# MODELING TAIL RISKS AND SYSTEMIC RISKS USING COPULAS

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# DECLARATION AND APPROVAL

I the undersigned declare that this project report is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

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In my capacity as a supervisor of the candidate, I certify that this report has my approval for submission.

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# DEDICATION

I dedicate this study to Family and my Friends.

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My heart goes out to the Mighty God for good health and great life during this work epoch. Furthermore, I would want to acknowledge the great help I have received in preparing this study. I am indebted to my family for the unwavering support that they have showed me during the time I have been taking my studies.

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# ABSTRACT

Understanding the relationship between financial variables is an essential part of managing financial risk. Such relationships in the context of risk management are of sensation interest in the modern world in which modeling risk has become a popular idea. The 2007/08 financial meltdown has accentuated the importance of this because it showed that with a distant probability of an event happening, say 0.01%, it should not be overlooked. The existence of dependency between the two variables is underscored in their established joint distribution. Conventional financial models intensely depend on a relationship to elucidation the dependence that exists between tow variables. Nevertheless, as a means of measuring dependence, correlation entirely defines the dependence structure for normal distributions. This is more so in the case of elliptical distributions as opposed to other classes. Even within elliptical distributions, correlation remains delicate because it is easily distorted by outliers.

One of the biggest shortfalls of current financial models is in their assumption and over-simplification that leads to false parameterization. Many financial models assume that asset returns and risk variables follow normal distribution and as such pay attention the judiciously possible (central moments), undermining the value of what is remotely probable (extreme events). Yet, the financial crisis showed that remote events are understated. Thus, this paper focuses on copula model- that characterized the dependence structure across the entire distribution. Also, copulas have the advantage of being able to portray more information concerning the dependence structures between the two variables because they are distributions themselves.

From the viewpoint systemic and tail risk, the copula is motivating. In this regard, it enables us to disengage the dependence structure (linked with systemic risk) from the marginal distribution (linked with tail risk) and archetypal both distinctly with a bigger degree of precision. Fundamentally, copulas act as magnifying glasses that permit us to examine and model financial variables with greater precision- in this case the connections between systemic and tail risk

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# ABBREVIATIONS AND ACRONYMS

| | |
|---|---|
| IRM: | Integrated Risk Management |
| MES: | Marginal Expected Shortfall |
| BSMD: | Banking System's Multivariate Density |
| SIFI: | Systematically Important Financial Institution |
| VAR: | Value at Risk |
| LTCM: | Long-Term Capital Management |
| CVaR: | Conditional Value-at-Risk |
| EVT: | Extreme Value Theory |
| HFTR: | Hedge Fund Tail Risk |
| CDO: | Collateralized Debt Obligations |
| IFM: | Inference for Margins |
| FML: | Full Maximum Likelihood Estimation |
| EML: | Exact Maximum Likelihood Method |
| AIC: | Akaike Information Criterion |
| SIC: | Schwarz Information Criterion |
| BIC: | Bayesian Information Criterion |
| JB: | Jarque-Bera Test |
| LB: | Ljung-Box Q-test |
| ACF: | Auto-Correlation Function |
| PACF: | Partial Auto-Correlation Function |

# 1 INTRODUCTION

## 1.1 Background

Globally, actuaries are at the forefront in providing/building financial models that are used to mitigate risks in various fields especially in the financial service industry in an attempt to make the global financial systems safer. In this regard, actuaries have become more involved in managing risk; more recently integrated risk management (IRM) has been the focus of many firms' especially insurance, pricing securities, and risk management. To attain efficient in risk management, the actuaries ought to respond to the critical question of whether the correlation structure dangerous or not. In case it is found to be dangerous, the degree dangerous-ness ought to be established based on the situation. In this respect, the, tools used quantify, carry out comparison and contrast, and also to model the existing strength of dependence amongst various risks become very crucial.(Denuit et al., 2005).

Even so, the biggest challenge actuaries face when modeling risks is dependence (dependency is extremely difficult to model given its volatility and uncertainty). In modern portfolio and risk theory, the dependency structure plays an increasingly important role which cannot be overlooked. Unlike independence that is defined in a singular way, dependence can be explained in many ways leading to complicated and far less convenient models. Risk can be defined as an event or a situation that may or may not take place, and exposes a firm to adverse financial consequence as a result of internal and or external vulnerabilities. Modeling of risks uses probability theory due to the probabilistic nature of risks; for instance, in the insurance sector risk is a situation where the probability of an event taking place is known but the intensity and time of occurrence is unknown.

Traditional financial models were anchored on the supposition of independence, and the popular law. Not so long ago, an emergence of complex financial products has stirred the interest on modeling dependency. Modeling dependence involves the rigorous processes of analyzing dependency between different risks faced by a company and postulating a credible model. In the insurance industry, dependency is present in all aspects from portfolio structure, capital allocation, pricing, and reserve calculations. Insurance is the act of transferring (totally or partially) the economic impact of risk to another party for an agreed amount of premium.

One of the biggest shortfalls of current financial models is in their assumption and over-simplification that leads to false parameterization. Many financial models use or assume a Gaussian process because of their desired tractable properties in calculation. In reality, many loss distributions are skewed, non-linear, and heavy tailed. Additionally, asset returns are not Gaussian and this presents a problem. The problem is unit-dimensional financial systems are not Gaussian while their multivariate counterparts are Gaussian. It has taken the market time to realize how best to use and model multivariate financial models, and for the first time, copula-based modeling provide a rigorous and intuitive concept that adequately addresses the fundamental important issue of dependence at the heart of financial products.

The reminder of this article is structure in the following order. The reminder of section 1 explains the problem or objective, hypothesis, and justification or limitation of the study; in the part marked section 2 there is a review of theoretical and empirical texts; in the part 3 we examine a relationship between copula and dependence measures; Section 4 analyses the result based on numerical results of data from financial data. The last section 5 converses the experimental outcomes and makes a conclusion.

## 1.2 Statement of Problem

The main problem that this paper seeks to address is; finding a way to understand tail and systemic risk. Such a relationship would help improve current financial models. As aforementioned, most of the existing financial models used in finance industry fail to explain the extreme dependence. This mistake in the context of risk management (risk management focus on central moments over tail events) is very costly because these tail risks can make or break a company; it's the unknown unknowns that hurt every time.

Over the last ten years, large and well-respected financial institutions have experienced failure in the face of tail risks, revealing the inadequate protection provided by current practice. To exemplify this, the recent financial crisis (2008), illustrated the devastating effect that tail risk have on a company. In the face of a systemic crisis, a resilient financial model can provide strong line of defense against failure.

This thesis proposes using copulas to model the correlation between the tail risk and systemic risk. Sklar (1959) stated that copulas are the best probability tools for separating the joint distributions into marginals (component associated with tail risks)and the correlation component (often related with systemic risk). Moreover, according to Schweizer and Wolff (1981) the copula approach is invariant whereas its margins are changeable at will. Consequently, this means it is the copula that effectively takes into account those aspects of the joint distribution that are proved to be invariant when subjected to increasingly strict changes.

Furthermore, copula is suitable for modeling Tail Risks and Systemic Risks because it focuses on modeling the complete dispersal instead of just the fundamental moments. Even so, the selection of copula function is important because one must look at the correlation, marginal distributions, tail dependence, and asymmetry/symmetry of the events to be modeled. For instance, assume that $\Lambda$ represents a prospect that a given risk is great because the other is also great. Bouye et al. (2000) contend that Gaussian copula extreme levels are asymptotically independent for $\rho \neq 1$, i.e. $\lambda = 0$ for $\rho < 1$ while their counterpart student copula extreme levels are asymptotically dependent for $\rho \neq -1$".Therefore, results are governed by the choice of copula family. In this article, we adopt previous approaches to calibrate copula based on assumptions and historical data

## 1.3 Objectives

The objective of this research study is twofold: first we will show that the copula function is a powerful tool because it focuses on modeling the distribution in its entirety than just focusing of the essential moments of the random variables. Secondly, we will examine the simulation and estimation of copula functions, provide examples where necessary, and outline the theoretical and computational difficulties when dealing with copula and large data sets.

### 1.3.1 General objectives

The expansive objective of this study is to define a suitable copula that effectively describes the dependence arrangement between the tail risk and the systemic risk in the financial markets.

### 1.3.2 Definite objectives

Some of the define objectives for this study are;

i. To fit marginal distributions to the tail and systemic risk in financial returns

ii. To fit different copulas to the tail and systemic risk using the marginal distributions obtained.

iii. To investigation the appropriate of the copulas in order to establish the best copula that fits the data

## 1.4 Hypothesis

This study proposed the null hypotheses that; for a typical financial data,

i. The reported tail and systemic risk can be fit to individual marginal distributions

ii. There exists dependence between the tail and systemic risks that can be modeled using copulas

iii. Different copulas represent the tail and systemic risks dependence to different extents and therefore some copulas can be considered to explain the dependence better than others.

## 1.5 Justification of the study

This study sought to find a suitable explicit copula that can be used to explain the relationship between tail and systemic risks and therefore would be useful for further research in trying to find expected tail risk and systemic risk from past financial information.

## 1.6 Limitation and Scope of the study

This study focuses on particularly on the financial sector, specifically the financial markets which affects the two main sectors the insurance and banking industry. Even so, this research can be applied to other financial industries particularly the financial markets. The main objective of the research is to establish a connection between the tail risks and systemic risks incurred by financial markets. For this reason, the study was not restricted geographically. Therefore, the concept of this research can be applied anywhere in a financial setting as well as any other non-financial sector. Nonetheless, researchers should be wary of the difference in the different financial sectors as a result of different risks experiences. To exemplify the relationship between the tail risks and systemic risks, this investigation considered data from the global financial market in which the methodology was applied.

The first challenge experienced in the study was determining and defining of the tail risks. An ideal definition of the tail risk for the study is the statistical probability that represents the least likely outcome. However; in practice it is quite difficult to ascertain this exact probabilities and as such most financial companies use the normal (bell-shaped) curve to rank risks. Secondly, data used was retrieved from initial estimates of the risk amount because it is difficult to get the actual/ revised amounts because the revision occur after the financial companies have accessed the impact of the risk at a future time. Nonetheless, the aim of this research is to carry out an empirical investigation of the existing connection between systemic and tail risk based one on the initial observation of risks; thus the initial estimates were useful in establishing the relationship between the risks.

Moreover, it is imperative to appreciate that the research did not put into consideration any human error that may have been present in the data entry stages and as such the model may be subject to parametric errors.

# 2 LITERATURE REVIEW

## 2.1 Introduction

The existence of dependence among random variables is indicated through joint distributions. Therefore, the relations of dependency between variables X and Y is wholly defined by their shared dispersal function F(x, y). The concept of dependency is widely used in actuarial mathematics particularly in measuring and modeling risks in the insurance industry. McNeil et al. (2005) asserted that it is very important to correctly account for all aspects of dependencies and model them in case of measuring or aggregating risks. However, unquantifiable uncertainties will always exist, and traditional risk management approaches are not adequate.

Embrechts et al. (1999) stated that although quantitative methods are used most often in risk assessment, their usefulness wanes in the realm of tail risk management. This is particularly pertinent in the perspective of contemporary risk-based solvency and capital schemes, which among many others may include as the Basel requirements and Swiss Solvency Test. As mentioned, oversimplified assumptions or oversimplified models used in the traditional financial models lead to catastrophic losses for a financial company. To exemplify this, the Basel Committee on Banking Supervision assumes implicitly that various types of losses are seamlessly dependent; this is not always the case.

Various models have been proposed to model the interaction as well as the relationship between two variables. Although not a new concept, the copulas are a powerful instruments used by risk managers because they are well-matched with univariate elliptical financial modeling. Not surprisingly, this method has received a lot of attention from the field of insurance risk management particularly when modeling dependency. This chapter reviews the role of copula in modeling dependency in the financial industry. In addition, it provides a broad review of previous literature systemic risk and tail risk models and measures.

### 2.1.1 Theoretic Literature Review

The notion of dependency is central to many statistical models and and quantifies the extent of the linkage in two random variables involved. Over the years, many theories have been put forward to explain assumptions and measure the level of dependency in statistical sense. Though the subject of many financial and statistical articles, there is a lot of misunderstanding in describing and distinguishing the dependence structure and correlation of random variables (Staudt, 2010).

Most of these literatures use dependence and correlation interchangeably even though they have different meanings. Moreover, it is critical to appreciate that the correlation figure does not completely defines the association of variables in specific cases (e.g. when the distribution is multivariate Gaussian) otherwise information given by the coefficient is not enough to determine the dependency structure of the random variables (Embrechts, McNeil and Straumann, 2002).

The focus of this study will be on two theories that that have been the focus of a number of studies in the recent past. These theories are namely

1. the Sklars theorem (Copula theorem)
2. the Bayesian inference theorem.

The following is a review of applications of these theorems in the financial industry.

### 2.1.2 Sklar's Theorem

Abe Sklar first proposed Sklar's theorem in 1959. According to Sklar (1959), the joint distribution of n-variables contains enough evidence on the issues of dependence structure between variables and also the marginal behavior of the individual random variables. Therefore, the theorem recommended copula as tools for decoupling the marginal properties of the variables and understanding the dependency structure of two or multiple variables.

In the recent, copula centered approaches have become more popular in insurance and finance, taking over these sectors by storm. As Per Fisher (1997) asserts, Copulas play are vital role to the statisticians because there are used for studying scale-free measures with regard to dependence and also serve as a preparatory point for them when constructing various families of two variable distributions.

The research make a contribution to expand the already existing the literature on risk management and Archimedean copula modeling as the aim is to find an appropriate copula to explain dependence between the systemic risk and tail risk in insurance. Dependency modeling and extreme value theory in the financial industry did received immense attention in the recent past. Popular texts on dependence modeling using copula in finance consist of an introduction by Embrechts et al. (1999), Joe (1997), Jaworski et al. (2010), Trivedi and Zimmer (2005), Kojadinovic and Yan (2010), and McNeil et al. (2005).

Subsequent works showing dependency in finance include the pioneer works of Embrechts et al. (2001), Tibiletti (2000), and Bouye (2000). A large number of contributions of copulas in modeling extreme risks and degree of interdependence among risks/variables in actuarial science include the works of Barnett et al. (2007), (Wang 1997), Wang (1998), and Venter (2002, 2003).

Furthermore, some important literature concerning copula revolve around the statistical aspect of finding the best copulas that fits the data. And to the essential real-world dilemma some studies that sufficiently address the issue. An actuarial viewpoint is given by Valdez (1998) while Genest and Rivest (1993) supplied a more general statistical theory of fitting copulas.

An alternative strand of literature focuses on Computing with Copulas. McNeil and Ulman (2010) and Azzalini (2010) demonstrated how to compute copula models using the R-package. In line with the theory in Embrechts et al.(1999), several studies investigate the theory of multivariate association and dependence in risk management; see Schmid et al. (2010), Venter (2003), and Li (2000).

### 2.1.3 Bayesian Inference Theorem

The Bayesian theorem which was postulated by Reverend Thomas Bayes in 1763 offers a way of systematically evaluating the complex resulting models. Financial models are increasingly becoming complex as the financial systems become more complex. There is need for the new sophisticated models to reveal the true magnitude of the risk insurance company's face. Just recently, the implementation of new regulatory work (solvency II) demanded that insurance companies must have enough funds that match their liabilities as well as that which will insure against credit and market risks. Such requirements require means of quantifying and evaluating model and parameters uncertainty. The Bayesian inference theorem presents a way that simply extent the computations to more superior systems. Several posterior predictive simulations of the original data can be compared with the original data set to check the model fit.

In particular, the Bayesian inference is attractive when evaluating insurance models because it is exact in finite samples. According to Puustelli, Koskinen, and Luoma (2008) it is critical to have an exact description of sample being used in order to avoid sample errors. Likewise Cairns (2000) stated that a benefit of the theorem is found in its ability to incorporates prior information in the model. According to Cairns (2000), most actuarial problems arising from the underlying model, parameter values, and the stochastic nature of a model are solved by using a Bayesian framework. Modern techniques of Bayesian Inference are useful in constructing models for extreme value of large claims.

Equally, Hardy (2002) used Bayesian approach in valuing equity-linked insurance contracts. Recently, Bayesian approach has been used for evaluating equity-linked saving contracts in the previous model to deal with estimation of model uncertainty and estimation in in valuing equity-linked insurance contracts. Bunnin et al. (2002) while using Bayesian Inference to price European Call Options mentioned that Bayesian approach clearly acknowledged the risks associated to parameter estimation and model choice.

Bayesian mortality model are used in modeling parametric curves. In the same way, Bayesian inference theorem has been applied in smoothing spline methods in mortality modeling. Additionally, Bayesian Inference has also been used to forecast insurance loss reserving. Bornhuetter-Ferguson reserving and credibility theories heavily rely on Bayesian methodology, Gisler and Buhlmann (2005) proposed that the insurance practice mathematically motivates Bayesian Statistics, that is the concept of credibility theory. Zhang et al. (2010) recommended a this model for predicting losses in insurance industry while Pacakova (1997) used Bayesian inference to estimate credibility premium for short-term insurance contracts.

## 2.2 Empirical Literature Review

Some applications of dependence modeling are outlined in this section. But first, we look at a general introduction to modeling dependence in financial institutions offered by Joe (1997); this literature is very informative as it gives the readers an overview of multivariate models. The multivariate normal model explained by Joe (1997) is not the best model for insurance and financial data fitting because their dependency structure is not only too restrictive but also their marginal distributions tails are too short and symmetric.

Similarly, Dhaene and Goovaerts (1996) offered an illustration of application of dependence in the insurance sector; they presented a model for calculating stop loss premiums that reflects the possibility of association between risks in the portfolio. Stop loss insurance are designed to limit the claim amount beyond a specific value so that the insurance financial reserves is protected against excess claim amounts that may result from catastrophic or numerous claim amount at the same time. Based on the assumption that when a given individual survives, all other individuals with a probability of survival greater than that of the individual will survive and vice-versa, the scholars undertook to estimate the expected maximum stop loss amounts.

Dhaene and Goovaerts (1996) extended the same issue and offered a statistical model for stop-loss insurance. These ideas provide us with a means of modifying individual events of risks in a portfolio and allow us to create patterns that can help deduce the distributions of the modified risks. These authors determined the stop loss premiums using the modified risks. This paper uses the same concept to calculate the dependence between individual risk in the insurance and financial industry.

Nonetheless, dependencies between risks can be classified as empirical or structural. Structural dependencies is a situation where the loss variable is determined by common variables such as common shocks (e.g. an automobile accident that causes numerous related claims), economic factors (e.g. inflation determines costs in various lines of insurance), catastrophic events (e.g. 9/11 that distresses many lines of insurances such as property, healthy, and life), and uncertain risk variables (e.g. long term changes in mortality that affect life-annuities).

On the other hand, empirical dependencies refer to observed co-movements between different lines of business. These relationships do not have well-defined cause-effect and occur at a macro level. What is more, such relationships are complex and data on these relationships is often unavailable. However, data on their marginal distributions is richly available. Before we discuss the applications of copula modeling, some knowledge on literature on extreme risk and systemic risk is required. A discussion of systemic and tail risk causes and measure in empirical literature is given below.

**Systemic Risk**

Whether structural or empirical dependency, they display key characteristics of systemic risk- the possibility of movement from positive equilibrium to appositive equilibrium. Although varying definitions have been put forward, Basias et al. (2012) offered a definition that captures the essence of other disparities, they defined systemic risk as a disruption that affects the course of a given financial services which is triggered by an injury of part or an entire financial system as well as can likely cause a serious adverse impact on the real economic performance. Ostalecka (2012) accentuates that regardless of the varying definitions, systemic risk evolve along with new developments of the financial sector. A lot of scholars and institutions have offered varying definitions of systemic risk in financial literature, yet no consensus has been reached. Majority of their opinions resolve that more focus should be on consequences of systemic risk to the real economy; they unanimously agree that current research should look at the interlink-ages or dependency between firms in the financial industry and common exposures.

However, current causes of systemic risks differ from those observed in the past. As such, there is need to define systemic risk to reflect recent trends that focus on impact of firms on the real economy. Ostalecka (2012) observed that current systemic risks to financial institutions include poor monetary or fiscal policies, poor regulations, and catastrophic events. However, firms in the insurance industry are less likely to suffer from systemic risk when compared to those in the asset and banking industries. This is because insurance firms design products that restrict access to assets under management, require ongoing premiums, and use proactive ALM techniques. In fact, insurance firms have strict regulations that focus on solvency which shield against insolvency whereas banks seek return optimization, for this reason banks try to keep capital regulations at a minimum.

One notable contribution to the rich literature of systemic risk in financial, insurance and reinsurance was made by Rudolph (2017); he offers a deep review of systemic risk within the insurance industry. The scholar argues that systemic risk build up over time and the regime switching models cannot capture the complex and higher order interaction across many scenarios. Besides, most of these relationships not only expand exponentially but also these variables are rarely linear. Similarly, Ostalecka (2012) contents that although some existing models may reflect the higher volatility; they only replace extreme values of the multivariate distribution with fat tails. Such models only make the mathematics tractability of the model easy to compute but do not reflect the actual empirical distribution. Rudolph (2017) asserts that copula; particularly its covariance matrix reflects higher order dependencies between variables.

Hendricks et al. (2006) in their paper provide a synopsis of general risk sources and discussed the propagation of systemic risk. The scholars shared the opinion that existing systemic shock models in the financial system are no longer adequate to apprehend the spread of major disruption. Recent studies show that the increasing complexity of the financial system structures and the diverse nature of activities in the financial sector and so new models are desirable (Hendricks et al., 2006). They asserted that systemic risk in the banking system and establish that bank "runs"- the decision by majority of the depositors to withdraw their funds causes' instability. The concern for liquidity (insolvency) in banks is contagious in the banking sector; a self-fulfilling prophecy that a run at one bank will cause runs in many other banks.

The consequences of systemic risk in the banking sector lead to social costs that are borne by the nonfinancial portion of the economy; because banks will be reluctant to extend financial loans amid a series of bank runs. Additionally, Acharya and Richardson (2009) suggested that public policy/regulations and high leverage positions to prevent bank runs may in turn generate systemic risks. The burden of supervision/ regulation falls heavily on traditional institutions when compared to non-traditional institutions such as private equity and hedge funds (Basias et al., 2012). In hindsight, innovation is endemic in the deregulated non-traditional institutions and as their activities become so intertwined with global financial systems that a single demise such as the Long Term Capital Management hedge fund has severe consequences to the financial stability.

According to Hendricks et al. (2006) such policies pay little attention to the "moral hazard" of the firms to pay attention to firm risk as they engaged in high-risk projects that endanger the financial health of the banks. Significantly, liquidity-based models that can result to "runs" is not exclusively used by banks only, this model is relevant to hedge funds and security firms. Basias et al. (2012) discussed systemic risk in the financial asset markets that caused disruptions to financial markets. Hendricks et al. (2006) stressed the need shift emphasis from liquidity- based run models to "market gridlock" because recent financial crises manifested themselves in financial markets rather than at institutional level. Basias et al. (2012) propose that "disintermediation" of financial activity resulted from growth of mutual funds and capital markets in recent decades. As a result, large financial institutions shifted their activities from bank-based to market-based financial systems that partake in issuance of stocks and bonds as well as investing on securities on behalf of firms and households (mutual funds and pension funds).

Acharya and Richardson (2009) found that although market-based systems are superior to bank-based models, they are vulnerable because it requires the market to be liquid/ tradability. Therefore, in the absent of liquidity (market-gridlock crisis) - investors are unable or unwilling to buy assets and as a result market liquidity dries up and market grid-lock takes hold (Hendricks et al., 2006). Ex post, a market-gridlock crisis is characterized by a several investors simultaneously halting trading activities in an attempt to protect capital and reduce losses. In aggregate, the combined actions of such investors paralyze and eventually stop financial activity. An illustration of the aforementioned action was experience in the periods leading to the 1987 stock market crash.

Additional discussion on sources of systemic risks caused by market-based models is offered by Minsky (1977). Both papers propose that some changes macro-economic conditions can open new profitable investment opportunities which in turn attract capital facilitated by investors. However, after sometime profits begin to run out and prices fall stimulating creating panic as investors cut off credit. Another source of systemic risk in financial market is clearance and settlement. Hendricks et al. (2006) found that clearance and settlement mechanisms put a strain to investors who hedge across different market;

this is because the urgency to transfer large cash can create gridlock in payment systems. This situation is exemplified by the failure of the Gemant bank- Herstatt Bank.

Basias et al. (2012) paper emphasis the need to expand systemic risk models to incorporate potential endogenous variables created by current trends. New trends like gradual deregulation of markets, ongoing financial innovation, and disintermediation make financial systems more vulnerable. Smaga (2014) echoed the same sentiments when they analyzed the term systemic risk and pro-cyclicality in view of the 2007/2008 financial crisis that wronged. He recommends a philosophical approach of systemic risks that combines the process of building up of negative externalities (imbalances) towards their materialization. Smaga (2014) acknowledges that when dealing with systemic risk, researchers must incorporate the aspect of uncertainty of occurrence and timing of the events. The model consist of four important elements namely (1) shock, (2) channel of contagion, (3) institutions affected, and (4) structural vulnerabilities.

Equally, Allen and Carletti (2011) formulated six classes of systemic risk viz.; (i) conjoint exposure to the drop in the price of assert, (ii) contagion/ domino effect, (iii) currency mismatch, (iv) assets bubble, (v) multiple equilibrium panic, and (vi) sovereign debt. Systemic risks vary considerably because it can originate within or outside the system and can be endogenous or exogenous. Scott (2012) argued that contagion, correlation, and connectedness are key features of systemic risk that aid the distribution and increase the level of risk. Anand et al. (2013) arrived at a similar conclusion and found that many financial systems are robust but fragile. Even with diversification at micro-levels, the probability of systemic risk is still very high due to correlation that amplifies contagion and shock effects.

Hitherto, construction of systemic risk measures followed two approaches bottom-up or top-down. The bottom-up approach emulates the portfolio approach to assign systemic risk measure whereas the top-down approach utilizes historical behavior and time series data to arrive at a quantitative measure (Chan-Lau et al., 2007). Danielsson et al. (2005) points out that systemic risk measures are more theoretical in nature and it is impossible to fit the models to real market data. Moreover, there is lack of sufficiently large datasets due to shortage of financial data in crisis periods.

Eisenberg and Noe (2001) devised a mechanism to measure systemic risk in complex financial systems. They examined the cyclical obligations and inter-linkage in financial institution and proposed a model that used clearing vectors. The clearing vector helps in used to calculate the amount of a firm's debt as dependent/interlinked to other firms. A simulated sequential default algorithm is used to compute the clearing vector- the neutral measure of systemic risk. This systemic risk measure is based on how many 'waves' of induced default a system has before it fails. However, a shortcoming of this model is that it does not allow true dynamics (clearing for more than one date).

Several probability distributions measures of systemic risk have been put forward, most of these measures are direct because they employ the joint distribution of negative outcomes of SIFI. What is more, most of these models are a-theoretical and are used to provide estimates of correlated losses. Examples of probability distribution systemic threat measures include the Marginal anticipated loss (MES) of Acharya et al. (2010), the monetary system's of many variables density (BSMD) function suggested by Segoviano and Goodhart (2009), the conditional value at Risk (CoVaR) suggested by Adrian and Brunnermeier (2011), instability in the monetary system approach method presented by Kritzman and Li (2010), and option implied probability of default (iPoD) of Capuano (2008).

Of these models, the most conspicuous are the conditional value at threat measure that is suggested by Adrian and Brunnermeier (2010) and the Marginal expected shortfall (MES) of Acharya et al. (2010). Adrian and Brunnermeier (2010) model measures the systemic importance of firm A and distress in firm A given that another firm B is in distress. On the contrary, Acharya et al. (2010) model uses the marginal expected shortfall to quantify a single firm's contribution to total systemic risk; and arrived at the conclusion that firms with higher MES contribute more to systemic risk.

Nevertheless Basias et al. (2012) after employing model risk analytics points out the empirical methods (the proposed statistical systemic risk measures) do not provide consistent signals to legislators. They investigated the model risk when calculating the marginal expected shortfall and the conditional value at risk measures using of daily returns for a sample of 92 large financial institutions. The models are subject to high degrees of parameter uncertainty and concluded that systemic risks using these models are undependable.

### 2.2.1 Tail Risk

The global financial crisis rekindled the debate on tail risk and tail dependency both within the academic world and the financial service industry. However, interest in tail risk has been an interesting area of research since academia's proved that market returns violate the normality assumption. Currently, the concept of tail risk has found a niche in hedge funds. This offers the liquidity and enhances the effectiveness of the monetary markets. Nevertheless, Hedge-funds are exposed in a structural manner to tail risk and in times of crisis they amplify market volatilities and financial risks on the global economy.

Almeida and Leal (2016) defined tail related risk as the likelihood of tremendously large damages in portfolio returns. Subsequently, every investor and fund manager fears the possibility of their portfolios incurring extreme large losses. Nonetheless, some early studies by Rietz (1988) and Barro (2006) revealed that positive relationships exist between portfolio returns and tail risk; this implies that portfolio returns increases when tail risk increases. Still, many empirical findings show that offloading tail risk comes at the cost of lower portfolio returns. There are many ways to measure tail risk, which include using option prices, using cross-section of returns, and stochastic discount factor approach and risk-neutral information.

The first literature on tail risk dates back to the early 1960s when Mandelbrot (1963) challenged the assumption that financial returns followed a Gaussian distribution. He applied power law to show the skewed distributions of financial returns. Consistent with Mandelbrot's argument, Fama (1963) assessed that the abrupt price movements experienced in certain markets do not follow the usual normal distribution. Blattberg and Gonedes (1974) examined the earlier studies and suggested the students (t) distribution to account for the fat tails in asset returns.Other subsequent studies just to mention a few Akgiray and Booth (1988), Janse and De Vries (1991), and Hols and De Vries (1991) confirmed that the distribution of returns is asymmetrical at the tails.

In light of this important development, economists have proposed different measures (distributions) to capture the tail returns/risks. Among them is downside deviation as an alternative to the standard deviation and beta measures of the traditional Gaussian distribution. Still, the use of this measure was disputed because it does not consider the full distribution of financial returns. The Value at Risk (VAR) measure advocated by Beder (1995) was and is still widely accepted and used by policymakers. Yet as mentioned earlier, the VAR measure has some drawbacks; Artzner et al. (1999) established that the measure is not coherent.

Despite the flaws many policymakers used VaR as a measure of downside risk exposure until the collapse of LTCM in 1998 when researchers began to criticize it as inaccurate. Li (1999) proposed a new approach based on skewedness and kurtosis in addition to volatility were proposed to estimate VaR. Farve and Galeano (2002) used a similar approach when they developed the Modified Value at Risk. Likewise, Rockafella and Uryasev (2000) came up with another measure called the expected shortfall also known as Conditional VaR (CVaR). When Alexander and Baptista (2004) compared VaR and CVaR, they proved that CVaR is more effective. In a similar approach, Agarwal and Naik (2004) showed that the mean-variance approach underestimates the left-tails and that the CVaR measure is more resourceful.

Bollerlev and Todorov (2011) employed time series analysis which yields the cross-sectional approaches for estimating tail risk. They investigated this model using high-frequency data from S & P 500 futures in estimating model-free index of investors fear. The scholars reached the conclusion that investors who take on more risks are compensated for tail events. However, their definition of tail risk varies from the above-mention one, for them tail risk is the jumps in asset prices. Distaso et al.(2009) used quantile regression investigated the link between systemic crisis and tail risk in hedge funds. They established that tail inter-dependencies are higher in times of distress.

There is a growing interest in quantitative theories in particular the application of Extreme Value Theory (EVT) - that appears to offer accurate measures of extreme risks. Bali (2003) and Gençay and Selcuk (2004) found that EVT models not only outperform traditional VaR models but they give better estimates of tail risk. The implication of these findings is that the standard VaR methodologies can be improved by exploiting EVT. This is demonstrated by Kelly and Jiang (2014) who used extreme value theory and suggested a global tail risk measure for common fluctuations; this approach uses to estimate lower tail. They used a dynamic power law structure that takes into account systematic tail risk and asset specific tail risk and demonstrated that tail risk has strong prognostic power for portfolio returns.

Santos (2015) crated a Hedge Fund Tail Risk measure by extending the extreme value theory to incorporate a other factors that controls for tail situations, this measures is superior to the one offered by Kelly and Jiang (2014). More recently, Almeida et al. (2016) using stock returns proposed a novel model of estimating tail risk that incorporates risk-neutral information. One notable advantage of this model is that it precludes the use of option data.

In addition to tail risk, scholars have shown an interest in tail dependence and time-varying tail distributions. The copula theory has received a lot of attention because of its effectiveness in capturing different patterns of tail dependence. Patton (2006) used copula theory to study the correlation between exchange rates during depreciations and during appreciation. Equally, by conditioning variables using Joe-Clayton copula, Michelis and Ning (2010) considered the dependence between exchange rates and stock returns.

Besides copula, Markov switching processes have been used to examine time-varying tail distributions. For instance, Ballio et al. (2007) and Litzenberger and Modest (2008) applied models of Markov regimes to determine time varying exposure in varying market conditions. Researchers like Brown and Spitzer (2006), Bacmann and Gawron (2004), and Distaso et al. (2009) have also scrutinized tail dependency in hedge funds. These studies observed time-varying tail dependency in the structures of hedge funds and concluded that normality assumption is not appropriate for hedge funds.

### 2.2.2 Copula Models

The application of copula theory to risk modeling has increased because copula models take into account the dependency between individual risks when aggregating them (Joe, 1997). In addition, copulas models capture the correlation between extreme events, thus replicating the probability of sporadic events buildup. This is because disruptive events do not just happen as a result of short, sharp shocks but rather are drawn out over long periods when disconnected risks become connected over time via intermediate linkages. Thus, copulas are robust mathematical tools to model such dependencies. Notably, the Norma copula is a benchmark model for many financial applications. However, massive evidence has been cumulated against the Normal copula, see Nelsen (1999).

The natures of copula application vary from risk management, financial markets, derivative contracts, and portfolio decision problems. Nonetheless, the pioneer users of copulas in finance were the field of risk management. In risk management, the challenge is to estimate VAR and probability of large losses (tail events). Cherubini and Luciano (2011) and Embrechts et al. (2000) are just a few of researchers who studied portfolio value at risk using copula models. Li (2000) proposed Gaussian copula models in pricing of credit derivatives specifically to price collateralized debt obligations (CDOs). His model gained wide-spread popularity in correlating associations between securities. The mathematical elegance in Li's model allowed complex investments like mortgages to be correctly priced.

Others like Embrechts et al. (2003) and Bouye et al. (2000) pioneered the use of copula to measure dependencies in financial markets. These scholars were interested on the idea that non-normal joint distributions could be used to estimate portfolio shortfalls. Rosenberg and Schuermann (2006) suggested the use of copula to solve integrated risk management problems that are considered jointly. Copula is also used to model portfolio decision problems; determining the correct weights that maximize expected returns.

Patton (2004) used copula (the multivariate distribution of assets) to solve the bivariate equity portfolio problem. Using the same approach Hong and White (2005) used copula to determine the weights of eleven equity portfolios. Other notable uses of copula in finance include studies by Kang (2007) who examined the dependence structure of hedge funds and Zimmer (2012) who explore the role of copula in the recent housing crisis.

Regarding systemic risks, copula-based dynamic models are proposed to estimate the measures of systemic risk. Oh and Patton (2016) employed a copula model to probability of individual firm distress as well as the joint probability of distress (systemic risk measure). Recently, vine copulas have been proposed to model the complex interdependence between financial institutions due to their flexible structure that overcomes the limitation of copula in higher dimensions. Such studies include works by Aas et al. (2009) who tipped the use of vine copula in finance and insurance. A similar study used vine copula model to analyze interdependencies between different borrowers.

On tail dependence, Bhatti and Nguyen (2012a) modeled the dependency structure of international stock markets using extreme value theory as well as time-varying (dynamic) copula. Nguyen and Bhatti (2012b) employed semi-parametric copula to illustrate the relationship between stock markets and oil prices. Likewise, Naifar (2012) investigated dependence in presence of jump-risk between equity returns, risk premium, and volatility.

Taking into account the conclusions reached on this topic, this paper focus on co-movement of tail risk and systemic risk using a non-simple correlation analysis: the copula approach. Per Staudt (2010) copula are "powerful mathematical tool for modeling the joint distribution of simultaneous events". In the context of systemic and tail risk, the copula is motivating because it permits us to decouple the dependence structure (that which is associated with systemic risk) from the marginal distribution (that which is associated with tail risk) and model each one distinctly with more degree of accuracy. Staudt (2010) clearly argues that copula are critical in understanding higher- dimension dependence due to their invariant nature using strictly increasing transformations of random variables. Likewise, Schuermann (2006) pointed out that copula are especially useful when dealing with financial information because it lets us examine tail dependency.

As mentioned, this article acknowledges that for financial appliances the model construction should include strong dependencies between extreme tail events (losses) in place of extreme gains. Staudt (2010) states that financial markets are described by this effect, therefore, simple Archimedean copulas specifically Gumbel's and Clayton's that effectively take into account asymmetric tail dependency is used to evaluate tail risk and systemic risk in this case. The Gumbel copula takes into account upper tail dependence whereas the Clayton Copula takes account of the lower tail dependence. In addition, this paper includes the benchmark Gaussian Copula- symmetric with tail independence- as a comparative estimate to the other two copulas.

## 2.3 Conceptual Frame work

As mentioned earlier, the focus of this study is to examine the association between systemic risk and tail risk using copula. This involved obtaining the marginal distribution of tail risk and dependency structure of systemic risk. Copula allows us to predict each one differently with a bigger degree of accuracy in order construct a resilient financial model that takes into account the correlation between systemic and tail risk.

# 3 Copula Models for Tail and Systemic Risk

## 3.1 Introduction

Kort (2007) asserted that the association among more than one random variable is completely determined by their joint distribution function. Currently, the most popular tool for modeling dependency is the correlation coefficient. But correlation being among the universal concepts in financial theory, correlation is also taken the wrong way in most literature (Embrechts et al., 1999). Per Artzner et al., (1999), the widespread assumption and use of the Gaussian distribution led to the interpretation of dependence as correlation. This misunderstanding stems from the use of correlation in literature to imply dependency. For instance, Modern portfolio theory pioneered by Harry Markowitz (1969) assumes that asset returns follow Gaussian distribution. However, this is but only one of the canonical measures of dependency in the field of multivariate joint distributions.

Mathematically, measures of dependency provide a solution to the complicated nature of dependency in the bivariate case. Thus the main task is to construct multivariate function which is dependable with its corresponding marginal function and correlations (Embrechts et al., 1999). Over the last decades, different distributions have been proposed to replace the normality assumption because empirical research revealed that distributions of actual financial data is rarely normally distributed and as such, the correlation coefficient became inadequate in modeling dependency (Embrechts et al. 1999).

### 3.1.1 Linear Correlation

#### Pearson's correlation coefficient

This correlation coefficient determines the degree of dependency (correlation) amid two linear variables. It is only a reasonable measure when the distribution of risk variables is bivariate normal. A correlation of +1 indicates positive dependence (rise or fall together), -1 implies negative dependence (one rise and the other falls), and a zero correlation suggests the risk variables are independent (Joe, 1997). The correlation measure among X and Y is

$$\rho(X,Y) = Cov(X,Y) \operatorname{div} \sqrt{VarX \times VarY} \tag{3.1}$$

The interpretation of linear correlation to more than two variables is forthright; the summary of the variance-covariance matrix vector yield the corresponding correlations. The acceptance of the linear correlation is because of its ability to offer straight forward calculations.

### 3.1.2 Rank correlation

#### Spearman's rank correlation

Assumed two indiscriminate variables X and Y with marginal distributions G1 and G2 in that order and joint distribution F, then, the Spearman's rho measure is specified as:

$$\rho_s(X,Y) = \rho(F_x(X), F_y(Y)) \tag{3.2}$$

(Schweizer and Wolff, 1981)

#### Kendall's rank correlation

Equally given (X1,Y1) and (X2,Y2) pairwise independent random variables with joint function F, the Kendall's tau coefficient is specified as:

$$\tau = prob[(X_1 - X_2)(Y_1 - Y_2) > 0] - prob[(X_1 - X_2)(Y_1 - Y_2) < 0] \tag{3.3}$$

(Schweizer and Wolff, 1981)

Staudt (2010) suggested that to overcome the problem of outliers, rank dependency coefficients like Kendall's tau and Spearman's rho are suitable because it works on ranks of the data instead of using the nominal values. Embrechts et al. (1999) emphasized that while the Pearson's linear coefficient only determines the linear relationship, both rank correlation measures can quantify the amount of monotonic association between binary set of random variables. Both Spearman's rho and Kendall's tau are symmetric measures of association that take values between [-1, 1].

And unlike linear correlation measure, a value of zero (for independent risk variables) in the case of rank measures does not automatically suggest unconventionality. Additionally, it has been used in cases of nonlinear relationship and are invariant to some transformations such as the natural logarithm. The invariant property of rank correlation is very useful in modeling probabilities; thus rank correlation (gives a way of fitting copula to data) have a natural place in copula mathematics. Hitherto, it is not as easy to compute linear correlation as rank correlations since they are not moment-based correlations (Joe, 1997).

### 3.1.3 Tail dependency

The measures of tail relationship determines the level of association that exist in extreme tails of the joint functions (i.e. measures any co-movements in extremes of the distribution of a pair of random variables). Extreme tail association is useful when examining extreme tails of a set of risk variables G and H that exhibit no correlation (Kort, 2007).

However, it is important to understand that tail dependency is not the same as dependency and it is possible for two risk variables to be dependent but exhibit no dependence in their extreme tails. In situations of continuous marginal functions, extreme relationship amid the two risk variables can be measured using the copula approach due to its invariant property under severely increasing transformations (Embrechts et al. 1999). Staudt (2010) claimed that the tail dependence is a good alternative that measures the systemic risk between the risk variables. *"An interesting result from the tail dependence statistic shows the normal copula lack tail dependence"*.

**Upper tail dependency**

The measure of extreme upper association between binary random variants W and X with cumulative densities Y and Z respectively is shown to be:

$$\lambda_U = Lim_{u \to 0}(Prob[X > Z^{-}1(u)/W > Y^{-}1(u)]) \tag{3.4}$$

as long as a limit $\Lambda \exists [0,1]$ exists.

If $\Lambda \in [0,1]$, then the two variants M and N are asymptotically related (in the upper extreme) otherwise when $\lambda = 0$ it is said they are asymptotically non-dependent (Embrechts et al. 1999).

**Lower tail dependency**

Likewise, the coefficient for corresponding lower extreme dependency is specified as:

$$\lambda_L = Lim_{l \to 0}(Prob[X \leqslant Z^-1(U)/W \leqslant Y^-1(u)]) \quad (3.5)$$

as long as a limit $\Lambda \in [0,1]$ exists.

Similarly, lower tail dependence is present if and only if $\lambda \neq 0$.

### 3.1.4 Positive Quadrant Dependence

This is a specific kind of dependence, given two random variables X and Y, X and Y are positive quadrant dependent (PQD) if:

$$Prob(X > x, Y > y) \geqslant Prob(X > x)Prob(Y > y)$$

This means the probability that the two random variables are simultaneously large is at least greater as it would if they were independent.

### 3.1.5 Limitation of correlation

Per (Bouye et al., 2000), correlation is only a sensible measure when dealing with multivariate normal distribution and when this is not the case, the following problems are encountered;

- Its inability to comprehend the dependence in non-linear relationships- correlations do not work with nonlinear dependencies and there is need for alternative ways for understanding the complete joint function of the multivariate functions.
- Correlation is easily distorted by outliers
- Correlation does not exist without the variances of both variables are determined
- Correlation is not invariant under transformation of the risk variables (Kort, 2007).
- The traditional preposition that independence implies zero correlation is flawed because zilch association does not necessarily mean non-association (unless the normality assumption is assumed)

## 3.2 Copula

A comprehensive meaning of the copula function was offered in Nelsen (1999) in which he states that copulas are are special tools that act as a connection between the marginal functions to their corresponding multivariate distribution functions. A copula function allows us to detach the marginal component from the dependency construct. A novelty and important discovery in mathematics, in fact, Fisher (1997) noted that copulas are important in mathematicians for two main reasons namely (a) they pave way for studying scale-free measures of dependency, and (b) are foundation for constructing bivariate distributions.

### 3.2.1 Skalar's theorem

Skalar's (1959) stated that if $F$ is a d-dimensional distribution function with marginals

$$F_1, F_2, F_3, ..., F_n$$

Then there is a d-dimensional copula $C$ such that for all $x \in R^d$:

$$F(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n))$$

If $F_1, ..., F_n$ are all continuous, then $C$ is unique.

Conversely, if $C$ is a d-dimensional copula, and $F_1, ..., F_n$ are distribution functions, then $F$ defined by:

$$F(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_{n)})$$

is a d-dimensional distribution with margins $F_1, ..., F_n$.

**Definitions**

According to Nelsen (1999) both of the following copula definitions are 'coupled' by Sklar's theorem (1959), the main finding in copula theory.

**Informal Copula Definition**

Let X and Y be continuous random variables with distribution functions

$$F(x) = P(X \leq x)$$

and

$$G(y) = P(Y \leq y)$$

,

With the joint distribution function be given as

$$H(x,y) = Prob(X \leq x, Y \leq y). \forall (x,y) \in [-\infty, +\infty]^2$$

Consider the point in $I^2(I = [0,1])$ with coordinates $(F(x), G(y), H(x,y))$.

This mapping from $[0;1]^2 \to [0;1]$ is a 2-dimensional copula (Nelsen, 1999).

Similarly, an n-dimensional copula is a function C that maps $[0;1]^n \to [0;1]$.

**Formal Copula Definition**

An n-dimensional copula is a function $C : [0;1]^n \to [0;1]$ with specified properties. To expound the properties of copulas functions; let us consider the bivariate case where the copula is derived from two variables. As mentioned above, the 2-dimensional copula C is a function $C : I^2 \to I$ such that;

1. 1. Copula functions are grounded.

   This property implies that $C(0,v) = C(u,0) = 0$

2. Copula functions are 2-increasing.

   This is basically a two dimension version of non-decreasing functions in one dimension.Mathematically, this property can be represented as; C is 2-increasing: for a, b, c, d in I with and if;

   $$C\ (b, d) - C\ (a, d) - C\ (b, c) + C\ (a, c) > 0$$

3. When one variable is equal to 1, the copula is equal to the other variable I.e.

$$C(1,x) = C(x,1) = x.\forall I$$

4. The copula functions should lie between the Frechet-Hoeffding bounds:

$$W(u,v) = \max(u+v-1,0) \leqslant C(u,v) \leqslant \min(u,v) = M(u,v)$$

.

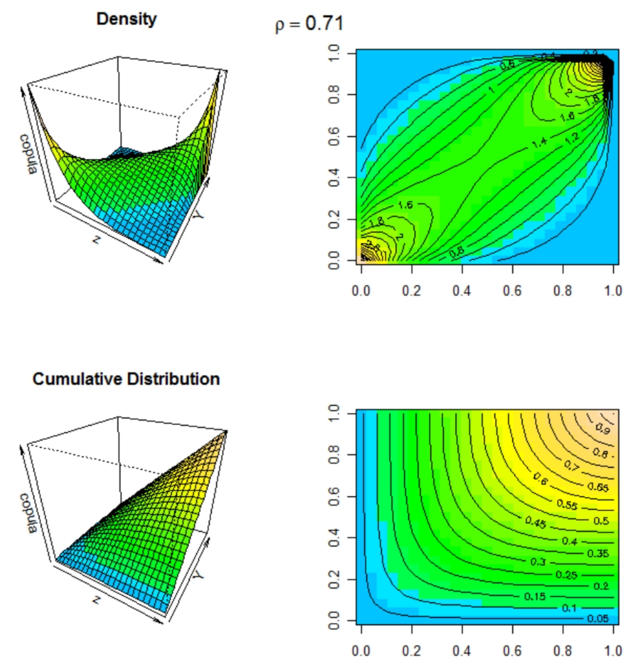
**Frechet-Hoeffding bounds**

Hoeffding (1940) and Frechet (1951) suggested that for a copula to be considered valid, it must lie within the $W(u,v) = \max(u+v-1,0) \leqslant C(u,v) \leqslant \min(u,v) = M(u,v)$ bounds. This inequality is known as the Frechet- Hoeffding bounds where W and M are the lower and upper bounds respectively. What is intriguing is that the bounds W and M are also copulas by themselves. These copulas together with the product copula $\Pi = UV$ plays a significant role in explaining copulas. Per Nelsen (1999) these three copulas have important statistical interpretations;

- The copula of U and V is M (u, v) if and only if U and V is almost surely an increasing function of the other.
- The copula of U and V is W (u, v) if and only if U and V is almost surely a decreasing function of the other.
- The copula of U and V is $\Pi$ if and only if U and V are independent.

All aforementioned copulas form a crucial class of copulas well known as Fundamental copulas (Joe 1997). Because of the requirement that the copulas have to be within the Frechet-Hoeffding bounds, it is understandable that while some copulas will cover both bounds, others can fail to cover both bounds. Comprehensive copulas refer to a family of copulas that are capable of modeling the three aforementioned copulas.

**Figure 3.1. Fundamental Copula density plot**




Per Patton (2002), copulas are powerful tools to model dependence because they are not only flexible but there are also many types (variety) of dependence they allow for.

### 3.2.2 Copula and Dependence Measures

Staudt (2010) stated that copula are either symmetric or not. Therefore, they are the best tools for measuring dependency and can be easily connected to the dependency measures discussed earlier. Nonetheless, the main objective here is to understand the connection in specific cases of Archimedean copulas (Franks, Clayton, t-copula and Normal (Gaussian) copula). Considering the dependency measures already discussed, their links to copula are defined as follows:

a. Kendall's Tau: is linked to copula by the following formula

$$\rho_\tau(X,Y) = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1$$

.

b. Spearman's rho is related to copula in general through the following ways:

$$\rho_s(X,Y) = 12\int_0^1\int_0^1 C(u,v)dudv - 3$$

$$\rho_s(X,Y) = 12\int_0^1\int_0^1 uvdC(u,v) - 3$$

$$\rho_s(X,Y) = 12\int_0^1\int_0^1 C(u,v) - uvdudv$$

c. Positive Quadrant dependence: in relation to copula, two variables are said to be positive quadrant dependent if;

$$C(u,v) \geq uv$$

d. Upper tail dependence: where $C$ is the copula function for $(U,V)$, it follows that

$$\Lambda_u = Lim_{u\to 1}\frac{1-2u+C(u,u)}{(1-u}$$

If $\Lambda_u$ exists and is greater than 0 but less than or equal to 1 then C has upper tail dependence and if $\Lambda_u$ is 0 then C has no upper tail independence.

e. Lower tail dependence: similarly,

$$\Lambda_l = Lim_{u\to 0}\frac{C(u,u)}{u}$$

Equally, if $\Lambda_L$ exists then C has lower tail dependence as long as $\Lambda_L$ is greater than 0 but less than or equal to 1 and has no lower tail independence if $\Lambda_L$ is equal to 0

### 3.2.3 Types of Copula

This section provides a brief overview of parametric families of copula. However, we will concentrate on copulas that portray deep tail dependency; the Archimedean family. Generally, copulas are classified into three main categories namely;

- **Frechet family or Fundamental copula** - they are constructed by taking affine linear combinations of the Frechet-Hoeffding bounds and the product copula. This family is can depict positive dependency, negative dependency, and independence. The coefficient of tail dependence is found by solving the weights in linear combination.
- **Implicit or elliptical copula** – this family of copulas stems from elliptical distributions and is symmetric nature. This implies that their lower and upper tail dependency coefficients are equal. Examples include the Normal copula and the student’s t copula.
- **Explicit copulas** - this family satisfying certain properties and are constructed using general mathematical constructions. They are also known as the Archimedean copulas and take a variety of forms. As a result, they depict distinct lower and upper tail dependence coefficients and are very suitable for modeling market data (such as asset prices) with profound tail dependence.

## 3.3 The Archimedean copulas

Nelsen (1999) define bivariate Archimedean copulas as the following:

$$C(u_1, u_2) = \psi^{-1}(\psi(u_1) + \psi(u_2)) \tag{3.6}$$

where $\psi$ is a strict generator and $\psi^{-1}$ is completely monotonic on the intervals $[0, \infty)$ These copulas are resultants of parametric generators/ generator functions. These generator functions represented as $\psi$(t) have precise properties discussed below.

**Properties of the generator function;**

- Continuous
- Decreasing
- Convex with $\psi(1) = 0$
- The generator function is said to be strict if $\psi(0) = +\infty$

**Definition:Pseudo-inverse**

The pseudo-inverse of $\psi$ is given as;

$$\psi^{-1}(t) = \psi^{-1}(t), 0 \leqslant t \leqslant \psi(0) \tag{3.7}$$

Now, let us consider $\psi(t)$ a continuous, decreasing, convex function $\psi : [0,1] \to [0, \infty)$ such that $\psi(1) = 0$. Then, the Archimedean copulas function C is defined as;

$$C^A(u_1, u_2) = \psi^{-1}(\psi(u) + \psi(v)) \tag{3.8}$$

The density function of the Archimedean copula with generator $\psi$ is given by;

$$C^A(u, v) = \frac{-\psi''(C(u,v))\psi'(u)\psi'(v)}{[\psi'(C(u,v))]3} \tag{3.9}$$

Note that,in the case of a strict generator function $\psi^{[-1]} = \psi^{-1}$

Coles and Tawn (1991) found that Archimedean copulas are popular and desirable because they present several desired properties (symmetric, associative) as well as the fact that they simplify calculus.

### Construction of the Generator Functions

There are various methods of constructing generator functions. However, the most popular method is the use of Laplace transform. Consider $\psi^{[-1]}(t)$ the inverse of the generator function $\psi(t)$ and let G be the distribution function of a non-negative random variable (Z), Marshall and Olkin [1988] showed that the relationship between the two is the Laplace transform. Therefore, the inverse of the Laplace transform is a generator of Archimedean copula, i.e. the Laplace transform is given as;

$$\psi^{[-1]}(t) = \int_0^\infty \exp^{tz} dG(z)$$

, then $\psi = \psi^{-1}$ generates a strict Archimedean copula.

This generator function has the following properties;

- $\psi^{[-1]}(-t)$ is a moment generating function for the random variable Z.
- $\psi^{[-1]}(t)$ is continuous and strictly decreasing with $\psi^{[-1]}(0) = 1$ and $\psi^{[-1]}(\infty) = 0$
- $\psi^{[-1]}(t)$ is a completely monotonic function.

A good example of a generator function that is derived from Laplace transform is the Clayton copula generator where the random variable follows gamma distribution with parameters $(\gamma, 1/\gamma)$

### Properties of Archimedean copula

- They are symmetric in nature; i.e. C (u, v) = C (v, u)
- Archimedean copula are associative i.e. C (C (u, v), w) = C (u, C (v, w))
- For any constant k>0, k$\psi(t)$ is also a generator

### Archimedean copulas and dependence measures

As mentioned above, Archimedean copulas are popular because of their tractable properties that simplify calculus. This means they are easy to link with dependence measures such as the Kendall's tau, upper tail dependence, and lower tail dependence.

Genest and MacKay (1986) proved the relation between Kendall's tau and the copula generator function in the bivariate case as;

$$\tau = 1 + 4\int_0^1 (\psi(u)/\psi'(u))du$$

Correspondingly, the relation between tail dependency and Archimedean copula generator function was demonstrated by Joe (1997). He argued that if $\psi$ is a strict generator such that$\psi^{[-1]}$ belongs to the class of Laplace transforms for strictly positive random variables. If $\psi'(0)$ is finite and different from zero, then the copula does not have tail dependence.

$$C(u,v) = \psi^{[-1]}(\psi(u) + \psi(v))$$

if $\psi^{-1}(0)$ is finite, then the copula from that generator function does not have upper tail dependence. But where $1/\psi'(0)$ or $\psi^{[-1]}(0) = -\infty$ the copula has upper tail dependence with coefficient given by;

$$\Lambda_u = 2 - 2Lim_{\rightarrow +0} + [\psi'(s)/\psi'(2s)$$

Equally, the coefficient of lower tail dependence is given as;

$$\Lambda_l = 2Lim_{\rightarrow +\infty} + [\psi'(s)/\psi'(2s)$$

In the next section, we will discuss the closed formula for the Clayton, Franks, and Gumbel copulas. Additionally, we will discuss the explicit formulas between the dependence measures and the copula parameter.

### 3.3.1 The Frank Copula

The Frank Copula (1979) is a member of the Archimedean family whose generator function is given by:

$$\psi(t) = -\log[\exp^{-\theta t} - 1/\exp^{-\theta} - 1] \quad \textbf{(3.10)}$$

The Frank's Copula has the following properties;

- Is comprehensive
- Is a strict copula
- Is radially symmetric
- Its probability distribution function is given by;

$$C^{Fr}(u, v; \theta) = -\theta^{[-1]} \frac{\log 1 + (\exp^{[-\theta u]} - 1)(\exp^{[-\theta v]} - 1)}{(\exp^{[-\theta]} - 1)} \quad \textbf{(3.11)}$$

In a scenario where $\theta$ is the copula parameter that assumes any value in the Euclidean space of dimension one. In contrast to the Gumbel and the Clayton Copula, the Frank copula offers a highest attainable range of dependence. Consequently, the dependence parameter of the Frank's copula permits an estimation of the lower and upper Frechet-Hoeffding bounds, implying that the Frank copula allows for modeling of positive and negative dependence in a data set.

When $\theta$ approaches $-\infty$ and $+\infty$ the lower and upper Frechet- Hoeffding bounds while the independence case is attained when $\theta$ approaches zero. This implies that the value of $\theta$ affects the level of dependence, consequently, a higher level of $\alpha$ indicates more dependence.

Nevertheless, it is important to point out that the Frank copula has neither upper nor lower tail dependence $\lambda_L = \lambda_U = 0$. This concept is known as radial symmetry i.e. $C(u, v) = u + v\check{~}1 + C(1 - u, 1 - v)$. Thus, the Frank's copula is appropriate for modeling information is characterized by weak tail dependence. The Spearman's rho and Kendall's tau are linked to the Franks copula by Debye function and yields,

$$\tau_k = 1 + 4[D_1(\alpha) - 1]/\alpha$$

$$\rho_s = 1 - 12[D_2 - \alpha - D_1 - \alpha]/\alpha$$

The Debye function given by;

$$D_k(\alpha) = \frac{k}{\alpha^k}\int_0^{\alpha} \frac{t^k}{\exp^1 - 1} \quad \textbf{(3.12)}$$

### 3.3.2 The Clayton Copula

Introduced by Clayton (1978), the Clayton copula is also known as the Pareto family of copulas. It is widely used in modeling correlated risks due to its ability to display lower tail dependence. The Clayton generator function is;

$$\psi(t) = 1/\theta(t^{-\theta} - 1) \quad \textbf{(3.13)}$$

The Pareto family of copulas has the following properties;

- Is strict for $\alpha > 0$
- Is comprehensive
- With a probability distribution function given by;

$$C^{CL}(u, v; \theta) = \max[0, u^{-\theta} + v^{-\theta} - 1]^{-1/\theta} \quad \textbf{(3.14)}$$

Where $\theta$ the copula parameter is strictly in the interval $(0, \infty)$, when $\theta = 0$ the marginal distributions become independent, when $\theta \to \infty$ the copula approximates the upper Frechet- Hoeffing bound. The Frechet- Hoeffing lower bound cannot be approximated due to the restriction on dependence parameters. The implication of this assumption is that the Clayton copula does not account for negative dependence. The relationship between the Clayton copula and Kendall's tau is stated as;

$$\tau = \alpha/(\alpha + 2)$$

Equally, Clayton copula is related to lower tail dependence in the following way;

$$\Lambda_l = 2 - 1/\alpha$$

(Cherubini, Luciano, and Vecchiato, 2004).

Note that the link between Clayton copula and the Spearman's rho is very complicated.

### 3.3.3 Gumbel Copula

The Gumbel (1960) is popular for modeling asymmetric dependence in data. According to Staudt (2010), this Archimedean copula is capable of capturing strong upper tail dependence and weak lower tail dependence. Similar arguments were made by Cherubini, Luciano, and Vecchiato (2004) who suggested that the Gumbel copula should be used when expected outcome have less correlation at low values but strong association at higher values.

The Gumbel Copula generator function is;

$$\psi(t) = (-\log(t))\theta \tag{3.15}$$

The properties of the Gumble copula are;

- Is strict
- With the distribution function given as;

$$C^{GU}(u, v; \theta) = \exp^{-[-\log(u)\theta]+[-\log(v)\theta]^{1/\theta}} \tag{3.16}$$

Where the copula parameter $\theta$ is restricted on the interval $[1, \infty)$, when $\theta \to 1$,the marginal distributions become independent and approaches the Frechet- Hoeffding upper bounds when $\theta \to \infty$. The Gumbel copula is similar to the Clayton copula in that they only portray cases of positive dependence and independence but fail to account for negative association. The link between the Gumbel parameter and the Kendall's tau is shown to be;

$$\tau = (\theta - 1)/\theta$$

or

$$\tau = 1 - \theta^{-1}$$

Similarly, the relationship with upper tail dependence is $\lambda_u = 2 - 2^{1/\theta}$ while the lower tail dependence is $\lambda_l = 0$. Note that the connection Gumbel copula parameter and Spearman's rho have is not in a closed form.

### 3.3.4 Copula Parameter estimation

Just like other copulas, Archimedean copulas involves several underlying functions namely the joint cumulative distribution functions and the marginal cumulative distribution functions. According to Charpentier, Fermanian, and Scaillet (2006), copula parameter estimation involves how to estimate separately the joint law and the margins. The copulas discussed above are characterized by a dependence parameter that we need to estimate. Depending on the assumptions made, different approaches can be taken to estimate the parameters (we can use parametric, semi-parametric, or non-parametric).

#### Non-Parametric (Empirical) method

Although not widely used, in this method, the practitioner may choose to use 'empirical counterparts' or invoke well known smoothing statistical methods. This method is simple and straight forward because parameters are estimated from the observations instead of making initial assumptions about the distribution of the data and later verifying the validity of assumptions.

#### Parametric methods

Although there are numerous methods of parametric estimation, the most widely used estimation methods are the Inference for margins (IFM) and the full maximum likelihood estimation (FML) also known as exact maximum likelihood method (EML). Below is a detailed discuss of these methods

#### Full maximum likelihood estimation (FML)/Exact maximum likelihood method

The FLM is the simplest and most direct method of estimation that maximizes the log likelihood. This method not only yields the copula parameter but also gives the marginal distribution parameter (i.e. provides the parameters of the marginal distributions and the copula simultaneously).

Considering Sklar's theorem, the canonical form of joint density of random variables X1 and X2;

$$F(X_1, X_2) = C(F(X_1)F(X_2))$$

Differentiating this expression gives the canonical form:

$$f(x_1, x_2) = c(F_1(x_1)F_2(x_2))f_1(x_1)f_2(x_2)$$

Where $c$ is the copula density function $f_i$ and $F_i$ are the marginal density and distribution function respectively.

Our aim is to find $\theta \in \Theta$ that maximizes the likelihood function

$$l(\theta) = cF_1(x_1)F_2(x_2)f_1(x_1)f_2(x_2)$$

This$\theta$ also maximimes the log-likelihood;

$$\log l(\theta) = \log c(F_1(x_1)F_2(x_2) + \log f_1(x_1)f_2(x_2)$$

Then the maximum likelihood estimator (MLE) is given by:

$$\Theta_{ML} = argmax \log(\theta)$$

**Inference for margins (IFM)**

Joe and Xu (1996) proposed this simpler method in which the first step is to estimate the margins parameters and then the second step is to estimate the copula parameters. This method significantly reduces the computation of finding the optimal estimators (Joe, 2005).

$$\frac{\partial^2}{\partial_x \partial_y} C_\theta F_\alpha(x), G_\beta(y) = [c_\theta(x), G_\beta(y)] f_\alpha(x) g_\beta(y)$$

The first marginal are estimator is:

$$\theta_1 = argmax \sum_i^n \sum_i^n \log f_i(x_i; \theta_1$$

The copula estimator given the marginal estimator is:

$$\Theta_2 = argmax \sum_i^n \log c(F_1(x_1), F_2(x_2); \theta_1, \theta_2)$$

This results in the IFM estimator:

$$\Theta_{IFM} = (\theta_1, \theta_2)$$

Joe (1997) found that under certain conditions, the IFM estimator verifies property of asymptotic normality and can be shown that the estimator is consistent. However, if the margins are incorrectly specified, the estimation of $\Theta_{IFM}$ may be seriously affected (Kim et al., 2007).

### 3.3.5 Goodness of Fit Tests/ Copula Selection

In order to establish the best copula for the data, a goodness of fit test must be conducted. Although there is no universally acceptable statistic standard for selecting the best copula, we consider two tests namely the Akaike Information Criterion (AIC) and the Schwarz Information Criterion (SIC) (Palaro and Hotta, 2006). Dias and Embrechts (2004) found that the AIC tends to be superior in small samples whereas the Schwarz Information Criterion (SIC or BIC)) works best in larger samples. In this research, both goodness-of-fit-tests shall be applied to choose the copula that offers the suitable fit, i.e. the copula that matches the lowermost values in these criterions.

### The Akaike Information Criterion

Proposed by Akaike (1973), the Akaike information criterion is simple yet very flexible for determining the best statistical model. This method is based on earlier models proposed by Kullback (1951) and Kullback (1959) who suggested using the Kullback-Leibler (1951) information quantity that measures the distance between the true model (observations) and the model of interest. The task is to minimize this distance in order to arrive at the best model.

Initially, the Kullback-Leibler (1951) information quantity recommended the expectation of the log likelihood of the distribution proposed and that of the log likelihood of the true distribution but Akaike (1977) suggested the use of mean log likelihood as an estimate of the Kullback-Leibler information quantity. From these developments, the Akaike Information Criterion statistic was introduced defined as;

$$AIC = -2Likelihood + 2k$$

; k is the number of parameters

### The Schwarz Information Criterion

Also known as the Bayesian Information Criterion, the Schwarz Information Criterion was derived by Schwarz (1978). This criterio is more favorable because it is not only linked to Bayesian statistics but it also does not require the input of the prior information. It is given as;

$$SIC = -2Likelihood + K \times \ln(n)$$

;k is the number of parameters

# 4 DATA ANALYSIS AND INTERPRETATION

## 4.1 Data Description

Here, we evaluate the relevance of copula model in assessing the systemic and tail risk within a stock market. The dataset consist of 2 stock indices from strongest economies globally. The daily closing prices in Euro from was obtained from the stock market official website. The stock price indices are converted to log-returns (continuously compounded returns). This results in 2910 observations per stock index. The estimation of parameters and model building were performed using 12-year time frame, specifically from January 2007 to July 2018. The analysis was conducted using the software package R.

This data-set is interesting from a number of reasons. First, it displays presence of systemic risk (i.e., a peril/financial threat which affect the FTSE 100 index is also very likely to affect the S& P 500 (US) stock). In addition, there is also evidence of tail risk in the fat right tails, of the kernel densities fit to historic FTSE 100 (UK), and the S& P 500 (US) stock indices (see Figure 4.1). Figure 4.1 shows the distribution plots and the time plots of the FTSE 100 (UK), and the S& P 500 (US) stock indices. The returns trend clearly hints a positive dependence between the stocks in the sense that whenever one of the two prices goes up or down, so does the other one.

**Figure 4.2. The time plots and the distribution plots of the FTSE 100 (UK), and the S&P 500 (US) stock indices**

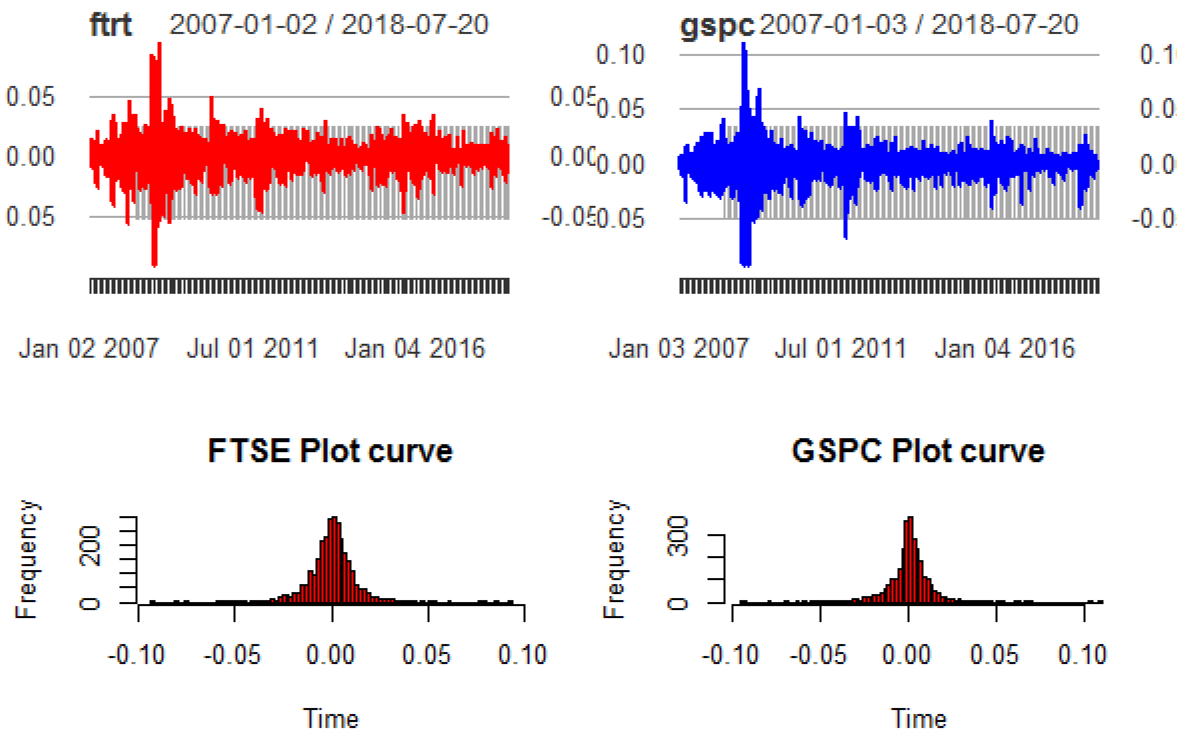



Figure 4.1 shows volatility clustering in which large (small) returns is followed by large (small) returns. Likewise, the distribution plots indicate that the returns are not normally distributed but tend to be more leptokurtic. In close inspection, the time plots shows those durations of high volatility are associated with financial crisis such as the 2008-2009, late 2011, and more recently in mid-2015.

Table 4.1 indicates that the return series have a positive mean. Moreover, the distributions are slightly left-tailed as shown by their negative skewness. Therefore, as expected from the distribution plots, all kurtosis values were higher than the kurtosis value of the normal distribution (3).This further ascertains that the distributions are leptokurtic.

| Stock Index | Mean | Median | Maximum | Minimum | Std. deviation | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| FTSE | 0.00005 | 0.00004 | 0.0094 | -0.0093 | 0.0121 | -0.1333 | 7.7979 |
| S&P 500 | 0.0002 | 0.00064 | 0.1096 | -0.0945 | 0.0125 | -0.3682 | 10.8618 |

**Table 4.1. Descriptive statistics of the FTSE and the S&P 500 stock indices**

Nonetheless, we carried out non-normality test to test the supposition that the returns series are typically dispersed at the 1% meaningful level. The JarqueBera (JB) test is used to assess the non-normality of returns. As expected, the JB test affirmatively rejects the non-substantiated hypothesis that returns are usually distributed at the 1% significance level.

Similarly, we used Ljung-Box Q-test (LB) to measure the volatility clustering observed in the time plots. We employed the Ljung-Box Q-test (LB) to test the null hypothesis of no serial correlation at the 1% significance level. We performed the test on the first 12 lags of the squared log-returns; the p value is low. Therefore we can reject the null hypothesis (no serial correlation) and say that there is serial correlation in the log returns.

| Stock Index | Jarque-Bera | JB P-Value | Ljung Box for squared log returns | LB P-Value |
|---|---|---|---|---|
| FTSE | 7413.3 | 0 | 2366.2 | 0 |
| S&P 500 | 14381 | 0 | 3373.8 | 0 |

**Table 4.2. Jarque-Bera and Liung Box tests to test respectively for the normality of the log returns and for the ARCH effect in the squared log returns**

Previous studies showed that financial returns (stock and stock index) display heavy-tails and leptokurtosis. Thus, modeling the data that is not normally distributed and exhibit serial correlation requires ARMA-GARCH model that helps in filter the log-returns to provide serially independent innovations.

## 4.2 Model

To specify the multivariate model, the models must be specified for the marginal distributions as well as the general model that describes the dependence structure between the marginal distributions.

### 4.2.1 Modeling the marginal distributions

As previously mentioned, the ARIMA-GARCH is applied extensively in the financial industry because many financial returns exhibit some degree of autocorrelation and conditional heteroskedasticity. The premier step in modeling of the datasets encompasses a repeated application of ARIMA-GARCH filtration with student-t distributed error terms to find the marginal distribution of individual stock index. This step helps us understand the details of each country. The sieving the log-returns using ARIMA- GARCH model because each of the indexes offers a series that is nearly independent as well as identically distributed.

The next step involves estimating the piece-wise probability distributions to model the daily returns of each index. Note that we do not make assumptions (i.e. normal distribution or any other simple parametric) regarding the distribution of the data; instead, we allow the data to speak for itself (gives a more flexible empirical distribution). Thus, if$X_{[i,t]}$ is the log return of index I at time T, then the ARMA(p,q)-GARCH(m,s) model is given by:

$$X(i,t) = \mu_i + \sum_{j=1}^{p} \Theta_{i,j} X_{i,t-j} + \sum_{k=1}^{q} \Psi_{i,k} \varepsilon_{i,t-q} + \varepsilon_{i,t}$$

where

$$\varepsilon_{i,t} = \delta_{i,t} \varepsilon_{i,t}$$

$$\delta^2(i,t) = \alpha_{i,0} + \sum_{j=1}^{max(m+s)} (\alpha_{i,j} + \gamma_{i,j} < 1$$

*where i= FTSE 100, and US S& P 500* and

$$\alpha_{i,0} > 0, \alpha_{i,j} \geqslant 0, and \sum_{j=1}^{max(m+s)} (\alpha_{i,j} + \gamma_{i,j} < 1$$

We use the conditional likelihood approach to approximate the parameters of the ARMA(p,q)-GARCH(m,s) model. The log-likelihood function is given as:

$$l(\theta_i) = -\frac{n}{2} \log 2\pi - \frac{1}{2} \sum_{t=1}^{n} \log \delta^2(i,t) - \frac{1}{2} \sum_{t=1}^{n} \frac{\beta^2(i,t)}{\delta^2(i,t)}$$

where

$$\theta_i = \mu_i \psi_{i,1}, \ldots, \psi_{i,p} \theta_{i,1}, \ldots, \theta_{i,q} \alpha_{i,0} \alpha_{i,1}, \ldots, \alpha_{i,m} \beta_{i,1}, \ldots, \beta_{i,s}$$

Consequently, the maximum likelihood estimator maximizes the log likelihood function:

$$\overline{\theta_i} = argmaxl(\theta_i)$$

We use numerical maximization approach to calculate the maximum likelihood estimator. The next step is to plug the estimator into the model in order to obtain the one-step next-period forecasts of the volatility and the value of the log-likelihood.

In ARMA modeling, the autocorrelation function (ACF) and the partial autocorrelation function (PACF) play a central role, for they serve as the primary tools that are used to detect the relationships that may occur between and within the return series at different lags that may apply. The ACF and PACF plots are explore to recognize the structuring of the ARMA model. As a result, when the model with the identified lags is estimated, a check is done to make sure that it correctly describes the return series. The model fits if the p-value is statistically significant (should be less than 0.05). Likewise, the lag parameters of the GARCH model but for squared log-returns were also identified using the ACF and PACF functions. The sufficiency of the final model is tested by means of LB test and can be graphically assessed by means of ACF plots.

The optimal models that seem to fit the FTSE 100, and S& P 500 respectively are ARMA(1,1)-GARCH(1,1)and ARMA(0,2)-GARCH(1,1)

| | FTSE | | S&P | |
|---|---|---|---|---|
| Parameter | Estimate | Error | Estimate | Error |
| $\mu$ | 0.00002 | 0.00 | 0.00006 | 0.00001 |
| ar1 | 0.09 | 0.0017 | -0.007 | 0.002 |
| ma1 | -0.09 | 0.001 | -0.0009 | 0.002 |
| $\omega$ | 0.000002 | 0 | 0 | 0 |
| $\alpha_1$ | 0.012 | 0.0014 | 0.127 | 0.00134 |
| $\beta_1$ | 0.086 | 0.0015 | 0.085 | 0.00137 |

**Table 4.3. The estimated parameters and as well as their standard errors**

Moreover, the results presented the table demonstrates that the figure of the appraised parameters$\beta$ and $\alpha$ for both stock indices is lower than 1. As indicated by Tsay (2005), this is an appropriate property that point toward the variance of the error with regard of finite while conditional variance fluctuates over time.

From the information gathered, the effectiveness of the ARMA(1,1)-GARCH(1,1)and ARMA(0,2)-GARCH(1,1) models is graphically evaluated by plotting ACF plots of the consistent residuals and the squared standardized residuals. The ACF plots shows that there is no major serial correlation in the squared standardized residuals and standardized residuals. Moreover, we used Ljung Box-test on the squared standardized residuals and standardized residuals as well as test the null hypothesis of no serial at the first 12 lags of the standardized residuals of stock indices at the 5% significance level. At the same time, the LB test did not manage to reject the null hypothesis. Consequently, the p-values are registered are 0.023 and 0.013 for the FTSE 100 and S&P 500 in that order.

Moreover, the LB test failed to castoff the null hypothesis of no serial correlation in the squared standard residual for FTSE 100, and S&P 500 indices. In this respect, the LB test on the squared standardized residuals for the FTSE 100 indicated that the LB (12) = 16.206 with a p-value = 0.023. Consequently, the equation unpredictability is only suitable at the 1% significance level. Equally, the LB test does flops and fails to reject the null hypothesis of no serial correlation in the first 12 lags at the 1% significance level for the S&P 500 index with a LB (12)= 14.665 with a p-value= 0.013.

The results explicitly show that the ARMA(1,1)-GARCH(1,1)and ARMA(0,2)-GARCH(1,1) models are appropriate to use since they offer serially independent innovations. The implication of this results is that the attained innovation can be applied in modeling the dependence organization of the marginal distributions of the for FTSE 100, and S&P 500 stock indices.

### 4.2.2 Modeling the dependence structure

Once information has been filtered through the ARCH-GARCH model, transforming the standardized residuals to uniform variates by student-t distributions is the next stage. We fit the Clayton, Gumbel, and Frank to the transformed data. The maximum likelihood approach is used to estimate the copula parameters, which is a vital point to note.

Table 4.4 below shows the Clayton, Gumbel, and Frank estimated parameters alongside their respective confidence intervals (CI), SIC values, and the AIC values.

| Copula | Estimated parameter | 95% Confidence Interval | SIC Value | AIC Value |
|---|---|---|---|---|
| Clayton | 1.042194 | [3.611,3.825] | -5301.488 | -5295.407 |
| Gumbel | 1.661397 | [3.746,3.935] | -5445.510 | -5439.430 |
| Frank | 4.302434 | [13.370,14.171] | -4591.805 | -4585.725 |

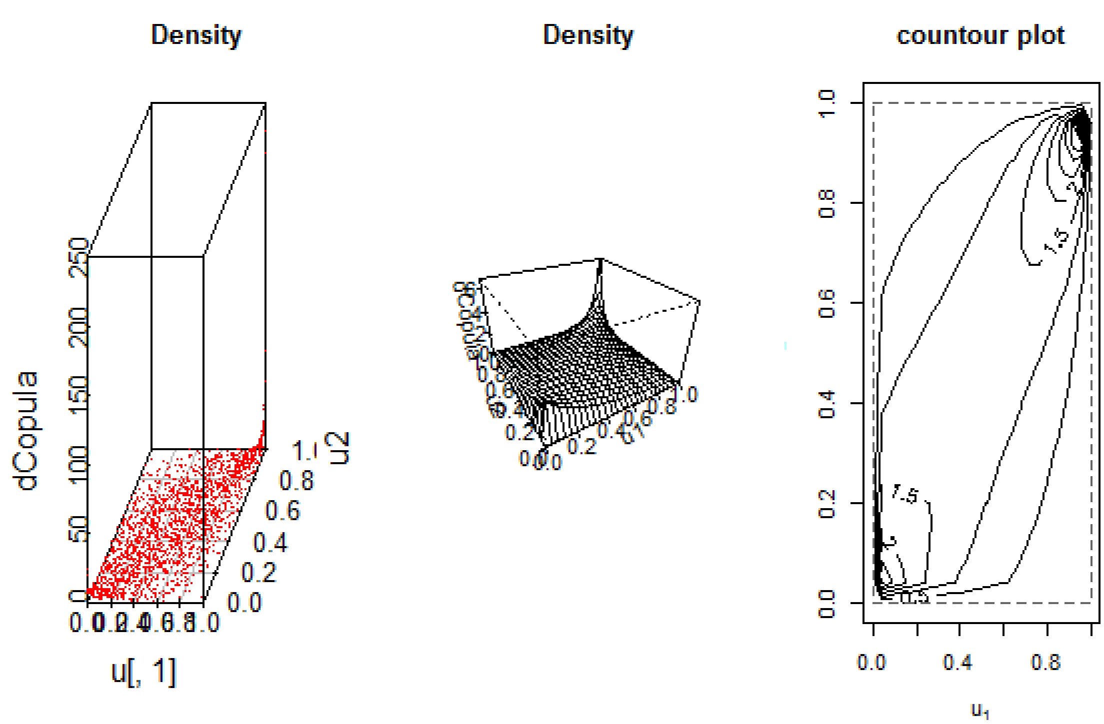
**Table 4.4. Copula parameter estimation**

The results in figure 4.4 show that the Gumbel copula offer the most suitable to the standard residual data based on the factor that it provide the lowermost values for both criterions used, the AIC and the SIC.

According to information before, in the hypothetical context, the Gumbel copula is affirmed to be disproportionately extreme, appropriate for the modeling the upper tail dependence. Nonetheless, the dependence structure in the marginal of the for FTSE 100, and S&P 500 stock indices is shown by the copula parameter. As discussed earlier, in the event that the copula approximation is equal to 1 then the marginal distributions are independent but when the estimated value methods infinitude then the Gumbel copula methods the Fréchet-Hoeffding upper bounds and the marginal distributions become perfectly dependent. In this case, the estimated Gumbel parameter indicates the presence of positive dependence between FTSE 100, and S&P 500 stock indices.

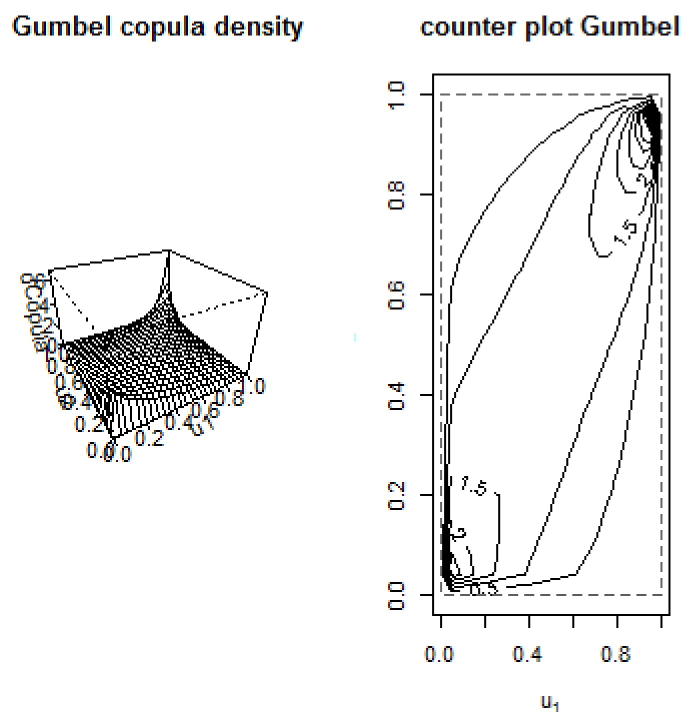
The graphs in figure 4.2 illustrate in more details the dependence structure given by the Gumbel copula. The density, scatter, and contour plots of random sample simulated from Gumbel copula are given in Figure 4.2. As anticipated from the projected copula parameter, the density plot show strong upper tail dependence in the data. Likewise, the simulated values in the scatter plot are heavily concentrated in the upper tail. We compare the contour plots with the cases of independence and perfect dependence. The contour plot for the approximation Gumbel parameter has more similarity to the contour plot for the comonotonic case

**Figure 4.3. Copula density plot (upper left), copula plot (upper right), 1000 simulated points, and the copula contour plot of the Gumbel copula with parameter 1.661397**



Consequently, the manner in which the Gumbel (1.661397) captured dependence triggers the joint distribution to create an upper tail dependence is quite fascinating. As shown in Figure 4.3 , illustrations of the density and the contour plot of the distribution is attained by linking the approximated Gumbel copula and the student-t margins that were acquired from the GARCH approximation.. Subsequently, the figure demonstrates a clear upper tail dependence, which shows that large gains from the FTSE 100 index and the S&P 500 index creates further propensities to occur concurrently than large losses.

**Figure 4.4. Density and contour plot of the joint distribution obtained by coupling the Gumbel (1.661397) with the estimated student-t margins**



### 4.2.3 Dependence Measures

Applying the use of the projected copula and understanding of the theory as discussed in section 3.2.2, we are in a position to calculate the copula based dependence measures namely; tail dependency, Kendall's tau, and Spearman's rho. The calculations of the aforementioned dependence measure are tabulated in Table 4.5 below. The results show the rank correlation measures (Spearman's rho and Kendall's tau) have positive values that are closer to 1. This implies that FTSE 100 index and the S& P 500 index are strongly positively correlated. It is important to point out that the estimated Spearman's rho and Kendall's tau values are very close to their sample equivalents, which are 0.8315 and 0.753492 respectively.

The fact that the estimated values are close to their sample counterparts points out the adequacy and flexibility of modeling dependency using copula. Thus, if the dependence structure between risks can be taken into account using a particular Archimedean copula, then the example of Kendall's tau or Spearman's rho approximations might be rightly employed to calculate the joint probability that may exist between the risk factors. The outcomes entered in Table 4.5 endorse that FTSE 100 and the S&P 500 stock indices show greater upper tail dependence. Conversely, based on the fact that Gumbel copula does not capture any lower tail dependence, the captured value of the lower tail dependence equals to zero

| Kendall's tau | Spearman's rho | Upper tail | Lower tail |
|---|---|---|---|
| 0.8097 | 0.753492 | 0.802 | 0.000 |

**Table 4.5. Estimated Kendall's tau, Spearman's rho, and tail dependence based on the estimated copula**

# 5 CONCLUSIONS AND RECOMMENDATIONS

## 5.1 Conclusions

The research paper empirically highlights several issues that are linked to our indulgent of dependence structure in financial risk models, taking into account the varying market environments involved. It also demonstrates the statistical value of copula-based models using risk management lens. First, we discuss the shortcomings of traditional measures of dependency particularly Pearson correlation which lacks the desired properties of dependence measures. The concept dependence is dynamic; correlation is only a substitute for expressing the dependence structure.

Therefore, in given settings correlation works as a suitable measure but popularly in risk management dynamics, correlation is a weak supposition and is dangerous to completely rely on correlation to understand the dependence between risks (correlation tells one side of the story). The three major shortcomings of Pearson correlation are: 1) correlation is easily distorted and is not able to take into account the non-linear dependencies. 2) The use of correlation lacks clarity when handling extreme events. 3) In strictly snowballing transformations, correlation is not invariant of the risks.

Nonetheless, it is important to understand that while dependence is a measure of the connection between the random variables in respect to their total range and tail dependence is the association in the extreme tails. In this respect, tail dependence is defined as a systemic risk- the relationship in the extreme tails of the combined distribution. Consequently, tail dependence and dependence are not one thing or the same but distinct from one another. In this regard, there may be a possibility for any given financial variable to be dependent without there being dependence in the tail of its distributions.

Secondly, we discuss copula based models specifically this paper focused on the Archimedean class as alternative statistical techniques that capture the entire dependence structure between risks. Financial models based around assumptions of Gaussian distribution mostly underestimate tail risks. Nonetheless, we show that Archimedean Copula models not only capture the central dependence but also adequate for modeling tail dependence (extreme events). In addition, Archimedean copula models put into consideration the dynamic nature of tail dependence and helps risk managers improve extreme event forecasts.

Still, there are caveats associated with copula-based models. To exemplify this, there are a number of copula roles that do not appropriately explain the conduct in the extreme tails of different distributions and as such seriously minimize tail and systemic risk because parameter estimation affects the selection of the copula function. Consequently, several factors must be considered for example using better dependence measures more robust than Pearson correlation, using extreme value theory to fit the marginal distributions, and selecting a copula function with natural explanation.

Thirdly, the 2007-2009 and other financial crises emphasized the need for better tools to measure systemic and tail risk. The crisis proved that extreme events impact the overall financial system largely causing uncertainty which can lead to financial crisis. In this paper we propose copula models to measure systemic tail dependence. The model is more robust and precise and can help regulators and investors choose diversification possibilities.

Taken together, the empirical findings of this paper show the statistical importance of incorporating asymmetric copula models in managing financial risk. Such dynamic models can assist the stakeholders to have a better understanding of the dependency structure between risk variables with diverse characteristics under diverse market environments. Moreover, copula models are more powerful and precise tools for forecasting risk and as such should be useful to both investors and regulatory bodies in determining capital requirements and allocation.

## 5.2 Future Research

This thesis modeled the systemic and tail risk between a portfolio of two risk variables using two-dimensional copulas. Nonetheless, many portfolios in the financial industry consist of more than two random variables. Thus, there is need to extend this analysis to a multivariate case.

In addition, the only copulas used in this study are the Archimedean class but only Frank Copula accounts for both lower and upper tail dependence. This means further research in the use of other copulas that allow modeling extreme tail dependencies such as the use extreme-value copulas (e.g. the Symmetrical Joe Clayton copula).

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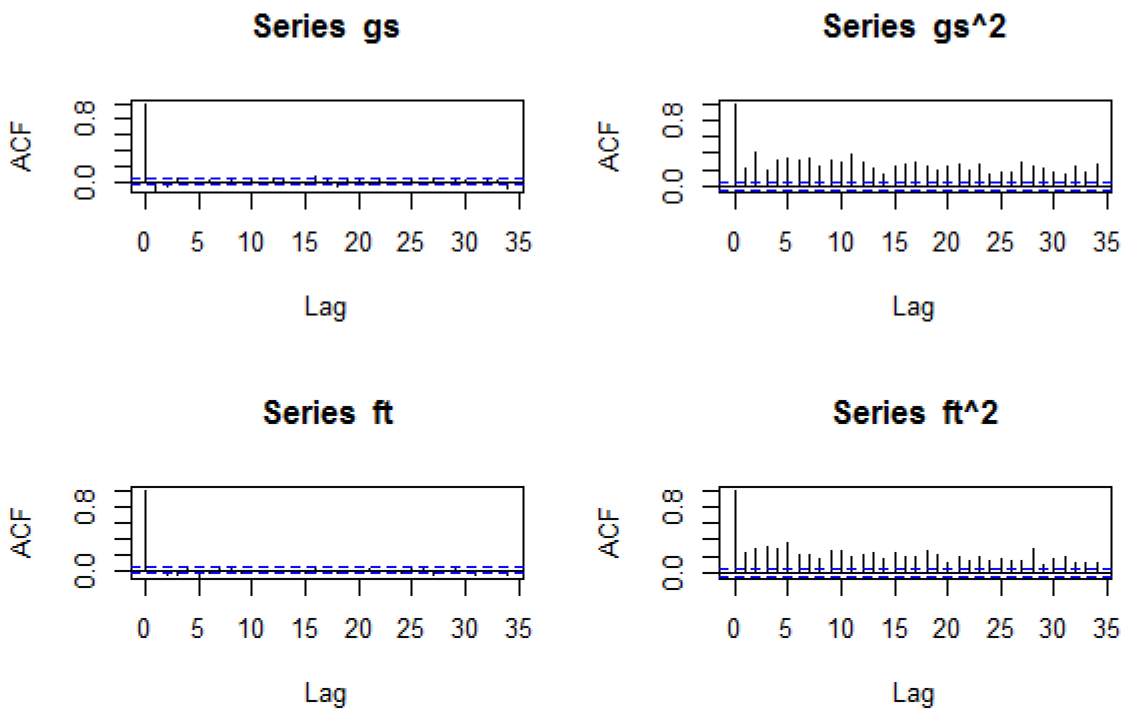
# A ACF and PACF plots of S& P 500 and FTSE 100 Indices



# B ARMA-GARCH OUTPUT

### FTSE 100

```
Title:
 GARCH Modelling

Call:
 garchFit(formula = ~arma(1, 1) + garch(1, 1), data = ft, trace = F)

Mean and Variance Equation:
 data ~ arma(1, 1) + garch(1, 1)
<environment: 0x00000000269b41a8>
 [data = ft]

Conditional Distribution:
 norm

Coefficient(s):
         mu          ar1          ma1        omega       alpha1        beta1
 2.0893e-05   9.4600e-01  -9.6807e-01   2.2944e-06   1.2097e-01   8.6341e-01

Std. Errors:
 based on Hessian

Error Analysis:
         Estimate  Std. Error  t value Pr(>|t|)
mu      2.089e-05   9.293e-06    2.248   0.0246 *
ar1     9.460e-01   1.773e-02   53.352  < 2e-16 ***
ma1    -9.681e-01   1.326e-02  -72.990  < 2e-16 ***
omega   2.294e-06   4.992e-07    4.596 4.30e-06 ***
alpha1  1.210e-01   1.465e-02    8.259 2.22e-16 ***
beta1   8.634e-01   1.568e-02   55.064  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Log Likelihood:
 9363.084    normalized:  3.209833

Description:
 Tue Jul 24 20:38:34 2018 by user: jackiepc


Standardised Residuals Tests:
                                Statistic p-Value
 Jarque-Bera Test   R    Chi^2  165.3569  0
 Shapiro-Wilk Test  R    W      0.9887548 1.999068e-14
 Ljung-Box Test     R    Q(10)  4.607277  0.9158234
 Ljung-Box Test     R    Q(15)  6.141508  0.9772714
 Ljung-Box Test     R    Q(20)  8.399444  0.9888777
 Ljung-Box Test     R^2  Q(10)  6.187002  0.799315
 Ljung-Box Test     R^2  Q(15)  18.44372  0.2400564
 Ljung-Box Test     R^2  Q(20)  21.47741  0.3695271
 LM Arch Test       R    TR^2   11.71896  0.4685063

Information Criterion Statistics:
      AIC       BIC       SIC      HQIC
-6.415553 -6.403256 -6.415562 -6.411124
```

## S&P 500

```
Title:
 GARCH Modelling

Call:
 garchFit(formula = ~arma(0, 2) + garch(1, 1), data = gs, trace = F)

Mean and Variance Equation:
 data ~ arma(0, 2) + garch(1, 1)
<environment: 0x0000000025a840d0>
 [data = gs]

Conditional Distribution:
 norm

Coefficient(s):
         mu          ma1           ma2         omega       alpha1         beta1
 6.5060e-04  -7.6456e-02  -9.9045e-03   2.4381e-06   1.2788e-01   8.5397e-01

Std. Errors:
 based on Hessian

Error Analysis:
         Estimate  Std. Error  t value Pr(>|t|)
mu      6.506e-04   1.317e-04    4.940 7.82e-07 ***
ma1    -7.646e-02   2.050e-02   -3.729 0.000192 ***
ma2    -9.905e-03   2.047e-02   -0.484 0.628536
omega   2.438e-06   4.013e-07    6.075 1.24e-09 ***
alpha1  1.279e-01   1.342e-02    9.529  < 2e-16 ***
beta1   8.540e-01   1.373e-02   62.182  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Log Likelihood:
 9457.855    normalized:  3.253476

Description:
 Tue Jul 24 20:05:35 2018 by user: jackiepc


Standardised Residuals Tests:
                                Statistic p-Value
 Jarque-Bera Test   R    Chi^2  696.9781  0
 Shapiro-Wilk Test  R    W      0.9726012 0
 Ljung-Box Test     R    Q(10)  13.4565   0.1992559
 Ljung-Box Test     R    Q(15)  22.37649  0.09830533
 Ljung-Box Test     R    Q(20)  26.17749  0.1600298
 Ljung-Box Test     R^2  Q(10)  10.3048   0.4141716
 Ljung-Box Test     R^2  Q(15)  15.90831  0.3881712
 Ljung-Box Test     R^2  Q(20)  18.96377  0.524182
 LM Arch Test       R    TR^2   11.13845  0.5170924

Information Criterion Statistics:
      AIC       BIC       SIC      HQIC
-6.502824 -6.490492 -6.502833 -6.498381

>
```