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PRICING CURRENCY OPTIONS USING PARABOLIC PDE

Research Report in Mathematics,2018

Kamande John Munene

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Submitted to the School of Mathematics in partial fulfilment for a degree in Master of Science in Actuarial Science

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Kamande John Munene

School of Mathematics
College of Biological and Physical sciences
Chiromo, off Riverside Drive
30197-00100 Nairobi, Kenya

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Abstract

The amount spent on option contract is the main problem in option pricing. The problem further gets complicated when there is need to project the future possible price of the option. This is achievable if one can be able to correctly determine the probability of the price increasing, decreasing or remaining constant. Any investor wishing to invest in the stock exchange would wish to make a profit thus the need for good formulas that give very close solutions to the market prices.

This project aims at using finite difference method to price options using partial differential equations.

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Declaration and Approval

I the undersigned declare that this project report is my original work and to the best of my knowledge, it has not been submitted in support of an award of a degree in any other university or institution of learning.

Signature Date

John Munene Kamande
Reg No. I56/74494/2014

In my capacity as a supervisor of the candidate, I certify that this report has my approval for submission.

Signature Date

Dr Philip Ngare
School of Mathematics,
University of Nairobi,
Box 30197, 00100 Nairobi, Kenya.
E-mail: pngare@uonbi.ac.ke

Dedication

This project is dedicated to my mom Beth for the unwavering support.

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Acknowledgments

I thank God for giving me the zeal, health, strength and inspiration throughout my studies.
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Kamande John Munene

Nairobi, 2018.

1 Introduction

1.1 Background of the Study

In the world of finance, pricing option's is a major problem that's encountered in financial mathematics. The idea of option contract draws back prior to 1973. These contracts were seen as Over-the-Counter (OTC). (Wilmott, 1995) .Thus an intermediary often referred to as option broker was necessary for option trading. The role of an option broker was to negotiate the price of an option between a seller and a buyer. Since contracts were not standardized in expressions of conditions, this kind of option contracts were not handled properly. However, since only few companies were involved, the OTC could manage.

In 1973, with the celebration of Merton, Black and Scholes and their Nobel consecration later an official exchange began when the current modern financial market came into place and replaced OTC. While some use derivatives to hedge against risks, others use options as a strategy to maximize their investments income. In a typical financial market, options traded vary significantly according to the type and each has its unique features. An investor will typically choose the type of option that favours their portfolio and is inclined more to their investment objectives.

In financial mathematics, pricing options has been a major field of research with the main method of pricing being the Black and Scholes. However, various other methods have come about which include FDM, FEM, Monte Carlo, Binomial and Trinomial.

FDM has been used in pricing of American and European Options and also in exotic options which include, basket, barrier, Asian and Bermudian options. Among these exotic options is the Russian option which is rare in the market.

Numerical methods can be classified into three classes in the math's of finance: Monte Carlo simulation, decision tree methods (binomial and trinomial), deterministic(non-random) method which includes pricing options through use of PDE. In this paper we focus on the latter.

Our objective of this paper is to discuss the application of partial differential equations in finance and their numerical methods. For instance, one may opt to choose from FDM, FDM, FVM and spectral methods .In this report, focus is on FDM since its very flexible and have a strong theoretical support. In financial mathematics, PDEs are of parabolic nature.

1.2 STATEMENT OF THE PROBLEM

Any investment made by an individual or an organization in any business is done with the aim of making a profit. The same applies to an investor who invests in the stock exchange with the aim of reaping profits at the end of the investment period. An example would be purchase of an option at current time with the aim of selling it at a higher price at a future date. However in business, there is a probability of either making a profit or a loss. Good decision making is therefore of paramount importance to achieve expected investment result.

Important knowledge an investor needs to put into account before purchase of an option in the stock exchange is being able the purchase price at current time. The investor will ask himself "How much should I pay for an option?" This factor is vital when selecting the shares to purchase by the factor of optimal prices. An ideal stock would be very marketable and with less risk of price dropping from the current price. This will lead to profit making and thus the need for the right judgment of the current price.

1.3 Objective

The general objective of this study if to find the optimal price of an option contract at the beginning of the contract by use of PDEs and their numerical solutions.

1.3.1 Specific objectives

1. Comprehend the working of the finite difference method.

2. To compare pricing of options using various numerical methods.

2 Chapter 2

2.1 Literature Review

For many years, option pricing has remained a main area of study in financial studies. The most famous model used is the Black and Scholes model (1973).The model however is purely mathematical thus someone without basic mathematical background would have a hard time understanding it. Later in (1973) Merton came up with article "Theory of rational option pricing" which gave a breakthrough to the pricing of the value of options by use of the "Black-Scholes-Merton" formula. Thereafter there was introduction of Cox, Ross and Rubenstain (1979) who introduced the binomial option pricing model which is much easier to understand since it has simple mathematics and has implicit economic importance thus used in a vast way in financial markets. However, this model could lead to large errors since it has only allows for two states. Ie. The stock price of the underlying asset move up or down from one step to another. The binomial option pricing model is a discrete time model and it contains the continuous Black Scholes model as a special limiting case.

2.1.1 PRICING CURRENCY OPTIONS

Garman and Kohlhagen (1983) expounded of the reknown Black scholes model to include foreign exchange model by suggesting that currency options could be treated as non-dividend paying stock. They allowed the fact that currency pricing involved two rates of interest (domestic interest rate and the rate of interest of foreign currency) and trading currency can be at forward discount or premium dependent on the rate of interest differential. However, the formula by Garman and Kohlhagen is only applicable to European options. The model suggest that it's easy to convert domestic currency to foreign currency and that one can put money with the hope of getting some profit in foreign bonds without any limitation.

Biger and Hull (1983) provided the variation formulas for the direct derivation of a European put and call options of currency using the Black, Scholes and Merton formula. Their idea was to show that the formulas could be derived by assuming the Capital Asset Pricing Model and the Expectation Theory.

Bodurtha and Courtadon (1987) use the assumption of lognormal probability to predict the prices of options by use of implied volatilities and the distribution of foreign exchange rates. The implied probabilities were chosen in such a way that the maximized the correspondence between predictions from the model and the observed prices from the previous day. The findings of this procedure are rather small biases on the price at the average which we predict in almost all categories of the options. Normally, there was a consistent overestimation of put and call option prices. The model also produces pricing errors with enormous dispersion with the ratio of absolute forecast error to that of the authentic price being about 113 per centum of both calls and put. Melino and Turnbull (1987) cross examine the lognormal assumption.

Fabbozzi, Hauser and Yaari (1990) carried out a research to point out the differences between the Garman Kohlhagen model and the BaroneAdesi Whaley (BA-W) model for currency options in calculating the price of American currency options. The results indicated that the BaroneAdesi Whaley (BA-W) model was inferior to Garman Kohlhagen model in pricing the worth of the primary asset is greater than the worth of the price of the strike in short term put options and superior in pricing the value of the underlying asset is less than the value of the price of the strike in long term put options. Both models preformed relatively well when the calculation of the price call options was done. The research further revealed the differential rate of interest in various countries had a large influence on the probability of lucrative initial exercise in foreign exchange put to call. On the American model, there was identification of superior chances in the trial for lucrative early exercise amid the pricing of the worth of the primary asset being greater than the worth of the price of the strike options that mature in less than 45 days. The (BA-W) model showed the need to research more on the impact of exercising early decisions. This investigation was useful while checking for the efficiency of the market and also coming up with strategies of trade in development.
Jabbour, Petrescu and Onayov (2006) incorporated higher order moments in valuing currency options through Edgeworth series expansion functions. The value of options was calculated and a comparison done with the Black Scholes Method. There was an indication of significant differences and mostly for put options where the price of the current share price and price of the strike are the same and a call option that has greater strike price as compared to the underlying asset.

2.1.2 PRICING ASIAN OPTIONS

There is difficulty in pricing of Asian options as not much is known about the distribution of the underlying assets average price and we can view this as a sum of lognormal random variables and the density function of a sum of lognormal random variables is unavailable currently. It is for this cause that there are no closed form results to price Asian options. However, there exist various approximation methods which can be grouped into three categories, that is, Monte Carlo Simulation, analytical methods and the lattice methods which are closely related to PDE approach.

2.1.3 ANALYTICAL METHODS

The value of an option can be approximated using semi-closed and closed form formulae in the analytical approach. Bouaziz, Briys and Crouchy (1994) assessed Asian options by use of a derived closed form solution where the strike price was the mean. Both the forward start average options and the average rate options were considered. The results were compared to the Monte Carlo simulation and they discovered that the results were fairly accurate even when the volatility level was high. Pudet and Barraquand (1996) solved degenerate diffusion PDE under the pricing problem for a contingent claim that follow a dependent path on a space occupied by variables that follow a dependent path. They did this through a numerical method referred to as forward shooting grid method for a derivative whose payoff depends on the realization of a future event that is uncertain. The method was faster in execution to the Monte Carlo and accurate. American options were also put into account in the forward shooting grid method. Rogers Shi (1995) lessened the dimension of the PDE through the change of numeraire for the fixed and floating option whose payoff be determined by the average of the risky asset. This minimized the problem of solving options whose payoff be determined by the average of the risky asset to a problem of solving PDEs with two variables. Vecer (2002) priced Asian options in a simpler unifying approach for discrete and continuous arithmetic mean options. He introduced a PDE of one dimension which when executed gave speedy and accurate results and provided stability under all volatilities. Forsyth et al (2002) in his research showed the convergent nature of various used to price path dependent options. A conclusion was arrived at showing that Pudet and Barraquand (1996) and Hull and White (1996) provided results that are an approximate of the exact solution if interpolation is carried out backward in time. This method of pricing options using partial differential equations converges in the continuous limit of time for options whose payoff be determined by the average of the risky asset. Fujiwara (2006) combined three different methods developed in computational fluid dynamics to come up with a fast, accurate and simple numerical method to price American Asian options. The results were consistent with the rest of the analytical methods.

3 Chapter 3

3.1 METHODOLOGY

3.2 PARTIAL DIFFERENTIAL EQUATION IN FDM

The order of the highest derivative is usually the order of the partial differential equation. In a linear PDE with U as the value of the dependent variable, the partial derivatives arise to the power of one only and there exists no result that involves more than one of these terms. The number of spatial variables that are independent in a PDE are representative of the dimension of the partial differential equation. That is, the equation is 2D if and only if x and y are spatial variables.

$$a(t,x,y)U_t + b(t,x,y)U_x + c(t,x,y)U_{yy} = f(t,x,y)$$

Where t,x,y are independent variables of time and space. a,b,c,f are independent variable functions and U is the value of dependent variable function. Partial derivatives are denoted as $U_t = \frac{\partial U}{\partial t}$, $U_x = \frac{\partial U}{\partial x}$,$U_{yy} = \frac{\partial^2 U}{\partial y^2}$ and the equation is second order and linear.

3.3 SOLUTION TO A PDE

In order to solve a PDE, we ought to solve the unknown function U.We refer to functions that satisfy a partial differential equations and also that satisfy the boundary conditions, right had side and initial conditions as the exact solution of the PDE. It's usually very challenging to get the exact solution to a PDE; we use numerical procedures to come up with approximate solutions. We approximate using computer systems and these approximations are done at discontinuous values of the variables that are independent. The working of the finite difference method involves approximations taking the place of all partial derivatives and other terms in the partial differential equation. A finite difference scheme is then generated that approximates the solution through the use of Taylor's theorem.

3.4 INITIAL AND BOUNDARY CONDITIONS

For Partial differential equations to be well posed, they require initial and boundary conditions in order to define well-posed problem. Too many conditions lead to no solutions while too few conditions lead to a solution that's not unique. Specifying boundary conditions (either initial or boundary) in wrong places and time lead to non-smooth dependence on initial and boundary conditions and thus small errors in initial and boundary conditions lead to the solution changing in enormous ways which we refer to as ill-posed problem.

Associated values and ghost points in a computational region might occur at or next to the boundaries of the area. How we treat the boundary conditions is of vital importance for precise problem simulation. For example, if we have points on grid with indexes $i = 1, 2, \ldots N$ for the x-axis and $j = 1, 2 \ldots M$ for the y-axis then the indices $0, N+1$ and $M+1$ are ghost points.

3.5 DIRICHLET (FIRST-TYPE) BOUNDARY CONDITIONS

The value of the dependent variable symbolized as U_0^n in first type boundary condition at a ghost point is defined in some way.eg.

U_0^n=Constant, $U_0^n = 0$

$U_0^n = f(n)$ time dependent boundary condition

$U_0^n = U_N^n$ Periodic boundary condition.ie. What passes out on the right passes out on the left as though the two boundaries are joined.

3.6 DERIVATIVE (NEUMANN) BOUNDARY CONDITIONS

It describes the rate of change of the variable that is dependent on derivative boundary condition at a grid point next to the ghost point. ie. $i = 1$ We can show this in two ways:

$U_x = f(U)$ the derivative of U in the course of x is identified at the boundary grid point $i = 1$.That is,U_0^n can be calculated and we estimate U_x at $i = 1$ by central difference

$U_x = f(U)\dfrac{(U_2^n - U_0^n)}{2x}$ rearranges to $U_0^n = U_2^n - 2xf(U)\ldots\ldots$(a)

$U_n = f(U)$ the derivative of U in the course of n the outward aiming normal to the boundary is given at the grid point adjacent to the ghost point. We note that this bearing is opposite to the x bearing at the left hand boundary $(i = 1)$. U_0^n can be calculated by central difference.

$U_x = f(U)\frac{(U_0^n - U_2^n)}{2x}$ rearranges to $U_0^n = U_2^n - 2xf(U)\ldots\ldots$(b)

Note; Because of the opposite directions of the derivatives, the two equations are different (a) and (b). Central difference is mostly used in Neumann Boundary condition but we could easily use other estimates such as first order backward difference.

3.6.1 FUNDAMENTAL WORKING OF FINITE DIFFERENCE METHOD

A finite grid mesh of points replaces the region over which the independent variables of the PDE are located, and then the dependent variables are approximated. Taylor approximation is then used to approximate the partial derivatives in the partial differential equation at each point from the adjacent values.

3.6.2 TAYLORS THEOREM

If $U(x)$ have n continuous adjectives over the interval (a,b).Then for $a < X_0, X_0 + h < b$

$U(X_0 + h) = U(X_0) + hU_x(X_0) + h^2\frac{(U_{xx}(X_0))}{2!} + +h^{n-1}\frac{(U_{n-1}(X_0))}{(n-1)!} + O(h^n)$ 2.1

Where $U_x = \frac{dU}{dx}, U_{xx} = \frac{(d^2U)}{(dx^2)}, \ldots, U_{n-1} = \frac{(d^{n-1}U)}{dx^{n-1}}$.$U_x(X_0)$ is a derivative of U with reference to X evaluated at $X = X_0$ and $O(h^n)$ is an unknown error term.

Knowledge of the value of U and that of the derivatives at X_0 can lead us to writing the above equation with reference to a nearby point X_{0+h}.If we discard the term $O(h^n)$, we truncate the right hand side to get an approximation to $U(X_{0+h})$.The error term is $O(h^n)$.

3.6.3 APPLICATION OF TAYLORS THEOREM TO FINITE DIFFERENCE METHOD

In finite difference method the values of U grid points are known and the objective is to solve approximations at the grid points in the Taylors formula by replacing the partial differential equations. In finite difference method both X_0 and X_{0+h} are known. We thus rearrange the equality to get the finite difference approximations to the derivatives which have $O(h^n)$ errors.

3.6.4 GRID CONVERGENCE

It's important to note the precision an accuracy of different numerical methods used in FDM.This process is highly dependent on the size of the computational grid.
An ideal solution will have a grid that converges Ie. The solution does not converge significantly if we use more grid points.

3.6.5 PSOR METHOD

This is an iterative method and the principle formula behind it is that any new iteration is dependent on the old point value plus an error or residual at that point.

$U_{i,j}^{m+1} = U_{i,j}^{m} + R_{i,j}^{m}$ where $R_{i,j}^{m}$ symbolizes the difference between iterations that are successful of $U_{i,j}$ and the error. An appropriate weight for $R_{i,j}^{m}$ can greatly speed up the rate of convergence of the iterative scheme.

$U_{i,j}^{m+1} = U_{i,j}^{m} + wR_{i,j}^{m}$ where we refer to w as the relaxation parameter where $0 < w < 1$ is referred to as under-relaxation and $1 < w < 2$ is referred to as over-relaxation.

3.6.6 EXPLICIT SCHEMES

In explicit schemes, data on the preceding level is got from data form the prior level through use of forward difference with an explicit formula. This hints to a stability (restraint) on the maximum acceptable time step $\triangle t$.

3.6.7 IMPLICIT SCHEMES

Data on subsequent time level arises on both sides of the difference scheme thus the need to solve a scheme of linear equations. In this case there is no stability constraint on the maximum time step which may be greater than an explicit scheme for identical problem. The time step is selected on the basis of correctness.

3.6.8 FINITE ELEMENT METHOD

In solving a PDE using the finite element method, there's a sequence of steps that one takes which include;

Discretize the continuum-here we divide the solution into smaller regions referred to as elements. Inside this elements are a number of points we refer to as nodes. Elements can take various shapes from segments of lines, triangles, squares etc. The type of the problem dictates the shape of the element. However, linear segments are the simplest for a 1D problem and triangular segments for 2D problems.

Select the kind of test function to use and the shape functions-we then select the type of functions we will take to define the variation of the function ϕ inside each trial function. That is we select basis set functions to describe the solution.

The formulation-To solve the PDE in mind, you find a system of algebraic equations for each element 'e' such that by solving it you get the values of ϕ at the position of nodes of the element 'e' such that $([\Phi_1, \Phi_2, \ldots, \Phi_N][\Phi_e])$.That is, you find for each element 'e' the matrix $[k]_e and$ a vector $[f]_e such$ that $[k]_e.[\Phi]_e = [f]_e$.

We assemble the equations into different elements.We then assemble the equations for all elements.It's important to note that contiguous elements usually have more nodes in common.For example, you may have a total of five elements with two nodes each but the number of effective nodes is say six and not ten.

Solving the system of equations—In this stage, it's important to note that any method is applicable and the higher the number of nodes, the better the quality of the solution.

Compute secondary quantities.Once you calculate the value of Φ you can compute the value of the other magnitudes using Φ.

3.7 PRICING OF OPTIONS USING PARTIAL DIFFERENTIAL EQUATIONS

Deriving the Black Scholes Merton PDE for European options.
We begin this segment by recalling the basics of the derivation of a PDE in European option pricing without providing the detailed mathematical presentation of the derivation. For a detailed and accurate mathematical presentation, we refer to an example in [7].

We assume a standard Black and Scholes model by way of a risky asset priced at time t and its price is S_t and a risk free asset priced at a time t and its price is S_t^0, that is

$$dS_t = S_t(dt + dB_t), dS_t^0 = rS_t^0 dt$$

The process B_t is a one dimensional Weiner process well-defined on the sample space (W, F, F_t, Q) where μ is the average rate of return while r is the rate of interest and $\sigma > 0$ is the gradation of variation where all are constants. We can generalize this to the case where r, μ and $\sigma > 0$ are functions of S and t where appropriate assumptions of smoothness apply. Further,there is an introduction of a random process $W_t = B_t + \frac{(\mu - r)}{\sigma} t$ in the Risk-neutral probability P in the determination of a fair price and we define it under the Radon Nikodym derivative w.r.t Q, that is

$$\frac{dP}{dQ}|_{F_t} = (\int_0^t \frac{(r-\mu)}{\sigma} B_s - \frac{1}{2} \int_0^t (\frac{r-\mu}{\sigma})^2 ds) \quad \textbf{(1)}$$

W_t is a Brownian motion and $\frac{S_t}{S_t^0}$ is a martingale. This is a basic principle of the above stochastic process. The stochastic differential equation below falls under P and satisfies the process S_t

$\mathrm{dS}_t = S_t(rdt + \sigma dB_t)$ **(2)**

Supposing we now examine a portfolio with risky assets H_t and risk free assets H_t^0, at time t the value would be

$\mathrm{P}_t = H_t S_t + H_t^0 S_t^0$ **(3)**

If we further assume that in the portfolio above is considered to be self-financing this equation will now be

$\mathrm{dP}_t = H_t dS_t + H_t^0 dS_t^0$ **(4)**

it's important to note if the worth of the risky asset changes so will the worth of the self-financing portfolio.

We can use (4) above to convey that $\frac{P_t}{S_t^0}$ is a martingale.

Suppose we contemplate a certain problem such that: having ϕ as the payoff function at maturity time $T > 0$ we can build a portfolio that is self-financing of the form $P_T = \phi(S_T)$. An ideal example of ϕ payoff function would be a vanilla call of the form $\phi(S) = (S-K)_+$ or a vanilla put of the form $\phi(S) = (S-K)_-$.In this case, the define any real x as $x, x_+ = max(x,0)$ and/or $x_- = max(-x,0)$.Basing the solution on a martingale representation theorem, we can observe that it's positive and that $(\frac{P_t}{S_t^0})$
Is a martingale while the payoff $\phi(S_T) is\ F_t$ measurable. The worth of the group of assets at time t is:

P$_t = (exp(-\int_t^T rds)\phi(S_T)|F_t)$ **(5)**

The investor is able to seize a payoff of $\phi(S_T)$ at time T since by application of the 'no-arbitrage' principle, we can show that P_t is a fair price of the option at time t.If r and σ are known to be constants, then the value of vanilla options is well known under the renown Black and Scholes formula. In the event that r and σ are functions of t or S_t, we estimate the value of the vanilla options through use of a numerical method. This section will only cover numerical methods that are deterministic and based on equation(5) above

Markov property is the second fundamental property essential to get a PDE formulation from the stochastic property S_t.This can be stated basically as, an expectation to a function of $(S_t)_{0 \leq t \leq T}$ conditional to F_t is a function of the price of S_t at time t of the risky asset. Under our circumstance, we can write P_t as

P$_t = p(t, S_t)$ **(6)**

We introduce p as a function of $t\varepsilon[0,T]$ and $S\varepsilon[0,\infty]$ referred to as the option pricing function. The values of $S\varepsilon[0,\infty]$ and $S \geq 0$ define the deterministic function of the pricing function p which comprises the price of p at point(t, S_t).The markov property that defines S_t is used to get

p(t,x)=(exp(-$\int_t^T rds)\phi S_T^{t,x}))$ **(7)**

Where $(S_\theta^{t,x})_{t \leq \theta \leq T}$ is the equality to (1) beginning from x at time t.

$$\begin{cases} dS_\theta^{t,x} = S_\theta^{t,x}(rd\theta + dW_\theta), \theta \geq t, \\ S_t^{t,x} = x, \end{cases} \tag{8}$$

The fact that $\frac{P_t}{S_t^0}$ is a martingale and by use of the Itos calculus, p satisfies the following partial differential equation

$$\begin{cases} \frac{\partial p}{\partial t} + rS\frac{\partial p}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 p}{\partial S^2} - rp = 0, \\ p(T,S) = \phi(S), \end{cases} \tag{9}$$

By use of the martingale representation theorem, we can show that the worth of a self-financing portfolio $p(t,S_t)$ is equivalent to $\phi(S_T)$ at time T if p satisfies (8). To obtain one and only one solution to problem (8), we apply boundary conditions to the system at $S=0$ or $S \to \infty$, that is, make precise the functional space for p.

REMARK (MAXIMUM PRINCIPLE) This principle states that the result to the partial differential equation of the European option is positive if the data given for the initial condition, boundary condition and the right hand side of the PDE is positive. This property is vital in pricing of options.

3.8 OTHER OPTIONS

From (7) above, the derivation of a PDE above is prototypical. The martingale principle and the Markov property are the two important properties in pricing function of an option using PDE. We now provide the PDE for various other exotic options without getting into detailed derivation.

3.8.1 BASKET OPTIONS

A basket option is a financial derivative where the payoff of the asset is dependent on the value of more than one asset.An ideal case would be two dependent assets evolving following a stochastic differential equation on a risk neutral probability.

$$\begin{cases} dS_t^1 = S_t^1(rdt + \sigma_1 dW_t^1) \\ dS_t^2 = S_t^2(rdt + \sigma_2 dW_t^2), \end{cases} \tag{10}$$

Where W_t^1 and W_t^2 could be correlated one dimensional Weiner processes. Where ρ is the correlation between $W_t^1$and W_t^2 : $d\langle W_1, W_2\rangle_t = \rho dt$. $T > 0$ is the maturity time and $\phi(S_T^1, S_T^2)$ is the payoff and ϕ is a given function. If the option price is $p(t, S_t^1, S_t^2)$ at time t,we can reveal that the option price at time t is $p(t, S_t^1, S_t^2)$ where p satisfies

$$\begin{cases} \frac{\partial p}{\partial t} + rS_1 \frac{\partial p}{\partial S_1} + rS_2 \frac{\partial p}{\partial S_2} + \frac{(\sigma_1^2 S_1^2)}{2} \frac{\partial^2 p}{\partial S_1^2} + \frac{(\sigma_2^2 S_2^2)}{2} \frac{\partial^2 p}{\partial S_2^2} + \rho_1 \sigma_2 S_1 S_2 \frac{\partial^2 p}{\partial S_1 \partial S_2} - rp = 0 \\ p(T, S_1, S_2) = \phi(S_1, S_2), \end{cases} \tag{11}$$

where r, σ_1, σ_2 are funtions of t and (S_1, S_2) and this PDE can be solved by standard numerical methods. .

3.8.2 BARRIER OPTIONS

The payoff is depends on the value of the asset reaching some worth that's determined by the boundary conditions set.That is, if we consider a single option asset, the payoff of some options becomes 0 if for time $t\varepsilon[0,T]S_t$ is less than a or greater than b which we can write as $0 < a < b$. When $a = 0$ or $b = \infty$ are treated in a similar way.
We can show mathematically that the payoff is $1_{\forall\varepsilon[0,T],S_t\varepsilon[a,b]}\phi(S_T)$ under any occurrence $A \subset \Omega, 1_A$ indicative of a characteristic function of S_t and A satisfy (1).To derive the PDE, an admissible random process of the form $S_{t\wedge\tau}$ is required where $\tau = inf t\varepsilon[0,T]$,S$_t \geq b$ or$S_t \leq a$ and is the time to stop the process.
For all real x and y variables,x$\wedge y = inf(x,y)$. Through mathematical calculations, we can show that $S_{t\wedge\tau}$ is a markov process while $\frac{S_{t\wedge\tau}}{S^0_{t\wedge\tau}}$ is a martingale. At time t, the option price is $p(t\wedge\tau, S_{t\wedge\tau})$ and we can define p for $t\varepsilon[0,T]$ and $S\varepsilon[a,b]$ to satisfy the following PDE

$$\begin{cases} \frac{\partial p}{\partial t} + rS\frac{\partial p}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 p}{\partial S^2} - rp = 0, \\ p(T,S) = \phi(S), \\ p(t,a) = p(t,b) = 0 \end{cases} \tag{12}$$

Where r and σ are funtions of t and S.We can consider general barrier options where the payoff is of the form $1_{\forall\varepsilon[0.T],(S_t^1,S_t^2,.......,S_t^d)\varepsilon D}\phi(S_T)$ and d is indicative of the quantity of underlying assets while D is a simple domain denoted by R^d.The best method to discretize this PDE with the domain D is through the finite element method.The appropriate discretization for general domain D is the finite element method.

3.8.3 OPTIONS OF THE MAXIMUM (LOOKBACK OPTIONS)

Lookback Options are options whose payoff of the option involves the maximum of risky asset. Such a case would be, if $\phi(S_t, M_t)$ where $M_t = max_{0 \leq r \leq t} \mathrm{S}_r$ and S_t satisfies (1).We can verify that (S_t, M_t) is a Markov process. The price of the option can be revealed to $p(t, S_t, M_t)$ at time t and we can define p for $t\varepsilon[0,T]$ and $(S,M)\varepsilon(S,M)\varepsilon R^2, 0 \leq S \leq M$ for the following PDE

$$\begin{cases} \frac{\partial p}{\partial t} + rS\frac{\partial p}{\partial S} + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial S^2}S^2 - rp = 0, \\ p(T,S,M) = \phi(S,M), \\ \frac{\partial p}{\partial M}(t,S,S) = 0 \end{cases} \tag{13}$$

The form of this payoff can be written as $\phi(S,M) = M\phi(\frac{S}{M})$ giving us a chance to reduce from a two dimensional equation to a one dimensional equation inclusive of the time variable. $p(t,S,M) = M\omega(t, \frac{S}{M})$ can be shown to satisfy this PDE

$$\begin{cases} \frac{\partial w}{\partial t} + rS\frac{\partial w}{\partial S} + \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial S^2}\varepsilon^2 - rp = 0, \\ \omega(T,\varepsilon) = \phi(\varepsilon), \\ \frac{\partial w}{\partial S}(t,1) = \omega(t,1) \end{cases} \tag{14}$$

where ω is defined as a function of $t\varepsilon[0,T]$ and $\xi\varepsilon[0,1]$. For (t,S,M) rate of interest and volatility are dependent, such a reduction is impossible.

3.8.4 ASIAN OPTIONS (OPTIONS ON THE AVERAGE)

An Asian option is an option whose payout is depends on the average of the risky asset. That is, $\phi(S_T, A_T)$ is the payoff where $A_T = \frac{1}{t}\int_0^t S_r dr$ and S_t satisfy (1).The option price at time t is $p(t, S_t, A_t)$ and p is defined on $t\varepsilon[0,T]$ and $(S,A)\varepsilon[0,\infty)^2$ and we can verify that (S_t, A_t) is a Markov process to satisfy

$$\begin{cases} \frac{\partial p}{\partial t} + rS\frac{\partial p}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 p}{\partial S^2} + \frac{1}{t}(S-A)\frac{\partial p}{\partial A} - rV = 0, \\ p(T,S,A) = \phi(S,A), \end{cases} \tag{15}$$

Just as in the case above,we can reduce the problem to a one dimensional PDE .That is, f satisfies

$$\begin{cases} \frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 f}{\partial S^2} + (\frac{1}{T} + r\varepsilon) \frac{\partial f}{\partial s} = 0, \\ f(T,\varepsilon) = \phi(\varepsilon), \end{cases} \tag{16}$$

on $p(t,S,A) = (t, \frac{K - \frac{tA}{T}}{S})$ for a fixed strike put $\phi(S,A) = (K-A)_+$ and a fixed strike call fixed strike call $\phi(S,A) = (A-K)_+$ and $\phi(\varepsilon) = \varepsilon_-$ (resp. $\phi(\varepsilon) = \varepsilon_+$.

NOTE If we set $p(t,S,A) = Sf(t, \frac{-tA}{TS})$ and $(\phi(\varepsilon) = (1+\varepsilon)_+$(resp. $(() = (1+\varepsilon)_-)$ we can reduce the above equation (13) to (14).However, except in cases where the dependencies are peculiar it's impossible to carry out the reduction for (t,S,A).That is, where the rates of interest and volatility are dependent.

3.9 EXCHANGE OPTIONS

The Hull-White pricing model efficiently prices exchange options through partial differential equations. Ways used to price foreign exchange hybrids that are long dated are at the center of this research and these hybrids are usually referred to as Power Reverse Dual Currency and are categorized under the Bermudian exotic options. Three correlated procedures formulate the problem and incorporate FX skew through the function of local volatility. Grid mesh of uniform interval is used for the finite difference method while for the time discretization we use Crank Nicolson method.

Mathematical results reveal that the numerical methods in use are convergent in second order. Monte Carlo simulation is the most popular choice of pricing PDRC but it has its disadvantages as which include decelerate level of convergence and hedging parameters being unfavorable. The current standard modeling of term structures is composed of two factor Gaussian models with two factors and the spot FX rate has one factor log normal model with one factor. This choice has several advantages which include;

I.A minimum number of factors which is three.
II.The spot FX rate uses closed form calibration that's very efficient.

Nonetheless, challenges are also experienced which include;

I.FX option exhibit skew thus cannot be captured in log normal distribution.
II.Sensitive to FX volatility skew.

The model incorporates smiles to counter the problem through the function of local volatility which in turn prevents the model from acquiring more stochastic factors.
Ie. Under a model with three factors, the derivatives of interest rate depend on the variables of three stochastic states thus the value function of the partial differential equation must fulfill variables of three states and variables of time.

3.9.1 PDE FOR CROSS CURRENCY OPTION (FX OPTION)

Suppose we examine an economy with domestic and foreign currency. The spot rate of the number of units that equate to a unit of foreign currency and can be written as $S(t)$.The domestic and foreign short rates can be written as $r_i(t)$,$i = d, f$.The description of $S(t), r_d(t), r_f(t)$ on the risk neutral measure can be described as

$$\frac{ds(t)}{s(t)} = (r_d(t) - r_f(t))dt + \gamma(t, s(t))dW_s(t) \tag{17}$$

$$dr_d(t) = (\theta_d(t) - k_d(t)r_d(t))dt + \sigma_d(t)dW_d(t) \tag{18}$$

$$dr_f(t) = (\theta_f(t) - k_f(t)r_f(t)) - \rho_f s(t)\sigma_f(t)\gamma(t, s(t))) + \sigma_f(t)dW_f(t), \tag{19}$$

$W_s(t), W_d(t), W_f(t)$, are correlated Brownian motions with $dW_d(t)dW_s(t) = \rho_d sdt, dW_f(t)dW_s(t) = \rho_f sdt$ and $dW_d(t)dW_f(t) = \rho_d f dt$.

The volatility $\sigma_i(t)$ and mean reversion rate $k_i(t)$ for $i = d, f$ are representative of the mean reverting Hull White model while the term structure is represented by $\theta_i(t)$,$i = d, f$.The alteration of the amount from foreign to domestic represents the drift adjustment which can be denoted as $-\rho_f s(t)\sigma_f(t)\gamma(t, s(t))$ for $dr_f(t)$ where $\sigma_i(t), k_i(t)$, and $\theta_i(t)$ are all deterministic. The form for the function of local volatility $(t, s(t))$ in the spot rate FX can be denoted as $\gamma(t, s(t)) = \varepsilon(t)\ \frac{(s(t)}{(L(t)}^{c(t)-1}$

In this equation, the function of relative volatility is $\varepsilon(t)$, the constant of elasticity of variance(CEV) dependent on time is $c(t)$ and the scaling constant dependent on time can be set to forward foreign exchange rate expiring at time t written as $F(0, t)$ for the purpose of calibration.

3.9.2 THEOREM

Suppose the value of the domestic function is $u \equiv u(s, r_d, r_f, t)$ representative of a security whose measurable terminal payoff with respect to σ-algebra at the time to maturity is T_{end} and there are no immediate payments. We can assume further that $u\varepsilon C^{2,1}$ on $R_+^3[T_{start}, T_{end})$, then u satisfied the PDE

$$\frac{\partial u}{\partial t}+Lu\equiv\frac{\partial u}{\partial t}+(r_d-r_f)s\frac{\partial u}{\partial s}+(\theta_d(t)-k_d(t)r_d)\frac{\partial u}{\partial r_d}+(\theta_f(t)-k_f(t)r_f-\rho_f s\sigma_f(t)\gamma(t,s(t)))\frac{\partial u}{\partial r_f}+ \frac{1}{2}\gamma^2(t,s(t))s^2\frac{\partial^2 u}{\partial s^2}+\frac{1}{2}\sigma_d^2(t)\frac{\partial^2 u}{\partial r_d^2}+\frac{1}{2}\sigma_f^2(t)\frac{\partial^2 u}{\partial r_f^2}+\rho_d s\sigma_d(t)\gamma(t,s(t))s\frac{\partial^2 u}{\partial s\partial r_d}+\rho_f s\sigma_f(t)\gamma(t,s(t))\frac{\partial^2 u}{\partial s\partial r_f}+ \rho_d f\sigma_d(t)\sigma_f(t)\frac{\partial^2 u}{\partial r_d\partial r_f}-r_d u=0$$

3.9.3 DISCRETIZE THE PDE

Suppose the sub intervals are represented by $n+1, p+1, q+1, l+1$ for the $s-, r_d-, r_f-$ and $\tau-$ bearings respectively. The respective step sizes that have a uniform grid are written as $\Delta s=\frac{s}{(n+1)}, \Delta r_d=\frac{R_d}{(p+1)}, \Delta r_f=\frac{R_f}{(q+1)},, \Delta\tau=\frac{T}{(l+1)}$.The values at the grid point for approximating the finite difference are written as $u_(i,j,k)^m\approx u(s_i, r_{dj}, r_{fk}, \tau_m)=u(i\Delta s, j\Delta r_d, k\Delta r_f, m\Delta\tau)$ and $i=1\ldots,n, j=1\ldots,p, k=1\ldots,q, m=1\ldots,l+1$.We use central differences to approximate the values of the first and second partial derivatives. The value of the cross derivatives is approximated using the four point FD stencil.ie.we approximate $\frac{\partial u}{\partial s}$ and $\frac{\partial^2 u}{\partial s^2}$ by

$$\frac{\partial u}{\partial s}\approx\frac{(u^m_{(i+1,j,k)}-u^m_{(i-1,j,k)})}{2\Delta s}, \frac{\partial^2 u}{\partial s^2}\approx\frac{(u^m_{(i+1,j,k)}-2u^m_{(i,j,k)}+u^m_{(i-1,j,k)})}{(\Delta s)^2}$$

While we approximate the cross derivative $\frac{(\partial^2 u)}{(\partial s\partial r_d)}$ by

$$\frac{(\partial^2 u)}{(\partial s\partial r_d)}\approx\frac{(u^m_{(i+1,j+1,k)}+u^m_{(i-1,j-1,k)}-u^m_{(i-1,j+1,k)}-u^m_{(i+1,j-1,k)})}{(4\Delta s\Delta r_d)}$$

We omit the derivations for brevity but we can show through Taylor expansion that these formulas have a second error truncation error as long as u is sufficiently smooth.

3.9.4 APPLICATION OF THE CRANK NICOLSON SCHEME

The crank Nicolson scheme is applied to move from τ_{m-1} to τ_m

$$\frac{(u^m_{(i,j,k)} - u^{(m-1)}_{(i,j,k)}}{\Delta\tau} = \frac{1}{2}Lu^m_{(i,j,k)} + \frac{1}{2}Lu^{m-1}_{i,j,k}$$

Where $i = 1,...,n, k = 1,...,q, j = 1,...,p$ and the mesh points are assumed to be ordered in the following directions $s-, r_{d-}$ and r_{f-} .Assume that the vector of values be U^m at time τ_m on a mesh that approximates the analytical solution to $u^m = u(s, r_d, r_f, \tau_m)$ .By Crank Nicolson method, we approximate U^m successively for $m = 1, 2, ..., l+1$ by

$$(I - \frac{1}{2}\tau A^m)u^m = (I - \frac{1}{2}\Delta\tau A^{m-1})u^{m-1} + \frac{1}{2}\Delta\tau(g^m + g^{m-1})$$

The identity matrix $npq \times npq$ can be written as I while A^m is representative of a matrix of the same size. g^m and g^{m-1} can be obtained from the boundary conditions. To solve the equation via direct methods such as lower upper factorization have high costs computationally since;

I.The matrix $I - \frac{1}{2}\Delta\tau A^m$ holds a bandwidth proportionate to $\min(np, nq, pq)$ subject to the organization of the grid points.
II. Scant solvers undergo substantial fill in when unraveling systems resultant from PDEs of the method.
III.These matrix necessities to be factored at every time step since its reliance on the time step index m of the function of local volatility.

3.10 USING PDEs TO PRICE EUROPEAN OPTIONS

In the previous sections we focused on presentation of PDEs of various exotic options. We will now show a process to solve these PDEs in finance and their applications. The method is the finite difference method (FDM) which is solely founded on Taylor's expansion.

3.10.1 The finite difference method

We shall carry out the application of the finite difference method(FDM) on the European option PDE (8).The main step is to discretize the PDE w.r.t the variable S.The interval is then divided into $[0, S_{max}/I]$ where we choose S_{max} to be large enough and carry out an approximation through the finite differences. A workable semi-discretization of (8) would be $i\varepsilon 0, 1, \ldots.., I$,

$$\begin{cases} \frac{\partial P_i}{\partial t} + rS_i \frac{P_{i+1} - P_{i-1}}{2\delta S} + \frac{\sigma^2 S_i^2}{2} \frac{P_{i+1} - 2P_i + P_{i-1}}{\delta S^2} - rP_i = 0 \\ P_i(T) = \phi(S_i), \end{cases} \tag{20}$$

The i-th discretization point can be written as $S_i = i\delta S$ and we approximate $p(t, S_i)$ by $P_i(t)$.The time and spot dependent r and σ are direct for this system of coupled ODEs. We can solve P_0 independently for $S = 0$, since $(S_0 = 0) : P_0(t) = \phi(0)exp(-\int_t^T rds$. To solve this ODE we define the boundary condition $S = S_{max}$. P_{I+1} is a priori and undefined and at $i = I$ can be solved in two ways.

We use prior information on the value of $p(t, S)$ if S is large to solve in the first method then use approximations of $p(t, S_{max})$. The value of P_I in this case is given as a data ie. Often referred to as Dirichlet boundary condition and the unknowns are $(P_i)_{0 \leq i \leq I-1}$.For example, a put $\phi(S) = (S - K)_{-}o$ or a call $\phi(S) = (S - K)_{+}$, we know that $\lim_{s \to \infty} p(t, S) = 0$ and $\lim_{s \to \infty} p(t, S) \sim Kexp(-\int_t^T rds)$respectively, we can set $P_I(t) = 0$ and $P_I(t) = S_{max} - Kexp(-\int_t^T rds)$.We can estimate the error introduced by this boundary conditions.

The asymptotic behavior of p sets the basis of the second method we can use to solve the PDE.The use of homogeneous Neumann boundary condition can be used to solve the put above and this can either be continuous $\frac{\partial p}{(\partial S(t, S_{max})) = 0}$ or discrete $\frac{P_{I+1}(t)}{P_I(t)} \delta S = 0$ We can define the unknowns as $(P_i)_{0 \leq i \leq I}$.It's important to note that S_{max} would be selected adequately large for both methods. The time discretization can be considered by dividing the time interval $[0, T]$ into N intervals of length $\delta t = T/N$. The time derivatives can further be replaced by finite differences.The following classical numerical techniques are used:

$$\begin{cases} \frac{(P_i^{n+1}-P_i^n)}{t} + rS_i \frac{P_{i+1}^{n+1}-P_{i-1}^{n+1}}{2\delta S} + \frac{(\sigma^2 S_i^2)}{2} \frac{P_{i+1}^{n+1}-2P_i^{n+1}+P_{i-1}^{n+1}}{\delta S^2} - rP_i^{n+1} = 0 \\ P_i^N = \phi(S_i), \end{cases} \tag{21}$$

$$\begin{cases} \frac{(P_i^{n+1}-P_i^n)}{t} + rS_i \frac{P_{i+1}^{n}-P_{i-1}^{n}}{2\delta S} + \frac{(\sigma^2 S_i^2)}{2} \frac{P_{i+1}^{n}-2P_i^{n}+P_{i-1}^{n}}{\delta S^2} - rP_i^{n} = 0 \\ P_i^N = \phi(S_i), \end{cases} \tag{22}$$

$$\begin{cases} \frac{(P_i^{n+1}-P_i^n)}{t} + \frac{1}{2}(rS_i \frac{P_{i+1}^{n}-P_{i-1}^{n}}{2\delta S} + \frac{(\sigma^2 S_i^2)}{2} \frac{P_{i+1}^{n}-2P_i^{n}+P_{i-1}^{n}}{\delta S^2} - rP_i^{n} + rS_i \\ \frac{P_{i+1}^{n+1}-P_{i-1}^{n+1}}{2\delta S} + \frac{(\sigma^2 S_i^2)}{2} \frac{P_{i+1}^{n+1}-2P_i^{n+1}+P_{i-1}^{n+1}}{\delta S^2} - rP_i^{n+1}) = 0 \\ P_i^N = \phi(S_i), \end{cases} \tag{23}$$

Where $P_(i)^n$ ought to be the approximation of $p(t_n, S_i)$ with $t_n = n\delta t$. The prices of $(P_i)_{0\leq i\leq I}$ in equation (21) are explicitly attained from the prices of$(P_i^{n+1})_{0\leq i\leq I}$ in the explicit Euler scheme. The prices of $(P_i)_{0\leq i\leq I}$ in equation (22) (implicit Euler scheme) and (23) (Crank-Nicolson scheme) are gotten from the prices of$(P_i^{n+1})_{0\leq i\leq I}$ over resolve of a linear system. To solve the linear system, various methods can be used such as the iterative method which consists of computation of the solution at the bound of order of estimates where matrix vector approximations are used and direct methods based on Gaussian elimination.

3.10.2 DENOTING STABILITY AND CONSISTENCY

Having come up with the discretization schemes for the explicit Euler(EE), Implicit Euler(IE) and the Crank Nicolson(CN), we then evaluate the convergence of the three schemes then comprehend their dissimilarities. In order to do that, we have to introduce two very important notations.

Consistency-In a consistent numerical method, once the precise solution is plowed in the numerical scheme, as the discretization parameters incline to zero the error term tends to zero. This means that in (21), (22) and (23) you replace P_i^n with $p(t_n, S_i)$ where p satisfies

equation (8).You then need to verify that the other expressions incline to zero if δt and δS incline to zero.
The discretization schemes are then checked using Taylor expansion that they are bound by $C(\delta t + \delta S^2)$ where C is a constant dependent on some derivatives of p.Thus from the bound, $C(\delta t + \delta S^2)$ we can see that the discretization schemes are are consistent with spot variable of command 2 and command 1 in time.
The second notation is stability. In a steady arithmetic method, the norm of the result to the scheme is bound by a constant that is free of the constraints multiplied by the norm of the data, that is, the boundary conditions, initial conditions or the right hand side. Thus we can say that a numerical approximation is converging to a solution of a partial differential equation if the discretization parameters approaches zero. We can estimate convergence by estimating the convergence error. We shall look at the following example to show this, the error for explicit Euler scheme (EE) is bound from above by $C(\delta t + \delta S^2)$ where the constant C is dependent on the solution p: There needs to be more regularity on p for higher order schemes. To show this we shall refer to some parameters under the Crank-Nicolson scheme at $t = T$ are better obtained in the CN scheme rather than the (IE or EE) with command 1 as the solution isn't adequately regular around $t = T$.

Conclusion of the properties of the schemes

The three discretization schemes described above have been observed to be consistent. We can also show that the Explicit Euler scheme described is said to be stable if there's an additional CFL condition described in of this nature $(\delta t \leq C\delta S^2)$ and C is described to be a positive constant. The other conditions are unconditionally stable. The prices of $(P_i^{n+1})_{0\leq i\leq I}$ can be acquired from the prices of $(P_i^{n+1})_{0\leq i\leq I}$ in the Explicit Euler scheme but we need small time steps in relation to spot step for us to achieve stability and convergence. However for the implicit Euler(IE) and the Crank Nicolson(IE), we require a determination of linear scheme at each time step but meet without restrictions at the time steps.In parabolic PDE in finance, this is very common. Thus it's preferable to compute implicit schemes in relation to costs as CFL condition is usually very stringent.

3.10.3 APPLICATIONS TO ASIAN OPTIONS

In this segment we present the use of the finite difference method relative to valuing of Asian options.We calculate the numerical results to for a call with a fixed strike.

$\phi(\varepsilon) = \varepsilon_-$ **(24)**

In the discretization above, we used finite difference method to solve the solution to a renowned black Scholes equation of European PDE. If we used the same method to solve

for the American PDE we would get unwanted results especially with a small volatility. This is because with ε closer to zero, the diffusion term $(\sigma^2\varepsilon^2)/2$ is smaller than the $(1/T + r\varepsilon)$ term. This causes a deterioration to the numerical scheme stability w.r.t L^∞-norm. There exist some oscillations as the principle of discrete maximum is unsatisfied in the numerical solution. Further, there is an introduction of arithmetical diffusion which points to outcomes that aren't satisfactory for purely advective equations.
In practice, the numerical solution does not fulfill the discrete maximum principle as it reveals some oscillations. Besides, the finite difference method presents numerical diffusion which points to inadequate outcomes for purely advective equivalences.
This can be solved through the characteristic method centering on the solution $(d\varepsilon/dt = -1/T)$ with the aim eliminate $1/T$. We introduce the following change in variable.

g(t,x)=f(t,x- $\frac{t}{T}$) **(25)**

g is the solution to the PDE below

$$\begin{cases} \frac{\partial g}{\partial t} + \frac{\sigma^2(x-\frac{t}{T})^2}{2}\frac{\partial^2 g}{\partial g} - r(x-\frac{t}{T})\frac{\partial g}{\partial x} = 0 \\ g(T,x) = \phi(x-1) = (1-x)_+, \end{cases} \quad \textbf{(26)}$$

If g satisfies (26), when the advective term $r(x-\frac{t}{T})$ is insignificant, then the diffusion term $\frac{\sigma^2(x-\frac{t}{T})^2}{2}$ is insignificant as well. This result is satisfactory. Another core characteristic of the result to PDE (26) for $\phi(\varepsilon) = \varepsilon_-$ is when $\forall \leq 0$

f(t,ε) $= \frac{1}{rT}(1-e^{-r(T-t)}) - \varepsilon e^{-r(T-t)}$ **(27)**

Thus $\forall \leq \frac{t}{T}$

then,

g(t,x)= $\frac{1}{rT}(1-e^{-r(T-t)}) - (x-\frac{t}{T})e^{-r(T-t)}$ **(28)**

The f given by PDE (27) if the solution to the American PDE with $\phi(\varepsilon) = -\varepsilon$.Thus, for us to prove (28), we notice that the advective term is negative and diffusion term is null for $\varepsilon = 0$, the solution to the American PDE for $\phi(\varepsilon) = \varepsilon_-$ where $\varepsilon \leq 0$ is the same as the solution to Asian PDE for $\phi(\varepsilon) = -\varepsilon$ on $\varepsilon \leq 0$.

4 Chapter 4

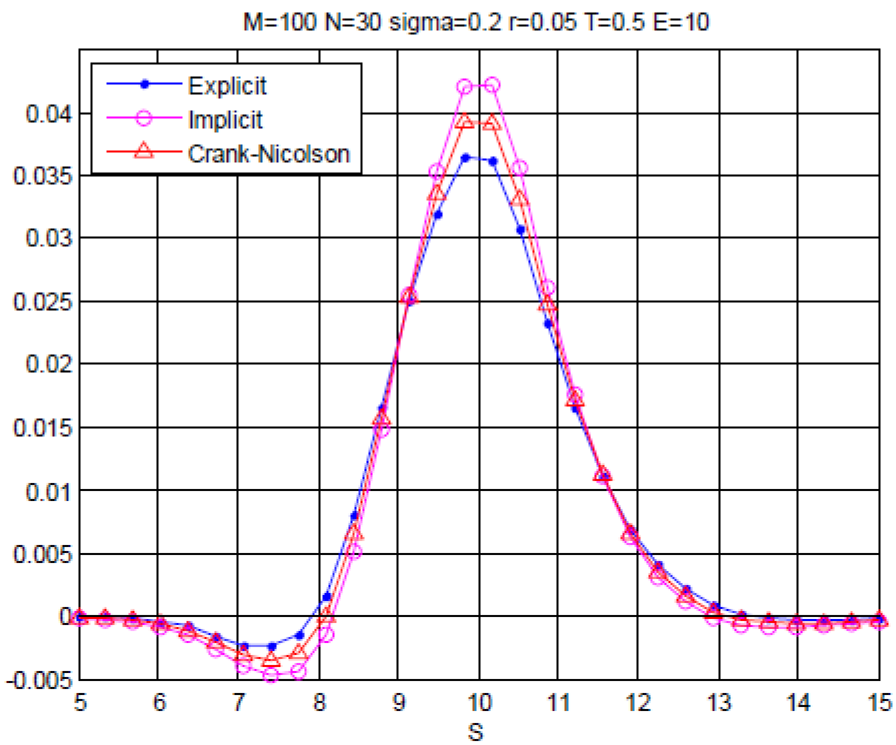
4.1 STATISTICS AND RESULTS

4.2 INTRODUCTION

This section presents outcomes obtained from calculation of American Put Option and Currency Options using both the finite difference methods. We also segment the time period for the American put option into intervals then carry out European put pricing of every interval independently while assuming that every interval is the expiry date of a European Put Option. We draw line graphs from the prices obtained at different points which give us the best prices for an American put option. The results we obtain are then compared to Black-Scholes, and Monte Carlo simulation. We implement the numerical solution using R program.

4.2.1 IMPLEMENTATION AND RESULTS

The figures below shows the errors of European Put option under different parameters



M=100 N=30 sigma=0.2 r=0.05 T=0.5 E=10
Explicit
Implicit
Crank-Nicolson
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
-0.005
5
6
7
8
9
10
11
12
13
14
15
S

Figure 2.3.5.1

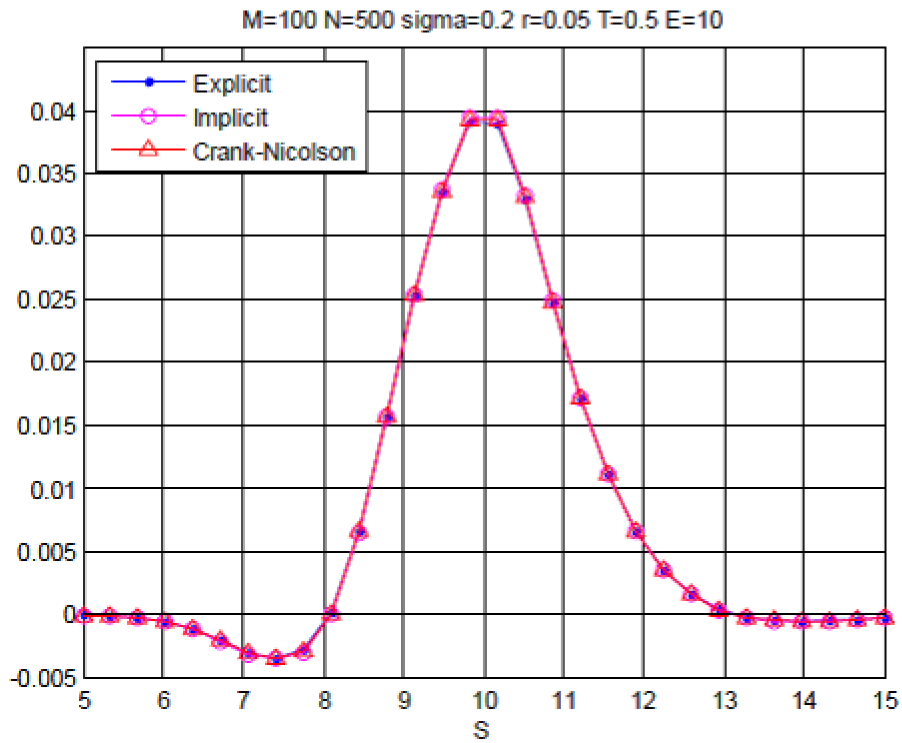

M=100 N=500 sigma=0.2 r=0.05 T=0.5 E=10
Explicit
Implicit
Crank-Nicolson
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
-0.005
5
6
7
8
9
10
11
12
13
14
15
S

Figure 2.3.5.2

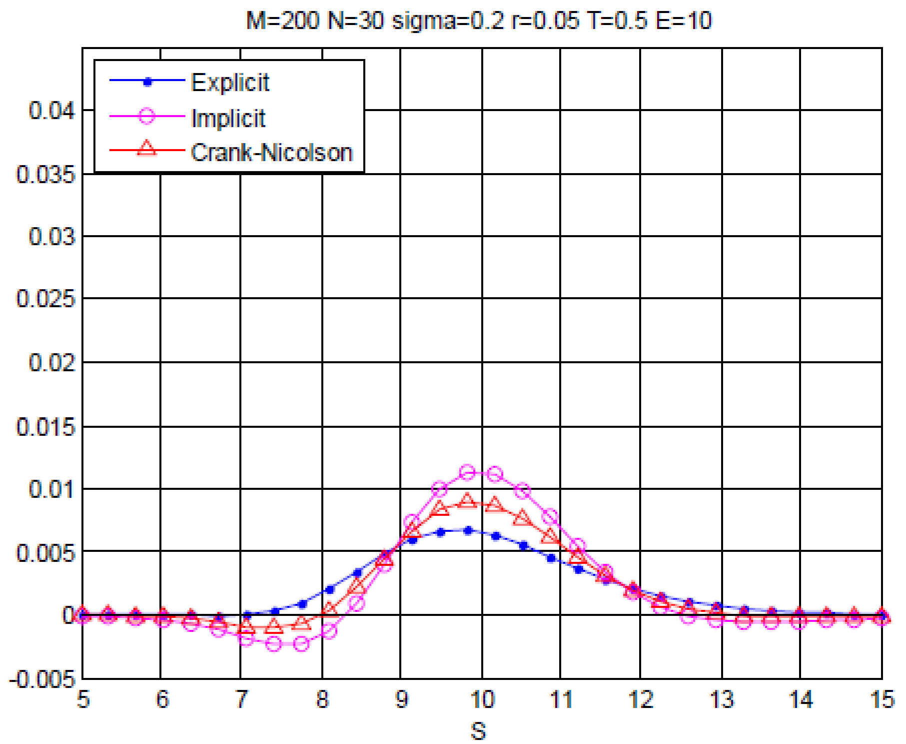


Figure 2.3.5.3

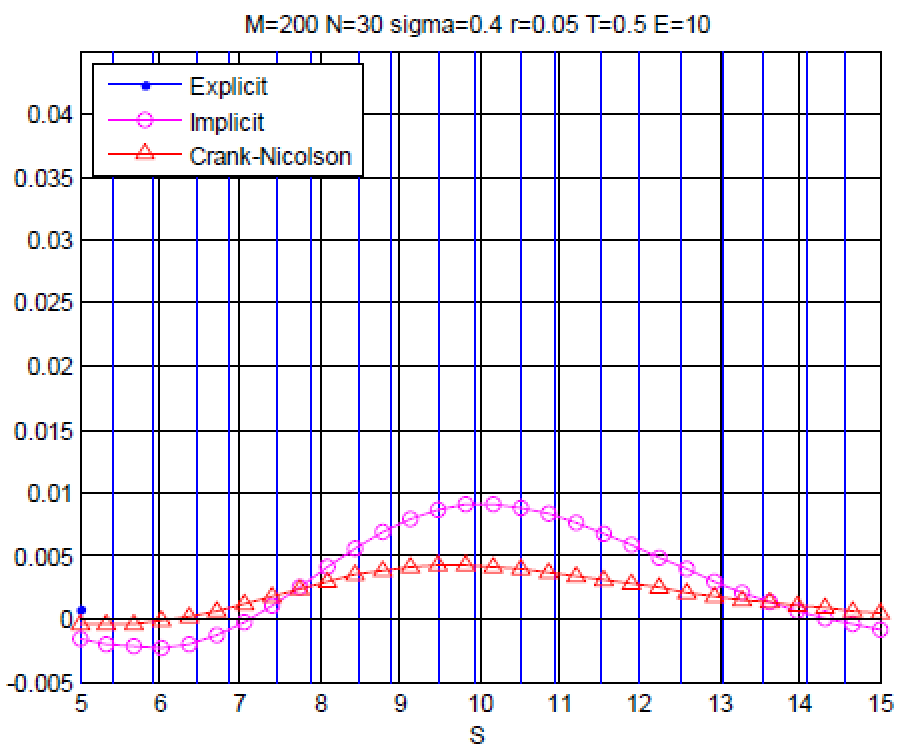


Figure 2.3.5.4

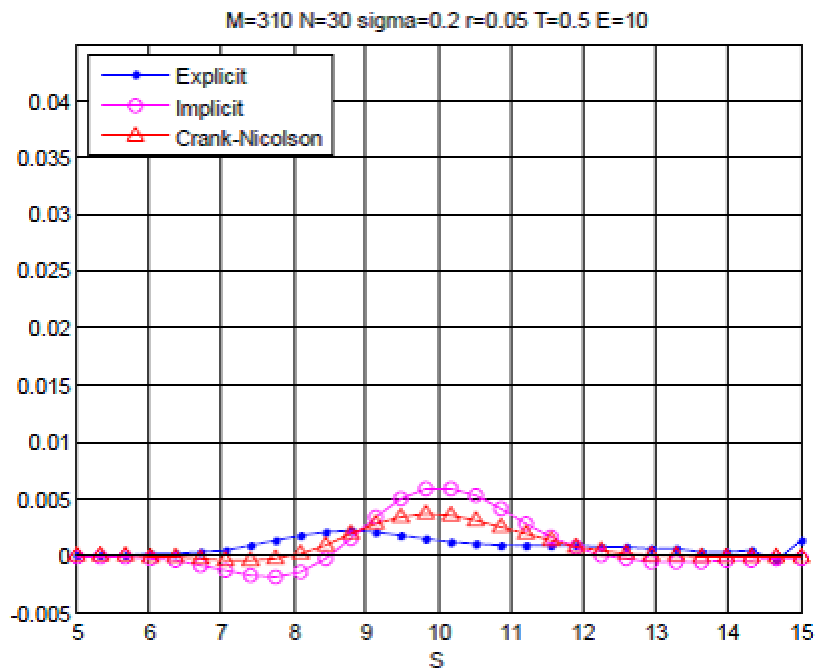


Figure 2.3.5.5

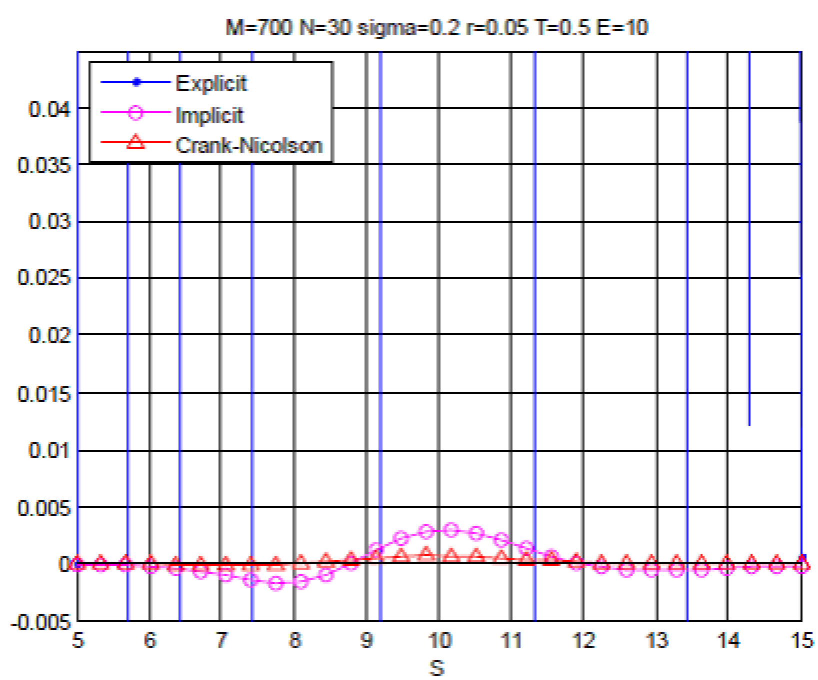


Figure 2.3.5.6

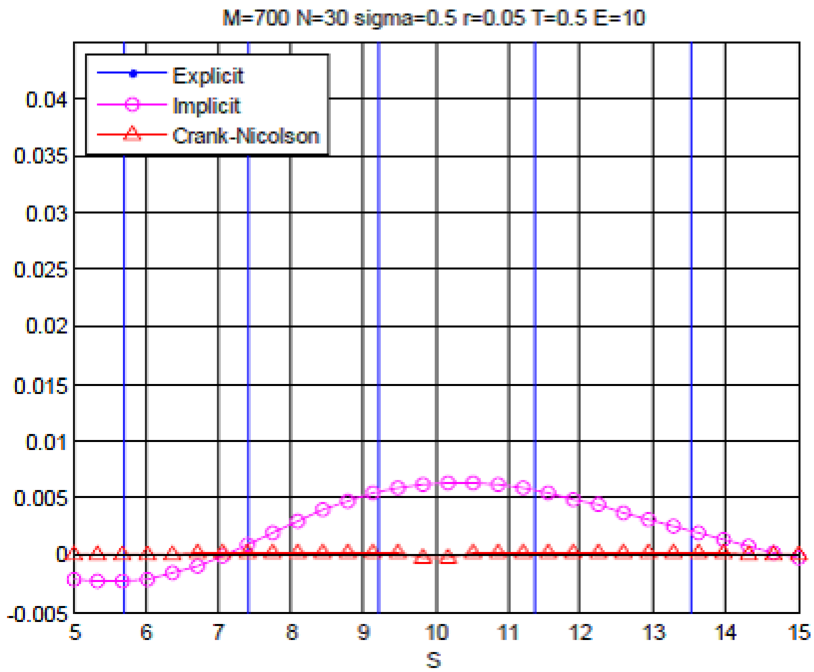
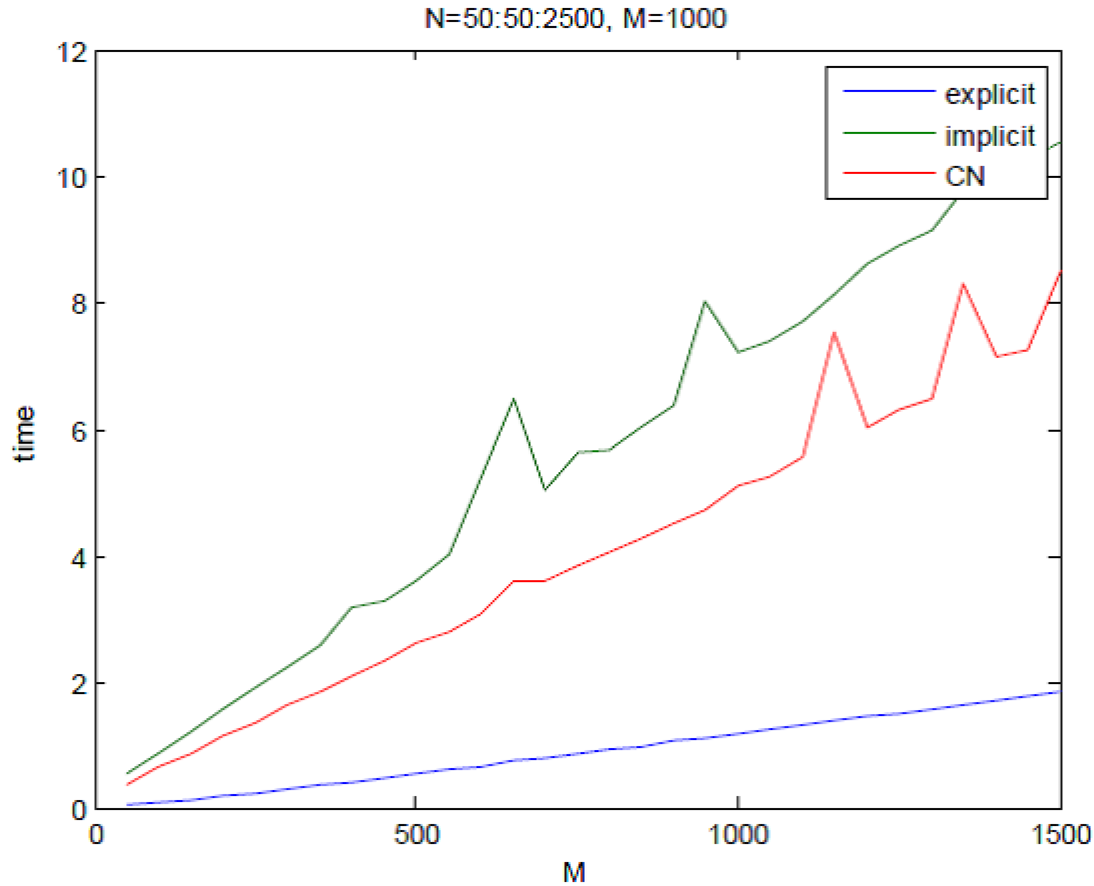
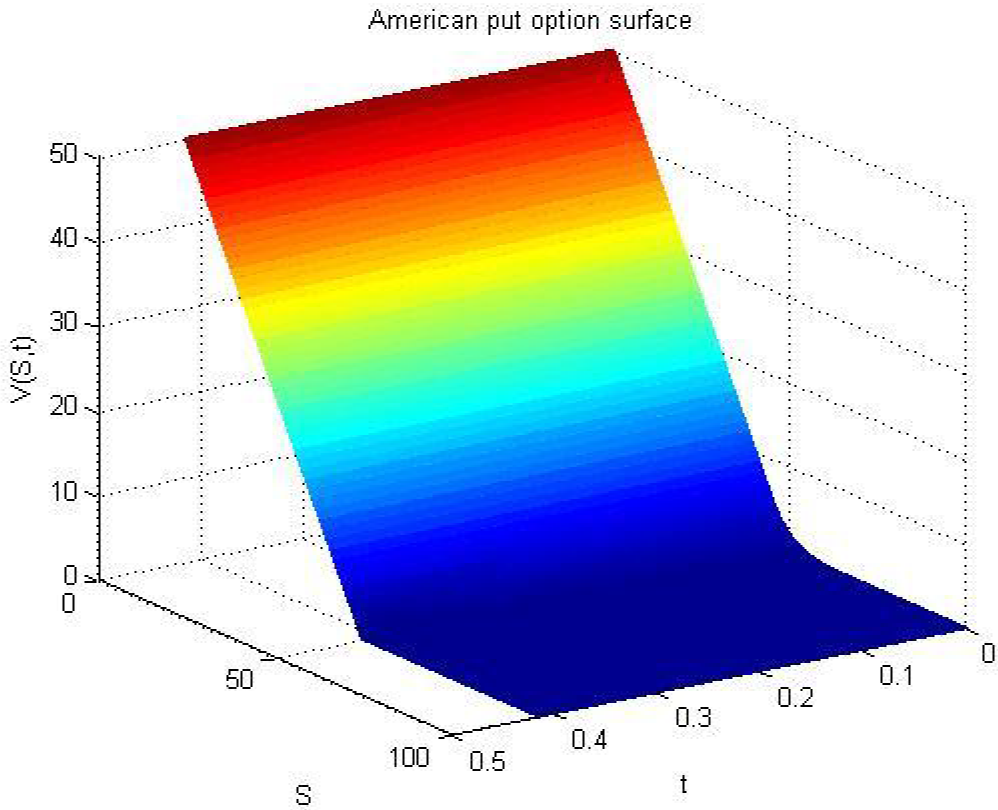


Figure 2.3.5.7

The graphs above show the comparison of errors relative to the logical result of the European put option obtained through the three FDM methods. The first conclusion we make is that the error is large when the stock price of the option is at the money. The error is dependent on r and and it impacts the stability of the explicit method.The stability of the explicit method is also affected by the relationship between dS and dt.We can finally conclude that Crank Nicolson method is more precise in outcome with increase in M while N constant. In our following graph we compare the speed of the three FDM methods.



The results in the explicit method are consistent with the principle. The explicit method gives modest solutions to every portion in explicit form. However, for the implicit Euler(IE) and the Crank Nicolson(CN) method, we obtain the solution of one element system of linear equations through solving. Further, Crank Nicolson method necessitates one matrix multiplication extra in every time step resulting in the method being slower than the implicit method. However, Matlab and R solve this problem since they multiply matrices very fast. This results in Crank Nicolson method being better method and with better precision.



The benefits of explicit method consist of;

- Programming is easy and it's difficult to make errors.
- Relatively fast
- Easily applicable to American options
- It manages fine with coefficients that depends on time and asset.
- If its unstable its easily noticeable

The main drawback is the restraint on time step because of matters with stability which at times affects the stability of the algorithm.

Implicit Euler finite difference method however outdos the stability issue. The need for small time steps is not necessary anymore but the solution is not as straightforward as there is necessity to solve a set of linear equations. Crank Nicolson method is a further improvement of the implicit finite difference method. It also requires one to solve a set of linear equations. For this reason,LU and PSOR methods are satisfactory.LU decomposition is very proficient. Limited time steps are adequate for implicit method going together with by LU decomposition and are generally more effective than explicit method.LU nonetheless can only be used European Options and not American Option. The purpose for the introduction of PSOR algorithm is applicable to every exercise possibility but is sluggish than LU in concept.

The main disadvantage of FDM is that when confronted by a problem that's 3-4 dimensions, they tend to be slow. In this case Monte Carlo is more preferable.

4.2.2 American Put Options results

The table below gives the prices of an American Put Option by subsequent bounds:

$k = 50, vol = 0.3, r = 0.1, T = 1, S(0) = 25 : 5 : 75$ Tree based binomial with 5000 steps

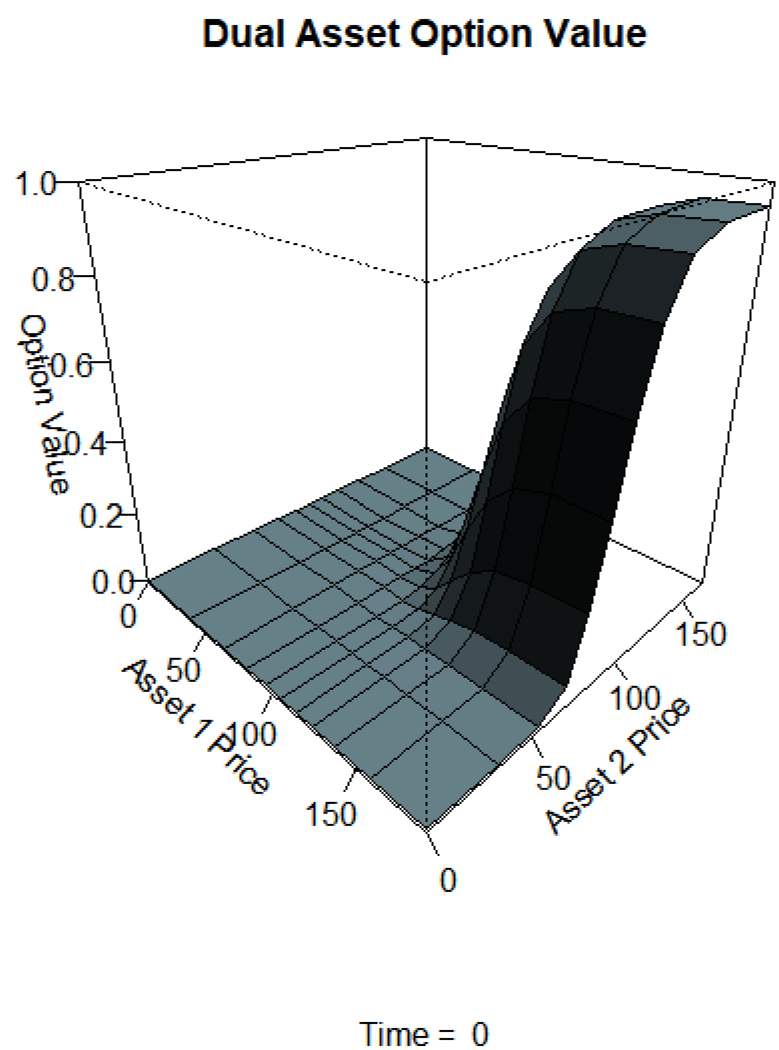
Monte Carlo (MC) with 80 time steps and 200000 paths

Finite difference: Project Successive Over Relaxation-price of stock $M = 200, N = 200$ time steps, Explicit $M = 300$ and $N = 15000$

Stock price	binomial tree	LSM AT	LSM	PSOR	EXPLICIT
25.0	25.0101	24.93849	24.97687	25	25
30.0	20.0101	19.98753	19.94946	20	20
35.0	15.07011	14.97748	14.93551	15	15
40.0	10.16439	10.16537	10.12188	10.14345	10.234
45.0	6.55042	6.54333	6.51467	6.54925	6.76
50.0	4.17877	4.14574	4.19085	4.14781	4.16549
55.0	2.684434	2.58549	2.69043	2.63345	2.64407
60.0	1.66399	1.5892	1.67317	1.63305	1.64356
65.0	0.93647	0.96871	0.96182	0.97374	0.94617
70.0	0.52955	0.59526	0.57642	0.58346	0.54885
75.0	0.39371	0.35585	0.35266	0.35692	0.35234
Average(time)	1.51s	12.3-5.4s	12.3-5.4s	14.7s	4.14s

Although the speed of the algorithms is dependent on the parameters in use, we can see that the obtained time is per expectation in theory. The binomial method is the fastest followed by the explicit method which is the fastest FDM method and it provides the value of options for each stock value and time in a specified array. Monte Carlo simulations are considerably slow but PSOR proved to be the slowest method. The reason PSOR is the slowest is due to the fact that it's an iterative method and is dependent on limitations of tolerance, relaxation limitations and maximum number of reiterations that can be castoff. The use of different parameters can provide better precision.

The graph below is a describes the pricing of basket options using various asset values.



5 Chapter 5

5.1 Conclusion

We can conclude that option pricing is a main achievement in modern investment which has incited the expansion of conversant financial options such as puts and calls as well as exotic options. As much as the major advantage of option pricing is to offer us with the "optimal price", market price is the greatest pricing option since a well-organized market offers the superlative price for options. Thus the major benefit of option pricing is the fact that it offers an accurate "snapshot" of the present market prices.eg. volatility that's has been implied. This hypothesis only scratches the surface of a wide area of arithmetical option pricing. A starter into the area can be done by comparing the fundamental methods of valuation of prevalent derivatives. We present the finite difference method and discuss three methods; the explicit euler (EE), implicit euler (IE) and crank Nicolson(CN) method. Explicit method is a very easy method to apply for both European and American options. In the evaluation table above, it comes right behind binomial pricing model. Further, finite difference methods give provision for option values for specified stock price and time array as similar to binomial model. This allows for interpolation in cases where prices have not been directly calculated. Finite difference method is however easily adapted to many problems thus better than binomial model in the Explicit. For us to overcome the stability issues, we use the implicit method and even further the Crank Nicolson method. Figure 2.3.5.7 clearly explains the precision. Crank Nicolson method improved the accuracy much faster while the price step decreased (M increased) which confirms our theory. When comparing the speeds of the three FDM methods, Crank Nicolson method was the fastest and most precise while Implicit method was the slowest. To value the American Put Option, we introduced PSOR algorithm. The outcome was as follows; Finite difference method was the most sluggish but had enhanced correctness to the Monte Carlo. We ought to choose a relaxation parameter carefully in order to achieve convergence at the right price. It's important to say that FDM is used in almost seventy percent of all valuation processes currently in place.

We can conclude by saying that different options require different kinds of solutions thus there is no universal recommended method. Exotic options however have analytical solutions and we use them depending on their availability. Monte Carlo is used if the problem has more than three underlying variables or for strong path dependent options.

5.2 Future Research

The major step would be designing arithmetic ways of calculating of exotic options, greeks and prizing options in non constant volatility or interest rates.

Finite difference methods can be enhanced in various ways. The first technique is on working on the error which can be solved by creation of a denser grid when the asset price is close to the money.Next, since the Explicit Euler method is not usually stable, we can solve this by using the alternating method explicit method[10].Finally FDM can be executed in HPC environment for both GPU and multi CPU[9].The apparent trend currently is use of GPUs in most large financial institutions but it's still a large research area.

Bibliography

[1] Merton, R.C.(1973), *Theory of rational option pricing*. The bell Journal of economics and management science, 4:141-183.

[2] Hull,J.C.(2006), *Options, futures, and other derivatives*. Pearson Education India.

[3] Y.Achdou and O.Pironneau,(2005) *Computational methods for option pricing*. Frontiers in applied mathematics, SIAM.

[4] G.Barles and P.E.Souganidis,(1991) *Convergence of approximation schemes for fully nonlinear second order equations*. Asymptotic Anal., 4(3):271-283.

[5] J.Cox,S.Ross, and M.Rubinstein,(1979) *Option Pricing: A simplified approach.J. Financial Economics*. 7:44-50,

[6] F.Dubois and T.Levievre,(2005) *Efficient pricing of Asian Options by the PDE approach.*. Journal of computational finance, 8(2):55-64.

[7] I.Karatzas and S.E. Shreve.(1988) *Brownian motion and stochastic calculus.*. Springer-Verlag.

[8] R.C.Merton,(1976) *Option pricing when underlying stock returns are discontinuous*. Journal of Financial economics, 3:125-144.

[9] Matthew F Dixon,(2002) *"Parallel solution of High Order Finite difference schemes for pricing Multi-Dimensional American Options"*. Maste thesis.

[10] Gulliaume Pealat,Daniel J.Duffy,(2011) *"The alternating direction explicit(ADE) method for one factor problems,*. Journal of Financial economics, Wilmott, issue 54,pages 54-60