\right] &= 1 \\ \pi_2\left[\frac{1-pq+1+1-2pq}{(1-pq)}\right] &= 1 \\ \pi_2\left[\frac{3-3pq}{(1-pq)}\right] &= 1 \\ 3\pi_2\frac{(1-pq)}{(1-pq)} &= 1 \\ \pi_2 &= \frac{1}{3}\end{aligned}$$

therefore

$$\begin{aligned}\pi_1 &= \frac{1}{3}\left(\frac{p^2+q}{1-pq}\right) \\ &= \frac{1}{3}\left(\frac{p^2+q}{1-p(1-p)}\right) \\ &= \frac{1}{3}\end{aligned}$$

$$\begin{aligned}
\pi_0 &= \frac{1}{3}\left[\frac{qp^2+q^2}{(1-pq)}+p\right] \\
&= \frac{1}{3}\left[\frac{qp^2+q^2+p(1-pq)}{(1-pq)}\right] \\
&= \frac{1}{3}\left[\frac{qp^2+q^2+p-p^2q}{(1-pq)}\right] \\
&= \frac{1}{3}\left[\frac{q^2+p}{(1-pq)}\right] \\
&= \frac{1}{3}\left[\frac{q^2+1-q}{(1-(1-q)q)}\right] \\
&= \frac{1}{3}\left[\frac{q^2+1-q}{1-q+q^2}\right] \\
&= \frac{1}{3}
\end{aligned}$$

In general

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\begin{aligned}
\mu_0 &= \sum_{n=1}^{\infty} n f_{00}^{(n)} \\
&= \frac{1}{\mu_k} \\
&= 3
\end{aligned}$$

hence μ is finite and hence E_1 is non-null.

Periodicity

$$\begin{aligned}
d &= GCD\left\{n : f_{jj}^{(n)} > 0\right\} \\
&= 1
\end{aligned}$$

thus $E_0, E_1,$ and E_2 are persistent, non-null, and aperiodic therefore ergodic.

6.2.4 The n^{th} power $\mathbf{P}^n$

Using the Eigen value technique

We determine the corresponding eigenvalues by solbing teh characteristic function

$$|\mathbf{P}-\lambda\mathbf{I}|=\mathbf{0}$$

$$\left|\begin{bmatrix} 0 & p & q \\ q & 0 & p \\ p & q & 0 \end{bmatrix}-\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right|=\begin{vmatrix} -\lambda & p & q \\ q & -\lambda & p \\ p & q & -\lambda \end{vmatrix}$$

$$\begin{vmatrix} -\lambda & p & q \\ q & -\lambda & p \\ p & q & -\lambda \end{vmatrix}=\begin{vmatrix} -\lambda+p+q & -\lambda+p+q & -\lambda+p+q \\ q & -\lambda & p \\ p & q & -\lambda \end{vmatrix}=0$$

$$(-\lambda+p+q)\begin{vmatrix} 1 & 1 & 1 \\ q & -\lambda & p \\ p & q & -\lambda \end{vmatrix}=(-\lambda+p+q)\begin{vmatrix} 1 & 0 & 0 \\ q & -\lambda-q & p-q \\ p & q-p & -\lambda-p \end{vmatrix}=0$$

$$(-\lambda+p+q)\left[(-\lambda-q)(-\lambda-p)-(p-q)(q-p)\right]=0$$

$$(-\lambda+p+q)=0$$

or

$$(-\lambda-q)(-\lambda-p)-(p-q)(q-p)=0$$

$$\lambda_1=1$$

$$\begin{aligned}
(-\lambda-q)(-\lambda-p)-(p-q)(q-p) &= \lambda^2+\lambda+pq-\left[pq-p^2-q^2+qp\right] \\
&= \lambda^2+\lambda+pq-2pq+p^2+q^2 \\
&= \lambda^2+\lambda+pq-2pq+1-2pq \\
&= \lambda^2+\lambda-3pq+1=0
\end{aligned}$$

$$\begin{aligned}
\lambda &= \frac{-1\pm\sqrt{1-4(1-3pq)}}{2} \\
&= \frac{-1\pm\sqrt{1-4+12pq}}{2} \\
&= \frac{-1\pm\sqrt{12pq-3}}{2}
\end{aligned}$$

6.2.5 Special case when $\mathrm{p} = 1$

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right] \end{array}$$

Classification of the Markov chain

$\mathbf{P}$ is an irreducible Markov Chain since all the states can be reached from every other state; hence all the states are of the same type.

Classification of the states

$$\begin{aligned} f_{11}^{(1)} &= \Pr[E_1 \to E_1] = 0 \\ f_{11}^{(2)} &= 0 \\ f_{11}^{(3)} &= \Pr[E_1 \to E_2 \to E_3 \to E_1] = 1 \\ f_{11}^{(4)} &= 0 \\ f_{11}^{(5)} &= 0 \end{aligned}$$

$$\begin{aligned} d &= GCD\{n : p_{jj}^{(n)} > 0 \\ GCD\{3\} &= 3 \end{aligned}$$

Hence the Markov chain has a period of 3.

The n^{th} power

$$\mathbf{P}^3 = \mathbf{P}^2\mathbf{P} = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{P}^4 = \mathbf{P}^3\mathbf{P} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right]$$

$$\mathbf{P}^5 = \mathbf{P}^4\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{P}^6 = \mathbf{P}^5\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In general

$$\mathbf{P}^n = \left\{ \begin{array}{l} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ where } n = 1+3x,\ x = 0,1,2\dots \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ where } n = 2+3x,\ x = 0,1,2\dots \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } n = 3+3x,\ x = 0,1,2\dots \end{array} \right\}$$

6.2.6 Special case when $p = q = \frac{1}{2}$

$$\mathbf{P} = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

Consider E_1, to show that $f_1 = \sum_{n=1}^{\infty} f_{11}^{(n)} \leq 1$

$$\begin{aligned}
f_{11}^{(1)} &= 0 \\
f_{11}^{(2)} &= \Pr[E_1 \to E_2 \to E_1] + \Pr[E_1 \to E_3 \to E_1] = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} \\
f_{11}^{(3)} &= \Pr[E_1 \to E_2 \to E_3 \to E_1] + \Pr[E_1 \to E_3 \to E_2 \to E_1] \\
&= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^2 \\
f_{11}^{(3)} &= \Pr[E_1 \to E_2 \to E_3 \to E_2 \to E_1] + \Pr[E_1 \to E_3 \to E_2 \to E_3 \to E_1] \\
&= \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^3
\end{aligned}$$

In general

$$f_{11}^{(n)} = \left\{ \begin{array}{c} 0,\ n = 1 \\ \left(\frac{1}{2}\right)^{n-1},\ n \geq 2 \end{array} \right\}$$

$$\begin{aligned}
f_1 &= \sum_{n=1}^{\infty} f_{11}^{(n)} \\
&= 0 + \sum_{n=2}^{\infty} f_{11}^{(n)} \\
&= \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^{n-1} \\
&= \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots \\
&= \left(\frac{1}{2}\right) \frac{1}{1 - \left(\frac{1}{2}\right)} \\
&= \left(\frac{1}{2}\right) \frac{1}{\left(\frac{1}{2}\right)} \\
&= 1
\end{aligned}$$

since it is a geometric series. Therefore, E_1 is persistent since $f_1 = 1$.

$$\begin{aligned}
\mu_1 &= \sum_{n=2}^{\infty} n f_{11}^{(n)} \\
&= 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{1-x}\right) &= \frac{d}{dx}(1-x)^{-1} \\
&= \frac{d}{dx}\left(1 + x + x^2 + x^3 + \right)
\end{aligned}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots$$

$$\begin{aligned}
\mu_1 &= -1 + 1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \cdots \\
&= -1 + \frac{1}{\left(1-\frac{1}{2}\right)^2} \\
&= 4 - 1 \\
&= 3
\end{aligned}$$

Periodicity

since

$$\begin{aligned}
p_{jj}^{(n)} &= \sum_{v=1}^{n} f_{jj}^{(v)} p_{jj}^{(n-v)} \\
&= \sum_{v=1}^{n-1} f_{jj}^{(v)} p_{jj}^{(n-v)} + f_{jj}^{(n)} p_{jj}^{(0)} \\
&= \sum_{v=1}^{n-1} f_{jj}^{(v)} p_{jj}^{(n-v)} + f_{jj}^{(n)}
\end{aligned}$$

if

$$f_{jj}^{(n)} > 0 \text{ then } \implies p_{jj}^{(n)} > 0$$

$$\begin{aligned}
d &= GCD\{n : p_{jj}^{(n)} > 0 \\
&= GCD\{n : f_{jj}^{(n)} > 0 \\
&= GCD\{2, 3, 4 \ldots\} \\
&= 1
\end{aligned}$$

since

$$f_{jj}^{(n)} = \left(\frac{1}{2}\right)^{n-1} > 0, \text{ for } n \geq 2 \text{ and } f_{11}^{(1)} = 0$$

then E_j is

(i) persistent

(ii) non-null

(iii) aperiodic

Hence E_1 is ergodic

The n^{th} power

$$\mathbf{P} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$\mathbf{P}^2 = \mathbf{P}\mathbf{P} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$\mathbf{P}^3 = \mathbf{P}^2\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{bmatrix}$$

$$\mathbf{P}^4 = \mathbf{P}^3\mathbf{P} = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{5}{16} & \frac{5}{16} \\ \frac{5}{16} & \frac{3}{8} & \frac{5}{16} \\ \frac{5}{16} & \frac{5}{16} & \frac{3}{8} \end{bmatrix}$$

$$\mathbf{P}^5 = \mathbf{P}^4\mathbf{P} = \begin{bmatrix} \frac{3}{8} & \frac{5}{16} & \frac{5}{16} \\ \frac{5}{16} & \frac{3}{8} & \frac{5}{16} \\ \frac{5}{16} & \frac{5}{16} & \frac{3}{8} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{16} & \frac{11}{32} & \frac{11}{32} \\ \frac{11}{32} & \frac{5}{16} & \frac{11}{32} \\ \frac{11}{32} & \frac{11}{32} & \frac{5}{16} \end{bmatrix}$$

$$\mathbf{P}^6 = \mathbf{P}^5\mathbf{P} = \begin{bmatrix} \frac{5}{16} & \frac{11}{32} & \frac{11}{32} \\ \frac{11}{32} & \frac{5}{16} & \frac{11}{32} \\ \frac{11}{32} & \frac{11}{32} & \frac{5}{16} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{11}{32} & \frac{21}{64} & \frac{21}{64} \\ \frac{21}{64} & \frac{11}{32} & \frac{21}{64} \\ \frac{21}{64} & \frac{21}{64} & \frac{11}{32} \end{bmatrix}$$

$$\mathbf{P}^7 = \mathbf{P}^6\mathbf{P} = \begin{bmatrix} \frac{11}{32} & \frac{21}{64} & \frac{21}{64} \\ \frac{21}{64} & \frac{11}{32} & \frac{21}{64} \\ \frac{21}{64} & \frac{21}{64} & \frac{11}{32} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{21}{64} & \frac{43}{128} & \frac{43}{128} \\ \frac{43}{128} & \frac{21}{64} & \frac{43}{128} \\ \frac{43}{128} & \frac{43}{128} & \frac{21}{64} \end{bmatrix}$$

In general, when n is odd

$$\mathbf{P}^n = \left\{ \begin{array}{c} \frac{2^n+1}{2^n \times 3}, \text{ for the diagonal elements} \\ \frac{2^n-2}{2^n \times 3}, \text{ for the other elements} \end{array} \right\}$$

and when n is even

$$\mathbf{P}^n = \left\{ \begin{array}{c} \frac{2^n-1}{2^n \times 3}, \text{ for the diagonal elements} \\ \frac{2^n+2}{2^n \times 3}, \text{ for the other elements} \end{array} \right\}$$

6.3 A 4×4 Transition Probability Matrix

$$\mathbf{P} = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{cccc} E_0 & E_1 & E_2 & E_3 \end{array} \\ \left[\begin{array}{cccc} 0 & p & 0 & q \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ p & 0 & q & 0 \end{array}\right] \end{array}$$

6.3.1 Classification of the states

$$\begin{aligned}
f_{00}^{(1)} &= \Pr[E_0 \to E_0] = 0 \\
f_{00}^{(2)} &= \Pr[E_0 \to E_1 \to E_0] + \Pr[E_0 \to E_3 \to E_0] = pq + qp = 2pq \\
f_{00}^{(3)} &= 0 \\
f_{00}^{(4)} &= \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_0] + \Pr[E_0 \to E_1 \to E_2 \to E_3 \to E_0] \\
&\quad + \Pr[E_0 \to E_3 \to E_2 \to E_3 \to E_0] + \Pr[E_0 \to E_3 \to E_2 \to E_1 \to E_0] \\
&= 2p^2q^2 + p^4 + q^4 \\
f_{00}^{(5)} &= 0
\end{aligned}$$

$$\begin{aligned}
f_{00}^{(6)} &= \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_3 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_3 \to E_2 \to E_3 \to E_0] \\
&\quad + \Pr[E_0 \to E_3 \to E_2 \to E_3 \to E_2 \to E_3 \to E_0] \\
&\quad + \Pr[E_0 \to E_3 \to E_2 \to E_1 \to E_2 \to E_3 \to E_0] \\
&\quad + \Pr[E_0 \to E_3 \to E_2 \to E_3 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_2 \to E_3 \to E_0] \\
&\quad + \Pr[E_0 \to E_3 \to E_2 \to E_1 \to E_2 \to E_1 \to E_0] \\
&= 4p^3q^3 + 2p^5q^2 + 2pq^5
\end{aligned}$$

$$f_{00}^{(7)} = 0$$

$$
\begin{aligned}
f_{00}^{(8)} &= \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_2 \to E_1 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_2 \to E_1 \to E_2 \to E_3 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_2 \to E_3 \to E_2 \to E_3 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_1 \to E_2 \to E_3 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_3 \to E_2 \to E_3 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_3 \to E_2 \to E_3 \to E_2 \to E_3 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_3 \to E_2 \to E_1 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_1 \to E_2 \to E_3 \to E_2 \to E_1 \to E_2 \to E_3 \to E_0] \\
&\quad + \Pr[E_0 \to E_3 \to E_2 \to E_3 \to E_2 \to E_3 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_3 \to E_2 \to E_3 \to E_2 \to E_3 \to E_2 \to E_3 \to E_0] \\
&\quad + \Pr[E_0 \to E_3 \to E_2 \to E_3 \to E_2 \to E_1 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_3 \to E_2 \to E_3 \to E_2 \to E_1 \to E_2 \to E_3 \to E_0] \\
&\quad + \Pr[E_0 \to E_3 \to E_2 \to E_1 \to E_2 \to E_1 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_3 \to E_2 \to E_1 \to E_2 \to E_1 \to E_2 \to E_3 \to E_0] \\
&\quad + \Pr[E_0 \to E_3 \to E_2 \to E_1 \to E_2 \to E_3 \to E_2 \to E_1 \to E_0] \\
&\quad + \Pr[E_0 \to E_3 \to E_2 \to E_1 \to E_2 \to E_3 \to E_2 \to E_3 \to E_0] \\
&= 8p^4q^4 + 4p^6q^2 + 4p^2q^4
\end{aligned}
$$

$$
\begin{aligned}
f_0 &= \sum_{n=1}^{\infty} f_{00}^{(n)} \\
&= 2pq + 2p^2q^2 + p^4 + q^4 + 4p^3q^3 + 2p^5q + 2pq^5 + 8p^4q^4 + 4p^6q^2 + 4p^2q^4 + \cdots \\
&= pq + p^4 + q^4 \\
&\quad + pq + 2p^2q^2 + 4p^3q^3 + 8p^4q^4 + \cdots \\
&\quad + 2p^5q + 4p^6q^2 + 8p^7q^3 + \cdots \\
&\quad + 2q^5p + 4q^6p^2 + 8q^7p^3 + \cdots \\
&= 1
\end{aligned}
$$

The asymptotic behavior

$$\pi = \mathbf{P}'\pi$$

$$
\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}
=
\begin{bmatrix} 0 & p & 0 & q \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ p & 0 & q & 0 \end{bmatrix}
\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}
$$

$$\begin{aligned} \pi_0 &= q\pi_1 + p\pi_3 && \text{(i)} \\ \pi_1 &= p\pi_0 + q\pi_2 && \text{(ii)} \\ \pi_2 &= p\pi_1 + q\pi_3 && \text{(iii)} \\ \pi_3 &= q\pi_0 + p\pi_2 && \text{(iv)} \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 &= 1 && \text{(v)} \end{aligned}$$

Using (i)

$$\pi_0 = q\pi_1 + p\pi_3$$

but from (ii)

$$\pi_1 = p\pi_0 + q\pi_2$$

$$\begin{aligned} \pi_0 &= q(p\pi_0 + q\pi_2) + p\pi_3 \\ &= qp\pi_0 + q^2\pi_2 + p\pi_3 \end{aligned}$$

but

$$\pi_3 = q\pi_0 + p\pi_2$$

$\Longrightarrow$

$$\pi_0 = qp\pi_0 + q^2\pi_2 + p(q\pi_0 + p\pi_2)$$

$$\begin{aligned} (q^2 + p^2)\pi_2 &= \pi_0(1 - 2pq) \\ \pi_2 &= \pi_0 \end{aligned}$$

using (iv)

$$\begin{aligned} \pi_3 &= q\pi_0 + p\pi_0 \\ &= (p+q)\pi_0 \\ &= \pi_0 \end{aligned}$$

using (i)

$$\begin{aligned} \pi_0 &= q\pi_1 + p\pi_3 \\ &= q\pi_1 + p\pi_0 \\ (1-p)\pi_0 &= q\pi_1 \\ \pi_0 &= \pi_1 \end{aligned}$$

$\Longrightarrow$

$$\pi_0 = \pi_1 = \pi_2 = \pi_3 = \frac{1}{4}$$

that is

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

$$\begin{aligned} \mu_0 &= \sum_{n=1}^{\infty} n f_{00}^{(n)} \\ &= \frac{1}{\pi_k} \\ &= 4 < \infty \end{aligned}$$

μ is finite and hence E_1 is non-null.

Periodicity

$$\begin{aligned} d &= GCD\{n : f_{jj}^{(n)} > 0\} \\ &= 1 \end{aligned}$$

Thus $E_0, E_1, E_2,$ and E_3 are persistent, non-null, and aperiodic hence they are ergodic. In general, the asymptotic behavior of an $n \times n$ transition probability matrix of a cyclic random walk is given by

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \vdots \\ \pi_n \end{bmatrix} = \begin{bmatrix} \frac{1}{n} \\ \frac{1}{n} \\ \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix}$$

$$\begin{aligned} \mu_0 &= \sum_{n=1}^{\infty} n f_{00}^{(n)} \\ &= \frac{1}{\pi_k} \\ &= n < \infty \end{aligned}$$

6.3.2 The n^{th} power

We determine the corresponding eigenvalues by solving the characteristic function

$$|\mathbf{P}-\lambda\mathbf{I}|=0$$

$$\left|\begin{bmatrix} 0 & p & 0 & q \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ p & 0 & q & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}\right| = \begin{vmatrix} -\lambda & p & 0 & q \\ q & -\lambda & p & 0 \\ 0 & q & -\lambda & p \\ p & 0 & q & -\lambda \end{vmatrix}$$

$$\begin{vmatrix} -\lambda & p & 0 & q \\ q & -\lambda & p & 0 \\ 0 & q & -\lambda & p \\ p & 0 & q & -\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} -\lambda\ +p+q & -\lambda\ +p+q & -\lambda\ +p+q & -\lambda\ +p+q \\ q & -\lambda & p & 0 \\ 0 & q & -\lambda & p \\ p & 0 & q & -\lambda \end{vmatrix} = 0$$

$$(-\lambda\ +p+q)\begin{vmatrix} 1 & 1 & 1 & 1 \\ q & -\lambda & p & 0 \\ 0 & q & -\lambda & p \\ p & 0 & q & -\lambda \end{vmatrix} = 0$$

$$(-\lambda\ +p+q)\begin{vmatrix} 1 & 0 & 0 & 0 \\ q & -\lambda & p & 0 \\ 0 & q & -\lambda & p \\ p & 0 & q & -\lambda \end{vmatrix} = 0$$

$$p+q=1$$

$\Longrightarrow$

$$\lambda_1 = 1$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ q & -\lambda & p & 0 \\ 0 & q & -\lambda & p \\ p & 0 & q & -\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} -\lambda-q & p-q & -q \\ q & -\lambda & p \\ -p & q-p & -\lambda-p \end{vmatrix} = 0$$

$$\begin{vmatrix} -\lambda-q-p & 0 & -\lambda-q-p \\ q & -\lambda & p \\ -p & q-p & -\lambda-p \end{vmatrix} = 0$$

$$(-\lambda-1)\left\{\begin{vmatrix} -\lambda & p \\ q-p & -\lambda-p \end{vmatrix} + \begin{vmatrix} q & -\lambda \\ -p & q-p \end{vmatrix}\right\}$$

$$\lambda_2 = -1$$

$$\begin{vmatrix} -\lambda & p \\ q-p & -\lambda-p \end{vmatrix} + \begin{vmatrix} q & -\lambda \\ -p & q-p \end{vmatrix} = 0$$

$$\lambda^2 + p^2 + q^2 - 2pq$$

using

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

we have

$$\begin{aligned} \lambda &= \frac{-0 \pm \sqrt{-4\left(p^2+q^2-2pq\right)}}{2} \\ &= \frac{\pm 2\sqrt{-\left(q-p\right)^2}}{2} \\ \lambda_3 &= \sqrt{(-)}\left(q-p\right) \\ \lambda_4 &= -\sqrt{(-)}\left(q-p\right) \end{aligned}$$

The corresponding eigenvalues are

$$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = \sqrt{(-)}\left(q-p\right), \quad \text{and } \lambda_4 = -\sqrt{(-)}\left(q-p\right)$$

$$\begin{aligned} |\mathbf{A}(\lambda)| &= |\lambda\mathbf{I}-\mathbf{P}| \\ &= \begin{vmatrix} \lambda & -p & 0 & -q \\ -q & \lambda & -p & 0 \\ 0 & -q & \lambda & -p \\ -p & 0 & -q & \lambda \end{vmatrix} \end{aligned}$$

When $\lambda = 1$,

$$\begin{aligned}|\mathbf{A}(\lambda)| &= |\lambda\mathbf{I} - \mathbf{P}| \\ &= \begin{vmatrix} 1 & -p & 0 & -q \\ -q & 1 & -p & 0 \\ 0 & -q & 1 & -p \\ -p & 0 & -q & 1 \end{vmatrix}\end{aligned}$$

Minors of the matrix are

$$\begin{vmatrix} 1 & -p & 0 \\ -q & 1 & -p \\ 0 & -q & 1 \end{vmatrix}\begin{vmatrix} -q & -p & 0 \\ 0 & 1 & -p \\ -p & -q & 1 \end{vmatrix}\begin{vmatrix} -q & 1 & 0 \\ 0 & -q & -p \\ -p & 0 & 1 \end{vmatrix}\begin{vmatrix} -q & 1 & -p \\ 0 & -q & 1 \\ -p & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & -q \\ -q & 1 & -p \\ 0 & -q & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 & -q \\ 0 & 1 & -p \\ -p & -q & 1 \end{vmatrix}\begin{vmatrix} 1 & -p & -q \\ 0 & -q & -p \\ -p & 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & -p & 0 \\ 0 & -q & 1 \\ -p & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & -q \\ 1 & -p & 0 \\ 0 & -q & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 & -q \\ -q & -p & 0 \\ -p & -q & 1 \end{vmatrix}\begin{vmatrix} 1 & -p & -q \\ -q & 1 & 0 \\ -p & 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & -p & 0 \\ -q & 1 & -p \\ -p & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & -q \\ 1 & -p & 0 \\ -q & 1 & -p \end{vmatrix}\begin{vmatrix} 1 & 0 & -q \\ -q & -p & 0 \\ 0 & 1 & -p \end{vmatrix}\begin{vmatrix} 1 & -p & -q \\ -q & 1 & 0 \\ 0 & -q & -p \end{vmatrix}\begin{vmatrix} 1 & -p & 0 \\ -q & 1 & -p \\ 0 & -q & 1 \end{vmatrix}$$

$$\begin{aligned}\begin{vmatrix} 1 & -p & 0 \\ -q & 1 & -p \\ 0 & -q & 1 \end{vmatrix} &= 1\begin{vmatrix} 1 & -p \\ -q & 1 \end{vmatrix} + p\begin{vmatrix} -q & -p \\ 0 & 1 \end{vmatrix} \\ &= (1 - pq) + (-pq) = 1 - 2pq = p^2 + q^2\end{aligned}$$

$$\begin{aligned}\begin{vmatrix} -q & -p & 0 \\ 0 & 1 & -p \\ -p & -q & 1 \end{vmatrix} &= -q\begin{vmatrix} 1 & -p \\ -q & 1 \end{vmatrix} + p\begin{vmatrix} 0 & -p \\ -p & 1 \end{vmatrix} \\ &= -q(1 - pq) + p(-p^2) = q + pq^2p^3 = pq^2 - q - p^3 \\ &= p(q^2 - p^2) - q = p(q - p)(q + p) - q = -(p^2 + q^2)\end{aligned}$$

$$\begin{vmatrix} -q & 1 & 0 \\ 0 & -q & -p \\ -p & 0 & 1 \end{vmatrix} = -q \begin{vmatrix} -q & -p \\ 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & -p \\ -p & 1 \end{vmatrix} = p^2 + q^2$$

$$\begin{aligned} \begin{vmatrix} -q & 1 & -p \\ 0 & -q & 1 \\ -p & 0 & -q \end{vmatrix} &= -q \begin{vmatrix} -q & 1 \\ 0 & -q \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ -p & -q \end{vmatrix} - p \begin{vmatrix} 0 & -q \\ -p & 0 \end{vmatrix} \\ &= -q^3 - p + p^2 q = p^2 q - p - q^3 = -(p^2 + q^2) \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} -p & 0 & -q \\ -q & 1 & -p \\ 0 & -q & 1 \end{vmatrix} &= -p \begin{vmatrix} 1 & -p \\ -q & 1 \end{vmatrix} - q \begin{vmatrix} -q & 1 \\ 0 & -q \end{vmatrix} \\ &= -p(1 - pq) - q(q^2) = p^2 q - p - q^3 = -(p^2 + q^2) \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} 1 & 0 & -q \\ 0 & 1 & -p \\ -p & -q & 1 \end{vmatrix} &= 1 \begin{vmatrix} 1 & -p \\ -q & 1 \end{vmatrix} - q \begin{vmatrix} 0 & 1 \\ -p & -q \end{vmatrix} \\ &= 1 - pq - pq = p^2 + q^2 \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} 1 & -p & -q \\ 0 & -q & -p \\ -p & 0 & 1 \end{vmatrix} &= 1 \begin{vmatrix} -q & -p \\ 0 & 1 \end{vmatrix} + p \begin{vmatrix} -p & -q \\ -q & -p \end{vmatrix} \\ &= q + p(p^2 - q^2) = q + (1 - q)(p^2 - q^2) \\ &= -(p^2 + q^2) \end{aligned}$$

$$\begin{vmatrix} 1 & -p & 0 \\ 0 & -q & 1 \\ -p & 0 & -q \end{vmatrix} = 1 \begin{vmatrix} -q & 1 \\ 0 & -q \end{vmatrix} + p \begin{vmatrix} 0 & 1 \\ -p & -q \end{vmatrix} = (p^2 + q^2)$$

$$\begin{vmatrix} -p & 0 & -q \\ 1 & -p & 0 \\ 0 & -q & 1 \end{vmatrix} = -p \begin{vmatrix} -p & 0 \\ -q & 1 \end{vmatrix} - q \begin{vmatrix} 1 & -p \\ 0 & -q \end{vmatrix} = (p^2 + q^2)$$

$$\begin{vmatrix} 1 & 0 & -q \\ -q & -p & 0 \\ -p & -q & 1 \end{vmatrix} = 1\begin{vmatrix} -p & 0 \\ -q & 1 \end{vmatrix} - q\begin{vmatrix} -q & -p \\ -p & -q \end{vmatrix}$$

$$-p - q(q^2 - p) = p^2q - p - q^3 = -(p^2+q^2)$$

$$\begin{vmatrix} 1 & -p & -q \\ -q & 1 & 0 \\ -p & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + p\begin{vmatrix} -q & 0 \\ -p & 1 \end{vmatrix} - q\begin{vmatrix} -q & 1 \\ -p & 0 \end{vmatrix}$$

$$= 1 + p(-q) - q(p) = (p^2+q^2)$$

$$\begin{vmatrix} 1 & -p & 0 \\ -q & 1 & -p \\ -p & 0 & -q \end{vmatrix} = 1\begin{vmatrix} 1 & -p \\ 0 & -q \end{vmatrix} + p\begin{vmatrix} -q & -p \\ -p & -q \end{vmatrix}$$

$$= -q + p(q^2 - p^2) = pq^2 - q - p^3 = -(p^2+q^2)$$

$$\begin{vmatrix} -p & 0 & -q \\ 1 & -p & 0 \\ -q & 1 & -p \end{vmatrix} = -p\begin{vmatrix} -p & 0 \\ 1 & -p \end{vmatrix} - q\begin{vmatrix} 1 & -p \\ -q & 1 \end{vmatrix}$$

$$= pq^2 - q - p^3 = -(p^2+q^2)$$

$$\begin{vmatrix} 1 & 0 & -q \\ -q & -p & 0 \\ 0 & 1 & -p \end{vmatrix} = 1\begin{vmatrix} -p & 0 \\ 1 & -p \end{vmatrix} - q\begin{vmatrix} -q & -p \\ 0 & 1 \end{vmatrix}$$

$$= (p^2+q^2)$$

$$\begin{vmatrix} 1 & -p & -q \\ -q & 1 & 0 \\ 0 & -q & -p \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ -q & -p \end{vmatrix} + p\begin{vmatrix} -q & 0 \\ 0 & -p \end{vmatrix} - q\begin{vmatrix} -q & 1 \\ 0 & -q \end{vmatrix}$$

$$= p^2q - p - q^3 = -(p^2+q^2)$$

$$\begin{vmatrix} 1 & -p & 0 \\ -q & 1 & -p \\ 0 & -q & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & -p \\ -q & 1 \end{vmatrix} + p\begin{vmatrix} -q & -p \\ 0 & 1 \end{vmatrix}$$

$$= 1 - pq - pq = (p^2+q^2)$$

$$= \begin{bmatrix} (p^2+q^2) & -(p^2+q^2) & (p^2+q^2) & -(p^2+q^2) \\ -(p^2+q^2) & (p^2+q^2) & -(p^2+q^2) & (p^2+q^2) \\ (p^2+q^2) & -(p^2+q^2) & (p^2+q^2) & -(p^2+q^2) \\ -(p^2+q^2) & (p^2+q^2) & -(p^2+q^2) & (p^2+q^2) \end{bmatrix}$$

Hence

$$\text{the adjoint of } |A(\lambda)| = \begin{bmatrix} (p^2+q^2) & -(p^2+q^2) & (p^2+q^2) & -(p^2+q^2) \\ -(p^2+q^2) & (p^2+q^2) & -(p^2+q^2) & (p^2+q^2) \\ (p^2+q^2) & -(p^2+q^2) & (p^2+q^2) & -(p^2+q^2) \\ -(p^2+q^2) & (p^2+q^2) & -(p^2+q^2) & (p^2+q^2) \end{bmatrix}$$

When $\lambda = -1$,

$$\begin{aligned} |\mathbf{A}(\lambda)| &= |\lambda\mathbf{I} - \mathbf{P}| \\ &= \begin{vmatrix} -1 & -p & 0 & -q \\ -q & -1 & -p & 0 \\ 0 & -q & -1 & -p \\ -p & 0 & -q & -1 \end{vmatrix} \end{aligned}$$

Minors of the matrix are

$$\begin{vmatrix} -1 & -p & 0 \\ -q & -1 & -p \\ 0 & -q & -1 \end{vmatrix} \begin{vmatrix} -q & -p & 0 \\ 0 & -1 & -p \\ -p & -q & -1 \end{vmatrix} \begin{vmatrix} -q & -1 & 0 \\ 0 & -q & -p \\ -p & 0 & -1 \end{vmatrix} \begin{vmatrix} -q & -1 & -p \\ 0 & -q & -1 \\ -p & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & -q \\ -q & -1 & -p \\ 0 & -q & -1 \end{vmatrix} \begin{vmatrix} -1 & 0 & -q \\ 0 & -1 & -p \\ -p & -q & -1 \end{vmatrix} \begin{vmatrix} -1 & -p & -q \\ 0 & -q & -p \\ -p & 0 & -1 \end{vmatrix} \begin{vmatrix} -1 & -p & 0 \\ 0 & -q & -1 \\ -p & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & -q \\ -1 & -p & 0 \\ 0 & -q & -1 \end{vmatrix} \begin{vmatrix} -1 & 0 & -q \\ -q & -p & 0 \\ -p & -q & -1 \end{vmatrix} \begin{vmatrix} -1 & -p & -q \\ -q & -1 & 0 \\ -p & 0 & -1 \end{vmatrix} \begin{vmatrix} -1 & -p & 0 \\ -q & -1 & -p \\ -p & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & -q \\ -1 & -p & 0 \\ -q & -1 & -p \end{vmatrix} \begin{vmatrix} -1 & 0 & -q \\ -q & -p & 0 \\ 0 & -1 & -p \end{vmatrix} \begin{vmatrix} -1 & -p & -q \\ -q & -1 & 0 \\ 0 & -q & -p \end{vmatrix} \begin{vmatrix} -1 & -p & 0 \\ -q & -1 & -p \\ 0 & -q & -1 \end{vmatrix}$$

$$\begin{vmatrix} -1 & -p & 0 \\ -q & -1 & -p \\ 0 & -q & -1 \end{vmatrix} = -1\begin{vmatrix} -1 & -p \\ -q & -1 \end{vmatrix} + p\begin{vmatrix} -q & -p \\ 0 & -1 \end{vmatrix}$$
$$= -(1-pq)+(pq) = -(1-2pq) = -\left(p^2+q^2\right)$$

$$\begin{vmatrix} -q & -p & 0 \\ 0 & -1 & -p \\ -p & -q & -1 \end{vmatrix} = -q\begin{vmatrix} -1 & -p \\ -q & -1 \end{vmatrix} + p\begin{vmatrix} 0 & -p \\ -p & -1 \end{vmatrix}$$
$$= -q(1-pq)+p(-p^2) = q+pq^2p^3 = pq^2-q-p^3$$
$$= p(q^2-p^2)-q = p(q-p)(q+p)-q = -(p^2+q^2)$$

$$\begin{vmatrix} -q & -1 & 0 \\ 0 & -q & -p \\ -p & 0 & -1 \end{vmatrix} = -q\begin{vmatrix} -q & -p \\ 0 & -1 \end{vmatrix} + 1\begin{vmatrix} 0 & -p \\ -p & -1 \end{vmatrix} = -\left(p^2+q^2\right)$$

$$\begin{vmatrix} -q & -1 & -p \\ 0 & -q & -1 \\ -p & 0 & -q \end{vmatrix} = -q\begin{vmatrix} -q & -1 \\ 0 & -q \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ -p & -q \end{vmatrix} - p\begin{vmatrix} 0 & -q \\ -p & 0 \end{vmatrix}$$
$$= -q^3-p+p^2q = p^2q-p-q^3 = -(p^2+q^2)$$

$$\begin{vmatrix} -p & 0 & -q \\ -q & -1 & -p \\ 0 & -q & -1 \end{vmatrix} = -p\begin{vmatrix} -1 & -p \\ -q & -1 \end{vmatrix} - q\begin{vmatrix} -q & -1 \\ 0 & -q \end{vmatrix}$$
$$= -p(1-pq)-q(q^2) = p^2q-p-q^3 = -(p^2+q^2)$$

$$\begin{vmatrix} -1 & 0 & -q \\ 0 & -1 & -p \\ -p & -q & -1 \end{vmatrix} = -1\begin{vmatrix} -1 & -p \\ -q & -1 \end{vmatrix} - q\begin{vmatrix} 0 & -1 \\ -p & -q \end{vmatrix}$$
$$= -(1-pq-pq) = -\left(p^2+q^2\right)$$

$$\begin{vmatrix} -1 & -p & -q \\ 0 & -q & -p \\ -p & 0 & -1 \end{vmatrix} = -1\begin{vmatrix} -q & -p \\ 0 & -1 \end{vmatrix} + p\begin{vmatrix} -p & -q \\ -q & -p \end{vmatrix}$$
$$= q + p(p^2 - q^2) = q + (1-q)(p^2-q^2)$$
$$= -(p^2+q^2)$$

$$\begin{vmatrix} -1 & -p & 0 \\ 0 & -q & -1 \\ -p & 0 & -q \end{vmatrix} = -1\begin{vmatrix} -q & -1 \\ 0 & -q \end{vmatrix} + p\begin{vmatrix} 0 & -1 \\ -p & -q \end{vmatrix} = -(p^2+q^2)$$

$$\begin{vmatrix} -p & 0 & -q \\ -1 & -p & 0 \\ 0 & -q & -1 \end{vmatrix} = -p\begin{vmatrix} -p & 0 \\ -q & -1 \end{vmatrix} - q\begin{vmatrix} -1 & -p \\ 0 & -q \end{vmatrix} = -(p^2+q^2)$$

$$\begin{vmatrix} -1 & 0 & -q \\ -q & -p & 0 \\ -p & -q & -1 \end{vmatrix} = -1\begin{vmatrix} -p & 0 \\ -q & -1 \end{vmatrix} - q\begin{vmatrix} -q & -p \\ -p & -q \end{vmatrix}$$
$$-p - q(q^2 - p) = p^2q - p - q^3 = -(p^2+q^2)$$

$$\begin{vmatrix} -1 & -p & -q \\ -q & -1 & 0 \\ -p & 0 & -1 \end{vmatrix} = -1\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} + p\begin{vmatrix} -q & 0 \\ -p & -1 \end{vmatrix} - q\begin{vmatrix} -q & -1 \\ -p & 0 \end{vmatrix}$$
$$= -(1 + p(-q) - q(p)) = -(p^2+q^2)$$

$$\begin{vmatrix} -1 & -p & 0 \\ -q & -1 & -p \\ -p & 0 & -q \end{vmatrix} = -1\begin{vmatrix} -1 & -p \\ 0 & -q \end{vmatrix} + p\begin{vmatrix} -q & -p \\ -p & -q \end{vmatrix}$$
$$= -q + p(q^2 - p^2) = pq^2 - q - p^3 = -(p^2+q^2)$$

$$\begin{vmatrix} -p & 0 & -q \\ -1 & -p & 0 \\ -q & -1 & -p \end{vmatrix} = -p\begin{vmatrix} -p & 0 \\ -1 & -p \end{vmatrix} - q\begin{vmatrix} -1 & -p \\ -q & -1 \end{vmatrix}$$
$$= pq^2 - q - p^3 = -(p^2+q^2)$$

$$\begin{vmatrix} -1 & 0 & -q \\ -q & -p & 0 \\ 0 & -1 & -p \end{vmatrix} = -1\begin{vmatrix} -p & 0 \\ -1 & -p \end{vmatrix} - q\begin{vmatrix} -q & -p \\ 0 & -1 \end{vmatrix}$$
$$= -(p^2+q^2)$$

$$\begin{vmatrix} -1 & -p & -q \\ -q & -1 & 0 \\ 0 & -q & -p \end{vmatrix} = -1\begin{vmatrix} -1 & 0 \\ -q & -p \end{vmatrix} + p\begin{vmatrix} -q & 0 \\ 0 & -p \end{vmatrix} - q\begin{vmatrix} -q & -1 \\ 0 & -q \end{vmatrix}$$
$$= p^2q - p - q^3 = -(p^2+q^2)$$

$$\begin{vmatrix} -1 & -p & 0 \\ -q & -1 & -p \\ 0 & -q & -1 \end{vmatrix} = -1\begin{vmatrix} -1 & -p \\ -q & -1 \end{vmatrix} + p\begin{vmatrix} -q & -p \\ 0 & -1 \end{vmatrix}$$
$$= -(1-pq-pq) = -(p^2+q^2)$$

$$= \begin{bmatrix} -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) \\ -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) \\ -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) \\ -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) \end{bmatrix}$$

Hence

$$\text{the adjoint of } |A(\lambda)| = \begin{bmatrix} -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) \\ -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) \\ -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) \\ -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) & -(p^2+q^2) \end{bmatrix}$$

When $\lambda = i(q-p)$,

$$|\mathbf{A}(\lambda)| = |\lambda \mathbf{I} - \mathbf{P}|$$
$$= \begin{vmatrix} i(q-p) & -p & 0 & -q \\ -q & i(q-p) & -p & 0 \\ 0 & -q & i(q-p) & -p \\ -p & 0 & -q & i(q-p) \end{vmatrix}$$

Minors of the matrix are

$$\begin{vmatrix} i(q-p) & -p & 0 \\ -q & i(q-p) & -p \\ 0 & -q & i(q-p) \end{vmatrix} \begin{vmatrix} -q & -p & 0 \\ 0 & i(q-p) & -p \\ -p & -q & i(q-p) \end{vmatrix} \begin{vmatrix} -q & i(q-p) & 0 \\ 0 & -q & -p \\ -p & 0 & i(q-p) \end{vmatrix}$$

$$\begin{vmatrix} -q & i(q-p) & -p \\ 0 & -q & i(q-p) \\ -p & 0 & -q \end{vmatrix} \begin{vmatrix} -p & 0 & -q \\ -q & i(q-p) & -p \\ 0 & -q & i(q-p) \end{vmatrix} \begin{vmatrix} i(q-p) & 0 & -q \\ 0 & i(q-p) & -p \\ -p & -q & i(q-p) \end{vmatrix}$$

$$\begin{vmatrix} i(q-p) & -p & -q \\ 0 & -q & -p \\ -p & 0 & i(q-p) \end{vmatrix} \begin{vmatrix} i(q-p) & -p & 0 \\ 0 & -q & i(q-p) \\ -p & 0 & -q \end{vmatrix} \begin{vmatrix} -p & 0 & -q \\ i(q-p) & -p & 0 \\ 0 & -q & i(q-p) \end{vmatrix}$$

$$\begin{vmatrix} i(q-p) & 0 & -q \\ -q & -p & 0 \\ -p & -q & i(q-p) \end{vmatrix} \begin{vmatrix} i(q-p) & -p & -q \\ -q & i(q-p) & 0 \\ -p & 0 & i(q-p) \end{vmatrix} \begin{vmatrix} i(q-p) & -p & 0 \\ -q & i(q-p) & -p \\ -p & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & -q \\ i(q-p) & -p & 0 \\ -q & i(q-p) & -p \end{vmatrix} \begin{vmatrix} i(q-p) & 0 & -q \\ -q & -p & 0 \\ 0 & i(q-p) & -p \end{vmatrix} \begin{vmatrix} i(q-p) & -p & -q \\ -q & i(q-p) & 0 \\ 0 & -q & -p \end{vmatrix}$$

$$\begin{vmatrix} i(q-p) & -p & 0 \\ -q & i(q-p) & -p \\ 0 & -q & i(q-p) \end{vmatrix}$$

$$\begin{aligned}
\begin{vmatrix} i(q-p) & -p & 0 \\ -q & i(q-p) & -p \\ 0 & -q & i(q-p) \end{vmatrix} &= i(q-p)\begin{vmatrix} i(q-p) & -p \\ -q & i(q-p) \end{vmatrix} + p\begin{vmatrix} -q & -p \\ 0 & i(q-p) \end{vmatrix} \\
&= i(q-p)[-1(q-p)^2 - pq] + p[-iq(q-p)] \\
&= i(q-p)[-1(q^2-2pq+p^2) - pq] + [-ipq(q-p)] \\
&= i(q-p)(p+q)(-p^2-q^2) = -i(q^4-p^4) \\
&= i(p^4-q^4)
\end{aligned}$$

$$\begin{vmatrix} -q & -p & 0 \\ 0 & i(q-p) & -p \\ -p & -q & i(q-p) \end{vmatrix} = -q\begin{vmatrix} i(q-p) & -p \\ -q & i(q-p) \end{vmatrix} + p\begin{vmatrix} 0 & -p \\ -p & i(q-p) \end{vmatrix}$$

$$\begin{aligned} &= -q[-1(q-p)-pq]-p^3 = -q[-1(q^2-2pq+p^2)-pq]-p^3 \\ &= (q^3-pq^2+p^2q-p^3)(p+q) \\ &= q^4-pq^3+p^2q^2-p^3q+pq^3-p^2q^2+p^3q-p^4 \\ &= -(p^4-q^4) \end{aligned}$$

$$\begin{vmatrix} -q & i(q-p) & 0 \\ 0 & -q & -p \\ -p & 0 & i(q-p) \end{vmatrix} = -q\begin{vmatrix} -q & -p \\ 0 & i(q-p) \end{vmatrix} - i(q-p)\begin{vmatrix} 0 & -p \\ -p & i(q-p) \end{vmatrix}$$

$$\begin{aligned} &= -q[-iq^2+ipq]+ip^2q-ip^3 \\ &= i(q^3-pq^2+p^2q-p^3)(p+q) \\ &= -i(p^4-q^4) \end{aligned}$$

$$\begin{vmatrix} -q & i(q-p) & -p \\ 0 & -q & i(q-p) \\ -p & 0 & -q \end{vmatrix} = -q\begin{vmatrix} -q & i(q-p) \\ 0 & -q \end{vmatrix} - i(q-p)\begin{vmatrix} 0 & i(q-p) \\ -p & -q \end{vmatrix} - p\begin{vmatrix} 0 & -q \\ -p & 0 \end{vmatrix}$$

$$\begin{aligned} &= -q^3-p[-(q-p)^2-pq] \\ &= (-q^3+pq^2-p^2q+p^3)(p+q) \\ &= (p^4-q^4) \end{aligned}$$

$$\begin{vmatrix} -p & 0 & -q \\ -q & i(q-p) & -p \\ 0 & -q & i(q-p) \end{vmatrix} = -p\begin{vmatrix} i(q-p) & -p \\ -q & i(q-p) \end{vmatrix} - q\begin{vmatrix} -q & i(q-p) \\ 0 & -q \end{vmatrix}$$

$$= -p[-(q-p)^2-pq]-q^3 = (p^4-q^4)$$

$$\begin{aligned}
\begin{vmatrix} i(q-p) & 0 & -q \\ 0 & i(q-p) & -p \\ -p & -q & i(q-p) \end{vmatrix} &= i(q-p)\begin{vmatrix} i(q-p) & -p \\ -q & i(q-p) \end{vmatrix} - q\begin{vmatrix} 0 & i(q-p) \\ -p & -q \end{vmatrix} \\
&= i(q-p)[-(q-p)^2-pq]-p[iq(q-p)] \\
&= i(q-p)[-(q^2-2pq+p^2)-pq]-[ipq(q-p)] \\
&= i(q-p)[-q^2+2pq-p^2-pq]-ipq^2+ip^2q)] \\
&= i(-q^3+pq^2-p^2q+p^3)(p+q) \\
&= i(p^4-q^4)
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} i(q-p) & -p & -q \\ 0 & -q & -p \\ -p & 0 & i(q-p) \end{vmatrix} &= i(q-p)\begin{vmatrix} -q & -p \\ 0 & i(q-p) \end{vmatrix} + p\begin{vmatrix} -p & -q \\ -q & -p \end{vmatrix} \\
&= i(q-p)[-iq(q-p)]-p(p^2-q^2) \\
&= -i^2q(q-p)^2-p^3+pq^2 \\
&= q(q-p)^2-p^3+pq^2 \\
&= -(-q^3+pq^2-p^2q+p^3)(p+q) \\
&= -(p^4-q^4)
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} i(q-p) & -p & 0 \\ 0 & -q & i(q-p) \\ -p & 0 & -q \end{vmatrix} &= i(q-p)\begin{vmatrix} -q & i(q-p) \\ 0 & -q \end{vmatrix} + p\begin{vmatrix} 0 & i(q-p) \\ -p & -q \end{vmatrix} \\
&= iq^2(q-p)+ip^1(q-p) \\
&= -i(-q^3+pq^2-p^2q+p^3)(p+q) \\
&= -i(p^4-q^4)
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} -p & 0 & -q \\ i(q-p) & -p & 0 \\ 0 & -q & i(q-p) \end{vmatrix} &= -p\begin{vmatrix} -p & 0 \\ -q & i(q-p) \end{vmatrix} - q\begin{vmatrix} i(q-p) & -p \\ 0 & -q \end{vmatrix} \\
&= -i(-q^3+pq^2-p^2q+p^3)(p+q) \\
&= -i(p^4-q^4)
\end{aligned}$$

$$\begin{vmatrix} i(q-p) & 0 & -q \\ -q & -p & 0 \\ -p & -q & i(q-p) \end{vmatrix} = i(q-p)\begin{vmatrix} -p & 0 \\ -q & i(q-p) \end{vmatrix} - q\begin{vmatrix} -q & -p \\ -p & -q \end{vmatrix}$$

$$\begin{aligned} &= -ip(q-p)[i(q-p) - q(q^2-p^2) \\ &= (pq^2 - p^2q + p^3 - q^3)(p+q) \\ &= (p^4 - q^4) \end{aligned}$$

$$\begin{vmatrix} i(q-p) & -p & -q \\ -q & i(q-p) & 0 \\ -p & 0 & i(q-p) \end{vmatrix} = i(q-p)\begin{vmatrix} i(q-p) & 0 \\ 0 & i(q-p) \end{vmatrix} + p\begin{vmatrix} -q & 0 \\ -p & i(q-p) \end{vmatrix}$$

$$-q\begin{vmatrix} -q & i(q-p) \\ -p & 0 \end{vmatrix}$$

$$\begin{aligned} &= -ip(q-p)[i(q-p)[i(q-p) - q(q^2-p^2) \\ &= p(q-p)^2 - q^3 + p^2q \\ &= (-q^3 + pq^2 - p^2q + p^3)(p+q) \\ &= (p^4 - q^4) \end{aligned}$$

$$\begin{vmatrix} i(q-p) & -p & 0 \\ -q & i(q-p) & -p \\ -p & 0 & -q \end{vmatrix} = i(q-p)\begin{vmatrix} i(q-p) & -p \\ 0 & -q \end{vmatrix} + p\begin{vmatrix} -q & -p \\ -p & -q \end{vmatrix}$$

$$\begin{aligned} &= -iq(q-p)i(q-p) + p(p^2-q^2) \\ &= q(q-p)^2 + pq^2 - p^3 \\ &= (q^3 - pq^2 + p^2q - p^3)(p+q) \\ &= -(p^4 - q^4) \end{aligned}$$

$$\begin{vmatrix} -p & 0 & -q \\ i(q-p) & -p & 0 \\ -q & i(q-p) & -p \end{vmatrix} = -p\begin{vmatrix} -p & 0 \\ i(q-p) & -p \end{vmatrix} - q\begin{vmatrix} i(q-p) & -p \\ -q & i(q-p) \end{vmatrix}$$

$$\begin{aligned} &= -p^3 - q[-(q-p)^2 - pq] \\ &= -p^3 + (q^3 - pq^2 + p^2q) \\ &= (q^3 - pq^2 + p^2q - p^3)(p+q) \\ &= -(p^4 - q^4) \end{aligned}$$

$$\begin{vmatrix} i(q-p) & 0 & -q \\ -q & -p & 0 \\ 0 & i(q-p) & -p \end{vmatrix} = i(q-p)\begin{vmatrix} -p & 0 \\ i(q-p) & -p \end{vmatrix} - q\begin{vmatrix} -q & -p \\ 0 & i(q-p) \end{vmatrix}$$

$$\begin{aligned} &= ip^2(q-p) - iq^2(q-p) \\ &= i(q^3 - pq^2 + p^2q - p^3)(p+q) \\ &= -i(p^4 - q^4) \end{aligned}$$

$$\begin{vmatrix} i(q-p) & -p & -q \\ -q & i(q-p) & 0 \\ 0 & -q & -p \end{vmatrix} = i(q-p)\begin{vmatrix} i(q-p) & 0 \\ -q & -p \end{vmatrix} + p\begin{vmatrix} -q & 0 \\ 0 & -p \end{vmatrix}$$

$$-q\begin{vmatrix} -q & i(q-p) \\ 0 & -q \end{vmatrix}$$

$$\begin{aligned} &= i(q-p)[-ip(q-p)] + q(p^2 - q^2) \\ &= -(q^3 - pq^2 + p^2q - p^3)(p+q) \\ &= (p^4 - q^4) \end{aligned}$$

$$\begin{vmatrix} i(q-p) & -p & 0 \\ -q & i(q-p) & -p \\ 0 & -q & i(q-p) \end{vmatrix} = i(q-p)\begin{vmatrix} i(q-p) & -p \\ -q & i(q-p) \end{vmatrix} + p\begin{vmatrix} -q & -p \\ 0 & i(q-p) \end{vmatrix}$$

$$\begin{aligned} &= i(q-p)[-(q^2 - 2pq + p^2) - pq] - ipq^2 + ip^2q \\ &= -i(q^3 - pq^2 + p^2q - p^3)(p+q) \\ &= i(p^4 - q^4) \end{aligned}$$

$$= \begin{bmatrix} i(p^4-q^4) & -(p^4-q^4) & -i(p^4-q^4) & (p^4-q^4) \\ (p^4-q^4) & i(p^4-q^4) & -(p^4-q^4) & -i(p^4-q^4) \\ -i(p^4-q^4) & (p^4-q^4) & i(p^4-q^4) & i(p^4-q^4) \\ -(p^4-q^4) & -i(p^4-q^4) & (p^4-q^4) & -(p^4-q^4) \end{bmatrix}$$

When $\lambda = -i(q-p)$,

$$
\begin{aligned}
|\mathbf{A}(\lambda)| &= |\lambda\mathbf{I} - \mathbf{P}| \\
&= \begin{vmatrix} -i(q-p) & -p & 0 & -q \\ -q & -i(q-p) & -p & 0 \\ 0 & -q & -i(q-p) & -p \\ -p & 0 & -q & -i(q-p) \end{vmatrix}
\end{aligned}
$$

Minors of the matrix are

$$\begin{vmatrix} -i(q-p) & -p & 0 \\ -q & -i(q-p) & -p \\ 0 & -q & -i(q-p) \end{vmatrix} \begin{vmatrix} -q & -p & 0 \\ 0 & -i(q-p) & -p \\ -p & -q & -i(q-p) \end{vmatrix}$$

$$\begin{vmatrix} -q & -i(q-p) & 0 \\ 0 & -q & -p \\ -p & 0 & -i(q-p) \end{vmatrix} \begin{vmatrix} -q & -i(q-p) & -p \\ 0 & -q & -i(q-p) \\ -p & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & -q \\ -q & -i(q-p) & -p \\ 0 & -q & -i(q-p) \end{vmatrix} \begin{vmatrix} -i(q-p) & 0 & -q \\ 0 & -i(q-p) & -p \\ -p & -q & -i(q-p) \end{vmatrix}$$

$$\begin{vmatrix} -i(q-p) & -p & -q \\ 0 & -q & -p \\ -p & 0 & -i(q-p) \end{vmatrix} \begin{vmatrix} -i(q-p) & -p & 0 \\ 0 & -q & -i(q-p) \\ -p & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & -q \\ -i(q-p) & -p & 0 \\ 0 & -q & -i(q-p) \end{vmatrix} \begin{vmatrix} -i(q-p) & 0 & -q \\ -q & -p & 0 \\ -p & -q & -i(q-p) \end{vmatrix}$$

$$\begin{vmatrix} -i(q-p) & -p & -q \\ -q & -i(q-p) & 0 \\ -p & 0 & -i(q-p) \end{vmatrix} \begin{vmatrix} -i(q-p) & -p & 0 \\ -q & -i(q-p) & -p \\ -p & 0 & -q \end{vmatrix}$$

$$\begin{vmatrix} -p & 0 & -q \\ -i(q-p) & -p & 0 \\ -q & -i(q-p) & -p \end{vmatrix} \begin{vmatrix} -i(q-p) & 0 & -q \\ -q & -p & 0 \\ 0 & -i(q-p) & -p \end{vmatrix}$$

$$\begin{vmatrix} -i(q-p) & -p & -q \\ -q & -i(q-p) & 0 \\ 0 & -q & -p \end{vmatrix} \begin{vmatrix} -i(q-p) & -p & 0 \\ -q & -i(q-p) & -p \\ 0 & -q & -i(q-p) \end{vmatrix}$$

$$\begin{vmatrix} -i(q-p) & -p & 0 \\ -q & -i(q-p) & -p \\ 0 & -q & -i(q-p) \end{vmatrix} = -i(q-p)\begin{vmatrix} -i(q-p) & -p \\ -q & -i(q-p) \end{vmatrix}$$

$$\begin{aligned} & +p\begin{vmatrix} -q & -p \\ 0 & -i(q-p) \end{vmatrix} \\ &= -i(q-p)[-1(q-p)^2-pq]-p[-iq(q-p)] \\ &= -i(q-p)[-1(q^2-2pq+p^2)-pq]-[-ipq(q-p)] \\ &= -i(q-p)(p+q)(-p^2-q^2) = -i(q^4-p^4) \\ &= -i(p^4-q^4) \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} -q & -p & 0 \\ 0 & -i(q-p) & -p \\ -p & -q & -i(q-p) \end{vmatrix} &= -q\begin{vmatrix} -i(q-p) & -p \\ -q & -i(q-p) \end{vmatrix} + p\begin{vmatrix} 0 & -p \\ -p & -i(q-p) \end{vmatrix} \\ &= -q[-1(q-p)-pq]-p^3 \\ &= -q[-1(q^2-2pq+p^2)-pq]-p^3 \\ &= (q^3-pq^2+p^2q-p^3)(p+q) \\ &= q^4-pq^3+p^2q^2-p^3q+pq^3-p^2q^2+p^3q-p^4 \\ &= -(p^4-q^4) \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} -q & -i(q-p) & 0 \\ 0 & -q & -p \\ -p & 0 & -i(q-p) \end{vmatrix} &= -q\begin{vmatrix} -q & -p \\ 0 & -i(q-p) \end{vmatrix} + i(q-p)\begin{vmatrix} 0 & -p \\ -p & -i(q-p) \end{vmatrix} \\ &= q[-iq^2+ipq]+ip^2q-ip^3 \\ &= -i(q^3-pq^2+p^2q-p^3)(p+q) \\ &= i(p^4-q^4) \end{aligned}$$

$$\begin{vmatrix} -q & -i(q-p) & -p \\ 0 & -q & -i(q-p) \\ -p & 0 & -q \end{vmatrix} = -q\begin{vmatrix} -q & -i(q-p) \\ 0 & -q \end{vmatrix} + i(q-p)\begin{vmatrix} 0 & -i(q-p) \\ -p & -q \end{vmatrix}$$

$$-p\begin{vmatrix} 0 & -q \\ -p & 0 \end{vmatrix}$$

$$= -q^3 - p[-(q-p)^2 - pq]$$

$$= (-q^3 + pq^2 - p^2q + p^3)(p+q)$$

$$= (p^4 - q^4)$$

$$\begin{vmatrix} -p & 0 & -q \\ -q & -i(q-p) & -p \\ 0 & -q & -i(q-p) \end{vmatrix} = -p\begin{vmatrix} -i(q-p) & -p \\ -q & -i(q-p) \end{vmatrix} - q\begin{vmatrix} -q & -i(q-p) \\ 0 & -q \end{vmatrix}$$

$$= -p[-(q-p)^2 - pq] - q^3 = (p^4 - q^4)$$

$$\begin{vmatrix} -i(q-p) & 0 & -q \\ 0 & -i(q-p) & -p \\ -p & -q & -i(q-p) \end{vmatrix} = -i(q-p)\begin{vmatrix} -i(q-p) & -p \\ -q & -i(q-p) \end{vmatrix}$$

$$-q\begin{vmatrix} 0 & -i(q-p) \\ -p & -q \end{vmatrix}$$

$$= -i(q-p)[-(q-p)^2 - pq] + p[iq(q-p)]$$

$$= -i(q-p)[-(q^2 - 2pq + p^2) - pq] + [ipq(q-p)]$$

$$= -i(q-p)[-q^2 + 2pq - p^2 - pq] + ipq^2 - ip^2q)]$$

$$= -i(-q^3 + pq^2 - p^2q + p^3)(p+q)$$

$$= -i(p^4 - q^4)$$

$$\begin{vmatrix} -i(q-p) & -p & -q \\ 0 & -q & -p \\ -p & 0 & -i(q-p) \end{vmatrix} = -i(q-p)\begin{vmatrix} -q & -p \\ 0 & -i(q-p) \end{vmatrix} + p\begin{vmatrix} -p & -q \\ -q & -p \end{vmatrix}$$

$$= i(q-p)[-iq(q-p)] - p(p^2 - q^2)$$

$$= -i^2q(q-p)^2 - p^3 + pq^2$$

$$= q(q-p)^2 - p^3 + pq^2$$

$$= -(-q^3 + pq^2 - p^2q + p^3)(p+q)$$

$$= -(p^4 - q^4)$$

$$\begin{vmatrix} -i(q-p) & -p & 0 \\ 0 & -q & -i(q-p) \\ -p & 0 & -q \end{vmatrix} = -i(q-p)\begin{vmatrix} -q & -i(q-p) \\ 0 & -q \end{vmatrix} + p\begin{vmatrix} 0 & -i(q-p) \\ -p & -q \end{vmatrix}$$
$$= -iq^2(q-p) - ip^1(q-p)$$
$$= i(-q^3+pq^2-p^2q+p^3)(p+q)$$
$$= i(p^4-q^4)$$

$$\begin{vmatrix} -p & 0 & -q \\ -i(q-p) & -p & 0 \\ 0 & -q & -i(q-p) \end{vmatrix} = -p\begin{vmatrix} -p & 0 \\ -q & -i(q-p) \end{vmatrix} - q\begin{vmatrix} -i(q-p) & -p \\ 0 & -q \end{vmatrix}$$
$$= i(-q^3+pq^2-p^2q+p^3)(p+q)$$
$$= i(p^4-q^4)$$

$$\begin{vmatrix} -i(q-p) & 0 & -q \\ -q & -p & 0 \\ -p & -q & -i(q-p) \end{vmatrix} = -i(q-p)\begin{vmatrix} -p & 0 \\ -q & -i(q-p) \end{vmatrix} - q\begin{vmatrix} -q & -p \\ -p & -q \end{vmatrix}$$
$$= -ip(q-p)[i(q-p) - q(q^2-p^2)$$
$$= (pq^2-p^2q+p^3-q^3)(p+q)$$
$$= (p^4-q^4)$$

$$\begin{vmatrix} -i(q-p) & -p & -q \\ -q & -i(q-p) & 0 \\ -p & 0 & -i(q-p) \end{vmatrix} = -i(q-p)\begin{vmatrix} -i(q-p) & 0 \\ 0 & -i(q-p) \end{vmatrix}$$
$$+p\begin{vmatrix} -q & 0 \\ -p & -i(q-p) \end{vmatrix} - q\begin{vmatrix} -q & -i(q-p) \\ -p & 0 \end{vmatrix}$$
$$= -i(q-p)[i(q-p)i(q-p) - ipq(q-p)$$
$$= p(q-p)^2 - q^3 + p^2q$$
$$= (-q^3+pq^2-p^2q+p^3)(p+q)$$
$$= -i(p^4-q^4)$$

$$\begin{vmatrix} -i(q-p) & -p & 0 \\ -q & -i(q-p) & -p \\ -p & 0 & -q \end{vmatrix} = -i(q-p)\begin{vmatrix} -i(q-p) & -p \\ 0 & -q \end{vmatrix} + p\begin{vmatrix} -q & -p \\ -p & -q \end{vmatrix}$$

$$\begin{aligned} &= -iq(q-p)i(q-p) + p(p^2-q^2) \\ &= q(q-p)^2 + pq^2 - p^3 \\ &= (q^3 - pq^2 + p^2q - p^3)(p+q) \\ &= -(p^4 - q^4) \end{aligned}$$

$$\begin{vmatrix} -p & 0 & -q \\ -i(q-p) & -p & 0 \\ -q & -i(q-p) & -p \end{vmatrix} = -p\begin{vmatrix} -p & 0 \\ -i(q-p) & -p \end{vmatrix} - q\begin{vmatrix} -i(q-p) & -p \\ -q & -i(q-p) \end{vmatrix}$$

$$\begin{aligned} &= -p^3 - q[-(q-p)^2 - pq] \\ &= -p^3 + (q^3 - pq^2 + p^2q) \\ &= (q^3 - pq^2 + p^2q - p^3)(p+q) \\ &= -(p^4 - q^4) \end{aligned}$$

$$\begin{vmatrix} -i(q-p) & 0 & -q \\ -q & -p & 0 \\ 0 & -i(q-p) & -p \end{vmatrix} = -i(q-p)\begin{vmatrix} -p & 0 \\ -i(q-p) & -p \end{vmatrix} - q\begin{vmatrix} -q & -p \\ 0 & -i(q-p) \end{vmatrix}$$

$$\begin{aligned} &= -\Big(ip^2(q-p) - iq^2(q-p)\Big) \\ &= -i(q^3 - pq^2 + p^2q - p^3)(p+q) \\ &= i(p^4 - q^4) \end{aligned}$$

$$\begin{vmatrix} -i(q-p) & -p & -q \\ -q & -i(q-p) & 0 \\ 0 & -q & -p \end{vmatrix} = -i(q-p)\begin{vmatrix} -i(q-p) & 0 \\ -q & -p \end{vmatrix} + p\begin{vmatrix} -q & 0 \\ 0 & -p \end{vmatrix}$$

$$\begin{aligned} & -q\begin{vmatrix} -q & i(q-p) \\ 0 & -q \end{vmatrix} \\ &= i(q-p)[-ip(q-p)] + q(p^2-q^2) \\ &= -(q^3 - pq^2 + p^2q - p^3)(p+q) \\ &= (p^4 - q^4) \end{aligned}$$

$$
\begin{aligned}
\begin{vmatrix} -i(q-p) & -p & 0 \\ -q & -i(q-p) & -p \\ 0 & -q & -i(q-p) \end{vmatrix} &= -i(q-p)\begin{vmatrix} -i(q-p) & -p \\ -q & -i(q-p) \end{vmatrix} \\
&\quad +p\begin{vmatrix} -q & -p \\ 0 & -i(q-p) \end{vmatrix} \\
&= -i(q-p)[-(q^2-2pq+p^2)-pq]+ipq^2-ip^2q \\
&= i(q^3-pq^2+p^2q-p^3)(p+q) \\
&= -i(p^4-q^4)
\end{aligned}
$$

$$
= \begin{bmatrix} -i(p^4-q^4) & -(p^4-q^4) & i(p^4-q^4) & (p^4-q^4) \\ (p^4-q^4) & -i(p^4-q^4) & -(p^4-q^4) & i(p^4-q^4) \\ i(p^4-q^4) & (p^4-q^4) & -i(p^4-q^4) & (p^4-q^4) \\ -(p^4-q^4) & i(p^4-q^4) & (p^4-q^4) & -i(p^4-q^4) \end{bmatrix}
$$

The corresponding eigenvalues which are the roots of the equation $(|\mathbf{P}-\lambda\mathbf{I}|=0)$ are

$$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i(q-p), \text{ and } \lambda_4 = -i(q-p)$$

and hence

$$
\begin{aligned}
\prod_{m=2}^{4}(\lambda_1-\lambda_m) &= (1+1)(1-i(q-p))(1+i(q-p)) \\
&= 2(1+p^2-2pq+q^2) \\
&= 2(p^2+p^2+q^2+q^2) \\
&= 4(p^2+q^2)
\end{aligned}
$$

$$
\begin{aligned}
\prod_{m=1, m\neq 2}^{4}(\lambda_2-\lambda_m) &= (-1-1)(-1-i(q-p))(-1+i(q-p)) \\
&= -2(1+p^2-2pq+q^2) \\
&= -2(p^2+p^2+q^2+q^2) \\
&= -4(p^2+q^2)
\end{aligned}
$$

$$
\begin{aligned}
\prod_{m=1, m\neq 3}^{4}(\lambda_3-\lambda_m) &= (i(q-p)-1)(i(q-p)+1)(2i(q-p)) \\
&= -2i[(q^2-2pq+p^2)+1][(q-p)(q+p)] \\
&= -2i(q^2-2pq+p^2+1)(q^2-p^2) \\
&= 4i(p^4-q^4)
\end{aligned}
$$

$$\begin{aligned}
\prod_{m=1}^{3}(\lambda_4-\lambda_m) &= (-i(q-p)-1)(-i(q-p)+1)(-2i(q-p)) \\
&= 2i[(q^2-2pq+p^2)+1][(q-p)(q+p)] \\
&= 2i(q^2-2pq+p^2+1)(q^2-p^2) \\
&= -4i(p^4-q^4)
\end{aligned}$$

For the characteristic matrices

$$A(\lambda_l)=\begin{bmatrix} \lambda_l & -p & 0 & -q \\ -q & \lambda_l & -p & 0 \\ 0 & -q & \lambda_l & -p \\ -p & 0 & -q & \lambda_l \end{bmatrix}$$

for $l=1,2,3,4$. We compute the cofactors as follows

$$A_{ji}(\lambda_1)=(p^2+q^2), \qquad i,j=1,2,3,4$$

$$A_{ji}(\lambda_2)=(-1)^{i+j+1}(p^2+q^2), \qquad i,j=1,2,3,4$$

$$A_{ji}(\lambda_3)=\left(\sqrt{(-1)^{j-i+1}}\right)(p^4-q^4), \qquad i,j=1,2,3,4$$

$$A_{ji}(\lambda_3)=\left(\sqrt{(-1)^{i-j-1}}\right)(p^4-q^4), \qquad i,j=1,2,3,4$$

Substituting the cofactors in the formula

$$p_{jj}^{(n)}=\sum_{l=1}^{4}A_{ji}(\lambda_l)\lambda_l^n\frac{1}{\prod_{m=1,m\neq l}^{4}(\lambda_l-\lambda_m)}$$

yields

$$\begin{aligned}
p_{ij}^{(n)} &= \frac{(p^2+q^2)(1)(1)^n}{4(p^2+q^2)}+\frac{(p^2+q^2)(-1)(-1)^n}{-4(p^2+q^2)}+\frac{(p^2+q^2)(i(q-p))(i(q-p))^n}{4i(p^4-q^4)} \\
&\quad +\frac{(p^2+q^2)(-i(q-p))(-i(q-p))^n}{-4i(p^4-q^4)} \\
&= \frac{1}{4}+\frac{1}{4}(-1)^n+\frac{(p^2+q^2)(i(q-p))(i(q-p))^n}{4i(p^2-q^2)(p^2+q^2)}+\frac{(p^2+q^2)(i(q-p))(-i(q-p))^n}{-4i(p^2-q^2)(p^2+q^2)} \\
&= \frac{1}{4}+\frac{1}{4}(-1)^n-\frac{(q-p)(i(q-p))^n}{4(p+q)(p-q)}+\frac{(q-p)(-i(q-p))^n}{4(p+q)(p-q)} \\
&= \frac{1}{4}+\frac{1}{4}(-1)^n+\frac{(i(q-p))^n}{4}+\frac{(-i(q-p))^n}{4}
\end{aligned}$$

$$p_{ij}^{(n)}=\frac{1}{4}\left[1+(-1)^{i+j+n}+\sqrt{(-1)^{j-i+n}}(q-p)^n+\sqrt{(-1)^{j-i-n}}(q-p)^n\right]$$

for $i,j = 1,2,3,4$ and $n = 1,2,3\ldots.$ The final formula of $p_{ij}^{(n)}$ assumes slightly different forms for different values of n. The following are matrices of $\mathbf{P}^n$ for $n = 4k+1, n = 4k+2, n = 4k+3$, and $n = 4k+4$ and $k = 0,1,2\ldots.$

$$\mathbf{P}^{(4k+1)} = \frac{1}{2}\begin{bmatrix} 0 & 1-(q-p)^{(4k+1)} & 0 & 1+(q-p)^{(4k+1)} \\ 1+(q-p)^{(4k+1)} & 0 & 1-(q-p)^{(4k+1)} & 0 \\ 0 & 1+(q-p)^{(4k+1)} & 0 & 1-(q-p)^{(4k+1)} \\ 1-(q-p)^{(4k+1)} & 0 & 1+(q-p)^{(4k+1)} & 0 \end{bmatrix}$$

$$\mathbf{P}^{(4k+2)} = \frac{1}{2}\begin{bmatrix} 1-(q-p)^{(4k+2)} & 0 & 1+(q-p)^{(4k+2)} & 0 \\ 0 & 1-(q-p)^{(4k+2)} & 0 & 1+(q-p)^{(4k+2)} \\ 1+(q-p)^{(4k+2)} & 0 & 1-(q-p)^{(4k+2)} & 0 \\ 0 & 1+(q-p)^{(4k+2)} & 0 & 1-(q-p)^{(4k+2)} \end{bmatrix}$$

$$\mathbf{P}^{(4k+3)} = \frac{1}{2}\begin{bmatrix} 0 & 1+(q-p)^{(4k+3)} & 0 & 1-(q-p)^{(4k+3)} \\ 1-(q-p)^{(4k+3)} & 0 & 1+(q-p)^{(4k+3)} & 0 \\ 0 & 1-(q-p)^{(4k+3)} & 0 & 1+(q-p)^{(4k+3)} \\ 1+(q-p)^{(4k+3)} & 0 & 1-(q-p)^{(4k+3)} & 0 \end{bmatrix}$$

$$\mathbf{P}^{(4k+4)} = \frac{1}{2}\begin{bmatrix} 1+(q-p)^{(4k+4)} & 0 & 1-(q-p)^{(4k+4)} & 0 \\ 0 & 1+(q-p)^{(4k+4)} & 0 & 1-(q-p)^{(4k+4)} \\ 1-(q-p)^{(4k+4)} & 0 & 1+(q-p)^{(4k+4)} & 0 \\ 0 & 1-(q-p)^{(4k+4)} & 0 & 1+(q-p)^{(4k+4)} \end{bmatrix}$$

6.4 Conclusion

The four methods of finding the nth power give the same results. Computation of the 2x2 transition probabilities give the results which are easily generalized. However, using the direct method of multiplication, it is difficult to identify the pattern easily to help to come up with the generalized pattern.

The 3x3 transition probability matrices onwards give complex patterns which are not easy to generalize especially in the case of the cyclic random walks.

The method of multiplication gives a visible pattern similar to that of the Pascal Triangle, but the generalization of the nth term needs to be investigated further from the pattern.

For the eigen value technique, it is not easy to find the eigenvectors and therefore again, this method needs further review to see whether it can give a result, otherwise we might have to explore whether there is another alternative method which can give us a result.

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