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Laplace Distribution and Its Generalizations

Research Report in Mathematics, Number 16, 2018

Gilbert Kiprotich

August 2018

Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Mathematical Statistics

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Master of Science Project

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Abstract

In this project construction of Laplace distribution has been reviewed using the difference method,method of mixture,product of Rayleigh and normal distributions.The properties of Laplace distribution including mgf,skewness and kurtosis have been determined.
A generalization of Laplace distribution has been studied based on Beta generated distribution,generalized Laplace distribution,Laplace mixtures and Exponential power mixtures.Both Laplace mixtures and Exponential power mixtures are obtained using 16 mixing distributions through the methods of modified Bessel function of the third kind and in terms of confluent hypergeometric function.The the rth moments for Laplace mixtures are looked into.In addition, the classical cases of EP when $r = 1$ giving rise to Laplace distribution and when $r = 2$ to yield normal distribution are also studied.

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Declaration and Approval

I the undersigned declare that this project report is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature　　　　Date

GILBERT KIPROTICH
Reg No. I56/87238/2016

In my capacity as a supervisor of the candidate, I certify that this report has my approval for submission.

Signature　　　　Date

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Dedication

This project is dedicated to my Dad, Joseph and Mum, Christine,who through their training to seek God‘s wisdom and how I should go,now cannot depart from it.

Contents

Acknowledgments

You,Oh Lord,are all I have,and you give me all I need ;my future is in your hands.And so I am thankful and glad and I feel completely secure.
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Gilbert Kiprotich

Nairobi, 2017.

1 General Introduction

1.1 Background Information

The methods of constructing probability distributions among others are mixtures,transformations and special functions.
When two or more distributions are combined,then we have a mixture.Three types of mixtures are finite,discrete and continuous mixtures.
Consider a pdf or pmf given by $f(m;p)$ where p is a constant parameter.We have situations however where p is varying and therefore can be considered as a random variable. In this case we have a conditional distribution $f(m/p)$.
The mixing distribution is given by $h(p)$ and therefore the mixed distribution is $f(m)$.
If the mixing distribution $h(p)$ is continuous or discrete,then we have continuous mixture or discrete mixture respectively.
In this study we will focus on Laplace distribution and its generalization based on beta generated distribution,exponentiated generator, Laplace mixtures and EP mixtures.

1.2 Notations,Definitions and Terminologies

cdf cummulative distribution function
pdf probability density function
mgf moment generating function
iid independent and identically distributed
r.v random variable
EP Exponential Power
$f(m/p)$ Probability density function of a conditional distribution
$h(p)$ Probability density function of a mixing distribution
BL Beta Laplace
MLE maximum likelihood estimation
GND Generalized normal distribution
GED Generalized error distribution
GGD Generalized Gaussian distribution
LHS Left Hand Side
RHS Right Hand Side

Laplace density function:

$$f(m;\mu,p)=\frac{1}{2p}e^{-\frac{|m-\mu|}{p}} \qquad p>0,-\infty<\mu<\infty,\ -\infty<m<\infty.$$

Exponential Power density function:

$$f(m;\mu,r,p)=\frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma(\frac{r+1}{r})}e^{-\frac{1}{p}|m-\mu|^{r}} \quad p>0,\ -\infty<\mu<\infty,\ -\infty<m<\infty.$$

1.3 Literature Review

In this section, we will review what has been done on Laplace distribution and mixtures,Beta generated distribution,exponentiated generator and Exponential Power distribution and mixtures.

1.3.1 Laplace Distribution and Mixtures

The Laplace distribution was named after Pierre-Simon Laplace (1749-1827).It arises from the difference of two exponential random variables,consequently also known as Double exponential distribution.

Rachev and Sengupta (1993) obtained a Laplace-Weibull mixture distribution and applied to model price change data.They also obtained the parameter estimation of this mixture distribution.

Kotz,Kozubowski and Podgorski (2001) did extensive work on Laplace distribution obtaining its properties such as moments,moment generating function and characteristic function.They also deduced that the characteristic function of the product of standard exponential and normal distributions coincides with the standard classical Laplace characteristic function.They gave a wide range of applications in the engineering sciences and financial data among others.

Linden (2001) used a mixture of normal-exponential distribution to obtain Laplace distribution.

Nadarajah and Kotz (2006) looked at mixtures of Laplace distribution using 16 mixing distributions.However,the properties of the obtained distributions are not considered.

1.3.2 Beta Generated Distribution and exponentiated generator

The beta generated distribution was introduced by Eugene et.al (2002) through the cdf of a classical beta distribution.He obtained the beta generator equation or beta generated distribution which can be used to generate a new family of beta distributions.
The beta generated distribution can be obtained based on Binomial expansion,in terms of Gauss hypergeometric function and as an infinite mixture.
He obtained the beta-normal distribution,a composition of the classical beta distribution and the normal distribution and discussed some of its properties.
Famoye et.al (2005) introduced the beta-Weibull distribution,composed of the classical beta distribution and the Weibull distribution.Lee et.al (2007) gave the hazard function,entopies and an application to censored data.
Nadarajah and Kotz (2006) obtained the beta-exponential distribution based on the classical beta distribution and the exponential distribution.They determined the rth moment,properties of the hazard,distribution of the sum of beta-exponential random variables and maximum likelihood estimation.
Kong et.al (2007) introduced the beta-gamma distribution,a compostion of the classical beta distribution and the gamma distribution.
Akinsete et.al (2008) introduced the beta-Pareto distribution consisting of the classical beta distribution and the Pareto distribution.He defined and studied the properties of the 4-parameter beta-Pareto distribution.
Akinsete and Lowe (2009) introduced the beta-Rayleigh distribution composed of the classical beta distribution and the Rayleigh distribution.
Cordeiro and Lemonte (2011) introduced the beta-Laplace distribution composed of the classical beta distribution and the standard Laplace distribution.They obtained its properties including moments,cumulants,generating function,determined the order statistics,entropy,estimation and application.All results were based on an infinite mixture.
Srivastava et.al (2006) obtained the cdf and pdf of generalized Laplace distribution by the method of exponentiated generator,but only considered the case when $z \geq 0$.

1.3.3 Exponential Power Distribution and Mixtures

The Exponential Power (EP) distribution was first introduced by Subbotin(1923).
Box (1953) used the distribution in robust inference where the parameters of the distribution were estimated via moments.
West (1987) showed the normal scale mixture property of EP distribution and put its mixing distribution in connection with stable population.
Choy and Smith (1997) used the normal scale mixture representation to simulate posterior distributions within the context of random effects linear model.
Gomez (1998) obtained the multivariate EP distribution and Lindsey (1999) used this multivariate distribution to model repeated measurements.

Tibor and Nadarajah (2009) focused on the characteristic function of EP distribution,deriving an explicit closed form expression for the characteristic function.
Zang,Wang and Liu (2012) looked at mixtures of EP distribution using 3 mixing distributions namely;Gamma,Inverse Gamma and GIG all obtained in terms of Modified Bessel Function of the third kind.

1.4 Research Problem

- Amongst the reviews on Laplace distribution,Nadarajah and Kotz (2006) focused only on construction of Laplace mixture distribution.However,their moments are not obtained.

- Cordeiro and Lemonte (2011) obtained the beta-Laplace distribution and its properties but all results were based on an infinite mixture.There is need to obtain this distribution based on binomial expansion and in terms of Gauss hypergeometric function.

- Zang,Wang and Liu (2012) used only 3 mixing distributions to construct EP mixtures.There is need to extend this to include more mixing distributions.

1.5 Objectives

The main objective of this study is to construct and obtain the properties of Laplace distribution and its generalizations.
The specific objectives include:

1 To use various methods to construct Laplace distribution and study its properties.

2 To describe the concept of Beta generated distribution in general and hence construct Beta-Laplace distribution and express in different forms,namely;Binomial expansion form,Gauss hypergeometric form and as infinite mixtures.

3 To construct generalized Laplace distribution,obtain its properties and study special cases.

4 To construct Laplace mixtures using 16 mixing distributions and obtain their moments.

5 To construct Exponential Power Mixtures using 16 mixing distributions and hence show that Laplace and Gaussian distributions are special cases that arise from EP distribution.

1.6 Methodology

This study has used various mathematical tools highlighted below to achieve the objectives.

1.6.1 Transformations

We have constructed some distributions using the Change of Variable technique:
$h(\lambda) = f(t)|J|, \quad J = \frac{dt}{d\lambda}$

1.6.2 Special Functions

Gamma Function

$$\Gamma\alpha = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$

Properties

$$\Gamma(\alpha+1) = \alpha\Gamma\alpha$$

Beta Function

$$B(x,y) = \int_0^1 a^{x-1}(1-a)^{y-1} da$$

Properties

$$B(x,y) = B(x,y)$$

$$B(x,y) = \int_0^\infty \frac{a^{x-1}}{(1+a)^{x+y}}\, da$$

Relationship Between Beta and Gamma Function

$$B(x,y) = \frac{\Gamma x\, \Gamma y}{\Gamma(x+y)}$$

$$\frac{\Gamma\alpha}{\beta^\alpha} = \int_0^\infty t^{\alpha-1} e^{-\beta t}\, dt$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

The Modified Bessel Function of the third kind

$$K_y(\omega) = \frac{1}{2}\int_0^\infty t^{y-1} e^{-\frac{\omega}{2}(t+\frac{1}{t})}\, dt$$

Properties

$$K_{-y}(\omega) = K_y(\omega)$$

$$K_{\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}\, e^{-\omega}$$

$$\begin{aligned} K_{\frac{3}{2}}(\omega) &= K_{\frac{1}{2}}(\omega)\{1+\frac{1}{\omega}\} \\ &= \sqrt{\frac{\pi}{2\omega}}\, e^{-\omega}\{1+\frac{1}{\omega}\} \end{aligned}$$

Kummer‘s Confluent Hypergeometric Function

$${}_1F_1(b;d;y) = \int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}}{B(b,d-b)} e^{y\,t}\, dt$$

Tricomi Confluent Hypergeometric Function

$$\Psi(b;d;y) = \int_0^\infty \frac{t^{b-1}(1+t)^{d-b-1}}{\Gamma b} e^{-y\,t}\, dt$$

The following relation holds

$$\Psi(b;d;y) = y^{1-d}\Psi(b-d+1;2-d;y)$$

and

$$\Psi(b;d;y) = \frac{\Gamma(1-d)}{\Gamma(b-d+1)}\ {}_1F_1(b;d;y) + \frac{\Gamma(1-d)}{\Gamma b} y^{1-d}\ {}_1F_1(b-d+1;2-d;y)$$

Gauss hypergeometric Function

$${}_2F_1(b,c;d;y) = 1 + \frac{b\,c}{d}\frac{y}{1!} + \frac{b(b+1)c(c+1)}{d(d+1)}\frac{y^2}{2!} + \frac{b(b+1)(b+2)c(c+1)(c+2)}{d(d+1)(d+2)}\frac{y^3}{3!} + \cdots$$

Incomplete Gamma Function

$$\begin{aligned} \gamma(a,x) &= \int_0^x e^{-t}t^{a-1}\, dt \\ &= a^{-1}x^a e^{-x}\ {}_1F_1(1;a+1;x) \end{aligned}$$

1.7 Significance of the Study

The Laplace distribution is used to model symmetric data with long tails.It is also used to model errors,thereby providing a motivation for the use of LAD regression,in which parameter estimates are based on minimizing the sum of absolute values of the residuals rather than on least squares.

The Exponential Power distribution,being a generalization of normal and Laplace distributions,has found applications in many areas including modelling the lifetimes of a wide variety of electronic as well as certain mechanical components,signal processing,quantitative finance,operations research and information systems.

2 CONSTRUCTION OF LAPLACE DISTRIBUTION

2.1 Introduction

In this chapter various methods have been used in constructing Laplace distribution.
We have considered the difference of two iid exponential random variables and its equivalents.
The normal mixture of an exponential distribution is another method.
The third method is the product of a standard Rayleigh variable and a standard normal variable and its equivalents.
An interesting connection between the normal and Laplace distribution has been established by Nyquist,Rice and Riordan (1954).
They showed that if U_1, U_2, U_3 and U_4 are independent unit normal variables,then the determinant

$$D = \begin{vmatrix} U_1 & U_2 \\ U_3 & U_4 \end{vmatrix}$$
$$= U_1U_4 - U_2U_3$$

has Laplace distribution.
It may be noted that $U_1U_4 - U_2U_3$ and $U_1U_4 + U_2U_3$ have the same distribution.
The case when the expected values of the $U'S$ are not equal to zero leads to a more complicated distribution which was considered by Nicholson (1958).Some additional remarks on this result were made by Missiakouli‘s and Darton (1985) and Mantel (1987).
Mantel and Pasternack (1966) and Mantel (1969) gave a heuristic demonstration that $U_1U_4 + U_2U_3$ follows a Laplace distribution.
A simple proof of this result through the use of characteristic function has been provided by Mantel (1970).

2.2 Construction of Laplace Distribution based on the difference method

If M_1 and M_2 are two (iid) exponential random variables with parameter $\frac{1}{p}$ i.e with mean p.
Then the probability density function is

$$f(m) = \frac{1}{p} e^{-\frac{m}{p}} \qquad m > 0; p > 0 \tag{2.1}$$

$$\Rightarrow F(m) = 1 - e^{-\frac{m}{p}} \tag{2.2}$$

Let
$Z = M_2 - M_1, \qquad M_2 > M_1 > 0$
and
$G(z)$=cdf of Z

$$\begin{aligned}
G(z) &= \text{Prob}\{Z \leq z\} \\
&= \text{Prob}\{M_2 - M_1 \leq z\} \\
&= \text{Prob}\{M_2 \leq z + m_1\} \\
G(z) &= \text{Prob}\{0 < M_1 < \infty, \quad M_2 \leq z + m_1\} \\
&= \int_0^\infty \int_0^{z+m_1} f(m_2) f(m_1)\, dm_2\, dm_1 \\
&= \int_0^\infty \left[\int_0^{z+m_1} f(m_2)\, dm_2 \right] f(m_1)\, dm_1 \\
&= \int_0^\infty F(z+m_1) f(m_1)\, dm_1 \\
&= \int_0^\infty \left[1 - e^{-\frac{z+m_1}{\lambda}} \right] \frac{1}{p} e^{-\frac{m_1}{p}}\, dm_1 \\
&= \int_0^\infty \frac{1}{p} e^{-\frac{m_1}{p}}\, dm_1 - \int_0^\infty \frac{1}{p} e^{-\frac{z}{p} - \frac{2m_1}{p}}\, dm_1 \\
&= 1 - \frac{1}{p} e^{-\frac{z}{p}} \int_0^\infty e^{-\frac{2m_1}{p}}\, dm_1 \\
&= 1 - \frac{1}{p} e^{-\frac{z}{p}} \left[-\frac{p}{2} e^{-\frac{2m_1}{p}} \right]_0^\infty \\
&= 1 - \frac{e^{-\frac{z}{p}}}{2} \left[-e^{-\frac{2m_1}{p}} \right]_0^\infty \\
&= 1 - \frac{e^{-\frac{z}{p}}}{2} [0 + 1]
\end{aligned}$$

$$\therefore G(z) = 1 - \frac{1}{2} e^{-\frac{z}{p}}, \qquad z \geq 0 \tag{2.3}$$

Next consider the case when $M_1 > M_2 > 0$
Then
$G(z)$=Prob$\{Z \leq z\}$
=Prob$\{M_1 - M_2 \leq z\}$
=Prob$\{M_2 \geq m_1 - z\}$

=1-Prob$\{M_2 \leq m_1 - z\}$

$\therefore$

$G(z)$=1-Prob$\{0 < M_1 < \infty, \quad M_2 \leq m_1 - z\}$

$$
\begin{aligned}
&= 1 - \int_0^\infty \left[\int_0^{m_1 - z} f(m_2)\,dm_2\right] f(m_1)\,dm_1 \\
&= 1 - \int_0^\infty F(m_1 - z) f(m_1)\,dm_1 \\
&= 1 - \left\{\int_0^\infty \left[1 - e^{-\frac{m_1 - z}{p}}\right] \frac{1}{p} e^{-\frac{m_1}{p}}\,dm_1\right\} \\
&= 1 - \left\{1 - \int_0^\infty \frac{1}{p} e^{-\left(\frac{m_1 - z}{p}\right)} e^{-\frac{m_1}{p}}\,dm_1\right\} \\
&= \frac{1}{p} e^{\frac{z}{p}} \int_0^\infty e^{-\frac{2m_1}{p}}\,dm_1 \\
&= \frac{1}{p} e^{\frac{z}{p}} \left[-\frac{p}{2} e^{-\frac{2m_1}{p}}\right]_0^\infty \\
&= \frac{e^{\frac{z}{p}}}{2}[0 + 1]
\end{aligned}
$$

$\therefore$

$$G(z) = \frac{1}{2} e^{\frac{z}{p}}, \qquad z < 0 \tag{2.4}$$

Therefore

$$G(z) = \begin{cases} \frac{1}{2} e^{\frac{z}{p}} & z < 0 \\ 1 - \frac{1}{2} e^{-\frac{z}{p}} & z \geq 0 \end{cases} \tag{2.5}$$

and

$$g(z) = \begin{cases} \frac{1}{2p} e^{\frac{z}{p}} & z < 0 \\ \frac{1}{2p} e^{-\frac{z}{p}} & z \geq 0 \end{cases} \tag{2.6}$$

$$\therefore\ g(z) = \frac{1}{2p} e^{\frac{-|z|}{p}}, \qquad z \geq 0 \tag{2.7}$$

<u>Moment Generating Function</u>

An exponential random variable with parameter $\frac{1}{p}$ has mgf

$$
\begin{aligned}
M_W(t) &= E[e^{tW}] \\
M_W(t) &= \int_0^\infty e^{tw}\frac{1}{p}e^{\frac{-w}{p}}\,dw \\
&= \frac{1}{p}\int_0^\infty e^{(t-\frac{1}{p})w}\,dw \\
&= \frac{1}{p}\int_0^\infty e^{-(\frac{1}{p}-t)w}\,dw \\
&= \frac{1}{p}\frac{1}{(\frac{1}{p}-t)} \\
&= \frac{1}{1-pt} \\
\text{Let } X &= X_2 - X_1 \\
\therefore M_X(t) &= E[e^{(X_2-X_1)t}] \\
&= E(e^{X_2t})E(e^{-X_1t}) \\
&= M_W(t)M_W(-t) \\
&= \frac{1}{1-pt}\frac{1}{1+pt} \\
&= \frac{1}{1-(pt)^2} \\
\therefore M_X(t) &= \frac{1}{1-(pt)^2}, \qquad -\frac{1}{p} < t < \frac{1}{p}
\end{aligned} \tag{2.8}
$$

Expanding $M_X(t)$ in Taylor Series we have

$$
\begin{aligned}
M_X(t) &= \sum_{r=0}^{\infty}[(pt)^2]^r \\
&= \sum_{r=0}^{\infty}p^{2r}t^{2r} \\
&= \sum_{r=0}^{\infty}p^{2r}(2r)!\frac{t^{2r}}{(2r)!} \\
\therefore E(X^{2r}) &= (2r)!p^{2r}, \qquad r = 0,1,2,\cdots
\end{aligned}
$$

i.e

$$
E(X^k) = \begin{cases} 0 & \text{if k is odd .} \\ p^k k! & \text{if k is even .} \end{cases} \tag{2.9}
$$

Thus

$$E(X) = 0$$

and

$$E(X^2) = 2p^2$$

$\therefore$

$$Var(X) = 2p^2$$

Skewness

$$= \frac{\mu_3}{(\sqrt{\mu_2})^3}$$
$$= 0$$

Skewness measures symmetry.Positive and negative values implies that pdf is skewed to the right or left respectively.If pdf is symmetric about mean,the skewness is zero.So Laplace distribution has 0 skewness.

Kurtosis

$$= \frac{\mu_4}{{\mu_2}^2}$$
$$= \frac{4!p^4}{(2p^2)^2}$$
$$= \frac{4!}{4}$$
$$= 6$$

It measures the peakedness or heaviness of tails.If kurtosis>0,the distribution is leptokurtic,platykurtic if kurtosis < 0 and mesokurtic if kurtosis$=0$.Positive values implies heavy tails(i.e more data in tails) while negative values implies light tails(less data in tails).The heaviness or lightness in tails means data looks flatter or less flat compared to the normal distribution.

Remark 2.1: Equivalent Result based on uniform distribution

Let $W = -\frac{1}{\lambda} lnX$ where $\lambda > 0$ is a constant and $X \sim U(0,1)$.
We obtain the pdf of W by the change of variable technique as follows:

$$x = e^{-\lambda w} \Rightarrow \frac{dx}{dw} = -\lambda e^{-\lambda w}$$

Given
$f(x) = 1, \quad 0 \leq x \leq 1$
then the pdf of W is

$$g(w) = f(x)|J| = f(x)|\frac{dx}{dw}|$$
$$g(w) = 1.|-\lambda e^{-\lambda w}|$$
$$g(w) = \lambda e^{-\lambda w}, \qquad w > 0, \lambda > 0$$

which is an exponential pdf with parameter λ
Therefore
$Z = W_2 - W_1 = -\frac{1}{\lambda} ln X_2 - (-\frac{1}{\lambda} ln X_1)$
where W_1 and W_2 are iid exponential random variables with parameter λ which results in a Laplace distribution as given above.

Remark 2.2: Equivalent Result based on Pareto distribution

Let $V = ln\frac{X}{\theta}$ where X is Pareto I random variable with parameters λ and θ

$$f(x) = \frac{\lambda \theta^{\lambda}}{x^{\lambda+1}} \qquad x > \theta > 0 \tag{2.10}$$

By the change of variable technique

$$x = \theta e^{v} \Rightarrow \frac{dx}{dv} = \theta e^{v}$$

$$\therefore g(v) = f(x)|J| = f(x)|\frac{dx}{dv}|$$
$$= \frac{\lambda \theta^{\lambda}}{x^{\lambda+1}} \ \theta e^{v}$$
$$= \frac{\lambda \theta^{(\lambda+1)} \ e^{v}}{(\theta e^{v})^{\lambda+1}}$$
$$g(v) = \lambda e^{-\lambda v}, \qquad v > 0, \lambda > 0$$

an exponential pdf

$$\therefore \text{ If } Z = V_2 - V_1$$
$$= ln\frac{X_2}{\theta} - ln\frac{X_1}{\theta}$$
$$= ln\frac{X_2}{X_1}$$

will again results to Laplace distribution.

2.3 Laplace distribution based on the mixture method

Let the conditional pdf be given by

$$f(x/v) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{x^2}{2v}}, \qquad -\infty < x < \infty;\ v > 0 \tag{2.11}$$

a normal pdf $\sim N(0,v)$.
Further let the mixing pdf be exponential given by

$$g(v) = \theta e^{-\theta v}, \qquad v > 0;\ \theta > 0 \tag{2.12}$$

Therefore mixed pdf becomes

$$\begin{aligned}
f(x) &= \int_0^\infty f(x/v)g(v)\,dv \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi v}} e^{-\frac{x^2}{2v}} \theta e^{-\theta v}\,dv \\
&= \frac{\theta}{\sqrt{2\pi}} \int_0^\infty v^{-\frac{1}{2}} \exp\{-\frac{2\theta}{2}(v + \frac{x^2}{2\theta}\frac{1}{v})\}\,dv \\
\text{Let } v &= \sqrt{\frac{x^2}{2\theta}} z \Rightarrow dv = \sqrt{\frac{x^2}{2\theta}}\,dz \\
\therefore f(x) &= \frac{\theta}{\sqrt{2\pi}} \int_0^\infty \left(\sqrt{\frac{x^2}{2\theta}} z\right)^{\frac{1}{2}-1} \left[\exp\left\{-\frac{2\theta}{2}\sqrt{\frac{x^2}{2\theta}}[z + \frac{1}{z}]\right\}\right] \sqrt{\frac{x^2}{2\theta}} z\,dz \\
&= \frac{2\theta}{\sqrt{2\pi}} \left(\sqrt{\frac{x^2}{2\theta}}\right)^{\frac{1}{2}} \frac{1}{2} \int_0^\infty z^{\frac{1}{2}-1} \exp\left\{-\frac{\sqrt{2\theta x^2}}{2}[z + \frac{1}{z}]\right\} dz \\
&= \frac{2\theta}{\sqrt{2\pi}} \left(\sqrt{\frac{x^2}{2\theta}}\right)^{\frac{1}{2}} K_{\frac{1}{2}} \sqrt{2\theta x^2} \\
\text{But } K_{\frac{1}{2}} \sqrt{2\theta x^2} &= \sqrt{\frac{\pi}{2\sqrt{2\theta x^2}}} e^{-\sqrt{2\theta x^2}} \\
\therefore f(x) &= \frac{2\theta}{\sqrt{2\pi}} \left(\sqrt{\frac{x^2}{2\theta}}\right)^{\frac{1}{2}} \frac{\sqrt{\pi}}{\sqrt{2\sqrt{2\theta x^2}}} e^{-\sqrt{2\theta x^2}} \\
&= \frac{2\theta}{\sqrt{2}\sqrt{\pi}} \left(\sqrt{\frac{x^2}{2\theta}}\right)^{\frac{1}{2}} \frac{\sqrt{\pi}}{\sqrt{2}\left(\sqrt{2\theta x^2}\right)^{\frac{1}{2}}} e^{-\sqrt{2\theta x^2}}
\end{aligned}$$

$$
\begin{aligned}
f(x) &= \theta\left(\sqrt{\frac{x^2}{2\theta}}\right)^{\frac{1}{2}}\left(\frac{1}{\sqrt{2\theta x^2}}\right)^{\frac{1}{2}} e^{-\sqrt{2\theta x^2}} \\
&= \theta\left(\sqrt{\frac{1}{2\theta}}\right)^{\frac{1}{2}}\left(\frac{1}{\sqrt{2\theta}}\right)^{\frac{1}{2}} e^{-\sqrt{2\theta x^2}} \\
&= \theta\sqrt{\frac{1}{2\theta}} e^{-\sqrt{2\theta x^2}} \\
f(x) &= \frac{\theta}{\sqrt{2\theta}} e^{-\sqrt{2\theta x^2}}
\end{aligned}
$$

Let $\sqrt{2\theta} = \frac{1}{p} \Rightarrow \theta = \frac{1}{2p^2}$

$$\therefore f(x) = \frac{1}{2p} e^{\frac{-|m|}{p}}$$

as obtained in(2.7)

Moment generating function

Let $X \sim N(\mu, \sigma^2) = N(\mu, y)$

$$
\begin{aligned}
\therefore f(x/y) &= \frac{1}{\sqrt{2\pi y}} e^{-\frac{(x-\mu)^2}{2y}}, \qquad -\infty < x < \infty;\ y > 0 \\
\therefore M_{X/Y}(t) &= E[e^{tX/Y}] \\
&= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi y}} e^{-\frac{(x-\mu)^2}{2y}}\, dx \\
&= \frac{1}{\sqrt{2\pi y}} \int_{-\infty}^{\infty} \exp[tx - \frac{1}{2}(\frac{x^2 - 2\mu x + \mu^2}{y})]\, dx \\
&= \frac{1}{\sqrt{2\pi y}} \int_{-\infty}^{\infty} \exp\{tx - \frac{x^2}{2y} + \frac{\mu x}{y} - \frac{\mu^2}{2y}\}\, dx \\
&= \frac{1}{\sqrt{2\pi y}} \int_{-\infty}^{\infty} \exp\{\frac{2ytx - x^2 + 2\mu x}{2y} - \frac{\mu^2}{2y}\}\, dx \\
&= \frac{e^{-\frac{\mu^2}{2y}}}{\sqrt{2\pi y}} \int_{-\infty}^{\infty} \exp\{\frac{-x^2 + 2(\mu + yt)x}{2y}\}\, dx \\
&= \frac{e^{-\frac{\mu^2}{2y}}}{\sqrt{2\pi y}} \int_{-\infty}^{\infty} \exp\{-\frac{1}{2y}[x^2 - 2(\mu + yt)x]\}\, dx \\
&= \frac{e^{-\frac{\mu^2}{2y}}}{\sqrt{2\pi y}} \int_{-\infty}^{\infty} \exp\Big\{-\frac{1}{2y}\Big[[x - (\mu + yt)]^2 - (\mu + yt)^2\Big]\Big\}\, dx
\end{aligned}
$$

$$
\begin{aligned}
M_{X/Y}(t) &= e^{-\frac{\mu^2}{2y}+\frac{(\mu+yt)^2}{2y}} \int_{-\infty}^{\infty} \frac{\exp\{-\frac{1}{2y}[x-(\mu+yt)]^2\}}{\sqrt{2\pi y}} \\
&= \exp\{\frac{(\mu+yt)^2}{2y}\} \\
&= \exp\left\{\frac{(\mu+yt+\mu)(\mu+yt-\mu)}{2y}\right\} \\
&= \exp\left\{\frac{(2\mu+yt)yt}{2y}\right\} \\
&= \exp\left\{\frac{2\mu t}{2}+\frac{yt^2}{2}\right\} \\
\therefore M_{X/Y}(t) &= \exp\{\mu t+\frac{\sigma^2}{2}t^2\}
\end{aligned}
$$

which is the mgf for X $\sim N(\mu, \sigma^2)$.

$$
\begin{aligned}
\therefore M_X(t) &= E(e^{tX}) \\
&= EE[e^{tX/Y}] \\
&= \int_0^\infty E[e^{tx/y}]g(y)\,dy \\
&= \int_0^\infty e^{\mu t+\frac{y}{2}t^2} g(y)\,dy \\
\text{But } g(y) &= \beta e^{-\beta y}, \qquad y>0;\ \beta>0 \\
\text{Then } M_M(t) &= \int_0^\infty e^{\mu t+\frac{y}{2}t^2}\beta e^{-\beta y}\,dy \\
&= \beta e^{\mu t}\int_0^\infty e^{\frac{y}{2}t^2-\beta y}\,dy \\
&= \beta e^{\mu t}\int_0^\infty e^{-(\beta-\frac{t^2}{2})y}\,dy \\
&= \beta e^{\mu t}\frac{1}{(\beta-\frac{t^2}{2})} \\
&= \frac{2\beta e^{\mu t}}{2\beta-t^2} \\
\text{Let } 2\beta &= \frac{1}{p^2} \\
\therefore M_X(t) &= \frac{e^{\mu t}}{p^2(\frac{1}{p^2}-t^2)} \\
&= \frac{e^{\mu t}}{1-(pt)^2} \\
\text{when } \mu &= 0 \\
\text{then } M_X(t) &= \frac{1}{1-(pt)^2}
\end{aligned}
$$

as obtained in (2.8).

2.4 Product of Rayleigh distribution and Normal Distribution

Rayleigh distribution

Rayleigh distribution is a special case of a Weibull distribution.
Let us therefore construct a Weibull distribution by considering

$$Y = \left(\frac{W}{\theta}\right)^c$$

where Y has exponential distribution with mean 1 .

$$\begin{aligned}\therefore g(w) &= e^{-y}\left|\frac{dy}{dw}\right| \\ &= e^{-(\frac{w}{\theta})^c}\frac{c}{\theta}\left(\frac{w}{\theta}\right)^{c-1} \\ &= \frac{c}{\theta}\left(\frac{w}{\theta}\right)^{c-1}e^{-(\frac{w}{\theta})^c},\ w>0;\theta>0,c>0 \end{aligned} \quad (2.13)$$

This is the Weibull pdf with parameters θ and c .
Put $c = 2$ and $\theta = 2$.
Then

$$g(w) = we^{-\frac{w^2}{2}},\ w>0 \quad (2.14)$$

which is the standard Rayleigh distribution we are going to consider.
Let us now consider the product
$U = WZ$
where W and Z are independent random variables.
W is a standard Rayleigh distributed and Z is a standard normal distributed.
We wish to find the pdf of U.
By change of variable technique,let
$U = WZ$ and $V = W$
$\therefore\ W = V$ and $Z = \frac{U}{V}$
The new joint pdf is

$$\begin{aligned} h(u,v) &= f(w,z)|W| \\ &= g(w)\phi(z)\begin{vmatrix}\frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\end{vmatrix}\end{aligned}$$

$$= g(w)\phi(z)\begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix}$$

$$= g(w)\phi(z)\left| -\frac{1}{v} \right|$$

$$= we^{-\frac{w^2}{2}} \ \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}} \ \frac{1}{v}$$

$$= ve^{-\frac{v^2}{2}} \ \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{u}{v})^2} \ \frac{1}{v}$$

$$= \frac{1}{\sqrt{2\pi}} \ e^{-\frac{v^2}{2}-\frac{u^2}{v^2}}, \ v > 0; |u| > 0$$

$$\therefore h(u) = \int_0^\infty h(u,v)dv$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}} \ e^{-\frac{v^2}{2}-\frac{u^2}{2v^2}} \ dv$$

Let $v^2 = s \Rightarrow 2vdv = ds \therefore dv = \dfrac{ds}{2v} = \dfrac{ds}{2\sqrt{s}}$

$$\therefore h(u) = \int_0^\infty \frac{1}{\sqrt{2\pi}} \ e^{-\frac{s}{2}-\frac{u^2}{2s}} \ \frac{ds}{2\sqrt{s}}$$

$$= \frac{1}{\sqrt{2\pi}} \ \frac{1}{2} \int_0^\infty \frac{1}{\sqrt{s}} e^{-\frac{1}{2}\left(s+\frac{u^2}{s}\right)} \ ds$$

$$= \frac{1}{\sqrt{2\pi}} \ \frac{1}{2} \int_0^\infty s^{-\frac{1}{2}} e^{-\frac{1}{2}\left(s+\frac{u^2}{s}\right)} \ ds$$

$$= \frac{1}{\sqrt{2\pi}} \ \frac{1}{2} \int_0^\infty s^{\frac{1}{2}-1} e^{-\frac{1}{2}\left(s+\frac{u^2}{s}\right)} \ ds$$

Let $s = |u|t \Rightarrow ds = |u|dt$

$$\therefore h(u) = \frac{1}{\sqrt{2\pi}} \ \frac{1}{2} \int_0^\infty (|u|t)^{\frac{1}{2}-1} e^{-\frac{1}{2}\left(|u|t+\frac{|u|^2}{|u|t}\right)} \ |u|dt$$

$$= \frac{1}{\sqrt{2\pi}} \ (|u|)^{\frac{1}{2}} \ \frac{1}{2} \int_0^\infty t^{\frac{1}{2}-1} e^{-\frac{1}{2}|u|\left(t+\frac{1}{t}\right)} \ dt$$

$$= \frac{|u|^{\frac{1}{2}}}{\sqrt{2\pi}} \ K_{\frac{1}{2}}(|u|)$$

$$\text{But } K_{\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}e^{-\omega}$$

$$\therefore h(u) = \sqrt{\frac{|u|}{2\pi}}\sqrt{\frac{\pi}{2|u|}}e^{-|u|}$$

$$\therefore h(u) = \frac{1}{2}e^{-|u|}$$

By cummulative distribution technique

$$\begin{aligned}
H(u) &= \text{Prob}\{U \leq u\} \\
&= \text{Prob}\{WZ \leq u\} \\
&= \text{Prob}\{Z \leq \frac{u}{w}\} \\
&= \text{Prob}\{Z \leq \frac{u}{w}, 0 \leq W < \infty\} \\
&= \int_0^\infty \int_{-\infty}^{\frac{u}{w}} \phi(z)g(w)dzdw \\
&= \int_0^\infty \Big[\int_{-\infty}^{\frac{u}{w}} \phi(z)dz\Big]g(w)dw
\end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty \Phi(\frac{u}{w})g(w)dw \\
\therefore h(u) &= \frac{d}{du}H(u) \\
&= \int_0^\infty \frac{1}{w}\phi(\frac{u}{w})g(w)\,dw \\
&= \int_0^\infty \frac{1}{w}\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{u}{w}\right)^2}\,we^{-\frac{w^2}{2}}\,dw \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}\left(\frac{u^2}{w^2+w^2}\right)}\,dw \\
&= \frac{1}{2}e^{-|u|}, \quad |u| > 0
\end{aligned}$$

as shown in the change of variable technique. Let $u = \frac{y}{p} \Rightarrow \frac{du}{dy} = \frac{1}{p}$
Therefore

$$g(y) = \frac{1}{2p}e^{-\frac{|y|}{p}}, \quad |y| > 0; p > 0$$

as obtained in (2.7).

Remark 2.3

A standard Rayleigh distribution is a special case of a generalized Rayleigh distribution derived as follows:
Let
$X = \sum_{i=1}^{N} Y_i^2$
where $Y_i \sim N(0, \sigma^2)$
and Y_i's are iid random variables.
(1) Therefore the pdf of Y_i is

$$g(y_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y_i^2}{2\sigma^2}}, \quad -\infty < y < \infty; \sigma^2 > 0$$

(2) Let
$R = Y^2$

$$\begin{aligned}
\therefore G(r) &= \text{Prob}\{R \leq r\} \\
&= \text{Prob}\{Y^2 \leq r\} \\
&= \text{Prob}\{-\sqrt{r} \leq Y \leq +\sqrt{r}\} \\
&= F(\sqrt{r}) - F(-\sqrt{r}) \\
\therefore g(r) &= \frac{d}{dr}\sqrt{r}F'(\sqrt{r}) - \frac{d}{dr}(-\sqrt{r})F'(-\sqrt{r}) \\
&= \frac{1}{2}r^{-\frac{1}{2}} f(\sqrt{r}) + \frac{1}{2}r^{-\frac{1}{2}} f(-\sqrt{r}) \\
&\text{If } f(-\sqrt{r}) = f(\sqrt{r}) \text{ then} \\
g(r) &= r^{-\frac{1}{2}} f(\sqrt{r}) \\
&= r^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{r}{2\sigma^2}} \\
&= \frac{1}{\sqrt{\pi}\sqrt{2\sigma^2}} e^{-\frac{r}{2\sigma^2}} r^{\frac{1}{2}-1} \\
&= \frac{1}{\Gamma\left(\frac{1}{2}\right)\left(2\sigma^2\right)^{\frac{1}{2}}} e^{-\frac{r}{2\sigma^2}} r^{\frac{1}{2}-1}
\end{aligned}$$

$$g(r) = \frac{\left(\frac{1}{2\sigma^2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} e^{-\frac{r}{2\sigma^2}} r^{\frac{1}{2}-1} \qquad r > 0, \sigma^2 > 0 \tag{2.15}$$

A gamma distribution with $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2\sigma^2}$

The mgf of R is

$$M_R(t) = \left(\frac{\beta}{\beta - t}\right)^{\alpha}$$
$$= \left(\frac{1}{1-\frac{t}{\beta}}\right)^{\alpha}$$
$$M_R(t) = \left(\frac{1}{1-2\sigma^2 t}\right)^{\frac{1}{2}} \quad \textbf{(2.16)}$$

(3) Let
$S_N = Z_1 + Z_2 + Z_3 + \cdots + Z_N$
where $Z_i's$ are iid random variables;
$Z_i \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right)$
Therefore the mgf of S_N is

$$\begin{aligned} M_{S_N}(t) &= E[e^{tS_N}] \\ &= E[e^{t(Z_1+Z_2+Z_3+\cdots+Z_N)}] \\ &= E(e^{tZ_1})E(e^{tZ_2})\cdots E(e^{tZ_N}) \\ &= \left(\frac{1}{1-2\sigma^2 t}\right)^{\frac{N}{2}} \end{aligned} \quad \textbf{(2.17)}$$

which is the mgf of Gamma $\left(\frac{N}{2}, \frac{1}{2\sigma^2}\right)$.

$$\therefore \text{Prob}\{S_N = x\} = \frac{1}{\Gamma\left(\frac{N}{2}\right)\left(2\sigma^2\right)^{\frac{N}{2}}} e^{-\frac{x}{2\sigma^2}} x^{\frac{N}{2}-1}, \qquad x > 0; \sigma^2 > 0, N = 1, 2, 3, \cdots \quad \textbf{(2.18)}$$

(4) Let
$P = \sqrt{S_N}$
$\Rightarrow P^2 = S_N \Rightarrow x = p^2$
$\therefore \frac{dS_N}{dP} = 2P$
Therefore the pdf of P is

$$
\begin{aligned}
h(p) &= f(x)|J| \\
&= f(x)\left|\frac{dx}{dp}\right| \\
&= f(x)2p \\
&= 2p\frac{e^{-\frac{x}{2\sigma^2}}x^{\frac{N}{2}-1}}{\Gamma\left(\frac{N}{2}\right)\left(2\sigma^2\right)^{\frac{N}{2}}} \\
&= \frac{2p}{\left(2\sigma^2\right)^{\frac{N}{2}}\Gamma\left(\frac{N}{2}\right)}e^{-\frac{p^2}{2\sigma^2}}(p^2)^{\frac{N}{2}-1} \\
h(p) &= \frac{2}{\left(2\sigma^2\right)^{\frac{N}{2}}\Gamma\left(\frac{N}{2}\right)}e^{-\frac{p^2}{2\sigma^2}}p^{N-1} \quad p>0;\sigma>0,N=1,2,3,\cdots
\end{aligned} \tag{2.19}
$$

When N=2 and $\sigma^2 = 1$ then we have
$g(p) = pe^{-\frac{p^2}{2}} \quad p > 0$
which is the standard Rayleigh distribution.
Therefore h(p) is a generalized Rayleigh distribution.
It should be noted that
When N=1,we have

$$
\begin{aligned}
h(p) &= \frac{2}{\sqrt{2\sigma^2}\,\Gamma\left(\frac{1}{2}\right)}e^{-\frac{p^2}{2\sigma^2}} \\
h(p) &= \frac{2}{\sqrt{2\pi\sigma^2}}e^{-\frac{p^2}{2\sigma^2}},\ p>0;\sigma^2>0
\end{aligned} \tag{2.20}
$$

which is the density function of a half-normal variable.
When N=3,

$$
\begin{aligned}
h(p) &= \frac{2e^{-\frac{p^2}{2\sigma^2}}p^2}{\left(2\sigma^2\right)^{\frac{3}{2}}\Gamma\left(\frac{3}{2}\right)} \\
&= \frac{2p^2e^{-\frac{p^2}{2\sigma^2}}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\left(2\sigma^2\right)^{\frac{3}{2}}} \\
&= \frac{2^2p^2e^{-\frac{p^2}{2\sigma^2}}}{2^{\frac{3}{2}}\sqrt{\pi}\sigma^3} \\
h(p) &= \frac{\sqrt{2}}{\sqrt{\pi}}\frac{p^2}{\sigma^3}e^{-\frac{p^2}{2\sigma^2}}, \quad p > 0; \sigma > 0 \qquad (2.21)
\end{aligned}
$$

which is Maxwell-Boltzmann pdf.

Remark 2.4

A standard Rayleigh distribution can also be obtained from a standard exponential distribution as follows:
Let $Y = \sqrt{2W} \Rightarrow Y^2 = 2W \Rightarrow W = \frac{Y^2}{2} \Rightarrow \frac{dw}{dy} = y$
W is standard exponential r.v with pdf $g(w) = e^{-w}$.
Thus the pdf of Y is

$$
\begin{aligned}
f(y) &= gw\left|\frac{dw}{dy}\right| \\
&= e^{-w}y \\
&= ye^{-\frac{y^2}{2}} \quad y > 0
\end{aligned}
$$

which is a standard Rayleigh distribution.

2.5 Sums or Differences of normal Products

Method 1:Direct Approach

Let M_1, M_2, M_3 and M_4 be iid normal random variables each with mean 0 and variance 1.
The aim is to obtain the pdf of $Y = M_1M_2 + M_3M_4$
Further let

$$
\begin{aligned}
U_1 &= \frac{M_1+M_2}{2} \Rightarrow 2U_1 = M_1+M_2 \\
U_2 &= \frac{M_1-M_2}{2} \Rightarrow 2U_2 = M_1-M_2 \\
U_3 &= \frac{M_3+M_4}{2} \Rightarrow 2U_3 = M_3+M_4 \\
U_4 &= \frac{M_3-M_4}{2} \Rightarrow 2U_4 = M_3-M_4
\end{aligned} \tag{2.22}
$$

Next consider U_1 and U_2.
$U_1+U_2 = M_1$. This implies that $U_1 \sim N(0,\frac{1}{2})$ and $U_2 \sim N(0,\frac{1}{2})$ so that $M_1 \sim N(0,1)$.
Again
$U_1^2 - U_2^2 = \left(\frac{M_1+M_2}{2}\right)^2 - \left(\frac{M_1-M_2}{2}\right)^2 = M_1M_2$
It follows that
$U_3^2 - U_4^2 = M_3M_4$
Therefore

$$
\begin{aligned}
Y &= M_1M_2 + M_3M_4 \\
&= U_1^2 - U_2^2 + U_3^2 - U_4^2 \\
&= (U_1^2+U_3^2) - (U_2^2+U_4^2)
\end{aligned}
$$

where $(U_1^2+U_3^2) \sim \chi^2(2)$ and $(U_2^2+U_4^2) \sim \chi^2(2)$.
Hence
$Y = \chi_2^2 - \chi_1^2$. We can represent it as $Y = M_2 - M_1$.
The pdf of a chi-square random variable with 2 df is
$f(m) = \frac{1}{2}e^{-\frac{m}{2}} \quad x > 0$
the cdf $F(m) = 1 - e^{-\frac{m}{2}}$

By method of cummulative distribution,let

$$
\begin{aligned}
Y &= M_2 - M_1,\ M_2 > M_1 > 0 \\
F(y) &= \text{Prob}\{Y \leq y\} \\
&= \text{Prob}\{M_2 - M_1 \leq y\} \\
&= \text{Prob}\{M_2 \leq y + m_1\} \\
&= \text{Prob}\{M_2 \leq y + m_1,\ 0 < M_1 < \infty\} \\
&= \int_0^\infty \int_0^{y+m_1} f(m_2) f(m_1)\, dm_2\, dm_1 \\
&= \int_0^\infty F(y+m_1) f(m_1)\, dm_1 \\
&= \int_0^\infty \left[1 - e^{-\frac{(y+m_1)}{2}}\right] \frac{1}{2} e^{-\frac{m_1}{2}}\, dm_1 \\
&= \int_0^\infty \frac{1}{2} e^{-\frac{m_1}{2}}\, dm_1 - \int_0^\infty \frac{1}{2} e^{-\frac{y}{2} - \frac{m_1}{2}}\, dm_1 \\
F(y) &= 1 - \frac{1}{2} e^{-\frac{y}{2}}\ y \geq 0 \\
\text{hence } f(y) &= \frac{1}{4} e^{-\frac{y}{2}}\ y \geq 0
\end{aligned}
$$

Next,when $M_1 > M_2 > 0$ then

$$
\begin{aligned}
F(y) &= \text{Prob}\{Y \leq y\} \\
&= \text{Prob}\{M_1 - M_2 \leq y\} \\
&= \text{Prob}\{X_2 \geq m_1 - y\} \\
&= 1 - \text{Prob}\{M_2 \leq m_1 - y\} \\
&= 1 - \text{Prob}\{0 < M_1 < \infty, \quad M_2 \leq m_1 - y\} \\
&= 1 - \int_0^\infty \left[\int_0^{m_1 - y} f(m_2) dm_2\right] f(m_1) dm_1 \\
&= 1 - \int_0^\infty F(m_1 - y) f(m_1) dm_1 \\
&= 1 - \int_0^\infty \left[1 - e^{-\frac{(m_1 - y)}{2}}\right] \frac{1}{2} e^{-\frac{m_1}{2}}\, dm_1 \\
&= \frac{1}{2} e^{\frac{y}{2}}, \quad y < 0 \\
\therefore f(y) &= \frac{1}{4} e^{\frac{y}{2}} \quad y < 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f(y) &= \frac{1}{4}e^{-\frac{|y|}{2}} \quad y \geq 0 \\
f(y) &= \frac{1}{2}e^{-|y|} \quad y \geq 0
\end{aligned}
\tag{2.23}
$$

which is a Laplace distribution.

Method 2:Using mgf

In $Y = M_1M_2 + M_3M_4$, we note that M_1m_2 and M_3M_4 are iid.Therefore with a foreknowledge of standard normal mgf as $e^{\frac{t^2}{2}}$ we have

$$
\begin{aligned}
M_{M_1M_2}(t) &= E[e^{tM_1M_2}] \\
&= EE[\exp(tM_2M_1/M_2)] \\
&= E[e^{\frac{(tM_2)^2}{2}}] \\
&= E[e^{\frac{t^2}{2}M_2^2}] \\
&= \int_{-\infty}^{\infty} e^{\frac{t^2}{2}m_2^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{m_2^2}{2}}\, dm_2 \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(1-t^2)m_2^2}\, dm_2 \\
&= \int_{-\infty}^{\infty} \frac{\sqrt{1-t^2}}{\sqrt{1-t^2}}\, \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(1-t^2)m_2}\, dm_2 \\
&= \frac{1}{\sqrt{1-t^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\, (\sqrt{1-t^2})^{-1}} e^{-\frac{m_2^2}{2(1-t^2)^{-1}}}\, dm_2 \\
&= \frac{1}{\sqrt{1-t^2}}
\end{aligned}
$$

$$
\begin{aligned}
\therefore M_Y(t) &= E[e^{t(M_1M_2+M_3M_4)}] \\
&= E(e^{tM_1M_2})E(e^{tM_3M_4}) \\
&= \left(\frac{1}{\sqrt{1-t^2}}\right)^2 \\
M_Y(t) &= (1-t^2)^{-1}
\end{aligned}
\tag{2.24}
$$

This is the mgf of a standard Laplace distribution.

3 BETA GENERATED DISTRIBUTION

3.1 Introduction

The concept of a beta generated distribution has been defined according to Eugene et.al (2002) and Jones (2004).
Special cases considered are Type I and Type II exponentiated distributions and the ith order statistic distribution along with its minimum and maximum cases.
Then the beta generated distribution and its special cases are expressed based on binomial expansion,Gauss hypergeometric function and as infinite mixtures of exponentiated distributions.
Shannon entropy has been derived according to Zografos and Balakrishnan (2009) in terms of digamma (psi) function.

3.2 The Concept of Beta Generated Distribution

The pdf of a classical beta distribution is given by

$$w(m) = \frac{m^{x-1}(1-m)^{y-1}}{B(x,y)} \qquad 0 < m < 1 \tag{3.1}$$

$$W(v) = \int_0^v \frac{m^{x-1}(1-m)^{y-1}}{B(x,y)}\,dm \qquad 0 \leq v \leq 1 \tag{3.2}$$

is the corresponding cdf.
We replace v by $G(v)$ the cdf of any random variable V taking values between $-\infty$ and ∞ since $0 \leq G(v) \leq 1$.

$$\therefore W[G(v)] = \int_0^{G(v)} \frac{m^{x-1}(1-m)^{y-1}}{B(x,y)}\,dm \tag{3.3}$$

which is a function of v .

Let

$$F(v) = W[G(v)] \tag{3.4}$$

Then

$$\begin{aligned} f(v) &= \frac{d}{dv}F(v) \\ &= \{W'[G(v)]\}G'(v) \\ &= \{w[G(v)]\}g(v) \end{aligned}$$

$$\therefore f(v) = \frac{[G(v)]^{x-1}[1-G(v)]^{y-1}}{B(x,y)}g(v) \tag{3.5}$$

where $g(v) = \frac{d}{dv}G(v)$

Also from (3.3)

$$F(v) = \int_0^{G(v)} \frac{m^{x-1}(1-m)^{y-1}}{B(x,y)}\, dm \tag{3.6}$$

Then by Leibintz principle of differentiation we have

$$\begin{aligned} f(v) &= \frac{d}{dv}\int_0^{G(v)} \frac{m^{x-1}(1-m)^{y-1}}{B(x,y)}\, dm \\ &= d\int_0^{G(v)} \frac{m^{x-1}(1-m)^{y-1}}{B(x,y)} \\ &= \frac{m^{x-1}(1-m)^{y-1}}{B(x,y)}\Bigg|_0^{G(v)} \frac{d}{dv}G(v) \\ &= \frac{[G(v)]^{x-1}[1-G(v)]^{y-1}}{B(x,y)}g(v) \end{aligned}$$

as in 3.5

We shall refer to $G(v)$ and $g(v)$ as old or parent cdf and pdf respectively;while $F(v)$ and $f(v)$ as new cdf and pdf respectively,forming a beta generated distribution.

3.3 Special Cases of Beta Generated Distribution

3.3.1 Type I Exponentiated Distribution

Let $y=1$ in (3.6)

$$\begin{aligned}
\therefore F(v) &= \int_0^{G(v)} \frac{m^{x-1}}{B(x,1)}\,dm \\
&= x\int_0^{G(v)} m^{x-1}\,dm \\
&= x\,\frac{m^x}{x}\Big|_0^{G(v)} \\
F(v) &= [G(v)]^x, \quad x>0 \qquad (3.7)
\end{aligned}$$

which we shall refer to as Type I exponentiated distribution obtained by expressing the new cdf $F(v)$ as a power of the parent cdf $G(v)$
The corresponding pdf is

$$f(v) = x[G(v)]^{x-1}g(v) \qquad (3.8)$$

3.3.2 Type II Exponentiated Distribution

This is the case when $x=1$ in (3.6)
Then

$$\begin{aligned}
F(v) &= \int_0^{G(v)} \frac{(1-m)^{y-1}}{B(1,y)}\,dm \\
&= y\int_0^{G(v)} (1-m)^{y-1}\,dm
\end{aligned}$$

Let $z=1-m \Rightarrow dz=-dm$

$$\begin{aligned}
\therefore F(v) &= y\int_1^{1-G(v)} z^{y-1}(-dz) \\
&= y\int_{1-G(v)}^{1} z^{y-1}dz \\
F(v) &= 1-[1-G(v)]^y \qquad (3.9)
\end{aligned}$$

and

$$f(v) = y[1-G(v)]^{y-1}g(v) \tag{3.10}$$

3.3.3 The ith order statistic distribution

Let $x = i$ and $y = s-i+1$ in (3.6) where i and s are positive integers and $1 \leq i \leq s$. Then we have the ith order statistic distribution given by

$$F_{i:s}(v) = \int_0^{G(v)} \frac{m^{i-1}(1-m)^{s-i}}{B(i,s-i+1)}\, dm \tag{3.11}$$

Hence

$$f_{i:s}(v) = \frac{[G(v)]^{i-1}[1-G(v)]^{s-i}}{B(i,s-i+1)} g(v) \tag{3.12}$$

When $i = 1$ then we have minimum distribution given by

$$\begin{aligned} F_{1:s}(v) &= \int_0^{G(v)} \frac{(1-m)^{s-1}}{B(1,s)}\, dm \\ &= s\int_0^{G(v)} (1-m)^{s-1}\, dm \\ &= s\int_{1-G(v)}^{1} z^{s-1} dz \\ F_{1:s}(v) &= 1-[1-G(v)]^s && (3.13) \\ \text{Hence } f_{1:s}(v) &= s[1-G(v)]^{s-1}g(v) && (3.14) \end{aligned}$$

when $i = s$ we have maximum distribution:

$$\begin{aligned} F_{s:s}(v) &= \int_0^{G(v)} \frac{m^{s-1}}{B(s,1)}\, dm \\ &= s\int_0^{G(v)} m^{s-1}\, dm \\ F_{s:s}(v) &= [G(v)]^s && (3.15) \\ \Rightarrow f_{s:s}(v) &= s[G(v)]^{s-1}g(v) && (3.16) \end{aligned}$$

3.4 Beta Generated Distribution and Its Special Cases Based on Binomial Expansion

3.4.1 Two versions

Applying binomial expansion to formula (3.6) we have:

Version 1

$$F(v) = \frac{1}{B(x,y)} \int_0^{G(v)} m^{x-1} \sum_{q=0}^{\infty} (-1)^q \binom{y-1}{q} m^q \, dm$$

$$= \frac{1}{B(x,y)} \int_0^{G(v)} \sum_{q=0}^{\infty} (-1)^q \binom{y-1}{q} m^{x+q-1} \, dm$$

$$= \frac{1}{B(x,y)} \sum_{q=0}^{\infty} (-1)^q \binom{y-1}{q} \int_0^{G(v)} m^{x+q-1} \, dm$$

$$\therefore F(v) = \frac{1}{B(x,y)} \sum_{q=0}^{\infty} (-1)^q \binom{y-1}{q} \frac{[G(v)]^{x+q}}{x+q} \tag{3.17}$$

and

$$f(v) = \frac{1}{B(x,y)} \sum_{q=0}^{\infty} (-1)^q \binom{y-1}{q} \, [G(v)]^{x+q-1} \, g(v) \tag{3.18}$$

Version 2

By letting $z = 1 - t \Rightarrow dz = -dt$ formula (3.6) changes to

$$F(v) = \frac{1}{B(x,y)} \int_1^{1-G(v)} (1-z)^{x-1} z^{y-1} (-dz)$$

$$= \frac{1}{B(x,y)} \int_{1-G(v)}^{1} (1-z)^{x-1} z^{y-1} dz$$

$$= \frac{1}{B(x,y)} \int_{1-G(v)}^{1} z^{y-1} \sum_{q=0}^{\infty} (-1)^q \binom{x-1}{q} z^q \, dz$$

$$= \frac{1}{B(x,y)} \sum_{q=0}^{\infty} (-1)^q \binom{x-1}{q} \int_{1-G(v)}^{1} z^{y+q-1} \, dz$$

$$\therefore F(v) = \frac{1}{B(x,y)} \sum_{q=0}^{\infty} (-1)^q \binom{x-1}{q} \frac{1}{y+q} \left\{ 1 - (1-G(v))^{y+q} \right\} \tag{3.19}$$

and

$$f(v) = \frac{1}{B(x,y)} \sum_{q=0}^{\infty} (-1)^q \binom{x-1}{q} \, (1-G(v))^{y+q-1} \, g(v) \tag{3.20}$$

3.5 Special Cases of Beta Generated Distribution Based on Binomial Expansion

3.5.1 Type I Exponentiated Distribution

Put $y = 1$ in (3.17) and (3.18)

Then

$$F(v) = \frac{1}{B(x,1)} \frac{(G(v))^x}{x}$$
$$= [G(v)]^x \tag{3.21}$$

and

$$f(v) = \frac{1}{B(x,1)} (G(v))^{x-1} g(v)$$
$$= x[G(v)]^{x-1} g(v) \tag{3.22}$$

3.5.2 Type II Exponentiated Distribution

Put $x = 1$ in (3.19) and (3.20) to get

$$F(v) = \frac{1}{B(1,y)} \frac{\left\{1-(1-G(v))^y\right\}}{y}$$
$$= 1-[1-G(v)]^y \tag{3.23}$$

and

$$f(v) = \frac{1}{B(1,y)} [1-G(v)]^{y-1} g(v)$$
$$= y[1-G(v)]^{y-1} g(v) \tag{3.24}$$

3.5.3 The ith order statistic distribution

Put $x = i$ and $y = s - i + 1$

Using version 1

$$F_{i:s}(v) = \frac{1}{B(i, s-i+1)} \sum_{q=0}^{s-i} (-1)^q \binom{s-i}{q} \frac{(G(v))^{i+q}}{i+q} \tag{3.25}$$

and

$$f_{i:s}(v) = \frac{1}{B(i, s-i+1)} \sum_{q=0}^{s-i} (-1)^q \binom{s-i}{q} (G(v))^{i+q-1} g(v) \tag{3.26}$$

Version 2

$$F_{i:s}(v) = \frac{1}{B(i, s-i+1)} \sum_{q=0}^{i-1} (-1)^q \binom{i-1}{q} \frac{1}{s-i+q+1} \left\{1 - (1-G(v))^{s-i+q+1}\right\} \tag{3.27}$$

and

$$f_{i:s}(v) = \frac{1}{B(i, s-i+1)} \sum_{q=0}^{i-1} (-1)^q \binom{i-1}{q} (1-G(v))^{s-i+q} g(v) \tag{3.28}$$

Let us now determine $F(v)$ when
(i) When $x = s,\ s = 1, 2, 3, \cdots$ and $y > 0$
(ii) When $y = s,\ s = 1, 2, 3, \cdots$ and $x > 0$

To do so we need:

Lemma 3.1

$$\sum_{p=q}^{s-1}(-1)^{p-q}\binom{c+p-1}{p-q}\binom{c+s-1}{s-p-1}=1, \quad c>0 \tag{3.29}$$

<u>*Proof*:</u>

$$\begin{aligned}
LHS &= \sum_{p=q}^{s-1}(-1)^{p-q}\frac{\Gamma(c+p)}{(p-q)!\Gamma(c+q)}\,\frac{\Gamma(c+s)}{(s-p-1)!\Gamma(c+p+1)}\\
&= \sum_{p=q}^{s-1}\frac{(-1)^{p-q}\Gamma(c+s)}{(p-q)!\Gamma(c+q)(s-p-1)!(c+p)}\\
&= \frac{\Gamma(c+s)}{\Gamma(c+q)}\sum_{p=q}^{s-1}\frac{(-1)^{p-q}}{(p-q)!(s-p-1)!}\frac{1}{c+p}\\
&= \frac{\Gamma(c+s)}{\Gamma(c+q)\Gamma(s-q)}\sum_{p=q}^{s-1}\frac{(-1)^{p-q}}{(p-q)!}\frac{\Gamma(s-q)}{(s-p-1)!}\frac{1}{c+p}\\
&= \frac{1}{B(c+q,s-q)}\sum_{p=q}^{s-1}(-1)^{p-q}\frac{\Gamma(s-q)}{(p-q)!\Gamma(s-p)}\frac{1}{c+p}\\
&= \frac{1}{B(c+q,s-q)}\sum_{p=q}^{s-1}(-1)^{p-q}\binom{s-q-1}{p-q}\frac{1}{c+p}\\
&= \frac{1}{B(c+q,s-q)}\sum_{p=q}^{s-1}(-1)^{p-q}\binom{s-q-1}{p-q}\int_0^1 m^{c+p-1}\,dm\\
&= \frac{1}{B(c+q,s-q)}\sum_{p=q}^{s-1}(-1)^{p-q}\binom{s-q-1}{p-q}\int_0^1 m^{c-1}m^{p}\,dm\\
&= \frac{1}{B(c+q,s-q)}\int_0^1 m^{c-1}\sum_{p=q}^{s-1}(-1)^{p-q}\binom{s-q-1}{p-q}m^{p}\,dm\\
&= \frac{1}{B(c+q,s-q)}\int_0^1 m^{c+q-1}\sum_{p=q}^{s-1}(-1)^{p-q}\binom{s-q-1}{p-q}m^{p-q}\,dm
\end{aligned}$$

Let $u=p-q \Rightarrow p=u+q$

$$\begin{aligned}
\therefore LHS &= \frac{1}{B(c+q,s-q)}\int_0^1 m^{c+q-1}\sum_{u=0}^{s-1-q}(-1)^{u}\binom{s-q-1}{u}m^{u}\,dm\\
&= \frac{1}{B(c+q,s-q)}\int_0^1 m^{c+q-1}(1-m)^{s-q-1}\,dm\\
&= \frac{1}{B(c+q,s-q)}B(c+q,s-q)\\
&= 1
\end{aligned}$$

Corollary 3.1.1

Put $q=0$ and we have

$$\sum_{p=0}^{s-1}(-1)^p\binom{c+p-1}{p}\binom{c+s-1}{s-p-1}=1 \tag{3.30}$$

We now obtain the $F(v)$ for the cases mentioned above.
(a) When $x=s,\ s=1,2,3,\cdots$ and $y>0$
Using version II

$$\begin{aligned}
F(v) &= \frac{1}{B(s,y)}\sum_{q=0}^{s-1}(-1)^q\binom{s-1}{q}\left\{\frac{1-(1-G(v))^{y+q}}{y+q}\right\}\\
&= \frac{1}{B(s,y)}\sum_{q=0}^{s-1}(-1)^q\binom{s-1}{q}\frac{1}{y+q}-\frac{1}{B(s,y)}\sum_{q=0}^{s-1}(-1)^q\binom{s-1}{q}\frac{(1-G(v))^{y+q}}{y+q}
\end{aligned}$$

First term of the RHS above

$$\begin{aligned}
&= \frac{1}{B(s,y)}\sum_{q=0}^{s-1}(-1)^q\binom{s-1}{q}\frac{1}{y+q}\\
&= \sum_{q=0}^{s-1}(-1)^q\frac{1}{B(s,y)}\binom{s-1}{q}\frac{1}{y+q}\\
&= \sum_{q=0}^{s-1}(-1)^q\frac{\Gamma(s+y)}{\Gamma(s)\Gamma(y)}\ \frac{\Gamma(s)}{q!\Gamma(s-q)}\ \frac{1}{y+q}\\
&= \sum_{q=0}^{s-1}(-1)^q\ \frac{\Gamma(s+y)}{\Gamma(y)}\ \frac{1}{q!\Gamma(s-q)}\frac{1}{y+q}\\
&= \sum_{q=0}^{s-1}(-1)^q\ \frac{\Gamma(s+y)}{q!\Gamma(y)}\ \frac{\Gamma(y+q)}{\Gamma(s-q)\Gamma(y+q+1)}\\
&= \sum_{q=0}^{s-1}(-1)^q\ \frac{\Gamma(s+y)}{(s-q-1)!\Gamma(y+q+1)}\ \frac{\Gamma(y+q)}{q!\Gamma(y)}\\
&= \sum_{q=0}^{s-1}(-1)^q\binom{s+y-1}{s-q-1}\binom{y+q-1}{q}\\
&= \sum_{q=0}^{s-1}(-1)^q\binom{y+q-1}{q}\binom{y+s-1}{s-q-1}\\
&= 1 \text{ by corollary 3.1.1}
\end{aligned}$$

The second term

$$= \frac{1}{B(s,y)} \sum_{q=0}^{s-1} (-1)^q \binom{s-1}{q} \frac{(1-G(v))^{y+q}}{y+q}$$

$$= \frac{(1-G(v))^y}{B(s,y)} \sum_{q=0}^{s-1} (-1)^q \binom{s-1}{q} \sum_{p=0}^{q} \frac{\binom{q}{p}(-G(v))^p}{y+q}$$

$$= \frac{(1-G(v))^y}{B(s,y)} \sum_{q=0}^{s-1} \sum_{p=0}^{q} (-1)^q \binom{s-1}{q} \binom{q}{p} \frac{(-1)^p \, (G(v))^p}{y+q}$$

$$= \frac{(1-G(v))^y}{B(s,y)} \sum_{q=0}^{s-1} \sum_{p=0}^{s-1} (-1)^{q+p} \frac{\Gamma(s)}{q!\Gamma(s-q)} \frac{q!}{p!(q-p)!} \frac{(G(v))^p}{y+q}$$

$$= \frac{(1-G(v))^y}{B(s,y)} \sum_{p=0}^{s-1} \frac{(G(v))^p}{p!} \sum_{q=0}^{s-1} (-1)^{q+p} \frac{\Gamma(s)}{(s-q-1)!} \frac{1}{(q-p)!} \frac{1}{y+q}$$

$$= \frac{(1-G(v))^y}{\Gamma(y)} \sum_{p=0}^{s-1} \frac{(G(v))^p}{p!} \sum_{q=0}^{s-1} (-1)^{q+p} \frac{\Gamma(y+s)}{(s-q-1)!} \frac{1}{(q-p)!} \frac{\Gamma(y+q)}{\Gamma(y+q+1)}$$

$$= \frac{(1-G(v))^y}{\Gamma(y)} \sum_{p=0}^{s-1} \frac{(G(v))^p}{p!} \sum_{q=0}^{s-1} (-1)^{q+p} \binom{y+s-1}{s-q-1} \frac{\Gamma(y+q)}{(q-p)!}$$

$$= \frac{(1-G(v))^y}{\Gamma(y)} \sum_{p=0}^{s-1} \Gamma(y+p) \frac{(G(v))^p}{p!} \sum_{q=0}^{s-1} (-1)^{q+p} \binom{y+s-1}{s-q-1} \frac{\Gamma(y+q)}{(q-p)!\Gamma(y+p)}$$

$$= \frac{(1-G(v))^y}{\Gamma(y)} \sum_{p=0}^{s-1} \Gamma(y+p) \frac{(G(v))^p}{p!} \sum_{q=p}^{s-1} (-1)^{q-p} \binom{y+s-1}{s-q-1} \binom{y+q-1}{q-p}$$

$$= \frac{(1-G(v))^y}{\Gamma(y)} \sum_{p=0}^{s-1} \Gamma(y+p) \frac{(G(v))^p}{p!} \; .1 \text{ ; using lemma 3.1}$$

$$\therefore F(v) = 1 - \left\{ \frac{(1-G(v))^y}{\Gamma(y)} \sum_{p=0}^{s-1} \Gamma(y+p) \frac{(G(v))^p}{p!} \right\} \tag{3.31}$$

Hence

$$f(v) = y \, \frac{(1-G(v))^{y-1}}{\Gamma(y)} \sum_{p=0}^{s-1} \Gamma(y+p) \frac{(G(v))^p}{p!} \, g(v) - \frac{(1-G(v))^y}{\Gamma(y)} \sum_{p=1}^{s-1} \Gamma(y+p) \frac{(G(v))^{p-1}}{(p-1)!} \, g(v) \tag{3.32}$$

(b) When $y = s,\ s = 1,2,3,\cdots$ and $x > 0$
Using version I

$$
\begin{aligned}
F(v) &= \frac{1}{B(x,s)}\sum_{q=0}^{s-1}(-1)^q\binom{s-1}{q}\frac{(G(v))^{x+q}}{x+q}\\
&= \frac{(G(v))^x}{\Gamma(x)}\sum_{p=0}^{s-1}(-1)^q\binom{s-1}{q}\frac{\Gamma(x+s)}{\Gamma(s)}\frac{(G(v))^q}{x+q}\\
&= \frac{(G(v))^x}{\Gamma(x)}\sum_{q=0}^{s-1}\frac{(-1)^q\Gamma(s)}{q!\Gamma(s-q)}\ \frac{\Gamma(x+s)}{\Gamma(s)}\frac{(G(v))^q}{x+q}\\
&= \frac{(G(v))^x}{\Gamma(x)}\sum_{q=0}^{s-1}\frac{(-1)^q}{q!}\ \frac{\Gamma(x+s)}{\Gamma(s-q)}\frac{(G(v))^q}{x+q}\\
&= \frac{(G(v))^x}{\Gamma(x)}\sum_{q=0}^{s-1}\frac{(-1)^q}{q!}\ \frac{\Gamma(x+s)\,\Gamma(x+q)}{(s-q-1)!\Gamma(x+q+1)}\ (G(v))^q\\
&= \frac{(G(v))^x}{\Gamma(x)}\sum_{q=0}^{s-1}\frac{(-1)^q}{q!}\ \binom{x+s-1}{s-q-1}\Gamma(x+q)\ (G(v))^q\\
&= \frac{(G(v))^x}{\Gamma(x)}\sum_{q=0}^{s-1}\frac{(-1)^q}{q!}\ \binom{x+s-1}{s-q-1}\Gamma(x+q)\ [1-(1-G(v))]^q\\
&= \frac{(G(v))^x}{\Gamma(x)}\sum_{q=0}^{s-1}\frac{(-1)^q}{q!}\ \binom{x+s-1}{s-q-1}\Gamma(x+q)\ \sum_{p=0}^{q}(-1)^p\ \binom{q}{p}(1-G(v))^p\\
&= \frac{(G(v))^x}{\Gamma(x)}\sum_{q=0}^{s-1}\sum_{p=0}^{s-1}\frac{(-1)^{q+p}}{q!}\ \binom{x+s-1}{s-q-1}\ \frac{q!\Gamma(x+q)}{p!(q-p)!}(1-G(v))^p\\
&= \frac{(G(v))^x}{\Gamma(x)}\sum_{p=0}^{s-1}\Gamma(x+p)\frac{(1-G(v))^p}{p!}\sum_{q=0}^{s-1}(-1)^{q+p}\ \binom{x+s-1}{s-q-1}\ \frac{\Gamma(x+q)}{(q-p)!\Gamma(x+p)}\\
&= \frac{(G(v))^x}{\Gamma(x)}\sum_{p=0}^{s-1}\Gamma(x+p)\frac{(1-G(v))^p}{p!}\sum_{q=0}^{s-1}(-1)^{q+p}\ \binom{x+s-1}{s-q-1}\ \binom{x+q-1}{q-p}\\
&= \frac{(G(v))^x}{\Gamma(x)}\sum_{p=0}^{s-1}\Gamma(x+p)\frac{(1-G(v))^p}{p!}\ \sum_{q=p}^{s-1}(-1)^{q-p}\ \binom{x+s-1}{s-q-1}\ \binom{x+q-1}{q-p}\\
&= \frac{(G(v))^x}{\Gamma(x)}\sum_{p=0}^{s-1}\Gamma(x+p)\frac{(1-G(v))^p}{p!}.1 \text{ using lemma 3.1}
\end{aligned}
$$

$$
\therefore F(v) = \frac{(G(v))^x}{\Gamma(x)}\sum_{p=0}^{s-1}\Gamma(x+p)\frac{(1-G(v))^p}{p!} \qquad \textbf{(3.33)}
$$

and

$$
f(v) = x\ \frac{(G(v))^{x-1}}{\Gamma(x)}\ g(v)\sum_{p=0}^{s-1}\Gamma(x+p)\frac{(1-G(v))^p}{p!} - \frac{(G(v))^x}{\Gamma(x)}\sum_{p=1}^{s-1}\Gamma(x+p)\frac{(1-G(v))^{p-1}}{(p-1)!}\ g(v) \qquad \textbf{(3.34)}
$$

3.6 Beta Generated Distribution and its Special Cases in terms of Gauss Hypergeometric Function

Gauss hypergeometric function is given by

$$_2F_1(b,c;d;y) = 1 + \frac{b\,c}{d}\frac{y}{1!} + \frac{b(b+1)c(c+1)}{d(d+1)}\frac{y^2}{2!} + \frac{b(b+1)(b+2)c(c+1)(c+2)}{d(d+1)(d+2)}\frac{y^3}{3!} + \cdots \tag{3.35}$$

Now starting from (3.17) and using the identity

$$\binom{-\alpha}{q} = (-1)^q \binom{\alpha+q-1}{q} \tag{3.36}$$

we have

$$\begin{aligned}
F(v) &= \frac{1}{B(x,y)} \sum_{q=0}^{\infty} (-1)^q \binom{y-q+q-1}{q} \frac{(G(v))^{x+q}}{x+q} \\
&= \frac{1}{B(x,y)} \sum_{q=0}^{\infty} \binom{-(y-q)}{q} \frac{(G(v))^{x+q}}{x+q} \\
&= \frac{(G(v))^x}{B(x,y)} \sum_{q=0}^{\infty} \binom{-y+q}{q} \frac{(G(v))^q}{x+q} \\
&= \frac{(G(v))^x}{B(x,y)} \sum_{q=0}^{\infty} \frac{\Gamma(1-y+q)}{q!\Gamma(1-y)} \frac{(G(v))^q}{x+q} \\
&= \frac{(G(v))^x}{B(x,y)} \sum_{q=0}^{\infty} \frac{\Gamma(1-y+q)}{(x+q)\Gamma(1-y)} \frac{(G(v))^q}{q!} \\
&= \frac{(G(v))^x}{B(x,y)} \left\{ \frac{1}{x} + \frac{\Gamma(1-y+1)}{(x+1)\Gamma(1-y)}\frac{G(v)}{1!} + \frac{\Gamma(1-y+2)}{(x+2)\Gamma(1-y)}\frac{(G(v))^2}{2!} + \cdots \right\} \\
&= \frac{(G(v))^x}{B(x,y)} \left\{ \frac{1}{x} + \frac{(1-y)}{(x+1)}\frac{G(v)}{1!} + \frac{(1-y+1)(1-y)}{(x+2)}\frac{(G(v))^2}{2!} + \cdots \right\} \\
&= \frac{(G(v))^x}{xB(x,y)} \left\{ 1 + \frac{x(1-y)}{(x+1)}\frac{G(v)}{1!} + \frac{x(1-y)(1-y+1)}{(x+2)}\frac{(G(v))^2}{2!} + \cdots \right\} \\
&= \frac{(G(v))^x}{xB(x,y)} \left\{ 1 + \frac{x(1-y)}{(x+1)}\frac{G(v)}{1!} + \frac{x(x+1)(1-y)(1-y+1)}{(x+1)(x+2)}\frac{(G(v))^2}{2!} + \cdots \right\}
\end{aligned}$$

$$\therefore F(v) = \frac{[G(v)]^x}{xB(x,y)}\, {}_2F_1(x,1-y;x+1;G(v)) \tag{3.37}$$

where

$$ {}_2F_1 = 1 + \sum_{s=1}^{\infty} \frac{x(x+1)(x+2)\cdots(x+s-1)(1-y)(1-y+1)\cdots(1-y+s-1)}{(x+1)(x+2)\cdots(x+s)} \frac{(G(v))^s}{s!} \tag{3.38} $$

$$ \frac{d}{dv}\,{}_2F_1 = \sum_{s=1}^{\infty} \frac{x(x+1)\cdots(x+s-1)(1-y)(1-y+1)\cdots(1-y+s-1)}{(x+1)(x+2)\cdots(x+s)} \frac{(G(v))^{s-1}}{(s-1)!} g(v) $$

where ${}_2F_1 = {}_2F_1(x, 1-y; x+1; G(v))$

$$ \therefore \frac{d}{dv}\,{}_2F_1 = \frac{x(1-y)}{x+1} \sum_{s=1}^{\infty} \frac{(x+1)(x+2)\cdots(x+s-1)(1-y+1)\cdots(1-y+s-1)}{(x+2)\cdots(x+s)} \frac{(G(v))^{s-1}}{(s-1)!}\, g(v) $$

$$ = \frac{x\,(1-y)}{x+1}\,{}_2F_1\Big(x+1, (1-y+1); x+2; G(v)\Big)\, g(v) $$

$$ \frac{d}{dv}\,{}_2F_1 = \frac{x\,(1-y)}{x+1}\,{}_2F_1\Big(x+1, 2-y; x+2; G(v)\Big)\, g(v) \tag{3.39} $$

hence $f(v) = \dfrac{[G(v)]^{x-1}}{B(x,y)}\, g(v)\,{}_2F_1\Big(x, 1-y; x+1; G(v)\Big) + \dfrac{(G(v))^x}{xB(x,y)}\, \dfrac{d}{dv}\,{}_2F_1\Big(x, 1-y; x+1; G(v)\Big)$

$$ = \frac{[G(v)]^{x-1}}{B(x,y)} g(v)\,{}_2F_1\Big(x, 1-y; x+1; G(v)\Big) + \frac{(G(v))^x}{xB(x,y)}\, \frac{x(1-y)}{x+1} g(v)\,{}_2F_1\Big(x+1, 2-y; x+2; G(v)\Big) $$

$$ \therefore f(v) = \frac{[G(v)]^{x-1}}{B(x,y)}\, g(v)\,{}_2F_1\Big(x, 1-y; x+1; G(v)\Big) + \frac{(G(v))^x}{B(x,y)}\, \frac{1-y}{x+1} g(v)\,{}_2F_1\Big(x+1, 2-y; x+2; G(v)\Big) \tag{3.40} $$

3.7 Special Cases of Beta Generated Distribution in terms of Gauss Hypergeometric Function

3.7.1 Type I Exponentiated Distribution

when $y=1$ we have

$$F(v)=\frac{[G(v)]^{x}}{xB(x,1)}\,{}_2F_1(x,0;x+1;G(v))$$
$$=[G(v)]^{x}\,{}_2F_1(x,0;x+1;G(v))$$

But

$${}_2F_1(x,0;x+1;G(v))=\left\{1+\frac{x.0}{(x+1)}\frac{G(v)}{1!}+\frac{x(x+1)\;0.1}{(x+1)(x+2)}\frac{(G(v))^2}{2!}+\cdots\right\}$$
$$=1$$
$$\therefore F(v)=[G(v)]^{x} \tag{3.41}$$

and

$$f(v)=\frac{[G(v)]^{x-1}}{B(x,1)}g(v)\,{}_2F_1(x,0;x+1;G(v))+\frac{(G(v))^{x}}{B(x,1)}\,\frac{1-1}{x+1}g(v)\,{}_2F_1(x+1,1;x+2;G(v))$$
$$=x\,[G(v)]^{x-1}\,g(v)\;.1+0$$
$$=x\,[G(v)]^{v-1}\,g(v) \tag{3.42}$$

3.7.2 Type II Exponentiated Distribution

when $x=1$ we have

$$F(v)=\frac{[G(v)]}{B(1,y)}\,{}_2F_1(1,1-y;2;G(v))$$
$$=y\,G(v)\left\{1+\frac{1(1-y)}{2}\,G(v)+\frac{1.2(1-y)(1-y+1)}{2.3}\frac{(G(v))^2}{2!}+\cdots\right\}$$
$$=y\left\{G(v)+\frac{1(1-y)}{2}\,(G(v))^2+\frac{1.2(1-y)(1-y+1)}{2.3}\frac{(G(v))^3}{2!}+\cdots\right\}$$
$$=yG(v)+\frac{y(1-y)}{2}\,(G(v))^2+\frac{2y(1-y)(2-y)}{2.3}\frac{(G(v))^3}{2!}+\cdots$$
$$=-1+1+yG(v)-\frac{y(y-1)}{2}\,(G(v))^2+\frac{y(y-1)(y-2)}{3!}(G(v))^3+\cdots$$

$$
\begin{aligned}
&= 1 - 1 + \frac{y}{1!}\, G(v) - \frac{y(y-1)}{2!}\, (G(v))^2 + \frac{y(y-1)(y-2)}{3!}(G(v))^3 + \cdots \\
&= 1 - \left\{ 1 - \frac{y}{1!}\, G(v) + \frac{y(y-1)}{2!}\, (G(v))^2 - \frac{y(y-1)(y-2)}{3!}(G(v))^3 + \cdots \right\} \\
&= 1 - \left\{ \binom{y}{0} + (-1)\binom{y}{1}\, G(v) + (-1)^2\binom{y}{2}\, (G(v))^2 + (-1)^3\binom{y}{3}(G(v))^3 + \cdots \right\} \\
&= 1 - \sum_{p=0}^{\infty} \binom{y}{p}(-1)^p[G(v)]^p \\
&= 1 - \sum_{p=0}^{\infty} \binom{y}{p}[-G(v)]^p \\
&= 1 - [1 - G(v)]^y \qquad \textbf{(3.43)}
\end{aligned}
$$

and

$$
f(v) = y[1 - G(v)]^{y-1}\, g(v) \qquad \textbf{(3.44)}
$$

3.7.3 The ith order statistic distribution

Let $x = i$ and $y = s - i + 1$

Then

$$
\begin{aligned}
F_{i:s}(v) &= \frac{[G(v)]^i}{iB(i, s-i+1)}\, {}_2F_1(i, 1-(s-i+1); i+1; G(v)) \\
&= \frac{[G(v)]^i}{iB(i, s-i+1)}\, {}_2F_1(i, -(s-i); i+1; G(v)) \\
&= \frac{[G(v)]^i}{iB(i, s-i+1)}\, {}_2F_1(-(s-i), i; i+1; G(v)) \qquad \textbf{(3.45)}
\end{aligned}
$$

where

$$
\begin{aligned}
{}_2F_1(-(s-i), i; i+1; G(v)) = \Bigg\{ & 1 + \frac{-(s-i)i}{(i+1)}\frac{G(v)}{1!} + \frac{(-1)^2(s-i)(s-i-1)i(i+1)}{(i+1)(i+2)}\frac{(G(v))^2}{2!} \\
& + \frac{(-1)^3(s-i)(s-i-1)(s-i-2)i(i+1)(i+2)}{(i+1)(i+2)(i+3)}\frac{(G(v))^3}{3!} + \cdots \Bigg\}
\end{aligned}
$$

$$+\left\{\frac{(-1)^{s-i}(s-i)(s-i-1)(s-i-2)\cdots[(s-i)-(s-i-1)]i(i+1)(i+2)\cdots(i+s-i)}{(i+1)(i+2)(i+3)\cdots(i+s-i)}\frac{(G(v))^{s-i}}{(s-i)!}\right\}$$

$$=1+(-1)(s-i)\frac{i}{i+1}+(-1)^2(s-i)(s-i-1)\frac{i}{i+2}\frac{(G(v))^2}{2!}$$

$$+(-1)^3(s-i)(s-i-1)(s-i-2)\frac{i}{i+3}\frac{(G(v))^3}{3!}$$

$$+\cdots(-1)^{s-i}(s-i)(s-i-1)(s-i-2)\cdots[(s-i)-(s-i-1)]\frac{i}{i+(s-i)}\frac{(G(x))^{s-i}}{(s-i)!}$$

$$\therefore\ {}_2F_1(-(s-i),i;i+1;G(v))=\sum_{q=0}^{s-i}(-1)^q\frac{i}{i+q}\binom{s-i}{q}[G(v)]^q$$

$$\therefore F_{i:s}(v)=\frac{[G(v)]^i}{iB(i,s-i+1)}\sum_{q=0}^{s-i}(-1)^q\frac{i}{i+q}\binom{s-i}{q}[G(v)]^q$$

$$=\frac{1}{B(i,s-i+1)}\sum_{q=0}^{s-i}(-1)^q\frac{1}{i+q}\binom{s-i}{q}[G(v)]^{i+q}$$

Now

$$\frac{d}{dv}\,{}_2F_1(-(s-i),i;i+1;G(v))=\frac{-(s-i)\,i}{i+1}\,{}_2F_1(-(s-i+1),i+1;i+2;G(v))\,g(v)\text{ i.e from (3.39)}$$

Therefore

$$f_{i:s}(v)=\frac{[G(v)]^{i-1}}{B(i,s-i+1)}\,g(v)\,{}_2F_1(-(s-i),i;i+1;G(v))$$

$$-\frac{(G(v))^i}{B(i,s-i+1)}\,\frac{s-i}{i+1}g(v)\,{}_2F_1(-(s-i+1),i+1;i+2;G(v))$$

(3.46)

3.8 Beta Generated Distribution and its Special Cases as Infinite Mixtures

Let

$$w_q = \frac{(-1)^q}{B(x,y)} \binom{y-1}{q} \frac{1}{x+q} \tag{3.47}$$

Now

$$\begin{aligned} B(x,y) &= \int_0^1 m^{x-1}(1-m)^{y-1}\, dm \\ &= \sum_{q=0}^{\infty} (-1)^q \binom{y-1}{q} \int_0^1 m^{x+q-1}\, dm \\ &= \sum_{q=0}^{\infty} (-1)^q \binom{y-1}{q} \frac{1}{x+q} \\ \therefore \sum_{q=0}^{\infty} w_q &= \frac{1}{B(x,y)} \sum_{q=0}^{\infty} (-1)^q \binom{y-1}{q} \frac{1}{x+q} \\ &= \frac{1}{B(x,y)} B(x,y) = 1 \end{aligned}$$

Hence (3.17) becomes

$$\begin{aligned} F(v) &= \frac{1}{B(x,y)} \sum_{q=0}^{\infty} (-1)^q \binom{y-1}{q} \frac{(G(v))^{x+q}}{x+q} \\ &= \sum_{q=0}^{\infty} w_q\, (G(v))^{x+q} \end{aligned}$$

$$F(v) = \sum_{q=0}^{\infty} w_q\, G_{x+q}(v) \tag{3.48}$$

where

$$G_{x+q}(v) = [G(v)]^{x+q}$$

is an exponentiated cdf.

Therefore $F(v)$ is an infinite mixture of type I exponentiated distribution. Therefore

$$f(v) = \sum_{q=0}^{\infty} w_q\, (x+q)\, [G(v)]^{x+q-1}\, g(v)$$

$$f(v) = \sum_{q=0}^{\infty} w_q\, g_{x+q}(v) \tag{3.49}$$

where

$$g_{x+q}(v) = (x+q)\, [G(v)]^{x+q-1}\, g(v)$$

Similarly,if we let

$$v_q = \frac{(-1)^q}{B(x,y)} \binom{x-1}{q} \frac{1}{y+q} \tag{3.50}$$

then

$$\sum_{q=0}^{\infty} v_q = 1$$

Therefore (3.19) becomes

$$F(v) = \sum_{q=0}^{\infty} v_q \{1-(1-G(v))^{y+q}\}$$

$$F(v) = \sum_{q=0}^{\infty} v_q \; G_{y+q}(v) \tag{3.51}$$

where

$$G_{y+q}(v) = 1-(1-G(v))^{y+q}$$

Therefore $F(v)$ is an infinite mixture of type II exponentiated distribution.

The pdf is given by

$$f(v) = \sum_{q=0}^{\infty} v_q \; (y+q) \; [1-G(v)]^{y+q-1} \; g(v)$$

$$f(v) = \sum_{q=0}^{\infty} v_q \; g_{y+q}(v) \tag{3.52}$$

where

$$g_{y+q}(v) = (y+q) \; [1-G(v)]^{y+q-1} \; g(v)$$

3.9 Special Cases of Beta Generated Distribution as Infinite Mixtures

3.9.1 Type I Exponentiated Distribution

$$\text{Let } y=1 \; \Rightarrow \; w_q = \begin{cases} 1 & \text{for } q=0 \\ 0 & \text{for } \; q \neq 0 \end{cases} \tag{3.53}$$

$$\therefore F(v) = \sum_{q=0}^{\infty} w_q \; [G(v)]^{x+q}$$

$$= w_0 \,[G(v)]^x + \sum_{q=1}^{\infty} w_q \,[G(v)]^{x+q}$$

$$= 1.\,[G(v)]^x + 0$$

$$F(v) = \Big(G(v)\Big)^x \tag{3.54}$$

$$\therefore f(v) = x\Big(G(v)\Big)^{x-1} g(v) \tag{3.55}$$

3.9.2 Type II Exponentiated Distribution

$$\text{Let } x = 1 \;\Rightarrow\; v_q = \begin{cases} 1 & \text{for } q = 0 \\ 0 & \text{for } q \neq 0 \end{cases} \tag{3.56}$$

$$\therefore F(v) = \sum_{q=0}^{\infty} v_q \,\{1-(1-G(v))^{y+q}\}$$

$$= v_0 \,\{1-(1-G(v))^{y}\} + \sum_{q=1}^{\infty} v_q \,\{1-(1-G(v))^{y+q}\}$$

$$= \{1-(1-G(v))^{y}\} + 0$$

$$F(v) = 1 - \Big(1-G(v)\Big)^y \tag{3.57}$$

therefore

$$f(v) = y\Big(1-G(v)\Big)^{y-1} g(v) \tag{3.58}$$

3.9.3 The ith order statistic distribution

When $x = i$ and $y = s - i + 1$

Then

$$w_q = \frac{(-1)^q}{B(i, s-i+1)} \binom{s-i}{q} \frac{1}{i+q}$$

where

$$
\begin{aligned}
B(i,s-i+1) &= \int_0^1 m^{i-1}(1-m)^{s-i}\,dm \\
&= \sum_{q=0}^{\infty}(-1)^q \binom{s-i}{q} \int_0^1 m^{i+q-1}\,dm \\
&= \sum_{q=0}^{s-i}(-1)^q \binom{s-i}{q} \frac{1}{i+q} \\
\therefore \sum_{q=0}^{s-i} w_q &= \frac{1}{B(i,s-i+1)} \sum_{q=0}^{s-i}(-1)^q \binom{s-i}{q} \frac{1}{i+q} \\
&= \frac{1}{B(i,s-i+1)}\, B(i,s-i+1) = 1 \\
\therefore F_{i:s}(v) &= \sum_{q=0}^{s-i} w_q\,(G(v))^{i+q} \qquad \textbf{(3.59)}
\end{aligned}
$$

which is a finite mixture of type I exponentiated distribution.
Therefore

$$
f_{i:s}(v) = \sum_{q=0}^{s-i} w_q\,(i+q)\,[G(v)]^{i+q-1}\,g(v) \qquad \textbf{(3.60)}
$$

Putting $i = s$ we have

$$
\begin{aligned}
F_{s:s}(v) &= w_0\,(G(v))^s \\
&= (G(v))^s \qquad \textbf{(3.61)}
\end{aligned}
$$

and

$$
\begin{aligned}
f_{s:s}(v) &= w_0\, s\,[G(v)]^{s-1}\,g(v) \\
&= s\,[G(v)]^{s-1}\,g(v) \qquad \textbf{(3.62)}
\end{aligned}
$$

Again consider

$x = i$ and $y = s-i+1$

Then

$$
\begin{aligned}
v_q &= \frac{(-1)^q}{B(i,s-i+1)} \binom{i-1}{q} \frac{1}{s-i+q} \\
\therefore \sum_{q=0}^{i-1} v_q &= 1
\end{aligned}
$$

Therefore

$$F_{i:s}(v) = \sum_{q=0}^{i-1} v_q \{1-(1-G(v))^{s-i+1+q}\} \tag{3.63}$$

which is a finite mixture of type II exponentiated distribution.

and

$$f_{i:s}(v) = \sum_{q=0}^{i-1} v_q \,(s-i+1+q)\,[1-G(v)]^{s-i+q}\, g(v) \tag{3.64}$$

hence

$$\begin{aligned} F_{1:s}(v) &= v_0\,\{1-(1-G(v))^{s}\} \\ &= 1-(1-G(v))^{s} \end{aligned} \tag{3.65}$$

and

$$f_{1:s}(v) = s\,[1-G(v)]^{s-1}\, g(v) \tag{3.66}$$

3.10 Entropy

Entropy is a measure of the uncertainty in a probability distribution.
It has been used in various situations in science as a measure of variation of the uncertainty.
Shannon entropy plays a similar role as the kurtosis measure in comparing the shapes of various densities and measuring heaviness of tails.
It is defined by

$$I_s = E[-log f(v)] \tag{3.67}$$

i.e $f(v)$ is a pdf.
Lemmas that will be useful in determining entropy for the beta-generated distribution are given below :

Lemma 3.1

Let $G(v)$ and $g(v)$ be the parent cdf and pdf respectively.Also let $f(v)$ be the pdf of beta generated distribution,with parameters x and y then

$$(1)\ E[ln\ G(v)] = \psi(x) - \psi(x+y) \tag{3.68}$$

$$(2)\ E[ln\ (1-G(v))] = \psi(y) - \psi(x+y) \tag{3.69}$$

$$(3)\ E[ln\ g(v)] = E_H[ln\ g(G^{-1}(H))] \tag{3.70}$$

where $H \sim Beta(x,y)$ and ψ denotes the digamma function.
Proof for part (1):

$$\begin{aligned} E[ln\ G(v)] &= \int_{-\infty}^{\infty} [ln\ G(v)] f(v)\ d(v) \\ &= \int_{-\infty}^{\infty} [ln\ G(v)] \frac{[G(v)]^{x-1}[1-G(v)]^{y-1} g(v)}{B(x,y)}\ d(v) \end{aligned}$$

Let $h = G(v) \Rightarrow dh = g(v)dv$
$v = -\infty \Rightarrow h = G(-\infty) = 0;\ v = \infty \Rightarrow h = G(\infty) = 1$

$$\therefore E[ln\,G(v)] = \int_0^1 \frac{(ln\,h)\,h^{x-1}(1-h)^{y-1}}{B(x,y)}\,dh$$

But

$$\begin{aligned}
B(x,y) &= \int_0^1 h^{x-1}(1-h)^{y-1}\,dh \\
\therefore \frac{\partial}{\partial x}B(x,y) &= \int_0^1 \left[\frac{\partial}{\partial x}h^{x-1}\right](1-h)^{y-1}\,dh \\
&= \int_0^1 (log\,h)h^{x-1}(1-h)^{y-1}\,dh \\
&= \int_0^1 (ln\,h)h^{x-1}(1-h)^{y-1}\,dh
\end{aligned}$$

Also

$$\begin{aligned}
B(x,y) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \\
\therefore \frac{\partial}{\partial x}B(x,y) &= \Gamma(y)\left\{\frac{\Gamma(x+y)\,\Gamma'(x) - \Gamma(x)\Gamma'(x+y)}{[\Gamma(x+y)]^2}\right\} \\
&= \Gamma(y)\left\{\frac{\Gamma'(x)}{\Gamma(x+y)} - \frac{\Gamma(x)\Gamma'(x+y)}{[\Gamma(x+y)]^2}\right\} \\
&= \Gamma(y)\left\{\frac{\Gamma(x)}{\Gamma(x+y)}\,\frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma(x)}{\Gamma(x+y)}\,\frac{\Gamma'(x+y)}{\Gamma(x+y)}\right\} \\
&= \frac{\Gamma(y)\Gamma(x)}{\Gamma(x+y)}\{\psi(x) - \psi(x+y)\} \\
&= \{\psi(x) - \psi(x+y)\}B(x,y)
\end{aligned}$$

Therefore

$$\begin{aligned}
E[ln\,G(v)] &= \frac{1}{B(x,y)}\,\frac{\partial}{\partial x}B(x,y) \\
&= \frac{1}{B(x,y)}\,\{\psi(x) - \psi(x+y)\}B(x,y) \\
&= \psi(x) - \psi(x+y)
\end{aligned}$$

Proof for part (2):

$$
\begin{aligned}
E[ln\,(1-G(v))] &= \int_{-\infty}^{\infty} [ln\,(1-G(v))] f(v)\, d(v) \\
&= \int_{-\infty}^{\infty} [ln\,(1-G(v))] \frac{[G(v)]^{x-1}[1-G(v)]^{y-1} g(v)}{B(x,y)}\, d(v) \\
&= \int_{0}^{1} \frac{ln\,(1-h))\; h^{x-1}(1-h)^{y-1}}{B(x,y)}\, dh
\end{aligned}
$$

Let $z = 1-h \Rightarrow dz = -dh$

$$
\begin{aligned}
\therefore E[ln\,(1-G(v))] &= \int_{1}^{0} \frac{(ln\,z)(1-z)^{x-1} z^{y-1}}{B(x,y)}\, (-dz) \\
&= \int_{0}^{1} \frac{(ln\,z)\; z^{y-1}\; (1-z)^{x-1}}{}\, dz \\
&= \psi(y) - \psi(x+y)
\end{aligned}
$$

Proof for part (3):

$$
\begin{aligned}
E[ln\,g(v)] &= \int_{-\infty}^{\infty} [ln\,g(v)] f(v)\, d(v) \\
&= \int_{-\infty}^{\infty} [ln\,g(v)] \frac{[G(v)]^{x-1}[1-G(v)]^{y-1} g(v)}{B(x,y)}\, d(v) \\
&= \int_{-\infty}^{\infty} [ln\,g(v)] \frac{h^{x-1}[1-h]^{y-1}}{B(x,y)}\, dh
\end{aligned}
$$

since $G(v) = h \Rightarrow G^{-1}G(v) = G^{-1}(h)$
therefore $v = G^{-1}(h)$

$$
\begin{aligned}
\therefore E[ln\,g(v)] &= \int_{-\infty}^{\infty} [G^{-1}(h)] \frac{h^{x-1}[1-h]^{y-1}}{B(x,y)}\, dh \\
&= E[ln(G^{-1}(H)]
\end{aligned}
$$

where $H \sim Beta(x,y)$

Lemma 3.2

Let V be a continuous r.v with distribution function $G(v)$, then $W = G(v)$ follows a uniform distribution $U(0,1)$.
Proof

$$\begin{aligned}
\text{Let } W &= G(v) \\
\text{If } W(h) = &\text{ the cdf of } H \text{ then} \\
W(h) &= Prob\{H \leq h\} \\
&= Prob\{G(v) \leq h\} \\
&= Prob\{G^{-1}G(v) \leq G^{-1}(h)\} \\
&= Prob\{V \leq G^{-1}(h)\} \\
&= GG^{-1}(h) \\
&= h \\
\therefore\ w(h) &= 1, \qquad 0 \leq h \leq 1
\end{aligned}$$

An extension of lemma 3.3 is given by

Lemma 3.3

Let a r.v V have a beta generated pdf with parent distibution $G(v)$ then the r.v $H = G(v)$ follows a beta distribution.

Proof

Let $H = G(V)$ where V has a beta generated density $f(v)$ and

$$\begin{aligned}
W(h) &= Prob\{H \leq h\} \\
&= Prob\{G(v) \leq h\} \\
&= Prob\{V \leq G^{-1}(h)\} \\
&= F[G^{-1}(h)] \\
&= F(v) \\
\therefore\ w(h) &= f(v)\,\frac{dv}{dh} \\
&= f(v)\,\frac{1}{\frac{dh}{dv}}
\end{aligned}$$

$$= \frac{[G(v)]^{x-1}[1-G(v)]^{y-1}g(v)}{B(x,y)} \frac{1}{g(v)}$$
$$= \frac{[G(v)]^{x-1}[1-G(v)]^{y-1}}{B(x,y)}$$
$$w(h) = \frac{h^{x-1}(1-h)^{y-1}}{B(x,y)}, \qquad 0 \leq h \leq 1;\ x,y > 0$$

, a beta I distribution.

Proposition 3.1

Shannon entropy for the beta generated density with parameters x and y is given by

$$\begin{aligned} E[-log\ f(v)] &= log\ B(x,y) - (x-1)[\psi(x) - \psi(x+y)] \\ &- (y-1)[\psi(y) - \psi(x+y)] \\ &- E_H[ln\ g(G^{-1}(H))] \\ &= log\ B(x,y) - (x-1)E[log\ G(v)] - (y-1)E[log(1-G(v))] - E[log\ g(v)] \end{aligned} \tag{3.71}$$

where H is $Beta(x,y)$ and ψ denotes the digamma function.
Proof

$$E[-log\ f(v)] = \int_{-\infty}^{\infty} [-log\ f(v)] f(v)\ dv$$

where

$$\begin{aligned} log\ f(v) &= log \frac{[G(v)]^{x-1}[1-G(v)]^{y-1}g(v)}{B(x,y)} \\ &= -log\ B(x,y) + (x-1)log\ G(v) + (b-1)log[1-G(v)] + log\ g(v) \\ \therefore E[-log\ f(v)] &= -\int_{-\infty}^{\infty} \left\{ -log\ B(x,y) + (x-1)log\ G(v) + (y-1)log[1-G(v)] + log\ g(v) \right\} f(v)\ dv \\ &= log\ B(x,y) - (x-1)E[log\ G(v)] - (y-1)E[log(1-G(v))] - E[log\ g(v)] \end{aligned}$$

Using Lemma 3.1,we get

$$
\begin{aligned}
E[-log\ f(v)] = & \ log\ B(x,y) - (x-1)[\psi(x) - \psi(x+y)] \\
& - (y-1)[\psi(y) - \psi(x+y)] \\
& - E_H[ln\ g(G^{-1}(H))]
\end{aligned}
$$

4 BETA-LAPLACE DISTRIBUTION

4.1 Introduction

In chapter 3 we have looked at the concept of Beta-generated distribution and obtained the special cases. In this chapter we apply this concept to obtain Beta-Laplace distribution. Then Beta-Laplace distribution and its special cases are expressed based on Binomial expansion,Gauss hypergeometric function and as infinite mixtures of exponentiated distributions.
The Shannon entropy is also determined.

4.1.1 The Beta-Laplace distribution

The Laplace cdf is:

$$K(b) = \begin{cases} \frac{1}{2}e^{\frac{b}{\lambda}} & \text{b} < 0 \\ 1-\frac{1}{2}e^{-\frac{b}{\lambda}} & \text{b} \geq 0 \end{cases}$$

Therefore the pdf is

$$k(b) = \begin{cases} \frac{1}{2\lambda}e^{\frac{b}{\lambda}} & \text{b} < 0 \\ \frac{1}{2\lambda}e^{-\frac{b}{\lambda}} & \text{b} \geq 0 \end{cases}$$

Therefore the corresponding Beta-Laplace cdf is:

$$F(b) = \int_0^{\frac{1}{2}e^{\frac{b}{\lambda}}} \frac{h^{m-1}(1-h)^{n-1}}{B(m,n)}\, dh \quad b<0 \tag{4.1}$$

hence

$$F(b) = \int_0^{1-\frac{1}{2}e^{-\frac{b}{\lambda}}} \frac{h^{m-1}(1-h)^{n-1}}{B(m,n)}\, dh \quad b \geq 0 \tag{4.2}$$

Therefore

$$f(b) = \frac{\left[\frac{1}{2}\, e^{\frac{b}{\lambda}}\right]^m \left[1-\frac{1}{2}e^{\frac{b}{\lambda}}\right]^{n-1}}{\lambda\, B(m,n)} \quad b<0 \tag{4.3}$$

and

$$f(b) = \frac{\left[1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right]^{m-1} \left[\frac{1}{2}e^{-\frac{b}{\lambda}}\right]^n}{\lambda\, B(m,n)} \quad b \geq 0 \tag{4.4}$$

The moment generating function is obtained as follows

$$
\begin{aligned}
M_B(h) = E[e^{hB}] &= \int_{-\infty}^{\infty} e^{hb}\, f(b)\, db \\
&= \frac{1}{\lambda B(m,n)} \left\{ \int_{-\infty}^{0} e^{hb} (e^{\frac{b}{2\lambda}})^m \, [1-e^{\frac{b}{2\lambda}}]^{n-1} db + \int_{0}^{\infty} e^{hb} (e^{-\frac{b}{2\lambda}})^n \, [1-e^{-\frac{b}{2\lambda}}]^{m-1} db \right\}
\end{aligned}
$$

The first term of the RHS above

$$
\begin{aligned}
&= \int_{-\infty}^{0} e^{hb+\frac{mb}{2\lambda}} \sum_{v=0}^{\infty} \binom{n-1}{v} (e^{-\frac{b}{2\lambda}})^v \, db \\
&= \sum_{v=0}^{\infty} \binom{n-1}{v} \int_{-\infty}^{0} e^{(\frac{2\lambda h+m-v}{2\lambda})b} \, db
\end{aligned}
$$

$$
= \sum_{v=0}^{\infty} \binom{n-1}{v} \frac{2\lambda}{2\lambda h+m-v} \, .1
$$

The second term

$$
\begin{aligned}
&= \int_{0}^{\infty} e^{hb-\frac{nb}{2\lambda}} \sum_{v=0}^{\infty} \binom{m-1}{v} (e^{\frac{b}{2\lambda}})^v \, db \\
&= \sum_{v=0}^{\infty} \binom{m-1}{v} \int_{0}^{\infty} b^{1-1} e^{-(\frac{n-2\lambda h-v}{2\lambda})b} \, db \\
&= \sum_{v=0}^{\infty} \binom{m-1}{v} \frac{2\lambda}{n-2\lambda h-v}
\end{aligned}
$$

$$
\therefore\ M_B(h) = \frac{2}{B(m,n)} \sum_{v=0}^{\infty} \left\{ \binom{n-1}{v} \frac{1}{2\lambda h+m-v} + \binom{m-1}{v} \frac{1}{n-2\lambda h-v} \right\} \qquad \textbf{(4.5)}
$$

The rth moment:

$$
\begin{aligned}
E(B^r) &= \int_{-\infty}^{\infty} f(b)\, d(b) \\
&= \frac{1}{\lambda B(m,n)} \left\{ \int_{-\infty}^{0} b^r (e^{\frac{b}{2\lambda}})^m \, [1-e^{\frac{b}{2\lambda}}]^{n-1} db + \int_{0}^{\infty} b^r (e^{-\frac{b}{2\lambda}})^n \, [1-e^{-\frac{b}{2\lambda}}]^{m-1} db \right\}
\end{aligned}
$$

First term of the RHS

$$
\begin{aligned}
&= \int_{-\infty}^{0} b^r (e^{\frac{b}{2\lambda}})^m \sum_{v=0}^{\infty} \binom{n-1}{v} (e^{-\frac{b}{2\lambda}})^v \, db \\
&= \sum_{uv=0}^{\infty} \binom{n-1}{v} \int_{-\infty}^{0} b^r e^{-(\frac{v-m}{2\lambda})b} \, db
\end{aligned}
$$

The second term

$$
\begin{aligned}
&= \int_{0}^{\infty} b^r (e^{-\frac{b}{2\lambda}})^n \sum_{v=0}^{\infty} \binom{m-1}{u} (e^{\frac{b}{2\lambda}})^v \, db \\
&= \sum_{v=0}^{\infty} \binom{m-1}{v} \int_{0}^{\infty} b^{(r+1)-1} e^{-(\frac{n-v}{2\lambda})b} \, db \\
&= \sum_{v=0}^{\infty} \binom{m-1}{v} \frac{(2\lambda)^{r+1}\Gamma(r+1)}{(n-v)^{r+1}}
\end{aligned}
$$

$\therefore$

$$
E(B^r) = \frac{1}{\lambda B(m,n)} \sum_{v=0}^{\infty} \left\{ \binom{n-1}{v} \int_{-\infty}^{0} b^r e^{-(\frac{v-m}{2\lambda})b} \, db + \binom{m-1}{v} \frac{(2\lambda)^{r+1}\Gamma(r+1)}{(n-v)^{r+1}} \right\} \quad \textbf{(4.6)}
$$

When r=1

$$
E(B) = \frac{1}{\lambda B(m,n)} \sum_{v=0}^{\infty} \left\{ \binom{n-1}{v} \int_{-\infty}^{0} b e^{(\frac{m-v}{2\lambda})b} \, db + \binom{m-1}{v} \frac{(2\lambda)^2}{(n-v)^2} \right\}
$$

Integrating $\int_{-\infty}^{0} b e^{(\frac{m-v}{2\lambda})b} \, db$ by parts we have

Let $j = b \Rightarrow dj = db$

$$
\begin{aligned}
db &= e^{(\frac{m-v}{2\lambda})b} \Rightarrow b = \frac{2\lambda}{m-v} e^{(\frac{m-v}{2\lambda})b} \\
&= b \frac{2\lambda}{m-v} e^{(\frac{m-v}{2\lambda})b} \Big|_{-\infty}^{0} - \frac{2\lambda}{m-v} \int_{-\infty}^{0} e^{(\frac{m-v}{2\lambda})b} \, db \\
&= -\left(\frac{2\lambda}{m-v}\right)^2
\end{aligned}
$$

$$
\therefore E(B) = \frac{4\lambda}{B(m,n)} \sum_{v=0}^{\infty} \left\{ \binom{m-1}{v} \frac{1}{(n-v)^2} - \binom{n-1}{v} \frac{1}{(m-v)^2} \right\} \quad \textbf{(4.7)}
$$

4.2 Beta-Laplace Distribution and Its Special Cases Based on Binomial Expansion

4.2.1 Two versions

From formula (3.17) and (3.18) we obtain Beta-Laplace distribution based on Binomial expansion as follows:

Version 1

$$F(b) = \frac{1}{B(m,n)} \sum_{v=0}^{\infty} (-1)^v \binom{n-1}{v} \frac{(\frac{1}{2} e^{\frac{b}{\lambda}})^{m+v}}{m+v}$$

$$F(b) = \frac{(e^{\frac{b}{\lambda}})^m}{2^m\, B(m,n)} \sum_{v=0}^{\infty} (-1)^v \binom{n-1}{v} \frac{(e^{\frac{b}{\lambda}})^v}{2^v\; m+v} \text{ for } b < 0 \tag{4.8}$$

and

$$F(b) = \frac{1}{B(m,n)} \sum_{v=0}^{\infty} (-1)^v \binom{n-1}{v} \frac{(1-\frac{1}{2} e^{-\frac{b}{\lambda}})^{m+v}}{m+v}$$

$$F(b) = \frac{(1-\frac{1}{2} e^{-\frac{b}{\lambda}})^m}{B(m,n)} \sum_{v=0}^{\infty} (-1)^v \binom{n-1}{v} \frac{(1-\frac{1}{2} e^{-\frac{b}{\lambda}})^{m+v}}{m+v} \text{ for } b \geq 0 \tag{4.9}$$

The pdf is

$$f(b) = \frac{1}{B(m,n)} \sum_{v=0}^{\infty} (-1)^v \binom{n-1}{v} \left(\frac{1}{2} e^{\frac{b}{\lambda}}\right)^{m+v-1} \frac{1}{2\lambda} e^{\frac{b}{\lambda}}$$

$$f(b) = \frac{(e^{\frac{b}{\lambda}})^m}{2^m\, \lambda\, B(m,n)} \sum_{v=0}^{\infty} (-1)^v \binom{n-1}{v} \left(\frac{1}{2} e^{\frac{b}{\lambda}}\right)^v \text{ for } b < 0 \tag{4.10}$$

and

$$f(b) = \frac{1}{B(m,n)} \sum_{v=0}^{\infty} (-1)^v \binom{n-1}{v} (1-\frac{1}{2} e^{-\frac{b}{\lambda}})^{m+v-1} \frac{1}{2\lambda} e^{-\frac{b}{\lambda}}$$

$$f(b) = \frac{1}{B(m,n)} (1-\frac{1}{2} e^{-\frac{b}{\lambda}})^{m-1} \frac{1}{2\lambda} e^{-\frac{b}{\lambda}} \sum_{v=0}^{\infty} (-1)^v \binom{n-1}{v} (1-\frac{1}{2} e^{-\frac{b}{\lambda}})^v \text{ for } b \geq 0 \tag{4.11}$$

Version 2

using formula (3.19) and (3.20) we obtain

$$F(b) = \frac{1}{B(m,n)} \sum_{v=0}^{\infty} (-1)^v \binom{m-1}{v} \frac{1}{n+v} \left\{ 1 - \left(1 - \frac{1}{2} e^{\frac{b}{\lambda}}\right)^{n+v} \right\} \text{ for } b < 0 \tag{4.12}$$

and

$$F(b) = \frac{1}{B(m,n)} \sum_{v=0}^{\infty} (-1)^v \binom{m-1}{v} \frac{1}{n+v} \left\{ 1 - (1 - [1 - \frac{1}{2} e^{-\frac{b}{\lambda}}])^{n+v} \right\}$$

$$F(b) = \frac{1}{B(m,n)} \sum_{v=0}^{\infty} (-1)^v \binom{m-1}{v} \frac{1}{n+v} \left\{ 1 - (\frac{1}{2} e^{-\frac{b}{\lambda}})^{n+v} \right\} \text{ for } b \geq 0 \tag{4.13}$$

The pdf is given by

$$f(b) = \frac{1}{B(m,n)} \sum_{v=0}^{\infty} (-1)^v \binom{m-1}{v} \, (1 - \frac{1}{2} e^{\frac{b}{\lambda}})^{n+v-1} \, \frac{1}{2\lambda} e^{\frac{b}{\lambda}}$$

$$f(b) = \frac{1}{B(m,n)} \, (1 - \frac{1}{2} e^{\frac{b}{\lambda}})^{n-1} \, \frac{1}{2\lambda} e^{\frac{b}{\lambda}} \sum_{v=0}^{\infty} (-1)^v \binom{m-1}{v} \, (1 - \frac{1}{2} e^{\frac{b}{\lambda}})^{v} \quad \text{for } b < 0 \tag{4.14}$$

and

$$f(b) = \frac{1}{B(m,n)} \sum_{v=0}^{\infty} (-1)^v \binom{m-1}{v} \, (1 - [1 - \frac{1}{2} e^{-\frac{b}{\lambda}}])^{n+v-1} \, \frac{1}{2\lambda} e^{-\frac{b}{\lambda}}$$

$$f(b) = \frac{(e^{-\frac{b}{\lambda}})^n}{2^n \, \lambda \, B(m,n)} \sum_{v=0}^{\infty} (-1)^v \binom{m-1}{v} \, \frac{1}{2\lambda} e^{-\frac{b}{\lambda}} \quad \text{for } b \geq 0 \tag{4.15}$$

4.3 Special Cases of Beta-Laplace Distribution Based on Binomial Expansion

4.3.1 Type I Exponentiated-Laplace Distribution

Using (3.21) and (3.22)

Then

$$F(b) = \left[\frac{1}{2}e^{\frac{b}{\lambda}}\right]^m \quad \text{for } b < 0 \tag{4.16}$$

and

$$F(b) = \left[1 - \frac{1}{2}e^{-\frac{b}{\lambda}}\right]^m \quad \text{for } b \geq 0 \tag{4.17}$$

Therefore

$$f(b) = m\left[\frac{1}{2}e^{\frac{b}{\lambda}}\right]^{m-1} \frac{1}{2\lambda}e^{\frac{b}{\lambda}}$$

$$f(b) = \frac{m}{\lambda}\left[\frac{1}{2}e^{\frac{b}{\lambda}}\right]^m \quad \text{for } b < 0 \tag{4.18}$$

and

$$f(b) = m\left[1 - \frac{1}{2}e^{-\frac{b}{\lambda}}\right]^{m-1} \frac{1}{2\lambda}e^{-\frac{b}{\lambda}} \quad \text{for } b \geq 0 \tag{4.19}$$

4.3.2 Type II Exponentiated Laplace-Distribution

By using (3.23) and (3.24) we have

$$F(b) = \left\{1 - (1 - \frac{1}{2}e^{\frac{b}{\lambda}})^n\right\} \quad \text{for } b < 0 \tag{4.20}$$

$$F(b) = \left\{1 - \left(1 - [1 - \frac{1}{2}e^{-\frac{b}{\lambda}}]\right)^n\right\}$$

$$\therefore F(b) = 1 - \left(\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^n \quad \text{for } b \geq 0 \tag{4.21}$$

The pdf is

$$f(b) = n\Big(1 - \frac{1}{2}e^{\frac{b}{\lambda}}\Big)^{n-1}\ \frac{1}{2\lambda}e^{\frac{b}{\lambda}} \quad \text{for } b < 0 \tag{4.22}$$

and

$$f(b) = n\Big(\frac{1}{2}e^{-\frac{b}{\lambda}}\Big)^{n-1}\ \frac{1}{2\lambda}e^{-\frac{b}{\lambda}}$$

$$f(b) = \frac{n}{\lambda}\ \Big(\frac{1}{2}e^{-\frac{b}{\lambda}}\Big)^{n} \quad \text{for } b \geq 0 \tag{4.23}$$

4.3.3 The ith order statistic distribution

Using version I together with (3.25) and (3.26) we have

$$F_{j:s}(b) = \frac{1}{B(j, s-j+1)} \sum_{v=0}^{s-j} (-1)^v \binom{s-j}{v} \frac{(\frac{1}{2}e^{\frac{b}{\lambda}})^{j+v}}{j+v} \quad \text{for } b < 0 \tag{4.24}$$

and

$$F_{j:s}(b) = \frac{1}{B(j, s-j+1)} \sum_{v=0}^{s-j} (-1)^v \binom{s-j}{v} \frac{(1-\frac{1}{2}e^{-\frac{b}{\lambda}})^{j+v}}{j+v} \quad \text{for } b \geq 0 \tag{4.25}$$

hence

$$f_{j:s}(b) = \frac{1}{B(j, s-j+1)} \sum_{v=0}^{s-j} (-1)^v \binom{s-j}{v}\ (\frac{1}{2}e^{\frac{b}{\lambda}})^{j+v-1}\ \frac{1}{2\lambda}e^{\frac{b}{\lambda}} \quad \text{for } b < 0$$

$$= \frac{1}{\lambda\ B(j, s-j+1)}\ \Big(\frac{1}{2}e^{\frac{b}{\lambda}}\Big)^{j} \sum_{v=0}^{s-j} (-1)^v \binom{s-j}{v}\ \Big(\frac{1}{2}e^{\frac{b}{\lambda}}\Big)^{v} \tag{4.26}$$

and

$$f_{j:s}(b) = \frac{1}{B(j, s-j+1)} \sum_{v=0}^{s-j} (-1)^v \binom{s-j}{v}\ (1 - \frac{1}{2}e^{-\frac{b}{\lambda}})^{j+v-1}\ \frac{1}{2\lambda}e^{-\frac{b}{\lambda}} \quad \text{for } b \geq 0 \tag{4.27}$$

Using version II with (3.27) and (3.28) we have

$$F_{j:s}(b)=\frac{1}{B(j,s-j+1)}\sum_{v=0}^{j-1}(-1)^v\binom{j-1}{v}\frac{1}{s-j+v+1}\left\{1-\left(1-\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{s-j+v+1}\right\}\quad \text{for } b<0$$

(4.28)

and

$$F_{j:s}(b)=\frac{1}{B(j,s-j+1)}\sum_{v=0}^{j-1}(-1)^v\binom{j-1}{v}\frac{1}{s-j+v+1}\left\{1-(1-[1-\frac{1}{2}e^{-\frac{b}{\lambda}}])^{s-j+v+1}\right\}$$

$$F_{j:s}(b)=\frac{1}{B(j,s-j+1)}\sum_{v=0}^{j-1}(-1)^v\binom{j-1}{v}\frac{1}{s-j+v+1}\left\{1-\left(\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{s-j+v+1}\right\}\quad \text{for } b\geq 0$$

(4.29)

hence

$$f_{j:s}(b)=\frac{1}{B(j,s-j+1)}\sum_{v=0}^{j-1}(-1)^v\binom{j-1}{v}\left(1-\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{s-j+v}\frac{1}{2\lambda}e^{\frac{b}{\lambda}}\quad \text{for } b<0$$

(4.30)

and

$$f_{j:s}(b)=\frac{1}{B(j,s-j+1)}\sum_{v=0}^{j-1}(-1)^v\binom{j-1}{v}(1-[1-\frac{1}{2}e^{-\frac{b}{\lambda}}])^{s-j+v}\frac{1}{2\lambda}e^{-\frac{b}{\lambda}}$$

$$f_{j:s}(b)=\frac{1}{\lambda\, B(j,s-j+1)}\left(\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{s-j+1}\sum_{v=0}^{j-1}(-1)^v\binom{j-1}{v}\left(\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{v}\quad \text{for } b\geq 0$$

(4.31)

4.3.4 When one parameter is a positive integer and the other is a real number

(a) When $m=s,\ s=1,2,3,\cdots$ and $n>0$

Using version II,then formula (3.31) and (3.32) becomes

$$F(b)=1-\left\{\frac{1}{\Gamma(n)}\left(1-\frac{1}{2}e^{\frac{b}{\lambda}}\right)^n\sum_{l=0}^{s-1}\frac{\Gamma(n+l)}{l!}\left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^l\right\}\quad \text{for } b<0 \qquad \textbf{(4.32)}$$

and

$$F(b)=1-\left\{\frac{(1-[1-\frac{1}{2}e^{-\frac{b}{\lambda}}])^n}{\Gamma(n)}\sum_{l=0}^{s-1}\Gamma(n+l)\frac{(1-\frac{1}{2}e^{-\frac{b}{\lambda}})^l}{l!}\right\}$$

$$F(b)=1-\left\{\frac{1}{\Gamma(n)}\left(\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^n\sum_{l=0}^{s-1}\frac{\Gamma(n+l)}{l!}\left(1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^l\right\}\quad \text{for } b\geq 0 \qquad \textbf{(4.33)}$$

Hence

$$f(b) = n\,\frac{(1-\frac{1}{2}e^{\frac{b}{\lambda}})^{n-1}}{\Gamma(n)}\,\sum_{l=0}^{s-1}\Gamma(n+l)\frac{(\frac{1}{2}e^{\frac{b}{\lambda}})^{l}}{l!}\,\frac{1}{2\lambda}e^{\frac{b}{\lambda}}$$

$$-\frac{(1-\frac{1}{2}e^{\frac{b}{\lambda}})^{n}}{\Gamma(n)}\,\sum_{l=1}^{s-1}\Gamma(n+l)\frac{(\frac{1}{2}e^{\frac{b}{\lambda}})^{l-1}}{(l-1)!}\,\frac{1}{2\lambda}e^{\frac{b}{\lambda}}$$

$$f(b) = \frac{n}{\lambda}\,\frac{(1-\frac{1}{2}e^{\frac{b}{\lambda}})^{n-1}}{\Gamma(n)}\,\sum_{l=0}^{s-1}\Gamma(n+l)\frac{1}{l!}\left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{l+1}$$

$$-\frac{(1-\frac{1}{2}e^{\frac{b}{\lambda}})^{n}}{\lambda\,\Gamma(n)}\,\sum_{l=1}^{s-1}\frac{\Gamma(n+l)}{(l-1)!}\left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{l}\quad \text{for } b<0 \tag{4.34}$$

Also

$$f(b) = n\,\frac{(1-[1-\frac{1}{2}e^{-\frac{b}{\lambda}}]))^{n-1}}{\Gamma(n)}\,\sum_{l=0}^{s-1}\Gamma(n+l)\frac{(1-\frac{1}{2}e^{-\frac{b}{\lambda}})^{l}}{l!}\,\frac{1}{2\lambda}e^{-\frac{b}{\lambda}}$$

$$-\frac{(1-[1-\frac{1}{2}e^{-\frac{b}{\lambda}}])^{n}}{\Gamma(n)}\,\sum_{l=1}^{s-1}\Gamma(n+l)\frac{(1-\frac{1}{2}e^{-\frac{b}{\lambda}})^{l-1}}{(l-1)!}\,\frac{1}{2\lambda}e^{-\frac{b}{\lambda}}$$

$$f(b) = \frac{n}{\lambda\,\Gamma(n)}\left(\frac{1}{2}e^{-\frac{b}{\lambda}}\right))^{n}\sum_{l=0}^{s-1}\Gamma(n+l)\frac{1}{l!}\left(1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{l}$$

$$-\frac{1}{\lambda\,\Gamma(n)}\,(\frac{1}{2}e^{-\frac{b}{\lambda}})^{n+1}\sum_{l=1}^{s-1}\frac{\Gamma(n+l)}{(l-1)!}\left(1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{l-1}\quad \text{for } b\geq 0 \tag{4.35}$$

(b) When $n=s,\ s=1,2,3,\cdots$ and $m>0$
Using version I,then formula (3.33) and (3.34) becomes

$$F(b) = \frac{1}{\Gamma(m)}\left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{m}\sum_{l=0}^{s-1}\frac{\Gamma(m+l)}{l!}\left(1-\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{l}\quad \text{for } b<0 \tag{4.36}$$

and

$$F(b) = \frac{1}{\Gamma(m)}\left(1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{m}\sum_{l=0}^{s-1}\frac{\Gamma(m+l)}{l!}\left(1-[1-\frac{1}{2}e^{-\frac{b}{\lambda}}]\right)^{l}$$

$$F(b) = \frac{1}{\Gamma(m)}\left(1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{m}\sum_{l=0}^{s-1}\frac{\Gamma(m+l)}{l!}\left(\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{l}\quad \text{for } b\geq 0 \tag{4.37}$$

hence

$$f(b) = m\,\frac{(\frac{1}{2}e^{\frac{b}{\lambda}})^{m-1}}{\Gamma(m)}\;\frac{1}{2\lambda}e^{\frac{b}{\lambda}}\sum_{l=0}^{s-1}\Gamma(m+l)\frac{(1-\frac{1}{2}e^{\frac{b}{\lambda}})^{l}}{l!}$$

$$-\frac{(\frac{1}{2}e^{\frac{b}{\lambda}})^{m}}{\Gamma(m)}\sum_{l=1}^{s-1}\Gamma(m+l)\frac{(1-\frac{1}{2}e^{\frac{b}{\lambda}})^{l-1}}{(l-1)!}\;\frac{1}{2\lambda}e^{\frac{b}{\lambda}}$$

$$f(b) = \frac{m}{\lambda\;\Gamma(m)}\left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{m}\sum_{l=0}^{s-1}\frac{\Gamma(m+l)}{l!}\left(1-\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{l}$$

$$-\frac{1}{\Gamma(m)}\left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{m+1}\sum_{l=1}^{s-1}\frac{\Gamma(m+l)}{(l-1)!}\,(1-\frac{1}{2}e^{\frac{b}{\lambda}})^{l-1}\quad \text{for } b<0 \tag{4.38}$$

and

$$f(b) = m\,\frac{(1-\frac{1}{2}e^{-\frac{b}{\lambda}})^{m-1}}{\Gamma(m)}\;\frac{1}{2\lambda}e^{-\frac{b}{\lambda}}\sum_{l=0}^{s-1}\Gamma(m+l)\frac{(1-[1-\frac{1}{2}e^{-\frac{b}{\lambda}}])^{l}}{l!}$$

$$-\frac{(1-\frac{1}{2}e^{-\frac{b}{\lambda}})^{m}}{\Gamma(m)}\sum_{l=1}^{s-1}\Gamma(m+l)\frac{(1-[1-\frac{1}{2}e^{-\frac{b}{\lambda}}])^{l-1}}{(l-1)!}\;\frac{1}{2\lambda}e^{-\frac{b}{\lambda}}$$

$$f(b) = \frac{m}{\lambda\;\Gamma(m)}\,(1-\frac{1}{2}e^{-\frac{b}{\lambda}})^{m-1}\sum_{l=0}^{s-1}\frac{\Gamma(m+l)}{l!}\left(\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{l+1}$$

$$-\frac{1}{\Gamma(m)}\left(1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{m}\sum_{l=1}^{s-1}\frac{\Gamma(m+l)}{(l-1)!}\left(\frac{1}{2\lambda}e^{-\frac{b}{\lambda}}\right)^{l}\quad \text{for } b\geq 0 \tag{4.39}$$

4.4 Beta Laplace Distribution and its Special Cases in terms of Gauss Hypergeometric Function

Using formula (3.37) and (3.40) we have:

$$F(b) = \frac{1}{mB(m,n)} \left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^m {}_2F_1\left(m, 1-n; m+1; \frac{1}{2}e^{\frac{b}{\lambda}}\right) \quad \text{for } b < 0 \tag{4.40}$$

and

$$F(b) = \frac{1}{mB(m,n)} \left(1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^m {}_2F_1\left(m, 1-n; m+1; 1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right) \quad \text{for } b \geq 0 \tag{4.41}$$

Therefore

$$f(b) = \frac{[\frac{1}{2}e^{\frac{b}{\lambda}}]^{m-1}}{B(m,n)} \frac{1}{2\lambda}e^{\frac{b}{\lambda}} {}_2F_1(m, 1-n; m+1; \frac{1}{2}e^{\frac{b}{\lambda}})$$
$$+ \frac{(\frac{1}{2}e^{\frac{b}{\lambda}})^m}{B(m,n)} \frac{1-n}{m+1} \frac{1}{2\lambda}e^{\frac{b}{\lambda}} {}_2F_1(m+1, 2-n; m+2; \frac{1}{2}e^{\frac{b}{\lambda}})$$

$$f(b) = \frac{1}{\lambda\, B(m,n)} \left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^m {}_2F_1\left(m, 1-n; m+1; \frac{1}{2}e^{\frac{b}{\lambda}}\right)$$
$$+ \frac{1}{\lambda\, B(m,n)} \left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{m+1} {}_2F_1\left(m+1, 2-n; m+2; \frac{1}{2}e^{\frac{b}{\lambda}}\right) \quad \text{for } b < 0 \tag{4.42}$$

and

$$f(b) = \frac{1}{B(m,n)} \left(1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{m-1} \frac{1}{2\lambda}e^{-\frac{b}{\lambda}} {}_2F_1\left(m, 1-n; m+1; 1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right)$$
$$+ \frac{1}{B(m,n)} \left(1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^m \frac{1-n}{m+1} \frac{1}{2\lambda}e^{-\frac{b}{\lambda}} {}_2F_1\left(m+1, 2-n; m+2; 1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right) \quad \text{for } b \geq 0 \tag{4.43}$$

4.5 Special Cases of Beta-Laplace Distribution in terms of Gauss Hypergeometric Function

Using type I exponentiated distribution given by (3.41) and (3.42) we obtain the type I exponentiated-Laplace distribution in terms of Gauss Hypergeometric function which is similar to type I exponentiated-Laplace distribution in terms of Binomial expansion given by (4.16),(4.17),(4.18) and (4.19).
Similarly,type II exponentiated distribution given by (3.43) and (3.44) yields type II exponentiated-Laplace distribution in terms of Gauss Hypergeometric function which is similar to type II exponentiated-Laplace distribution in terms of Binomial expansion given by (4.20),(4.21),(4.22)and (4.23).

4.5.1 The ith order statistic distribution

From (3.45) and (3.46) we have

$$F_{j:s}(b) = \frac{1}{jB(j,s-j+1)}\left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{j}{}_2F_1\left(-(s-j),j;j+1;\frac{1}{2}e^{\frac{b}{\lambda}}\right) \quad \text{for } b<0 \tag{4.44}$$

and

$$F_{j:s}(b) = \frac{1}{jB(j,s-j+1)}\left(1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{j}{}_2F_1\left(-(s-j),j;j+1;[1-\frac{1}{2}e^{-\frac{b}{\lambda}}]\right) \quad \text{for } b\geq 0 \tag{4.45}$$

Therefore

$$\begin{aligned} f_{j:s}(b) &= \frac{\left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{j-1}}{B(j,s-j+1)}\,\frac{1}{2\lambda}e^{\frac{b}{\lambda}}\,{}_2F_1\left(-(s-j),j;j+1;\frac{1}{2}e^{\frac{b}{\lambda}}\right) \\ &\quad -\frac{\left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{j}}{B(j,s-j+1)}\,\frac{s-j}{j+1}\frac{1}{2\lambda}e^{\frac{b}{\lambda}}\,{}_2F_1\left(-(s-j+1),j+1;j+2;\frac{1}{2}e^{\frac{b}{\lambda}}\right) \\ f_{j:s}(b) &= \frac{1}{\lambda\, B(j,s-j+1)}\left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{j}{}_2F_1\left(-(s-j),j;j+1;\frac{1}{2}e^{\frac{b}{\lambda}}\right) \\ &\quad -\frac{1}{\lambda\, B(j,s-j+1)}\,\frac{s-j}{j+1}\left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{j+1}{}_2F_1\left(-(s-j+1),j+1;j+2;\frac{1}{2}e^{\frac{b}{\lambda}}\right) \quad \text{for } b<0 \end{aligned} \tag{4.46}$$

and

$$\begin{aligned} f_{j:s}(b) &= \frac{\left(1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{j-1}}{B(j,s-j+1)}\,\frac{1}{2\lambda}e^{-\frac{b}{\lambda}}\,{}_2F_1\left(-(s-j),j;j+1;1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right) \\ &\quad -\frac{\left(1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{j}}{B(j,s-j+1)}\,\frac{s-j}{j+1}\frac{1}{2\lambda}e^{-\frac{b}{\lambda}}\,{}_2F_1\left(-(s-j+1),j+1;j+2;1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right) \quad \text{for } b\geq 0 \end{aligned} \tag{4.47}$$

4.6 Beta Laplace Distribution and its Special Cases as Infinite Mixtures

By letting

$$w_v = \frac{(-1)^v}{B(m,n)} \binom{n-1}{v} \frac{1}{m+v}$$

and using (3.48) and (3.49) we have

$$F(b) = \sum_{v=0}^{\infty} w_v \left(\frac{1}{2} e^{\frac{b}{\lambda}}\right)^{m+v} \quad \text{for } b < 0 \tag{4.48}$$

and

$$F(b) = \sum_{v=0}^{\infty} w_v \left(1 - \frac{1}{2} e^{-\frac{b}{\lambda}}\right)^{m+v} \quad \text{for } b \geq 0 \tag{4.49}$$

whose pdf is therefore:

$$f(b) = \sum_{v=0}^{\infty} w_v \,(m+v) \left(\frac{1}{2} e^{\frac{b}{\lambda}}\right)^{m+v-1} \frac{1}{2\lambda} e^{\frac{b}{\lambda}}$$

$$f(b) = \frac{1}{\lambda} \sum_{v=0}^{\infty} w_v \,(m+v) \left(\frac{1}{2} e^{\frac{b}{\lambda}}\right)^{m+v} \quad \text{for } b < 0 \tag{4.50}$$

and

$$f(b) = \sum_{v=0}^{\infty} w_v \,(m+v) \left(1 - \frac{1}{2} e^{-\frac{b}{\lambda}}\right)^{m+v-1} \frac{1}{2\lambda} e^{-\frac{b}{\lambda}} \quad \text{for } b \geq 0 \tag{4.51}$$

Similarly,if we let

$$b_v = \frac{(-1)^v}{B(m,n)} \binom{m-1}{v} \frac{1}{n+v}$$

and use (3.51) and (3.52)

Then

$$F(b) = \sum_{v=0}^{\infty} b_v \left\{1 - \left(1 - \frac{1}{2} e^{\frac{b}{\lambda}}\right)^{n+v}\right\} \quad \text{for } b < 0 \tag{4.52}$$

and

$$F(b) = \sum_{v=0}^{\infty} b_v \left\{1 - \left(1 - \left[1 - \frac{1}{2} e^{-\frac{b}{\lambda}}\right]\right)^{n+v}\right\}$$

$$F(b) = \sum_{v=0}^{\infty} b_v \left\{1 - \left(\frac{1}{2} e^{-\frac{b}{\lambda}}\right)^{n+v}\right\} \quad \text{for } b \geq 0 \tag{4.53}$$

The pdf is:

$$f(b)=\sum_{v=0}^{\infty} b_v\,(n+v)\left(1-\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{n+v-1}\frac{1}{2\lambda}e^{\frac{b}{\lambda}} \quad \text{for } b<0 \tag{4.54}$$

and

$$f(b)=\sum_{v=0}^{\infty} b_v\,(n+v)\left(1-\left[1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right]\right)^{n+v-1}\frac{1}{2\lambda}e^{-\frac{b}{\lambda}} \quad \text{for } b\geq 0$$

$$f(b)=\sum_{v=0}^{\infty} b_v\,(n+v)\left(\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{n+v} \quad \text{for } b\geq 0 \tag{4.55}$$

4.6.1 Special Cases of Beta-Laplace Distribution as Infinite Mixtures

Using type I exponentiated distribution given by (3.54) and (3.55) we obtain the type I exponentiated-Laplace distribution as infinite mixtures which is similar to type I exponentiated-Laplace distribution in terms of Binomial expansion gven by (4.16),(4.17),(4.18) and (4.19). Similarly,type II exponentiated distribution given by (3.57) and (3.58) yields type II exponentiated-Laplace distribution as infinite mixtures which is similar to type II exponentiated-Laplace distribution in terms of Binomial expansion given by (4.20),(4.21),(4.22)and (4.23).

4.6.2 The ith order statistic distribution

When $m=j$ and $n=s-j+1$

Then

$$w_v=\frac{(-1)^v}{B(j,s-j+1)}\binom{s-j}{v}\frac{1}{j+v}$$

Therefore using (3.59) and (3.60) implies

$$F_{j:s}(b)=\sum_{v=0}^{s-j} w_v\left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{j+v} \quad \text{for } b<0 \tag{4.56}$$

and

$$F_{j:s}(b)=\sum_{v=0}^{s-j} w_v\left(1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{j+v} \quad \text{for } b\geq 0 \tag{4.57}$$

which is a finite mixture of type I exponentiated-Laplace.

The pdf is given by

$$f_{j:s}(b) = \sum_{v=0}^{s-j} w_v\ (j+v)\ \left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{j+v-1}\ \frac{1}{2\lambda}e^{\frac{b}{\lambda}}$$

$$f_{j:s}(b) = \frac{1}{\lambda}\sum_{v=0}^{s-j} w_v\ (j+v)\ \left(\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{j+v} \quad \text{for } b<0 \tag{4.58}$$

and

$$f_{j:s}(b) = \sum_{v=0}^{s-j} w_v\ (j+v)\ \left(1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{j+v-1}\ \frac{1}{2\lambda}e^{-\frac{b}{\lambda}} \quad \text{for } b\geq 0 \tag{4.59}$$

Again consider

$$m=j \text{ and } n=s-j+1$$

Then

$$b_v = \frac{(-1)^v}{B(j,s-j+1)}\binom{j-1}{v}\frac{1}{s-j+v}$$

Therefore using (3.63) and (3.64) we have

$$F_{j:s}(b) = \sum_{v=0}^{\infty} b_v\left\{1-\left(1-\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{s-j+1+v}\right\} \quad \text{for } b<0 \tag{4.60}$$

and

$$F_{j:s}(b) = \sum_{v=0}^{\infty} b_v\left\{1-\left(1-\left[1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right]\right)^{s-j+1+v}\right\}$$

$$F_{j:s}(b) = \sum_{v=0}^{\infty} b_v\left\{1-\left(\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{s-j+1+v}\right\} \quad \text{for } b\geq 0 \tag{4.61}$$

hence

$$f_{j:s}(b) = \sum_{v=0}^{\infty} b_v\ (s-j+1+v)\ \left(1-\frac{1}{2}e^{\frac{b}{\lambda}}\right)^{s-j+v}\ \frac{1}{2\lambda}e^{\frac{b}{\lambda}} \quad \text{for } b<0 \tag{4.62}$$

and

$$f_{j:s}(b) = \sum_{v=0}^{\infty} b_v\ (s-j+1+v)\ \left(1-\left[1-\frac{1}{2}e^{-\frac{b}{\lambda}}\right]\right)^{s-j+v}\ \frac{1}{2\lambda}e^{-\frac{b}{\lambda}}$$

$$f_{j:s}(b) = \frac{1}{\lambda}\sum_{v=0}^{\infty} b_v\ (s-j+1+v)\ \left(\frac{1}{2}e^{-\frac{b}{\lambda}}\right)^{s-j+v+1} \quad \text{for } b\geq 0 \tag{4.63}$$

4.7 Entropy

Using formula (3.71) we have

$$
\begin{aligned}
E[-log\, f(b)] &= log\, B(m,n)-(m-1)E[log\, K(b)]-(n-1)E[log(1-K(b))]-E[log\, k(b)]\\
&= log\, B(m,n)-(m-1)E[log\, \frac{1}{2}e^{\frac{b}{\lambda}}]-(n-1)E[log(1-\frac{1}{2}e^{\frac{b}{\lambda}})]-E[log\, \frac{1}{2\lambda}e^{\frac{b}{\lambda}}]\\
&= log\, B(m,n)-(m-1)E[log\, \frac{1}{2}+log\, e^{\frac{b}{\lambda}}]-(n-1)E[log(1-\frac{1}{2}e^{\frac{b}{\lambda}})]\\
&-E[log\, \frac{1}{2\lambda}+log\, e^{\frac{b}{\lambda}}]\\
&= log\, B(m,n)-(m-1)log\, \frac{1}{2}-(m-1)\frac{E(B)}{\lambda}-(n-1)E[log(1-\frac{1}{2}e^{\frac{b}{\lambda}})]\\
&-log\, \frac{1}{2\lambda}-\frac{E(B)}{\lambda}\\
&= log\, B(m,n)+(m-1)log\, 2+log\, 2\lambda-\frac{mE(B)}{\lambda}-(n-1)E[log(1-\frac{1}{2}e^{\frac{b}{\lambda}})]\\
&= log\, B(m,n)+mlog\, 2+log\, \lambda-\frac{mE(B)}{\lambda}-(n-1)E[log(1-\frac{1}{2}e^{\frac{b}{\lambda}})]\\
&= log\, B(m,n)+mlog\, 2+log\, \lambda-\frac{mE(B)}{\lambda}-log[E(1-\frac{1}{2}e^{\frac{b}{\lambda}})]^{n-1}\\
&= log\, B(m,n)+mlog\, 2+log\, \lambda-\frac{mE(B)}{\lambda}-log[(1-E(e^{\frac{b}{2\lambda}}))]^{n-1}\\
&= log\, B(m,n)+mlog\, 2+log\, \lambda-\frac{mE(B)}{\lambda}-log\Big\{\sum_{v=0}^{\infty}(-1)^{v}\binom{n-1}{v}[E(e^{\frac{b}{2\lambda}})]^{v}\Big\}\\
&= log\, B(m,n)+mlog\, 2+log\, \lambda-\frac{mE(B)}{\lambda}-log\sum_{v=0}^{\infty}(-1)^{v}\binom{n-1}{v}-vElog\, e^{\frac{b}{2\lambda}}\\
&= log\, B(m,n)+mlog\, 2+log\, \lambda-\frac{mE(B)}{\lambda}-log\sum_{v=0}^{\infty}(-1)^{v}\binom{n-1}{v}-v\frac{E(B)}{2\lambda}\\
&= log\, B(m,n)+mlog\, 2+log\, \lambda-log\sum_{v=0}^{\infty}(-1)^{v}\binom{n-1}{v}-\Big(\frac{2m+v}{\lambda}\Big)E(B)\\
&= log\, B(m,n)+mlog\, 2+log\, \lambda-log\sum_{v=0}^{\infty}(-1)^{v}\binom{n-1}{v}\\
&-\Big(\frac{2m+v}{\lambda}\Big)\,\frac{1}{\lambda B(m,n)}\sum_{v=0}^{\infty}\Bigg\{\binom{m-1}{v}\frac{(2\lambda)^{2}}{(n-v)^{2}}-\binom{n-1}{v}\frac{(2\lambda)^{2}}{(m-v)^{2}}\Bigg\}
\end{aligned}
$$

Therefore

$$E[-log\ f(b)] = log\ B(m,n) + mlog\ 2 + log\ \lambda - log \sum_{v=0}^{\infty} (-1)^v \binom{n-1}{v}$$

$$- \frac{2m+v}{B(m,n)} \sum_{v=0}^{\infty} \left\{ \binom{m-1}{v} \frac{(2)^2}{(n-v)^2} - \binom{n-1}{v} \frac{(2)^2}{(m-v)^2} \right\} \text{ for } b < 0 \quad \textbf{(4.64)}$$

and

$$E[-log\ f(b)] = log\ B(m,n) - (m-1)E[log\ 1 - \frac{1}{2}e^{-\frac{b}{\lambda}}] - (n-1)E[log(\frac{1}{2}e^{-\frac{b}{\lambda}})] - E[log\ \frac{1}{2\lambda}e^{-\frac{b}{\lambda}}]$$

$$= log\ B(m,n) + (n-1)log\ 2 + \frac{(b-1)E(B)}{\lambda} + log\ 2\lambda + \frac{E(B)}{\lambda} - log[E(1-e^{-\frac{b}{2\lambda}})]^{m-1}$$

$$= log\ B(m,n) + nlog\ 2 + log\ \lambda + \frac{nE(B)}{\lambda} - log[E(1-e^{-\frac{b}{2\lambda}})]^{m-1}$$

$$= log\ B(m,n) + nlog\ 2 + log\ \lambda + \frac{nE(B)}{\lambda} - log\Big\{ \sum_{v=0}^{\infty} (-1)^v \binom{m-1}{v} [E(e^{-\frac{b}{2\lambda}})]^v \Big\}$$

$$= log\ B(m,n) + nlog\ 2 + log\ \lambda + \frac{nE(B)}{\lambda} - log \sum_{v=0}^{\infty} (-1)^v \binom{m-1}{v} + \frac{vE(B)}{2\lambda}$$

$$= log\ B(m,n) + nlog\ 2 + log\ \lambda - log \sum_{v=0}^{\infty} (-1)^v \binom{m-1}{v} + \Big(\frac{2n+v}{\lambda}\Big) E(B)$$

$$= log\ B(m,n) + nlog\ 2 + log\ \lambda - log \sum_{v=0}^{\infty} (-1)^v \binom{m-1}{v}$$

$$+ \Big(\frac{2n+n}{\lambda}\Big) \frac{1}{\lambda B(m,n)} \sum_{v=0}^{\infty} \left\{ \binom{m-1}{v} \frac{(2\lambda)^2}{(n-v)^2} - \binom{n-1}{v} \frac{(2\lambda)^2}{(m-v)^2} \right\}$$

Therefore

$$E[-log\ f(b)] = log\ B(m,n) + nlog\ 2 + log\ \lambda - log \sum_{v=0}^{\infty} (-1)^v \binom{m-1}{v}$$

$$+ \frac{2n+v}{B(m,n)} \sum_{v=0}^{\infty} \left\{ \binom{m-1}{v} \frac{(2)^2}{(n-v)^2} - \binom{n-1}{v} \frac{(2)^2}{(m-v)^2} \right\} \text{ for } b \geq 0 \quad \textbf{(4.65)}$$

5 LAPLACE GENERALIZATION BASED ON EXPONENTIATED GENERATOR

5.1 Introduction

Exponentiated exponential distribution is a generalization of a standard exponential distribution.
Generalized Laplace distribution and its properties including moment generating function is presented here.

5.2 Construction of Generalized Laplace Distribution based on the difference method

Consider the cdf of an exponentiated exponential r.v given by

$$F(v) = \left(1 - e^{-\tau v}\right)^{\theta} \qquad v > 0; \tau > 0 \tag{5.1}$$

Therefore

$$\begin{aligned} f(v) &= \theta \left(1 - e^{-\tau v}\right)^{\theta - 1} \tau e^{-\tau v} \\ &= \theta \tau e^{-\tau v} \left(1 - e^{-\tau v}\right)^{\theta - 1} \end{aligned} \tag{5.2}$$

is the pdf.
Our aim is to find the pdf of $J = V_1 - V_2$,
V_1 and V_2 are (iid) exponentiated exponential r.vs.
Srivastava et.al (2006) obtained the cdf and pdf for $j \geq 0$ as given in the following proposition

Proposition 5.1

For $j \geq 0$ the cdf and pdf are given by:

$$F(j) = 1 - \theta \sum_{p=0}^{\infty} (-1)^p \, e^{\tau p j} \, a_p(\theta) \, B_{e^{-\tau j}}(p+1, \theta) \tag{5.3}$$

and

$$f(j) = \theta^2 \tau \, e^{\tau j} \, \sum_{p=0}^{\infty} (-1)^p \, e^{\tau p j} \, a_p(\theta - 1) \, B_{e^{-\tau j}}(p+2, \theta) \tag{5.4}$$

Proof

$$J = V_1 - V_2, \qquad V_1 > V_2 > 0$$

$$\begin{aligned}
F(j) &= \text{Prob}\{J \leq j\} \\
&= \text{Prob}\{V_1 - V_2 \leq j\} \\
&= \text{Prob}\{V_2 \geq v_1 - j\} \\
&= 1 - \text{Prob}\{V_2 \leq v_1 - j\} \\
F(j) &= 1 - \text{Prob}\{0 < V_1 < \infty, \quad V_2 \leq v_1 - j\} \\
&= 1 - \int_0^{\infty} \int_0^{v_1 - j} f(v_2) f(v_1) \, dv_2 \, dv_1 \\
&= 1 - \int_0^{\infty} \left[\int_0^{v_1 - j} f(v_2) \, dv_2 \right] f(v_1) \, dv_1 \\
&= 1 - \int_j^{\infty} F(v_1 - j) f(v_1) \, dv_1
\end{aligned}$$

$$F(j) = 1 - \int_j^{\infty} F(v - j) f(v) \, dv \tag{5.5}$$

$$\begin{aligned}
&= 1 - \int_j^{\infty} \left(1 - e^{-\tau(v-j)}\right)^{\theta} \, \theta \tau e^{-\tau v} \left(1 - e^{-\tau v}\right)^{\theta - 1} dv \\
&= 1 - \theta \tau \int_j^{\infty} e^{-\tau v} \left(1 - e^{-\tau v}\right)^{\theta - 1} \left(1 - e^{-\tau(v-j)}\right)^{\theta} dv
\end{aligned}$$

Let $\qquad r = e^{-\tau v} \Rightarrow \dfrac{dr}{dv} = -\tau e^{-\tau v} \Rightarrow dv = \dfrac{dr}{-\tau e^{-\tau v}} = \dfrac{dr}{-\tau r}$

$$\therefore F(j) = 1 - \theta\tau \int_{e^{-\tau j}}^{0} r(1-r)^{\alpha-1}\,(1-re^{\tau j})^{\theta}\ \frac{dr}{-\tau r}$$

$$= 1 - \theta \int_{0}^{e^{-\tau j}} (1-r)^{\theta-1}\,(1-re^{\tau j})^{\theta}\ dr$$

$$= 1 - \theta \int_{0}^{e^{-\tau j}} (1-r)^{\theta-1}\ \sum_{p=0}^{\infty}(-1)^p \binom{\theta}{p} (re^{\tau j})^p\ dr$$

$$F(j) = 1 - \theta \sum_{p=0}^{\infty}(-1)^p \binom{\theta}{p} e^{\tau p j} \int_{0}^{e^{-\tau j}} r^{(p+1)-1}(1-r)^{\theta-1}\ dr \tag{5.6}$$

Therefore $\quad F(j) = 1 - \theta \sum_{p=0}^{\infty}(-1)^p\, e^{\tau p j}\, a_p(\theta)\, B_{e^{-\tau j}}(p+1,\theta) \qquad j \geq 0$

Where $\quad a_p(\theta) = \binom{\theta}{p} = \begin{cases} 1 & p = 0 \\ \frac{1}{p!}\prod_{j=0}^{p-1}(\theta - j) & p > 0 \end{cases}$

$$B_{e^{-\tau j}}(p+1,\theta) = \int_{0}^{e^{-\tau j}} r^{(p+1)-1}(1-r)^{\theta-1}\ dr$$

Its corresponding pdf is:

$$f(j) = \int_{j}^{\infty} f(v)f(v-j)\, dv \tag{5.7}$$

$$= \int_{j}^{\infty} \theta\tau e^{-\tau v}\left(1-e^{-\tau v}\right)^{\theta-1}\ \theta\tau e^{-\tau(v-j)}\left(1-e^{-\tau(v-j)}\right)^{\theta-1} dv$$

$$= (\theta\tau)^2\, e^{\tau j} \int_{j}^{\infty} e^{-2\tau v}\left(1-e^{-\tau v}\right)^{\theta-1}\ \left(1-e^{-\tau(v-j)}\right)^{\theta-1} dv$$

Let $\quad r = e^{-\tau v} \Rightarrow \frac{dr}{dv} = -\tau e^{-\tau v} \Rightarrow dv = \frac{dr}{-\tau e^{-\tau v}} = \frac{dr}{-\tau r}$

$$\therefore f(j) = (\theta\tau)^2\, e^{\tau j} \int_{e^{-\tau j}}^{0} r^2(1-r)^{\theta-1}\ \left(1-re^{\tau j}\right)^{\theta-1}\ \left(\frac{dr}{-\tau r}\right)$$

$$= \theta^2\tau\, e^{\tau j} \int_{0}^{e^{-\tau j}} r(1-r)^{\theta-1}\ \left(1-re^{\tau j}\right)^{\theta-1}\ dr$$

$$= \theta^2\tau\, e^{\tau j} \int_{0}^{e^{-\tau j}} r(1-r)^{\theta-1}\ \sum_{p=0}^{\infty}(-1)^p \binom{\theta-1}{p} (re^{\tau j})^p\ dr$$

$$= \theta^2\tau\, e^{\tau j}\ \sum_{p=0}^{\infty}(-1)^p \binom{\theta-1}{p}\ e^{\tau p j} \int_{0}^{e^{-\tau j}} r^{p+1}(1-r)^{\theta-1}\ dr$$

$$f(j) = \theta^2\tau\, e^{\tau j}\ \sum_{p=0}^{\infty}(-1)^p \binom{\theta-1}{p}\ e^{\tau p j} \int_{0}^{e^{-\tau j}} r^{(p+2)-1}(1-r)^{\theta-1}\ dr \tag{5.8}$$

$$\therefore f(j) = \theta^2\tau\, e^{\tau j}\ \sum_{p=0}^{\infty}(-1)^p\, e^{\tau p j}\, a_p(\theta-1)\, B_{e^{-\tau j}}(p+2,\theta) \qquad j \geq 0$$

Alternatively,beginning from (5.6) we can express $F(j)$ as follows:

$$\begin{aligned}
F(j) &= 1-\theta\sum_{p=0}^{\infty}(-1)^{p}\binom{\theta}{p}e^{\tau pj}\int_{0}^{e^{-\tau j}} r^{(p+1)-1}(1-r)^{\theta-1}\ dr \\
&= 1-\theta\sum_{p=0}^{\infty}(-1)^{p}\binom{\theta}{p}e^{\tau pj}\int_{0}^{e^{-\tau j}} r^{(p+1)-1}\sum_{m=0}^{\infty}(-1)^{m}\binom{\theta-1}{m}r^{m}\,dr \\
&= 1-\theta\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{m+p}\binom{\theta-1}{m}\binom{\theta}{p}e^{\tau pj}\int_{0}^{e^{-\tau q}} r^{m+p}\,dr \\
&= 1-\theta\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{m+p}\binom{\theta-1}{m}\binom{\theta}{p}e^{\tau pj}\frac{e^{-\tau(m+p+1)j}}{m+p+1}
\end{aligned}$$

$$F(j) = 1-\theta\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{m+p}\binom{\theta-1}{m}\binom{\theta}{p}\frac{e^{-\tau(m+1)j}}{m+p+1} \quad \textbf{(5.9)}$$

Alternatively,beginning from (5.8) we can express $f(j)$ as follows:

$$\begin{aligned}
f(j) &= \theta^{2}\tau\, e^{\tau j}\sum_{p=0}^{\infty}(-1)^{p}\binom{\theta-1}{p}e^{\tau pj}\int_{0}^{e^{-\tau j}} r^{(p+2)-1}(1-r)^{\theta-1}\,dr \\
&= \theta^{2}\tau\, e^{\tau j}\sum_{p=0}^{\infty}(-1)^{p}\binom{\theta-1}{p}e^{\tau pj}\int_{0}^{e^{-\tau j}} r^{(p+2)-1}\sum_{m=0}^{\infty}(-1)^{m}\binom{\theta-1}{m}r^{m}\,dr \\
&= \theta^{2}\tau\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{m+p}\binom{\theta-1}{m}\binom{\theta-1}{m}e^{\tau(p+1)j}\int_{0}^{e^{-\tau j}} r^{(m+p+1)}\,dr \\
&= \theta^{2}\tau\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{m+p}\binom{\theta-1}{m}\binom{\theta-1}{p}e^{\tau(p+1)j}\,\frac{e^{-\tau(m+p+2)j}}{m+p+2}
\end{aligned}$$

$$f(j) = \theta^{2}\tau\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{m+p}\binom{\theta-1}{m}\binom{\theta-1}{p}\frac{e^{-\tau(m+1)j}}{m+p+2} \quad \textbf{(5.10)}$$

However,Srivastava et al (2006) did not consider the case for $j<0$ which is shown below.

Theorem 5.1

For $j < 0$ the cdf and pdf are given by:

$$F(j) = \theta \sum_{p=0}^{\infty} (-1)^p \, e^{-\tau p j} \, a_p(\theta) \, B_{e^{\tau j}}(p+1, \theta) \tag{5.11}$$

and

$$f(j) = \theta^2 \tau \, e^{-\tau j} \, \sum_{p=0}^{\infty} (-1)^p \, e^{-\tau p j} \, a_p(\theta - 1) \, B_{e^{\tau j}}(p+2, \theta) \tag{5.12}$$

Proof

Let
$V_2 > V_1 > 0$
then
$J = V_2 - V_1$
Therefore

$$\begin{aligned}
F(j) &= \text{Prob}\{J \leq j\} \\
&= \text{Prob}\{V_2 - V_1 \leq j\} \\
&= \text{Prob}\{V_2 \leq v_1 + j\} \\
&= \text{Prob}\{0 < V_1 < \infty, \quad V_2 \leq v_1 + j\} \\
&= \int_0^{\infty} \int_0^{v_1 + j} f(v_2) f(v_1) \, dv_2 \, dv_1 \\
&= \int_0^{\infty} F(v_1 + j) f(v_1) \, dv_1
\end{aligned}$$

$$F(j) = \int_0^{\infty} F(v + j) f(v) \, dv \tag{5.13}$$

$$\begin{aligned}
F(j) &= \int_{-j}^{\infty} F(v + j) f(v) \, dv \qquad (5.14) \\
&= \int_{-j}^{\infty} \left(1 - e^{-\tau(v+j)}\right)^{\theta} \, \theta \tau e^{-\tau v} \, \left(1 - e^{-\tau v}\right)^{\theta - 1} dv \\
&= \theta \tau \int_{-j}^{\infty} e^{-\tau v} \, \left(1 - e^{-\tau v}\right)^{\theta - 1} \, \left(1 - e^{-\tau(v+j)}\right)^{\theta} dv
\end{aligned}$$

Let $r=e^{-\tau v}\Rightarrow \dfrac{dr}{dv}=-\tau e^{-\tau v}\Rightarrow dv=\dfrac{dr}{-\tau e^{-\tau v}}=\dfrac{dr}{-\tau r}$

$$\therefore F(j)=\theta\tau\int_{e^{\tau j}}^{0} r(1-r)^{\theta-1}\,(1-re^{-\tau j})^{\theta}\;\frac{dr}{-\tau r}$$

$$=\theta\int_{0}^{e^{\tau j}}(1-r)^{\theta-1}\,(1-re^{-\tau j})^{\theta}\,dr$$

$$=\theta\int_{0}^{e^{\tau j}}(1-r)^{\theta-1}\,\sum_{p=0}^{\infty}(-1)^{p}\binom{\theta}{p}(re^{-\tau j})^{p}\,dr$$

$$F(j)=\theta\sum_{p=0}^{\infty}(-1)^{p}\binom{\theta}{p}e^{-\tau pj}\int_{0}^{e^{\tau j}} r^{(p+1)-1}(1-r)^{\theta-1}\;dr \tag{5.15}$$

Therefore $F(j)=\theta\sum_{p=0}^{\infty}(-1)^{p}\,e^{-\tau pj}\,a_{p}(\theta)\,B_{e^{\tau j}}(p+1,\theta)\qquad j<0$

Hence

$$f(j)=\int_{-j}^{\infty}f(v)f(v+j)\,dv \tag{5.16}$$

$$=\int_{-j}^{\infty}\theta\tau e^{-\tau v}\left(1-e^{-\tau v}\right)^{\theta-1}\,\theta\tau e^{-\tau(v+j)}\left(1-e^{-\tau(v+j)}\right)^{\theta-1}dv$$

$$=(\theta\tau)^{2}\,e^{-\tau j}\int_{-j}^{\infty}e^{-2\tau v}\left(1-e^{-\tau v}\right)^{\theta-1}\,\left(1-e^{-\tau(v+j)}\right)^{\theta-1}dv$$

Let $r=e^{-\tau v}\Rightarrow \dfrac{dr}{dv}=-\tau e^{-\tau v}\Rightarrow dv=\dfrac{dr}{-\tau e^{-\tau v}}=\dfrac{dr}{-\tau r}$

$$\therefore f(j)=(\theta\tau)^{2}\,e^{-\tau j}\int_{e^{\tau j}}^{0}r^{2}(1-r)^{\theta-1}\,\left(1-re^{-\tau j}\right)^{\theta-1}\,\left(\frac{dr}{-\tau r}\right)$$

$$=\theta^{2}\tau\,e^{-\tau j}\int_{0}^{e^{\tau j}}r(1-r)^{\theta-1}\,\left(1-re^{-\tau j}\right)^{\theta-1}dr$$

$$=\theta^{2}\tau\,e^{-\tau j}\int_{0}^{e^{\tau j}}r(1-r)^{\theta-1}\,\sum_{p=0}^{\infty}(-1)^{p}\binom{\theta-1}{p}(re^{-\tau j})^{p}\,dr$$

$$f(j)=\theta^{2}\tau\,e^{-\tau j}\sum_{p=0}^{\infty}(-1)^{p}\binom{\theta-1}{p}\,e^{-\tau pj}\int_{0}^{e^{\tau j}}r^{(p+2)-1}(1-r)^{\theta-1}\,dr \tag{5.17}$$

$$\therefore f(j)=\theta^{2}\tau\,e^{-\tau j}\sum_{p=0}^{\infty}(-1)^{p}\,e^{-\tau pj}\,a_{p}(\theta-1)\,B_{e^{\tau j}}(p+2,\theta)\qquad j<0$$

Alternatively,beginning from (5.15) we can express $F(j)$ as follows:

$$\begin{aligned}
F(j) &= \theta \sum_{p=0}^{\infty} (-1)^p \binom{\theta}{p} e^{-\tau p j} \int_0^{e^{\tau j}} r^{(p+1)-1}(1-r)^{\theta-1} \; dr \\
&= \theta \sum_{p=0}^{\infty} (-1)^p \binom{\theta}{p} e^{-\tau p j} \int_0^{e^{\tau j}} r^{(p+1)-1} \sum_{m=0}^{\infty} (-1)^m \binom{\theta-1}{m} r^m \, dr \\
&= \theta \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} (-1)^{m+p} \binom{\theta-1}{m} \binom{\theta}{p} e^{-\tau p j} \int_0^{e^{\tau j}} r^{m+p} \, dr \\
&= \theta \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} (-1)^{m+p} \binom{\theta-1}{m} \binom{\theta}{p} e^{-\tau p j} \frac{e^{\tau(m+p+1)j}}{m+p+1} \\
F(j) &= \theta \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} (-1)^{m+p} \binom{\theta-1}{m} \binom{\theta}{p} \frac{e^{\tau(m+1)j}}{m+p+1} \qquad \textbf{(5.18)}
\end{aligned}$$

Alternatively,beginning from (5.17) we can express $f(j)$ as follows:

$$\begin{aligned}
f(j) &= \theta^2 \tau \, e^{-\tau j} \sum_{p=0}^{\infty} (-1)^p \binom{\theta-1}{p} e^{-\tau p j} \int_0^{e^{\tau j}} r^{(p+2)-1}(1-r)^{\theta-1} \, dr \\
&= \theta^2 \tau \, e^{-\tau j} \sum_{p=0}^{\infty} (-1)^p \binom{\theta-1}{p} e^{-\tau p j} \int_0^{e^{\tau j}} r^{(p+2)-1} \sum_{m=0}^{\infty} (-1)^m \binom{\theta-1}{m} r^d \, dr \\
&= \theta^2 \tau \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} (-1)^{m+p} \binom{\theta-1}{m} \binom{\theta-1}{p} e^{-\tau(p+1)j} \int_0^{e^{\tau j}} r^{(m+p+1)} \, dr \\
&= \theta^2 \tau \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} (-1)^{m+p} \binom{\theta-1}{m} \binom{\theta-1}{p} e^{-\tau(p+1)j} \frac{e^{\tau(m+p+2)j}}{m+p+2} \\
f(j) &= \theta^2 \tau \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} (-1)^{m+p} \binom{\theta-1}{m} \binom{\theta-1}{p} \frac{e^{\tau(m+1)j}}{m+p+2} \qquad \textbf{(5.19)}
\end{aligned}$$

Moment Generating function

An exponentiated exponential r.v with parameters θ and τ has mgf:

$$
\begin{aligned}
M_H(x) &= E[e^{xH}] \\
M_H(x) &= \int_0^{\infty} e^{xh} \theta \tau e^{-\tau h} \Big(1 - e^{-\tau h}\Big)^{\theta - 1} \, dh \\
&= \theta \tau \int_0^{\infty} e^{xh} e^{-\tau h} \Big(1 - e^{-\tau h}\Big)^{\theta - 1} \, dh
\end{aligned}
$$

Let $\quad r = e^{-\tau h} \Rightarrow h = \dfrac{-log\ r}{\tau}$

$$
\begin{aligned}
\therefore \frac{dr}{dh} &= -\tau e^{-\tau h} \Rightarrow dh = \frac{dr}{-\tau e^{-\tau h}} = \frac{dr}{-\tau r} \\
\therefore M_H(x) &= \theta \int_0^1 e^{\frac{-x}{\tau} log\ r} (1-r)^{\theta - 1} \, dr \\
&= \theta \int_0^1 r^{-\frac{x}{\tau}} (1-r)^{\theta - 1} \, dr \\
&= \theta \int_0^1 r^{(1-\frac{x}{\tau})-1} (1-r)^{\theta - 1} \, dr \\
\therefore M_H(x) &= \theta \ B\Big(1 - \frac{x}{\tau}, \theta\Big)
\end{aligned}
$$

Now let $V = V_1 - V_2$

$$
\begin{aligned}
\therefore M_V(x) &= E[e^{(V_1 - V_2)x}] \\
&= E(e^{xV_1}) E(e^{-xV_2}) \\
&= M_H(x) M_H(-x) \\
&= \theta \ B\Big(1 - \frac{x}{\tau}, \theta\Big) \ \theta \ B\Big(1 + \frac{x}{\tau}, \theta\Big) \\
\therefore M_V(x) &= \theta^2 \ B\Big(1 - \frac{x}{\tau}, \theta\Big) \ B\Big(1 + \frac{x}{\tau}, \theta\Big) \qquad \textbf{(5.20)}
\end{aligned}
$$

5.2.1 The Special case when $\theta = 1$

From (5.4) we have

$$
\begin{aligned}
f(j) &= \theta^2 \tau \, e^{\tau j} \sum_{p=0}^{\infty} (-1)^p \binom{\theta-1}{p} e^{\tau p j} \int_0^{e^{-\tau j}} r^{(p+2)-1}(1-r)^{\theta-1}\, dr \\
&= \tau \, e^{\tau j} \sum_{p=0}^{\infty} (-1)^p \, e^{\tau p j} \binom{1-1}{p} e^{\tau p j} \int_0^{e^{-\tau j}} r^{(p+2)-1}(1-r)^{1-1}\, dr \\
&= \tau \, e^{\tau j} \int_0^{e^{-\tau j}} r \, dr \\
&= \tau \, e^{\tau j} \left(\frac{e^{-2\tau j}}{2} \right) \\
&= \frac{\tau}{2} \, e^{-\tau(j)}
\end{aligned}
$$

If $\tau = \frac{1}{\tau}$ then

$$f(j) = \frac{1}{2\tau} \, e^{-\frac{j}{\tau}} \text{ for } j \geq 0$$

which is Laplace pdf for $j \geq 0$ as given in (2.6)
Furthermore,from (5.12)

$$
\begin{aligned}
f(j) &= \theta^2 \tau \, e^{-\tau j} \sum_{p=0}^{\infty} (-1)^p \binom{\theta-1}{p} e^{-\tau p j} \int_0^{e^{\tau j}} r^{(p+2)-1}(1-r)^{\theta-1}\, dr \\
&= \tau \, e^{-\tau j} \sum_{p=0}^{\infty} (-1)^p \, e^{-\tau p j} \binom{1-1}{p} e^{-\tau p j} \int_0^{e^{\tau j}} r^{(p+2)-1}(1-r)^{1-1}\, dr \\
&= \tau \, e^{-\tau j} \int_0^{e^{\tau j}} r \, dr \\
&= \tau \, e^{-\tau j} \left(\frac{e^{2\tau j}}{2} \right) \\
&= \frac{\tau}{2} \, e^{\tau(j)}
\end{aligned}
$$

If $\tau = \frac{1}{\tau}$ then

$$f(j) = \frac{1}{2\tau} \, e^{\frac{j}{\tau}} \text{ for } j < 0$$

which is Laplace pdf for $j < 0$ as given in (2.6)

5.2.2 The Special case when $\theta = s$

From (5.3) and (5.4) for the case when $j \geq 0$ we have

$$F(j) = 1-\theta\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{m+p}\binom{\theta-1}{m}\binom{\theta}{p}\frac{e^{-\tau(m+1)j}}{m+p+1}$$

$$= 1-s\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{m+p}\binom{s-1}{m}\binom{s}{p}\frac{e^{-\tau(m+1)j}}{m+p+1}$$

$$F(j) = 1-s\sum_{m=0}^{s-1}\sum_{p=0}^{s}(-1)^{m+p}\binom{s-1}{m}\binom{s}{p}\frac{e^{-\tau(m+1)j}}{m+p+1} \qquad \textbf{(5.21)}$$

Then the pdf becomes

$$f(j) = \theta^2\tau\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{m+p}\binom{\theta-1}{m}\binom{\theta-1}{p}\frac{e^{-\tau(m+1)j}}{m+p+2}$$

$$f(j) = s^2\tau\sum_{m=0}^{s-1}\sum_{p=0}^{s-1}(-1)^{m+p}\binom{s-1}{m}\binom{s-1}{p}\frac{e^{-\tau(m+1)j}}{m+p+2} \qquad \textbf{(5.22)}$$

In addition,from (5.11) and (5.12) for the case when $j < 0$ we have

$$F(j) = \theta\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{m+p}\binom{\theta-1}{m}\binom{\theta}{p}\frac{e^{\tau(m+1)j}}{m+p+1}$$

$$F(j) = s\sum_{m=0}^{s-1}\sum_{p=0}^{s}(-1)^{m+p}\binom{s-1}{m}\binom{s}{p}\frac{e^{\tau(m+1)j}}{m+p+1} \qquad \textbf{(5.23)}$$

and

$$f(j) = \theta^2\tau\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{m+p}\binom{\theta-1}{m}\binom{\theta-1}{p}\frac{e^{\tau(m+1)j}}{m+p+2}$$

$$f(j) = s^2\tau\sum_{m=0}^{s-1}\sum_{p=0}^{s-1}(-1)^{m+p}\binom{s-1}{m}\binom{s-1}{p}\frac{e^{\tau(m+1)j}}{m+p+2} \qquad \textbf{(5.24)}$$

6 LAPLACE MIXTURES

6.1 Introduction

When two or more distributions are combined then we have a mixture.
In this chapter we construct Laplace mixtures using 16 mixing distributions which are: exponential,gamma,transmuted exponential,half-logistic,Lindley,generalized III Lindley,inverse gaussian,reciprocal inverse gaussian and generalized inverse gaussian.These are obtained based on modified bessel function of the third kind.
Mixing distributions obtained based on confluent hypergeometric series include:Beta I,Beta II,scaled beta,full beta,Pareto I,Pareto II and generalized Pareto.

6.2 Problem in Mathematical Form

The pdf of a continuous mixture is given by

$$f(m) = \int_p f(m/p)h(p)\, dp \tag{6.1}$$

whereby
$f(m/p)$ is the conditional distribution i.e Laplace density function:

$$f(m/p) = \frac{1}{2p}e^{-\frac{|m|}{p}} \qquad p > 0,\ -\infty < m < \infty.$$

i.e μ fixed at 0
$h(p)$ is the mixing distribution
We determine $f(m)$for above mentioned mixing distributions.

The rth moment

Will be obtained using the following theorem:

Theorem 6.1

$$E(M^r) = r!E(P^r) \tag{6.2}$$

where $E(P^r)$ is the rth moment of the mixing distribution.

Proof

$$
\begin{aligned}
E(M^r) &= EE(M^r/P=p) \\
\text{But} & \\
E(M^r/P=p) &= \int_{-\infty}^{\infty} m^r f(m/p)\, dm \\
\therefore E(M^r) &= \int_0^{\infty} \left\{ \int_{-\infty}^{\infty} m^r \frac{1}{2p} e^{-\frac{|m|}{p}}\, dm \right\} h(p)\, dp \\
&= \frac{1}{2} \int_0^{\infty} p^{-1} \left\{ \int_{-\infty}^{\infty} m^r e^{-\frac{|m|}{p}}\, dm \right\} h(p)\, dp
\end{aligned}
$$

For r odd the integral

$$
\int_{-\infty}^{\infty} m^r e^{-\frac{|m|}{p}}\, dm = 0
$$

while for r even

$$
\int_{-\infty}^{\infty} m^r e^{-\frac{|m|}{p}}\, dm = 2 \int_0^{\infty} m^r e^{-\frac{m}{p}}\, dm
$$

Therefore we have

$$
\begin{aligned}
E(M^r) &= \frac{1}{2} \int_0^{\infty} \lambda^{-1} \left\{ 2 \int_0^{\infty} m^r e^{-\frac{m}{p}}\, dm \right\} h(p)\, dp \\
&= \int_0^{\infty} p^{-1}\, \Gamma(r+1)\, p^{r+1}\, h(p)\, dp \\
&= \Gamma(r+1) \int_0^{\infty} p^r\, h(p)\, dp \\
&= \Gamma(r+1)\, E(P^r) \\
E(M^r) &= r!\, E(P^r)
\end{aligned}
$$

6.3 Mixtures in terms of Modified Bessel Function of the Third Kind

6.3.1 Laplace-Exponential Distribution

$$
h(p) = \theta e^{-\theta p} \qquad p > 0, \theta > 0 \tag{6.2}
$$

Hence the mixture becomes

$$\begin{aligned} f(m) &= \int_0^\infty \frac{1}{2p} e^-\frac{|m|}{p} \theta e^{-\theta p}\, dp \\ &= \theta\, \frac{1}{2} \int_0^\infty p^{-1} e^{-p\theta - \frac{|m|}{p}}\, dp \\ &= \theta\, \frac{1}{2} \int_0^\infty p^{-1} e^{-\theta\{p + \frac{|m|}{\theta}\}\frac{1}{p}}\, dp \end{aligned}$$

Let $p = \sqrt{\frac{|m|}{\theta}}\, j \Rightarrow dp = \sqrt{\frac{|m|}{\theta}} dj$

$$\begin{aligned} &= \theta\, \frac{1}{2} \int_0^\infty \left(\sqrt{\frac{|m|}{\theta}} j\right)^{0-1} \exp -\theta \left\{ \sqrt{\frac{|m|}{\theta}} j + \frac{|m|}{\theta}\} \frac{1}{\sqrt{\frac{|m|}{\theta}} j} \right\} \sqrt{\frac{|m|}{\theta}} dj \\ &= \theta\, \frac{1}{2} \int_0^\infty j^{0-1} e^{-\frac{2\sqrt{\theta|m|}}{2}(j+\frac{1}{j})}\, dj \\ &= \theta K_0(2\sqrt{\theta|m|}) \end{aligned}$$

$$\therefore f(m) = \theta K_0(2\sqrt{\theta|m|}) \tag{6.3}$$

The rth moment

$$E(M^r) = r! E(P^r)$$

and

$$\begin{aligned} E(P^r) &= \int_0^\infty p^r \theta e^{-\theta p}\, dp \\ &= \theta \int_0^\infty p^{(r+1)-1} e^{-\theta p}\, dp \\ &= \theta \frac{\Gamma(r+1)}{\theta^{r+1}} \\ &= \frac{r!}{\theta^r} \end{aligned}$$

$$\therefore E(M^r) = \frac{(r!)^2}{\theta^r} \tag{6.4}$$

6.3.2 Gamma Mixing Distribution

The distribution is constructed as follows:
Let $T = \theta p$ where R is a one parameter gamma distribution given by $h(t) = \frac{e^{-t} t^{\beta-1}}{\Gamma\beta}$

Using change of variable technique

$$h(p)=h(t)|J|, J=\frac{dt}{dp}=\theta$$
$$\therefore h(p)=\frac{e^{-t}t^{\beta-1}}{\Gamma\beta}\theta$$
$$=\frac{e^{-\theta p}(\theta p)^{\beta-1}}{\Gamma\beta}\theta$$
$$h(p)=\frac{\theta^{\beta}}{\Gamma\beta}e^{-\theta p}p^{\beta-1} \tag{6.5}$$

Therefore

$$f(m)=\int_0^\infty \frac{1}{2p}e^{-\frac{|m|}{p}}\frac{\theta^{\beta}}{\Gamma\beta}e^{-\theta p}p^{\beta-1}\,dp$$
$$=\frac{\theta^{\beta}}{\Gamma\beta}\frac{1}{2}\int_0^\infty p^{(\beta-1)-1}e^{-\theta p-\frac{|m|}{p}}\,dp$$
$$=\frac{\theta^{\beta}}{\Gamma\beta}\frac{1}{2}\int_0^\infty p^{\beta-1)-1}\exp-\theta\left\{p+\frac{|m|}{\theta}\frac{1}{p}\right\}dp$$

Let $p=\sqrt{\frac{|m|}{\theta}}j \Rightarrow dp=\sqrt{\frac{|m|}{\theta}}dj$

$$\therefore f(m)=\frac{\theta^{\beta}}{\Gamma\beta}\frac{1}{2}\int_0^\infty \left(\sqrt{\frac{|m|}{\theta}}j\right)^{(\beta-1)-1}\exp-\theta\left\{\sqrt{\frac{|m|}{\theta}}j+\frac{|m|}{\theta}\frac{1}{\sqrt{\frac{|m|}{\theta}}j}\right\}\sqrt{\frac{|m|}{\theta}}dj$$
$$=\frac{\theta^{\beta}}{\Gamma\beta}\left(\sqrt{\frac{|m|}{\theta}}\right)^{\beta-1}\frac{1}{2}\int_0^\infty j^{(\beta-1)-1}e^{-\frac{2\sqrt{\theta|m|}}{2}(j+\frac{1}{j})}\,dj$$
$$=\frac{\theta^{\beta}}{\Gamma\beta}\left(\sqrt{\frac{|m|}{\theta}}\right)^{\beta-1}K_{\beta-1}(2\sqrt{\theta|m|})$$
$$=\frac{\left(\sqrt{\theta|m|}\right)^{\beta}}{\Gamma\beta\left(\sqrt{\frac{|m|}{\theta}}\right)}K_{\beta-1}(2\sqrt{\theta|m|})$$

$$\therefore f(m)=\frac{\left(\sqrt{\theta|m|}\right)^{\beta}}{\Gamma\beta\left(\sqrt{\frac{|m|}{\theta}}\right)}K_{\beta-1}(2\sqrt{\theta|m|}) \tag{6.6}$$

The rth moment

$$E(M^r) = r!E(P^r)$$

and

$$\begin{aligned} E(P^r) &= \int_0^\infty p^r \frac{\theta^\beta}{\Gamma\beta} e^{-\theta p} p^{\beta-1}\, dp \\ &= \frac{\theta^\beta}{\Gamma\beta} \int_0^\infty p^{(\beta+r)-1} e^{-\theta p}\, dp \\ &= \frac{\theta^\beta}{\Gamma\beta} \frac{\Gamma(\beta+r)}{\theta^{\beta+r}} \\ &= \frac{\Gamma(\beta+r)}{\theta^r \Gamma\beta} \end{aligned}$$

$$\therefore E(M^r) = \frac{r!\,\Gamma(\beta+r)}{\theta^r\,\Gamma\beta} \tag{6.7}$$

6.3.3 Transmuted Exponential Mixing Distribution

Consider an exponential cdf given by

$$G(v) = 1 - e^{-\tau v} \quad v > 0;\ \tau > 0$$

Let

$$D(v) = (1+\beta)G(v) - \beta[G(v)]^2 \qquad -1 \le \beta \le 1 \tag{6.8}$$

$$\begin{aligned} \therefore d(v) &= (1+\beta)g(v) - 2\beta G(v)g(v) \\ d(v) &= (1+\beta)\tau e^{-\tau v} - 2\beta(1-e^{-\tau v})\tau e^{-\tau v} \\ &= \tau e^{-\tau v}\{(1+\beta) - 2\beta + 2\beta e^{-\tau v}\} \\ &= \tau e^{-\tau v}\{(1-\beta) + 2\beta e^{-\tau v}\} \end{aligned}$$

$$d(v) = (1-\beta)\tau e^{-\tau v} + 2\beta\tau e^{-2\tau v} \qquad v > 0;\ \tau > 0 \tag{6.9}$$

which is a transmuted exponential distribution (Bhati,et.al,2015) a finite mixture of two exponential distributions with parameters $(1-\beta), \tau$ and $2\beta, 2\tau$ respectively.
Therefore

$$h(p) = (1-\beta)\tau e^{-\tau p} + 2\beta\tau e^{-2\tau p} \qquad p > 0;\ \tau > 0,\ -1 \le \beta \le 1 \tag{6.10}$$

and hence

$$f(m) = \int_0^\infty \frac{1}{2p} e^{-\frac{|m|}{p}} \left[(1-\beta)\tau e^{-\tau p} + 2\beta\tau e^{-2\tau p} \right] dp$$
$$= \int_0^\infty \left[\frac{1}{2p} e^{-\frac{|m|}{p}} (1-\beta)\tau e^{-\tau p}\, dp + \frac{1}{2p} e^{-\frac{|m|}{p}} 2\beta\tau e^{-2\tau p}\, dp \right]$$

The first part of the RHS above

$$= \tau(1-\beta)\frac{1}{2}\int_0^\infty p^{-1} e^{-\frac{|m|}{p} - \tau p}\, dp$$
$$= \tau(1-\beta)\frac{1}{2}\int_0^\infty p^{-1} e^{-\tau\{p + \frac{|m|}{\tau}\frac{1}{p}\}}\, dp$$

$$\text{Let } p = \sqrt{\frac{|m|}{\tau}}\, j \Rightarrow dp = \sqrt{\frac{|m|}{\tau}}\, dj$$

then we have

$$\tau(1-\beta)\frac{1}{2}\int_0^\infty \left(\sqrt{\frac{|m|}{\tau}}\, j \right)^{-1} \exp -\tau \left\{ \sqrt{\frac{|m|}{\tau}}\, j + \frac{|m|}{\tau} \frac{1}{\sqrt{\frac{|m|}{\tau}}\, j} \right\} \sqrt{\frac{|m|}{\tau}}\, dj$$
$$= \tau(1-\beta)\frac{1}{2}\int_0^\infty j^{0-1} e^{-\frac{2\sqrt{\tau|m|}}{2}(j + \frac{1}{j})}\, dj$$
$$= \tau(1-\beta)\, K_0(2\sqrt{\tau|m|})$$

The second part

$$= 2\beta\tau\frac{1}{2}\int_0^\infty p^{-1} e^{-2\tau\{p + \frac{|m|}{2\tau}\frac{1}{p}\}}\, dp$$

Let

$$p = \sqrt{\frac{|m|}{2\tau}}\, j \Rightarrow dp = \sqrt{\frac{|m|}{2\tau}}\, dj$$

therefore we have

$$2\tau\beta\frac{1}{2}\int_0^\infty \left(\sqrt{\frac{|m|}{2\tau}}\, j \right)^{-1} \exp -2\tau \left\{ \sqrt{\frac{|m|}{2\tau}}\, j + \frac{|m|}{2\tau} \frac{1}{\sqrt{\frac{|m|}{2\tau}}\, j} \right\} \sqrt{\frac{|m|}{2\tau}}\, dj$$
$$= 2\tau\beta\frac{1}{2}\int_0^\infty j^{0-1} e^{-\frac{2\sqrt{2\theta|m|}}{2}(j + \frac{1}{j})}\, dj$$
$$= 2\tau\beta\, K_0(2\sqrt{2\tau|m|})$$

hence

$$f(m) = \tau(1-\beta)\, K_0(2\sqrt{\tau|m|}) + 2\tau\beta\, K_0(2\sqrt{2\tau|m|}) \tag{6.11}$$

The rth moment

$$E(M^r) = r!E(P^r)$$

and

$$\begin{aligned}
E(P^r) &= \int_0^\infty p^r \left[\tau(1-\beta)e^{-\tau p} + 2\beta\tau e^{-2\tau p}\right] dp \\
&= \tau(1-\beta)\int_0^\infty p^r e^{-\tau p}\, dp + 2\beta\tau \int_0^\infty p^r e^{-2\tau p}\, dp \\
&= \tau(1-\beta)\frac{\Gamma(r+1)}{\tau^{r+1}} + 2\tau\beta\frac{\Gamma(r+1)}{(2\tau)^{r+1}} \\
&= \frac{(1-\beta)\Gamma(r+1)}{\tau^r} + \frac{\beta\Gamma(r+1)}{(2\tau)^r} \\
&= \frac{2^r\, r!\,(1-\beta) + r!\,\beta}{(2\tau)^r} \\
&= \frac{r!}{\tau^r}
\end{aligned}$$

$$\therefore E(M^r) = \frac{(r!)^2}{\tau^r} \tag{6.12}$$

6.3.4 Half Logistic Mixing Distribution

The pdf of half logistic distribution is

$$h(p) = \frac{2\theta e^{-\theta p}}{(1+e^{-\theta p})^2} \qquad p > 0,\ \theta > 0 \tag{6.13}$$

The pdf of the mixture is

$$\begin{aligned}
f(m) &= \int_0^\infty \frac{1}{2p} e^{-\frac{|m|}{p}} \frac{2\theta e^{-\theta p}}{(1+e^{-\theta p})^2}\, dp \\
&= 2\theta\, \frac{1}{2}\int_0^\infty p^{-1} e^{-\theta p - \frac{|m|}{p}} (1+e^{-\theta p})^{-2}\, dp
\end{aligned}$$

Given $(d+e)^{-p}$we use a binomial expansion to get

$$\begin{aligned}(d+e)^{-p} &= \binom{-p}{0}d^{-p} + \binom{-p}{1}d^{-p-1}e^{1} + \binom{-p}{2}d^{-p-2}e^{2} + \cdots \\ &= \sum_{v=0}^{\infty}\binom{-p}{v}d^{-p-v}e^{v}\end{aligned}$$

hence

$$\begin{aligned}(1+e^{-\theta w})^{-2} &= \sum_{w=0}^{\infty}\binom{-2}{w}1^{-2-w}e^{w} \\ &= \sum_{w=0}^{\infty}\binom{-2}{w}e^{-\theta w}\end{aligned}$$

$$\begin{aligned}\therefore f(m) &= 2\theta\,\frac{1}{2}\int_{0}^{\infty}p^{-1}e^{-\theta p-\frac{|m|}{p}}\sum_{w=0}^{\infty}\binom{-2}{w}e^{-\theta pw}\,dp \\ &= 2\theta\sum_{w=0}^{\infty}\binom{-2}{w}\,\frac{1}{2}\int_{0}^{\infty}p^{-1}e^{-\theta p-\theta pw-\frac{|m|}{p}}\,dp \\ &= 2\theta\sum_{w=0}^{\infty}\binom{-2}{w}\,\frac{1}{2}\int_{0}^{\infty}p^{-1}\exp-\theta(w+1)\Big\{p+\frac{|m|}{\theta(w+1)}\frac{1}{p}\Big\}\,dp\end{aligned}$$

Let $p=\sqrt{\frac{|m|}{\theta(w+1)}}j \Rightarrow dp=\sqrt{\frac{|m|}{\theta(w+1)}}dj$

$$\begin{aligned}\therefore f(m) &= 2\theta\sum_{w=0}^{\infty}\binom{-2}{w}\,\frac{1}{2}\int_{0}^{\infty}\left(\sqrt{\frac{|m|}{\theta(w+1)}}j\right)^{0-1} e^{-\theta(w+1)\left\{\sqrt{\frac{|m|}{\theta(w+1)}}j+\frac{|m|}{\theta(w+1)}\frac{1}{\sqrt{\frac{|m|}{\theta(w+1)}}j}\right\}}\sqrt{\frac{|m|}{\theta(w+1)}}dj \\ &= 2\theta\sum_{w=0}^{\infty}\binom{-2}{w}\,\frac{1}{2}\int_{0}^{\infty}j^{0-1}e^{-\frac{2\sqrt{\theta(w+1)|m|}}{2}\left(j+\frac{1}{j}\right)}\,dj \\ &= 2\theta\sum_{w=0}^{\infty}\binom{-2}{w}\,K_{0}\Big(2\sqrt{\theta(w+1)|m|}\Big)\end{aligned}$$

$$\therefore f(m) = 2\theta\sum_{w=0}^{\infty}\binom{-2}{w}\,K_{0}\Big(2\sqrt{\theta(w+1)|m|}\Big) \qquad \textbf{(6.14)}$$

The rth moment

$$E(M^r) = r!E(P^r)$$

and

$$\begin{aligned}
E(P^r) &= \int_0^\infty p^r \frac{2\theta e^{-\theta p}}{(1+e^{-\theta p})^2}\,dp \\
&= 2\theta \sum_{w=0}^{\infty} \binom{-2}{w} \int_0^\infty p^{(r+1)-1} e^{-\theta(1+w)p}\,dp \\
&= 2\theta \sum_{w=0}^{\infty} \binom{-2}{w} \frac{\Gamma(r+1)}{[\theta(1+w)]^{r+1}} \\
&= \frac{2r!}{\theta^r} \sum_{w=0}^{\infty} \binom{-2}{w} \frac{1}{(1+a)^{r+1}} \\
\therefore E(M^r) &= \frac{2(r!)^2}{\theta^r} \sum_{w=0}^{\infty} \binom{-2}{w} \frac{1}{(1+w)^{r+1}} \qquad (6.15)
\end{aligned}$$

6.3.5 Lindley Mixing Distribution

The pdf of Lindley distribution is

$$h(p) = \frac{\tau^2}{\tau+1}(p+1)e^{-\tau p} \quad p>0; \quad \tau>0 \qquad (6.16)$$

Therefore

$$\begin{aligned}
f(m) &= \int_0^\infty \frac{1}{2p} e^{-\frac{|m|}{p}} \frac{\tau^2}{\tau+1}(p+1)e^{-\tau p}\,dp \\
&= \frac{\tau^2}{\tau+1}\,\frac{1}{2}\int_0^\infty p^{-1}(p+1)e^{-\tau p-\frac{|m|}{p}}\,dp \\
&= \frac{\tau^2}{\tau+1}\,\frac{1}{2}\int_0^\infty (p^{1-1}+p^{0-1})\exp -\tau\Big\{p+\frac{|m|}{\tau}\frac{1}{p}\Big\}\,dp \\
&= \frac{\tau^2}{\tau+1}\left[\frac{1}{2}\int_0^\infty p^{1-1}\exp -\tau\Big\{p+\frac{|m|}{\tau}\frac{1}{p}\Big\}\,dp + \frac{1}{2}\int_0^\infty p^{0-1}\exp -\tau\Big\{p+\frac{|m|}{\tau}\frac{1}{p}\Big\}\,dp\right]
\end{aligned}$$

Let $p = \sqrt{\frac{|m|}{\tau}}\,j \Rightarrow dp = \sqrt{\frac{|m|}{\tau}}\,dj$

Then $\frac{1}{2}\int_0^\infty p^{1-1}\exp-\tau\Big\{p+\frac{|m|}{\tau}\frac{1}{p}\Big\}\,dp$

$$= \frac{1}{2}\int_0^\infty \left(\sqrt{\frac{|m|}{\tau}}j\right)^{1-1}\exp-\tau\left\{\sqrt{\frac{|m|}{\tau}}j+\frac{|m|}{\tau}\frac{1}{\sqrt{\frac{|m|}{\tau}}l}\right\}\sqrt{\frac{|m|}{\tau}}\,dj$$

$$= \sqrt{\frac{|m|}{\tau}}\,\frac{1}{2}\int_0^\infty j^{1-1}e^{-\frac{2\sqrt{\tau|m|}}{2}}\big(j+\frac{1}{j}\big)\,dj$$

$$= \sqrt{\frac{|m|}{\tau}}\,K_1(2\sqrt{\tau|m|})$$

and $\frac{1}{2}\int_0^\infty p^{0-1}\exp-\tau\Big\{p+\frac{|m|}{\tau}\frac{1}{p}\Big\}\,dp$

$$= \frac{1}{2}\int_0^\infty \left(\sqrt{\frac{|m|}{\tau}}j\right)^{0-1}\exp-\tau\left\{\sqrt{\frac{|m|}{\tau}}j+\frac{|m|}{\tau}\frac{1}{\sqrt{\frac{|m|}{\tau}}j}\right\}\sqrt{\frac{|m|}{\tau}}\,dj$$

$$= \frac{1}{2}\int_0^\infty j^{0-1}e^{-\frac{2\sqrt{\tau|m|}}{2}}\big(j+\frac{1}{j}\big)\,dj$$

$$= K_0(2\sqrt{\tau|m|})$$

hence

$$f(m) = \frac{\tau^2}{\tau+1}\left[\sqrt{\frac{|m|}{\tau}}\,K_1(2\sqrt{\tau|m|})+K_0(2\sqrt{\tau|m|})\right] \qquad \textbf{(6.17)}$$

The rth moment

$$E(M^r) = r!E(P^r)$$

and

$$E(P^r) = \int_0^\infty p^r\frac{\tau^2}{\tau+1}(p+1)e^{-\tau p}\,dp$$

$$= \frac{\tau^2}{\tau+1}\Big\{\int_0^\infty p^{(r+2)-1}e^{-\tau p}\,dp+\int_0^\infty p^{(r+1)-1}e^{-\tau p}\,dp\Big\}$$

$$= \frac{\tau^2}{\tau+1}\Big\{\frac{\Gamma(r+2)}{\tau^{r+2}}+\frac{\Gamma(r+1)}{\tau^{r+1}}\Big\}$$

$$= \frac{1}{\tau+1}\Big\{\frac{r!(r+1)!+r!\tau}{\tau^r}\Big\}$$

$$\therefore E(M^r) = \frac{(r!)^2}{\tau^r(\tau+1)}\,\{\tau+(r+1)!\} \qquad \textbf{(6.18)}$$

6.3.6 Generalized III Parameter Lindley Mixing Distribution

Let
$h(p) = k_1 j_1(p) + k_2 j_2(p)$
be a finite mixture
now $k_1 + k_2 = 1\ k_1,\ k_2 > 0$
Further let $k_1 = \frac{\tau}{\tau+\omega}$ implying $k_2 = \frac{\omega}{\tau+\omega}$
As a result

$$h(p) = \frac{\tau}{\tau+\omega} j_1(p) + \frac{\omega}{\tau+\omega} j_2(p)$$

Let $j_1(p) \sim \text{Gamma}(\eta, \tau) = \frac{\tau^\eta}{\Gamma(\eta)} e^{-\tau p} p^{\eta-1}$and
$j_2(p) \sim \text{Gamma}(\eta+1, \tau) = \frac{\tau^{\eta+1}}{\Gamma(\eta+1)} e^{-\tau p} p^{\eta}$
then

$$\begin{aligned}
h(p) &= \frac{\tau}{\tau+\omega}\frac{\tau^\eta}{\Gamma(\eta)} e^{-\tau p} p^{\eta-1} + \frac{\omega}{\tau+\omega}\frac{\tau^{\eta+1}}{\Gamma(\eta+1)} e^{-\tau p} p^{\eta} \\
&= \frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta)} e^{-\tau p}\left\{p^{\eta-1} + \frac{\omega}{\eta} p^\eta\right\} \\
&= \frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta)} e^{-\tau p}\left\{\frac{\eta p^{\eta-1} + \omega p^\eta}{\eta}\right\} \\
&= \frac{\tau^{\eta+1}}{(\tau+\omega)\eta\Gamma(\eta)} e^{-\tau p}\left\{(\eta+\omega p) p^{\eta-1}\right\} \\
\therefore h(p) &= \frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)} (\eta+\omega p) p^{\eta-1} e^{-\tau p} \quad p>0;\ \eta,\ \omega,\ \tau>0 \qquad \textbf{(6.19)}
\end{aligned}$$

which is a generalized III parameter Lindley distribution.
The pdf of the mixture is therefore

$$\begin{aligned}
f(m) &= \int_0^\infty \frac{1}{2p} e^{-\frac{|m|}{p}} \frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)} (\eta+\omega p) p^{\eta-1} e^{-\tau p}\, dp \\
&= \frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)}\ \frac{1}{2}\int_0^\infty p^{\eta-2}(\eta+\omega p) e^{-\tau p - \frac{|m|}{p}}\, dp \\
&= \frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)}\ \left[\eta\ \frac{1}{2}\int_0^\infty p^{(\eta-1)-1} e^{-\tau\{p+\frac{|m|}{\tau}\frac{1}{p}\}}\, dp + \omega\ \frac{1}{2}\int_0^\infty p^{\eta-1} e^{-\tau\{p+\frac{|m|}{\tau}\frac{1}{p}\}}\, dp\right]
\end{aligned}$$

Let $p=\sqrt{\frac{|m|}{\tau}}j \Rightarrow dp=\sqrt{\frac{|m|}{\tau}}dj$

Then $\eta\,\frac{1}{2}\int_0^\infty p^{(\eta-1)-1}e^{-\tau\{p+\frac{|m|}{\tau}\frac{1}{p}\}}dp$

$$=\eta\,\frac{1}{2}\int_0^\infty\left(\sqrt{\frac{|m|}{\tau}}j\right)^{(\eta-1)-1}\exp-\tau\left\{\sqrt{\frac{|m|}{\tau}}j+\frac{|m|}{\tau}\frac{1}{\sqrt{\frac{|m|}{\tau}}j}\right\}\sqrt{\frac{|m|}{\tau}}dj$$

$$=\eta\left(\sqrt{\frac{|m|}{\tau}}\right)^{\eta-1}\frac{1}{2}\int_0^\infty j^{(\eta-1)-1}e^{-\frac{2\sqrt{\tau|m|}}{2}}(j+\frac{1}{j})dj$$

$$=\eta\left(\sqrt{\frac{|m|}{\tau}}\right)^{\eta-1}K_{\eta-1}(2\sqrt{\tau|m|})$$

and $\omega\,\frac{1}{2}\int_0^\infty p^{\eta-1}e^{-\tau\{p+\frac{|m|}{\tau}\frac{1}{p}\}}dp$

$$=\omega\,\frac{1}{2}\int_0^\infty\left(\sqrt{\frac{|m|}{\tau}}j\right)^{\eta-1}\exp-\tau\left\{\sqrt{\frac{|m|}{\tau}}j+\frac{|m|}{\tau}\frac{1}{\sqrt{\frac{|m|}{\tau}}j}\right\}\sqrt{\frac{|m|}{\tau}}dj$$

$$=\omega\left(\sqrt{\frac{|m|}{\tau}}\right)^{\eta}\frac{1}{2}\int_0^\infty j^{\eta-1}e^{-\frac{2\sqrt{\tau|m|}}{2}}(j+\frac{1}{j})dj$$

$$=\omega\left(\sqrt{\frac{|m|}{\tau}}\right)^{\eta}K_{\eta}(2\sqrt{\tau|m|})$$

hence

$$f(m)=\frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)}\left[\eta\left(\sqrt{\frac{|m|}{\tau}}\right)^{\eta-1}K_{\eta-1}(2\sqrt{\tau|m|})+\omega\left(\sqrt{\frac{|m|}{\tau}}\right)^{\eta}K_{\eta}(2\sqrt{\tau|m|})\right]$$

$$f(m)=\frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)}\left(\sqrt{\frac{|m|}{\tau}}\right)^{\eta-1}\left[\eta K_{\eta-1}(2\sqrt{\tau|m|})+\omega\sqrt{\frac{|m|}{\tau}}K_{\eta}(2\sqrt{\tau|m|})\right] \tag{6.20}$$

The rth moment

$$
\begin{aligned}
E(M^r) &= r!E(P^r) \\
\text{and} & \\
E(P^r) &= \int_0^\infty p^r \frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)}(\eta+\omega p)p^{\eta-1}e^{-\tau p}\,dp \\
&= \frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)}\left\{\eta\int_0^\infty p^{r+\eta-1}e^{-\tau p}\,dp + \omega\int_0^\infty p^{(r+\eta+1)-1}e^{-\tau p}\,dp\right\} \\
&= \frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)}\left\{\frac{\eta\Gamma(r+\eta)}{\tau^{r+\eta}} + \frac{\omega\,\Gamma(r+\eta+1)}{\tau^{r+\eta+1}}\right\} \\
&= \frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)}\left\{\frac{\eta\ \tau\Gamma(r+\eta)+\omega\,\Gamma(r+\eta+1)}{\tau^{r+\eta+1}}\right\} \\
&= \frac{1}{\tau^r\ (\tau+\omega)\Gamma(\eta+1)}\left\{\eta\ \tau\Gamma(r+\eta)+\omega\,\Gamma(r+\eta+1)\right\} \\
&= \frac{1}{\tau^r\ (\tau+\omega)\Gamma\eta}\left\{\tau\Gamma(\eta+r)+\omega\ \frac{\Gamma(r+\eta+1)}{\eta}\right\} \\
\therefore E(M^r) &= \frac{r!}{\tau^r\ (\tau+\omega)\Gamma\eta}\left\{\tau\Gamma(\eta+r)+\omega\ \frac{\Gamma(r+\eta+1)}{\eta}\right\} \qquad \textbf{(6.21)}
\end{aligned}
$$

Special Cases

(i)When $\eta = \omega = 1$

$$h(p) = \frac{\tau^2}{\tau+1}(p+1)e^{-\tau p} \quad p > 0;\ \tau > 0$$

which is a one parameter Lindley distribution and

$$f(m) = \frac{\tau^2}{\tau+1}\left[\sqrt{\frac{|m|}{\tau}}\,K_1(2\sqrt{\tau|m|}) + K_0(2\sqrt{\tau|m|})\right]$$

which is Laplace-Lindley distribution as obtained in 6.17.
(ii)When $\omega = 1$

$$h(p) = \frac{\tau^{\eta+1}}{(\tau+1)\Gamma(\eta+1)}(\eta+p)p^{\eta-1}e^{-\tau p} \quad p > 0;\ \omega\,,\tau > 0$$

which is Generalized II parameter Lindley distribution,as given by Zakerzadeh and Dollati (2010).
(iii) When $\eta = 1$

$$h(p) = \frac{\tau^2}{(\tau+\omega)}(1+\omega p)e^{-\tau p} \quad p > 0;\ \omega\,,\tau > 0$$

which is Generalized II parameter Lindley distribution as obtained by Bhati et al (2015).

6.3.7 Laplace-Inverse Gaussian Distribution

$$h(p)=\sqrt{\frac{\omega}{2\pi p^3}}e^{-\frac{\omega}{2\theta^2 p}(p-\theta)^2}$$
$$h(p)=\sqrt{\frac{\omega}{2\pi}}p^{-\frac{3}{2}}e^{\frac{\omega}{\theta}}e^{-\frac{\omega p}{2\theta^2}}e^{-\frac{\omega}{2p}} \qquad p>0,\ \omega>0\ -\infty<\theta<\infty \tag{6.22}$$

hence

$$\begin{aligned}
f(m)&=\int_0^\infty \frac{1}{2p}e^{-\frac{|m|}{p}}\sqrt{\frac{\omega}{2\pi}}\,p^{-\frac{3}{2}}e^{\frac{\omega}{\theta}}e^{-\frac{\omega p}{2\theta^2}}e^{-\frac{\omega}{2p}}\,dp\\
&=\sqrt{\frac{\omega}{2\pi}}\,e^{\frac{\omega}{\theta}}\,\frac{1}{2}\int_0^\infty p^{-\frac{3}{2}-1}\,e^{-\frac{\omega}{2\theta^2}\left\{p+\frac{\theta^2(\omega+2|m|)}{\omega}\,\frac{1}{p}\right\}}\,dp\\
&=\sqrt{\frac{\omega}{2\pi}}\,e^{\frac{\omega}{\theta}}\,\frac{1}{2}\int_0^\infty p^{-\frac{3}{2}-1}\,e^{-\frac{\omega}{2\theta^2}\left\{p+\frac{\theta^2(\omega+2|m|)}{\omega}\,\frac{1}{p}\right\}}\,dp
\end{aligned}$$

Let

$$\omega=\sqrt{\frac{\theta^2(\omega+2|m|)}{\omega}}\,j\Rightarrow dp=\sqrt{\frac{\theta^2(\omega+2|m|)}{\omega}}\,dj$$

$$\begin{aligned}
\therefore f(m)&=\sqrt{\frac{\omega}{2\pi}}\,e^{\frac{\omega}{\theta}}\frac{1}{2}\int_0^\infty\left(\sqrt{\frac{\theta^2(\omega+2|m|)}{\omega}}\,j\right)^{-\frac{3}{2}-1}\exp-\frac{\omega}{2\theta^2}\left\{\sqrt{\frac{\theta^2(\omega+2|m|)}{\omega}}\,j+\right.\\
&\qquad\left.\frac{\theta^2(\omega+2|m|)}{\omega}\,\frac{1}{\sqrt{\frac{\theta^2(\omega+2|m|)}{\omega}}\,j}\right\}\sqrt{\frac{\theta^2(\omega+2|m|)}{\omega}}\,dj\\
f(m)&=\sqrt{\frac{\omega}{2\pi}}\,e^{\frac{\omega}{\theta}}\left(\sqrt{\frac{\theta^2(\omega+2|m|)}{\omega}}\right)^{-\frac{3}{2}}\frac{1}{2}\int_0^\infty j^{-\frac{3}{2}-1}\,e^{-\frac{\sqrt{\frac{\omega(\omega+2|m|)}{\theta^2}}}{2}\left(j+\frac{1}{j}\right)}\,dj\\
\therefore f(m)&=\sqrt{\frac{\omega}{2\pi}}\,e^{\frac{\omega}{\theta}}\left(\sqrt{\frac{\theta^2(\omega+2|m|)}{\omega}}\right)^{-\frac{3}{2}}K_{\frac{3}{2}}\left(\sqrt{\frac{\omega(\omega+2|m|)}{\theta^2}}\right)
\end{aligned} \tag{6.23}$$

The rth moment

$$E(M^r) = r!E(P^r)$$

and

$$E(P^r) = \int_0^\infty p^r \sqrt{\frac{\omega}{2\pi}}\, p^{-\frac{3}{2}} e^{\frac{\omega}{\theta}} e^{-\frac{\omega p}{2\theta^2}} e^{-\frac{\omega}{2p}}\, dp$$
$$= \sqrt{\frac{\omega}{2\pi}} e^{\frac{\omega}{\theta}} \int_0^\infty p^{r-\frac{1}{2}-1} e^{-\frac{\omega}{2\theta^2}\{p+\frac{\theta^2}{p}\}}\, dp$$

Let $p = \sqrt{\theta^2} j \Rightarrow dp = \sqrt{\theta^2} dj$

$$\therefore E(P^r) = 2\sqrt{\frac{\omega}{2\pi}} e^{\frac{\omega}{\theta}} \,\frac{1}{2} \int_0^\infty (\theta j)^{r-\frac{1}{2}-1} e^{-\frac{\omega}{2\theta^2}\{\theta j + \frac{\theta^2}{\theta j}\}} \theta dj$$
$$= 2\theta^{r-\frac{1}{2}} \sqrt{\frac{\omega}{2\pi}} e^{\frac{\omega}{\theta}} \frac{1}{2} \int_0^\infty j^{r-\frac{1}{2}-1} e^{-\frac{\omega}{2\theta}\{j+\frac{1}{j}\}}\, dj$$
$$= \theta^{r-\frac{1}{2}} \sqrt{\frac{2\,\omega}{\pi}} e^{\frac{\omega}{\theta}}\, K_{r-\frac{1}{2}}(\frac{\omega}{\theta})$$
$$\therefore E(M^r) = r!\, \theta^{r-\frac{1}{2}} \sqrt{\frac{2\,\omega}{\pi}} e^{\frac{\omega}{\theta}}\, K_{r-\frac{1}{2}}(\frac{\omega}{\theta}) \tag{6.24}$$

6.3.8 Reciprocal Inverse Gaussian Distribution

Let $T = \frac{1}{p}$, T has an Inverse Gaussian distribution
then $h(p) = h(t)|J|, J = \frac{dt}{dp} = -\frac{1}{p^2}$

$$\therefore h(p) = \frac{1}{p^2} \sqrt{\frac{\omega}{2\pi}}\, t^{-\frac{3}{2}}\, e^{\frac{\omega}{\theta}}\, e^{-\frac{\omega t}{2\theta^2}}\, e^{-\frac{\omega}{2t}}$$

replacing t by $\frac{1}{p}$ we have

$$h(p) = \sqrt{\frac{\omega}{2\pi}}\, p^{-\frac{1}{2}}\, e^{\frac{\omega}{\theta}}\, e^{-\frac{\omega p}{2}}\, e^{-\frac{\omega}{2\theta^2 p}} \tag{6.25}$$

Therefore

$$
\begin{aligned}
f(m) &= \int_0^\infty \frac{1}{2p} e^{-\frac{|m|}{p}} \sqrt{\frac{\omega}{2\pi}}\, p^{-\frac{1}{2}}\, e^{\frac{\omega}{\theta}}\, e^{-\frac{\omega p}{2}}\, e^{-\frac{\omega}{2\theta^2 p}}\, dp \\
&= \sqrt{\frac{\omega}{2\pi}}\, e^{\frac{\omega}{\theta}}\, \frac{1}{2} \int_0^\infty p^{-\frac{1}{2}-1}\, e^{-\frac{\omega}{2}\left\{p + \frac{\omega+2\theta^2|m|}{\omega\theta^2}\ \frac{1}{p}\right\}}\, dp \\
&= \sqrt{\frac{\omega}{2\pi}}\, e^{\frac{\omega}{\theta}}\, \frac{1}{2} \int_0^\infty p^{-\frac{1}{2}-1}\, e^{-\frac{\omega}{2}\left\{p + \frac{\omega+2\theta^2|m|}{\omega\theta^2}\ \frac{1}{p}\right\}}\, dp
\end{aligned}
$$

Let

$$
p = \sqrt{\frac{\omega + 2\theta^2|m|}{\omega\theta^2}}\ j \Rightarrow dp = \sqrt{\frac{\omega + 2\theta^2|m|}{\omega\theta^2}}\ dj
$$

$$
\begin{aligned}
\therefore f(m) &= \sqrt{\frac{\omega}{2\pi}}\, e^{\frac{\omega}{\theta}} \frac{1}{2} \int_0^\infty \left(\sqrt{\frac{\omega+2\theta^2|m|}{\omega\theta^2}}\ j\right)^{-\frac{1}{2}-1} \exp - \frac{\omega}{2} \left\{ \sqrt{\frac{\omega+2\theta^2|m|}{\omega\theta^2}}\ j + \right. \\
&\quad \left. \frac{\omega+2\theta^2|m|}{\omega\theta^2}\ \frac{1}{\sqrt{\frac{\omega+2\theta^2|m|}{\omega\theta^2}}\ j} \right\} \sqrt{\frac{\omega+2\theta^2|m|}{\omega\theta^2}}\ dj \\
&= \sqrt{\frac{\omega}{2\pi}}\, e^{\frac{\omega}{\theta}} \left(\sqrt{\frac{\omega+2\theta^2|m|}{\omega\theta^2}}\right)^{-\frac{1}{2}} \frac{1}{2} \int_0^\infty j^{-\frac{1}{2}-1}\, e^{-\frac{\sqrt{\frac{\omega(\omega+2\theta^2|m|)}{\omega\theta^2}}}{2}\left(j+\frac{1}{j}\right)}\, dj \\
\therefore f(m) &= \sqrt{\frac{\omega}{2\pi}}\, e^{\frac{\omega}{\theta}} \left(\sqrt{\frac{\omega+2\theta^2|m|}{\omega\theta^2}}\right)^{-\frac{1}{2}} K_{\frac{1}{2}} \sqrt{\frac{\omega(\omega+2\theta^2|m|)}{\theta^2}} \qquad \textbf{(6.26)}
\end{aligned}
$$

The rth moment

$$
E(M^r) = r!E(P^r)
$$

and

$$
\begin{aligned}
E(P^r) &= \int_0^\infty p^r \sqrt{\frac{\omega}{2\pi}}\, p^{-\frac{1}{2}}\, e^{\frac{\omega}{\theta}}\, e^{-\frac{\omega p}{2}}\, e^{-\frac{\omega}{2\theta^2 p}}\, dp \\
&= \sqrt{\frac{\omega}{2\pi}} e^{\frac{\omega}{\theta}} \int_0^\infty p^{r-\frac{1}{2}} e^{-\frac{\omega}{2}\{p + \frac{1}{\theta^2 p}\}}\, dp
\end{aligned}
$$

Let $p=\sqrt{\frac{1}{\theta^2}}j \Rightarrow dp=\sqrt{\frac{1}{\theta^2}}dj$

$$\therefore E(P^r)=2\theta^{-r-\frac{1}{2}}\sqrt{\frac{\omega}{2\pi}}e^{\frac{\omega}{\theta}}\frac{1}{2}\int_0^{\infty} j^{r+\frac{1}{2}-1}e^{-\frac{\omega}{2\theta}\{j+\frac{1}{j}\}}\,dj$$

$$\therefore E(M^r)=r!\theta^{-r-\frac{1}{2}}\sqrt{\frac{2\,\omega}{\pi}}\;e^{\frac{\omega}{\theta}}\;K_{r+\frac{1}{2}}\left(\frac{\omega}{\theta}\right) \tag{6.27}$$

6.3.9 Laplace-Generalized Inverse Gaussian Distribution

Consider

$$K_y(\omega)=\frac{1}{2}\int_0^{\infty} t^{y-1}e^{-\frac{\omega}{2}(t+\frac{1}{t})}\,dt$$

Which implies

$$1=\frac{\int_0^{\infty} t^{y-1}e^{-\frac{\omega}{2}(t+\frac{1}{t})}}{2K_y(\omega)}$$

Let $\omega=\sqrt{\psi\phi}$

$$\therefore 1=\int_0^{\infty}\frac{t^{y-1}e^{-\frac{\sqrt{\psi\phi}}{2}(t+\frac{1}{t})}}{2K_y(\sqrt{\psi\phi})}dt$$

Let $t=\sqrt{\frac{\psi}{\phi}}j \Rightarrow dt=\sqrt{\frac{\psi}{\phi}}\mathrm{dj}$

$$1=\int_0^{\infty}\frac{\left(\sqrt{\frac{\psi}{\phi}}j\right)^{y-1}\exp-\frac{\sqrt{\psi\phi}}{2}\left(\sqrt{\frac{\psi}{\phi}}j+\frac{1}{\sqrt{\frac{\psi}{\phi}}j}\right)}{2K_y(\sqrt{\psi\phi})}\sqrt{\frac{\psi}{\phi}}j\,dj$$

$$1=\left(\sqrt{\frac{\psi}{\phi}}\right)^{y}\int_0^{\infty}\frac{j^{y-1}e^{-\frac{1}{2}(\psi j+\frac{\phi}{j})}}{2K_y(\sqrt{\psi\phi})}dj$$

hence

$$h(p)=\frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{y}p^{y-1}e^{-\frac{1}{2}(\psi p+\frac{\phi}{p})}}{2K_y(\sqrt{\psi\phi})} \tag{6.28}$$

Therefore,

$$f(m) = \int_0^\infty \frac{1}{2p} e^{-\frac{|m|}{p}} \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^y p^{y-1} e^{-\frac{1}{2}(\psi p + \frac{\phi}{p})}}{2K_y(\sqrt{\psi\phi})} dp$$
$$= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^y}{2K_y(\sqrt{\psi\phi})} \frac{1}{2} \int_0^\infty p^{(y-1)-1} e^{-\frac{1}{2}\psi p - \frac{\phi}{2} - \frac{|m|}{p}} dp$$
$$= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^y}{2K_y(\sqrt{\psi\phi})} \frac{1}{2} \int_0^\infty p^{(y-1)-1} e^{-\frac{\psi}{2}(p + \frac{\phi+2|m|}{\psi}\frac{1}{p})} d\lambda$$

Let $p = \sqrt{\frac{\phi+2|m|}{\psi}} j \Rightarrow dp = \sqrt{\frac{\phi+2|m|}{\psi}} dj$

$$\therefore f(m) = \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^y}{2K_y(\sqrt{\psi\phi})} \frac{1}{2} \int_0^\infty \left(\sqrt{\frac{\phi+2|m|}{\psi}}\, j\right)^{(y-1)-1} \exp - \frac{\psi}{2} \left\{ \sqrt{\frac{\phi+2|m|}{\psi}}\, j + \frac{\phi+2|m|}{\psi} \frac{1}{\sqrt{\frac{\phi+2|m|}{c\phi}}\, j} \right\} \sqrt{\frac{\phi+2|m|}{\phi}} dj$$
$$= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^y}{2K_y(\sqrt{\psi\phi})} \left(\sqrt{\frac{\phi+2|m|}{\psi}}\right)^{y-1} \frac{1}{2} \int_0^\infty j^{(y-1)-1} e^{-\frac{\sqrt{\psi(\phi+2|m|)}}{2}} \left(j + \frac{1}{j}\right) dj$$
$$= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^y}{2K_y(\sqrt{\psi\phi})} \left(\sqrt{\frac{\phi+2|m|}{\psi}}\right)^{y-1} K_{y-1}\left(\sqrt{\psi(\phi+2|m|)}\right)$$

Hence

$$f(m) = \frac{\sqrt{\psi}\left(\sqrt{\phi+2|m|}\right)^{y-1}}{\left(\sqrt{\phi}\right)^y} \frac{K_{y-1}\left(\sqrt{\psi(\phi+2|m|)}\right)}{2K_y(\sqrt{\psi\phi})} \quad \textbf{(6.29)}$$

The rth moment

$$E(M^r) = r!E(P^r)$$

and

$$\begin{aligned}E(P^r) &= \int_0^\infty p^r \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^y p^{y-1} e^{-\frac{1}{2}(\psi p + \frac{\phi}{p})}}{2K_y(\sqrt{\psi\phi})}\, dp \\ &= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^y}{K_y(\sqrt{\psi\phi})} \frac{1}{2}\int_0^\infty p^{y+r-1} e^{-\frac{\psi}{2}\{p + \frac{\phi}{\psi p}\}}\, dp\end{aligned}$$

Let $p = \sqrt{\frac{\phi}{\psi}} j \Rightarrow dp = \sqrt{\frac{\phi}{\psi}} dj$

$$\begin{aligned}\therefore E(P^r) &= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^y}{K_y(\sqrt{\psi\phi})} \frac{1}{2}\int_0^\infty \left(\sqrt{\frac{\phi}{\psi}} j\right)^{y+r-1} \exp -\frac{\psi}{2}\left\{\sqrt{\frac{\phi}{\psi}} j + \frac{\phi}{\psi\sqrt{\frac{\phi}{\psi}} j}\right\} \sqrt{\frac{\phi}{\psi}} dj \\ &= \left(\sqrt{\frac{\psi}{\phi}}\right)^y \left(\sqrt{\frac{\phi}{\psi}}\right)^{y+r} \frac{K_{y+r}(\sqrt{\psi\phi})}{K_y(\sqrt{\psi\phi})} \\ &= \left(\sqrt{\frac{\phi}{\psi}}\right)^r \frac{K_{y+r}(\sqrt{\psi\phi})}{K_y(\sqrt{\psi\phi})}\end{aligned}$$

$$\therefore E(M^r) = r!\left(\sqrt{\frac{\phi}{\psi}}\right)^r \frac{K_{y+r}(\sqrt{\psi\phi})}{K_y(\sqrt{\psi\phi})} \tag{6.30}$$

6.4 Mixtures in terms of Confluent Hypergeometric Series

6.4.1 Laplace-Beta I Mixing Distribution

$$h(p) = \frac{p^{u-1}(1-p)^{v-1}}{B(u,v)} \qquad 0 < p < 1, \quad u,v > 0 \tag{6.31}$$

Hence

$$\begin{aligned}f(m) &= \int_0^1 \frac{1}{2\lambda} e^{-\frac{|m|}{p}} \frac{p^{u-1}(1-p)^{v-1}}{B(u,v)}\, dp \\ &= \frac{1}{2B(u,v)}\int_0^1 p^{(u-1)-1}(1-p)^{v-1} e^{-\frac{|m|}{p}}\, dp \\ &= \frac{1}{2B(u,v)}\int_0^1 \left(\frac{1}{p}\right)^{2-u}\left(\frac{1}{1-p}\right)^{1-v} e^{-\frac{|m|}{p}}\, dp\end{aligned}$$

If $q=\frac{1}{p}\Rightarrow p=\frac{1}{q}\Rightarrow dp=-\frac{dq}{q^2}$

$$\begin{aligned}
\therefore f(m) &= \frac{1}{2B(u,v)}\int_{\infty}^{1} q^{2-u}\left(\frac{1}{1-\frac{1}{q}}\right)^{1-v} e^{-|m|q}\left(-\frac{dq}{q^2}\right)\\
&= \frac{1}{2B(u,v)}\int_{1}^{\infty}\frac{1}{q^u}\left(\frac{q}{q-1}\right)^{1-v} e^{-|m|q}\,dq\\
&= \frac{1}{2B(u,v)}\int_{1}^{\infty} q^{1-v-u}(q-1)^{v-1}e^{-|m|q}\,dq
\end{aligned}$$

Let $j=q-1\Rightarrow q=1+j\Rightarrow dj=dq$

$$\begin{aligned}
f(m) &= \frac{1}{2B(u,v)}\int_{0}^{\infty}(1+j)^{1-v-u}\; j^{v-1}e^{-|m|(1+j)}\,dj\\
&= \frac{e^{-|m|}}{2B(u,v)}\int_{0}^{\infty} j^{v-1}(1+j)^{2-u-v-1}e^{-|m|j}\,dj\\
&= \frac{e^{-|m|}}{2B(u,v)}\Gamma v\int_{0}^{\infty}\frac{j^{v-1}}{\Gamma v}(1+j)^{2-u-v-1}e^{-|m|j}\,dj\\
\therefore f(m) &= \frac{e^{-|m|}}{2B(u,v)}\,\Gamma v\,\Psi(v;\,2-u;\,|m|) \qquad \textbf{(6.32)}
\end{aligned}$$

The rth moment

$$E(M^r)=r!E(P^r)$$

and

$$\begin{aligned}
E(P^r) &= \int_{0}^{1} p^r\,\frac{p^{u-1}(1-p)^{v-1}}{B(u,v)}\,dp\\
&= \frac{1}{B(u,v)}\int_{0}^{1} p^{u+r-1}(1-p)^{v-1}\,dp\\
&= \frac{B(u+r,v)}{B(u,v)}\\
\therefore E(M^r) &= r!\frac{B(u+r,v)}{B(u,v)} \qquad \textbf{(6.33)}
\end{aligned}$$

6.4.2 Beta II Mixing Distribution

Consider M which is Beta distributed with $f(m) = \frac{m^{u-1}(1-m)^{v-1}}{B(u,v)}$.If $P = \frac{M}{1-M}$

$$
\begin{aligned}
\text{Then } h(p) &= f(m)|J| \text{ where } J = \frac{dm}{dp} \\
p =& \frac{m}{1-m} \Rightarrow m = \frac{p}{1+p} \\
\Rightarrow \frac{dm}{dp} &= \frac{1}{(1+p)^2} \\
\text{Hence} \quad h(p) &= \frac{m^{u-1}(1-m)^{v-1}}{B(u,v)} \left| \frac{1}{(1+p)} \right|^2 \\
&= \frac{\left(\frac{p}{1+p}\right)^{u-1}\left(1-\frac{p}{1+p}\right)^{v-1}}{B(u,v)} \left| \frac{1}{(1+p)} \right|^2 \\
&= \frac{p^{u-1}}{(1+p)^{u+v-2}B(u,v)(1+p)^2}
\end{aligned}
$$

Therefore

$$
h(p) = \frac{p^{u-1}}{B(u,v)(1+p)^{u+v}} \qquad p > 0, \quad u,v > 0 \tag{6.34}
$$

The pdf of the mixture is thus

$$
\begin{aligned}
f(m) &= \int_0^\infty \frac{1}{2p} e^{-\frac{|m|}{p}} \frac{p^{u-1}}{B(u,v)(1+p)^{u+v}} \, dp \\
&= \frac{1}{2B(u+v)} \int_0^\infty p^{(u-1)-1}(1+p)^{-u-v} e^{-\frac{|m|}{p}} \, dp
\end{aligned}
$$

$$
\text{Let } p = \frac{1}{j} \Rightarrow dp = \left(-\frac{dj}{j^2}\right)
$$

$$
\begin{aligned}
&= \frac{1}{2B(u+v)} \int_\infty^0 \left(\frac{1}{j}\right)^{(u-1)-1} \left(1+\frac{1}{j}\right)^{-u-v} e^{-|m|j} \left(-\frac{dj}{j^2}\right) \\
&= \frac{1}{2B(u+v)} \int_0^\infty \frac{1}{j^u} \left(\frac{j}{1+j}\right)^{u+v} e^{-|m|j} \, dj
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2B(u+v)} \int_0^\infty j^v (1+j)^{-u-v} e^{-|m|j}\, dj \\
&= \frac{1}{2B(u+v)} \int_0^\infty j^{(v+1)-1} (1+j)^{2-u-v-1-1} e^{-|x|z}\, dz \\
&= \frac{\Gamma(v+1)}{2B(u+v)} \int_0^\infty \frac{j^{(v+1)-1}}{\Gamma(v+1)} (1+j)^{2-u-(v+1)-1} e^{-|m|j}\, dj \\
\therefore f(m) &= \frac{\Gamma(v+1)}{2B(u+v)}\, \Psi(v+1;\, 2-u;\, |m|)
\end{aligned}
\tag{6.35}
$$

The rth moment

$$
\begin{aligned}
E(M^r) &= r! E(P^r) \\
\text{and} & \\
E(P^r) &= \int_0^\infty p^r \frac{p^{u-1}}{B(u,v)(1+p)^{u+v}}\, dp \\
&= \frac{1}{B(u,v)} \int_0^\infty \frac{p^{u+r-1}}{(1+p)^{u+v}}\, dp \\
&= \frac{1}{B(u,v)} \int_0^\infty \frac{p^{u+r-1}}{(1+p)^{u+r+v-r}}\, dp \\
\therefore E(M^r) &= r! \frac{B(u+r, v-r)}{B(u,v)}
\end{aligned}
\tag{6.36}
$$

6.4.3 Scaled Beta Mixing Distribution

It is constructed from a Beta distribution as follows.
Let $p = \theta M$,M has a beta distribution given by
$f(m) = \frac{m^{u-1}(1-m)^{v-1}}{B(u,v)} \qquad 0 < m < 1 \quad u,\ v > 0$
then $h(p)$ the pdf of p is obtained as
$h(p) = f(m)|J|, \quad J = \frac{dm}{dp}$
Now $p = \theta m \Rightarrow m = \frac{p}{\theta} \Rightarrow \frac{dm}{dp} = \frac{1}{\theta}$

$$
\begin{aligned}
\therefore h(p) &= \frac{m^{u-1}(1-m)^{v-1}}{B(u,v)} |\frac{1}{\theta}| \\
&= \frac{(\frac{p}{\theta})^{u-1}(1-\frac{p}{\theta})^{v-1}}{\theta\, B(u,v)} \\
h(p) &= \frac{p^{u-1}(\theta-p)^{v-1}}{\theta^{u+v-1}\, B(u,v)} \qquad 0 < p < \theta,\ u,\ v > 0
\end{aligned}
\tag{6.37}
$$

$$\therefore f(m) = \int_0^{\theta} \frac{1}{2p} e^{-\dfrac{|m|}{p}} \frac{p^{u-1}(\theta - p)^{v-1}}{\theta^{u+v-1}\, B(u,v)}\, dp$$
$$= \frac{1}{2\theta^{u+v-1}\, B(u,v)} \int_0^{\infty} p^{(u-1)-1} (\theta - p)^{v-1} e^{-\dfrac{|m|}{p}}\, dp$$

If $q = \frac{1}{p} \Rightarrow p = \frac{1}{q} \Rightarrow dp = -\frac{dq}{q^2}$

$$\therefore f(m) = \frac{1}{2\theta^{u+v-1}\, B(u,v)} \int_{\infty}^{\frac{1}{\theta}} \left(\frac{1}{q}\right)^{(u-1)-1} (\theta - \frac{1}{q})^{v-1} e^{-|m|q} \left(-\frac{dq}{q^2}\right)$$
$$= \frac{1}{2\theta^{u+v-1}\, B(u,v)} \int_{\frac{1}{\theta}}^{\infty} \left(\frac{1}{q}\right)^{u+v-1} (\theta q - 1)^{v-1} e^{-|m|q}\, dq$$

Let $j = \theta q - 1 \Rightarrow q = \frac{1+j}{\theta} \Rightarrow dq = \frac{dj}{\theta}$

$$\therefore f(m) = \frac{1}{2\theta^{u+v-1} B(u,v)} \int_0^{\infty} \left(\frac{1+j}{\theta}\right)^{-u-v+1} j^{v-1} e^{-|m|(\frac{1+j}{\theta})} \frac{dj}{\theta}$$

$$= \frac{1}{2\theta^{u+v-1} B(u,v)} \theta^{u+v-1-1} e^{-\frac{|m|}{\theta}} \int_0^{\infty} j^{v-1} (1+j)^{2-u-v-1} e^{-\frac{|m|}{\theta} j}\, dj$$
$$= \frac{1}{2\theta B(u,v)} e^{-\frac{|m|}{\theta}} \Gamma v \int_0^{\infty} \frac{j^{v-1}}{\Gamma v} (1+j)^{2-u-v-1} e^{-\frac{|m|}{\theta} j}\, dj$$

Therefore

$$f(m) = \frac{e^{-\frac{|m|}{\theta}}\, \Gamma v}{2\theta B(u,v)}\, \Psi\left(v; 2-u; \frac{|m|}{\theta}\right) \tag{6.38}$$

The rth moment

$$E(M^r) = r!E(P^r)$$

and

$$E(P^r) = \int_0^\theta p^r \frac{p^{u-1}(\theta-p)^{v-1}}{\theta^{u+v-1}\,B(u,v)}\,dp$$

$$= \frac{1}{\theta^{u+v-1}\,B(u,v)}\int_0^\theta p^{u+r-1}(\theta-p)^{v-1}\,dp$$

Let $p = \theta j \Rightarrow dp = \theta dj$

$$\therefore E(P^r) = \frac{\theta^{u+r+v-1}}{\theta^{u+v-1}\,B(u,v)}\int_0^1 j^{u+r-1}(1-j)^{v-1}\,dj$$

$$= \theta^r \frac{B(u+r,v)}{B(u,v)}$$

$$\therefore E(M^r) = r!\theta^r \frac{B(u+r,v)}{B(u,v)} \tag{6.39}$$

6.4.4 Full Beta Mixing Distribution

Full Beta distribution is obtained by mixing two gamma distributions.It is constructed as follows:

Let

$$f(p/j) = \frac{j^u}{\Gamma u}e^{-jp}p^{u-1} \qquad p>0, \quad u,\ j\ >0$$

and

$$r(j) = \frac{1}{y^v\Gamma v}e^{-\frac{j}{y}}j^{v-1} \qquad j>0, \quad v,\ y\ >0$$

The pdf of the mixture is

$$
\begin{aligned}
h(p) &= \int_0^\infty f(p/j)r(j)\,dj \\
&= \int_0^\infty \frac{j^u}{\Gamma u} e^{-jp} p^{u-1} \frac{1}{y^v \Gamma v} e^{-\frac{j}{y}} j^{v-1}\,dj \\
&= \frac{p^{u-1}}{\Gamma u\,\Gamma v\, y^v} \int_0^\infty j^{u+v-1} e^{-\frac{j}{y}-jp}\,dj \\
&= \frac{p^{u-1}}{\Gamma u\,\Gamma v\, y^v} \int_0^\infty j^{u+v-1} e^{-(\frac{1}{y}+p)j}\,dj \\
&= \frac{p^{u-1}}{\Gamma u\,\Gamma v\, y^v}\, \frac{\Gamma(u+v)}{\left(\frac{1}{y}+p\right)^{u+v}} \\
\therefore h(p) &= \frac{y^u}{B(u,v)}\, \frac{p^{u-1}}{(1+yp)^{u+v}} \qquad \textbf{(6.40)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f(m) &= \int_0^\infty \frac{1}{2p} e^{-\frac{|m|}{p}} \frac{y^u}{B(u,v)}\, \frac{p^{u-1}}{(1+yp)^{u+v}}\,dp \\
&= \frac{y^u}{2B(u,v)} \int_0^\infty p^{(u-1)-1}(1+yp)^{-u-v} e^{-\frac{|m|}{p}}\,dp
\end{aligned}
$$

If $p = \frac{1}{jy} \Rightarrow dp = \left(-\frac{dj}{j^2 y}\right)$

$$
\begin{aligned}
\therefore f(m) &= \frac{y^u}{2B(u,v)} \int_\infty^0 \left(\frac{1}{jy}\right)^{(u-1)-1} \left(1+\frac{y}{jy}\right)^{-u-v} e^{-|m|jy} \left(-\frac{dj}{j^2 y}\right) \\
&= \frac{y^u}{2B(u,v)y^{u-1}} \int_0^\infty \left(\frac{1}{j}\right)^{(u-1)-1} \left(1+\frac{1}{j}\right)^{-u-v} e^{-(|m|yj)}\, \frac{dj}{j^2} \\
&= \frac{y}{2B(u,v)} \int_0^\infty \left(\frac{1}{j}\right)^{(u-1)-1} \left(\frac{1+j}{j}\right)^{-u-v} e^{-(|m|yj)}\, \frac{dj}{j^2} \\
&= \frac{y}{2B(u,v)} \int_0^\infty j^{-u} \left(\frac{j}{1+j}\right)^{u+v} e^{-(|m|yj)}\,dj \\
&= \frac{y\,\Gamma(v+1)}{2B(u,v)} \int_0^\infty \frac{j^{(v+1)-1}(1+j)^{2-u-(v+1)-1} e^{-(|m|yj)}}{\Gamma(v+1)}\,dj \\
&= \frac{y\,\Gamma(v+1)}{2B(u,v)}\, \Psi(v+1;2-u;|m|y) \\
\therefore f(m) &= \frac{y\,\Gamma(v+1)}{2B(u,v)}\, \Psi(v+1;2-u;|m|y) \qquad \textbf{(6.41)}
\end{aligned}
$$

The rth moment

$$
\begin{aligned}
E(M^r) &= r!E(P^r) \\
\text{and} & \\
E(P^r) &= \int_0^\infty p^r \frac{y^u}{B(u,v)} \frac{p^{u-1}}{(1+yp)^{u+v}} \, dp \\
&= \frac{y^u}{B(u,v)} \int_0^\infty \frac{p^{u+r-1}}{(1+yp)^{u+v}} \, dp \\
\text{Let} \quad & j = yp \Rightarrow \frac{dj}{dp} = y \\
\therefore E(P^r) &= \frac{y^u}{B(u,v)} \int_0^\infty \frac{(\frac{j}{y})^{u+r-1}}{(1+j)^{u+v}} \frac{dj}{y} \\
&= \frac{1}{B(u,v)\, y^r} \int_0^\infty \frac{j^{u+r-1}}{(1+j)^{u+r+v-r}} \, dj \\
\therefore E(M^r) &= r! \frac{B(u+r, v-r)}{y^r B(u,v)}
\end{aligned}
\tag{6.42}
$$

6.4.5 Laplace-Pareto I Distribution

The pdf of Pareto I is:

$$
h(p) = \frac{\omega \tau^\omega}{p^{\omega+1}} \qquad p > \tau; \omega,\ \tau > 0 \tag{6.43}
$$

Hence

$$
\begin{aligned}
f(m) &= \int_\tau^\infty \frac{1}{2p} e^{-\frac{|m|}{p}} \frac{\omega \tau^\omega}{p^{\omega+1}} \, dp \\
&= \frac{\omega \tau^\omega}{2} \int_\tau^\infty p^{-1-\omega-1} e^{-\frac{|m|}{p}} \, dp
\end{aligned}
$$

If $q=\frac{1}{p} \Rightarrow p=\frac{1}{q} \Rightarrow dp=-\frac{dq}{q^2}$

$$
\begin{aligned}
\therefore f(m) &= \frac{\omega\tau^{\omega}}{2}\int_{\frac{1}{\tau}}^{0}\Big(\frac{1}{q}\Big)^{-\omega-1-1} e^{-|m|q}\Big(-\frac{dq}{q^2}\Big) \\
&= \frac{\omega\tau^{\omega}}{2}\int_{0}^{\frac{1}{\tau}} q^{\omega+1-1}\, e^{-|m|q}\, dq \\
&= \frac{\omega\tau^{\omega}}{2}\gamma\Big(\omega+1,\frac{1}{\tau}\Big) \\
&= \frac{\omega\tau^{\omega}}{2}(\omega+1)^{-1}\Big(\frac{1}{\tau}\Big)^{\omega+1} e^{-\frac{1}{\tau}}\ {}_1F_1\Big(1;\omega+2;\frac{1}{\tau}\Big) \\
\therefore f(m) &= \frac{\omega}{2\tau(\omega+1)} e^{-\frac{1}{\tau}}\ {}_1F_1\Big(1;\omega+2;\frac{1}{\tau}\Big) \qquad \textbf{(6.44)}
\end{aligned}
$$

The rth moment

$$E(M^r) = r!E(P^r)$$

and

$$
\begin{aligned}
E(P^r) &= \int_{\beta}^{\infty} p^r \frac{\omega\tau^{\omega}}{p^{\omega+1}}\, dp \\
&= \omega\tau^{\omega}\int_{\tau}^{\infty} p^{r-\omega-1}\, dp \\
&= \omega\tau^{\omega}\left[\frac{1}{(r-\omega)p^{\omega-r}}\Bigg|_{\tau}^{\infty}\right] \\
&= \frac{v\ \tau^r}{(\omega-r)} \\
\therefore E(M^r) &= r!\frac{\omega\ \tau^r}{(\omega-r)} \qquad \textbf{(6.45)}
\end{aligned}
$$

6.4.6 Pareto II (Lomax) Mixing Distribution

Let $f(\lambda/v) = ve^{-v\lambda} \quad \lambda>0\ v>0$
and $h(v)=\frac{\beta^{\alpha}}{\Gamma\alpha}e^{-\beta v}v^{\alpha-1}$
then

$$\begin{aligned}
g(\lambda) &= \int_0^\infty f(\lambda/v)h(v)\,dv \\
&= \int_0^\infty ve^{-v\lambda}\frac{\beta^\alpha}{\Gamma\alpha}e^{-\beta v}v^{\alpha-1}\,dv \\
&= \frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty v^{(\alpha+1)-1}e^{-(\beta+\lambda)v}\,dv \\
&= \frac{\beta^\alpha}{\Gamma\alpha}\frac{\Gamma(\alpha+1)}{(\beta+\lambda)^{\alpha+1}} \\
g(\lambda) &= \frac{\alpha\beta^\alpha}{(\beta+\lambda)^{\alpha+1}}
\end{aligned}$$

Therefore by replacing $\alpha=\omega, \beta=\tau, \lambda=p$ we have

$$h(p) = \frac{\omega\tau^\omega}{(\tau+p)^{\omega+1}} \tag{6.46}$$

which is Lomax density function.
Hence

$$\begin{aligned}
f(m) &= \int_0^\infty \frac{1}{2p}e^{-\frac{|m|}{p}}\frac{\omega\tau^\omega}{(\tau+p)^{\omega+1}}\,dp \\
&= \frac{\omega\tau^\omega}{2}\int_0^\infty p^{-1}(\tau+p)^{-\omega-1}e^{-\frac{|m|}{p}}\,dp
\end{aligned}$$

Let $t=\frac{1}{p}\Rightarrow p=\frac{1}{t}\Rightarrow dp=-\frac{dt}{t^2}$

$$\begin{aligned}
\therefore f(m) &= \frac{\omega\tau^\omega}{2}\int_\infty^0\left(\frac{1}{t}\right)^{-1}\left(\tau+\frac{1}{t}\right)^{-\omega-1}e^{-|m|t}\left(-\frac{dt}{t^2}\right) \\
&= \frac{\omega\tau^\omega}{2}\int_0^\infty t^\omega(1+\tau t)^{-\omega-1}e^{-|m|t}\,dt
\end{aligned}$$

Let $z=\tau t\Rightarrow t=\frac{z}{\tau}\Rightarrow = dt=\frac{dz}{\tau}$

$$\begin{aligned}
\therefore f(m) &= \frac{\omega\tau^\omega}{2}\int_0^\infty\left(\frac{z}{\tau}\right)^{(\omega+1)-1}(1+z)^{-\omega-1}e^{-\frac{|m|}{\tau}z}\frac{dz}{\tau} \\
&= \frac{\omega\tau^\omega}{2\tau^{(\omega+2)-1}}\int_0^\infty z^{(\omega+1)-1}(1+z)^{1-(\omega+1)-1}e^{-\frac{|m|}{\tau}z}\,dz \\
&= \frac{\omega\Gamma(\omega+1)}{2\tau}\int_0^\infty\frac{z^{(\omega+1)-1}}{\Gamma(\omega+1)}(1+z)^{1-(\omega+1)-1}e^{-\frac{|m|}{\tau}z}\,dz
\end{aligned}$$

Therefore
$$f(m) = \frac{\omega\Gamma(\omega+1)}{2\tau}\Psi\Big(\omega+1;1;\frac{|m|}{\tau}\Big) \tag{6.47}$$

The rth moment

$$E(M^r) = r!E(P^r)$$
and
$$E(P^r) = \int_0^\infty p^r \frac{\omega\tau^\omega}{(\tau+p)^{\omega+1}}\,dp$$

Let $z = \frac{p}{\tau} \Rightarrow dz = \frac{1}{\tau}dp$

$$\begin{aligned}\therefore E(P^r) &= \omega\tau^\omega \int_0^\infty \frac{(\tau z)^r}{(1+z)^{\omega+1}}\,\tau dz \\ &= \omega\tau^{1+\omega+r}B(1+r,\omega-r) \\ \therefore E(M^r) &= r!\omega\tau^{\omega+r+1}B(r+1,\omega-r)\end{aligned} \tag{6.48}$$

6.4.7 Generalized Pareto Mixing Distribution

It is constructed from a mixture of gamma distributions. Let $f(\lambda/v) = \frac{v^\alpha}{\Gamma\alpha}e^{-v\lambda}\lambda^{\alpha-1}$ and
$h(v) = \frac{\theta^\beta}{\Gamma\beta}e^{-\theta v}v^{\beta-1}$
hence

$$\begin{aligned}g(\lambda) &= \int_0^\infty \frac{v^\alpha}{\Gamma\alpha}e^{-v\lambda}\lambda^{\alpha-1}\frac{\theta^\beta}{\Gamma\beta}e^{-\theta v}v^{\beta-1}\,dv \\ &= \frac{\lambda^{\alpha-1}\theta^\beta}{\Gamma\alpha\Gamma\beta}\int_0^\infty v^{(\alpha+\beta)-1}e^{-(\theta+\lambda)v}\,dv \\ &= \frac{\lambda^{\alpha-1}\theta^\beta}{\Gamma\alpha\Gamma\beta}\,\frac{(\Gamma\alpha+\beta)}{(\theta+\lambda)^{\alpha+\beta}} \\ g(\lambda) &= \frac{\theta^\beta\lambda^{\alpha-1}}{B(\alpha,\beta)\,(\theta+\lambda)^{\alpha+\beta}} \quad \lambda>0;\ \alpha,\,\beta,\,\theta>0\end{aligned}$$

If we replace $\theta=\eta, \beta=\tau, \lambda=p, \alpha=\omega$ we have

$$h(p) = \frac{\eta^\tau p^{\omega-1}}{B(\omega,\tau)\,(\eta+p)^{\omega+\tau}} \quad p>0;\ \omega,\,\tau,\,\eta>0 \tag{6.49}$$

Therefore

$$
\begin{aligned}
f(m) &= \int_0^\infty \frac{1}{2p} e^{-\frac{|m|}{p}} \frac{\eta^\tau p^{\omega-1}}{B(\omega,\tau)\,(\eta+p)^{\omega+\tau}}\,dp \\
&= \frac{\eta^\tau}{2B(\omega,\tau)} \int_0^\infty p^{\omega-1-1}(\eta+p)^{-\omega-\tau} e^{-\frac{|m|}{p}}\,dp
\end{aligned}
$$

Let $t = \frac{1}{p} \Rightarrow p = \frac{1}{t} \Rightarrow dp = -\frac{dt}{t^2}$

$$
\begin{aligned}
\therefore f(m) &= \frac{\eta^\tau}{2B(\omega,\tau)} \int_\infty^0 \Big(\frac{1}{t}\Big)^{\omega-1-1} \Big(\eta+\frac{1}{t}\Big)^{-\omega-\tau} e^{-|m|t} \Big(-\frac{dt}{t^2}\Big) \\
&= \frac{\eta^\tau}{2B(\omega,\tau)} \int_0^\infty t^\tau (1+\eta t)^{-\omega-\tau} e^{-|m|t}\,dt
\end{aligned}
$$

Let $z = \eta t \Rightarrow t = \frac{z}{\eta} \Rightarrow dt = \frac{dz}{\eta}$

$$
\begin{aligned}
\therefore f(m) &= \frac{\eta^\tau}{2B(\omega,\tau)} \int_0^\infty \Big(\frac{z}{\eta}\Big)^\tau (1+z)^{-\omega-\tau} e^{-\frac{|m|}{\eta}z}\,\frac{dz}{\eta} \\
&= \frac{\eta^\tau}{2B(\omega,\tau)\eta^{\tau+1}} \int_0^\infty z^{(\tau+1)-1}(1+z)^{-\omega+2-(\tau+1)-1} e^{-\frac{|m|}{\eta}z}\,\frac{dz}{\eta} \\
&= \frac{\Gamma(\tau+1)}{\eta\, B(\omega,\tau)} \int_0^\infty \frac{z^{(\tau+1)-1}}{\Gamma(\tau+1)}(1+z)^{-\omega+2-(\tau+1)-1} e^{-\frac{|m|}{\eta}z}\,dz
\end{aligned}
$$

Therefore

$$
f(m) = \frac{\Gamma(\tau+1)}{2\eta\, B(\omega,\tau)} \Psi\Big(\tau+1; 2-\omega; \frac{|m|}{\eta}\Big) \tag{6.50}
$$

The rth moment

$$
E(M^r) = r!E(P^r)
$$

and

$$
\begin{aligned}
E(P^r) &= \int_0^\infty p^r \frac{\eta^\tau p^{\omega-1}}{B(\omega,\tau)\,(\eta+p)^{\omega+\tau}}\,dp \\
&= \frac{\eta^\tau}{B(\omega,\tau)} \int_0^\infty \frac{p^{\omega+r-1}}{(\eta+p)^{\omega+\tau}}\,dp
\end{aligned}
$$

Let $z = \frac{p}{\eta} \Rightarrow dz = \frac{1}{\eta} dp$

$$
\begin{aligned}
E(Pr) &= \frac{\eta^{\tau}}{B(\omega,\tau)} \int_0^{\infty} \frac{(\eta z)^{\omega+r-1}}{(1+z)^{\omega+\tau}} \eta dz \\
&= \frac{\eta^{\omega+\tau+r}}{B(\omega,\tau)} B(\omega+r,\tau-r)
\end{aligned}
$$

Therefore

$$
E(M^r) = r! \frac{\eta^{\omega+\tau+r}}{B(\omega,\tau)} B(\omega+r,\tau-r) \tag{6.51}
$$

7 MIXTURES OF EXPONENTIAL POWER DISTRIBUTION

7.1 Introduction

In this chapter we will construct mixtures of EP distribution in terms of Modified Bessel function of the third kind and in terms of confluent hypergeometric function using 16 mixing distributions.
The mixing distributions derived in terms of modified Bessel function of the third kind are:exponential,gamma,transmuted exponential,half-logistic,Lindley,generalized III Lindley,inverse gaussian,reciprocal inverse gaussian and generalized inverse gaussian.
Mixing distributions obtained in terms of confluent hypergeometric series include:Beta I,Beta II,scaled beta,full beta,Pareto I,Pareto II and generalized Pareto.
Furthermore,we will also show that Laplace and Gaussian distributions are special cases that arise from EP distribution when $r = 1$ and $r = 2$ respectively.

7.2 Problem in Mathematical form

A random variable $M \in R$ follows an EP distribution with parameters μ, r, η and the density is given by

$$f(m) = \frac{\eta^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma(\frac{r+1}{r})} e^{-\frac{1}{2\eta}|m-\mu|^r} \quad -\infty < m < \infty; \quad \eta > 0,\ -\infty < \mu < \infty \tag{7.1}$$

Taking $2\eta = p$ and fixing $\mu = 0$ we have:

$$f(m/p) = \frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma(\frac{r+1}{r})} e^{-\frac{1}{p}|m|^r} \quad p > 0 \tag{7.2}$$

The problem is to find $f(m)$ given by

$$f(m) = \int_p f(m/p)h(p)\,dp$$

where $f(m/p)$= EP distribution
$h(p)$=mixing distribution.

7.3 Mixtures in terms of Modified Bessel Function of the third kind

7.3.1 EP-Exponential Distribution

$h(p) = \theta e^{-\theta p} \qquad p > 0,\ \theta > 0$
Hence

$$f(m) = \int_0^\infty \frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)} e^{-\frac{|m|^r}{p}} \theta e^{-\theta p}\, dp$$
$$= \frac{\theta}{2^{-\frac{1}{r}} 2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)} \int_0^\infty p^{-\frac{1}{r}} e^{-\theta p - \frac{|m|^r}{p}}\, dp$$
$$= \frac{\theta}{\Gamma\left(\frac{r+1}{r}\right)} \frac{1}{2} \int_0^\infty p^{-\frac{1}{r}} e^{-\theta\left\{p + \frac{|m|^r}{\theta}\frac{1}{p}\right\}}\, dp$$
$$= \frac{\theta}{\Gamma\left(\frac{r+1}{r}\right)} \frac{1}{2} \int_0^\infty p^{-\frac{1}{r}} e^{-\theta\left\{p + \frac{|m|^r}{\theta}\frac{1}{p}\right\}}\, dp$$

Let $p = \sqrt{\frac{|m|^r}{\theta}}\, j \Rightarrow dp = \sqrt{\frac{|m|^r}{\theta}}\, dj$

$$\therefore f(m) = \frac{\theta}{\Gamma\left(\frac{r+1}{r}\right)} \frac{1}{2} \int_0^\infty \left(\sqrt{\frac{|m|^r}{\theta}}\, j\right)^{1-\frac{1}{r}-1} \exp -\theta\left\{\sqrt{\frac{|m|^r}{\theta}}\, j + \frac{|m|^r}{\theta} \frac{1}{\sqrt{\frac{|m|^r}{\theta}}\, j}\right\} \sqrt{\frac{|m|^r}{\theta}}\, dj$$
$$= \frac{\theta}{\Gamma\left(\frac{r+1}{r}\right)} \left(\sqrt{\frac{|m|^r}{\theta}}\right)^{1-\frac{1}{r}} \frac{1}{2} \int_0^\infty j^{1-\frac{1}{p}-1}\, e^{-\frac{2\sqrt{\theta|m|^r}}{2}\left(j+\frac{1}{j}\right)}\, dj$$
$$= \frac{\theta}{\Gamma\left(\frac{r+1}{r}\right)} \left(\sqrt{\frac{|m|^r}{\theta}}\right)^{1-\frac{1}{r}} K_{1-\frac{1}{r}}\left(2\sqrt{\theta|m|^r}\right)$$
$$\therefore f(m) = \frac{\theta}{\Gamma\left(\frac{r+1}{r}\right)} \left(\sqrt{\frac{|m|^r}{\theta}}\right)^{1-\frac{1}{r}} K_{1-\frac{1}{r}}\left(2\sqrt{\theta|m|^r}\right) \tag{7.3}$$

Special Cases

(i)When $r = 1$

$$f(m) = \theta\, K_0(2\sqrt{\theta|m|}) \tag{7.4}$$

which is Laplace exponential distribution obtained in (6.3).
(ii)When $r = 2$

$$f(m) = \frac{\theta}{\Gamma(\frac{3}{2})}\left(\sqrt{\frac{|m|^2}{\theta}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(2\sqrt{\theta|m|^2}\right)$$

Now $\Gamma(\frac{3}{2}) = \Gamma(\frac{1}{2}+1) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$

$$\begin{aligned}
\therefore f(m) &= \frac{2\theta}{\sqrt{\pi}}\left(\sqrt{\frac{|m|^2}{\theta}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(2\sqrt{\theta|m|^2}\right) \\
&= \frac{2\theta}{\sqrt{\pi}}\left(\frac{m}{\sqrt{\theta}}\right)^{\frac{1}{2}} \sqrt{\frac{\pi}{2\left(2\sqrt{\theta|m|^2}\right)}}\, e^{-2\sqrt{\theta|m|^2}} \\
&= \frac{2\theta\sqrt{m}}{(\sqrt{\theta})^{\frac{1}{2}}}\,\frac{1}{\sqrt{4m}(\sqrt{\theta})^{\frac{1}{2}}} e^{-2\sqrt{\theta|m|^2}} \\
&= \sqrt{\theta} e^{-m\,2\sqrt{\theta}}
\end{aligned}$$

Let $\frac{\beta}{2} = \theta$
then

$$f(m) = \sqrt{\frac{\beta}{2}}\, e^{-m\sqrt{2\beta}} \tag{7.5}$$

which is normal exponential distribution as given by Odhiambo (2016).

7.3.2 Gamma Mixing Distribution

$$h(p) = \frac{\theta^{\beta}}{\Gamma\beta} e^{-\theta p} p^{\beta-1}$$

Therefore

$$\begin{aligned}
f(m) &= \int_0^\infty \frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)} e^{-\frac{|m|^r}{p}} \frac{\theta^\beta}{\Gamma\beta} e^{-\theta p} p^{\beta-1}\, dp \\
&= \frac{\theta^\beta}{2^{-\frac{1}{r}} 2^{\frac{r+1}{r}} \Gamma\beta\, \Gamma\left(\frac{r+1}{r}\right)} \int_0^\infty p^{(\beta-\frac{1}{r})-1} e^{-\theta p - \frac{|m|^r}{p}}\, dp \\
&= \frac{\theta^\beta}{\Gamma\beta\, \Gamma\left(\frac{r+1}{r}\right)} \frac{1}{2} \int_0^\infty p^{(\beta-\frac{1}{r})-1} e^{-\theta\left\{p + \frac{|m|^r}{\theta}\frac{1}{p}\right\}}\, dp
\end{aligned}$$

Let $p = \sqrt{\frac{|m|^r}{\theta}}\, j \Rightarrow dp = \sqrt{\frac{|m|^r}{\theta}}\, dj$

$$\begin{aligned}
\therefore f(m) &= \frac{\theta^\beta}{\Gamma\beta\, \Gamma\left(\frac{r+1}{r}\right)} \frac{1}{2} \int_0^\infty \left(\sqrt{\frac{|m|^r}{\theta}}\, j\right)^{(\beta-\frac{1}{r})-1} \exp -\theta\left\{\sqrt{\frac{|m|^r}{\theta}}\, j + \frac{|m|^r}{\theta} \frac{1}{\sqrt{\frac{|m|^r}{\theta}}\, j}\right\} \sqrt{\frac{|m|^r}{\theta}}\, dj \\
&= \frac{\theta^\beta}{\Gamma\beta\, \Gamma\left(\frac{r+1}{r}\right)} \left(\sqrt{\frac{|m|^r}{\theta}}\right)^{\beta-\frac{1}{r}} \frac{1}{2} \int_0^\infty j^{(\beta-\frac{1}{r})-1}\, e^{-\frac{2\sqrt{\theta|m|^r}}{2}\left(j+\frac{1}{j}\right)}\, dj \\
&= \frac{\theta^\beta}{\Gamma\beta\, \Gamma\left(\frac{r+1}{r}\right)} \left(\sqrt{\frac{|m|^r}{\theta}}\right)^{\beta-\frac{1}{r}} K_{\beta-\frac{1}{r}}\left(2\sqrt{\theta|m|^r}\right)
\end{aligned}$$

$$\therefore f(m) = \frac{\theta^\beta}{\Gamma\beta\, \Gamma\left(\frac{r+1}{r}\right)} \left(\sqrt{\frac{|m|^r}{\theta}}\right)^{\beta-\frac{1}{r}} K_{\beta-\frac{1}{r}}\left(2\sqrt{\theta|m|^r}\right) \tag{7.6}$$

Special Cases

(i)When $r = 1$

$$f(m) = \frac{\left(\sqrt{\theta|m|}\right)^\beta}{\Gamma\beta\left(\sqrt{\frac{|m|}{\theta}}\right)} K_{\beta-1}\left(2\sqrt{\theta|m|}\right) \tag{7.7}$$

which is Laplace-Gamma Distribution as obtained in (6.6).

(ii)When $r = 2$

$$f(m) = \frac{\theta^{\beta}}{\Gamma\beta\,\Gamma\left(\frac{3}{2}\right)}\left(\sqrt{\frac{|m|^2}{\theta}}\right)^{\beta-\frac{1}{2}} K_{\beta-\frac{1}{2}}\left(2\sqrt{\theta|m|^2}\right)$$

$$= \frac{2\theta^{\beta}}{\sqrt{\pi}\,\Gamma\beta}\left(\sqrt{\frac{|m|^2}{\theta}}\right)^{\beta-\frac{1}{2}} K_{\beta-\frac{1}{2}}\left(2\sqrt{\theta|m|^2}\right) \qquad \textbf{(7.8)}$$

7.3.3 Transmuted Exponential Mixing Distribution

$$h(p) = (1-\beta)\tau e^{-\tau p} + 2\beta\tau e^{-2\tau p} \qquad p > 0;\ \tau > 0,\ -1 \leq \beta \leq 1$$

Therefore

$$f(m) = \int_0^{\infty} \frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)} e^{-\frac{|m|^r}{p}}\left[(1-\beta)\tau e^{-\tau p} + 2\beta\tau e^{-2\tau p}\right] dp$$

$$= \frac{1}{\Gamma\left(\frac{r+1}{r}\right)}\left[\tau(1-\beta)\frac{1}{2}\int_0^{\infty} p^{-\frac{1}{r}} e^{-\tau p - \frac{|m|^r}{p}}\, dp + \right.$$

$$\left. 2\tau\beta\frac{1}{2}\int_0^{\infty} p^{-\frac{1}{r}} e^{-2\tau p - \frac{|m|^r}{p}}\, dp\right]$$

From

$$\tau(1-\beta)\frac{1}{2}\int_0^{\infty} p^{-\frac{1}{r}} e^{-\tau\{p - \frac{|m|^r}{\tau}\frac{1}{p}\}}\, dp$$

Let

$$p = \sqrt{\frac{|m|^r}{\tau}}\, j \Rightarrow dp = \sqrt{\frac{|m|^r}{\tau}} dj$$

We then have

$$= \tau(1-\beta)\left(\sqrt{\frac{|m|^r}{\tau}}\right)^{1-\frac{1}{r}} \frac{1}{2}\int_0^{\infty} j^{1-\frac{1}{r}-1} e^{-\frac{2\sqrt{\tau|m|^r}}{2}(j+\frac{1}{j})}\, dj$$

$$= \tau(1-\beta)\left(\sqrt{\frac{|m|^r}{\tau}}\right)^{1-\frac{1}{r}} K_{1-\frac{1}{r}}(2\sqrt{\tau|m|^r})$$

and from

$$2\tau\beta\frac{1}{2}\int_0^\infty p^{-\frac{1}{r}}e^{-2\tau\{p-\frac{|m|^r}{2\tau}\frac{1}{p}\}}\,dp$$

We let

$$p=\sqrt{\frac{|m|^r}{2\tau}}j \Rightarrow dp=\sqrt{\frac{|m|^r}{2\tau}}dj$$

We then have

$$=2\tau\beta\left(\sqrt{\frac{|m|^r}{2\tau}}\right)^{1-\frac{1}{r}}\frac{1}{2}\int_0^\infty j^{1-\frac{1}{r}-1}e^{-\frac{2\sqrt{2\tau|m|^r}}{2}(j+\frac{1}{j})}\,dj$$
$$=2\tau\alpha\left(\sqrt{\frac{|m|^r}{2\tau}}\right)^{1-\frac{1}{r}}K_{1-\frac{1}{r}}(2\sqrt{2\tau|m|^r})$$

Therefore

$$f(m)=\frac{1}{\Gamma\left(\frac{r+1}{r}\right)}\left[\tau(1-\beta)\left(\sqrt{\frac{|m|^r}{\tau}}\right)^{1-\frac{1}{r}}K_{1-\frac{1}{r}}(2\sqrt{\tau|m|^r})+2\tau\beta\left(\sqrt{\frac{|m|^r}{2\tau}}\right)^{1-\frac{1}{r}}K_{1-\frac{1}{r}}(2\sqrt{2\tau|m|^r})\right] \quad \textbf{(7.9)}$$

Special Cases

(i) When r=1

$$f(m)=\tau(1-\beta)\,K_0(2\sqrt{\tau|m|})+2\tau\beta\,K_0(2\sqrt{2\tau|m|}) \quad \textbf{(7.10)}$$

which is Laplace-transmuted exponential distribution obtained in (6.11).
(ii) When r=2

$$f(m)=\frac{2}{\sqrt{\pi}}\left[\tau(1-\beta)\left(\sqrt{\frac{|m|^2}{\tau}}\right)^{1-\frac{1}{2}}K_{1-\frac{1}{2}}(2\sqrt{\tau|m|^2})+2\tau\beta\left(\sqrt{\frac{|m|^2}{2\tau}}\right)^{1-\frac{1}{2}}K_{1-\frac{1}{2}}(2\sqrt{2\tau|m|^2})\right] \quad \textbf{(7.11)}$$

7.3.4 Half Logistic Mixing Distribution

$$h(p)=\frac{2\theta e^{-\theta p}}{(1+e^{-\theta p})^2} \qquad p>0;\ \theta>0$$

Therefore

$$f(m)=\int_0^\infty \frac{\left(\frac{r}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)} \exp\{-\frac{|m|^r}{p}\}\frac{2\theta e^{-\theta p}}{(1+e^{-\theta p})^2}\,dp$$

$$=\frac{2\theta}{2^{-\frac{1}{r}}2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)}\int_0^\infty \frac{p^{-\frac{1}{r}}e^{-\theta p-\frac{|m|^r}{p}}}{(1+e^{-\theta p})^2}\,dp$$

$$=\frac{2\theta}{\Gamma\left(\frac{r+1}{r}\right)}\frac{1}{2}\int_0^\infty p^{-\frac{1}{r}}e^{-\theta p-\frac{|m|^r}{p}}\sum_{a=0}^{\infty}\binom{-2}{a}e^{-\theta pa}\,dp$$

$$=\frac{2\theta}{\Gamma\left(\frac{r+1}{r}\right)}\sum_{a=0}^{\infty}\binom{-2}{a}\frac{1}{2}\int_0^\infty p^{-\frac{1}{r}}e^{-\theta pa-\theta p-\frac{|m|^r}{p}}\,dp$$

$$=\frac{2\theta}{\Gamma\left(\frac{r+1}{r}\right)}\sum_{a=0}^{\infty}\binom{-2}{a}\frac{1}{2}\int_0^\infty p^{(1-\frac{1}{r})-1}e^{-\theta(a+1)\left\{p+\frac{|m|^r}{\theta(a+1)}\frac{1}{p}\right\}}\,dp$$

Let

$$p=\sqrt{\frac{|m|^r}{\theta(a+1)}}\,j\Rightarrow dp=\sqrt{\frac{|m|^r}{\theta(a+1)}}\,dj$$

$$\therefore f(m)=\frac{2\theta}{\Gamma\left(\frac{r+1}{r}\right)}\sum_{a=0}^{\infty}\binom{-2}{a}\frac{1}{2}\int_0^\infty\left(\sqrt{\frac{|m|^r}{\theta(a+1)}}\,j\right)^{(1-\frac{1}{r})-1}\exp(-\theta(a+1))\left\{\sqrt{\frac{|m|^r}{\theta(a+1)}}\,j+\frac{|m|^r}{\theta(a+1)}\frac{1}{\sqrt{\frac{|m|^r}{\theta(a+1)}}\,j}\right\}\sqrt{\frac{|m|^r}{\theta(a+1)}}\,dj$$

$$=\frac{2\theta}{\Gamma\left(\frac{r+1}{r}\right)}\sum_{a=0}^{\infty}\binom{-2}{a}\left(\sqrt{\frac{|m|^r}{\theta(a+1)}}\right)^{1-\frac{1}{r}}\frac{1}{2}\int_0^\infty j^{(1-\frac{1}{r})-1}\,e^{-\frac{2\sqrt{\theta(a+1)|m|^r}}{2}\left(j+\frac{1}{j}\right)}\,dj$$

$$=\frac{2\theta}{\Gamma\left(\frac{r+1}{r}\right)}\sum_{a=0}^{\infty}\binom{-2}{a}\left(\sqrt{\frac{|m|^r}{\theta(a+1)}}\right)^{1-\frac{1}{r}}K_{1-\frac{1}{r}}\left(2\sqrt{\theta(a+1)|m|^r}\right)$$

$$\therefore f(m)=\frac{2\theta}{\Gamma\left(\frac{r+1}{r}\right)}\sum_{a=0}^{\infty}\binom{-2}{a}\left(\sqrt{\frac{|m|^r}{\theta(a+1)}}\right)^{1-\frac{1}{r}}K_{1-\frac{1}{r}}\left(2\sqrt{\theta(a+1)|m|^r}\right) \qquad \textbf{(7.12)}$$

Special Cases

(i) When $r=1$

$$f(m)=2\theta\sum_{a=0}^{\infty}\binom{-2}{a}K_0\left(2\sqrt{\theta(a+1)|m|}\right) \tag{7.13}$$

which is Laplace-Half logistic distribution as obtained in (6.14)

(ii) When $r=2$

$$f(m)=\frac{2^2\theta}{\sqrt{\pi}}\sum_{a=0}^{\infty}\binom{-2}{a}\left(\sqrt{\frac{|m|^2}{\theta(a+1)}}\right)^{\frac{1}{2}}K_{\frac{1}{2}}\left(2\sqrt{\theta(a+1)|m|^2}\right) \tag{7.14}$$

7.3.5 Lindley Mixing Distribution

$$\begin{aligned}
h(p)&=\frac{\tau^2}{\tau+1}(p+1)e^{-\tau p}\\
\therefore f(m)&=\int_0^{\infty}\frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)}e^{-\frac{|m|^r}{p}}\frac{\tau^2}{\tau+1}(p+1)e^{-\tau p}\,dp\\
&=\frac{\tau^2}{2^{-\frac{1}{r}}2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)(\tau+1)}\int_0^{\infty}p^{-\frac{1}{r}}(p+1)e^{-\tau p-\frac{|m|^r}{p}}\,dp
\end{aligned}$$

$$=\frac{\tau^2}{\Gamma\left(\frac{r+1}{r}\right)(\tau+1)}\frac{1}{2}\int_0^{\infty}(p^{1-\frac{1}{r}}+p^{-\frac{1}{r}})\,e^{-\tau\left\{p+\frac{|m|^r}{\tau}\frac{1}{p}\right\}}\,dp$$

$$=\frac{\tau^2}{\Gamma\left(\frac{r+1}{r}\right)(\tau+1)}\left[\frac{1}{2}\int_0^{\infty}p^{(2-\frac{1}{r})-1}e^{-\tau\left(p+\frac{|m|^r}{\tau}\frac{1}{p}\right)}\,dp+\frac{1}{2}\int_0^{\infty}p^{(1-\frac{1}{q})-1}e^{-\tau\left(p+\frac{|m|^r}{\tau}\frac{1}{p}\right)}\,dp\right]$$

Let $p=\sqrt{\frac{|m|^r}{\tau}}\,j\Rightarrow dp=\sqrt{\frac{|m|^r}{\tau}}\,dj$
Then

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty p^{(2-\frac{1}{r})-1} e^{-\tau\left(p+\frac{|m|^r}{\tau}\ \frac{1}{p}\right)}\, dp \\
&= \frac{1}{2}\int_0^\infty \left(\sqrt{\frac{|m|^r}{\tau}}\, j\right)^{(2-\frac{1}{r})-1} \exp-\tau\left\{\sqrt{\frac{|m|^r}{\tau}}\, j + \right. \\
&\left.\frac{|m|^r}{\tau}\ \frac{1}{\sqrt{\frac{|m|^r}{\tau}}\, j}\right\}\sqrt{\frac{|m|^r}{\tau}}\, dj \\
&= \left(\sqrt{\frac{|m|^r}{\tau}}\right)^{2-\frac{1}{r}} \frac{1}{2}\int_0^\infty j^{(2-\frac{1}{r})-1}\, e^{-\frac{2\sqrt{\tau|m|^r}}{2}\left(j+\frac{1}{j}\right)}\, dj \\
&= \left(\sqrt{\frac{|m|^r}{\tau}}\right)^{2-\frac{1}{r}} K_{2-\frac{1}{r}}\left(2\sqrt{\tau|m|^r}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty p^{(1-\frac{1}{r})-1} e^{-\tau\left(p+\frac{|m|^r}{\tau}\ \frac{1}{p}\right)}\, dp \\
&= \frac{1}{2}\int_0^\infty \left(\sqrt{\frac{|m|^r}{\tau}}\, j\right)^{(1-\frac{1}{r})-1} \exp-\tau\left\{\sqrt{\frac{|m|^r}{\tau}}\, j + \right. \\
&\left.\frac{|m|^r}{\tau}\ \frac{1}{\sqrt{\frac{|m|^r}{\tau}}\, j}\right\}\sqrt{\frac{|m|^r}{\tau}}\, dj \\
&= \left(\sqrt{\frac{|m|^r}{\tau}}\right)^{1-\frac{1}{r}} \frac{1}{2}\int_0^\infty j^{(1-\frac{1}{r})-1}\, e^{-\frac{2\sqrt{\tau|m|^r}}{2}\left(j+\frac{1}{j}\right)}\, dj \\
&= \left(\sqrt{\frac{|m|^r}{\tau}}\right)^{1-\frac{1}{r}} K_{1-\frac{1}{r}}\left(2\sqrt{\tau|m|^r}\right)
\end{aligned}
$$

Therefore

$$
f(m) = \frac{\tau^2}{\Gamma\left(\frac{r+1}{r}\right)(\tau+1)}\left[\left(\sqrt{\frac{|m|^r}{\tau}}\right)^{2-\frac{1}{r}} K_{2-\frac{1}{r}}\left(2\sqrt{\tau|m|^r}\right) + \left(\sqrt{\frac{|m|^r}{\tau}}\right)^{1-\frac{1}{r}} K_{1-\frac{1}{r}}\left(2\sqrt{\tau|m|^r}\right)\right]
$$

$$
f(m) = \frac{\tau^2}{(\tau+1)\Gamma\left(\frac{r+1}{r}\right)}\left(\sqrt{\frac{|m|^r}{\tau}}\right)^{1-\frac{1}{r}}\left[\sqrt{\frac{|m|^r}{\tau}}\, K_{2-\frac{1}{r}}\left(2\sqrt{\tau|m|^r}\right) + K_{1-\frac{1}{r}}\left(2\sqrt{\tau|m|^r}\right)\right] \tag{7.15}
$$

Special Cases

(i) When $r=1$

$$f(m)=\frac{\tau^2}{\tau+1}\left[\sqrt{\frac{|m|}{\tau}}K_1\left(2\sqrt{\tau|m|}\right)+K_0\left(2\sqrt{\tau|m|}\right)\right] \tag{7.16}$$

which is Laplace Lindley Distribution obtained in (6.17)
(ii) When $r=2$

$$f(m)=\frac{2\tau^2}{(\tau+1)\sqrt{\pi}}\left(\sqrt{\frac{|m|^2}{\tau}}\right)^{\frac{1}{2}}\left[\sqrt{\frac{|m|^2}{\tau}}\,K_{\frac{3}{2}}\left(2\sqrt{\tau|m|^2}\right)+K_{\frac{1}{2}}\left(2\sqrt{\tau|m|^2}\right)\right] \tag{7.17}$$

7.3.6 Generalized III Parameter Lindley Distribution

$$h(p)=\frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)}(\eta+\omega p)p^{\eta-1}e^{-\tau p} \quad p>0;\ \eta,\ \omega,\ \tau>0$$

The pdf of the mixture is

$$\begin{aligned}
f(m)&=\int_0^\infty \frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)}e^{-\frac{|m|^r}{p}}\frac{\tau^{\eta+1}(\eta+\omega p)p^{\eta-1}e^{-\tau p}}{(\tau+\omega)\Gamma(\eta+1)}\,dp\\
&=\frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)\Gamma\left(\frac{r+1}{r}\right)}\,\frac{1}{2}\int_0^\infty\left(\eta p^{(\eta-\frac{1}{r})-1}+\omega p^{(1+\eta-\frac{1}{r})-1}\right)e^{-\tau\{p+\frac{|m|^r}{\tau}\,\frac{1}{p}\}}\,dp\\
&=\frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)\Gamma\left(\frac{r+1}{r}\right)}\left[\eta\,\frac{1}{2}\int_0^\infty p^{(\eta-\frac{1}{r})-1}e^{-\tau\{p+\frac{|m|^r}{\tau}\,\frac{1}{p}\}}dp+\omega\frac{1}{2}\int_0^\infty p^{(1+\eta-\frac{1}{r})-1}e^{-\tau\{p+\frac{|m|^r}{\tau}\,\frac{1}{p}\}}dp\right]
\end{aligned}$$

Let

$$p=\sqrt{\frac{|m|^r}{\tau}}\,j\Rightarrow dp=\sqrt{\frac{|m|^r}{\tau}}\,dj$$

Then

$$
\begin{aligned}
&\eta\,\frac{1}{2}\int_0^\infty p^{(\eta-\frac{1}{r})-1}e^{-\tau\{p+\frac{|m|^r}{\tau}\,\frac{1}{p}\}}\,dp\\
&=\eta\,\frac{1}{2}\int_0^\infty\left(\sqrt{\frac{|m|^r}{\tau}}\,j\right)^{(\eta-\frac{1}{r})-1}\exp-\tau\left\{\sqrt{\frac{|m|^r}{\psi}}\,j+\right.\\
&\left.\frac{|m|^r}{\tau}\,\frac{1}{\sqrt{\frac{|m|^r}{\tau}}\,j}\right\}\sqrt{\frac{|m|^r}{\tau}}\,dj\\
&=\eta\left(\sqrt{\frac{|m|^r}{\tau}}\right)^{\eta-\frac{1}{r}}\frac{1}{2}\int_0^\infty j^{(\eta-\frac{1}{r})-1}\,e^{-\frac{2\sqrt{\tau|m|^r}}{2}\left(j+\frac{1}{j}\right)}\,dj\\
&=\eta\left(\sqrt{\frac{|m|^r}{\tau}}\right)^{\eta-\frac{1}{r}}K_{\eta-\frac{1}{r}}\left(2\sqrt{\tau|m|^r}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\omega\,\frac{1}{2}\int_0^\infty p^{(1+\eta-\frac{1}{r})-1}\,e^{-\tau\{p+\frac{|m|^r}{\tau}\,\frac{1}{p}\}}\,dp\\
&=\omega\,\frac{1}{2}\int_0^\infty\left(\sqrt{\frac{|m|^r}{\tau}}\,j\right)^{(1+\eta-\frac{1}{r})-1}\exp-\tau\left\{\sqrt{\frac{|m|^r}{\tau}}\,j+\right.\\
&\left.\frac{|m|^r}{\tau}\,\frac{1}{\sqrt{\frac{|m|^r}{\tau}}\,j}\right\}\sqrt{\frac{|m|^r}{\tau}}\,dj\\
&=\omega\left(\sqrt{\frac{|m|^r}{\tau}}\right)^{1+\eta-\frac{1}{r}}\frac{1}{2}\int_0^\infty j^{(1+\eta-\frac{1}{r})-1}\,e^{-\frac{2\sqrt{\tau|m|^r}}{2}\left(j+\frac{1}{j}\right)}\,dj\\
&=\omega\left(\sqrt{\frac{|m|^r}{\tau}}\right)^{1+\eta-\frac{1}{r}}K_{1+\eta-\frac{1}{r}}\left(2\sqrt{\tau|m|^r}\right)
\end{aligned}
$$

hence

$$
\begin{aligned}
f(m)=\frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)\Gamma\left(\frac{r+1}{r}\right)}\left(\sqrt{\frac{|m|^r}{\tau}}\right)^{\eta-\frac{1}{r}}\Bigg[&\eta K_{\eta-\frac{1}{r}}(2\sqrt{\tau|m|^r})+\\
&\omega\sqrt{\frac{|m|^r}{\tau}}K_{\eta-\frac{1}{r}+1}(2\sqrt{\tau|m|^r})\Bigg]
\end{aligned}
\tag{7.18}
$$

Special Cases

(i)When $r=1$

$$f(m)=\frac{\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)}\left(\sqrt{\frac{|m|}{\tau}}\right)^{\eta-1}\left[\eta K_{\eta-1}(2\sqrt{\tau|m|})+\omega\sqrt{\frac{|m|}{\tau}}K_{\eta}(2\sqrt{\tau|m|})\right] \tag{7.19}$$

which is Laplace-Generalized III parameter Lindley distribution obtained in (6.20)

(ii)When $r=2$

$$f(m)=\frac{2\tau^{\eta+1}}{(\tau+\omega)\Gamma(\eta+1)\sqrt{\pi}}\left(\sqrt{\frac{|m|^2}{\tau}}\right)^{\eta-\frac{1}{2}}\left[\eta\, K_{\eta-\frac{1}{2}}\left(2\sqrt{\tau|m|^2}\right)+\right.$$
$$\left.\omega\sqrt{\frac{|m|^2}{\tau}}\,K_{1+\eta-\frac{1}{2}}\left(2\sqrt{\tau|m|^2}\right)\right] \tag{7.20}$$

7.3.7 Inverse Gaussian Mixing Distribution

$$h(p)=\sqrt{\frac{\omega}{2\pi}}p^{-\frac{3}{2}}e^{\frac{\omega}{\theta}}e^{-\frac{\omega p}{2\theta^2}}e^{-\frac{\omega}{2p}} \qquad p>0,\ \omega>0\ -\infty<\theta<\infty$$

hence

$$\begin{aligned}
f(m)&=\int_0^\infty\frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)}\exp\{-\frac{|m|^r}{p}\}\sqrt{\frac{\omega}{2\pi}}\;p^{-\frac{3}{2}}e^{\frac{\omega}{\theta}}e^{-\frac{\omega p}{2\theta^2}}e^{-\frac{\omega}{2p}}\;dp\\
&=\frac{\sqrt{\frac{\omega}{2\pi}}\;e^{\frac{\omega}{\theta}}}{2^{-\frac{1}{r}}2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)}\int_0^\infty p^{-\frac{3}{2}-\frac{1}{r}}\;e^{-\frac{\omega}{2\theta^2}\left\{p+\frac{\theta^2(\omega+2|m|^r)}{\omega}\;\frac{1}{p}\right\}}\;dp\\
&=\frac{\sqrt{\frac{\omega}{2\pi}}\;e^{\frac{\omega}{\theta}}}{\Gamma\left(\frac{r+1}{r}\right)}\;\frac{1}{2}\int_0^\infty p^{-\frac{3}{2}-\frac{1}{r}}\;e^{-\frac{\omega}{2\theta^2}\left\{p+\frac{\theta^2(\omega+2|m|^r)}{\omega}\;\frac{1}{p}\right\}}\;dp
\end{aligned}$$

Let

$$p=\sqrt{\frac{\theta^2(\omega+2|m|^r)}{\omega}}\;j\Rightarrow dp=\sqrt{\frac{\theta^2(\omega+2|m|^r)}{\omega}}\;dj$$

$$\therefore f(m)=\frac{\sqrt{\frac{\omega}{2\pi}}\,e^{\frac{\omega}{\theta}}}{\Gamma\left(\frac{r+1}{r}\right)}\frac{1}{2}\int_0^\infty\left(\sqrt{\frac{\theta^2(\omega+2|m|^r)}{\omega}}\,j\right)^{-(\frac{1}{2}+\frac{1}{r})-1}\exp-\frac{\omega}{2\theta^2}\left\{\sqrt{\frac{\theta^2(\omega+2|m|^r)}{\omega}}\,j+\right.$$

$$\left.\frac{\theta^2(\omega+2|m|^r)}{\omega}\,\frac{1}{\sqrt{\frac{\theta^2(\omega+2|m|^r)}{\omega}}\,j}\right\}\sqrt{\frac{\theta^2(\omega+2|m|^r)}{\omega}}\,dj$$

$$=\frac{\sqrt{\frac{\omega}{2\pi}}\,e^{\frac{\omega}{\theta}}}{\Gamma\left(\frac{r+1}{r}\right)}\left(\sqrt{\frac{\theta^2(\omega+2|m|^r)}{\omega}}\right)^{-(\frac{1}{2}+\frac{1}{r})}\frac{1}{2}\int_0^\infty j^{-(\frac{1}{2}+\frac{1}{r})-1}\,e^{-\frac{\sqrt{\frac{\omega(\omega+2|m|^r)}{\theta^2}}}{2}\left(j+\frac{1}{j}\right)}\,dj$$

$$\therefore f(m)=\frac{\sqrt{\frac{\omega}{2\pi}}\,e^{\frac{\omega}{\theta}}}{\Gamma\left(\frac{r+1}{r}\right)}\left(\sqrt{\frac{\theta^2(\omega+2|m|^r)}{\omega}}\right)^{-(\frac{1}{2}+\frac{1}{r})}K_{\frac{1}{2}+\frac{1}{r}}\left(\sqrt{\frac{\omega(\omega+2|m|^r)}{\theta^2}}\right) \qquad \textbf{(7.21)}$$

Special Cases

(i) When $r=1$

$$f(m)=\sqrt{\frac{\omega}{2\pi}}\,e^{\frac{\omega}{\theta}}\left(\sqrt{\frac{\theta^2(\omega+2|m|)}{\omega}}\right)^{-\frac{3}{2}}K_{\frac{3}{2}}\left(\sqrt{\frac{\omega(\omega+2|m|)}{\theta^2}}\right) \qquad \textbf{(7.22)}$$

which is Laplace-Inverse Gaussian Distribution as obtained in (6.23)

(ii)When $r=2$

$$f(m)=\frac{2\sqrt{\frac{\omega}{2\pi}}\,e^{\frac{\omega}{\theta}}}{\sqrt{\pi}}\left(\sqrt{\frac{\theta^2(\omega+2|m|^2)}{\omega}}\right)^{-1}K_1\left(\sqrt{\frac{\omega(\omega+2|m|)}{\theta^2}}\right)$$

Let $\omega=\dfrac{2}{\beta}$ and $\theta=2p$

$$\therefore f(m)=\frac{2\sqrt{2}e^{\frac{2}{\beta\theta}}}{\pi\beta}\,\frac{1}{\theta\sqrt{\frac{2}{\beta}+2x^2}}\,K_1\sqrt{\frac{2(\frac{2}{\beta}+2m^2)}{\beta\theta^2}}$$

$$=\frac{2e^{\frac{2}{\beta\theta}}}{\pi\beta}\,\frac{1}{\theta\sqrt{\frac{1}{\beta}+m^2}}\,K_1\sqrt{\frac{4(\frac{1}{\beta}+m^2)}{\beta\theta^2}}$$

$$= \frac{2e^{\frac{2}{\beta(2p)}}}{\pi\beta}\,\frac{1}{2p\sqrt{\frac{1}{\beta}+m^2}}\,K_1\sqrt{\frac{4(\frac{1}{\beta}+m^2)}{4\beta p^2}}$$

$$f(m) = \frac{e^{\frac{1}{\beta p}}}{\pi\beta}\,\frac{1}{p\sqrt{\frac{1}{\beta}+m^2}}\,K_1\sqrt{\frac{\frac{1}{\beta}+m^2}{\beta p^2}} \qquad \textbf{(7.23)}$$

as obtained by Odhiambo (2016) where $\beta = \frac{1}{\theta}$ and $p = \mu$.

7.3.8 EP-Reciprocal Inverse Gaussian Distribution

$$h(p) = \sqrt{\frac{\omega}{2\pi}}\, p^{-\frac{1}{2}}\, e^{\frac{\omega}{\theta}}\, e^{-\frac{\omega p}{2}}\, e^{-\frac{\omega}{2\theta^2 p}}$$

Then

$$f(m) = \int_0^\infty \frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)} \exp\{-\frac{|m|^r}{p}\}\sqrt{\frac{\omega}{2\pi}}\, p^{-\frac{1}{2}}\, e^{\frac{\omega}{\theta}}\, e^{-\frac{\omega p}{2}}\, e^{-\frac{\omega}{2\theta^2 p}}\, dp$$

$$= \frac{\sqrt{\frac{\omega}{2\pi}}\, e^{\frac{\omega}{\theta}}}{\Gamma\left(\frac{r+1}{r}\right)}\,\frac{1}{2}\int_0^\infty p^{-\frac{1}{2}-\frac{1}{r}}\, e^{-\frac{\omega}{2}\left\{p+\frac{\omega+2\theta^2|m|^r}{\omega\theta^2}\,\frac{1}{p}\right\}}\, dp$$

$$= \frac{\sqrt{\frac{\omega}{2\pi}}\, e^{\frac{\omega}{\theta}}}{\Gamma\left(\frac{r+1}{r}\right)}\,\frac{1}{2}\int_0^\infty p^{-\frac{1}{2}-\frac{1}{r}}\, e^{-\frac{\omega}{2}\left\{p+\frac{\omega+2\theta^2|m|^r}{\omega\theta^2}\,\frac{1}{p}\right\}}\, dp$$

Let $p = \sqrt{\frac{\omega+2\theta^2|m|^r}{\omega\theta^2}}\, j \Rightarrow dp = \sqrt{\frac{\omega+2\theta^2|m|^r}{\omega\theta^2}}\, dj$

Then $$f(m) = \frac{\sqrt{\frac{\omega}{2\pi}}\, e^{\frac{\omega}{\theta}}}{\Gamma\left(\frac{r+1}{r}\right)}\frac{1}{2}\int_0^\infty \left(\sqrt{\frac{\omega+2\theta^2|m|^r}{\omega\theta^2}}\, j\right)^{\frac{1}{2}-\frac{1}{j}-1} \exp-\frac{\omega}{2}\left\{\sqrt{\frac{\omega+2\theta^2|m|^r}{\omega\theta^2}}\, j + \frac{\omega+2\theta^2|m|^q}{\omega\theta^2}\,\frac{1}{\sqrt{\frac{\omega+2\theta^2|m|^r}{\omega\theta^2}}\, j}\right\}\sqrt{\frac{\omega+2\theta^2|m|^r}{\omega\theta^2}}\, dj$$

$$= \frac{\sqrt{\frac{\omega}{2\pi}}\, e^{\frac{\omega}{\theta}}}{\Gamma\left(\frac{r+1}{r}\right)} \left(\sqrt{\frac{\omega+2\theta^2|m|^r}{\omega\theta^2}}\right)^{\frac{1}{2}-\frac{1}{r}} \frac{1}{2}\int_0^\infty j^{\frac{1}{2}-\frac{1}{r}-1}\, e^{-\frac{\sqrt{\frac{\omega(\omega+2\theta^2|m|^r)}{\theta^2}}}{2}\left(j+\frac{1}{j}\right)}\, dj$$

$$\therefore f(m) = \frac{\sqrt{\frac{\omega}{2\pi}}\, e^{\frac{\omega}{\theta}}}{\Gamma\left(\frac{r+1}{r}\right)} \left(\sqrt{\frac{\omega+2\theta^2|m|^r}{\omega\theta^2}}\right)^{\frac{1}{2}-\frac{1}{r}} K_{\frac{1}{2}-\frac{1}{r}}\left(\sqrt{\frac{\omega(\omega+2\theta^2|m|^r)}{\theta^2}}\right) \tag{7.24}$$

Special Cases

(i) When $r = 1$

$$f(m) = \sqrt{\frac{\omega}{2\pi}}\, e^{\frac{\omega}{\theta}} \left(\sqrt{\frac{\omega+2\theta^2|m|}{\omega\theta^2}}\right)^{-\frac{1}{2}} K_{\frac{1}{2}}\left(\sqrt{\frac{\omega(\omega+2\theta^2|m|)}{\theta^2}}\right) \tag{7.25}$$

which is Laplace-Reciprocal Inverse Gaussian distribution as obtained in (6.26)

(ii)When $r = 2$

$$f(m) = \frac{\sqrt{2\omega}}{\pi}\, e^{\frac{\omega}{\theta}}\, K_0\left(\sqrt{\frac{\omega(\omega+2\theta^2|m|^r)}{\theta^2}}\right)$$

Let $\theta = \frac{\beta}{2}$and $2\theta = p$

$$\therefore f(m) = \frac{\sqrt{\beta}}{\pi}\, e^{\frac{\beta}{2\theta}}\, K_0\left(\sqrt{\frac{\beta(\frac{\beta}{2}+2\theta^2 m^2)}{2\theta^2}}\right)$$

$$f(m) = \frac{\sqrt{\beta}}{\pi}\, e^{\frac{\beta}{p}}\, K_0 \frac{\sqrt{\beta(\beta+p^2m^2)}}{p} \tag{7.26}$$

as given by Odhiambo (2016),where $\theta = \beta$ and $p = \mu$

7.3.9 GIG Mixing Distribution

$$h(p) = \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^y p^{y-1} e^{-\frac{1}{2}(\psi p+\frac{\phi}{p})}}{2K_y(\sqrt{\psi\phi})}$$

then

$$f(m)=\int_0^{\infty}\frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)}\exp\{-\frac{|m|^r}{p}\}\left(\sqrt{\frac{\psi}{\phi}}\right)^{y}p^{y-1}e^{-\frac{1}{2}(\psi p+\frac{\phi}{p})}\,dp$$

$$=\frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{y}}{\Gamma\left(\frac{r+1}{r}\right)2K_y(\sqrt{\psi\phi})}\,\frac{1}{2}\int_0^{\infty}p^{y-\frac{1}{r}-1}\,e^{-\frac{\psi}{2}\left\{p+\frac{\phi+2|m|^r}{\psi}\,\frac{1}{p}\right\}}\,dp$$

$$=\frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{y}}{\Gamma\left(\frac{r+1}{r}\right)2K_y(\sqrt{\psi\phi})}\,\frac{1}{2}\int_0^{\infty}p^{y-\frac{1}{r}-1}\,e^{-\frac{\psi}{2}\left\{p+\frac{\phi+2|m|^r}{\psi}\,\frac{1}{p}\right\}}\,dp$$

Let

$$p=\sqrt{\frac{\phi+2|m|^r}{\psi}}\,j\Rightarrow dp=\sqrt{\frac{\phi+2|m|^r}{\psi}}\,dj$$

$$\therefore f(m)=\frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{y}}{\Gamma\left(\frac{r+1}{r}\right)2K_y(\sqrt{\psi\phi})}\frac{1}{2}\int_0^{\infty}\left(\sqrt{\frac{\phi+2|m|^r}{\psi}}\,j\right)^{y-\frac{1}{r}-1}\exp-\frac{\psi}{2}\left\{\sqrt{\frac{\phi+2|m|^r}{\psi}}\,j+\right.$$

$$\left.\frac{\phi+2|m|^r}{\psi}\,\frac{1}{\sqrt{\frac{\phi+2|m|^r}{\psi}}\,j}\right\}\sqrt{\frac{\phi+2|m|^r}{\psi}}\,dj$$

$$=\frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{y}}{\Gamma\left(\frac{r+1}{r}\right)2K_y(\sqrt{\psi\phi})}\left(\sqrt{\frac{\phi+2|m|^r}{\psi}}\right)^{y-\frac{1}{r}}\,\frac{1}{2}\int_0^{\infty}j^{y-\frac{1}{r}-1}\,e^{-\frac{\sqrt{\psi(\phi+2|m|^r)}}{2}\left(j+\frac{1}{j}\right)}\,dj$$

$$\therefore f(m)=\frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{y}\left(\sqrt{\frac{\phi+2|m|^r}{\psi}}\right)^{y-\frac{1}{r}}}{\Gamma\left(\frac{r+1}{r}\right)}\,\frac{K_{y-\frac{1}{r}}\left(\sqrt{\psi(\phi+2|m|^r)}\right)}{2\,K_y(\sqrt{\psi\phi})}\tag{7.27}$$

Let $y=\gamma, \phi=2\beta, \psi=\frac{\alpha}{2}$ and $|m|=|b|$

$$\therefore f(m)=\frac{\left(\sqrt{\frac{\alpha}{4\beta}}\right)^{\gamma}\left(\sqrt{\frac{4(\beta+|b|^r)}{\alpha}}\right)^{\gamma-\frac{1}{r}}}{\Gamma\left(\frac{r+1}{r}\right)}\ \frac{K_{\frac{\gamma r-1}{r}}\left(\sqrt{\alpha(\beta+|b|^r)}\right)}{2\,K_{\gamma}(\sqrt{\alpha\beta})}$$

$$=\frac{\alpha^{\frac{1}{2r}}\left(2\sqrt{\beta+|b|^r}\right)^{\frac{\gamma r-1}{r}}}{\left(2\sqrt{\beta}\right)^{\gamma}\Gamma\left(\frac{r+1}{r}\right)}\ \frac{K_{\frac{\gamma r-1}{r}}\left(\sqrt{\alpha(\beta+|b|^r)}\right)}{2\,K_{\gamma}(\sqrt{\alpha\beta})}$$

$$=\frac{\alpha^{\frac{1}{2r}}\left(\beta+|b|^r\right)^{\frac{\gamma r-1}{2r}}}{\beta^{\frac{\gamma}{2}}\,2^{\gamma}\,2^{1}\,2^{\frac{1-\gamma r}{r}}\Gamma\left(\frac{r+1}{r}\right)}\ \frac{K_{\frac{\gamma r-1}{r}}\left(\sqrt{\alpha(\beta+|b|^r)}\right)}{K_{\gamma}(\sqrt{\alpha\beta})}$$

$$\therefore f(m)=\frac{\alpha^{\frac{1}{2r}}\left(\beta+|b|^r\right)^{\frac{\gamma r-1}{2r}}}{\beta^{\frac{\gamma}{2}}\,2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)}\ \frac{K_{\frac{\gamma r-1}{r}}\left(\sqrt{\alpha(\beta+|b|^r)}\right)}{K_{\gamma}(\sqrt{\alpha\beta})} \tag{7.28}$$

as obtained by Zhang,Wang and Liu (2012),where $r=q$.

Special Cases

(i) When $r=1$ then from (7.27)

$$f(m)=\frac{\sqrt{\psi}\left(\sqrt{\phi+2|m|}\right)^{y-1}}{\left(\sqrt{\phi}\right)^{v}}\ \frac{K_{y-1}\left(\sqrt{\psi(\phi+2|m|)}\right)}{2K_{y}(\sqrt{\psi\phi})} \tag{7.29}$$

which is Laplace-GIG distribution as obtained in (6.29)

(ii) When $r=2$ then from (7.27)

$$f(m)=\frac{2\left(\sqrt{\frac{\psi}{\phi}}\right)^{y}\left(\sqrt{\frac{\phi+2|m|^2}{\psi}}\right)^{y-\frac{1}{2}}}{2\sqrt{\pi}}\ \frac{K_{y-\frac{1}{2}}\left(\sqrt{\psi(\phi+2|m|^2)}\right)}{K_{y}(\sqrt{\psi\phi})}$$

Let $\phi = 2\theta$ and $\psi = \frac{p}{2}$

$$\therefore f(m) = \frac{\left(\sqrt{\frac{p}{4\theta}}\right)^{y}\left(\sqrt{\frac{2(2\theta+2m^2)}{p}}\right)^{y-\frac{1}{2}}}{\sqrt{\pi}}\;\frac{K_{y-\frac{1}{2}}\left(\sqrt{\frac{p}{2}(2\theta+2m^2)}\right)}{K_y(\sqrt{\frac{p}{2}(2\theta)})}$$

$$f(m) = \frac{\left(\sqrt{p}\right)^{\frac{1}{2}}\left(\sqrt{\theta+m^2}\right)^{y-\frac{1}{2}}}{\sqrt{2\pi}\left(\sqrt{\theta}\right)^{y}}\;\frac{K_{y-\frac{1}{2}}\left(\sqrt{p(\theta+m^2)}\right)}{K_y(\sqrt{p\theta})} \qquad \textbf{(7.30)}$$

as obtained by Odhiambo (2016) where $\theta = \phi$ and $p = \psi$.

(iii)When $y = -\frac{1}{2}$ then from (7.27)

$$f(m) = \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{-\frac{1}{2}}\left(\sqrt{\frac{\phi+2|m|^r}{\psi}}\right)^{-\frac{1}{2}-\frac{1}{r}}}{\Gamma\left(\frac{r+1}{r}\right)}\;\frac{K_{-\frac{1}{2}-\frac{1}{r}}\left(\sqrt{\psi(\phi+2|m|^r)}\right)}{2\,K_{-\frac{1}{2}}(\sqrt{\psi\phi})}$$

$$= \frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{-\frac{1}{2}}\left(\sqrt{\frac{\phi+2|m|^r}{\psi}}\right)^{-\frac{1}{2}-\frac{1}{r}}}{\Gamma\left(\frac{r+1}{r}\right)}\;\frac{K_{\frac{1}{2}+\frac{1}{r}}\left(\sqrt{\psi(\phi+2|m|^r)}\right)}{\sqrt{\frac{2\pi}{\sqrt{\psi\phi}}}e^{-\sqrt{\psi\phi}}}$$

$$= \frac{\left(\sqrt{\frac{\phi}{\psi}}\right)^{\frac{1}{2}}(\sqrt{\psi\phi})^{\frac{1}{2}}e^{\sqrt{\psi\phi}}\left(\sqrt{\frac{\phi+2|m|^r}{\psi}}\right)^{-\frac{1}{2}-\frac{1}{r}}}{\Gamma\left(\frac{r+1}{r}\right)\sqrt{2\pi}}\;K_{\frac{1}{2}+\frac{1}{r}}\left(\sqrt{\psi(\phi+2|m|^r)}\right)$$

Let $\psi = \frac{\phi}{\theta^2}$
then

$$f(m) = \frac{\sqrt{\frac{\phi}{2\pi}}\,e^{\frac{\phi}{\theta}}}{\Gamma\left(\frac{r+1}{r}\right)}\left(\sqrt{\frac{\theta^2(\phi+2|m|^r)}{\phi}}\right)^{-(\frac{1}{2}+\frac{1}{r})}K_{\frac{1}{2}+\frac{1}{r}}\left(\sqrt{\frac{\phi(\phi+2|m|^r)}{\theta^2}}\right) \qquad \textbf{(7.31)}$$

which is EP-Inverse Gaussian distribution where $\phi = \omega$ as obtained in (7.21)

(iv) When $y=\frac{1}{2}$ then from (7.27)

$$f(m)=\frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{\frac{1}{2}}\left(\sqrt{\frac{\phi+2|m|^r}{\psi}}\right)^{\frac{1}{2}-\frac{1}{r}}}{\Gamma\left(\frac{r+1}{r}\right)}\;\frac{K_{\frac{1}{2}-\frac{1}{r}}\left(\sqrt{\psi(\phi+2|m|^r)}\right)}{2\,K_{\frac{1}{2}}(\sqrt{\psi\phi})}$$

$$=\frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{\frac{1}{2}}\left(\sqrt{\frac{\phi+2|m|^r}{\psi}}\right)^{\frac{1}{2}-\frac{1}{r}}}{\Gamma\left(\frac{r+1}{r}\right)}\;\frac{K_{\frac{1}{2}-\frac{1}{r}}\left(\sqrt{\psi(\phi+2|m|^r)}\right)}{\sqrt{\frac{2\pi}{\sqrt{\psi\phi}}}e^{-\sqrt{\psi\phi}}}$$

$$=\frac{\left(\sqrt{\frac{\psi}{\phi}}\right)^{\frac{1}{2}}(\sqrt{\psi\phi})^{\frac{1}{2}}e^{\sqrt{\psi\phi}}\left(\sqrt{\frac{\phi+2|m|^r}{\psi}}\right)^{\frac{1}{2}-\frac{1}{r}}}{\Gamma\left(\frac{r+1}{r}\right)\sqrt{2\pi}}\;K_{\frac{1}{2}-\frac{1}{r}}\left(\sqrt{\psi(\phi+2|m|^r)}\right)$$

$$=\frac{\sqrt{\frac{\psi}{2\pi}}e^{\sqrt{\psi\phi}}\left(\sqrt{\frac{\phi+2|m|^r}{\psi}}\right)^{\frac{1}{2}-\frac{1}{r}}}{\Gamma\left(\frac{r+1}{r}\right)}\;K_{\frac{1}{2}-\frac{1}{r}}\left(\sqrt{\psi(\phi+2|m|^r)}\right)$$

Let $\phi=\frac{\psi}{\theta^2}$
Therefore

$$f(m)=\frac{\sqrt{\frac{\psi}{2\pi}}e^{\frac{\psi}{\theta}}\left(\sqrt{\frac{\psi+2\theta^2|m|^r}{\psi\theta^2}}\right)^{\frac{1}{2}-\frac{1}{r}}}{\Gamma\left(\frac{r+1}{r}\right)}\;K_{\frac{1}{2}-\frac{1}{r}}\left(\sqrt{\psi(\phi+2|m|^r)}\right)$$

Replacing ψ by ϕ
we have

$$f(m)=\frac{\sqrt{\frac{\phi}{2\pi}}\;e^{\frac{\phi}{\theta}}}{\Gamma\left(\frac{r+1}{r}\right)}\left(\sqrt{\frac{\phi+2\theta^2|m|^r}{\phi\theta^2}}\right)^{\frac{1}{2}-\frac{1}{r}}\;K_{\frac{1}{2}-\frac{1}{r}}\left(\sqrt{\frac{\phi(\phi+2\theta^2|m|^r)}{\theta^2}}\right)\qquad\textbf{(7.32)}$$

which is EP-Reciprocal Inverse Gaussian distribution where $\phi=\omega$ obtained in (7.24)

7.4 Mixtures in terms of Confluent Hypergeometric Series

7.4.1 EP-Beta I Distribution

$$h(p) = \frac{p^{u-1}(1-p)^{v-1}}{B(u,v)} \qquad 0 < p < 1, \quad u,v > 0$$

then

$$\begin{aligned} f(m) &= \int_0^1 \frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)} e^{-\frac{|m|^r}{p}} \ \frac{p^{a-1}(1-p)^{v-1}}{B(u,v)}\, dp \\ &= \frac{1}{2B(u,v)\Gamma\left(\frac{r+1}{r}\right)} \int_0^1 p^{u-\frac{1}{r}-1}(1-p)^{v-1}\, e^{-\frac{|m|^r}{p}}\, dp \end{aligned}$$

Let

$$p = \frac{1}{q} \Rightarrow dp = -\frac{dq}{q^2}$$

Therefore

$$\begin{aligned} f(m) &= \frac{1}{2B(u,v)\Gamma\left(\frac{r+1}{r}\right)} \int_\infty^1 \left(\frac{1}{q}\right)^{u-\frac{1}{r}-1}\left(1-\frac{1}{q}\right)^{v-1} e^{-|m|^r q}\left(-\frac{dq}{q^2}\right) \\ &= \frac{1}{2B(u,v)\Gamma\left(\frac{r+1}{r}\right)} \int_1^\infty q^{\frac{1}{r}-u-v}(q-1)^{v-1}e^{-|m|^r q}\, dq \end{aligned}$$

Let

$$j = q-1 \Rightarrow q = 1+j \Rightarrow dq = dj$$

$$\begin{aligned} \therefore f(m) &= \frac{1}{2B(u,v)\Gamma\left(\frac{r+1}{r}\right)} \int_0^\infty (1+j)^{\frac{1}{r}-u-v} j^{v-1} e^{-|m|^r(1+j)}\, dj \\ &= \frac{1}{2B(u,v)\Gamma\left(\frac{r+1}{r}\right)}\, e^{-|m|^r} \int_0^\infty j^{v-1}(1+j)^{\frac{1}{r}-u-v}e^{-|m|^r j}\, dj \\ &= \frac{\Gamma v\, e^{-|m|^r}}{2B(u,v)\Gamma\left(\frac{r+1}{r}\right)} \int_0^\infty \frac{j^{v-1}(1+j)^{2-u+\frac{1}{r}-1-v-1}e^{-|m|^r j}}{\Gamma v}\, dj \end{aligned}$$

Therefore

$$f(m) = \frac{\Gamma v\, e^{-|m|^r}}{2B(u,v)\Gamma\left(\frac{r+1}{r}\right)}\Psi(v; 2-u+\frac{1}{r}-1; |m|^r) \tag{7.33}$$

Special Cases

(i) When $r=1$

$$f(m)=\frac{e^{-|m|}\,\Gamma v}{2B(u,v)}\Psi(v;2-u;|m|) \tag{7.34}$$

which is Laplace-Beta I distribution as obtained in (6.32)

(ii) When $r=2$

$$f(m)=\frac{e^{-|m|^2}\,\Gamma v}{B(u,v)\sqrt{\pi}}\Psi(v;\frac{3}{2}-u;|m|^2)$$

Let
$m^2=\frac{p}{2}$ and $\Gamma v=\frac{\Gamma d}{\sqrt{2\pi}}$
Then

$$f(m)=\frac{e^{-\frac{p}{2}}\,\Gamma d}{B(u,v)\sqrt{2\pi}}\Psi(v;\frac{3}{2}-u;\frac{p}{2}) \tag{7.35}$$

as obtained by Odhiambo (2016) where $p=m^2$ and $d=v$

7.4.2 Beta II Mixing Distribution

$$h(p)=\frac{p^{u-1}}{B(u,v)(1+p)^{u+v}} \qquad p>0, \quad u,v>0$$

Hence

$$\begin{aligned}
f(m) &= \int_0^\infty \frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)} e^{-\frac{|m|^r}{p}}\ \frac{p^{u-1}}{B(u,v)(1+p)^{u+v}}\,dp \\
&= \frac{1}{2B(u,v)\Gamma\left(\frac{r+1}{r}\right)}\int_0^\infty p^{u-\frac{1}{r}-1}(1+p)^{-u-v}e^{-\frac{|m|^r}{p}}\ dp
\end{aligned}$$

Let

$$p=\frac{1}{j}\Rightarrow dp=-\frac{dj}{j^2}$$

Therefore

$$f(m) = \frac{1}{2B(u,v)\Gamma\Big(\frac{r+1}{r}\Big)} \int_{\infty}^{0} \Big(\frac{1}{j}\Big)^{u-\frac{1}{r}-1} (1+\frac{1}{j})^{-u-v}\, e^{-|m|^r j} \Big(-\frac{dj}{j^2}\Big)$$

$$= \frac{1}{2B(u,v)\Gamma\Big(\frac{r+1}{r}\Big)} \int_{0}^{\infty} j^{v+\frac{1}{r}-1} (1+j)^{-u-v} e^{-|m|^r j}\, dj$$

$$= \frac{1}{2B(u,v)\Gamma\Big(\frac{r+1}{r}\Big)} \Gamma\Big(v+\frac{1}{r}\Big) \int_{0}^{\infty} \frac{j^{v+\frac{1}{r}-1}(1+j)^{2-u+\frac{1}{r}-1-(v+\frac{1}{r})-1} e^{-|m|^r j}}{\Gamma\Big(v+\frac{1}{r}\Big)}\, dj$$

$$\therefore f(m) = \frac{\Gamma\Big(v+\frac{1}{r}\Big)}{2B(u,v)\Gamma\Big(\frac{r+1}{r}\Big)} \Psi(v+\frac{1}{r}; 2-u+\frac{1}{r}-1; |m|^r) \tag{7.36}$$

Special Cases

(i) When $r = 1$

$$f(m) = \frac{\Gamma(v+1)}{2B(u,v)} \Psi(v+1; 2-u; |m|) \tag{7.37}$$

which is Laplace-Beta II Mixing distribution obtained in (6.35)

(ii) When $r = 2$

$$f(m) = \frac{\Gamma\Big(v+\frac{1}{2}\Big)}{B(u,v)\sqrt{\pi}} \Psi(v+\frac{1}{2}; \frac{3}{2}-u; |m|^2)$$

Let $m^2 = \frac{p}{2}$ and $\Gamma\Big(v+\frac{1}{2}\Big) = \frac{\Gamma\Big(d+\frac{1}{2}\Big)}{\sqrt{2}}$
then

$$f(m) = \frac{\Gamma\Big(d+\frac{1}{2}\Big)}{B(u,v)\sqrt{2\pi}} \Psi(v+\frac{1}{2}; \frac{3}{2}-u; \frac{p}{2}) \tag{7.38}$$

as obtained by Odhiambo (2016) where $p = m^2$ and $d = v$

7.4.3 EP-Scaled Beta Distribution

$$h(p) = \frac{p^{u-1}(\theta - p)^{v-1}}{\theta^{u+v-1}\, B(u,v)} \quad 0 < p < \theta,\ u,\ v > 0$$

Then

$$\begin{aligned} f(m) &= \int_0^\infty \frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)} e^{-\frac{|m|^r}{p}}\ \frac{p^{u-1}(\theta-p)^{v-1}}{\theta^{u+v-1}\, B(u,v)}\, dp \\ &= \frac{1}{2B(u,v)\ \theta^{u+v-1}\Gamma\left(\frac{r+1}{r}\right)} \int_0^\theta p^{u-\frac{1}{r}-1}(\theta-p)^{v-1} e^{-\frac{|m|^r}{p}}\, dp \end{aligned}$$

If

$$p = \frac{1}{q} \Rightarrow dp = -\frac{dq}{q^2}$$

$$\begin{aligned} \therefore f(m) &= \frac{1}{2B(u,v)\ \theta^{u+v-1}\Gamma\left(\frac{r+1}{r}\right)} \int_\infty^{\frac{1}{\theta}} \left(\frac{1}{q}\right)^{u-\frac{1}{r}-1} \left(\theta - \frac{1}{q}\right)^{v-1} e^{-|m|^r q}\left(-\frac{dq}{q^2}\right) \\ &= \frac{1}{2B(u,v)\ \theta^{u+v-1}\Gamma\left(\frac{r+1}{r}\right)} \int_{\frac{1}{\theta}}^\infty q^{\frac{1}{r}-u-v}(\theta q - 1)^{v-1} e^{-|m|^r q}\, dq \end{aligned}$$

Let

$$j = \theta q - 1 \Rightarrow q = \frac{j+1}{\theta} \Rightarrow dq = \frac{dj}{\theta}$$

$$\begin{aligned} \therefore f(m) &= \frac{1}{2B(u,v)\ \theta^{u+v-1}\Gamma\left(\frac{r+1}{r}\right)} \int_0^\infty \left(\frac{j+1}{\theta}\right)^{\frac{1}{r}-u-v} j^{v-1} e^{-|m|^r\left(\frac{j+1}{\theta}\right)} \frac{dj}{\theta} \\ &= \frac{e^{-\frac{|m|^r}{\theta}}\ \Gamma v}{2\theta^{\frac{1}{r}} B(u,v)\Gamma\left(\frac{r+1}{r}\right)} \int_0^\infty \frac{j^{v-1}(1+j)^{2-u-1+\frac{1}{r}-v-1}\ e^{-\frac{|m|^r}{\theta}j}}{\Gamma v}\, dj \\ \therefore f(m) &= \frac{e^{-\frac{|m|^r}{\theta}}\ \Gamma v}{2\theta^{\frac{1}{r}} B(u,v)\Gamma\left(\frac{r+1}{r}\right)} \Psi\left(v; 2-u+\frac{1}{r}-1; \frac{|m|^r}{\theta}\right) \end{aligned} \tag{7.39}$$

Special Cases

(i) When $r = 1$

$$f(m) = \frac{e^{-\frac{|m|}{\theta}}\,\Gamma v}{2\theta B(u,v)}\Psi\Big(v; 2-u; \frac{|m|}{\theta}\Big) \tag{7.40}$$

which is Laplace-Scaled Beta distribution as obtained in (6.38)

(ii) When $r = 2$

$$f(m) = \frac{e^{-\frac{|m|^2}{\theta}}\,\Gamma v}{\sqrt{\pi\theta}B(u,v)}\Psi\Big(v; \frac{3}{2}-u; \frac{|m|^2}{\theta}\Big)$$

Let $m^2 = \frac{p}{2}$ and $\Gamma v = \frac{\Gamma d}{\sqrt{2}}$
Then

$$f(m) = \frac{e^{-\frac{p}{2\theta}}\,\Gamma v}{\sqrt{2\pi\theta}B(u,v)}\Psi\Big(v; \frac{3}{2}-u; \frac{p}{2\theta}\Big) \tag{7.41}$$

as obtained by Odhiambo (2016) where $d = v, \theta = \mu$ and $p = m^2$

7.4.4 EP-Full Beta Distribution

The pdf of full beta is:

$$h(p) = \frac{y^u}{B(u,v)}\,\frac{p^{u-1}}{(1+yp)^{u+v}} \quad p > 0;\ u,\ v,\ y\ > 0$$

Hence

$$\begin{aligned} f(m) &= \int_0^\infty \frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)} e^{-\frac{|m|^r}{p}}\,\frac{y^u}{B(u,v)}\,\frac{p^{u-1}}{(1+yp)^{u+v}}\,dp \\ &= \frac{y^u}{2B(u,v)\Gamma\left(\frac{r+1}{r}\right)}\int_0^\infty p^{u-\frac{1}{r}-1}(1+yp)^{-u-v}e^{-\frac{|m|^r}{p}}\,dp \end{aligned}$$

Let

$$j = \frac{1}{yp} \Rightarrow p = \frac{1}{yj} \Rightarrow dp = -\frac{dj}{yj^2}$$

Then

$$f(m) = \frac{y^u}{2B(u,v)\Gamma\left(\frac{r+1}{r}\right)} \int_{\infty}^{0} \left(\frac{1}{yj}\right)^{u-\frac{1}{r}-1} \left(1+\frac{1}{j}\right)^{-u-v} e^{-|m|^r yj} \left(-\frac{dj}{yj^2}\right)$$

$$= \frac{y^{\frac{1}{r}}}{2B(u,v)\Gamma\left(\frac{r+1}{r}\right)} \int_{0}^{\infty} j^{v+\frac{1}{r}-1}(1+j)^{-u-v} e^{-|m|^r yj}\, dj$$

$$= \frac{y^{\frac{1}{r}}\Gamma\left(v+\frac{1}{r}\right)}{2B(u,v)\Gamma\left(\frac{r+1}{r}\right)} \int_{0}^{\infty} \frac{j^{v+\frac{1}{r}-1}(1+j)^{-u-v} e^{-|m|^r yj}}{\Gamma\left(v+\frac{1}{r}\right)}\, dj$$

$$f(m) = \frac{y^{\frac{1}{r}}\Gamma\left(v+\frac{1}{r}\right)}{2B(u,v)\Gamma\left(\frac{r+1}{r}\right)} \Psi\left(v+\frac{1}{r}; 2-u+\frac{1}{r}-1; |m|^r y\right) \tag{7.42}$$

Special Cases

(i) When $r = 1$

$$f(m) = \frac{y\,\Gamma(v+1)}{2B(u,v)}\, \Psi(v+1; 2-u; |m| y) \tag{7.43}$$

which is Laplace-Full Beta distribution as obtained in (6.41)

(ii) When $r = 2$

$$f(m) = \frac{y^{\frac{1}{2}}\,\Gamma(v+\frac{1}{2})}{B(u,v)\sqrt{\pi}}\, \Psi(v+\frac{1}{2}; 1+\frac{1}{2}-u; |m|^2 y) \tag{7.44}$$

7.4.5 EP-Pareto I Distribution

The pdf of Pareto I is:

$$h(p) = \frac{\omega\tau^{\omega}}{p^{\omega+1}} \qquad p > \tau; \omega,\ \tau > 0$$

Then

$$f(m) = \int_{\tau}^{\infty} \frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)} e^{-\frac{|m|^r}{p}} \frac{\omega\tau^{\omega}}{p^{\omega+1}}\, dp$$
$$= \frac{\omega\tau^{\omega}}{2\Gamma\left(\frac{r+1}{r}\right)} \int_{\tau}^{\infty} p^{-\omega-\frac{1}{r}-1} e^{-\frac{|m|^r}{p}}\, dp$$

Let

$$q = \frac{1}{p} \Rightarrow p = \frac{1}{q} \Rightarrow dq = -\frac{dq}{q^2}$$

$$\therefore f(m) = \frac{\omega\tau^{\omega}}{2\Gamma\left(\frac{r+1}{r}\right)} \int_{\frac{1}{\tau}}^{0} \left(\frac{1}{q}\right)^{-\omega-\frac{1}{r}-1} e^{-|m|^r q}\left(-\frac{dq}{q^2}\right)$$
$$= \frac{\omega\tau^{\omega}}{2\Gamma\left(\frac{r+1}{r}\right)} \int_{0}^{\frac{1}{\tau}} q^{\omega+\frac{1}{r}-1} e^{-|m|^r q}\, dq$$
$$= \frac{\omega\tau^{\omega}}{2\Gamma\left(\frac{r+1}{r}\right)}\, \gamma\left(\omega+\frac{1}{r}, \frac{1}{\tau}\right)$$

Therefore

$$f(m) = \frac{\omega}{2(\omega+\frac{1}{r})\tau^{\frac{1}{r}}\Gamma\left(\frac{r+1}{r}\right)} e^{-\frac{1}{\tau}}\ {}_1F_1\left(1; \omega+\frac{1}{r}+1; \frac{1}{\tau}\right) \tag{7.45}$$

Special Cases

(i) When $r = 1$

$$f(m) = \frac{\omega}{2\tau(\omega+1)} e^{-\frac{1}{\tau}}\ {}_1F_1\left(1; \omega+2; \frac{1}{\tau}\right) \tag{7.46}$$

which is Laplace-Pareto I distfribution obtained in (6.44)

(ii) When $r = 2$

$$f(m) = \frac{\omega}{\omega+\frac{1}{2}\sqrt{\pi\tau}} e^{-\frac{1}{\tau}}\ {}_1F_1\left(1; \omega+\frac{3}{2}; \frac{1}{\tau}\right) \tag{7.47}$$

7.4.6 EP-Lomax Distribution

$$h(p) = \frac{\omega\tau^{\omega}}{(\tau+p)^{\omega+1}}$$

Therefore

$$\begin{aligned} f(m) &= \int_0^{\infty} \frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)} e^{-\frac{|m|^r}{p}} \frac{\omega\tau^{\omega}}{(\tau+p)^{\omega+1}}\, dp \\ &= \frac{\omega\tau^{\omega}}{2\Gamma\left(\frac{r+1}{r}\right)} \int_0^{\infty} p^{-\frac{1}{r}} (\tau+p)^{-\omega-1} e^{-\frac{|m|^r}{p}}\, dp \end{aligned}$$

Let

$$t = \frac{1}{p} \Rightarrow p = \frac{1}{t} \Rightarrow dp = -\frac{dt}{t^2}$$

Then

$$\begin{aligned} f(m) &= \frac{\omega\tau^{\omega}}{2\Gamma\left(\frac{r+1}{r}\right)} \int_{\infty}^{0} \left(\frac{1}{t}\right)^{-\frac{1}{r}} \left(\tau+\frac{1}{t}\right)^{-\omega-1} e^{-|m|^r t} \left(-\frac{dt}{t^2}\right) \\ &= \frac{\omega\tau^{\omega}}{2\Gamma\left(\frac{r+1}{r}\right)} \int_0^{\infty} t^{\omega+\frac{1}{r}-1} (\tau t+1)^{-\omega-1} e^{-|m|^r t}\, dt \end{aligned}$$

Let

$$z = \tau t \Rightarrow t = \frac{z}{\tau} \Rightarrow dt = \frac{dz}{\tau}$$

Hence

$$\begin{aligned} f(m) &= \frac{\omega\tau^{\omega}}{2\Gamma\left(\frac{r+1}{r}\right)} \int_0^{\infty} \left(\frac{z}{\tau}\right)^{\omega+\frac{1}{r}-1} (1+z)^{-\omega-1} e^{\frac{-|m|^r}{\tau} z} \frac{dz}{\tau} \\ &= \frac{\omega\Gamma\left(\omega+\frac{1}{r}\right)}{2\tau^{\frac{1}{r}}\Gamma\left(\frac{r+1}{r}\right)} \int_0^{\infty} \frac{z^{\omega+\frac{1}{r}-1} (1+z)^{\frac{1}{r}-(\omega+\frac{1}{r})-1} e^{\frac{-|m|^r}{\tau} z}}{\Gamma\left(\omega+\frac{1}{r}\right)}\, dz \end{aligned}$$

Therefore

$$f(m) = \frac{\omega\Gamma\left(\omega+\frac{1}{r}\right)}{2\tau^{\frac{1}{r}}\Gamma\left(\frac{r+1}{r}\right)} \Psi\left(\omega+\frac{1}{r}; \frac{1}{r}; \frac{|m|^r}{\tau}\right) \tag{7.48}$$

Special Cases

(i) When $r = 1$

$$f(m) = \frac{\omega\Gamma(\omega+1)}{2\tau}\Psi\Big(\omega+1;1;\frac{|m|}{\tau}\Big) \tag{7.49}$$

which is Laplace-Pareto II distribution as given in (6.47)

(ii) When $r = 2$

$$f(m) = \frac{\omega\Gamma(\omega+\frac{1}{2})}{\tau^{\frac{1}{2}}\sqrt{\pi}}\Psi\Big(\omega+\frac{1}{2};\frac{1}{2};\frac{|m|^2}{\tau}\Big)$$

Let $\frac{p}{2} = m^2$ and $\Gamma(\omega+\frac{1}{2}) = \frac{\Gamma(\theta+\frac{1}{2})}{\sqrt{2}}$

$$\therefore f(m) = \frac{\omega\Gamma(\theta+\frac{1}{2})}{\sqrt{2\tau\pi}}\Psi\Big(\omega+\frac{1}{2};\frac{1}{2};\frac{p}{2\tau}\Big) \tag{7.50}$$

as obtained by Odhiambo (2016) where $\theta = \omega$ and $p = m^2$

7.4.7 EP-Generalized Pareto Distribution

$$h(p) = \frac{\eta^{\tau}p^{\omega-1}}{B(\omega,\tau)\,(\eta+p)^{\omega+\tau}} \quad p > 0;\ \omega,\,\tau,\,\eta > 0$$

Hence

$$\begin{aligned} f(m) &= \int_0^{\infty} \frac{\left(\frac{p}{2}\right)^{-\frac{1}{r}}}{2^{\frac{r+1}{r}}\Gamma\left(\frac{r+1}{r}\right)} e^{-\frac{|m|^r}{p}}\ \frac{\eta^{\tau}p^{\omega-1}}{B(\omega,\tau)\,(\eta+p)^{\omega+\tau}}\,dp \\ &= \frac{\eta^{\tau}}{2B(\omega,\tau)\Gamma\left(\frac{r+1}{r}\right)}\int_0^{\infty} p^{\omega-\frac{1}{r}-1}(\eta+p)^{-\omega-\tau}e^{-\frac{|m|^r}{p}}\,dp \end{aligned}$$

Let

$$t = \frac{1}{p} \Rightarrow p = \frac{1}{t} \Rightarrow dp = -\frac{dt}{t^2}$$

Then

$$f(m) = \frac{\eta^\tau}{2B(\omega,\tau)\Gamma\left(\frac{r+1}{r}\right)} \int_\infty^0 \left(\frac{1}{t}\right)^{\omega-\frac{1}{r}-1} \left(\eta+\frac{1}{t}\right)^{-\omega-\tau} e^{-|m|^r t} \left(-\frac{dt}{t^2}\right)$$

$$= \frac{\eta^\tau}{2B(\omega,\tau)\Gamma\left(\frac{r+1}{r}\right)} \int_0^\infty t^{\tau+\frac{1}{r}-1}(\eta t+1)^{-\omega-\tau} e^{-|m|^r t}\, dt$$

Let

$$z = \eta t \Rightarrow t = \frac{z}{\eta} \Rightarrow dt = \frac{dz}{\eta}$$

Hence

$$f(m) = \frac{\eta^\tau}{2B(\omega,\tau)\Gamma\left(\frac{r+1}{r}\right)\eta^{\tau+\frac{1}{r}}} \int_0^\infty z^{\tau+\frac{1}{r}-1}(1+z)^{-\omega-\tau} e^{-\frac{|m|^r}{\eta}z}\, dz$$

$$= \frac{\eta^\tau \Gamma\left(\tau+\frac{1}{r}\right)}{2B(\omega,\tau)\Gamma\left(\frac{r+1}{r}\right)\eta^{\tau+\frac{1}{r}}} \int_0^\infty \frac{z^{\tau+\frac{1}{r}-1}(1+z)^{2-\omega+\frac{1}{r}-1-(\tau+\frac{1}{r})-1} e^{-\frac{|m|^r}{\eta}z}}{\Gamma\left(\tau+\frac{1}{r}\right)}\, dz$$

Therefore

$$f(m) = \frac{\Gamma\left(\tau+\frac{1}{r}\right)}{2B(\omega,\tau)\Gamma\left(\frac{r+1}{r}\right)\eta^{\frac{1}{r}}} \Psi\left(\tau+\frac{1}{r}; 2-\omega+\frac{1}{r}-1; \frac{|m|^r}{\eta}\right) \quad \textbf{(7.51)}$$

Special Cases

(i) When $r = 1$

$$f(m) = \frac{\Gamma(\tau+1)}{2\eta\, B(\omega,\tau)} \Psi\left(\tau+1; 2-\omega; \frac{|m|}{\eta}\right) \quad \textbf{(7.52)}$$

which is Laplace-Generalized Pareto distribution as obtained in (6.50)

(ii) When $r = 2$

$$f(m) = \frac{\Gamma(\tau+\frac{1}{2})}{B(\omega,\tau)\sqrt{\pi\eta}} \Psi\left(\tau+\frac{1}{2}; \frac{3}{2}-\omega; \frac{|m|^2}{\eta}\right)$$

Let $\frac{p}{2} = m^2$ and $\Gamma(\tau + \frac{1}{2}) = \frac{\Gamma(\eta+\frac{1}{2})}{\sqrt{2}}$
Therefore

$$f(m) = \frac{\Gamma(\eta + \frac{1}{2})}{B(\omega, \tau)\sqrt{2\pi\eta}} \Psi\Big(\tau + \frac{1}{2}; \frac{3}{2} - \omega; \frac{p}{2\eta}\Big) \tag{7.53}$$

as obtained by Odhiambo (2016).

8 CONCLUSION

The main objective of our study was to construct Laplace distribution and obtain their properties,the specific objectives were to construct Beta-Laplace distribution,generalized Laplace distribution, Laplace mixtures and obtain moments,and finally to construct EP mixtures and obtain special cases when r=1 and when r=2.
We summarise our results as follows:
We have managed to construct Laplace distribution using the difference of two iid exponential random variables, by the mixture of normal-exponential distribution,product of Rayleigh and normal distributions.All methods gave the same result.Moments of Laplace distribution have also been obtained including;mgf,mean,variance,
skewness and kurtosis.
Furthermore,we have looked at Beta generated distribution,a generalization of Laplace distribution,and we have managed to obtain their special cases which are type I,type II and ith order statistic distribution.Beta-Laplace distribution and its special cases has then been derived from the Beta generated distribution in form of binomial expansion,Gauss hypergeometric function and as infinite mixtures.
Moreover,we have constructed generalized Laplace distribution using the difference of two iid exponentiated exponential random variables and have looked at special cases when $\alpha = 1$ which reduces to Laplace distribution and when $\alpha = n$.
Then Laplace mixtures have been constructed using 16 mixing distributions and have obtained their moments,Exponential power mixtures have also been obtained.The special cases when $r = 1$ resulted in Laplace distributions obtained in chapter 5 and when $r = 2$ we have obtained normal distributions.
Distributions obtained in terms of modified Bessel function of the third kind are;Exponential, Gamma,Transmuted exponential,Half logistic,Lindley,Generalized III parameter Lindley,Inverse Gaussian,Reciprocal Inverse Gaussian and GIG.Those obtained in terms of Confluent Hypergeometric function are;Beta I,Beta II,Scaled Beta,Full Beta,Pareto I,Pareto II and Generalized Pareto distributions.
Therefore,we have achieved the main and the specific objectives of our study.

8.1 Future Research

In this research our focus has been construction and obtaining the properties of Laplace distribution and their mixtures.A future research could look at the estimation of these distributions and their applications.
Furthermore,moments of EP mixtures could be explored.

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