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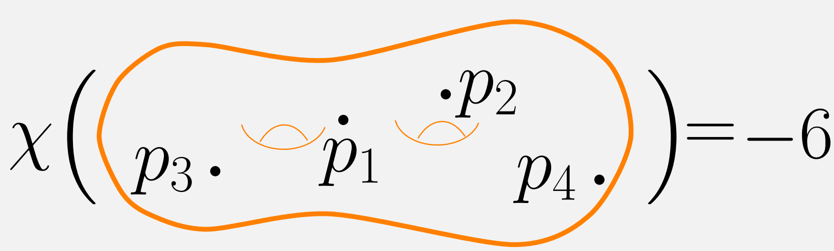
Master Project in Mathematics

# Euler Characteristic of the Moduli of Riemann Surfaces

Research Report in Mathematics, Number 02, 2018

Geoffrey Otieno Mboya

August 2018




$$\chi(\mathcal{M}_{2,4}) = -\frac{1}{2}, \chi(\overline{\mathcal{M}}_{2,4}) = 1004$$

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

**Master Project in Mathematics** **University of Nairobi**
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# Euler Characteristic of the Moduli of Riemann Surfaces

**Research Report in Mathematics, Number 02, 2018**

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Master of Science Project
Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Pure Mathematics

Prepared for The Director
Graduate School
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Monitored by School of Mathematics

# Abstract

In this thesis we study the topology of the Deligne-Mumford compactified moduli space $\overline{\mathscr{M}}_{g,n}$ of $n$ pointed genus $g$ stable Riemann surfaces. We decode the combinatorial information of the space through dual graphs of decorated Riemann surfaces in it. Finally, we extend the work of Harer and Zagier in [HZ86] to obtain the generating series of (and hence calculate) the Euler characteristic of the space in terms of Euler characteristics of $\mathscr{M}_{g,n}$.

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# Declaration and Approval

I the undersigned declare that this project report is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

Geoffrey Otieno Mboya
Reg No. I56/89490/2016

In my capacity as a supervisor of the candidate, I certify that this report has my approval for submission.

Signature Date

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# Dedication

I dedicate this dissertation to Winnie and Yolanda.

# Contents

# Acknowledgments

I would like to recognise each person whose support and contribution was crucial in putting up this dissertation. I uniquely express my gratitude to my supervisor Jared Ongaro for rigorously and competently guiding me throughout this project. His infectious enthusiasm for clarity in communicating Mathematics has been extremely inspirational.

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Am thankful to the University of Nairobi for awarding me the Masters Scholarship to pursue this degree. Many thanks to EAUMP for giving me opportunity to attend the ICTP-EAUMP Summer Schools in 2016 , 2017 and 2018 and for making me comfortable by contributing towards my stipend. The Annual University of Nairobi Algebraic Geometry Workshop is the fountain where I drank my first drop of Algebraic Geometry and is the source of this project.

To Tessler, your correspondence was extremely useful, thank you. To my classmates Vincent, Wilfred, Gladys, Gilbert, Kevin, Daniel and Mathias; our many discussions and your kind and healthy suggestions improved this dissertation.

I am deeply beholden to my loving wife Winnie and daughter Yolanda. Thank you so much for experiencing madness and trauma associated with long hours I spent putting this dissertation together. Words can't express how much am grateful. To my mother Agnes Mboya, brother Ben, sisters Phoebe, Lilian and Irene; thank you for your prayers and support. You have been so understanding. *Mama, odong' higni ang'wen kuom ng'uono mar Nyasaye!*

GEOFFREY OTIENO MBOYA

Nairobi, 2018.

# 1 Introduction

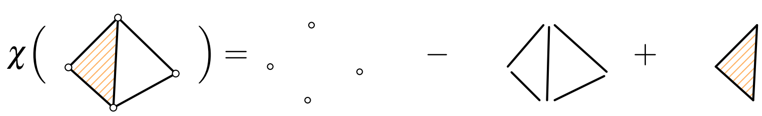
## 1.1 Why the Euler Characteristic?

Generally, following [MP91], Euler characteristic of a **topological manifold** $\Sigma$ is calculated by first embeding on $\Sigma$ a graph $\Gamma$ whose vertices are distinct and edges are Riemann surfaces on the surface of $\Sigma$. The surface is then dissected along edges and vertices of $\Gamma$ so that $\Sigma \setminus \Gamma$ is a disjoint union of $k-$dimensional cells (k-simplices) isormorphic to open subsets of $\mathbb{R}^n$. We then have that $\mathcal{X}(\Sigma)$ is the Euler characteristic of the resulting **cellular chain complex**

$$\chi(\Sigma) := \sum_{k \in \mathbb{N}_{\geq 0}} (-1)^k \#(\text{k-simplices}).$$

It is a famous old result due to Euler (1750) that **Euler characteristic** $\chi(\Sigma) \in \mathbb{Z}$ of a convex polyhedron $\Sigma$, calculated as $\mathcal{X}(\Sigma) :=|$ Vertices of $\Gamma \mid - \mid$ edges of $\Gamma \mid + \mid$ Faces of $\Gamma \mid$ is equal to $2-2g$ if $\Gamma$ can be embedded into a sphere with $g$ handles $\Sigma$.

$$\begin{aligned} \chi(X) &= \#(\text{0-simplices}) - \#(\text{1-simplices}) + \#(\text{2-simplices}) \\ &= \#\text{ of vertices} \quad - \# \quad \text{of edges} \quad + \# \quad \text{of faces} \end{aligned}$$



$$= \quad 4 \quad - \quad 5 \quad + \quad 1 \quad = \quad 0.$$

The number $\mathcal{X}(\Sigma)$ is independent of the choice of the graph $\Gamma$. These result was later generalized by Poincaré who chose a ring, say $A = \mathbb{Z}$ or $A = \mathbb{Q}$, and defined $\mathcal{X}(\Sigma)$ over $A$ through the **homology chain complex** as the alternating sum of **Betti numbers**

$$\mathcal{X}(\Sigma) := \sum_{n \in \mathbb{N}} (-1)^n \text{rank}\, H_n(\Sigma, A),$$

where $H_n(\Sigma, A)$ is the $n-$th homology group.

Euler characteristics is the oldest and one of the most important topological invariant. $\mathcal{X}(\Sigma)$ is useful in classifying topological spaces in the negative sense. If another topological space $\Sigma'$ is homeomorphic to $\Sigma$ then $\chi(\Sigma) = \chi(\Sigma')$ but spaces with same Euler Characteristics need not be topologically equivalent.

Further, in differential geometry, as reflected by the Gauss-Bonnet theorem

$$\iint_{\Sigma} Kd(\Sigma) = 2\pi\chi(\Sigma),$$

curvature $K$, a geometric property, of a topological manifold $\Sigma$ is related to it's Euler characteristic $\chi(\Sigma)$, a topological property.

## 1.2 Dissertations outline

We endow our main object $\overline{\mathscr{M}}_{g,n}$, the Deligne-Mumford compactified moduli space of stable Riemann surfaces, with an orbifold structure. This structure will be related to the combinatorial data obtained from its stratification through Strebel Theory as discussed in [Kon92]. By highlighting the work of Harer and Zagier in [HZ86] and [MP91] on $\chi(\mathscr{M}_{g,n})$, we extend these results to the computation of the Euler Characteristic $\chi(\overline{\mathscr{M}}_{g,n})$.

The outline is as follows:

**Chapter 2**: In this chapter, we introduce Riemann surfaces whose moduli is our object of interest. Metric trees and their inflated versions (ribbons) are also introduced here. The orbifold structure of the space of inflated graphs is mentioned owing to its usefulness in understanding our object of study in subsequent chapters.

**Chapter 3**: Here, an introduction to the notion of moduli spaces is presented. We focus on discussing the compactification and stratification of the moduli space of Riemann surfaces $\overline{\mathscr{M}}_{g,n}$ and present natural maps between them.

**Chapter 4**: We start off by a brief discussion of Stebel's Theory before presenting a brief proof of Harer and Zagier on $\chi(\mathscr{M}_{g,n})$. Finally, we discuss the generating function of $\chi(\overline{\mathscr{M}}_{g,n})$ in terms of $\chi(\mathscr{M}_{g,n})$.

# 2 Preliminaries

This chapter is intended to provide a basic background to the content of this project as well as serve to fix notations. The content follows [Mir95] , [MP91],[Zvo11] and [Ong14].

## 2.1 Riemann Surfaces

**Definition 2.1.1.** *A $1-$dimensional complex manifold $\Sigma$ equipped with complex atlas of charts $\mathscr{A} = \{(U_i, \phi_i) \mid i \in I, \phi_i : U_i \to V_i \subset \mathbb{C}\}$ is called a* ***Riemann surface*** *if*

*(i).* $\Sigma = \cup_{i\in I} U_i$

*(ii).* $\phi_i : U_i \to V_i$ *is a homeomorphism.*

*(iii). The transition function* $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ *is biholomorphic for all* $i, j \in I$*.*

Under the definition above, we say $\mathscr{A}$ is a complex structure on the manifold $\Sigma$.

**Definition 2.1.2.** *Let $\Sigma$ be a Riemann surface whose atlas is $\mathscr{A} = \{(U_i, \phi_i) \mid i \in I, \phi_i : U_i \to V_i \subset \mathbb{C}\}$. A function $f : \Sigma \to \mathbb{C}$ is called* ***holomorphic*** *if*

$$f \circ \phi_i^{-1} : V_i \to \mathbb{C}$$

*is holomorphic for every $i \in I$. We denote by*

$$\mathscr{O}(\Sigma) := \{f : \Sigma \to \mathbb{C} \mid f \text{ is holomorphic function}\}.$$

**Example 2.1.3.** *Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials in $n$ variables over $\mathbb{C}$. We denote the affine space associated to subsets of $\mathbb{C}^n$ by $\mathbb{A}^n$. For every polynomial $f \in R$ there is an* ***evaluation map***

$$f : \mathbb{A}^n \to \mathbb{C}$$

*defined by $(a_1, \ldots, a_n) \to f(a_1, \ldots, a_n)$. The evaluation map is an holomorphic function on the affine space.*

**Example 2.1.4** (Riemann Surfaces). *(i). Denote by $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ the* ***complex projective line****.*

*By taking the charts*

$$U_1 = \mathbb{CP}^1 \setminus \{\infty\} = \mathbb{C},$$
$$U_2 = \mathbb{CP}^1 \setminus \{0\} = \mathbb{C}^* \cup \{\infty\}.$$

*Define the transition functions,*

$$\phi_1(z) = z \neq \infty, \quad \phi_2(z) = \begin{cases} \frac{1}{z} & \text{if } z \in U_1 \\ 0 & \text{if } z = \infty. \end{cases}$$

*Then $\phi_2 \circ \phi_1^{-1} : \mathbb{C}^* \to \mathbb{C}$ defined by $z \mapsto \frac{1}{z}$ is biholomorphic. The Riemann surface $\mathbb{CP}^1$ is also called **Riemann sphere**.*

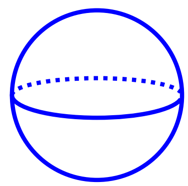


**Figure 1. Riemann Sphere**

(ii). *Pick $\tau_1, \tau_2 \in \mathbb{C}^*$ which are linearly independent over $\mathbb{R}$. We denote by $L$ the **lattice** generated by $\tau_1$, $\tau_2$ and defined by*

$$L := \tau_1\mathbb{Z} + \tau_2\mathbb{Z} = \Big\{ n\tau_1 + m\tau_2 \mid n, m \in \mathbb{Z} \Big\} \cong \mathbb{Z} \times \mathbb{Z}.$$

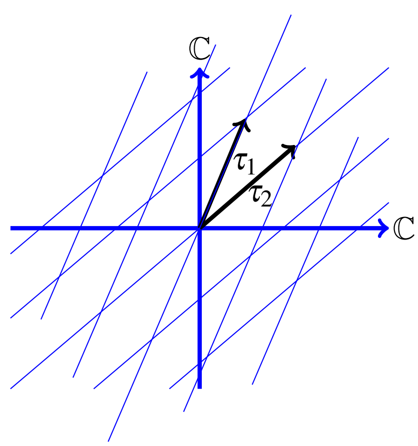



**Figure 2. Latttices of a torus**

*The relation $\sim$ on $\mathbb{C}^*$ defined by $z_1 \sim z_2$ whenever $z_1 - z_2 \in L$ for $z_1, z_2 \in \mathbb{C}^*$ is an equivalence relation. The set of equivalence classes $E = \mathbb{C}/L$ endowed with the quotient topology such that $U \subset E$ is open exactly when $p^{-1}(U) \subset \mathbb{C}$ is open for the projection $\mathbb{C} \xrightarrow{p} E$ defined by $z \mapsto [z]$ where $[z]$ is the $L-$orbit of $z$ or the equivalence class of $z$. The quotient topology therefore makes $p$ continuous open map. We then have that $E$*

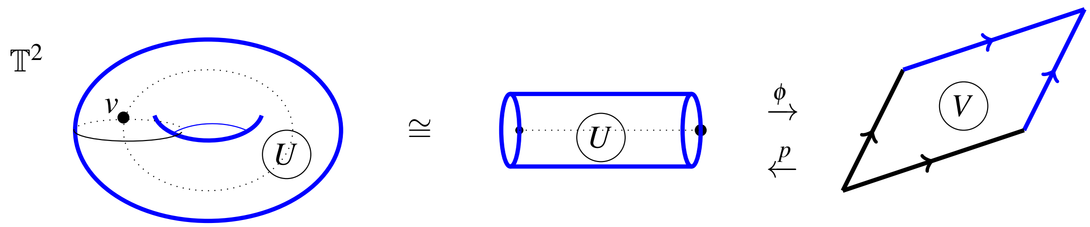



**Figure 3. Torus as a Riemann Surface.**

*is homeomorphic to a torus $\mathbb{T}^2 \cong S^1 \times S^1$. Let $V \subset \mathbb{C}$ be an open set such that for all $z_1, z_2 \in V, \quad z_1 - z_2 \notin L$. Equivalently, $V$ is in the interior of parallelograms spanned by $\tau_1$ and $\tau_2$. We then have that $p(V) = U \subset E$ is open and $\phi : U \to V$ the inverse of $p$. $(U, \phi)$ is then a complex chart on $E$. Any two such charts are compatible; the transition functions are translations (of lattices) which are **holomorphic**.*

**Definition 2.1.5.** *Let $\Sigma_1$ and $\Sigma_2$ be two Riemann surfaces. A continuous map $f : \Sigma_1 \to \Sigma_2$ is called **holomorphic map** if for every chart $(U_1, \phi_1)$ on $\Sigma_1$ and $(U_2, \phi_2)$ on $\Sigma_2$ with $f(U_1) \subset U_2$, we have that*

$$\phi_2 \circ f \circ \phi_1^{-1} : V_1 \to V_2$$

*is holomorphic. Further, if $f$ is bijective and both $f$ and $f^{-1}$ are holomorphic then $f$ is **biholomorphic**. In such a case, we say $\Sigma_1$ and $\Sigma_2$ are **isomorphic**.*

**Example 2.1.6.**

*(i). $f \in Aut\mathbb{CP}^1) = PSL(2,\mathbb{C})$ is a holomorphic map defined by $f(z) = \frac{az+b}{cz+d}$.*

*(ii). A degree $d \geq 1$ holomorphic map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ defined by $f(z) = z^d$ gives a covering of $\mathbb{CP}^1$ with $d$ sheets and which branch at $0$ and $\infty$ with multiplicity $d$.*

**Proposition 2.1.7.** *The automorphism group $PSL(2,\mathbb{C})$ acts $3-$ transitively on $\mathbb{CP}^1$. Moreover, any $3$ different points determine a unique Möbius map taking them to $0, 1$ and $\infty$.*

`Proof.` Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a biholomorphism. This implies that there exists $u \in \mathbb{CP}^1$ such that $f(u) = \infty$. Let $\mu$ be a Möbius element such that $\mu(\infty) = u$. Then $f \circ \mu := h$ is also a biholomorphism with property that $h(\infty) = \infty$. Such $h$ therefore has a pole at infinity. Hence,

$$h = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots$$

has no essential singularity at infinity which implies that

$$h = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m.$$

If $m = 0$ then $h$ is not surjective. If $m > 1$ then $h$ is injective. If $m = 1$ we have that $f \circ \mu := h = a + bz = \frac{bz+a}{0z+1}$. This is equivalent to $h = \begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix}$. Hence, for $f = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

and the Möbius element $\mu = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$, we have that $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}^{-1}$ □

Herein, we shall concentrate on connected, compact orientable Riemann surfaces. From the classification theorem, for every genus $g \geq 0$ there exists exactly one such surface. We usually draw cartoons with $g-$handles to represent such surfaces.

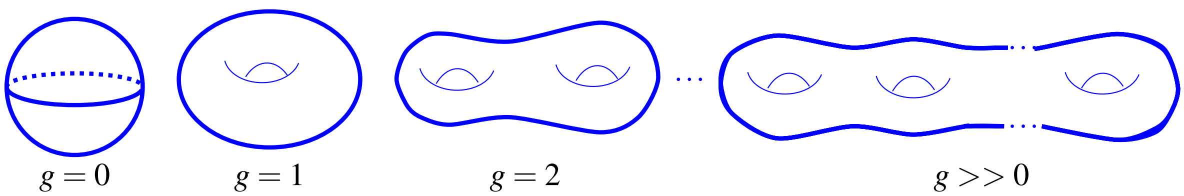



**Figure 4. Cartoons to represent such smooth connected compact Riemann surfaces.**

Consider a non-constant holomorphic map $f : \Sigma_1 \to \Sigma_2$ between smooth Riemann surface. We denote by $v_p(f)$ the *vanishing order* of $f$ at a point $p \in \Sigma_1$. By taking local coordinates at $p \in \Sigma_1$ and $q = f(p) \in \Sigma_2$, $f$ can be written in its normal form

$$w = z^{v_p(f)}.$$

If $f$ has degree $d$ then each $q \in \Sigma_2$ determines an effective divisor on $\Sigma_1$ through *pullback*

$$f^*(q) = \sum_{p \in f^{-1}(q)} v_p(f).q$$

**Theorem 2.1.8** (Riemann-Hurwitz Formula). *Let $f : \Sigma_1 \to \Sigma_2$ be a degree $d$ non-constant holomorphic map between smooth Riemann surfaces $\Sigma_1$ and $\Sigma_2$ of genus $g_1$ and $g_2$ respectively. Then*

$$2g_1 - 2 = d(2g_2 - 2) + \sum_{p \in \Sigma} (v_p(f) - 1).$$

## 2.2 Metric Trees

In this section, we discuss some results on the space of metric trees which will be useful in encoding the boundary strata of moduli spaces in the next chapter.

**Definition 2.2.1.** *A connected undirected graph with no cycles is called a **tree**.*

**Definition 2.2.2.** *Let $p, q \in \Sigma$ where $(\Sigma, \delta)$ is a metric space. A **metric segment** joining $p$ to $q$ denoted by $[p,q]$ is the image of $\gamma : [a,b] \to \Sigma$ which is an isometric embedding with $[a,b]$ a closed subset of $\mathbb{R}$, $p = \gamma(a)$ and $q = \gamma(b)$.*

**Definition 2.2.3** ([AO00]). *A **metric tree** is a metric space $(\Sigma, \delta)$ such that for all $p, q, r \in \Sigma$ the following are satisfied.*

1. *The metric segment $[p, q]$ joining $p$ to $q$ exists and is unique.*
2. *If $[p, r] \cap [r, q] = \{r\}$ then $[p, r] \cup [r, q] = [p, q]$.*

**Remark 2.2.4.**

- *A **spanning tree** $T_n$ of a graph $\Gamma$ is a **subgraph** $\Gamma' \subseteq \Gamma$ consisting of all the vertices $V_\Gamma$ (but not necessarily all edges) and is not a **circuit** (i.e edges used should be followed exactly once).*
- *An **external point** of a (metric) tree is one whose valence is $1$. An edge whose terminal or initial point is an external point is called **leaf** or **leg** of $T_n$ otherwise the edge is called **internal** edge.*

**Example 2.2.5.** *We shall consider, in this thesis, a tree $T_n$ called **spider with** $n$ **legs** consisting of a centre $O$ and points $(k, t) \in \{1, \ldots, n\} \times (0, a_k]$ where $n \in \mathbb{N}$ is fixed and $(a_k)_{k=1}^n \subset \mathbb{R}_+$.*

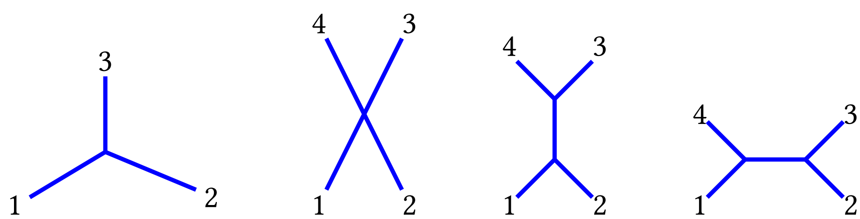



**Figure 5. Spiders with 3 and 4 legs.**

*Spiders with $n$ legs form a subtree of the metric tree $(\mathbb{R}^2, \delta)$ where the metric*

$$\delta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$$

*is defined by setting $\delta(O, (k, t)) = t$ and*

$$\delta((k, t), (l, s)) = \begin{cases} |\, t - s \,| & k = l \\ t + s & k \neq l. \end{cases}$$

**Proposition 2.2.6.**

1. *Every connected graph $\Gamma$ has a spanning tree $T$.*
2. *A spanning tree $T$ of $\Gamma$ has $v - 1$ edges where $v = |\, V_\Gamma \,|$ is the number of vertices of $\Gamma$.*
3. *Whenever any of the $e - (v - 1) = e - v + 1$ unused edge of $\Gamma$ is added to $T$, we get an 'independent' cycle.*
4. *A metric tree with $n$ external points (hence $n$ legs) has at most $n - 2$ internal vertices.*

### 2.2.1 Space of Metric Trees

**Theorem 2.2.7.** *The space of metric trees with n external points is a manifold whose points each have an alkene structure with a real dimension* $2n-3$.

Proof. Let $T_n$ be a tree with $n$ points. We know from 4 of 2.2.6 that such $T$ has $n-2$ internal vertices. Now by 2 of 2.2.6 , other than the $n$ legs, $T$ has $(n-2)-1$ internal edges. It is then clear that $T_n$ has $n+(n-2)-1 = 2n-3$ non-compact edges with boundaries or corners. Therefore, the space of possible choices of lengths is $\mathbb{R}_+^{2n-3}$. For instance, for $n=4$ and the lengths $\lambda, \alpha, \gamma, \beta \in \mathbb{R}_+^5$. we have:

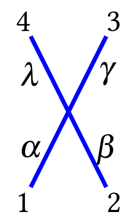



**Figure 6. An example of $T_4$**

□

#### Maps between Spaces of Metric Trees

The following two maps are natural between spaces of metric trees

1. The map $f : \mathbb{R}_+^{2n-3} \to \mathbb{R}_+^{2n-3} (n \geq 3)$ between metric trees defined by *fusing* vertices $v_i$ and $v_j$ followed by *moving* them apart in a different direction. This is equivalent to permutation of leaves on $T_n$.
2. The continuous map $f : \mathbb{R}_+^{2n-3} \to \mathbb{R}_+^{2n-3-k} (n \geq 4)$ defined by *contracting* $k$ $(1 \leq k \leq n-3)$ internal edge(s) of $T_n$ to zero.

### 2.2.2 Ribbon Graphs

**Definition 2.2.8.** *[MP91] A connected* ***ribbon graph*** $\Gamma$ *is a graph drawn on a compact connected oriented surface* $\Sigma$ *without a degree one vertex and which induces a cell-decomposition of* $\Sigma$. *We shall denote by* $RG_{g,n}$ *the space of all ribbon graphs.*

Now, for

$$\begin{cases} n \geq 3 & \text{if } g = 0 \\ n \geq 1 & \text{if } g \neq 0 \end{cases}$$

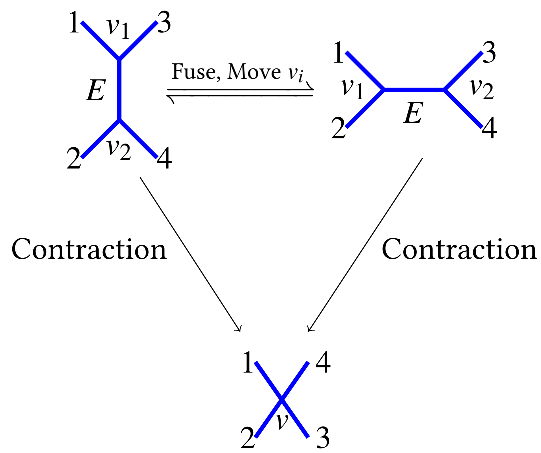



**Figure 7. An illustration of natural maps between spaces of metric trees**

(i.e $2-2g-n<0$) with $g \in \mathbb{Z}_{\geq 0}$ and $n \geq 1$. We define the space of **metric ribbon graphs** by

$$RG^{\text{met}}_{g,n} = \coprod_{\Gamma \in RG_{g,n}} \frac{\mathbb{R}_+^{e(\Gamma)}}{Aut(\Gamma)}. \tag{1}$$

Observe that $RG^{\text{met}}_{g,n}$ has a differentiable orbifold structure and
$Aut(\Gamma)$ is the automorphism group of the set of vertices $V_\Gamma$.

### 2.2.3 Orbifold Structure of the Space of Metric Ribbon Graphs

**Definition 2.2.9.** *Let $\Sigma$ be a Hausdorff topological space. An orbifold chart*

$$U/G \xrightarrow{\varphi} V \subset \Sigma$$

*where $U \subset \mathbb{C}^n$ is a contractable open set equipped with a biholomorphic action of a finite group $G$. $V \subset \Sigma$ is an open set and $\varphi$ is an homeomorphism.*

If $\Sigma$ is covered by compatible orbifold charts it is called **smooth complex orbifold.**

**Definition 2.2.10.** *Let $\Gamma$ be a ribbon graph whose set of edges is $E_\Gamma$. $\Gamma$ is said to be **exceptional** if the natural homomorphism $\phi_\Gamma$ :*
*$Aut(\Gamma) \to S_{|E_\Gamma|}$ of the automorphism group of $\Gamma$ to the permutation group of edges is not injective.*

For a ribbon graph $\Gamma$,
$Aut(\Gamma)$ acts on $\mathbb{R}_+^{e(\Gamma)}$ through a natural homomorphism

$$\phi : \text{Aut}(\Gamma) \to S_{e(\Gamma)}, \text{ where } e(\Gamma) = \mid E_\Gamma \mid$$

provided $\Gamma$ is not exceptional. We then conclude from Equation (1) that the space of ribbon graphs has an orbifold structure.

The following diagram shows the result of finite group action on metric graphs corresponding to an open set $B(r,x)$

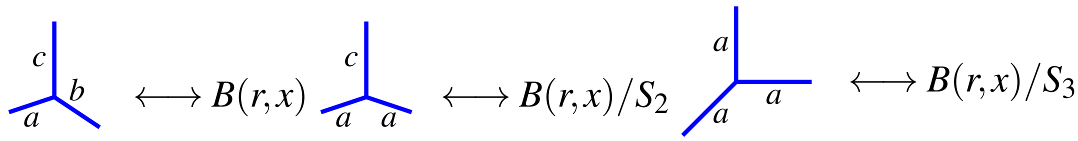



**Figure 8.** $T_3$ **as an orbifold**

# 3 Moduli Space of Pointed Riemann Surfaces

This chapter will introduce moduli spaces of Riemann surfaces and we also discuss the compactification of the moduli of marked Riemann surfaces.

A moduli space of a given geometric object is one which parametrises such object. Namely, a point in a moduli space is a particular geometric object. A subset of a moduli space is a set of those geometric objects satisfying a property(ies) $\mathscr{P}' \subset \mathscr{P}$. Moving from one point in a moduli space to another is a continuous deformation of the geometric object to another geometric object. We refer to this as modulation provided $\mathscr{P}$ is preserved.

We now discuss some important examples of moduli spaces.

### 3.0.1 Grassmannian

**Definition 3.0.1.** *The Grassmannian denoted by $G(k,n)$ is the space parametrizing all $k$ dimensional linear subspaces $(0 \leq k \leq n-1)$ of $\mathbb{C}^n$. That is to say*

$$G(k,n) := \{W \subset \mathbb{C}^n : W \text{ is a } k \text{ dimensional subspace of } \mathbb{C}^n \text{ with } \quad 0 \leq k \leq n-1\}.$$

To understand the modulation of $W \subset G(k,n)$ we present the following facts from [Has07] and [Ran10]

**Theorem 3.0.2.** *The structure of the Grassmannian $G(k,n)$ is that of an irreducible, algebraic variety of dimension $k(n-k)$. Furthermore, $G(k,n) \cong G(n-k,n)$.*

Let $v_1, \ldots, v_S \in \mathbb{C}^n$. The **wedge product** $v_1 \wedge \ldots \wedge v_S$ is defined by antisymmetric blade $\wedge$ given by

$$v_1 \wedge \ldots \wedge v_i \wedge v_j \ldots \wedge v_S = -v_1 \wedge \ldots \wedge v_j \wedge v_i \ldots \wedge v_S \in \bigwedge^S \mathbb{C}^n$$

where $\bigwedge^S \mathbb{C}^n = span\{v_1 \wedge \ldots \wedge v_S \mid v_i \in \mathbb{C}^n\}$ and $\dim(\bigwedge^S \mathbb{C}^n) = \binom{n}{S}$.

**Lemma 3.0.3.** *Let $\mathscr{B}_1 = \{v_1, \ldots, v_k\}$ and $\mathscr{B}_2 = \{w_1, \ldots, w_k\}$ be any two bases of a $k-$dimensional subspace $W$ of a $\mathbb{C}^n$. Then for some $\lambda \in \mathbb{C}^*$ we have that*

$$v_1 \wedge \ldots \wedge v_k = \lambda w_1 \wedge \ldots \wedge w_k.$$

**Proposition 3.0.4.** *By Plücker embedding $G(k,n) \hookrightarrow \mathbb{CP}(\bigwedge^k \mathbb{C}^n) \cong \mathbb{CP}^{\binom{n}{k}-1}$, we then have that $G(k,n)$ is a closed subset of a projective space.*

### 3.0.2 Decorated Riemann surfaces

**Definition 3.0.5.** *Let $n \geq 0$ be an integer. We shall call the $n+1-$tuple $(\Sigma, p_1, \ldots, p_n)$ a $n-$**pointed Riemann surface** where the elements of $\{p_1, \ldots, p_n\} \subset \Sigma$ are called **marked points** of the smooth Riemann surface $\Sigma$.*

The genus $g(\Sigma, p_1, \ldots, p_n) := g(\Sigma)$. We define $\mathscr{M}_{g,n}$ the **moduli space of $n-$ pointed genus $g$ Riemann surfaces** by

$$\mathscr{M}_{g,n} := \left\{ (\Sigma, p_1, \ldots, p_n) \middle| \begin{array}{c} \Sigma \text{ is a genus } g \text{ compact Riemann surface} \\ \text{with } n \text{ distinct marked points} \end{array} \right\} \Big/ \sim$$

with $(\Sigma, p_1, \ldots, p_n) \sim (\Sigma', p'_1, \ldots, p'_n)$ exactly when there exists an isomorphism $\varphi : \Sigma \to \Sigma'$ defined by $\varphi(p_i) = p'_i$, with $1 \leq i \leq n$. For $(g,n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$.

For instance when $g = 0$ and $n \geq 3$ we take $(\Sigma, p_1, \ldots, p_n) \in \mathscr{M}_{0,n}$ and use cross-ratio $\lambda$ to send $(p_1, p_2, p_3, p_4, \ldots, p_n)$ to $(0, 1, \infty, \lambda(p_4), \ldots, \lambda(p_n))$ where each of the $n-3$ distinct points $t_{i-3} := \lambda(p_i) \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ $\forall i = 4, \ldots, n$. In this way we can uniquely identify $(\Sigma, p_1, p_2, p_3, p_4, \ldots, p_n)$ by $(\mathbb{CP}^1, 0, 1, \infty, t_1, \ldots, t_{n-3})$. Hence

$$\mathscr{M}_{0,n} = (\mathbb{CP}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \cup \text{ All Diagonals.}$$

## 3.1 Compactification of the Moduli space $\mathscr{M}_{g,n}$

The following example illustrates why the moduli space $\mathscr{M}_{g,n}$ is not compact

**Example 3.1.1.** *Comparing the cross-ratio of two Riemann surfaces*

$$(\Sigma_t, 0, 1, \infty, t), (\Sigma'_t, 0, t^{-1}, \infty, 1) \in \mathscr{M}_{0,4}.$$

*For $\Sigma_t$, the cross-ratio $\lambda(p_4) = t$. For $\Sigma'_t$ we first note that $\lambda(x) = tx$ maps $(0, t^{-1}, \infty)$ to $(0, 1, \infty)$ so that the cross-ratio of $\Sigma'$ is $\lambda(1) = t.1 = t$. Hence the Riemann surfaces are essentially the same i.e. $\Sigma_t = \Sigma'_t \in \mathscr{M}_{0,4}$. But*

$$\lim_{t \to 0} \Sigma_t = (\Sigma, 0, 1, \infty, 0) \text{ i.e. } p_4 \to p_1 \text{ and } \lim_{t \to 0} \Sigma'_t = (\Sigma', 0, \infty, \infty, 1) \text{ i.e } p_2 \to p_3$$

*We observe that neither of the limits is in $\mathscr{M}_{0,4}$. This shows that $\mathscr{M}_{0,4}$ is not compact.*

**Definition 3.1.2.** *A **node** on a Riemann surface $\Sigma$ is a point whose neighbourhood looks like that of the origin in the equation $xy = \varepsilon$ as $\varepsilon$ tends to zero. $\Sigma$ is called **nodal Riemann surface** if its points are either nodes or smooth points.*

An $n$ **pointed nodal Riemann surface** is a nodal Riemann surface with $n$ distinct marked points. A marked point or a node on a nodal Riemann surface shall be refered to as a **special point**.

**Definition 3.1.3.** *A* ***stable*** $n$ ***pointed nodal Riemann surface*** $(\Sigma, p_1, \ldots, p_n)$ *is one whose each connected component satisfies:*

1. *The number of special points $n$ on a genus $g$ connected component satisfies the stability condition $n \geq 3-2g$. Such pair $(g,n)$ is called* ***stable pair.***
2. $|$
   $Aut\Sigma, p_1, \ldots, p_n) \, | < \infty$.
3. *The number of automorphism fixing the special points in $(\Sigma, p_1, \ldots, p_n)$ are not infinetesimal.*

We compactify $\mathscr{M}_{g,n}$ by adding nodal Riemann surfaces to get the Deligne-Mumford compactified moduli space of $n-$pointed genus $g$ stable Riemann surfaces $\overline{\mathscr{M}}_{g,n}$.

**Example 3.1.4.** *Consider $\mathscr{M}_{0,4} = \mathbb{CP}^1 \setminus \{0,1,\infty\}$. Take an element with the following representation.*

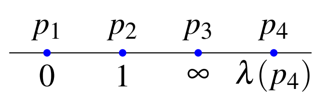



**Figure 9.** $(\Sigma_\lambda, 0, 1, \infty, \lambda(p_4)) \in \mathscr{M}_{0,4}$

*There are 3 limits $(0,1$ and $\infty)$ of $\Sigma_\lambda$ corresponding to $\lambda(p_4) \to 0, \lambda(p_4) \to 1$ and $\lambda \to \infty$ respectively. Hence $\overline{\mathscr{M}}_{0,4} = \mathbb{CP}^1$.*

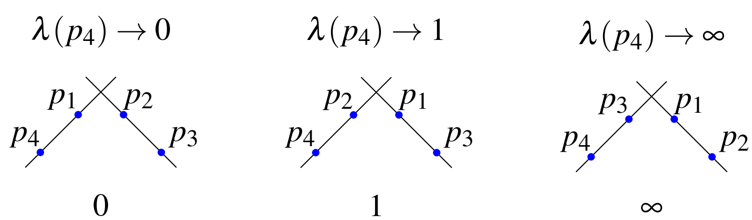



**Figure 10. The 3 limits $(0,1$ and $\infty)$ of $\Sigma_\lambda$**

Generally, $\overline{\mathscr{M}}_{g,n} = \mathscr{M}_{g,n} \cup \partial \overline{\mathscr{M}}_{g,n}$ with $\Sigma_g \in \partial \overline{\mathscr{M}}_{g,n}$ being stable Riemann surface with a single node. We call $\partial \overline{\mathscr{M}}_{g,n} \subset \overline{\mathscr{M}}_{g,n}$ **boundary** of $\overline{\mathscr{M}}_{g,n}$.

$$\overline{\mathscr{M}}_{g,n} := \left\{ (\Sigma_g, p_1, \ldots, p_n) \middle| \begin{array}{c} \Sigma_g \text{ is an } n \text{ pointed genus } g \text{ stable Riemann surface} \\ \text{with } p_i \neq p_j \text{ for all } i \neq j \end{array} \right\} \Big/ \sim$$

**Example 3.1.5.** *We then have that $\overline{\mathscr{M}}_{0,n}$ is the collection of such $\Sigma$ which is a connected projective Riemann surface with (at the extreme) nodal singularity ; and whose twigs (components) each is isomorphic $\mathbb{CP}^1$. The points of $\Sigma$ are either smooth or nodal or marked so that* $|\operatorname{Aut}\Sigma| < \infty$.

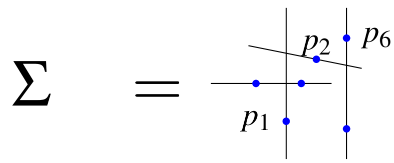



**Figure 11. An element in $\partial\overline{\mathscr{M}}_{0,6}$**

**Theorem 3.1.6.** *For stable pair $(g,n)$, the orbifold $\overline{\mathscr{M}}_{g,n}$ of dimension $n-3+3g$ parametrizes $n-$ pointed genus $g$ Riemann surfaces while $\mathscr{M}_{g,n}$ is its open Zariski dense subvariety.*

$\overline{\mathscr{M}}_{0,n}$ has a boundary stratification as will we shall discuss in the following section.

## 3.2 $\overline{\mathscr{M}}_{g,n}$ is a Stratified Space

Every $n$ pointed Riemann surface $\Sigma$ of genus $g$ has a corresponding **dual graph** $\Gamma$ which satisfies:

- $V_\Gamma :=$ Set of vertices of $\Gamma$ is in one-to-one correspondence to Smooth components of $\Sigma$.
- $E_\Gamma :=$ Set of edges of is in one-to-one correspondence to Nodes of $\Sigma$.
- $L_\Gamma :=$ Set of external legs of $\Gamma$ is in one-to-one correspondence to Marked points of $\Sigma$.

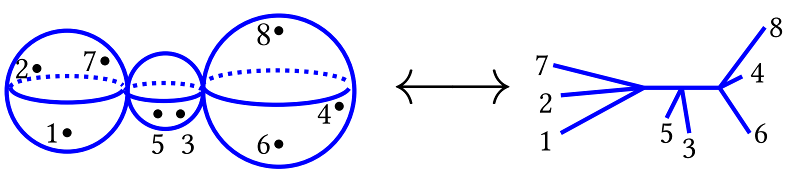



**Figure 12. Dual graph of an element in $\partial\overline{\mathscr{M}}_{0,8}$.**

**Example 3.2.1.**

The dual graph $\Gamma$ encodes the stratification of $\Sigma \in \partial\overline{\mathscr{M}}_{g,n}$ with $g = g(\Sigma) := g(\Gamma)$ computed as

$$g(\Gamma) = 1 - v(\Gamma) + e(\Gamma) + \sum_{v \in V_\Gamma} g_v = h^1(\Gamma) + \sum_{v \in V_\Gamma} g_v. \tag{2}$$

Here $g_v$ is the genus of the component represented by vertex $v$ while $v(\Gamma), e(\Gamma)$ and $h^1(\Gamma)$ are the number of vertices, edges and the first Betti number (rank$H_1(\Gamma)$ of the first homology group of $\Gamma$) of $\Gamma$ respectively. In **Figure** 13 above, for instance, the dimension of the **cell corresponding to** $\Gamma$, $\mathscr{M}_\Gamma \subset \overline{\mathscr{M}}_{0,n}$, is given by

$$dim_{\mathbb{C}}\mathscr{M}_\Gamma := n-3-e(\Gamma).$$

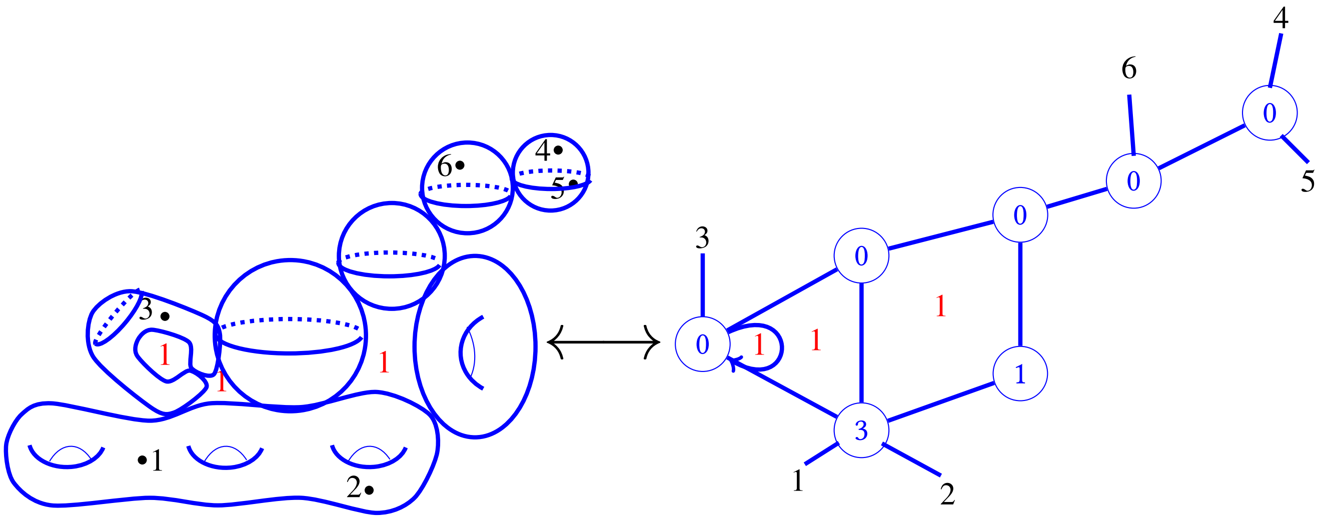



**Figure 13. Dual graph of an element in** $\partial\overline{\mathscr{M}}_{7,6}$.

**Example 3.2.2.** *We can compute the genus of the graph,*

$$g = g(\Gamma) = (3+1+0+0+0+0+0)+(\#\text{of holes in graph(genus of graph)}) = 4+(1+1+1) = 7.$$

*Alternatively, we can employ equation (2) for the dual graph* $\Gamma$ *of* $\Sigma$ *in Figure 16 above we find the same result*

$$g = g(\Gamma) = 1-7+9+(3+1+0+0+0+0+0) = 7.$$

Note that the genus of the dual graph $\Gamma$ on a compact Riemann surface $\Sigma$ is the same as the genus of the Riemann surface $\Sigma$.

We note that, $\overline{\mathscr{M}}_{g,n}$ is connected, irreducible and equipped with a universal stable Riemann surface

$$\mathscr{C}_{g,n} = \{(X,p) \mid X \in \overline{\mathscr{M}}_{g,n+1}, p \in \Sigma_X\} \xrightarrow{f(X,p)\to X} \overline{\mathscr{M}}_{g,n}.$$

This is called a **forgetful map** and it is a projection map. We have that $dim_{\mathbb{C}}(\mathscr{C}_{g,n}) = n-2+3g$. Whereas

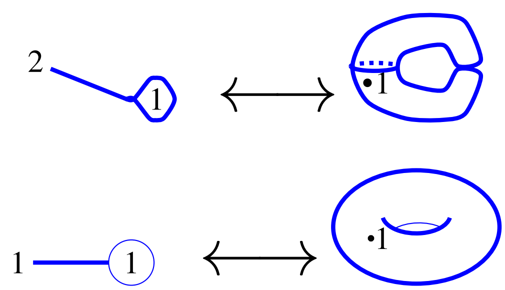
$$f^{-1}\mathscr{M}_\Gamma := f\mid_{\mathscr{M}_\Gamma} = \mathscr{M}_\Gamma$$

We can look at $\mathscr{M}_\Gamma$ as :

$$\mathscr{M}_\Gamma = (\prod_{v \in V_\Gamma} \mathscr{M}_v)/Aut(\Gamma). \tag{3}$$

Now for $g \geq 2$, the space immediately above is an orbifold. Consider the moduli space $M_{0,4} = \mathbb{CP}^1 \setminus \{0,1,\infty\}$ and its Deligne-mumford compactification $\overline{\mathscr{M}}_{0,4} = \mathbb{CP}^1$. Stratification above can be extended to other classes of Riemann surfaces as shown below:

For $g = 1, n = 1$we have the following configurations




and the configurations for $g = 1$ and $n = 2$ are illustrated below

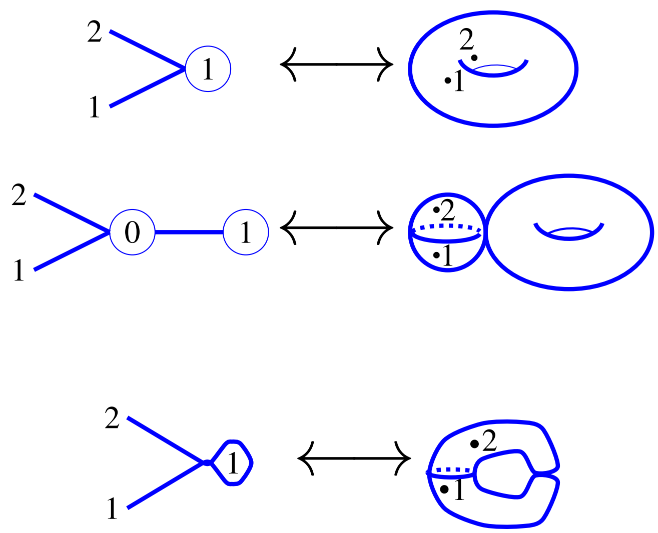



**Figure 14. Dual graph corresponding to some elements in $\partial\overline{\mathscr{M}}_{g,n}$ where $n = 1,2$.**

Hence for genus $g = 1$ there are exactly two configurations: The first one from the graph type with one vertex of genus one and all other vertices of genus zero with at least zero leg from each vertex; the second one having a loop. One could draw dual graphs for genus $g = 2$ to realise that there are exactly 7 configurations.

There are 3 different automorphisms of elements of $\overline{\mathscr{M}}_{0,4}$ when the marked points aproach the neighbouring points. These corresponds to the metric trees in Figure 16.

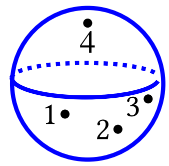



**Figure 15.** $\Sigma \in \overline{\mathscr{M}}_{0,4}$

These are the boundary strata of $\overline{\mathscr{M}}_{0,4}$.

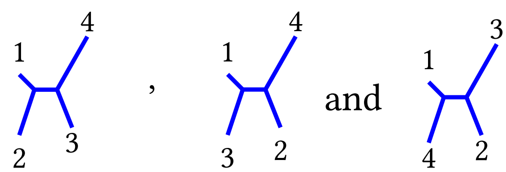



**Figure 16. Three Boundary strata of $\Sigma \in \mathscr{M}_{0,4}$**

**Definition 3.2.3.** *For an algebraic variety $B$, a* ***flat morphism*** $\pi : \mathscr{C} \to B$ *is a morphism for which there is an embedding*

$$\begin{array}{ccc} \mathscr{C} & \hookrightarrow & \mathbb{CP}^N \times B \\ \pi \downarrow & \swarrow p & \\ B & & \end{array}$$

*for some* $N \in \mathbb{N}$ *such that* $\mathscr{C}_b := \pi^{-1}(b). \subset \mathbb{CP}^N \times \{b\}$.

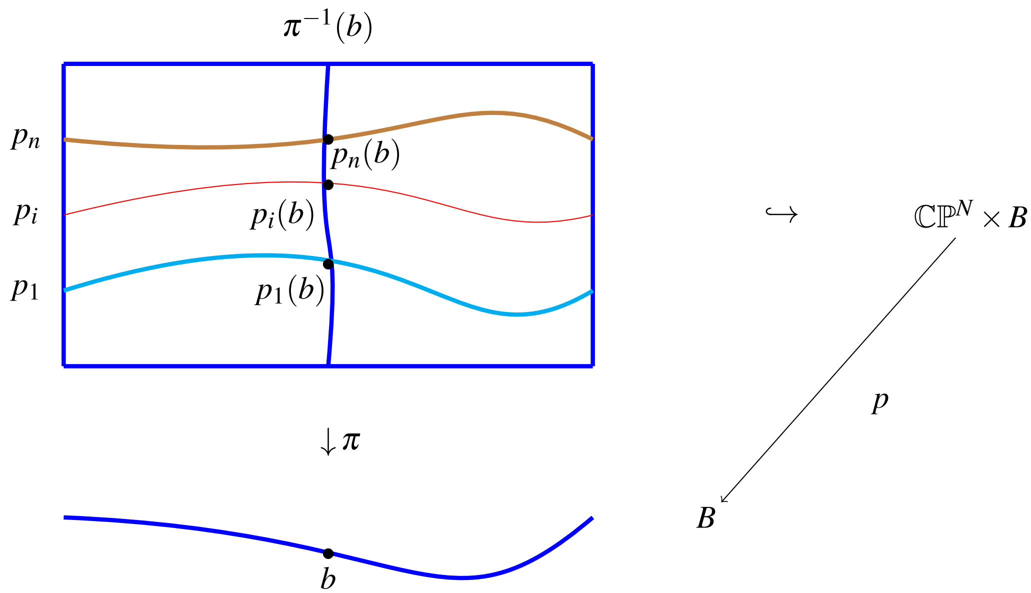



**Figure 17. An illustration of flat morphism over an algebraic variety $B$**

**Definition 3.2.4.** *A family over an algebraic variety $B$ of genus $g$ Riemann surfaces on which there are $n$ distinct marked points is a flat morphism $\pi : \mathscr{C} \to B$ with $n$ sections corresponding to marked points $p_i$ such that every fibre $\mathscr{C}_b := (\pi^{-1}(b), p_1(b), \ldots, p_n(b)) \in \overline{\mathscr{M}}_{g,n}$.*

We end this discussion on stratification of $\overline{\mathscr{M}}_{g,n}$ above by the following definition from [BH08].

**Definition 3.2.5.** *A type $(g,n)$ **stable graph** $\Gamma$ with $g \geq 0, n \geq 1$ satisfying $2-2g-n<0$ is given by the following data:*

S$\Gamma$1. *two finite sets $V_\Gamma$ and $L_\Gamma$;*

S$\Gamma$2. *a partition $\mathscr{P}$ of $L_\Gamma$ into singletons (called **legs** or **leaves** of $\Gamma$) or subsets of pairs of elements(called **edges** of $\Gamma$);*

S$\Gamma$3. *a map $\gamma : V_\Gamma \to \{0,1,\ldots,g\}$ satisfying*

$$g(\Gamma) = h^1(\Gamma) + \sum_{v \in V_\Gamma} \gamma(v);$$

$S\Gamma 4$. *for $v \in V_\Gamma$, a subset $L(v) \subset L_\Gamma$ is such that $2\gamma(v) - 2 + \mid L(v) \mid > 0$;*

$S\Gamma 5$. *a map $\eta : \{v : v \in V_\Gamma\} \to \{1, 2, \ldots, n\}$.*

By choosing an ordering of $L(v)$ for each of the $l(v) = \mid L(v) \mid$ vertices $v \in L(v)$,we obtain a morphism

$$\varepsilon_\Gamma : \prod_{v \in V_\Gamma} \overline{\mathscr{M}}_{l(v),\gamma(v)} \to \overline{\mathscr{M}}_{g,n}$$

which maps a genus $\gamma(v)$ Riemann surface $\Sigma_{\gamma(v)}$ with $l(v)$ distinct marked points into an $n-$pointed genus $g$ Riemann surface $\Sigma_g$. The morphism $\varepsilon_\Gamma$ does not depend on the choice of ordering of $L(v)$.

## 3.3 Morphisms Between Moduli Spaces $\overline{\mathscr{M}}_{g,n}$

The following are natural morphisms between moduli spaces.

1. Consider the action $\gamma : S_n \times \overline{\mathscr{M}}_{g,n} \to \overline{\mathscr{M}}_{g,n}$ defined by

$$\gamma((\tau, (\Sigma_g, p_1, \ldots, p_n))) = (\Sigma_g, p_{\tau(1)}, \ldots, p_{\tau(n)}).$$

This morphism is called **permutation morphism.** This is simply renaming the marked points.

**Example 3.3.1.** *The isomorphism of the corresponding trees with $n$ external leaves is through cyclic permutation $\tau \in S_n$. In particular if $n = 3$, $\tau \in \mathbb{R}^3_+/\mathbb{S}_3$ we have for instance $(a, b, c) \sim (b, a, c)$ and the corresponding morphism;*

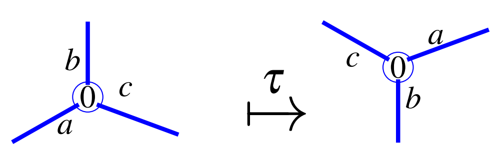



**Figure 18. An illustration of a permutation morphism**

2. We could also decide to forget the $(n+1)^{th}$ point, this gives a morphism

$$f : \overline{\mathscr{M}}_{g,n+1} \to \overline{\mathscr{M}}_{g,n} \ ,$$

referred to as **forgetful morphism**.

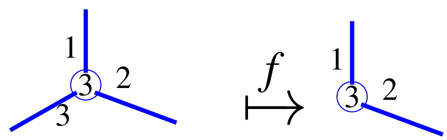



**Figure 19. An example of forgetful morphism for $g = 3$ and $n = 2$.**

However, if in rubbing out a marked point the stability of a component of the Riemann surface is compromised, we view $f$ as the **'inverse of bubbling.'** The points on $\mathbb{CP}^1$ come together as the $\mathbb{CP}^1$ copy shrinks into a scar (a point) or the scar acts as a node between 2 components.

**Example 3.3.2.** *With $g = 3$ and $n = 2$, we have that*

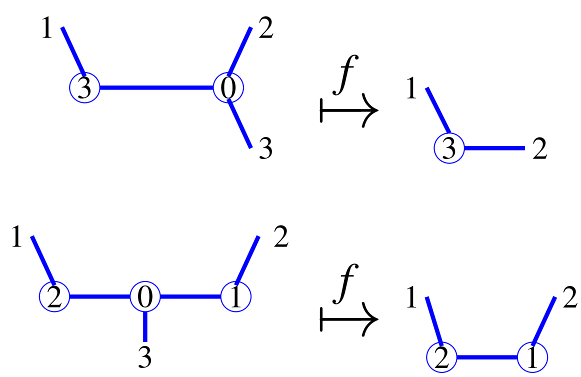



**Figure 20. An illustration of "inverse of bubbling" to restore stability.**

3. Consider the **gluing morphisms**

   i) $g : \overline{\mathscr{M}}_{g_1,n_1+1} \times \overline{\mathscr{M}}_{g_2,n_2+1} \to \overline{\mathscr{M}}_{g_1+g_2,n_1+n_2}$ is defined in such a way that the $(n_1 + 1)^{th}$ marked point on $\Sigma_{g_1} \in \overline{\mathscr{M}}_{g_1,n_1+1}$ is glued to the $(n_2+1)^{th}$ point on $\Sigma_{g_2} \in \overline{\mathscr{M}}_{g_2,n_2+1}$. The resulting Riemann surface now has genus $g_1 + g_2$, a node and at least two components. The two clumped marked points and 'killed' so that the new Riemann surface has $(n_1+1)+(n_2+1)-2 = n_1+n_2$ marked points i.e $(\Sigma_{g_1+g+2}, p_1, \dots, p_{n_1+n_2}) \in \overline{\mathscr{M}}_{g_1+g_2,n_1+n_2}$.

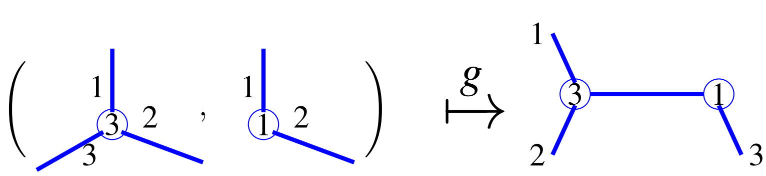



**Figure 21. An illustration of gluing morphism for $g_1 = 3, n_1 = 2$ and $g_2 = 1, n_2 = 1$.**

**Example 3.3.3.**

ii) $h : \overline{\mathscr{M}}_{g,n+2} \to \overline{\mathscr{M}}_{g+1,n}$ is defined by gluing the the marked points $p_{n+1}, p_{n+2} \in \Sigma_{g,n+2} \in \overline{\mathscr{M}}_{g,n+2}$ to form one more handle. The resulting stable Riemann surface has 2 points less and of genus 1 more than the original stable Riemann surface.

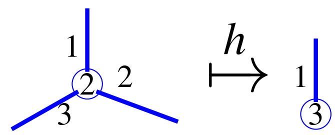



**Figure 22. An example to illustrate a gluing morphism for $g = 2$ and $n = 3$.**

# 4 Euler Characteristic of Moduli Space

## 4.1 Strebel Theory and the Topology of Metric Graphs

This section is dedicated to demonstrate how the **combinatorial information** of **ribbon graphs** $\Gamma$ can be used to cypher the holomorphic structure of a Riemann surface through Strebel differential theory. We will be mostly referring to [MP91].

**Definition 4.1.1.** *A **tiling** on a compact surface $\Sigma$ is an arrangement of finitely many polygons (called **faces of the tilling**) which cover $\Sigma$ and which meet (if they meet at all) at vertices (called **vertices of the tilling** ) or along complete edges (called **edges of the tilling**).*

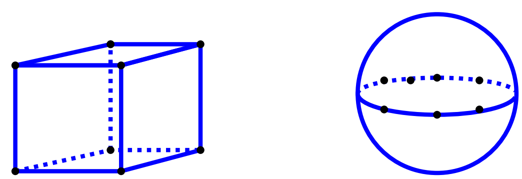


**Figure 23. Two tillings on a sphere**

**Remark 4.1.2.** *A **cell-decomposition** of the surface $\Sigma$ induced by the ribbon graph $\Gamma$ are the separate pieces of $\Sigma$ obtained by cutting $\Sigma$ along the edges and vertices of $\Gamma$. Each of these pieces look like a **point, line segment** or an **open disc** in an $n-$space. The **Euler characteristics** of $\Sigma$,*

$$\chi(\Sigma) := v(\Gamma) - e(\Gamma) + f(\Gamma) = 2 - 2g(\Gamma), \tag{4}$$

*where $g(\Gamma), v(\Gamma), e(\Gamma)$ and $f(\Gamma)$ are the genus of $\Sigma$, number of dimension $0$ (vertices), number of dimension $1$ (edges) and number of dimension $2$ (faces) respectively.*

Given a Riemann surface $\Sigma$, to calculate the Euler characteristic we fix a tiling on $\Sigma$ and write:

$$\chi(\Sigma) := \sum_{i=0}^{2} (-1)^k \#(\text{cells of dimension k}).$$

**Definition 4.1.3.** *Let $\Sigma$ be a Riemann surface. By a **ribbon graph** $\Gamma \in \Sigma$, we mean that $\Gamma$ is an embedded in $\Sigma$ so that each component of $\Sigma \setminus \Gamma$ is a disc.*

**Theorem 4.1.4** (Strebel). *Consider a genus g Riemann Surface* $(\Sigma, p_1, \dots, p_n)$ *with n distinct points on it such that*

$$\begin{cases} n \geq 3 & \text{if } g = 0 \\ n \geq 1 & \text{if } g \neq 0 \end{cases}$$

*with* $(g,n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$. *For any choice* $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n_+$ *of ordered positive real n tuple, there exists unique meromorphic quadratic differential* $\varphi$ *whose poles* $z_i$ *at* $p_i$ *for all* $i = 1, \dots, n$ *are of order* 2 *and the residue of* $\varphi$ *at* $z_i$ *is* $(\frac{a_i}{2\pi})^2 = A$. *Also, near* $z_i$, $\varphi$ *can be expressed as*

$$\varphi(z) = \frac{A}{z - z_i} dz_i^2 + \frac{B}{z - z_i} dz^2 + Cdz^2 + D(z - z_i)dz^2 + \dots$$

**Remark 4.1.5.** *Res*$(z_i, \varphi) = A$. *Let* $z_0 \in \Sigma \setminus \{p_1, p_2, \dots, p_n\}$ *such that* $\sqrt{\varphi}$ *is a neighbourhood of* $z_0 \in U$. *So that* $\pm\sqrt{f(z)dz^2} = \sqrt{f(z)}dz = \sqrt{\varphi}$.

$$F : U \to \mathbb{C} \text{ with } F(z_0) = 0,$$

$$F(z) = \int_\gamma \sqrt{\varphi} = \int_\gamma \sqrt{f(z)}dz.$$

*The choice of* $\pm$ *above is to make* $F(z) > 0$ *with respect to traversing the loop* $\gamma$ *in counter clockwise sense provided by complex structure on* $\Sigma$.

Around each marked point $z_i$ there is a foliated disk of compact horizontal leaves $\gamma_{i_1}, \dots, \gamma_{i_m}$ whose lengths are $p_{i_1}, \dots, p_{i_m}$ respectively. As the loop $\gamma$ enlarges outwards from $z_i$, it hits zeros of $\varphi$ and the shape becomes an $m$-gon whose circumference is $p_i = p_{i_1} + \dots + p_{i_m}$ given by

$$\int_\gamma \sqrt{\varphi} = p_i.$$

All such $m-$gon cover the whole surface $\Sigma$. Below, $p_i = p_{i_1} + p_{i_2} + p_{i_3} + p_{i_4}$, giving the following $4-$gon,

Consider the following 4 properties of a graph $\Gamma$

1. $\Gamma$ is a metric graph.
2. Each vertex is of degree at least 3 i.e $\partial(V) \geq 3$ for all vertices $V$.
3. Faces are isomorphic to discs and each face contains exactly one marking.
4. The perimeter of the face $D_i$ containing $z_i$ is $p_i$ for finitely many $i$, say $n$.

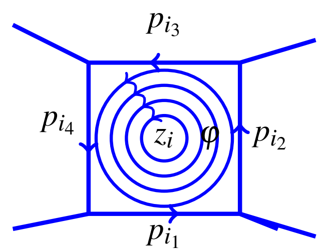



**Figure 24. Horizontal lines near marked points $z_i$ surround and converge to $z_i$.**

Now, for any choice $(a_1,\ldots,a_n)\in\mathbb{R}^n_+$, $M_{g,n}\xrightarrow{\cong}M^{Comb}_{g,n}(a_1,\ldots,a_n)$. With $M^{Comb}_{g,n}$ being the moduli of graphs on a topological surface of genus $g$ with $n$ faces satisfying properties $1-4$ in 4.1 above. This implies that there exists that the discs have a topological structure.

$$\mathbb{R}^n_+\times M_{g,n}\cong M^{Comb}_{g,n} \tag{5}$$

**Remark 4.1.6.** *The isomorphism in equation (5) above is constructed as follows:*

1. *First, we construct a mapping*

$$\coprod_{\Gamma\in RGB_{g,n}}\mathbb{R}^{e(\Gamma)}_+\xrightarrow{\alpha}M_{g,n}\times\mathbb{R}^n_+$$

*and show that the mapping*

$$RGB^{met}_{g,n}\xrightarrow{\beta}M_{g,n}\times\mathbb{R}^n_+$$

*is its descent. This is done by considering the boundary-preserving action of* $\mathrm{Aut}\,\Gamma$.

2. *Next, show that $\beta$ is a right-inverse of the mapping*

$$M_{g,n}\times\mathbb{R}^n_+\xrightarrow{\sigma}\coprod_{\Gamma\in RGB_{g,n}}\mathbb{R}^{e(\Gamma)}_+.$$

3. *The task ends by showing that $\sigma$ is a left-inverse of $\beta$ by demonstrating that*

$$\sigma\circ\beta=id\in M_{g,n}\times\mathbb{R}^n_+.$$

## 4.2 Calculating the Euler Characteristics of $\chi(\mathscr{M}_{g,n})$.

In computing the Euler characteristic $\chi(\mathscr{M}_{g,n})$ we will need the following properties of Euler characteristics.

1. $\chi(X\coprod Y)=\chi(X)+\chi(Y)$ where $X$ is open and $Y$ is closed.
2. $\chi(X\times Y)=\chi(X)\times\chi(Y)$. $\quad\chi(X\times \mathrm{A})=\chi(X)$ where A is a contractable face.
3. $\chi(X\setminus Y)=\chi(X)-\chi(Y)$.
4. If $p:E\to B$ is a continuous map then for $b\in B$, $p^{-1}(b)$ is a fibre and

$$\chi(E)=\chi(B)\times\chi(F).$$

We now start enumeration of $\chi(\mathscr{M}_{0,n})$. First, we show that

$$(\mathbb{CP}^1\setminus\{p_1,\ldots,p_n\})=2-n.$$

We proceed inductively. Since $\mathscr{M}_{0,3}=\{\mathrm{pt}\}$, thus we have $\chi(\mathscr{M}_{0,3})=1$.

Now from property 3 in 4.2 we have that

$$\chi(\mathbb{CP}^1)=\chi(\mathbb{CP}^{-1}\setminus\{p_1,\ldots,p_n\})+\{\chi(p_1)+\ldots+\chi(p_n)\}.$$

We then have that,

$$2=\chi(\mathbb{CP}^1\setminus\{p_1,\ldots,p_n\})+n\implies\chi(\mathbb{CP}^1\setminus\{p_1,\ldots,p_n\})=2-n.$$

By property 2 in 4.2 we get, $\chi(\mathscr{M}_{0,n})=\chi(\mathscr{M}_{0,n-1})\times\chi(\mathbb{CP}^1\setminus\{p_1,\ldots,p_{n-1}\})$ which implies

$$\chi(\overline{\mathscr{M}}_{0,n})=\chi(\mathscr{M}_{0,n-1})\times(3-n).$$

Recursively we get,

$$\chi(\mathscr{M}_{0,n})=(3-n)(4-n)(5-n)\ldots.(-1)=(-1)^{n-1}(n-3)!. \tag{6}$$

Further, we define $\chi_{g,n}:=\chi(\Sigma_g\setminus\{p_1,\ldots,p_n\})=2-2g-n$ to be the Euler Characteristic of open Riemann surfaces of corresponding type. So that,

$$\chi^{Orb}(\mathscr{M}_{g,n+1})=\chi_{g,n}\times\chi(\mathscr{M}_{g,n})=(2-2g-n)\chi(\mathscr{M}_{g,n}). \tag{7}$$

Let $X$ be an **orbifold**. Denote by $X_G\subset X$ the set of **points of $X$ which are invariant to a finite group $G$** but not to any larger group. It is a well known consequence that whenever

$p \in X$ has a neighbourhood $\cong \frac{U}{G}$ then $p \in X_G$. The **orbifold Euler characteristics** of $X$ is given by

$$\chi^{Orb}(X) = \sum_G \frac{\chi(X_G)}{|G|}. \tag{8}$$

**Lemma 4.2.1.** *[Mir95] Let $G < S_n$ be a permutation group acting on $\mathbb{R}^n_+$ naturally by permuting coordinate axes. Then $\mathbb{R}^n_+/G$ has a differentiable orbifold structure with orbifold Euler Characteristic*

$$\chi^{Orb}(\mathbb{R}^n_+/G) = \frac{(-1)^n}{|G|}. \tag{9}$$

**Example 4.2.2.** *Let $X = \{pt\}/G$ and $G < S_n$. Then the orbifold Euler characteristic of $X$ is given by*

$$\chi^{Orb}(X) = \frac{1}{|G|}.$$

**Theorem 4.2.3** (Harer-Zagier [HZ86])**.**

$$\chi^{Orb}(\mathscr{M}_{g,n}) := \chi(M_{g,n} \times \mathbb{R}_+{}^n) = (-)^{n-1}\frac{(2g+n-3)!(2g)(2g-1)}{(2g)!}\zeta(1-2g) \tag{10}$$

*with integers $g \geq 0, n \geq 1$ satisfying $2-2g-n < 0$ and $(g,n) \neq (1,1)$. Where*

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

*is the Riemann zeta function.*

`Proof.` Define the following generating function of $\chi^{Orb}(\mathscr{M}_{g,n})$ by

$$X(t) := \sum_{g,n\geq 1:\chi_{n,g}=2-2g-n<0} \frac{\chi^{Orb}(\mathscr{M}_{g,n})}{n!} t^{\chi_{g,n}}. \tag{11}$$

**Remark 4.2.4.** *The generating function $X(t)$ is obtained from Hermitian matrix integral whose asymptotic expansion is possible to calculate through the Penner model.*

Now, from the above identity given in (11), we have that

$$\frac{\chi^{Orb}(\mathscr{M}_{g,n+1})}{(n+1)!} t^{\chi_{g,n+1}} = \frac{d}{dt}\left(\frac{1}{n+1}\frac{\chi^{Orb}(\mathscr{M}_{g,n})}{n!} t^{\chi_{g,n}}\right). \tag{12}$$

As in [Kon92] let us write,

$A_1(t) := \sum_{g,n\geq 1} \chi(\mathscr{M}_{g,1}) t^{\chi_{g,n}}$, $A_0(t) := t\log t - t$ and $A(t) := A_1(t) - A_0(t)$.

We then have that,

$$\sum_{g,n\geq 1}\frac{\chi^{Orb}(M_{g,n})}{n!}t^{\chi_{g,n}} = A_1(t)+\frac{A_1'(t)}{2!}+\frac{A_1''(t)}{3!}+\ldots+\frac{A_0''(t)}{3!}+\frac{A_0'''(t)}{4!}+\ldots$$

$$=(A(t)+\frac{A'(t)}{2!}+\frac{A''(t)}{3!}+\ldots)-A_0(t)-\frac{A_0'(t)}{2},$$

where $\frac{A_k^{(k)}(t)}{k!}=\sum_{g\geq 1}\frac{\chi^{Orb}(M_{g,k})}{k!}t^{\chi_{g,k}}$.
As the we have from (6), with $g=0$,

$$\sum_{g=0,n\geq 3}\frac{\chi^{Orb}(\mathscr{M}_{0,n})}{n!}t^{\chi_{0,n}} = \sum_{n\geq 3}\frac{(-1)^{n-1}(n-3)!}{n!}t^{2-n}$$
$$=\frac{A_0''(t)}{3!}+\frac{A_0'''(t)}{4!}+\ldots,$$

we have that

$$X(t)=\sum_{k\geq 0}\frac{A_1^{(k)}(t)}{k!}+\sum_{k\geq 2}\frac{A_0^{(k)}(t)}{k!}=\sum_{k\geq 0}\frac{A^{(k)}(t)}{k!}-A(t)-\frac{A_0'(t)}{2} \tag{13}$$
$$=\int_t^{t+1}A(s)ds-(t\log t-t)-\frac{1}{2}\log t. \tag{14}$$

The integral in the equation above is given by

$$\int_t^{t+1}A(s)ds=\int_t^{t+1}\left(A(t)+(s-t)A'(t)+\frac{(s-t)^2}{2}A''+\ldots\right)ds$$
$$=A(t)+A'(t)\int_t^{t+1}(s-t)ds+A''(t)\int_t^{t+1}(s-t)^2ds$$
$$=A+\frac{A'}{2}+\frac{A''}{6}+\ldots=\log\left(\frac{\Gamma(t+1)}{\sqrt{2}\pi}\right).$$

Hence,

$$X(t)=\int_t^{t+1}A(s)ds=t\log t+t-\frac{1}{2}\log t. \tag{15}$$

We then have that for $\Gamma$ satisfying properties $1-4$ in 4.1 ,

$$\chi(\mathscr{M}_{g,n}) = \chi(\mathscr{M}_{g,n}\times{\mathbb{R}_+}^n)$$
$$=\chi(\mathscr{M}_{g,n}^{\text{Comb}})=\sum(-1)^{Codim\Gamma}\chi(M_\Gamma^{\text{Comb}}).$$

Here,

$$\frac{(\mathbb{R}_+)^{e(\Gamma)}}{\mathrm{Aut}(\Gamma)} := \mathscr{M}_{g,n}^{\mathrm{Comb}}.$$

But

$$\chi\left(\frac{(\mathbb{R}_+)^{e(\Gamma)}}{\mathrm{Aut}(\Gamma)}\right) = \frac{1}{|\,\mathrm{Aut}(\Gamma)\,|}.$$

As a result of the above fact, we get

$$\begin{aligned}\chi(\mathscr{M}_{g,n}) =& \chi(\mathscr{M}_{g,n}^{\mathrm{Comb}})\\ =& \sum_{\Gamma}(-1)^{6g-6+3n-E(\Gamma)}\frac{1}{|\,\mathrm{Aut}(\Gamma)\,|}.\end{aligned}$$

Therefore, $X(t) = \sum\limits_{\Gamma \text{ as before}} (-1)^{V(\Gamma)} \frac{t^{\chi_{g,n}}}{|\mathrm{Aut}(\Gamma)|}$ with

$$\begin{aligned}\chi_{g,n} =& \frac{2-2g}{V(\Gamma)-E(\Gamma)+E(\Gamma)-E(\Gamma)}\\ =& \frac{2-2g}{V(\Gamma)-E(\Gamma)}\end{aligned}$$

$$\implies X(t) = \sum_{\Gamma}(-1)^{V(\Gamma)}\frac{t^{V(\Gamma)-E(\Gamma)}}{|\,\mathrm{Aut}(\Gamma)\,|}. \tag{16}$$

**Asymptotic Expansion of $X(t)$**

Following [Eti02], we say $f(s) \sim \sum\limits_{i\geq 0} a_i s^{-i}$ almost everywhere in a neighborhood of zero if for every $\delta > 0$, we can find $\varepsilon > 0$ such that $|\,f(s) - \sum a_i s^{-i}\,| << \delta$ whenever $s \in (-\varepsilon, \varepsilon)$.

**Theorem 4.2.5 (Gauss).**

$$\begin{aligned}\int_{-\infty}^{\infty} x^n e^{tx^2} dx =& \sqrt{\frac{2\pi}{t}}\, t^{\frac{-n}{2}}\\ =& \begin{cases} 0 & if\, n = 2k+1 \\ (n-1) & if\, n = 2k, k \in \mathbb{Z}. \end{cases}\end{aligned}$$

*Further,*

$$\exp\left(-t\sum_{k\geq 2}^{\infty}\frac{x^k}{k}\right) = e^{-t\frac{x^2}{2}} e^{-t}\sum_{k\geq 3}^{\infty} e^{-t\frac{x^2}{2}}\left(\sum_{m_3,m_4,\ldots}\frac{\prod_{m=3}^{\infty}\left(\frac{x^k}{k}\right)^{m_k}}{(m_k)!}\right).$$

**Example 4.2.6.** *With almost all* $m_k = 0$,

$$\int e^{-t\frac{x^2}{2}} \left( \frac{\prod_{m=3}^{\infty} \left(\frac{x^k}{k}\right)^{m_k}}{(m_k)!} \right) dx.$$

*Here division means that we have distinguished edges.*

Now, using

$$-t \sum_{k\geq 2} \left( \frac{x^k}{k} \right) = t(x + \log(1-x))$$

and considering $t \to \infty$, we integrate the exponential function whose power is the above expression near zero to get

$$\begin{aligned}
\int_{\text{Near } 0} \exp\left( -t \sum_{k\geq 2} \left( \frac{x^k}{k} \right) \right) dx &= \int \exp\left(t(x+\log(1-x)\right) dx \\
&= \int_{\text{Near } 0} e^{tx}(1-x)^t dx \\
&= e^t \int_{\text{Near } 1} y^t e^{-ty} dy = \frac{e^t}{t^{t+1}} \int_0^{\infty} z^t e^{-z} dz = \frac{e^t}{t^{t+1}} \Gamma(t-1) \\
X(t) &= \log\left( \frac{e^t}{t^{t+1}} \sqrt{\frac{e^t}{2\pi}} \Gamma(t+1) \right) \\
&= \sum_{k\geq 1} \frac{\zeta(-k)}{-k} t^{-k},
\end{aligned}$$

where $\zeta(s) = \sum\limits_{n\geq 1} \frac{1}{n^s}$. By analytic continuation,

$$\int_t^{t+1} \left( A(t) - t\log t + t - \frac{1}{2}\log t \right) dt = \sum_{k\geq 1} \frac{\zeta(-k)}{-k} t^{-k}. \tag{17}$$

Now, by uniqueness of asymptotic expansion we get the following closed formulae

$$\begin{aligned}
\chi^{Orb}(\mathscr{M}_{g,n}) := \chi(\mathscr{M}_{g,n} \times \mathbb{R}_+{}^n) &= n! \sum_{\Gamma \in RG_{g,n}} \frac{(-1)^{e(\Gamma)}}{|\operatorname{Aut}(\Gamma)|}. \\
&= (-)^{n-1} \frac{(2g+n-3)!(2g)(2g-1)}{(2g)!} \zeta(1-2g). \\
&= (-1)^n \frac{(2g+n-3)!(2g-1)}{(2g)!} b_{2g}.
\end{aligned}$$

□

We then infer the result in [HZ86] that, $A(t) = \frac{1}{2} - t + t\log\Gamma(t)$ and hence $\chi(M_{g,1}) = \zeta(1-2g) = -\frac{b_{2g}}{2g}$. Here, the numbers $b_{2g}$ are the $(2g)^{th}$ Bernoulli coefficients given by the polynomial $f(x) = \frac{x}{e^x-1} = \sum_{m=0}^{\infty} \frac{b_m}{m!}x^m$. The Bernoulli numbers are generated by the formulae

$$b_m = \sum_{k=0}^{m} \frac{1}{k+1} \sum_{i=0}^{k} (-1)^i \binom{k}{i}.$$

The first few being $b_0 = 1, b_1 = -\frac{1}{2}, b_{2(1)} = \frac{1}{6}, b_{2(2)} = -\frac{1}{30}, b_{2(3)} = \frac{1}{42}, b_{2(4)} = -\frac{1}{30}$, $b_{2(5)} = \frac{5}{66}, b_{2(6)} = -\frac{691}{2730}, \dots$

For instance, when $(g,n) = (1,1)$, we get

$$\chi(\mathscr{M}_{1,1} \times \mathbb{R}_+^1) = \zeta(-1) = -\frac{1}{12}.$$

More examples are given below in table 1 .

| $n\backslash g$ | 3 | 4 | 5 | 6 |
|---|---|---|---|---|
| 2 | 5/252 | -7/240 | 9/132 | -7601/32760 |
| 3 | -5/42 | 7/30 | -15/22 | 7601/2730 |
| 4 | 5/6 | -21/10 | 15/2 | -7601/210 |
| 5 | -20/3 | 21 | -90 | 7601/15 |

**Table 1. Euler characteristic $\chi(\mathscr{M}_{g,n})$ for $(3,2) \leq (g,n) \leq (6,5)$**

## 4.3 Enumerating the Euler Characteristics of $\overline{\mathscr{M}}_{g,n}$

We now extend the result in equation 4.2 to $\chi(\overline{\mathscr{M}}_{g,n})$. To avoid notational clutter, the notation $\chi(\overline{\mathscr{M}}_{g,n})$ shall be used for an orbifold Euler characteristic $\chi^{Orb}(\overline{\mathscr{M}}_{g,n})$. From definition 3.2.5 and 3 we have that

$$\overline{\mathscr{M}}_{g,n} = \bigcup_{\Gamma} \frac{\mathscr{M}_\Gamma}{Aut(\Gamma)}$$

so that

$$\chi(\overline{\mathscr{M}}_{g,n}) = \sum_{\Gamma} \frac{\prod_{v \in V_\Gamma} \mathscr{M}_{l(v),\gamma(v)}}{|\operatorname{Aut}\Gamma|}.$$

We arrange $\chi(\overline{\mathscr{M}}_{g,n})$ in a generating series and use asymptotic expansion techniques to obtain a formula computing it from its genus $g$ and number of distinct points $n$ using basic operation.

Let

$$F(x,\lambda) := \sum_{g\geq 0} F_g(x)\lambda^{g-1}$$

where

$$F_g(x) := \sum_{n\geq 0,\ n\geq 3-2g} \chi(\overline{\mathscr{M}}_{g,n})\frac{x^n}{n!} \tag{18}$$

so that the generating function of $\overline{\mathscr{M}}_{g,n}$ becomes:

$$F(x,\lambda) := \sum_{n\geq 0}\sum_{n\geq 3-2g} \chi(\overline{\mathscr{M}}_{g,n})\frac{x^n}{n!}\lambda^{g-1}. \tag{19}$$

The theorem below will give the power series expansion of $F(x,\lambda)$ involving the generating series given by:

$$\Omega(x,\lambda) := \sum_{g\geq 0}\sum_{n\geq 3-2g} \chi(\mathscr{M}_{g,n})\frac{x^n}{n!}.$$

We then seek to find a closed formula for $F_g(x)$ using the usual asymptotic theory techniques.

**Theorem 4.3.1.**

$$\exp(F(x,\lambda)) = \frac{1}{\sqrt{2\pi\lambda}}\int_{\mathbb{R}} \exp(-\frac{(y-x)^2}{2\lambda} + \Omega(y,\lambda))dy. \tag{20}$$

Proof. We put $y - x = z\sqrt{\lambda}$ on the right side of equation 4.3 which reduces to

$$\exp(F(y,\lambda)) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(\sum_{g\geq 0}\sum_{n\geq 3-2g} \chi(\mathscr{M}_{g,n})\frac{(x+z\sqrt{\lambda})^n}{n!})\exp(-\frac{z^2}{2})dz,$$

a one-dimensional Gaussian integral. Expanding the integrand as a power series yields

$$1+\sum_{k\geq 1}\sum_{g_1,\ldots,g_k\geq 0}\sum_{\substack{r_1,\ldots,r_k\\ r_i\geq 3-2g_i}}\prod_{i=1}^{k}\chi(\mathscr{M}_{g_i,r_i}).\sum_{\substack{t_1,\ldots,t_k=0\\ \sum t_i\ even}}^{r_1,\ldots,r_k}\frac{(t_1+\ldots+t_k-1)!!}{k!t_1!\ldots t_k!}\frac{x^{\sum_{i=1}^{k}(r_i-t_i)}}{\prod_i(r_i-t_i)!}\lambda^{\sum_{i=1}^{k}(g_i-1)+\frac{1}{2}\sum_{i=1}^{k}t_i}. \tag{21}$$

In terms of the stable graphs, the equation (21) has a meaning. For $k \geq 1$, consider stable graphs $\Gamma_1, \dots, \Gamma_k$ with the pair $(g_i, r_i)$ encoding the genus of a vertex and legs in the graph $\Gamma_i$ respectively. For every choice $t_i$ of legs of $\Gamma_i$ such that $0 \leq t_i \leq r_i$, there are double factorial $(t_1 + \ldots + t_k - 1)!!$ possible interconnections between them provided $\sum t_i$ is even.

For every pairing we get a stable graph (disconnected) of type $(g_k, n_k)$ with

$$g_k = \sum_{i=1}^{k} g_i + 1 - k + \frac{1}{2}\sum_{i=1}^{k} t_i,\ n_k = \sum_{i=1}^{k}(r_i - t_i).$$

On the other hand, if we fix a disconnected stable graph of type $(g,n)$ then we can find integers $k, t_1, \dots, t_k, r_1, \dots, r_k$ as used in equation (21) which could be written as

$$1 + \sum_{k\geq 1} \sum_{g_1,\ldots,g_k \geq 0} \sum_{\Gamma \in \Gamma_{g,n}} \frac{\chi(\mathscr{M}_\Gamma)}{|\, Aut(\Gamma)\,|} \lambda^{g-1}, \tag{22}$$

where $\Gamma_{g,n}$ is the set of disconnected stable graphs of type $(g,n)$. □

## 4.4 Asymptotic Formula for $F_g(x)$

Expansion of the right hand side of equation (20) helps us deduce a formula for $F_g(x)$. This is done by substituting

$$U(x,y,\lambda) := -\frac{(y-x)^2}{2\lambda} + \Omega(y,\lambda)$$

whose formal power series is centred at the solution of

$$\bar{y}' := x + \sum_{g\geq 0} \frac{\partial \Omega(\bar{y})}{\partial y} \lambda^g. \tag{23}$$

A solution to equation (23) which takes the form

$$\bar{y} := \bar{y}(x,\lambda) = \sum_{g\geq 0} y_g(x)\lambda^g$$

gives rise to a recursive formula

$$y_0(x) = x + \sum_{n\geq 2} \chi(\mathscr{M}_{0,n+1}) \frac{y_0^n(x)}{n!}$$

$$y_g(x) = x + \sum_{k=0}^{g} \sum_{\substack{n\geq 2-2k \\ n\geq 0}} \chi(\mathscr{M}_{k,n+1}) \sum_{\substack{m_1+2m_2+\ldots+gm_g=g-k \\ m_0+m_1+\ldots+mg=n}} \frac{y_0^{m_0}+\ldots+y_g^{m_g}}{m_0!\ldots m_g!}.$$

The function $y_0(x)$ can be computed via the differential equation

$$\frac{dy_0(x)}{dx}(1+\log(1+y_0(x))) = 1.$$

This gives rise to the power series

$$y_0(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{7}{24}x^4 + \frac{17}{60}x^5 + \frac{71}{240}x^6 + \frac{163}{504}x^7 + O(x^8)$$

from which the boundary condition $y_0'(0) = 0$ is realised. We then have that all $y_g(x)$ values are uniquely determined by the recursion below

$$\frac{y_g(x)}{y_0'(x)} = \sum_{k=0}^{g} \sum_{\substack{n\geq 2-2k \\ n\neq 0}} \chi(\mathscr{M}_{k,n+1}). \sum_{\substack{m_1+2m_2+\ldots+gm_g=g-k \\ m_0+m_1+\ldots+mg=n}} \frac{y_0^{m_0}+\ldots+y_g^{m_g}}{m_0!\ldots m_g!}.$$

Now, the expansion of $U(x,y,\lambda)$ at the point $\overline{y}(x,\lambda)$ and put $w = y - \overline{y}$ yields

$$-\frac{(x-\overline{y})^2}{2\lambda} + \Omega(\overline{y},\lambda) - \frac{1}{2\lambda}w^2(1 - \sum_{g\geq 0}\Omega^{(2)}(\overline{y})\lambda^g) + \sum_{k\geq 3}\frac{1}{k!}w^k(\sum_{g\geq 0}\Omega^{(k)}(\overline{y})\lambda^{g-1}),$$

where all $k-$derivatives are with respect to $y$.

Finally writing

$$G := G(x,\overline{y}(x,\lambda)) = \sum_{g\geq 0}\Omega_g^2(\overline{y}(x,\lambda))\lambda^g,$$

$$S_k := S_k(x,\overline{y}(x,\lambda)) = \sum_{g\geq 0}\Omega_g^k(\overline{y}(x,\lambda))\lambda^{g-1}$$

and

$$A := A(x,\overline{y}(x,\lambda)) = \sum_{r\geq 1} \sum_{\substack{t_1,\ldots,t_k=0 \\ \sum t_i \text{ even}}} \frac{(t_1+\ldots+t_k-1)!!}{k_1!\ldots k_r!} \frac{S_{k_1}\ldots S_{k_r}}{\sqrt{1-G^{1+k_1+\ldots+k_r}}}\lambda^{\frac{1}{2}(k_1+\ldots+k_r)},$$

we have the following theorem.

**Theorem 4.4.1.** *An asymptotic expansion of $F(x,\lambda)$ is given by*

$$-\frac{(x-\overline{y})^2}{2\lambda} + \sum_{g\geq 0}\Omega_g(\overline{y})\lambda^{g-1} - \frac{1}{2}\log(1-G) + \log(1+A).$$

The generating series are obtained by adding up all contributions from each configurations of the dual graphs associated to a fixed stable pair $(g,n)$.

**Example 4.4.2.** *For $g=2$ and any fixed $n$, we add up the contributions from all 7 dual graph configurations of $(\Sigma_2, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{2,n}$ to obtain*

$$\begin{aligned}
F_2 =& \frac{1}{1440(1+D)^2(E-1)^3}[-2D^8(E-1)^2(7+3E) - 24D^7(E-1)^2(-7+17E) \\
&+30D^5(E-1)^2(61E-221) - 3D^6(E-1)^2(259+201E) \\
&+360D(45E^3-167E^2+206E-84)+60(73E^3-270E^2+336E-144) \\
&+180D^2(138E^3-519E^2+635E-254)+60D^3(341E^3-1322E^2+1633E-652) \\
&+15D^4(631E^3-2640E^2+3395E-1386]+\frac{D^5}{4}.
\end{aligned}$$

*Whose power series expansion is*

$$F_2(x) = 6+13x+21x^2+\frac{181}{6}x^3+\frac{251}{6}t^4+\frac{6883}{120}x^5+\frac{28196}{360}x^6+O(x^7)$$

*from where we read off $\chi(\overline{\mathcal{M}}_{2,n})$ for $0 \le n \le 6$ by multiplying the coefficient of $x^n$ by $n!$.*

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $\chi(\overline{\mathcal{M}}_{2,n})$ | 6 | 13 | 42 | 181 | 1004 | 6883 | 56392 |

**Table 2. Euler characteristic $\chi(\overline{\mathcal{M}}_{2,n})$ for $0 \le n \le 6$.**

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