UNIVERSITY OF NAIROBI

SCHOOL OF MATHEMATICS

Curvature Tensors on Semi-Riemannian and Generalized Sasakian Space Forms admitting Semi-symmetric metric connection

By

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Declaration

This thesis is my original work and has never been submitted for registration in any other University.

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Abstract

The study deals with curvature tensors on Semi-Riemannian and Generalized Sasakian space forms admitting semi-symmetric metric connection. More specifically, the study shall be to investigate the geometry of Semi-Riemannian and generalized Sasakian space forms, when they are $W_8 - \text{flat}$, $W_8 - \text{symmetric}$, $W_8 - \text{semisymmetric}$ and $W_8 - \text{Recurrent}$ and compared to results of projectively semi- symmetric, Weyl semi- symmetric and concircularly semi-symmetric on these spaces. Further, the conditions that admit a second order parallel symmetric tensor on functions of such spaces, shall be studied.

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Dedication

I wish to dedicate the work of this dissertation to my wife Cecily Wanjiru and my children

Mercy, Edwin, Benjamin and Beatrice.

Table of Contents

Chapter 1

1. Introduction

1.1. Historical background

Riemannian geometry is the branch of differential geometry that studies Riemannian manifolds, smooth manifolds with a Riemannian metric. Riemannian geometry was first put forward in generality by Bernhard Riemann in the nineteenth century and originated with the vision expressed in his 1854 inaugurational lecture, "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen (On the Hypotheses which lie at the Bases of Geometry)."

Development of Riemannian geometry resulted in synthesis of diverse results concerning the geometry of surfaces and the behavior of geodesics on them, with techniques that can be applied to the study of differentiable manifolds of higher dimensions. It enabled Einstein's general relativity theory, made profound impact on group theory and representation theory, as well as analysis, and spurred the development of algebraic and differential topology.

Any smooth manifold admits a Riemannian metric, which often helps to solve problems of differential topology. It also serves as an entry level for the study of the more complicated structures of pseudo-Riemannian; almost complex structures and complex structures on a Riemannian manifold, almost contact and contact structures, symplectic manifolds, K¨ahler manifolds and Calabi-Yau manifolds among many others. Many of these structures appear in the context of string theory and other areas in theoretical physics.

While the notion of a metric tensor was known in some sense to mathematicians such as Carl Gauss from the early 19th century, it was not until the early 20th century that its properties as a tensor were understood by, in particular, Gregorio Ricci-Curbastro and Tullio Levi-Civita, who first codified the notion of a tensor.

In 1960, a Japanese, Shigeo Sasaki, began the study of almost contact structures in terms of certain tensor fields, but it was not until 1962 that what are now called Sasakian manifolds first appeared under the name of "normal contact metric structure." By 1965 the terms "Sasakian structure" and "Sasakian manifold" began to be used more frequently replacing the original expressions. After 1968 Sasaki himself was less active although he continued to publish until 1980. Yet he had already created a new subfield of Riemannian geometry which slowly started to attract attention worldwide, not just in Japan. In 1966 Brieskorn wrote his famous paper describing a beautiful geometric model for all homotopy spheres which bound parallelizable manifolds. In 1976 Sasaki realized that Brieskorn manifolds admit almost contact and contact structures. (This very important fact was independently observed by several other mathematicians: Abe–Erbacher, Lutz–Meckert, and Thomas).

There was not much activity in this field after the mid-1970s, until the advent of Super String theory in 1980s. Since then Sasakian manifolds have gained prominence in physics and algebraic geometry, mostly due to a string of papers by Boyer, Galicki and their co-authors.

Sasakian geometry is not a separate subfield of Riemannian geometry but rather it is interrelated to other geometries. This is perhaps the most important feature of the subject. The study of Sasakian manifolds brings together several different fields of mathematics from differential and algebraic topology through complex algebraic geometry to Riemannian manifolds with special holonomy.

More closely related to Sasakian geometry is K̈ahlerian geometry. The relationship is clearly seen in the study of several topics, including the theory of Riemannian foliations, compact complex and Kähler orbifolds, and the existence and obstruction theory of Kähler-Einstein metrics on complex compact orbifolds. The study of contact and almost contact structures in the Riemannian setting, in which compact quasi-regular Sasakian manifolds emerge as algebraic objects is a clear prove of this. The discussion of the symmetries of Sasakian manifolds, has lead to a study of Sasakian structures on links of isolated hypersurface singularities. This has led to an in-depth study of compact sasakian

manifolds in dimensions three and five, and properties of curvature tensors on the generalized Sasakian space forms among others.

1.2. Notations, Terminologies and Definitions

Definition 1.2.1: Consider an $n-$dimensional manifold M. Let p be a point of the manifold. Denote as V_p the set of all vector fields defined at p. V_p is an $n-$dimensional vector space.

Definition 1.2.2: A 1-form $\tilde{q}$ defined at p is a linear scalar operator acting on vector space V_p, to real number R. That is

> 1) $\tilde{q} : V_p \rightarrow R$;
>
> 2) For any $\underline{u}, \underline{u} \in V_p$ and $a,b \in R \Rightarrow \tilde{q}(a\underline{u}+b\underline{v})= a\tilde{q}(\underline{u})+b\tilde{q}(v)$

The set of all 1-forms defined at p is called covector or dual space to V_p and it is denoted by V_p^*. This is an $n-$dimensional vector space.

Definition 1.2.3: Any vector $\underline{u} \in V_p$ can be associated with a linear scalar operator acting on 1-forms V_p^* to R. i.e. $\underline{u}(\tilde{q})= \tilde{q}(\underline{u}):V_p^* \rightarrow R$

Definition 1.2.4: An $\begin{pmatrix} k \\ l \end{pmatrix}-$type tensor defined at point p is a linear scalar operator with l slots for 1-forms from V_p^* and k slots from V_p. Such tensor can also be called as $l-$times contravariant and k-times covariant. The total number of slots, $r=l+k$, is called the rank of the tensor.

Thus,

> 1) Any vector is a $\begin{pmatrix} 1 \\ 0 \end{pmatrix}-$type tensor
>
> 2) Any 1-form is a $\begin{pmatrix} 0 \\ 1 \end{pmatrix}-$type tensor

Remarks

Tensors therefore are a generalization of vectors and 1-forms (covectors).

A tensor of type (k,l) at p is a multi-linear map which takes k vectors and l covectors (1-forms) and gives a real number.

A tensor (or tensor field) T, of type (k,l) is denoted with k superscripts and l subscripts (T_l^k) and is said to be of rank $k+l$.

Definition 1.2.5: Let M be a smooth manifold. A tangent vector at a point $p \in M$ is a map

$X_p : C^\infty(M) \to R$ which satisfies

     i) $X_p(f + g) = X_p(f) + X_p(g)$

     ii) $X_p(\text{constant map}) = 0$

     iii) $X_p(fg) = f(p)Xg + g(p)X_p f$

for all $f, g \in C^\infty(M)$ on their common domain.

The set of all tangent vectors to an n-dimensional manifold M at a point $p \in M$ forms an n-dimensional vector space called the tangent space and is denoted by $T_p M$.

Definition 1.2.6: The disjoint union of all the tangent spaces $T_p M$ for all points $p \in M$ is a 2n-dimensional manifold known as the tangent bundle of M and denoted TM,

$$TM = \coprod_{p \in M} T_p M$$

The tangent spaces at different points are, by definition, different vector spaces, which cannot have common elements. Hence, the union is a disjoint union.

This is an example of a fibre bundle and is itself a *2n-dimensional manifold.*

Definition 1.2.7: Let M be an n-dimensional smooth manifold. A vector field v on M is a section of tangent bundle TM, i.e., $v : M \to TM$ such that for all $p \in M$, $v(p) \in T_p M$.

In other words, a vector field on M is a map which assigns to each point $p \in M$ a tangent vector $v(p) \in T_p M$.

Vector fields are traditionally denoted by boldface letters such as *v, u or w or* by capital letters such as *X, Y, or Z.*

Definition 1.2.8. Let M be a smooth manifold. For each $p \in M$, we define the cotangent **space** at p, denoted by $T_p^* M$, to be the dual space to $T_p M$:

$$T_p^* M = \left(T_p M\right)^*.$$

Elements of $T_p^* M$ are called tangent covectors at p.

Definition 1.2.9: The disjoint union

$$T^* M = \coprod_{p \in M} T_p^* M$$

is called the cotangent bundle of *M*.

Definition 1.2.10: The number and position of indices of tensor components reveal all the general information about tensors as operators. For example, if a tensor T has components T_{jl}^{ik}

This immediately tells us that

1) T is a 4ᵗʰ rank tensor,

2) T is $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ − type tensor,

3) Its 1ˢᵗ and 3ʳᵈ slots are for 1-forms whereas its 2ⁿᵈ and 4ᵗʰ slots are fo vectors.

Definition 1.2.11: Let M be a smooth manifold, then by a Riemannian metric **tensor** g on M we mean a smooth assignments of an inner product to each tangent space of M. This means that, for each $p \in M, g_p : T_p M \times T_p M \to R$ is symmetric, positive definite and bilinear map. For instance, for any smooth vector fields *X and Y on* $M, p \mapsto g_p(X_p, Y_p)$ is a smooth function.

It is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ − tensor $g \in T_0^2(M)$

In a coordinate frame we may write $g = g_{ij} dx^i \otimes dx^j$

The pair (M, g) then will be called Riemannian manifold.

Definition 1.2.12: By S and R we denote respectively the Ricci tensor and Riemannian curvature tensors of an $n-$dimensional Riemannian manifold (M, g). Then S is defined by

$$S(X, Y) = \sum_{i=1}^{n} g\left(R(e_i, X)Y, e_i\right)$$

Where $\{e_1, e_2, \ldots \ldots, e_n\}$ are orthonormal basis vector fields in TM, and $X, Y, Z \in TM$.

Definition 1.2.13: Let M be a smooth manifold. An affine connection (Levi-Civita) connection ∇ on M is a differential operator, sending smooth vector fields X and Y to a smooth vector field $\nabla_X Y$, which satisfies the following conditions:

$$\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z, \quad \nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z,$$
$$\nabla_{fX} Y = f \nabla_X Y, \qquad \nabla_X(fY) = X(f)Y + f(\nabla_X Y)$$

for all smooth vector fields X, Y and Z and real valued functions f on M.

A vector field $\nabla_X Y$ is known as the covariant derivative of the vector field Y along X (with respect to the affine connection ∇).

Definition 1.2.14: Let X and Y be vector fields on a space M. We define the *Lie bracket* sometimes known as the Jacobi-Lie bracket *, commutator* or just *bracket*) $[X,Y]$ to be operator

$$[X,Y] = XY - YX.$$

As it turns out, the bracket of two vector fields is again a vector field, meaning it is a first order differential operator. In components, letting $X = X^i \dfrac{\partial}{\partial x^i}$ and $Y = Y^j \dfrac{\partial}{\partial x^j}$, we have

$$[X,Y] = X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j} - Y^i \frac{\partial X^j}{\partial x^i}\frac{\partial}{\partial x^j}$$
$$= X(Y^j)\frac{\partial}{\partial x^j} - Y(X^j)\frac{\partial}{\partial x^j}$$
$$= XY - YX$$

Thus, $[X,Y]$ is the vector field

Definition 1.2.15: The torsion tensor T and the Riemannian curvature tensor R of the affine connection ∇ are the operators sending smooth vector fields X, Y and Z on M to a smooth vector fields $T(X,Y)$ and $R(X,Y)Z$ respectively given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

An affine connection ∇ on M is said to be torsion-free if its torsion tensor is zero everywhere (so that $[X,Y] = \nabla_X Y - \nabla_Y X$ for all smooth vector fields X and Y on M).

Definition 1.2.16: A curve $\gamma(s)$ is a geodesic if its tangent vectors $\dot{\gamma}(s)$ at each point are parallel.

Definition 1.2.17: Let X be a nonempty set. A collection τ of subsets of X is called a topology on X. We call the pair (X,τ) a topological space. Often, we denote the topological space by X instead of (X,τ).

Definition 1.2.18: A mapping $f:X \to Y$ between two topological spaces is called continuous if for every $U \subseteq Y$ open in Y the inverse image $f^{-1}(U)$ is open in X. We also say that f is a map**.**

Definition 1.2.19: A homeomorphism $f:X \to Y$ is continuous bijection whose inverse $f^{-1}:Y \to X$ is also continuous.

Definition 1.2.20: A topological space X is said to be Hausdorff if for any two distinct points $x,y \in X (x \neq y)$ there exists two disjoint open subsets U,V $(U \cap V = \phi)$ such that $x \in U$ and $y \in V$. This is an example of a separation axiom since one thinks of the open sets U,V as "separating" the two points x and y.

Definition 1.2.21: Let M be a topological Hausdorff space with a countable basis. M is called a topological manifold if there exists an $n \in N$ (natural number) and for every point $p \in M$ an open neighborhood U_p of p which is Homeomorphic to some open subset $V_p \in R^n$. The integer n is called the dimension of M and we write M^n to denote that M has dimension n.

Definition 1.2.22: Let M be a topological manifold. An open cover of M is a collection of open (subsets) $U \subset M$ whose union is M, i.e. $M = \bigcup_{\alpha \in I} U_\alpha$.

A chart of M is a pair (U,φ) such that $U \subset M$ is an open set in M and φ is a homeomorphism from U onto an open set in R^n, i.e. $\varphi:U \to R^n$.

An atlas for M means a collection of charts $\{(U_\alpha,\varphi_\alpha)|\alpha \in I\}$ such that $\{U_\alpha|\alpha \in I\}$ is an open cover of M.

Definition 1.2.23: A manifold M is called a differential manifold of class C^k if there is an atlas of M $\{(U_\alpha, \varphi_\alpha) | \alpha \in I\}$ such that, for any $\alpha, \beta \in I$, the composites

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to R^n$$

is differentiable of class C^k.

The atlas $\{(U_\alpha, \varphi_\alpha) | \alpha \in I\}$ is called a differential atlas of class C^k on M. If instead, the atlas is of class C^∞, then M is said to have a differentiable(smooth) structure and is called a smooth(differential) manifold.

Definition 1.2.24: Let M and N be two smooth manifolds. A smooth map $f : M \to N$ is called a diffeomorphism if f is one-to-one and onto, and if a smooth inverse $f^{-1} : N \to M$ exists.

Definition 1.2.25: Let M be an $n-$dimensional contact manifold with contact form η, i.e. $\eta \wedge (d\lambda)^n \neq 0$. It is well known that a contact manifold admits a vector field ξ, called characteristic vector field, such that $\eta(\xi) = 1$ and $\eta(\xi) = 1$ for every field $X \in \chi(M)$. Moreover, if M admits a Riemannian metric g and a tensor field ϕ of type (1,1) such that

$$\phi^2 X = X - \eta(X)\xi$$
$$g(X, \xi) = \eta(X)$$
$$g(X, \phi Y) = d\eta(X, Y)$$

then we say that (ϕ, η, ξ, g) is a contact metric structure.

Definition 1.2.26: A contact manifold is said to be Sasakian if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$$

In which case

$$\nabla_X \xi = -\phi X,$$
$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

for all vector fields X, Y on M.

Definition 1.2.27: Let (M^n, g) be a contact Riemannian manifold with a contact form η, the associated vector field ξ, (1,1)-tensor field ϕ, and associated Riemannian metric g.

If ξ is a Killing vector field, then M^n is called a $K-$contact Riemannian manifold.

A $K-$contact Riemannian manifold is called Sasakian manifold if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$$

holds, where ∇ denotes the operator of covariant differentiation with respect to g.

Definition 1.2.28: A $(2n+1)$-dimensional Riemannian manifold (M, g) is called an almost contact metric manifold if the following results hold:

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0$$
$$g(X, \xi) = \eta(X),$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\,\eta(Y)$$
$$g(\phi X, Y) = -g(X, \phi Y), \qquad g(X,X) = 0$$
$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y)$$

Definition 1.2.29: An almost contact metric manifold (structure) is called

- a contact metric manifold if

$$d\eta(X,Y) = \Phi(X,Y) = g(X, \phi Y)$$

  where Φ is called the fundamental two form of the manifold.

- K-contact manifold if it is contact metric manifold and ξ is killing vector field

- Sasakian manifold if and only if it is a contact manifold satisfying

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$$

- a normal contact structure if and only if

$$[\phi, \phi](X,Y) = -2d\eta(X,Y)\xi$$

where $[\phi, \phi](X,Y)$ denotes the Nijenhuis tensor field of ϕ given by

$$[\phi, \phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

Definition 1.2.30: An almost contact metric manifold M is a trans-Sasakian manifold if there exists two smooth functions α and β on M such that

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X)$$

for all vector fields X and Y on M. We say that the trans-Sasakian structure is of type (α, β).

If in an $n-$dimensional trans-Sasakian manifold of type (α, β), then we have

$$\phi(grad\,\alpha = (n-2)grad\,\beta$$

Definition 1.2.31: Given an almost contact metric manifold $M(\phi, \xi, \eta, g)$ we say that M is a generalized Sasakian space –form if the curvature tensor R is given by

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\}$$
$$+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}$$
$$+ f_3\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}$$

where f_1, f_2, and f_3 are differentiable functions on M and X,Y,Z are vector fields on M.

Definition 1.2.32: An almost contact metric manifold is called Kenmotsu manifold if

$$(\nabla_X \phi)Y = -g(X,\phi Y)\xi - \eta(Y)\phi X$$
$$(\nabla_X \xi) = X - \eta(X)\xi$$
$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y)$$

where ∇ is Levi-Civita connection of g for any vector fields X,Y on M.

Definition 1.2.33: An $n-$dimensional differentiable manifold M is said to admit an almost para-contact Riemannian structure (ϕ, η, ξ, g) such that

$$\phi^2 X = X - \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0$$
$$g(X,\xi) = \eta(X)$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for all vector fields X,Y on M.

If (ϕ, η, ξ, g) satisfy the equations

$$d\eta = 0, \nabla_X \xi = \phi X$$
$$(\nabla_X \phi)\, Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

then M is called Para-Sasakian manifold or briefly, P-Sasakian manifold.

A P-Sasakian manifold is called a special para-Sasakian manifold or briefly SP-Sasakian manifold if M admits 1-form η such that

$$(\nabla_X \eta) Y = -g(X,Y) + \eta(X)\eta(Y)$$

for any vector fields X, Y on M.

Definition 1.2.34: An $n-$dimensional differentiable manifold M^n is Lorentzian para-Sasakian (LP-Sasakian) manifold, if it admits a (1,1)-tensor field ϕ, vector field ξ, 1-form η and a Loretzian metric g, which satisfies

$$\phi^2 X = X + \eta(X)\xi, \quad \phi\xi = 0 \ \ \eta(\xi) = -1, \quad \eta(\phi X) = 0$$
$$g(X,\xi) = \eta(X)$$
$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)$$
$$(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi$$
$$\nabla_X \xi = \phi X$$

for arbitrary vector fields X and Y; where ∇_X denotes covariant differentiation in the direction of X with respect to g.

Definition 1.2.35: Let (M^n, g) be an $n-$dimensional Riemannian manifold of class C^∞ with metric tensor g and let ∇ be the Levi-Civita connection on M^n. A linear connection $\tilde{\nabla}$ on (M^n, g) is said to be a semi-symmetric if the torsion T of the connection $\tilde{\nabla}$ satisfies

$$T(X,Y) = \pi(Y)X - \pi(X)Y,$$

where π is a 1-form on M^n with ρ as associated vector field, i.e., $\pi(X) = g(X,\rho)$ for any differentiable vector field X on M^n.

A semi-symmetric connection $\tilde{\nabla}$ is called semi-symmetric metric connection if it further satisfies

$$\tilde{\nabla} g = 0.$$

In an almost contact manifold, semi-symmetric metric connection is defined by identifying the 1-form π with the contact form η, and by setting

$$T(X,Y) = \eta(Y)X - \eta(X)Y$$

ρ with ξ as the associated vector field giving $g(X,\xi) = \eta(X)$.

Definition 1.2.36: An almost contact metric manifold is said to be η – Einstein manifold if the Ricci tensor S satisfies the condition

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a and b are certain scalars. If b=0, it is called Einstein manifold.

1.3. Statement of the problem.

The aim of this study is to investigate W_8 - Curvature Tensors on Semi-Riemannian and Generalized Sasakian space forms endowed with semi-symmetric metric connection. The motivation is to generate some fresh ideas with emphasis on producing new geometric results having physical meaning.

1.4.1 Overall Objective

The overall objective of the study is to generate some fresh ideas with emphasis on producing new geometric results having physical meaning.

1.4.2 Specific Objectives

The study shall be guided by the following specific objectives;

    i). Investigate the basic properties of various Riemannian and Sasakian spaces,

    ii). Investigate the results obtained and develop new results,

    iii). Find possible applications of the new results.

1.5 Significance of the study

This study will add to the existing applicable knowledge in mathematics, physics and chemistry in the analysis of curvature tensors to generate equations which describe the nature of forces existing in

1. Black holes, that is regions of spacetime from which gravity prevents anything, including light, from escaping.
2. Spinning planets and their shapes as they traverse their orbits in the space.
3. Electrons and protons in an atom and the shapes of atomic orbitals.

4. Bermuda triangle i.e. region in the western part of the North Atlantic Ocean where a number of aircrafts and ships are said to have disappeared under mysterious circumstances.

Discussion

The phenomena above are as consequences of gravitational force.

- More understanding of gravitational force can be enhanced by treating gravity as a curvature tensor and perform the necessary transformations.

- Gravity could be treated as a spacetime curvature tensor because gravity affects all the bodies in the same way irrespective of the coordinate system just like any curvature tensor.

- In the presence of mass density ρ the Newtonian potential ϕ obeys Poissons equation.

$$\nabla^2 \phi = 4\pi G \rho.$$

The left hand side is a covariant derivative acting on a tensor which describes a curvature tensor in the spacetime while the right hand side represent matter distribution and G being a constant known as universal gravitational force.

This is clearly shows that gravitational force is in itself a consequence of curvature tensor and should be treated as an outcome of one of the curvature tensors we have used in our study.

From this brief hypothesis, it is apparent that the above phenomena can also be explained using covariant derivatives on curvature tensors.

CHAPTER 2

2   Literature Review

The idea of semi-symmetric linear connection on a differentiable manifold was introduced by Friedman and Schouten (1924). Then later, Hayden (1932) introduced the idea of metric connection with a torsion on a Riemannian manifold. A systematic study of semi-symmetric metric connection on a Riemannian manifold has been given by Yano (1970) and later studied by Sharafuddin and Hussain (1976), Amur K.S.and Pujar S.S. (1978), Bagewadi C.S. (1982), De U.C.et al (1997) and others. In their papers, the authors,De U.C. (1997) and Bagewadi C.S.et al (2003, 2006) have obtained results on the conservativeness of Projective, Pseudo projective, Conformal, Concircular, Quasi conformal curvature tensors on K-contact, Kenmotsu and trans-sasakian manifolds.

In their study on Kenmotsu manifolds C.S. Bagewadi C. S. , Prakasha D.G, and Venkatesha (2007) established that if the projective curvature tensor of a Kenmotsu manifold M^n (n>2) admitting semi-symmetric metric connection vanishes, then M^n reduces to an Einstein manifold with constant scalar curvature $-n(n-1)$.

 In their paper (2008), Bagewadi Channabasappa et al, extended the conservativeness of Pseudo projective curvature tensor to K-contact and trans-Sasakian manifolds admitting semi-symmetric metric connection

Ingalahalli G. and Bagewadi C. S. (2012) dealt with the study on conservative C-Bochner curvature tensor in K-contact and Kenmotsu manifolds admitting semi-symmetric metric connection, and has shown that these manifolds are η-Einstein with respect to Levi-Civita connection.

On the other hand, Bochner S. introduced a Kahler analogue of the Weyl conformal curvature tensor by purely formal considerations, which is now well known as the Bochner curvature tensor (1949). A geometric meaning of the Bochner curvature tensor is given by Blair D. in (1975). By using the Boothby-Wang's fibration (1958), Matsumoto M. and Chuman G. constructed C-Bochner curvature tensor (1969) from the Bochner curvature tensor. In (1991), Endo H. defined E-Bochner curvature tensor as an

extended C-Bochner curvature tensor and showed that a K-contact manifold with vanishing E-Bochner curvature tensor is a Sasakian manifold.

These two important classes of contact manifolds; K-contact manifolds and Sasakian manifolds have been studied by many authors and several results established.

Motivated by the findings of earlier authors, De U. C. and De A. (2012) studied on the projective curvature tensor on K-contact and proved that a projectively flat K-contact manifold is isometric to a unit sphere.

As a generalization of locally symmetric manifolds, the notion of semisymmetry of Riemannian manifolds was introduced and first studied by Cartan (1926).

De U. C. (1976) studied projective curvature tensors on K-contact and proved that a projectively semisymmetric K-contact is a Sasakian manifold. In a later study (2011), De U. C. and De A. proved that a projectively pseudosymmetric K-contact manifold and pseudoprojectively flat K-contact manifolds are Sasakian manifoldsrespectively.

A K-contact manifold is always a contact metric manifold, but the converse is not true in general. Pradip M. and DE U. C. (2013) studied on concircular curvature tensor on K-contact manifolds and established that a $(2n+1)$ – dimensional ϕ – concircularly flat K-contact manifold ($n \geq 1$) is Einstein manifold of a scalar curvature equal to $2n(2n+1)$. In the same study, they proved that, a concircularly semisymmetric K-contact manifold of dimension $(2n+1)$, $n \geq 1$, is a Sasakian manifold.

Dwivedi M. K. and Kim J. (2011) studied on conharmonic curvature tensor in K-contact and Sasakian manifolds. They showed that a quasi-conharmonically flat K-contact manifold of dimension $(2n+1)$ has a vanishing scalar curvature. They established that $(2n+1)$-dimensional quasi-projectively flat K-contact is an Einstein manifold while a quasi-conharmonically flat K-contact is not. A quasi-conharmonically flat Sasakian manifold was shown to be η-Einstein though.

Tanno (1988) studied Ricci symmetric $(i.e.\ \nabla S = 0)$ K-contacts manifolds. Also later Zhen G. (1992) studied conformally symmetric $(i.e.\ \nabla C = 0)$ K-contact manifolds and proved that, if a Riemannian manifold is conformally symmetric, then the manifold satisfies the harmonic Weyl conformal curvature tensor. But, the converse is not true.

Ahmet Y. and Erhan A. (2011) in their paper generalized the results of Tanno and Zhen by establishing that, if a K-contact manifold is of harmonic conformal curvature tensor, that is, $divC = 0$, then the manifold is an Einstein manifold.

Gray A. (1978) introduced two classes of Riemannian manifolds determined by the covariant derivatives of the Ricci tensor, the class A consisting of all Riemannian manifolds whose Ricci tensor S is a Codazzi tensor, that is,

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z).$$

The class B consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel, that is,

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(X,Z) + (\nabla_Z S)(X,Y) = 0$$

Ahmet Y. and Erhan A. (2011) in their study on K-contacts deduced that an η-Einstein K-contact manifold satisfied the cyclic parallel Ricci tensor.

On the other hand in Pokhariyal(1982) defined a tensor field W^* on a Riemannian manifold as

$$W^*(X,Y,Z,U) = `R(X,Y,Z,U) - \frac{1}{2(n-1)}\big[S(Y,Z)g(X,U) - S(X,Z)g(Y,U)\big]$$

Where $W^*(X,Y,Z,U) = g\big(W^*(X,Y)Z,U\big)$ and $`R(X,Y,Z,U) = g\big(`R(X,Y)Z,U\big)$ such a tensor field W^* is known as m-projective curvature tensor. Later, Ojha (1986) defined and studied the properties of m-projective curvature tensor in Sasakian and K¨ahler manifolds. He also showed that it bridges the gap between the conformal curvature tensor, conharmonic curvature tensor, and concircular curvature tensor on one side and H-projective curvature tensor on the other. Recently m-projective curvature tensor has been studied by Chaubey and Ojha (2010), Singh et al. (2012), and many others. Motivated by the above studies, Singh R. N. and Shravan K. Pandey (2013), studied flatness and symmetry property of $N(k)$-contact metric manifolds regarding m-projective curvature tensor.

The notion of a Lorentzian Para Sasakian manifold was introduced by

Matsumoto K. (1989). Mihai I. and Rosca R. (1992) defined the same notion independently and they obtained several results on this manifold. Also LP-Sasakian manifolds have been studied by Matsumoto and Mihai I. (1988), De U.C. (1999) and shaikh A. A. (2004).

Taleshian A. and Asghari N. (2011), investigated the properties of the LP-Sasakian manifold equipped with projective curvature tensor.

Generalized Sasakian-space-forms was defined by Alegre et al. (2004) as the almost contact metric manifold $M^{2n+1}(\phi, \eta, \xi, g)$ whose curvature tensor R is given by

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3$$

where f_1, f_2, f_3 are some differential functions on M.

Kim U. K.(2006) studied generalized Sasakian-space-forms and proved that if a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ of dimension greater than three is conformally flat and ξ is Killing, then it is locally symmetric. De and Sarkar (2010) have studied generalized Sasakian space-forms regarding projective curvature tensor. Moreover, they proved that if $M(f_1, f_2, f_3)$ is locally symmetric, then $f_1 - f_3$ is constant. In the same study they established that,

    i)   In an η − recurrent generalized Sasakian space form, the 1-form A is closed, that

$$\text{is,} \quad (\nabla_W A)(Y) - (\nabla_Y A)(W) = 0$$

   ii) A $(2n+1)$ − dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ has

       η − parallel Ricci tensor if and only if $2nf_1 + 3f_2 - f_3$ is a constant.

Motivated by these findings Singh A. and Shyam K. (2017) studied on Semisymmetric metric connection on generalized Sasakian space forms and proved that a $(2n+1)$ - dimensional generalized Sasakian space forms $M(f_1, f_2, f_3)$ with respect to semisymmetric metric connection is always a ξ − conformally flat. On the same study, they established that a generalized Sasakian space form $M(f_1, f_2, f_3)$ whose curvature tensor of manifold is covariant constant with respect to the semi-symmetric metric connection and M, is recurrent with respect to the Levi-Civita.

Motivated by the studies of conformal curvature tensor in K-contact manifolds, and the studies of projective curvature tensor in K-contact, Sasakian manifolds (2003) and Lorentzian para-Sasakian manifolds, Ghosh S., DE. U. C., and Taleshian A. (2011), studied conharmonic curvature tensor in N(k)-contact metric manifolds. Later, DE. U. C., Singh R. N., and Shravan K. Pandey (2012) studied flatness and symmetry property of generalized Sasakian-space-forms regarding conharmonic curvature tensor. They also studied and characterized generalized Sasakian-space-forms satisfying certain curvature conditions on conharmonic curvature tensor. Also studied were the conharmonically semisymmetric, conharmonically flat, ξ-conharmonically flat, and conharmonically recurrent generalized Sasakian-space-forms. Also generalized Sasakian-space-forms satisfying $C \cdot S = 0$ and $C \cdot R = 0$ have been studied.

The notion of the quasi-conformal curvature tensor was given by Yano and Sawaki (1968). According to them a quasi-conformal curvature tensor C is defined by

$$C(X,Y)Z = aR(X,Y)Z + b\big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\big]$$
$$-\frac{r}{n}\left[\frac{a}{n-1} + 2b\big[g(Y,Z)X - g(X,Z)Y\big]\right]$$

where a and b are constants and R, S, Q and r are the Riemannian curvature tensor of type (1,3), the Ricci tensor of type (0,2), the Ricci operator defined by $g(QX, Y) = S(X, Y)$ and the scalar curvature of the manifold respectively.

It is known that a conformally flat Sasakian manifold is of constant curvature and a Weyl semi-symmetric Sasakian manifold is locally isometric with the unit sphere $S^n(1)$.

Gatti and Bagewadi (2003) have studied irrotational quasi-conformal curvature tensor in K-contact, Kenmotsu and trans-Sasakian manifolds and they have shown that these manifolds are Einsteinian. S. Bagewadi1, E. Girish Kumar ,and Venkatesha (2005) in their paper extended these results to irrotational D-Conformal curvature tensor in K-contact, Kenmotsu and trans-Sasakian manifolds.

In their paper De U.C., Jae Bok Jun, and Abul Kalam Gazi (2008) have studied quasi-conformally flat and quasi-conformally semi-symmetric Sasakian manifolds and proved that a Sasakian manifold is quasi-conformally flat if and only if it is locally isometric with the unit sphere $S^n(1)$ .And that, a compact orientable quasi-conformally flat Sasakian manifold cannot admit a non-isometric conformal transformation. Finally,they have shown that a Sasakian manifold is quasi-conformally flat if and only if it is quasi-conformally semi-symmetric.

Blair, Koufogiorgos and Papantoniou (1995) introduced the class of contact metric manifolds M^{2n+1} with contact metric structures (φ,ξ,η,g),in which the curvature tensor R satisfies the equation;

$$R(X,Y)\xi = k\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \forall X, Y \in \chi(M).$$

A contact Riemannian manifold belonging to this class is called a (k, μ)-contact manifold Here, (k,μ) are real constants and 2h denotes the Lie derivative ($2h = L_\xi\phi$)in the direction of ξ.

In this case we say that the characteristic vector field ξ belongs to the (k,μ) -nullity distribution and the class of contact metric manifolds satisfying this condition are called (k,μ) -contact metric manifolds. The class of (k, μ)-contact metric manifolds encloses both Sasakian and non-Sasakian manifolds.

In case the vector field ξ is Killing, this class of manifolds are called Sasakian manifolds. Before Boeckx (2000), two classes of non-Sasakian (k, μ)-contact metric manifolds were known. The first class consists of the unit tangent sphere bundles of spaces of constant curvature, equipped with their natural contact metric structure, and the second class contains all the three-dimensional unimodular Lie groups, except the commutative one, admitting the structure of a left invariant (k, μ)-contact metric manifold. A full classification of (k, μ)-contact metric manifolds was given by E. Boeckx (2000). Extending the result of Endo H. (1991) to a (k, μ))-manifold, Jeong-Sik Kim, Tripathi M. M., and Choi J. (2005) proved that a (k, μ)-manifold with vanishing E-Bochner curvature tensor is a Sasakian manifold. They drew several interesting corollaries of this result. They classified non-Sasakian (k, μ)-manifolds with C-Bochner curvature tensor B satisfying B (ξ, X).S = 0, where S is the Ricci tensor. The N(k)-contact metric manifolds

M^{2n+1}, satisfying B (ξ ,X).R = 0 or B (ξ, X).B = 0 were also classified and studied by Tripathi M. M.Et al (2005).

The (k, μ)-contact metric manifolds are invariant under D-homothetic transformations. Recently, the authors Ghosh A., Sharma R. and Cho J.T.,(2008) in their book, proved that a non-Sasakian contact metric manifold with η-parallel torsion tensor and sectional curvatures, of plane sections containing the Reeb vector field, different from 1 at some point, is a (k, μ)-contact manifold. In another recent paper (2008), Sharma R. showed that if a (k, μ)-contact metric manifold admits a nonzero holomorphically planer conformal vector field, then it is either Sasakian,or, locally isometric to E^3 or
E^{n+1} $\times$ S^n (4).

Cho J.T. (2001) studied a conformally flat contact Riemannian (k, μ)-space and such a space with vanishing C-Bochner curvature tensor.

In (2007), the authors studied extended pseudo projective curvature tensor on a contact metric manifold. Recently, quasi-conformal curvature tensor on a Sasakian manifold has been studied by De U.C., Jun J.B. and Gazi A.K. (2008).
 DE U.C. and Sarkar A. (2012) in their paper studied a quasi-conformally flat (k, μ)-contact metric manifold. Also studied is a (k, μ)-contact metric manifold with vanishing extended quasi-conformal curvature tensor (2010).

The notion of Lorentzian para- contact manifold was introduced by Matsumoto K. (1989). The properties of Lorentzian para -contact manifolds and their different classes, viz LP-Sasakian and LSP-Sasakian manifolds, have been studied by several authors since then. In (2000), M. Tarafdar and A. Bhattacharya proved that a LP-Sasakian manifold with conformally flat and quasi-conformally flat curvature tensor is locally isometric with a unit sphere S^n (1). Further, they obtained that a LP-Sasakian manifold with R(X, Y).C = 0 is locally isometric with a unit sphere S^n (1),where C is the conformal curvature tensor of type (1976, 1989) and R(X, Y) denotes the derivation of the tensor algebra at each point of the tangent space. J. P. Singh (2008) proved that an

M-projectively flat Para-Sasakian manifold is an Einstein manifold. He has also shown that, if in an Einstein P-Sasakian manifold $R(\xi,X).W^* = 0$ holds, then it is locally

isometric with a unit sphere $H^n(1)$. Also, an n-dimensional η-Einstein P-Sasakian manifold satisfies

$W^*(\xi,X).R = 0$ if and only if either the manifold is locally isometric to the hyperbolic

space $H^n(1)$ or the scalar curvature tensor r of the manifold is -n(n-1). LP-Sasakian manifolds have also studied by Matsumoto and Mihai (1988), Takahashi (1977), De, Matsumoto and Shaikh (1999), Prasad and Ojha (1994), Shaikh and De (2000), Venkatesha and Bagewadi (2008).

In their paper, DE. U. C. and S. K. Chaubeys (2011) studied the properties of the m-projective curvature tensor in LP-Sasakian, Einstein LP-Sasakian and η-Einstein LP-Sasakian manifolds.

Taleshian A. and Asghari N. (2012) in their paper, investigated the properties of the P-Sasakian manifold equipped with m-projective curvature tensor. An n -dimensional P-Sasakian manifold is a said to be m-projectively flat if $P = 0$, where P is the m-projective curvature tensor.

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation (1986), (1940). A concircular transformation is always a conformal transformation (1986). Blair D., Koufogiorgos T. and Papantoniou B. J. (1995) considered the (k, μ)-nullity condition on a contact metric manifold and introduced the class of contact metric manifolds M with contact metric structures. Papantoniou B. J. (1993) and Perrone D. (1992) included the studies of contact metric manifolds satisfying $R(X, \xi) . S = 0$, where S is the Ricci tensor. Motivated by these studies, Tripathi M. M. and Kim (2004) continued this study and classified (k, μ)-manifolds with concircular curvature tensor Z and Ricci tensor S satisfying $Z(\xi ,X) . S = 0$.

Tripathi M. M.and et al. (2011) introduced the τ -tensor which in particular cases reduces to known curvatures like conformal, concircular and projective curvature tensors and some recently introduced curvature tensors like M-projective curvature tensor,

W$_i$-curvature tensor (i = 0, ..., 9) and W$^*_j$-curvature tensors(j = 0, 1). Tripathi M. M.and et al., (2011), (2012) studied τ -curvature tensor in K-contact, Sasakian and Semi-Riemannian manifolds. Blair D.E., Koufogiogors T. and Papantoniou B.J. (1995) studied the (k, μ)-nullity conditions on a contact metric manifold and gave several examples. The study of (k, μ)-contact manifolds is interesting as it contains both Sasakian and non-Sasakian manifolds.

Motivated by the above studies, Nagaraja H. G. and Somashekhara G. (2012) in their paper studied τ- curvature tensor in (k, μ) manifold. Also studied were τ -flat and a ξ-τ - flat (k, μ)-contact metric manifolds and obtaining conditions for τ -flat (k, μ)-contact metric manifold to be ϕ-symmetric. They considered in their study, ϕ- τ - symmetric and ϕ- τ - Ricci recurrent (k, μ)-contact metric manifolds and (k, μ)-contact metric manifolds satisfying semi-symmetry condition τ .S = 0.

Motivated by these studies, Tripathi M. M. and P. Gupta (2011) studied the τ -curvature tensor in K-contact and Sasakian manifolds. In particular, some properties of quasi-τ-flat, ξ-τ- flat and φ-τ- flat K-contact and Sasakian manifolds were obtained. They gave the necessary and sufficient condition for the K-contact manifold to be ξ-τ- flat under some algebraic condition. Among others, they proved that a compact φ-τ- flat K-contact manifold with regular contact vector field, under an algebraic condition, is a principal S^1-bundle over an almost Kaehler space of constant holomorphic sectional curvature.

The notion of local symmetry of Riemannian manifolds have been weakened by many authors in several ways to the different extent. As a weaker version of local symmetry, Takahashi (1977), introduced the notion of locally ϕ-symmetry on sasakian manifolds. In respect of contact Geometry, the notion of ϕ-symmetry was introduced and studied by Boeckx, Buecken and Vanhecke (1999), with several examples. De (2009) studied the notion of ϕ-symmetry with several examples for Kenmotsu manifolds. Adati and Matsumoto (1977), defined Para-sasakian manifold and special Para-Sasakian manifolds (2008), which are special classes of an almost para contact manifold introduced by sato (1976).

Taleshian A. and Asghari N. (2012) studied Ricci-semi symmetric, ϕ-Ricci semi-symmetric and ϕ-symmetric Lorentzian α-Sasakian manifolds. They Also studied a Lorentzian α-Sasakian manifold satisfying S(X, ξ).R=0.

Pokhariyal and Mishra (1970) have introduced new tensor fields, called W_2 –curvature tensor as

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\big[g(X,Z)QY - g(Y,Z)QX\big]$$

in a Riemannian manifold, and studied their properties. Further, Pokhariyal (1982) has studied some properties of these tensor fields in a Sasakian manifold. Matsumoto, Ianus and Mihai (1986) have studied P-Sasakian manifolds admitting W_2 and E-tensor fields. On the other hand, Ahmet Yildiz and De U. C. (2010) and Venkate-sha, Bagewadi, C. S. and Pradeep Kumar K. T. (2011), have studied these tensor fields in Kenmotsu and Lorentzian para-Sasakian manifolds respectively. Pokhariyal (2001) studied W_2-curvature tensor, its associated symmetric and skew-symmetric tensors in an Einstein Sasakian manifold. Motivated by these studies, De U.C. and Sarkar A. studied and generalized some results of Matsumoto, Ianus and Mihai (1986) to prove that a P-Sasakian manifold is Ricci-semi-symmetric if and only if it is an Einstein manifold.

De. U. C.and Sarkar A. (2009) made a detailed study on P-Sasakian admitting W_2-curvature tenor and established that W_2-symmetric P-Sasakian and a manifold is of constant curvature, hence it is an SP-Sasakian manifold same case with W_2-recurrent P-Sasakian.

Moindi S.K., Pokhariyal G.P. and Nzimbi B.M. (2010) studied W_2-curvature tensor and E- curvature tensor and proved the theorem for W_2 - recurrent P-Sasakian manifolds. Mohit Kumar (2010), carried out the study of W2 -curvature tensor in N(k) –quasi Einstein manifolds. Pradeep Kumar (2012) in his paper made a detailed study on Lorentzian α-Sasakian manifolds satisfying certain conditions on the W_2 - curvature tensor.

Fu¨ sun O¨ Zen Zengin (2011), studied the properties of flat spacetimes under some conditions regarding the $W_2 -$ curvature tensor. Several results were obtained on the geometrical symmetries of this curvature tensor. It has been shown that in a spacetime with $W_2 -$ curvature tensor filled with a perfect fluid, the energy momentum tensor satisfying the Einstein's equations with a cosmological constant is a quadratic conformal Killing tensor. It has also been proved that a necessary and sufficient condition for the energy momentum tensor to be a quadratic Killing tensor is that the scalar curvature of this space must be constant. In a radiactive perfect fluid, it has been proved that the sectional curvature is constant.

Mallick S. and De C. (2014) studied spacetime adimmiting W_2 -curvature tensor and established that a W_2-flat spacetime is conformally flat and hence is of Petrov type O.. They also proved that if the perfect fluid space time with vanishing W_2-curvature tensor obeys Einsteins field equation without cosmological constant, then the spacetime has a vanishing acceleration vector and expansion scalar and hence the perfect fluid behaves like a cosmological constant. In the same they proved that, in a perfect fluid spacetime of constant scalar curvature with divergence-free W_2-curvature tensor, the energy-momentum tensor is of Codazzi type and the possible local cosmological structure of such a spacetime is of type I, D, or O.

Pokhariyal G. P. in 1982 gave a new tensor curvature tensor known as W_5-Curvature tensor as

$$W_5(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\left[g(X,Z)QY - S(X,Z)Y\right]$$

Prakasha D. G., Vasant C. And Kakasab Mirji (2016) established that a $\phi - W_5$ -flat generalized Sasakian -space-form is conformally flat and that it is $\phi - W_5$ - semi-symmetric if and only if it is $W_5 - flat$. Several other authors have made intense study of Riemannian manifolds admitting W_5 -curvature tensor.

CHAPTER 3

3 A STUDY OF W_8 − CURVATURE TENSOR IN K-CONTACT RIEMANNIAN MANIFOLD

3.1 Introduction.

The study is on W_8 − curvature tensor on K-contact manifold. The following geometrical properties of W_8 − curvature tensor are being investigated; flatness, semi-symmetric, symmetric and recurrence on the K-contact manifold.
In section 3.3, the flatness property is being investigated and observed that a W_8 -flat K-contact Riemannian manifold is a flat manifold. Section 3.4 investigates the semisymmetric, section 3.5 the symmetric and section 3.6 the recurrence properties of the curvature tensor on K-contact manifold. The results in the respective sections show that a W_8 − semisymmetric, and symmetric are W_8 − flat manifolds while W_8 − flat and recurrent manifold is a flat manifold.

3.2 Preliminaries

Let (M, ϕ, ξ, η, g) be $n = (2m+1)$ -dimensional almost contact Riemannian manifold consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g.
Then the following results hold

$$\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \quad \eta(\phi X) = 0, \quad (3.2.1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(Y)\eta(X) \quad (3.2.2)$$

where X, and Y are arbitrary vector fields on M
If moreover,

$$g(X, \phi Y) = -g(\phi X, Y)$$
$$g(X, \nabla_Y \xi) = -g(\nabla_X \xi, Y) \Leftrightarrow \nabla_X \xi = -\phi X \quad (3.2.3)$$

then M is a K-contact Riemannian manifold.
Where ∇ denotes the Riemannian connection of g.
In a K-contact manifold the following relations hold:

$$\nabla_X \xi = -\phi X \quad (3.2.4)$$
$$S(X, \xi) = (n-1)\eta(X) \quad (3.2.5)$$
$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \quad (3.2.6)$$

The following statements are true about K-contact manifold. If in an almost contact manifold M^n,

i) $\nabla_X \xi = -\phi X$ (3.2.7)

   then M^n is a K-contact manifold.

ii) $g(X, \nabla_Y \xi) = -g(\nabla_X \xi, Y)$ (3.2.8)

   then M^n is a K-contact manifold .

iii) $g(X, \phi Y) = -g(\phi X, Y)$ (3.2.9)

   then M^n is a K-contact manifold .

iv) is both contact manifold and ξ is a Killing vector, then M^n is a

   K-contact   manifold.

3.3   W$_8$-curvature tensor in K-contact Riemannian manifold

Pokhariyal (1982) gave definition of $W_8 -$ curvature tensor as

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\big[S(X,Y)Z - S(Y,Z)X\big]$$ (3.3.1)

Definition 3.3.1: A K-contact Riemannian manifold M^n is said to be flat if the
  Riemannian curvature tensor vanishes identically, i.e. $R(X,Y)Z = 0$

Definition 3.3 .2: A K-contact Riemannian manifold M^n is said to be W_8 -flat if the
  $W_8 -$ curvature tensor vanishes identically, i.e. $W_8(X,Y)Z = 0$

Theorem 3.3.3: A W_8 -flat K-contact Riemannian manifold is a flat manifold.

Proof; If W_8 -flat

   If our hypothesis is true, then $W_8 = 0$ in

   $$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\big[S(X,Y)Z - S(Y,Z)X\big]$$

   Expanding (3.3.1) with respect to variable U

   $$W_8'(X,Y,Z,U) = R'(X,Y,Z,U) + \frac{1}{n-1}\big[S(X,Y)g(Z,U) - S(Y,Z)g(X,U)\big]$$ (3.3.2)

   Therefore, if K-contact manifold M is $W_8 -$ flat then, we have,

   $$R'(X,Y,Z,U) = \frac{1}{n-1}\big[S(Y,Z)g(X,U) - S(X,Y)g(Z,U)\big]$$ (3.3.3)

   Where, $S(X,Y) = Ric(X,Y) = (n-1)g(X,Y)$

Then, using $S(X,Y) = (n-1)g(X,Y)$ in (3.3.3) we get,

$$R'(X,Y,Z,U) = \frac{n-1}{n-1}\big[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\big]$$

$$R'(X,Y,Z,U) = \big[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\big] \quad (3.3.4)$$

But, in K-contact manifold, we have

$$R'(X,Y,Z,U) = \big[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\big]$$

Referring to (3.3.4) we get

$$\Rightarrow R'(X,Y,Z,U) = \big[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\big]$$

Thus, for this to hold, we must have

$$R'(X,Y,Z,U) = 0 \text{ since,}$$

$$R'(X,Y,Z,U) \neq \big[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\big] \quad (3.3.5)$$

by definition.
This completes the theorem.

Corollary 3.3.4: A $W_8 -$ flat K-contact manifold is neither Einstein nor $\eta - Einstein$ Manifold

Proof:
From **definition 1.2.36**, a manifold is said to be Einstein manifold if $a \neq 0$ and

$b = 0$ in the given Ricci tensor relation

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

and $\eta - Einstein$ if both a and b are none zero.

From the results of (3.3.5) it is clear that Riemannian curvature tensor R is equal to

zero and consequently the Ricci tensor S is also equal to zero.

This therefore, means both a and b are are zero hence, it's neither Einstein nor

$\eta - Einstein$.

3.4 W₈-Semi-symmetric K-contact Riemannian manifold

De and Guha (1992) gave the definition of semisymmetric as

$$R(X,Y)R(Z,U)V = 0 \quad (3.4.1)$$

Definition 3.4.1: A K-contact manifold is said to be $W_8 -$ semisymmetric if

$$R(X,Y)W_8(Z,U)V = 0 \quad (3.4.2)$$

Theorem 3.4.2: A $W_8 -$ semisymmetric K-contact manifold is a $W_8 -$ flat manifold.

Proof:
If the K-Contact manifold is a $W_8 -$ semisymmetric then $R(X,Y)W_8(Z,U)V = 0$

$$R(X,Y)W_8(Z,U)V = g(Y,W_8(Z,U)V)X - g(X,W_8(Z,U)V)Y = 0$$
$$\Rightarrow g(Y,W_8(Z,U)V)X - g(X,W_8(Z,U)V)Y = 0$$
$$\Rightarrow W_8'(Y,Z,U,V)X - W_8'(X,Z,U,V)Y = 0$$
$$\Rightarrow g\big(W_8(Y,Z,U,V)X,\xi\big) - g\big(W_8(X,Z,U,V)Y,\xi\big) = 0$$

$$\Rightarrow W_8'(Y,Z,U,V)\eta(X) - W_8'(X,Z,U,V)\eta(Y) = 0 \qquad (3.4.3)$$

Note, this is only possible if $W_8'(Y,Z,U,V) = 0$ and $W_8'(X,Z,U,V) = 0$ since $\eta(X) \neq 0$ and $\eta(Y) \neq 0$ and thus follows the theorem.

Corollary 3.4.3: A $W_8 -$ semisymmetric K-contact manifold is neither Einstein nor η-Einstein manifold.

Proof:
From **definition 1.2.36**, a manifold is said to be Einstein manifold if $a \neq 0$ and $b = 0$ in the given Ricci tensor relation

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

and $\eta - Einstein$ if both *a and b are none zero* .

From the results of (3.3.5) it is clear that Riemannian curvature tensor R is equal to zero and consequently the Ricci tensor S is also equal to zero.

This therefore, means both *a and b are are zero* hence, it's neither Einstein nor $\eta - Einstein$.

3.5   W8-symmetric K-Contact Riemannian manifold

Chaki and Gupta (1963) gave the definition of a conformally symmetric manifold

as $\nabla_U C = 0$ which is said to be conformally symmetric (where C is conformal curvature tensor).

Definition 3.5.1: A K-contact Riemannian manifold M is said to be $W_8 -$ symmetric if

$$\nabla_U W_8(X,Y)Z = 0 \qquad (3.5.1)$$

Theorem 3.5.2: A $W_8 -$ symmetric and a $W_8 -$ flat K-contact Riemannian manifold is a flat-manifold.

Proof: If the K-contact space is a W_8 — symmetric and W_8 — semisymmetric then it follows

$$0 = R(X,Y)W_8(Z,U)V - W_8(R(X,Y)Z,U)V - W_8(Z,R(X,Y)U)V$$
$$- W_8(Z,U)R(X,Y)V \qquad (3.5.2)$$

Computing each of the above four terms separately yields

$$R(X,Y)W_8(Z,U)V = g(Y,W_8(Z,U)V)X - g(X,W_8(Z,U)V)Y$$
$$= W_8'(Y,Z,U,V)X - W_8'(X,Z,U,V)Y \qquad (3.5.3)$$

which yields

$$g\big(R(X,Y,W_8(Z,U)V,\xi\big) = g\big(W_8'(Y,Z,U,V)X,\xi) - W_8'(X,Z,U,V)Y,\xi)\big)$$
$$= \eta\big(W_8'(Y,Z,U,V)X\big) - \eta\big(W_8'(X,Z,U,V)Y\big)$$
$$= W_8'(Y,Z,U,V)\eta(X) - W_8'(X,Z,U,V)\eta(Y)$$

Again,

$$W_8(R(X,Y)Z,U)V = R(R(X,Y)Z,U)V + \frac{1}{n-1}\big[S(R(X,Y)Z,U)V - S(U,V)R(X,Y)Z\big]$$

$$= R(R(X,Y)Z,U)V + \frac{n-1}{n-1}\big[g(R(X,Y)Z,U)V - g(U,V)R(X,Y)Z\big] \qquad (3.5.4)$$

$$= R(R(X,Y)Z,U)V + \big[g(R(X,Y)Z,U)V - g(U,V)R(X,Y)Z\big]$$

$$= g(U,V)R(X,Y)Z - g(R(X,Y)Z,V)U + \big[R'(X,Y,Z,U)V - g(U,V)R(X,Y)Z\big]$$
$$= R'(X,Y,Z,U)V - g(R(X,Y)Z,V)U$$
$$= R'(X,Y,Z,U)V - R'(X,Y,Z,V)U$$

hence, we have

$$W_8'(R(X,Y)Z,U,V,\xi) = g\big(R'(X,Y,Z,U)V,\xi\big) - g\big(R'(X,Y,Z,V)U,\xi\big)$$
$$= R'(X,Y,Z,U)\eta(V) - R'(X,Y,Z,V)\eta(U)$$

Also,

$$W_8(Z,R(X,Y)U)V = R(Z,R(X,Y)U)V + \frac{1}{n-1}\big[S(Z,R(X,Y)U)V - S(R(X,Y)U,V)Z\big]$$

$$= g(R(X,Y)U,V)Z - g(Z,V)R(X,Y)U + \big[g(Z,R(X,Y)U)V - g(R(X,Y)U,V)Z\big]$$
$$= -g(Z,V)R(X,Y)U + g(Z,R(X,Y)U)V$$
$$= -g(Z,V)R(X,Y)U + R'(X,Y,U,Z)V$$

$$g\big(W_8'(Z,R(X,Y)U)V,\xi\big) = g\big(R'(Z,X,Y,U)V,\xi\big) - g\big(g(Z,V)R(X,Y)U,\xi\big)$$
$$= R'(X,Y,U,Z)\eta(V) - g(Z,V)R'(X,Y,U,\xi)$$

$$(3.5.5)$$

hence, we have

$$W_8(Z,U)R(X,Y)V = R(Z,U)R(X,Y)V + \frac{1}{n-1}\left[S(Z,U)R(X,Y)V - S(U,R(X,Y)V)Z\right]$$

$$= g(U,R(X,Y)V)Z - g(Z,R(X,Y)V)U + \left[g(Z,U)R(X,Y)V - g(U,R(X,Y)V)Z\right]$$

$$= g(Z,U)R(X,Y)V - g(Z,R(X,Y)V)U$$

$$= g(Z,U)R(X,Y)V - R'(X,Y,V,Z)U$$

$$g(W_8(Z,U)R(X,Y)V,\xi) = g(g(Z,U)R(X,Y)V,\xi) - g(R'(Z,X,Y,V)U,\xi)$$

$$= g(Z,U)R'(X,Y,V,\xi) - R'(X,Y,V,Z)\eta(U)$$

$$(3.5.6)$$

Next, we put together (3.5.3), (3.5.4), (3.5.5) and (3.5.6) to have

$$W_8'(Y,Z,U,V)\eta(X) - W_8'(X,Z,U,V)\eta(Y)$$

$$- \{R'(X,Y,Z,U)\eta(V) - R'(X,Y,Z,V)\eta(U)$$

$$+ R'(X,Y,U,Z)\eta(V) - g(Z,V)R'(X,Y,U,\xi)$$

$$+ g(Z,U)R'(X,Y,V,\xi) - R'(X,Y,V,Z)\eta(U)\} = 0$$

$$(3.5.7)$$

Terms which are coefficients of $\eta(V)$ and $\eta(U)$ cancel out since R' is skew-symmetric with respect to the last two variables. Hence, (4.5.7) reduces to

$$W_8'(Y,Z,U,V)\eta(X) - W_8'(X,Z,U,V)\eta(Y) + g(Z,V)R'(X,Y,U,\xi)$$

$$- g(Z,U)R'(X,Y,V,\xi) = 0$$

$$(3.5.8)$$

but it is a $W_8 -$ flat manifold, hence $W_8{}' = 0$

Therefore (3.5.8) reduces to

$$g(Z,V)R'(X,Y,U,\xi) - g(Z,U)R'(X,Y,V,\xi) = 0 \qquad (3.5.9)$$

But in (3.5.9) since,

$$g(Z,U) \neq g(Z,V) \neq 0$$

The above equation (3.5.9) reduces to

$$R' = 0 \qquad (3.5.10)$$

Thus, follows the theorem.

3.6 A W8-Recurrent K-contact Riemannian manifold.

Definition 3. 6.1: A K-contact Riemannian manifold is said to be recurrent if

$$(\nabla_U W_8)(X,Y)Z = B(U)W_8(X,Y)Z \qquad (3.6.1)$$

where B is a non-zero 1-form.

Theorem: 3.6.2: A W_8-recurrent and W_8-flat manifold is a flat manifold.

Proof;

We have

$$(\nabla_U W_8)(X,Y)Z = B(U)W_8(X,Y)Z \quad \text{where } B(U) \neq 0 \qquad (3.6.2)$$

but, if

$$W_8(X,Y)Z = 0$$

Then, (3.6.2) by definition becomes

$$0 = R'(X,Y,Z,U) + \frac{1}{n-1}\big[S(X,Y)g(Z,U) - S(Y,Z)g(X,U)\big] \qquad (3.6.3)$$

$$R'(X,Y,Z,U) = \big[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\big] \qquad (3.6.4)$$

But, for a K-contact manifold

$$R'(X,Y,Z,U) = \big[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\big] \qquad (3.6.5)$$

And this can only be true if and only if (3.6.4) is

$$R'(X,Y,Z,U) = 0 \qquad (3.6.6)$$

Hence, follows the theorem.

CHAPTER 4

4 A STUDY OF $W_8 -$-CURVATURE TENSOR IN SASAKIAN MANIFOLD

4.1 Introduction.

The study in this paper is on $W_8 -$ curvature tensor on Sasakian manifold. The
following geometrical properties of $W_8 -$ curvature tensor are being investigated;
flatness, semi-symmetric, symmetric and recurrence on the Sasakian manifold.
In section 4.3, the flatness property is being investigated and is being observed
that, a W_8 -flat Sasakian manifold is a flat manifold. Section 4.4 investigates the
semisymmetric, section 4.5 the symmetric and section 4.6 the recurrence
properties of the curvature tensor on Sasakian manifold. The results in the
respective sections show that a $W_8 -$ semisymmetric, and symmetric are
$W_8 -$ flat manifolds while $W_8 -$ recurrent manifold (under some set conditions)
is a symmetric and semisymmetric manifold.

4.2 Preliminaries

Let (M, F, T, A, g) be $(2n+1)$ -dimensional almost contact metric manifold
consisting of a (1, 1) tensor field F , a vector field T , a 1-form A and a
Riemannian metric g .
Then

$$\bar{\bar{X}} + X = A(X)T, \ A(T) = 1, \ \bar{T} = 0, \quad A(\bar{X}) = 0, \ \bar{X} \overset{def}{=} F(X), \ \bar{T} \overset{def}{=} F(T) \quad (4.2.1)$$

(Pokhariyal (1988))

$$g(\bar{X}, \bar{Y}) = g(X, Y) - A(X)A(Y) \quad (4.2.2)$$

From (4.2.1) and (4.2.2) we have

$$g(X, \bar{Y}) = -g(\bar{X}, Y) \ \text{ and } \ g(X, T) = A(X) \quad (4.2.3)$$

Where X , and Y are arbitrary vector fields on M
An almost contact metric manifold is a contact metric manifold if
$dA(X, Y) = g(X, \bar{Y})$ and an almost contact metric manifold is a K-Contact
manifold if $\nabla_X T = -\bar{X}$, where ∇ is Levi-Civita connection. An almost contact
metric manifold is a Sasakian manifold if $(\nabla_X F)Y = g(X, Y)T - A(Y)X$
A K-contact manifold is always contact metric manifold, but the converse is true
if $L_\xi F = 0$ that is, if the Lie derivative of F in the characteristic direction ξ
vanishes.
A Sasakian manifold is a K-contact but the converse is only true if dimension is 3.
A contact metric manifold is Sasakian if and only if

$$R(X,Y)T = A(Y)X - A(X)Y \qquad (4.2.4)$$

In Sasakian manifold (M, F, T, A, g), we easily get

$$R(T, X)Y = g(X,Y)T - A(Y)X \qquad (4.2.5)$$

In generally, in $(2n+1)$-dimensional Sasakian manifold with the structure (F, T, A, g), we have

$$\begin{aligned}
R'(X,Y,Z,U) &= g(R(X,Y)Z,U) \\
&= g(\{g(Y,Z)X - g(X,Z)Y\},U) \qquad (4.2.6) \\
&= g(Y,Z)g(X,U) - g(X,Z)g(Y,U)
\end{aligned}$$

where R is the Riemannian curvature tensor and rank(F)$=n-1$.

4.3 W₈-Curvature tensor in Sasakian manifold

Pokhariyal (1982) gave the definition of $W_8 - $ Curvature tensor as

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\big[S(X,Y)Z - S(Y,Z)X\big], \qquad (4.3.1)$$

Where $S(X,Y) = g(QX,Y) = (n-1)g(X,Y) = Ric(X,Y)$, and *Q is the Ricci-operator, that is,* the linear endomorphism of a tangent space at each of its points
Or equivalently,

$$W_8'(X,Y,Z,U) = R'(X,Y,Z,U) + \frac{1}{n-1}\big[S(X,Y)g(Z,U) - S(Y,Z)g(X,U)\big]$$

Definition 4.3.1: A Sasakian manifold M is said to be flat if the Riemannian curvature tensor vanishes identically i.e. $R(X,Y)Z = 0$

Definition 4.3.2: A Sasakian manifold M is said to be $W_8 - $ flat if

$W_8 - $ curvature tensor vanishes identically i.e. $W_8(X,Y)Z = 0$

Theorem 4.3.3. A $W_8 - $ flat Sasakian manifold is a flat manifold.

Proof:
If our hypothesis is true, then $W_8 = 0$ in

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\big[S(X,Y)Z - S(Y,Z)X\big]$$

Or equivalently,

$$W_8'(X,Y,Z,U) = R'(X,Y,Z,U) + \frac{1}{n-1}\big[S(X,Y)g(Z,U) - S(Y,Z)g(X,U)\big]$$

Therefore, if Sasakian manifold M is $W_8 -$ flat then, we have,

$$0 = R'(X,Y,Z,U) + \frac{1}{n-1}\left[S(X,Y)g(Z,U) - S(Y,Z)g(X,U)\right] \qquad (4.3.2)$$

or,

$$R'(X,Y,Z,U) = \frac{1}{n-1}\left[S(Y,Z)g(X,U) - S(X,Y)g(Z,U)\right]$$

Where, $S(X,Y) = Ric(X,Y) = (n-1)g(X,Y)$

Then, using

$$S(X,Y) = (n-1)g(X,Y)$$

In equation (4.3.2), we get

$$R'(X,Y,Z,U) = \frac{n-1}{n-1}\left[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\right]$$

or

$$R'(X,Y,Z,U) = \left[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\right] \qquad (4.3.3)$$

But, in Sasakian manifold, we have

$$R'(X,Y,Z,U) = \left[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\right]$$

From the computations, we get

$$R'(X,Y,Z,U) = \left[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\right]$$

Thus, for this to hold, we must have

$$R'(X,Y,Z,U) = 0 \text{ since,}$$

$$R'(X,Y,Z,U) \neq \left[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\right]$$

Thus, follows the theorem.

4.4   W8-Semi-symmetric Sasakian manifold

De and Guha (1992) gave the definition of semisymmetric as
$R(X,Y)R(Z,U)V = 0$

Definition 4.4.1: A Sasakian manifold is said to be $W_8 -$ semisymmetric if

$$R(X,Y)W_8(Z,U)V = 0 \qquad (4.4.1)$$

Theorem 4.4.2: A $W_8 -$ semisymmetric   Sasakian manifold is a $W_8 -$ flat manifold.

Proof:

If the Sasakian space is a $W_8 -$ semisymmetric then $R(X,Y)W_8(Z,U)V = 0$

$$R(X,Y)W_8(Z,U)V = g(Y,W_8(Z,U)V)X - g(X,W_8(Z,U)V)Y = 0$$

$$\Rightarrow g(Y,W_8(Z,U)V)X - g(X,W_8(Z,U)V)Y = 0$$

$$\Rightarrow W_8'(Y,Z,U,V)X - W_8'(X,Z,U,V)Y = 0$$

$$\Rightarrow g\{W_8'(Y,Z,U,V)X,T\} - g\{W_8'(X,Z,U,V)Y,T\} = 0$$

$$\Rightarrow W_8'(Y,Z,U,V)A(X) - W_8'(X,Z,U,V)A(Y) = 0 \qquad (4.4.2)$$

But since $A(X) \neq 0$ and $A(Y) \neq 0$, then it follows that $W_8'(Y,Z,U,V) = 0$ and

$$W_8'(X,Z,U,V) = 0$$

Hence, the theorem.

4.5   W₈-symmetric Sasakian manifold

Chaki and Gupta (1963) gave the definition of a conformally symmetric manifold as,

$$\nabla_U C = 0$$

(where C is conformal curvature tensor).

Definition 4.5.1: A Sasakian manifold M is said to be $W_8 -$ symmetric if

$$\nabla_U W_8(X,Y)Z = W_8'(U,X,Y)Z = 0. \tag{4.5.1}$$

Theorem 4.5.2: A $W_8 -$ symmetric and $W_8 -$ semisymmetric Sasakian manifold
is a flat manifold.

Proof:

If the Sasakian space is a $W_8 -$ symmetric then it follows

$$\nabla_U W_8(X,Y)Z = R(X,Y)W_8(Z,U)V - W_8(R(X,Y)Z,U)V - W_8(Z,R(X,Y)U)V$$
$$- W_8(Z,U)R(X,Y)V = 0.$$

$$\tag{4.5.2}$$

Computing each of the above four terms and subjecting them to the same conditions equivalently yields,

$$R(X,Y)W_8(Z,U)V = g(Y,W_8(Z,U)V)X - g(X,W_8(Z,U)V)Y$$
$$= W_8'(Y,Z,U,V)X - W_8'(X,Z,U,V)Y \tag{4.5.3}$$

$$g(R(X,Y,W_8(Z,U)V,T) = g(W_8'(Y,Z,U,V)X,T) - W_8'(X,Z,U,V)Y,T))$$
$$= A(W_8'(Y,Z,U,V)X) - A(W_8'(X,Z,U,V)Y)$$
$$= W_8'(Y,Z,U,V)A(X) - W_8'(X,Z,U,V)A(Y).$$

where $R'(X,Y,Z,U) = g(R(X,Y)Z,U)$ and
$$g(R(X,Y)Z,T) = A(R(X,Y)Z) = g(Y,Z)A(X) - g(X,Z)A(Y).$$

Again,

$$W_8(R(X,Y)Z,U)V = R(R(X,Y)Z,U)V + \frac{1}{n-1}\left[S(R(X,Y)Z,U)V - S(U,V)R(X,Y)Z\right]$$

$$= R(R(X,Y)Z,U)V + \frac{n-1}{n-1}\left[g(R(X,Y)Z,U)V - g(U,V)R(X,Y)Z\right] \tag{4.5.4}$$
$$= R(R(X,Y)Z,U)V + \left[g(R(X,Y)Z,U)V - g(U,V)R(X,Y)Z\right]$$

$$= g(U,V)R(X,Y)Z - g(R(X,Y)Z,V)U + \left[R'(X,Y,Z,U)V - g(U,V)R(X,Y)Z\right]$$

$$= R'(X,Y,Z,U)V - g(R(X,Y)Z,V)U$$

$$= R'(X,Y,Z,U)V - R'(X,Y,Z,V)U$$

$$W_8'(R(X,Y)Z,U,V,T) = g\big(R'(X,Y,Z,U)V,T\big) - g\big(R'(X,Y,Z,V)U,T\big)$$

$$= R'(X,Y,Z,U)A(V) - R'(X,Y,Z,V)A(U).$$

Also,

$$W_8(Z,R(X,Y)U)V = R(Z,R(X,Y)U)V + \frac{1}{n-1}\left[S(Z,R(X,Y)U)V - S(R(X,Y)U,V)Z\right]$$

$$= g(R(X,Y)U,V)Z - g(Z,V)R(X,Y)U + \left[g(Z,R(X,Y)U)V - g(R(X,Y)U,V)Z\right]$$

$$= -g(Z,V)R(X,Y)U + g(Z,R(X,Y)U)V$$

$$= -g(Z,V)R(X,Y)U + R'(X,Y,U,Z)V.$$

$$g\big(W_8'(Z,R(X,Y)U)V,T\big) = g\big(R'(Z,X,Y,U)V,T\big) - g\big(g(Z,V)R(X,Y)U,T\big)$$

$$= R'(X,Y,U,Z)A(V) - g(Z,V)R'(X,Y,U,T).$$

$$(4.5.5)$$

$$W_8(Z,U)R(X,Y)V = R(Z,U)R(X,Y)V + \frac{1}{n-1}\left[S(Z,U)R(X,Y)V - S(U,R(X,Y)V)Z\right]$$

$$= g(U,R(X,Y)V)Z - g(Z,R(X,Y)V)U + \left[g(Z,U)R(X,Y)V - g(U,R(X,Y)V)Z\right]$$

$$= g(Z,U)R(X,Y)V - g(Z,R(X,Y)V)U$$

$$= g(Z,U)R(X,Y)V - R'(X,Y,V,Z)U.$$

$$g\big(W_8(Z,U)R(X,Y)V,T\big) = g\big(g(Z,U)R(X,Y)V,T\big) - g\big(R'(Z,X,Y,V)U,T\big)$$

$$= g(Z,U)R'(X,Y,V,T) - R'(X,Y,V,Z)A(U).$$

$$(4.5.6)$$

Next, we put together (4.5.3), 4.5.4), (4.5.5) and (4.5.6) to have

$$W_8'(Y,Z,U,V)A(X) - W_8'(X,Z,U,V)A(Y)$$

$$-\{R'(X,Y,Z,U)A(V) - R'(X,Y,Z,V)A(U)$$

$$+ R'(X,Y,U,Z)A(V) - g(Z,V)R'(X,Y,U,T)$$

$$+ g(Z,U)R'(X,Y,V,T) - R'(X,Y,V,Z)A(U)\} = 0.$$

$$(4.5.7)$$

Terms which are coefficients of $A(V)$ and $A(U)$ cancel out since R' is skew-symmetric with respect to the last two variables. Hence, (4.5.7) reduces to

$$W_8'(Y,Z,U,V)A(X) - W_8'(X,Z,U,V)A(Y) + g(Z,V)R'(X,Y,U,T)$$

$$- g(Z,U)R'(X,Y,V,T) = 0.$$

$$(4.5.8)$$

But since $\nabla_X W_8'(Y,Z,U,V) = 0$ and

$$g(Z,U) \neq g(Z,V) \neq 0 \Rightarrow R'(X,Y,U,T) = 0.$$

Thus, follows the theorem.

Corollary 4.5.3: A W_8 – symmetric Sasakian manifold is always W_8 – semisymmetric Sasakian manifold.

That is, for $\nabla_X W_8'(Y,Z,U,V) = 0 \implies R(X,Y)W_8(Z,U)V = 0$

4.6   W₈-Recurrent Sasakian Manifold.

We study some of the geometrical properties of W_8-curvature tensor on a W_8-recurrent Sasakian manifold M . From (Adati and Miyazawa (1967), DC and Guha (1992), we have

$$\left(\nabla_U W_8\right)(X,Y)Z = B(U)W_8(X,Y)Z \qquad (4.6.1)$$

where B is a non-zero 1-form.

If we consider a Sasakian manifold M which is W_8-recurrent, then we have Pokhariyal (1988)

$$\nabla_U W_8(X,Y)Z = B(U)W_8(X,Y)Z \qquad (4.6.2)$$

where B is a non-zero 1-form and W_8-curvature tensor is given by

$$\begin{aligned}
W_8(X,Y)Z &= R(X,Y)Z + \frac{1}{n-1}\left[S(X,Y)Z - S(Y,Z)X\right] \\
&= R(X,Y)Z + \left[g(X,Y)Z - g(Y,Z)X\right] \\
&= g(Y,Z)X - g(X,Z)Y + \left[g(X,Y)Z - g(Y,Z)X\right] \\
&= g(X,Y)Z - g(X,Z)Y
\end{aligned}$$

and hence,

$$g\big(W_8(X,Y)Z,T\big) = g\big(g(X,Y)Z,T\big) - g\big(g(X,Z)Y,T\big) \qquad (4.6.3)$$

or

$$W_8'(X,Y,Z,T) = A(Z)g(X,Y) - A(Y)g(X,Z)$$

Note, $W_8'(X,Y,Z,T) = 0$ if and only if $A(Z)g(X,Y) - A(Y)g(X,Z) = 0$

Theorem 4.6.1: A W_8-Recurrent Sasakian Manifold with $R(X,Y)W_8(Z,U)V =0$ and $A(Z)g(X,Y) - A(Y)g(X,Z) = 0$ is a W_8-Symmetric and a semisymmetric space

Proof: From (4.6.1), we have

$$\nabla_U W_8(X,Y)Z = B(U)W_8(X,Y)Z$$
$$\begin{aligned}
\implies \nabla_X W_8(Z,U)V &= R(X,Y)W_8(Z,U)V - W_8(R(X,Y)Z,U)V \\
&\quad - W_8(Z,R(X,Y)U)V - W_8(Z,U)R(X,Y)V
\end{aligned} \qquad (4.6.4)$$

But, we are given $R(X,Y)W_8(Z,U)V = 0$ (semisymmetric space).

This implies that, we are left to show that the relation is symmetric under the given conditions.

Therefore, (4.6.4) becomes

$$\nabla_U W_8(X,Y)Z = B(U)W_8(X,Y)Z$$
$$= -W_8(R(X,Y)Z,U)V - W_8(Z,R(X,Y)U)V \qquad (4.6.5)$$
$$- W_8(Z,U)R(X,Y)V$$

Recall from (4.6.3)
$$W_8(X,Y)Z = g(X,Y)Z - g(X,Z)Y$$
Hence, expanding each term of (4.6.5) we get

$$W_8(R(X,Y)Z,U)V = g(R(X,Y)Z,U)V - g(R(X,Y)Z,V)U \qquad (4.6.6)$$
$$= R'(X,Y,Z,U)V - R'(X,Y,Z,V)U$$

Also,
$$W_8(Z,R(X,Y)U)V = g(Z,R(X,Y)U)V - g(Z,V)R(X,Y)U$$
$$= R'(X,Y,U,Z)V - g(Z,V)R(X,Y)U \qquad (4.6.7)$$

Again,

$$W_8(Z,U)R(X,Y)V = g(Z,U)R(X,Y)V - g(Z,R(X,Y)V)U \qquad (4.6.8)$$
$$= g(Z,U)R(X,Y)V - R'(X,Y,V,Z)U$$

Combining (4.6.6), (4.6.7) and (4.6.8) we get
$$\nabla_U W_8(X,Y)Z = B(U)W_8(X,Y)Z$$
$$= -W_8(R(X,Y)Z,U)V - W_8(Z,R(X,Y)U)V$$
$$- W_8(Z,U)R(X,Y)V$$

$$= -\{R'(X,Y,Z,U)V - R'(X,Y,Z,V)U + R'(X,Y,U,Z)V$$
$$- g(Z,V)R(X,Y)U + g(Z,U)R(X,Y)V - R'(X,Y,V,Z)U\} \qquad (4.6.9)$$

Terms which are coefficients of V and U cancel out since R' is skew-symmetric with respect to the last two variables hence, (5.6.9) reduces to
$$\nabla_X W_8(Z,U)V = B(X)W_8(Z,U)V$$
$$= g(Z,V)R(X,Y)U - g(Z,U)R(X,Y)V \qquad (4.6.10)$$

Expanding (4.6.10) gives
$$\nabla_X W_8(Z,U)V = B(X)W_8(Z,U)V$$
$$= g(Z,V)\{g(Y,U)X - g(X,U)Y\} \qquad (4.6.11)$$
$$- g(Z,U)\{g(Y,V)X - g(X,V)Y\}$$

Taking inner product of (4.6.11) with respect to T both sides yields
$$g(\nabla_X W_8(Z,U)V,T) = g(B(X)W_8(Z,U)V,T)$$
$$= g(Z,V)\{g(Y,U)g(X,T) - g(X,U)g(Y,T)\} \qquad (4.6.12)$$
$$- g(Z,U)\{g(Y,V)g(X,T) - g(X,V)g(Y,T)\}$$

Relation (4.6.12) reduces to

$$g\left(\nabla_X W_8(Z,U)V,T\right) = g\left(B(X)W_8(Z,U)V,T\right)$$
$$= g(Z,V)\{g(Y,U)A(X) - g(X,U)A(Y)\}$$
$$- g(Z,U)\{g(Y,V)A(X) - g(X,V)A(Y)\}$$

The coefficients for $g(Z,V)$ and $g(Z,U)$ from the initial given conditions given, are both equal to zero.

Hence,

$$\nabla_X W_8(Z,U)V = B(X)W_8(Z,U)V = 0 \qquad (4.6.13)$$

This completes the theorem.

CHAPTER 5

5 A STUDY OF W_8 − CURVATURE TENSOR IN LP-SASAKIAN MANIFOLD

5.1 Introduction.

The Paper studies on W_8 − curvature tensor on LP-Sasakian manifold. The following geometrical properties of W_8 − curvature tensor are being investigated; flatness, semi-symmetric, symmetric and recurrence on the LP-Sasakian manifold.

In section 5.3, the flatness property is being investigated and is being observed that, a W_8 -flat LP-Sasakian manifold is a flat manifold. Section 5.4 investigates the semisymmetric, section 5.5 the symmetric and section 5.6 the recurrence properties of the curvature tensor on LP-Sasakian manifold. The results in the respective sections show that a W_8 − semisymmetric, and symmetric are W_8 − flat manifolds while W_8 − recurrent manifold (under some set conditions) is a symmetric and semisymmetric manifold.

5.2 Preliminaries

An $n−$dimensional real differentiable manifold M_n is said to be Lorentzian para (LP)-LP-Sasakian manifold if it admits a $(1,1)$ tensor field F, a C^∞ vector field T, a C^∞ 1-form A and a Lorentzian metric g which satisfy (1972):

$$A(T) = -1 \quad , g(X, T) = A(X) \tag{5.2.1}$$

$$\overline{\overline{X}} = X + A(X)T, \quad \overline{X} \overset{def}{=} F(X), \quad \overline{T} = F(T) \tag{5.2.2}$$

$$g(\overline{X}, \overline{Y}) = g(X, Y) + A(X)A(Y), \tag{5.2.3}$$

$$g(X, Y) = A(X), \quad D_X T = \overline{X}, \tag{5.2.4}$$

$$(D_X F)Y = g(X, Y)T + A(Y)X + 2A(X)A(Y)T \tag{5.2.5}$$

Where D_X denotes the covariant differentiation with respect to g and X, and Y are any arbitrary vector fields on *M*.

In an LP-Sasakian manifold M_n, with structure (F, T, A, g), it can be seen that (Pokhariyal (1996))

$$\overline{T} = 0, \quad A(\overline{X}) = 0 \tag{5.2.6}$$

$rank(F) = n - 1$ (5.2.7)

If we put

$$F'(X,Y) = g(\overline{X},Y) \qquad (5.2.8)$$

then, the tensor field $F'(X,Y)$ is symmetric in X and Y, thus, we have

$$F'(X,Y) = F'(Y,X)$$

In an n-dimensional LP-Sasakian manifold with the structure (F,T,A,g), we have

$$
\begin{aligned}
R'(X,Y,Z,U) &= g(R(X,Y)Z,U) \\
&= g(\{g(Y,Z)X - g(X,Z)Y\},U) \qquad (5.2.9) \\
&= g(Y,Z)g(X,U) - g(X,Z)g(Y,U)
\end{aligned}
$$

Again, putting $U = T$, relation (5.2.9) becomes

$$R'(X,Y,Z,T) = g(Y,Z)g(X,T) - g(X,Z)g(Y,T) \qquad (5.2.10)$$

Using (5.2.1), relation (6.2.10) yields

$$
\begin{aligned}
R'(X,Y,Z,T) &= g(Y,Z)A(X) - g(X,Z)A(Y) \\
R'(X,Y,Z,T) &= A(R(X,Y)Z) \Leftrightarrow g(R(X,\ Y)Z,\ T) = A(R(X,\ Y)Z)
\end{aligned} \qquad (5.2.11)
$$

where $g(X,Y)Z$ is the metric tensor representing potential ,

$$
\begin{aligned}
S(X,Y) &= Ric(X,Y) \\
&= g(QX,Y) \\
&= (n-1)g(X,Y) \qquad (5.2.12)
\end{aligned}
$$

is the Ricci tensor representing the matter tensor,

$$
\begin{aligned}
S(T,T) &= (n-1)g(T,T) \\
&= (n-1)A(T) \\
&= -(n-1) \qquad (5.2.13)
\end{aligned}
$$

R' is the (0, 4) curvature tensor, and $S(X,Y) = Ric(X,Y)$ is the Ricci tensor.

5.3   W8-Curvature Tensor in LP-Sasakian Manifold

Pokhariyal (1982) gave the definition of W_8-curvature tensor as

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(X,Y)Z - S(Y,Z)X] \qquad (5.3.1)$$

or

$$W_8'(X,Y,Z,U) = R'(X,Y,Z,U) + \frac{1}{n-1}[S(X,Y)g(Z,U) - S(Y,Z)g(X,U)]$$

Definition 5.3.1: An LP-Sasakian manifold M_n is said to be flat if Riemannian-curvature tensor vanishes identically, i.e. $R(X,Y)Z = 0$.

Definition 5.3.2: An LP-Sasakian manifold M is said to be W_8-flat if W_8-curvature tensor vanishes identically, i.e. $W_8(X,Y)Z = 0$.

Theorem 5.3.3: A W_8-flat LP-Sasakian manifold is a flat manifold.

Proof.

If LP-Sasakian manifold is W_8-flat then $W_8 = 0$ in (5.3.1) and we have

$$0 = R(X,Y)Z + \frac{1}{n-1}\big[S(X,Y)Z - S(Y,Z)X\big]$$

Similarly, in equation

$$W_8'(X,Y,Z,U) = R'(X,Y,Z,U) + \frac{1}{n-1}\big[S(X,Y)g(Z,U) - S(Y,Z)g(X,U)\big]$$

If LP-Sasakian manifold M is W_8 – flat then, we have,

$$0 = R'(X,Y,Z,U) + \frac{1}{n-1}\big[S(X,Y)g(Z,U) - S(Y,Z)g(X,U)\big]$$

Which implies

$$R'(X,Y,Z,U) = \frac{1}{n-1}\big[S(Y,Z)g(X,U) - S(X,Y)g(Z,U)\big]$$

Where, $S(X,Y) = g(QX,Y) = (n-1)g(X,Y)$

Then, using $S(X,Y) = (n-1)g(X,Y)$ in the above equation, we have,

$$R'(X,Y,Z,U) = \frac{n-1}{n-1}\big[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\big]$$

which implies

$$R'(X,Y,Z,U) = \big[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\big]$$

But, in LP-Sasakian manifold, we have

$$R'(X,Y,Z,U) = \big[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\big]$$

$$\Rightarrow R'(X,Y,Z,U) \neq \big[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\big]$$

But from the computations, we get

$$\Rightarrow R'(X,Y,Z,U) = \big[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\big]$$

This can only be true if and only if $R'(X,Y,Z,U) = 0 \Rightarrow S(X,Y) = 0$ (5.3.2)

Hence, follows the theorem, a W_8-flat LP-Sasakian manifold is a flat manifold.

Corollary 5.3.4: A W_8-flat LP-Sasakian manifold is neither Einstein nor $n-$Einstein manifold.

Proof:

From **definition 5.3.5**, a manifold is said to be Einstein manifold   if $a \neq 0$ and

$b = 0$ in the given Ricci tensor (S) relation

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

and $\eta - Einstein$ if both *a and b are none zero*.

From the results of (5.3.2) it is clear that S is also equal to zero. This therefore, means both *a and b are are zero* hence, it's neither Einstein nor $\eta - Einstein$.

5.4 W_8- **Semi-symmetric LP-Sasakian manifold**

De and Guha (1992) gave the definition of semisymmetric as
$$R(X,Y)R(Z,U)V = 0$$

Definition 5.4.1: A LP-Sasakian manifold is said to be W_8- semisymmetric if
$$R(X,Y)W_8(Z,U)V = 0$$

Theorem 5.4.2: W_8- semisymmetric LP-Sasakian manifold is a W_8- flat manifold.

Proof:
If the LP-Sasakian space is a W_8- semisymmetric then

$$R(X,Y)W_8(Z,U)V = 0 \qquad\qquad (5.4.3)$$
$$R(X,Y)W_8(Z,U)V = g(Y,W_8(Z,U)V)X - g(X,W_8(Z,U)V)Y = 0$$
$$\Rightarrow g(Y,W_8(Z,U)V)X - g(X,W_8(Z,U)V)Y = 0$$
$$\Rightarrow W_8'(Y,Z,U,V)X - W_8'(X,Z,U,V)Y = 0$$
$$\Rightarrow g\big(W_8(Y,Z,U,V)X,T\big) - g\big(W_8(X,Z,U,V)Y,T\big) = 0$$
$$\Rightarrow W_8'(Y,Z,U,V)A(X) - W_8'(X,Z,U,V)A(Y) = 0$$

Note, this is only possible if $W_8'(Y,Z,U,V) = 0$ and $W_8'(X,Z,U,V) = 0$ since $A(X) \neq 0$ and $A(Y) \neq 0$ and thus follows the theorem 5.4.2.

5.5   **W8-symmetric LP- Sasakian manifold**

Definition 5.5.1: An LP-Sasakian manifold is said to be W_8- symmetric if
$$\nabla_U W_8(X,Y)Z = W_8'(U,X,Y)Z = 0 \qquad\qquad (5.5.1)$$

Theorem 5.5.2: A W_8- symmetric and W_8- semisymmetric LP-Sasakian manifold is a flat manifold.

Proof:
From the theorem 5.4.2, we found out that a W_8- semisymmetric LP- Sasakian manifold is a W_8- flat manifold and if LP-Sasakian space is a W_8- symmetric this implies,

$$\nabla_U W_8(X,Y)Z = R(X,Y)W_8(Z,U)V) - W_8(R(X,Y)Z,U)V$$
$$- W_8(Z,R(X,Y)U)V - W_8(Z,U)R(X,Y)V = 0 \qquad (5.5.2)$$

Computing each of the above four terms separately and subjecting them to equivalent conditions gives,

$$R(X,Y)W_8(Z,U)V) = g(Y,W_8(Z,U)V)X - g(X,W_8(Z,U)V)Y$$
$$= W_8'(Y,Z,U,V)X - W_8'(X,Z,U,V)Y \qquad (5.5.3)$$

$$g\big(R(X,Y,W_8(Z,U)V,T\big) = g\big(W_8'(Y,Z,U,V)X,T) - W_8'(X,Z,U,V)Y,T)\big)$$
$$= A\big(W_8'(Y,Z,U,V)X\big) - A\big(W_8'(X,Z,U,V)Y\big)$$
$$= W_8'(Y,Z,U,V)A(X) - W_8'(X,Z,U,V)A(Y)$$
where $R'(X,Y,Z,U) = g(R(X,Y)Z,U)$ and
$$g\big(R(X,Y)Z,T\big) = A(R(X,Y)Z) = g(Y,Z)A(X) - g(X,Z)A(Y)$$

Again,
$$W_8(R(X,Y)Z,U)V = R(R(X,Y)Z,U)V + \frac{1}{n-1}\big[S(R(X,Y)Z,U)V - S(U,V)R(X,Y)Z\big]$$
$$(5.5.4)$$
$$= R(R(X,Y)Z,U)V + \frac{n-1}{n-1}\big[g(R(X,Y)Z,U)V - g(U,V)R(X,Y)Z\big]$$
$$= R(R(X,Y)Z,U)V + \big[g(R(X,Y)Z,U)V - g(U,V)R(X,Y)Z\big]$$

$$= g(U,V)R(X,Y)Z - g(R(X,Y)Z,V)U + \big[R'(X,Y,Z,U)V - g(U,V)R(X,Y)Z\big]$$
$$= R'(X,Y,Z,U)V - g(R(X,Y)Z,V)U$$
$$= R'(X,Y,Z,U)V - R'(X,Y,Z,V)U$$
$$W_8'(R(X,Y)Z,U,V,T) = g\big(R'(X,Y,Z,U)V,T\big) - g\big(R'(X,Y,Z,V)U,T\big)$$
$$= R'(X,Y,Z,U)A(V) - R'(X,Y,Z,V)A(U)$$

Also,
$$W_8(Z,R(X,Y)U)V = R(Z,R(X,Y)U)V + \frac{1}{n-1}\big[S(Z,R(X,Y)U)V - S(R(X,Y)U,V)Z\big]$$
$$= g(R(X,Y)U,V)Z - g(Z,V)R(X,Y)U$$
$$+ \big[g(Z,R(X,Y)U)V - g(R(X,Y)U,V)Z\big]$$

$$= -g(Z,V)R(X,Y)U + g(Z,R(X,Y)U)V$$
$$= -g(Z,V)R(X,Y)U + R'(X,Y,U,Z)V$$
$$g\big(W_8'(Z,R(X,Y)U)V,T\big) = g\big(R'(Z,X,Y,U)V,T\big) - g\big(g(Z,V)R(X,Y)U,T\big)$$

$$= R'(X,Y,U,Z)A(V) - g(Z,V)R'(X,Y,U,T) \qquad (5.5.5)$$

44

We also observe that,

$$W_8(Z,U)R(X,Y)V = R(Z,U)R(X,Y)V + \frac{1}{n-1}\left[S(Z,U)R(X,Y)V - S(U,R(X,Y)V)Z\right]$$
$$= g(U,R(X,Y)V)Z - g(Z,R(X,Y)V)U$$
$$+\left[g(Z,U)R(X,Y)V - g(U,R(X,Y)V)Z\right]$$
$$= g(Z,U)R(X,Y)V - g(Z,R(X,Y)V)U$$
$$= g(Z,U)R(X,Y)V - R'(X,Y,V,Z)U$$

$$g\big(W_8(Z,U)R(X,Y)V,T\big)= g\big(g(Z,U)R(X,Y)V,T\big)- g\big(R'(Z,X,Y,V)U,T\big)$$
$$= g(Z,U)R'(X,Y,V,T) - R'(X,Y,V,Z)A(U) \tag{5.5.6}$$

Next, we put together (5.5.3), (5.5.4), (5.5.5) and (5.5.6) to have
$$W_8'(Y,Z,U,V)A(X) - W_8'(X,Z,U,V)A(Y)$$
$$-\{R'(X,Y,Z,U)A(V) - R'(X,Y,Z,V)A(U)$$
$$+ R'(X,Y,U,Z)A(V) - g(Z,V)R'(X,Y,U,T) \tag{5.5.7}$$
$$+ g(Z,U)R'(X,Y,V,T) - R'(X,Y,V,Z)A(U)\} = 0$$

Terms which are coefficients of $A(V)$ and $A(U)$ cancel out since R' is skew-symmetric with respect to the last two variables. Hence, (5.5.7) reduces to
$$W_8'(Y,Z,U,V)A(X) - W_8'(X,Z,U,V)A(Y) + g(Z,V)R'(X,Y,U,T)$$
$$- g(Z,U)R'(X,Y,V,T) = 0 \tag{5.5.8}$$

But since $\nabla_X W_8'(Y,Z,U,V) = 0$ and
$$g(Z,U) \neq g(Z,V) \neq 0 \Rightarrow R'(X,Y,U,T) = 0$$
Thus, follows the theorem.

5.6   W₈-Recurrent LP-Sasakian Manifold.

In this section, we study some of the geometrical properties of W_8-curvature tensor   which is recurrent on LP-Sasakian manifold M.

Definition 5.6.1: If we consider an LP-Sasakian manifold M which is W_8-recurrent, then we have (Pokhariyal (1996)),

$$\nabla_U W_8(X,Y)Z = B(U)W_8(X,Y)Z \tag{5.6.1}$$

where B is a non-zero 1-form and W_8-curvature tensor is given by

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\left[S(X,Y)Z - S(Y,Z)X\right] \tag{5.6.2}$$
$$= g(X,Y)Z - g(X,Z)Y$$
$$g\big(W_8(X,Y)Z,T\big)= g\big(g(X,Y)Z,T\big)- g\big(g(X,Z)Y,T\big)$$
$$W_8'(X,Y,Z,T) = A(Z)g(X,Y) - A(Y)g(X,Z)$$

Theorem 5.6.2: A W_8-Recurrent Sasakian Manifold with $R(X,Y)W_8(Z,U)V = 0$ and $A(Z)g(X,Y) - A(Y)g(X,Z) = 0$ is a W_8-Symmetric space

Proof: From definition 5.6.1, we have

$$\nabla_U W_8(X,Y)Z = B(U)W_8(X,Y)Z$$

$$\Rightarrow \nabla_X W_8(Z,U)V = R(X,Y)W_8(Z,U)V - W_8(R(X,Y)Z,U)V$$
$$- W_8(Z,R(X,Y)U)V - W_8(Z,U)R(X,Y)V \quad (5.6.3)$$

But, we are given $R(X,Y)W_8(Z,U)V = 0$ (semisymmetric space).

This implies that, we are left to show that the relation is symmetric under the stated conditions.

Therefore, (5.6.3) becomes

$$\nabla_U W_8(X,Y)Z = B(U)W_8(X,Y)Z$$

$$= -W_8(R(X,Y)Z,U)V - W_8(Z,R(X,Y)U)V \quad (5.6.4)$$
$$- W_8(Z,U)R(X,Y)V$$

Recall from (5.6.2)

$$W_8(X,Y)Z = g(X,Y)Z - g(X,Z)Y$$

Hence, expanding each term of (5.6.4) we get

$$W_8(R(X,Y)Z,U)V = g(R(X,Y)Z,U)V - g(R(X,Y)Z,V)U \quad (5.6.5)$$
$$= R'(X,Y,Z,U)V - R'(X,Y,Z,V)U.$$

Also,

$$W_8(Z,R(X,Y)U)V = g(Z,R(X,Y)U)V - g(Z,V)R(X,Y)U$$
$$= R'(X,Y,U,Z)V - g(Z,V)R(X,Y)U. \quad (5.6.6)$$

Again we have,

$$W_8(Z,U)R(X,Y)V = g(Z,U)R(X,Y)V - g(Z,R(X,Y)V)U \quad (5.6.7)$$
$$= g(Z,U)R(X,Y)V - R'(X,Y,V,Z)U.$$

Combining (5.6.5), (5.6.6) and (5.6.7) we get

$$\nabla_U W_8(X,Y)Z = B(U)W_8(X,Y)Z$$
$$= -\{R'(X,Y,Z,U)V - R'(X,Y,Z,V)U + R'(X,Y,U,Z)V$$
$$- g(Z,V)R(X,Y)U + g(Z,U)R(X,Y)V - R'(X,Y,V,Z)U\}. \quad (5.6.8)$$

Terms which are coefficients of V and U cancel out since R' is skew-symmetric with respect to the last two variables hence, (5.6.8) becomes

$$\nabla_X W_8(Z,U)V = B(X)W_8(Z,U)V$$
$$= g(Z,V)R(X,Y)U - g(Z,U)R(X,Y)V. \quad (5.6.9)$$

Expanding (5.6.9) gives

$$\nabla_X W_8(Z,U)V = B(X)W_8(Z,U)V$$
$$= g(Z,V)\{g(Y,U)X - g(X,U)Y\} \qquad (5.6.10)$$
$$- g(Z,U)\{g(Y,V)X - g(X,V)Y\}.$$

Taking inner product of (5.6.10) with respect to T both sides yields
$$g(\nabla_X W_8(Z,U)V,T) = g(B(X)W_8(Z,U)V,T)$$
$$= g(Z,V)\{g(Y,U)g(X,T) - g(X,U)g(Y,T)\} \qquad (5.6.11)$$
$$- g(Z,U)\{g(Y,V)g(X,T) - g(X,V)g(Y,T)\}.$$

Relation (5.6.11) reduces to

$$g(\nabla_X W_8(Z,U)V,T) = g(B(X)W_8(Z,U)V,T)$$
$$= g(Z,V)\{g(Y,U)A(X) - g(X,U)A(Y)\}$$
$$- g(Z,U)\{g(Y,V)A(X) - g(X,V)A(Y)\}.$$

The coefficients for $g(Z,V)$ and $g(Z,U)$ from the initial given conditions given, are both equal to zero. Hence,

$$\nabla_X W_8(Z,U)V = B(X)W_8(Z,U)V = 0$$
$$(5.6.12)$$

and thus follows the theorem.

CHAPTER 6

6   A STUDY OF W_8- CURVATURE TENSOR ON GENERALIZED SASAKIAN SPACE FORMS

6.1   Introduction.

The Paper is on W_8- curvature tensor on generalized Sasakian space-forms. The following geometrical properties of W_8- curvature tensor are being investigated; flatness, and geometrical properties of the three choice functions (f_1, f_2, f_3) making up the structure of the space.

In section 6.4.2, the flatness property has been investigated and observed that, a W_8-flat generalized Sasakian space-form satisfying

$$f_3 = f_1 = \frac{3f_2}{1-2n}$$

is a W_8-flat space.

In section 6.4.4, it has been established that a W_8-flat generalized Sasakian space form is a flat space.

Section 6.4.6 investigates the choice of the functions satisfying the condition $W_8(X,Y)\xi = 0$ and the following has been observed;

A generalized Sasakian space-form $M(f_1, f_2, f_3)$ satisfying the condition

$$W_8(X,Y)\xi = 0.$$

Results shows that, this can only be so only if

$$f_3 = 2nf_1 + 3f_2.$$

6.2   Preliminaries

A Sasakian manifold $M(\phi, \xi, \eta, g),$ is said to be a Sasakian-space form if all the $\phi-$ sectional curvatures $K(X \wedge \phi X)$ are equal to a constant $c,$ where $K(X \wedge \phi X)$ denotes the sectional curvature of the section spanned by the unit vector field $X,$ orthogonal to ξ and $\phi X.$ In such a case, the Riemannian curvature tensor of M is given by

$$R(X,Y)Z = \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\}$$

$$+ \frac{c-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}$$

$$+ \frac{c-1}{4}\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}$$

$$(6.2.1)$$

These spaces can be modeled depending on $c > -3$, $c = -3$, $c < -3$.

As a natural generalization of these manifolds, Alegre, Blair and Carriazo (2004) introduced and studied the notion of generalized Sasakian space forms. They replaced constant quantities $\frac{c+3}{4}$ and $\frac{c-1}{4}$ of relation (6.2.1) by differentiable functions, f_1, f_2, and f_3.

An almost contact metric manifold $M(\phi,\xi,\eta,g)$ is said to be a generalized Sasakian space –form if the curvature tensor R is given by (2004)

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\}$$
$$+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \quad (6.2.2)$$
$$+ f_3\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}$$

where f_1, f_2, and f_3 are differentiable functions on M and X,Y,Z are vector fields on M. In such a case the manifold is denoted by $M(f, f_2, f_3)$.

6.3 Generalized Sasakian space forms.

In an almost contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$, where ϕ is a (1, 1) tensor field, ξ is a contravariant vector field, η is a 1-form and g is the compatible Riemannian metric, we have by D. E. Blair (1976);

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \qquad (6.3.1)$$

$$g(X,\xi) = \eta(X), \qquad\qquad (6.3.2)$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad\qquad (6.3.3)$$
$$g(\phi X, X) = 0 \qquad\qquad (6.3.4)$$

$$g(\phi X,Y) = -g(X,\phi Y), \qquad\qquad (6.3.5)$$

$$\left(\nabla_X \eta\right)(Y) = g(\nabla_X \xi, Y) \tag{6.3.6}$$

In a $(2n+1)$ – dimensional generalized Sasakian space-form the following relations hold

$$\eta(R(X,Y)Z) = (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \tag{6.3.7}$$

$$\eta(R(X,Y)\xi) = 0 \tag{6.3.8}$$

$$\eta(R(\xi,X)Y) = (f_1 - f_3)\{g(X,Y) - \eta(X)\eta(Y)\} \tag{6.3.9}$$

$$R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} \tag{6.3.10}$$

$$R(\xi,Y)Z = (f_1 - f_3)\{g(Y,Z)\xi - \eta(Z)Y\} \tag{6.3.11}$$

$$g(R(\xi,X)Y,\xi) = (f_1 - f_3)\{g(\phi X, \phi Y)\} \tag{6.3.12}$$

$$R(\xi,X)\xi = (f_1 - f_3)\phi^2 X \tag{6.3.13}$$

$$R(\xi,X)\xi = (f_1 - f_3)\{\eta(X)\xi - X\} \tag{6.3.14}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y) \tag{6.3.15}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X) \tag{6.3.16}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y) \tag{6.3.17}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3 \tag{6.3.18}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n+1)f_3)\eta(X)\xi \tag{6.3.19}$$
$$Q\xi = 2n(f_1 - f_3) \tag{6.3.20}$$

Where Q is the Ricci operator, that is, $g(QX,Y) = S(X,Y)$

Here, S is the Ricci tensor and r is the scalar curvature of the space form.

It is well known from definition 1.2.36. that, a generalized Sasakian space form of dimension $(2n+1)$ with condition $(n > 1)$ is $\eta - Einstein$ space -form if its Ricci tensor S satisfies the condition;

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y). \tag{6.3.21}$$

For arbitrary vector fields X and $Y,$ where a and b are smooth functions on $M.$

From (6.3.15) and (6.3.13) we have

$$a = \left(2nf_1 + 3f_2 - f_3\right) \text{ and } b = -\left\{3f_2 + (2n-1)f_3\right\}.$$ (6.3.22)

Also from (7.2.1) we get

$$R(X,Y)\xi = \left(f_1 - f_3\right)\left[\eta(Y)X - \eta(X)Y\right].$$ (6.3.23)

Where $\left(\eta(Y)X - \eta(X)Y\right) = T(X,Y)$ from definition 1.2.35. is known as the torsion tensor which is non-zero for spaces admitting semisymmetric metric connections. This therefore, follows that for a flat manifold,

$$\left(f_1 - f_3\right) = 0.$$ (6.3.24)

6.4   W$_8$-Curvature Tensor in a Generalized-Sasakian space- form

Pokhariyal (1982) gave the definition of W_8-curvature tensor as

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\left[S(X,Y)Z - S(Y,Z)X\right].$$

Definition 6.4.1: A generalized Sasakian space form M of dimension $(2n+1)$ is said to be W_8-flat if W_8-curvature tensor vanishes identically, that is,

$$W_8(X,Y)Z = 0.$$

Definition 6.4.2: A generalized Sasakian space form M of dimension $(2n+1)$ is said to be flat if the Riemannian-curvature tensor vanishes identically, that is,

$$R(X,Y)Z = 0.$$

Theorem 6.4.3: If a $(2n+1)-$dimensional generalized Sasakian space-form $M(f,f_2,f_3)$ is W_8-flat, then

$$f_3 = f_1 = \frac{3f_2}{1-2n} \ .$$

Proof.
If generalized Sasakian space form is W_8-flat, then $W_8(X,Y)Z = 0$

Therefore,

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\left[S(X,Y)Z - S(Y,Z)X\right]$$

$$= R(X,Y)Z + \frac{1}{n-1}\left[a\{g(X,Y)Z - g(Y,Z)X\} + b\{\eta(X)\eta(Y)Z - \eta(Y)\eta(Z)X\}\right]$$

$$(6.4.1)$$

Taking inner product of (6.4.1) with V we get,

$$g(W_8(X,Y)Z,V) = g(R(X,Y)Z,V) + \frac{1}{n-1}\left[g(S(X,Y)Z,V) - g(S(Y,Z)X,V)\right]$$

$$(6.4.2)$$

$$= g(R(X,Y)Z,V) + \frac{1}{n-1}\left[(S(X,Y)g(Z,V)) - (S(Y,Z)g(X,V))\right]$$

Putting $V = \xi$ and with a vanishing $W_8 - $ curvature tensor vanishing we get

$$0 = \eta(R(X,Y)Z) + \frac{1}{n-1}\left[(S(X,Y)\eta(Z)) - (S(Y,Z)\eta(X))\right]$$

$$(6.4.3)$$

$$0 = f_1\{g(Y,Z)\eta(X) - g(X,Z)(Y)\} + f_3\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}$$

$$+ \frac{1}{n-1}\left[a\{g(X,Y)\eta(Z) - g(Y,Z)\eta(X)\} + b\{\eta(X)\eta(Y)\eta(Z) - \eta(Y)\eta(Z)\eta(X)\}\right]$$

$$0 = f_1\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} + f_3\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}$$

$$+ \frac{1}{n-1}\left[a\{g(X,Y)\eta(Z) - g(Y,Z)\eta(X)\}\right]$$

which implies,

$$0 = (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} + \frac{1}{n-1}\left[a\{g(X,Y)\eta(Z) - g(Y,Z)\eta(X)\}\right] \quad (6.4.4)$$

Putting $Y = \xi$ in (6.4.4) gives

$$0 = (f_1 - f_3)(\eta(Z)\eta(X) - g(X,Z)\xi). \quad\quad\quad\quad (6.4.5)$$

Since $(\eta(Z)\eta(X) - g(X,Z)\xi) \neq 0$ for a general Sasakian space-form, it implies that,

$$f_1 = f_3. \quad\quad\quad\quad\quad\quad\quad\quad (6.4.6)$$

Again, if instead we put $Z = \xi$ in (7.4.4) for a $W_8 - $ flat generalized Sasakian space form, we shall have

$$0 = (f_1 - f_3)\{\eta(Y)\eta(X) - \eta(X)\eta(Y)\} + \frac{1}{n-1}\left[a\{g(X,Y)\xi - \eta(Y)\eta(X)\}\right].$$

52

$$\Rightarrow 0 = [a\{g(X,Y)\xi - \eta(Y)\eta(X)\}].$$

$$0 = (2nf_1 + 3f_2 - f_3)(g(X,Y)\xi - \eta(Y)\eta(X)). \qquad (6.4.7)$$

Since $(g(X,Y)\xi - \eta(Y)\eta(X)) \neq 0$ for a generalized Sasakian space- form, it implies that,
$$0 = (2nf_1 + 3f_2 - f_3)$$
which yields,
$$f_3 = (2nf_1 + 3f_2). \qquad (6.4.8)$$
Putting (6.4.6) into (6.4.8) gives

$$(1 - 2n)f_3 = 3f_2$$
$$\Rightarrow f_1 = f_3 = \frac{3f_2}{1 - 2n}.$$

$$(6.4.9)$$

This completes the proof of the theorem.

Theorem 6.4.4: A W_8-flat generalized Sasakian space-form is a flat manifold.
Proof
From (6.3.10), that is

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y].$$
it is clear that when the generalized Sasakian space form is W_8 – flat as from (6.4.6) the relation (6.4.8) reduces to

$$R(X,Y)\xi = (0)[\eta(Y)X - \eta(X)Y]$$
$$R(X,Y)\xi = 0 \Rightarrow R(X,Y)Z = 0. \qquad (6.4.10)$$
Thus, the theorem.

Definition 6.4.5: A manifold is said to be $\xi - T - flat$ (2011) if
$$T(X,Y)\xi = 0.$$
Where T – is a curvature tensor.

Definition 6.4.6: A generalized Sasakian space-form $M(f_1, f_2, f_3)$ of dimension $(2n + 1)$ is said to be $\xi - W_8$ – flat if

$$W_8(X,Y)\xi = 0. \qquad (6.4.11)$$

Theorem 6.4.7:.If a $(2n + 1)$ –dimensional generalized Sasakian space-form $M(f_1, f_2, f_3)$ satisfies the condition $W_8(X,Y)\xi = 0$, then,

$$f_3 = 2nf_1 + 3f_2.$$

Proof. Suppose the condition $W_8(X,Y)\xi = 0$ holds in a $(2n+1)$–dimensional generalized Sasakian space-form. Then using (6.3.1) and (6.3.2) in (6.4.1) we get

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\left[S(X,Y)Z - S(Y,Z)X\right]$$

$$= R(X,Y)Z + \frac{1}{n-1}\left[a\{g(X,Y)Z - g(Y,Z)X\} + b\{\eta(X)\eta(Y)Z - \eta(Y)\eta(Z)X\}\right].$$

(6.4.12)

Putting $Z = \xi$ in (6.4.12) we get,

$$W_8(X,Y)\xi = R(X,Y)\xi + \frac{1}{n-1}\left[a\{g(X,Y)\xi - \eta(Y)X\} + b\{\eta(X)\eta(Y)\xi - \eta(Y)X\}\right] \quad (6.4.13)$$

$$\eta(W_8(X,Y)\xi) = \eta\big((f_1 - f_3)(\eta(Y)X - \eta(X)Y)\big)$$

$$+ \frac{1}{n-1}\left[a(g(X,Y) - \eta(Y)\eta(X)) + b(\eta(X)\eta(Y) - \eta(Y)\eta(X))\right].$$

And hence,

$$\eta(W_8(X,Y)\xi) = 0 + \frac{1}{n-1}\left[a(g(X,Y) - \eta(Y)\eta(X)) + 0\right].$$

But, if the space is a $\xi - W_8 -$ flat, then $W_8(X,Y)\xi = 0$.

Therefore,

$$0 = a\{g(X,Y) - \eta(Y)\eta(X)\}. \tag{6.4.14}$$

For a generalized Sasakian space-form $\{g(X,Y) - \eta(Y)\eta(X)\} \neq 0$, therefore $a = 0$

From (6.3.20), relation (6.4.14) becomes,

$$a = 2nf_1 + 3f_2 - f_3 = 0$$

$$\Rightarrow f_3 = 2nf_1 + 3f_2. \tag{6.4.15}$$

Hence, the proof of the theorem.

Corollary 6.4.8: A W_8-flat $(2n+1)$–dimensional generalized Sasakian space-form $M(f_1, f_2, f_3)$ has a vanishing scalar curvature tensor ($r = 0$).

Proof.
From (6.4.9) it is clear that for $W_8 -$ flat $(2n+1)$–dimensional generalized Sasakian space-form $M(f_1, f_2, f_3)$,

$$f_1 = f_3 = \frac{3f_2}{1 - 2n}.$$

Given the scalar curvature tensor (6.3.18) as

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3$$

Using (6.4.9) in (6.3.18) yields

$$r = 2n(2n+1)f_1 + \frac{6n(1-2n)f_1}{3} - 4nf_1$$

$$r = f_1\left[4n^2 + 2n + 2n - 4n^2 - 4n\right] \qquad (6.4.16)$$

$$r = 0.$$

Which proves the statement.

CHAPTER 7

7   A Study On W₆-Curvature Tensors and W₈-Curvature Tensors in Kenmotsu manifolds admitting semi-symmetric metric connection.

7.1.   Introduction.

The study is on W_8 – curvature tensor W_6 – flat Kenmotsu manifold with respect to semi-symmetric metric and W_8 – curvature tensor with respect to Levi-Civita connection. The geometrical relationship between the two is being investigated. In section 7.3, started by investigating the W_6 – flatness with respect to semi-symmetric connection on Kenmotsu manifold and established that a W_6 – flat Kenmotsu manifold is not a flat manifold. In section 7.4, a geometrical relationship between W_8 and W_6 – curvature tensors along the geodesic and semi-symmetric metric connections respectively. In deed an a relationship in geometrical equivalence between the two cases was observed in this manifold.

7.2.   Preliminaries

In 1924, Friendmann and Schouten introduced the idea of semi-symmetric linear connection on differentiable manifold. Hayden (1932) introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of the semi-symmetric metric connection on a Riemannian manifold was published by Yano (1970). After that the properties of semi-symmetric metric connection have been studied by many authors like Amur and Pujar (1978), Bagewadi (1982), Shardfuddin and Hussain (1976), De and Pathak (2002) and others.

In 1971, Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions. We call it Kenmotsu manifold. Kenmotsu manifolds have been studied by many authors such as J. B. Jun. U. C. De and G. Pathak (2005) and others.

7.3.   Kenmotsu manifolds

A smooth $n-$dimensional manifold $\left(M^n, g\right)$ is said to be almost contact metric manifold if it admits a (1, 1)-tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g which satisfy

$$\phi^2 X = -X + \eta(X)\xi, \tag{7.3.1}$$

$$g(X,\xi) = \eta(X), \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta(\phi X) = 0 \tag{7.3.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{7.3.3}$$

where X, and Y are arbitrary vector fields on M.

An almost contact manifold $M^n(\phi, \eta, \xi, g)$ is said to be a Kenmotsu manifold if the following conditions hold:

$$\left(\nabla_X \phi\right)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{7.3.4}$$

$$\nabla_X \xi = X - \eta(X)\xi \tag{7.3.5}$$

where ∇ is the Levi-Civita connection .

In Kenmotsu manifold the following relations are true;

$$\left(\nabla_X \eta\right)Y = g(X, Y) - \eta(X)\eta(Y) \tag{7.3.6}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X \tag{7.3.7}$$

$$R(\xi, X)Y = -R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi \tag{7.3.8}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \tag{7.3.9}$$

$$S(X, \xi) = -(n-1)\eta(X), \qquad Q\xi = -(n-1)\xi \tag{7.3.10}$$

for arbitrary vector fields X, Y and Z on M and R is Riemannian curvature tensor and S the Ricci tensor of type (0,2) and Q the Ricci operator such that

$$S(X, Y) = g(QX, Y) = -(n-1)g(X, Y) \tag{7.3.11}$$

A linear connection $\tilde{\nabla}$ in a Riemannian manifold M is said be a semisymmetric metric connection, (C. Ozgur (2010)), if its tensor T of the connection $\tilde{\nabla}$

$$T(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] \tag{7.3.12}$$

satisfies

$$T(X, Y) = \eta(Y)X - \eta(X)Y \tag{7.3.13}$$

where η is 1-form and ξ is the vector field given by
$$g(X, \xi) = \eta(X) \tag{7.3.14}$$
for all vector fields $X, Y \in \chi(M)$. Here $\chi(M)$ is the set of all differentiable vector fields on M .

A semi-symmetric connection $\tilde{\nabla}$ is called a semi-symmetric metric connection, if it further satisfies

$$\tilde{\nabla}g = 0 \qquad (7.3.15)$$

A relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ has been given by K. Yano (1970) which is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \qquad (7.3.16)$$

where $\eta(Y) = g(Y,\xi)$.

7.4 A W_6 – Curvature Tensor of a Kenmotsu manifold with respect to semisymmetric metric connection.

Pokhariyal (1982) have introduced new tensor fields W and studied their properties . Pokhariyal (1982) defined the W_6-curvature tensor field in Riemannian manifolds as

$$W_6(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\big[g(X,Y)QZ - S(Y,Z)X\big]. \qquad (7.4.1)$$

Definition 7.4.1: The W_6- curvature tensor in Kenmotsu manifold with respect to Levi-Civita connection ∇ is given by

$$W_6(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\big[g(X,Y)QZ - S(Y,Z)X\big].$$

Definition 7.4.2: The W_6-curvature tensor in Kenmotsu manifold with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is defined by

$$\tilde{W}_6(X,Y)Z = \tilde{R}(X,Y)Z + \frac{1}{n-1}\big[g(X,Y)\tilde{Q}Z - \tilde{S}(Y,Z)X\big]. \qquad (7.4.2)$$

Definition 7.4.3: In Kenmotsu manifolds, a relation between the curvature tensor R and $\tilde{R}$ of type (1, 3) of the connections ∇ and $\tilde{\nabla}$ respectively is given by I. Gurupadavva (2012)

$$
\begin{aligned}
\tilde{R}(X,Y)Z &= R(X,Y)Z + [g(X,Z)Y - g(Y,Z)X] \\
&\quad + 2[g(\phi X,\phi Z)Y - g(\phi Y,\phi Z)X] \\
&\quad + [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi]
\end{aligned} \qquad (7.4.3)
$$

From (7.4.3) it follows that

$$\tilde{S}(X,Y) = S(X,Y) - (n-1)g(X,Y) - 2(n-2)g(\phi Y,\phi Y). \qquad (7.4.4)$$

where $\tilde{S}$ denotes the Ricci tensor with respect to semi-symmetric metric connection and S the Ricci tensor with respect to Levi-Civita connection.

Definition 7.4.4: A semi-symmetric metric connection in a manifold is said to be flat if the Riemannian curvature tensor with respect to the connection vanishes that is,
$$\tilde{R}(X,Y)Z = 0.$$

Definition 7.4.5: A semi-symmetric metric connection in a manifold is said to be $W_6 -$ flat if the W_6-curvature tensor with respect to the connection vanishes that is,
$$\tilde{W}_6(X,Y)Z = 0.$$

Theorem 7.4.6: A $W_6 -$ flat Kenmotsu manifold with respect to semi-symmetric metric connection is not flat connection.

Proof: From (7.4.2) we have,

$$\tilde{W}_6(X,Y)Z = \tilde{R}(X,Y)Z + \frac{1}{n-1}\left[g(X,Y)\tilde{Q}Z - \tilde{S}(Y,Z)X\right].$$

Expanding (7.4.2) with respect to U yields
$$\tilde{W}_6'(X,Y,Z,U) = \tilde{R}'(X,Y,Z,U) + \frac{1}{n-1}\left[g(X,Y)\tilde{S}(Z,U) - \tilde{S}(Y,Z)g(X,U)\right]. \quad (7.4.5)$$

Suppose (7.4.5) is $W_6 -$ flat, then (7.4.5) reduces to

$$0 = \tilde{R}'(X,Y,Z,U) + \frac{1}{n-1}\left[g(X,Y)\tilde{S}(Z,U) - \tilde{S}(Y,Z)g(X,U)\right].$$

$$\tilde{R}'(X,Y,Z,U) = \frac{1}{n-1}\left[\tilde{S}(Y,Z)g(X,U) - g(X,Y)\tilde{S}(Z,U)\right]. \quad (7.4.6)$$

Putting $Z = \xi$ in (7.4.6) gives

$$\tilde{R}'(X,Y,\xi,U) = \frac{1}{n-1}\left[\tilde{S}(Y,\xi)g(X,U) - g(X,Y)\tilde{S}(\xi,U)\right]. \quad (7.4.7)$$

From (7.4.4), we have

$$\tilde{S}(Y,\xi) = -(n-1)g(Y,\xi) - (n-1)g(Y,\xi) \quad (7.4.8)$$

$$= -2(n-1)\eta(Y).$$

$$\tilde{S}(\xi,U) = -(n-1)g(\xi,U) - (n-1)g(\xi,U) \quad (7.4.9)$$
$$= -2(n-1)\eta(U).$$

Therefore (7.4.7) becomes

$$\tilde{R}'(X,Y,\xi,U) = 2g(X,Y)\eta(U) - 2g(X,U)\eta(Y) \neq 0. \qquad (7.4.10)$$

Hence, $R(X,Y)Z \neq 0$.

And thus, the proof of the theorem.

7.5 Geometrical relationship between W₆-curvature tensors and W₈-curvature tensors in Kenmotsu manifold admitting semi-symmetric metric connection.

The following theorem is a statement of the findings of the study investigating some of the geometrical relationships between curvature tensors.

Definition 7.5.1: Pokhariyal (1982) defined the W_8-curvature tensor field in Riemannian manifolds as

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(X,Y)Z - S(Y,Z)X]. \qquad (7.5.1)$$

Definition 7.5.2: Two geometrical objects are said to be geometrically equivalent if they are linearly dependent.

Theorem 7.5.3: A W_6-curvature tensor on a semi-symmetric metric connection is geometrically equivalent to W_8-curvature tensor along the Levi-Civita connection in a Kenmotsu manifold.

Proof:

The W_6-curvature tensor in Kenmotsu manifold with respect to semi-symmetric metric connection is given by (7.4.2) and expands with respect to U as

$$\tilde{W}_6{}'(X,Y,Z,U) = \tilde{R}'(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)\tilde{S}(Z,U) - \tilde{S}(Y,Z)g(X,U)]. \qquad (7.5.2)$$

Putting $Z = \xi$ in (7.5.2) gives

$$\tilde{W}_6{}'(X,Y,\xi,U) = \tilde{R}'(X,Y,\xi,U) + \frac{1}{n-1}[g(X,Y)\tilde{S}(\xi,U) - \tilde{S}(Y,\xi)g(X,U)]. \qquad (7.5.3)$$

But from (7.4.3) we have

$$\tilde{R}'(X,Y,\xi,U) = R'(X,Y,\xi,U) + [g(X,\xi)g(Y,U) - g(Y,\xi)g(X,U)] \\ + [g(Y,\xi)\eta(X)\eta(U) - g(X,\xi)\eta(Y)\eta(U)]. \qquad (7.5.4)$$

Equation (7.5.4) reduces to

$$\tilde{R}'(X,Y,\xi,U) = R'(X,Y,\xi,U) + [\eta(X)g(Y,U) - \eta(Y)g(X,U)]. \qquad (7.5.5)$$

Again, using (7.4.4) and (7.5.3) expression for $\tilde{S}$ can be easily made as
$$\tilde{S}(\xi,U) = -2(n-1)\eta(U) \qquad (7.5.6)$$
and

$$\tilde{S}(Y,\xi) = -2(n-1)\eta(Y). \qquad (7.5.7)$$

Hence, substituting $\tilde{S}$ and $\tilde{R}$ through (7.5.5), (7.5.6) and (7.5.7) into (7.5.2) gives

$$\tilde{W}_6{}'(X,Y,\xi,U) = R'(X,Y,\xi,U) + [\eta(X)g(Y,U) - \eta(Y)g(X,U)] \\ + 2[g(X,U)\eta(Y) - g(X,Y)\eta(U)]. \qquad (7.5.8)$$

Interchanging U and ξ makes the tensors skew symmetric. Therefore, dividing by minus one yields;

$$\tilde{W}_6{}'(X,Y,U,\xi) = R'(X,Y,U,\xi) - [\eta(X)g(Y,U) - \eta(Y)g(X,U)] \\ - 2[g(X,U)\eta(Y) - g(X,Y)\eta(U)]. \qquad (7.5.9)$$

Contracting (7.5.9) with respect to ξ gives

$$\tilde{W}_6(X,Y)U = R(X,Y)U - [g(Y,U)X - g(X,U)Y] \\ - 2[g(X,U)Y - g(X,Y)U]. \qquad (7.5.10)$$

Putting $Z = U$ in (7.5.10) gives

$$\tilde{W}_6(X,Y)Z = R(X,Y)Z - [g(Y,Z)X - g(X,Z)Y] \\ - 2[g(X,Z)Y - g(X,Y)Z]. \qquad (7.5.11)$$

Simplifying (7.5.11) becomes

$$\tilde{W}_6(X,Y)Z = 2[g(X,Z)Y - g(Y,Z)X] - 2[g(X,Z)Y - g(X,Y)Z]. \qquad (7.5.12)$$

Therefore, (7.5.12) reduces to

$$\tilde{W}_6(X,Y)Z = 2[g(X,Y)Z - g(Y,Z)X]. \qquad (7.5.13)$$

This Curvature tensor in a Kenmotsu manifold with respect to Levi-Civita connection has the given geometrical properties expressed in the expansion of (7.5.1) below.

$$W_8(X,Y)Z = R(X,Y)Z - [g(X,Y)Z - g(Y,Z)X]. \qquad (7.5.14)$$

Expanding (7.5.14) with respect to U becomes

$$W_8(X,Y,Z,U) = R(X,Y,Z,U) - \left[g(X,Y)g(Z,U) - g(Y,Z)g(X,U)\right]. \qquad (7.5.15)$$

Putting $Z = \xi$ in (7.5.15) gives

$$W_8(X,Y,\xi,U) = R(X,Y,\xi,U) - \left[g(X,Y)g(\xi,U) - g(Y,\xi)g(X,U)\right].$$

$$W_8(X,Y,\xi,U) = R(X,Y,\xi,U) - \left[g(X,Y)\eta(U) - g(X,U)\eta(Y)\right]. \qquad (7.5.16)$$

Interchanging U and ξ in (7.5.16) makes the tensors skew symmetric. Therefore, dividing by minus one yields;

$$W_8(X,Y,U,\xi) = R(X,Y,U,\xi) + \left[g(X,Y)\eta(U) - g(X,U)\eta(Y)\right]. \qquad (7.5.17)$$

Contracting (7.5.17) with respect to ξ gives

$$W_8(X,Y)U = R(X,Y)U + \left[g(X,Y)U - g(X,U)Y\right]. \qquad (7.5.18)$$

Replacing U with Z in equation (7.5.18) to get

$$W_8(X,Y)Z = R(X,Y)Z + \left[g(X,Y)Z - g(X,Z)Y\right].$$

$$W_8(X,Y)Z = \left[g(X,Z)Y - g(Y,Z)X\right] + \left[g(X,Y)Z - g(X,Z)Y\right]. \qquad (7.5.19)$$

Equation (7.5.19) simplifies to

$$W_8(X,Y)Z = \left[g(X,Y)Z - g(Y,Z)X\right]. \qquad (7.5.20)$$

Comparing (7.5.20) and (7.5.13)

$$\widetilde{W}_6(X,Y)Z = 2W_8(X,Y)Z. \qquad (7.5.21)$$

This completes the proof of the theorem.

CHAPTER 8

8. APPLICATIONS OF THIS STUDY AND POSSIBLE FUTURE RESEARCH AREAS.

Given the W_8 –curvature tensor

$$W_8'(X,Y,Z,U) = R'(X,Y,Z,U) + \frac{1}{n-1}[S(X,Y)g(Z,U) - S(Y,Z)g(X,U)], \qquad (8.1.1)$$

using $S(X,Y) = (n-1)g(X,Y)$ in (8.1.1), we have,

$$W_8'(X,Y,Z,U) = R'(X,Y,Z,U) + [g(X,Y)g(Z,U) - g(Y,Z)g(X,U)] \qquad (8.1.2)$$

putting

$$Z = X, \quad \text{and} \quad U = Y.$$

Equation (8.1.2) reduces to

$$W_8'(X,Y,X,Y) = R'(X,Y,X,Y) + [g(X,Y)g(X,Y) - g(Y,X)g(X,Y)]$$

$$\Rightarrow W_8'(X,Y,X,Y) = R'(X,Y,X,Y) = 0 \qquad (8.1.3)$$

Richard Hamilton has successfully classified the Ricci solitons on 3-dimension by considering Weyl's tensor together with one of the three tensors, that is Bach (B_{ij}), Cotton tensor (C_{ijk}) and three index tensor D_{ijk}.

All of them when contracted with respect to a pair of indices vanish.

That is,

$$W_{iik} = C_{kii} = B_{ii} = 0$$

Note:

i).   Ricci solitons are solutions of the evolution equations of Hamilton's.

ii).   The Riemannian manifold (M_n, g) is said to be of gradient Ricci soliton if there exist a smooth function f such that

$$Ric + \Delta f = \lambda f \, .$$

These Ricci solitons can be classified into three categories namely;

$$\lambda > 0,$$
$$\lambda = 0$$
$$\lambda < 0.$$

Note, when $\lambda > 0,$ solitons are said to be shrinking, when $\lambda = 0$ they are said to be steady and when $\lambda < 0$ they are expanding. All these have been considered on spaces of dimensions $2, 3$ and for $n \geq 4$.

Since the contracted part of W_8 –curvature tensor vanishes, then this curvature tensor can be used to replace Weyl's projective curvature tensor in classifying the gradient Ricci solitons on 3-dimension Riemannian manifold and also dimensions $n \geq 4$.

Again, since projective curvature tensor W was used to classify the nature of gravitational waves and it was found out that whenever its divergence was vanishing $(div\, W = 0)$ then the electromagnetic field was classified to be purely electrical. And since the $div\, W_8 = 0$, then it follows that Pirani formalism of gravitational waves can be extended using W_8-Curvature tensor.

From the study on W_8 –curvature tensors on the stated manifolds, it can be seen that the curvature tensor allows us to tell mathematically whether the space is flat or if curved, how much curvature takes place in any given surface of the manifold. This has been done by use of covariant derivative. Therefore, the established theorems in the W_8-flatness could be of great input in the field of general relativity.

From the generated theorems on W_8-symmetric and semi-symmetric properties on the said manifolds, it comes out clearly that this curvature tensor could be used in general relativity to analyze spacetime symmetries which are infinitesimally generated by vector fields that preserve some features of spacetime. The most common and evident is its symmetrical properties on LP-Sasakian, K-contact and Sasakain where the symmetry vector fields included the Killing vector field (ξ) which preserve the metric structure of the manifold.

Similarly, the results and further study on this tensor field on LP-Sasakian manifolds could make good contribution in the field of *Regge calculus*. *Regge calculus* is a formalism which chops up

Lorentzian manifold into discrete "chunk" (four dimensional *simplical blocks*) and the block edge lengths are taken as the basic variables.

A discrete version of the *Einstein-Hilbert action* is obtained by considering the so-called deficit angles of these blocks, a zero deficit angle corresponding to no curvature. This is equivalent to the already generated property of W_8 - flatness on the manifolds studied. This could be a novel field for future research with a view to establishing to what extent the two different versions of approach are equivalent.

In general relativity, it has been noted that under fairly generic conditions, gravitation collapse will inevitably result in a so-called singularity. A singularity is a point which the solution to the equations become infinite indicating that the theorem has been probed at an inappropriate ranges. Hence, future research should embark on finding out if such ranges exist in the W_8-flatness, symmetry, semi-symmetry and recurrence properties on the various manifolds. This could open room to investigate singularities arising in black holes space time, Bamunda triangle mystery, and in the neutron stars.

Finally, intriguing results were realized in the last chapter on studying the geometrical relationship between W_8 –curvature tensor acting along a geodesic and that of W_6 –curvature tensor along a semi-symmetric metric connection of a Kenmotsu manifold. The results indicated that they produced equivalent geometrical appearance when subjected to the respective connections on this manifold.

The results suggest strongly that these two curvature tensors have principle features in general relativity and could be an alternative method to solutions of geodesic equations. This would make a great contribution in determining paths of particles and radiations in gravitational fields. This too relates very well to the study on total matter (energy) distribution to the curvature of the space time.

This leaves a broad area of research where these two curvature tensor fields on manifolds such as the Kenmotsu could be alternative tool to solutions in the field equations and geodesic equations. This might open a new way of describing Einstein field equations which describe how mass and energy are related to the curvature of space time.

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APPENDICES